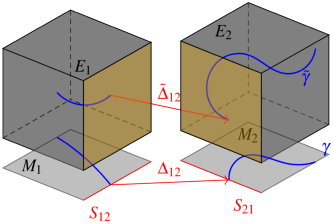
How do we walk?

Hybrid holonomy concepts



ADS, April 2022

Motivation

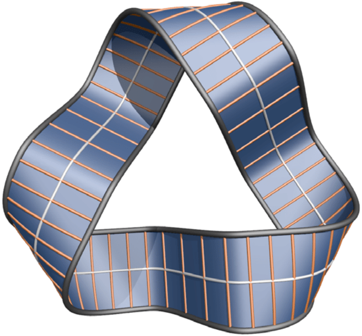
1. Periodic motion with a twist...



2. Collaboration, internal vs. external


Some math

A bundle (E, π, M) with $\pi : E \to M$ surjective, fibers $F_p := \pi^{-1}(p)$



With some add-ons

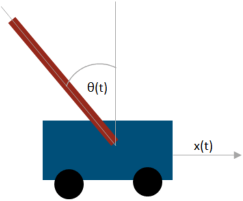
- principal bundle: $F_p \approx G$ Lie group. (assume isomorphic to $(\mathbb{R}^n, +)$)
- make it hybrid

How it works

Internal configuration $\implies$ many possible external configurations.

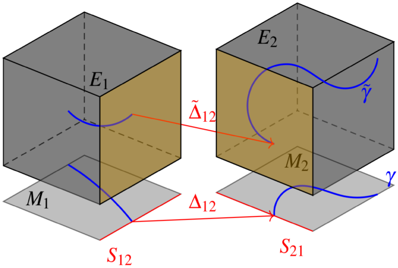
Example

Pendulum on a 1D cart:
$E = \mathbb{R} \times S^1$, $M = S^1$



Hybrid case

$(E_i, \pi_i, M_i, S_{ij}, D_{ij})$, with $D_{ij} = (\Delta_{ij}, \tilde{\Delta}_{ij})$



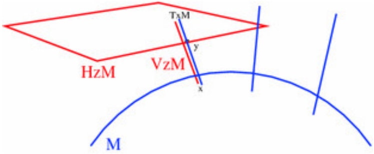
Example

1D walking robot: $E_1 = E_2 = \mathbb{R} \times S^1$, $M_1 = M_2 = S^1$,
$S_{12} = S_{21} = \frac{\delta}{2}$, $\Delta_{12} = -Id$.

Connections

How can we quantify "movement"? velocities $\in$ tangent spaces TM, TE.

- Specify a horizontal space $H_x M$ such that $T_x M = H_x M \oplus V_x M$
- Specify an infinitesimal horizontal direction $\omega : T_x M \to T_e G$.

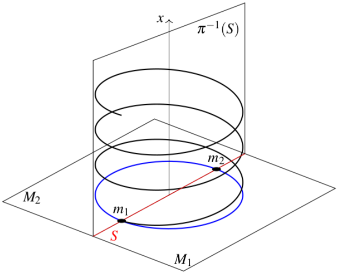


$$\omega = g^{-1} dg + A(m, g) dm$$

Lifts

Internal $\rightarrow$ external motion according to ω

1. Smooth case $\tilde{\gamma} : [0,1] \rightarrow E$ such that $\pi(\tilde{\gamma}) = \gamma$
2. Hybrid case $\tilde{\gamma}(t) = \tilde{\gamma_{i(n}}(t)$ for $t \in (i(n), i(n+1))$ where $\tilde{\gamma_{i(n)}}$ is the smooth lift through $\Delta_{i(n-1)i(n)}(\gamma(t_{i(n-1)}))$



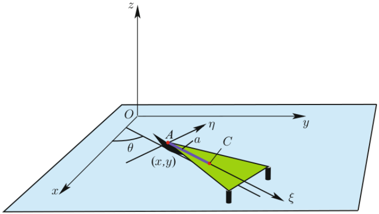
Connections from constraints

Allowable motion directions and velocities are given by the constraints.

Example

Chaplygin sleigh:

- constraint:
  $\dot{y}\cos\theta - \dot{x}\sin\theta = 0$
- Horizontal space
  $\xi = (\theta, x, y, \dot{\theta}, \dot{x}, \dot{y}) \in H_{\theta,x,y}$
  if $\omega(\xi) = 0$ where
  $\omega = \cos\theta dy - \sin\theta dx$

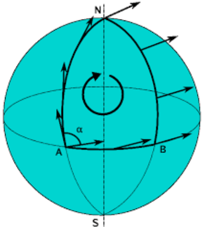


Holonomy group

- what is internal and external ✓
- what can we change ✓
- how are we allowed to move ✓

The big question

How much movement can we generate in the total space, by periodic motion in the base space? $\implies$ Find $Hol_m(\gamma)$.



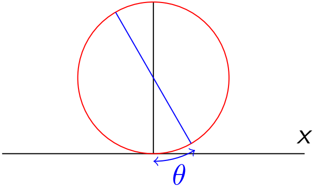
Holonomic vs non-holonomic

Periodic motion in the base $\implies$ net displacement in the fiber

$\iff$

$$Hol_m = \{g_\gamma | \gamma : [0,1] \to M, g_\gamma = \tilde{\gamma}(1) - \tilde{\gamma}(0)\}$$

- No net movement $\iff$ $Hol_m = 0$ $\iff$ holonomic systems
  <u>ex:</u> One leg robot: $\omega = dx - l\cos\theta d\theta$
- Positive displacement $\iff$ $Hol_m \neq 0$ $\iff$ non-holonomic system
  <u>ex:</u> Rolling coin: $\omega = dx - Rd\theta$



Hybrid holonomy

What is the holonomy group of a hybrid system?

$$\omega = g^{-1}dg + A(m,g)dm \implies g^{-1}dg(X_{hor}) = A(m,g)dm(X_{hor})$$

Some assumptions

- $A(m,g)dm = dF(m)$ exact
- G abelian
- $S = \cap S_{ij} \subset M_i \ \forall \ i$, $\Delta_{ij} = Id$.

Smooth:

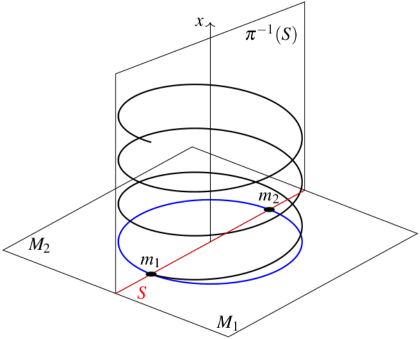
$$Hol_m = \int_{\tilde{\gamma}} g^{-1}dg = \int_0^1 dF(\gamma'(t))dt = F(\gamma(1)) - F(\gamma(0))$$

BUT γ is hybrid $\implies$ sum all the individual parts

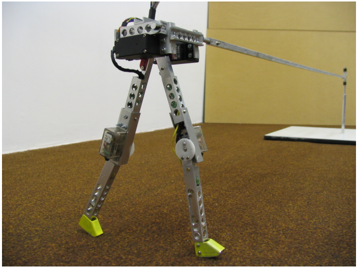
$$Hol_m = \sum_{k=0}^{m} F_{i(k)}(\gamma(t_{i(k+1)})^-) - F_{i(k)}(\gamma(t_{i(k)})^+)$$

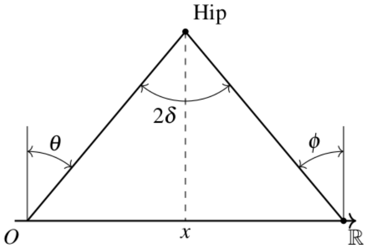
Some remarks

- $\gamma(t_{(i(k))}) \in S \implies$ Holonomy only depends on what happens on the guard!
- No term includes tangent vectors of the lifted space
- Exactness $\iff$ holonomic $\iff$ integrable
- Independent of the loop!



Taking the limit of a walking robot





- Variables: hip position x, angles of feet with the vertical θ, ϕ
- Structure of a hybrid bundle $E_1 = E_2 = \mathbb{R} \times S^1$, $M_1 = M_2 = S^1$, $S = \frac{\delta}{2}$, $\Delta(\frac{\delta}{2}) = -\frac{\delta}{2}$, $\tilde{\Delta}(x) = x$

Some more ingredients

- Connection coming from constraints: $\begin{cases} \dot{x} = l\cos\theta d\theta \\ \dot{x} = l\cos\phi d\phi \end{cases} \Longrightarrow$

$$\omega_1 = dx - \cos\theta d\theta, \quad \omega_2 = dx - \cos\phi d\phi$$

- Holonomy group for each leg individually $= 0$
- $\omega_{1,2}$ exact and $\cos\theta d\theta = d(\sin\theta)$, $\cos\phi d\phi = d(\sin\phi)$.

Question:

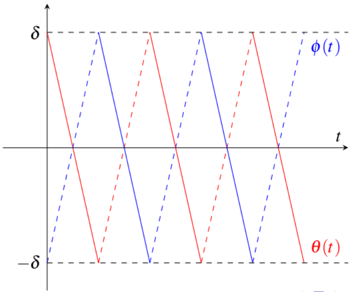
What is the holonomy group of the entire system?

Follow the recipe

1. Set a point in the base space $\theta = \delta/2$, $\phi = -\delta/2$
2. Pick a nice loop

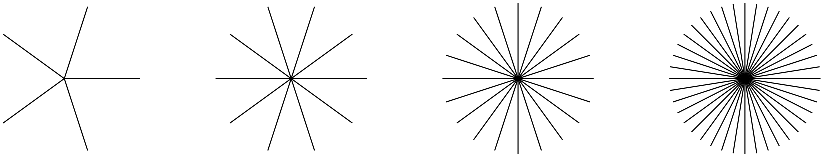
$$\theta(t) = \begin{cases} \delta - 4\delta t, & \text{if } t < \frac{1}{2} \\ -\delta + 4\delta\left(t - \frac{1}{2}\right), & \text{if } t \geq \frac{1}{2} \end{cases}$$

and define $\gamma(t) = \begin{cases} \theta(t), & \text{if } t \leq \frac{1}{2} \\ -\theta(t)), & \text{if } t > \frac{1}{2} \end{cases}$



Keep going...

3. Integrate to find the local holonomy $\Delta x = 4lN \sin \delta/2$, $N =$ number of steps
4. Take limit keeping $N\delta/2 \to C$.
5. Result $\Delta x \to lC \iff$ infinitesimally $dx = ldC$.



The future

- A fully general hybrid system where $\Delta_{ij} \neq Id$
- How about the dynamics? Changes the admissible loops
- what if the forms are not exact?
- What if the Lie group is not abellian?
- Some cool applications – Any ideas?