Optimality of Zeno executions for Hybrid Systems

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Hybrid Systems

$$\begin{cases} \dot{x} = f(x), & x \in M \setminus S \\ x^+ = \Delta(x^-), & x \in S \end{cases}$$

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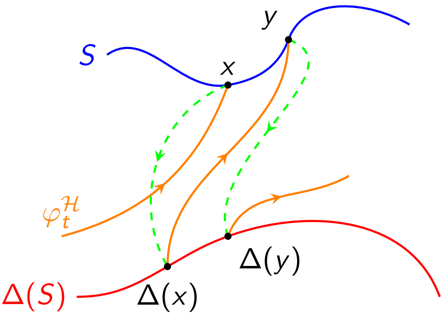

Hybrid dynamical system (M, S, f, Δ)

Zeno

Definition

Let $\varphi_t^{\mathcal{H}}$ be a hybrid flow. A point $x \in M$ has a Zeno trajectory if there exists an increasing sequence of times $\{t_i\}_{i=1}^{\infty}$ such that $\varphi_{t_i}^{\mathcal{H}}(x) \in S$ for all i and $t_i \to t_\infty < \infty$.

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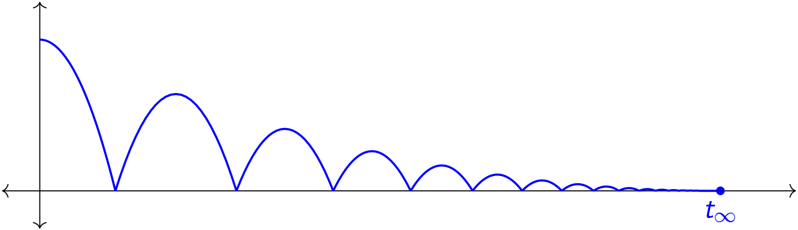


Hybrid control system

Definition

A hybrid control system is a 5-tuple $(M, \mathcal{U}, S, X, \Delta)$ where

- $M \approx$ state space
- $\mathcal{U} \approx$ admissible controls
- $X : M \times \mathcal{U} \to TM \approx$ vector field
- $S \subset M \approx$ guard $\approx h^{-1}(0)$
- $\Delta : S \to M \approx$ reset

Optimal control

Goal: find $u(\cdot) \in \mathcal{F}(\mathcal{U})$ that minimizes

$$J(x_0, u(\cdot)) = \int_0^{T_f} \ell(x(s), u(s))\, ds,$$

subject to $\begin{cases} \dot{x} = X(x, u), x \in M/S \\ x^+ = \Delta(x^-), x \in S \end{cases}$

and boundary conditions $x(0) = x_0$ and $x(T_f) = x_f$

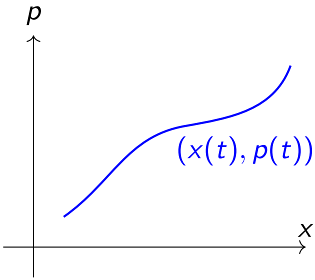
Hybrid maximum principle

Continuous part

$$\hat{H}(x,p) = \min_u H(x,p,u) = \min_u \ell(x,u) + \langle p, f(x,u) \rangle$$

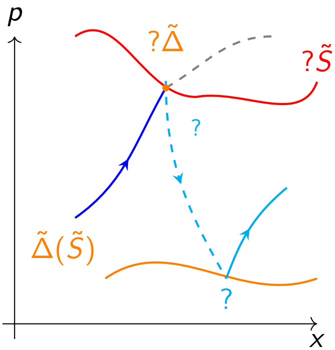
Hamilton's e.o.m. :

$$\dot{x} = \frac{\partial \hat{H}}{\partial p}, \quad \dot{p} = -\frac{\partial \hat{H}}{\partial x}$$



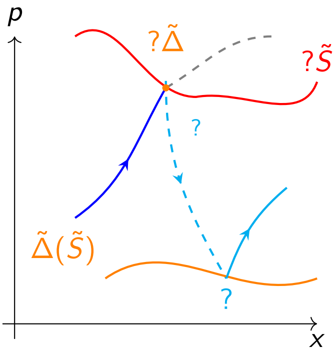
Hybrid maximum principle

Discrete part



Hybrid maximum principle

Discrete part

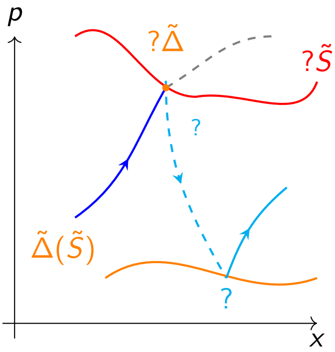


- inward pointing momentum

$$\tilde{S} = \left\{ (x, p) \in T^*M|_S : dh_x \left(\frac{\partial H}{\partial p} \right) < 0 \right\}$$

Hybrid maximum principle

Discrete part



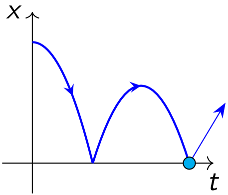
- $\mu(Zeno) = 0$

- inward pointing momentum

$$\tilde{S} = \left\{ (x,p) \in T^*M|_S : dh_x\left(\frac{\partial H}{\partial p}\right) < 0 \right\}$$

- $x^+ = \Delta(x^-)$ and p^+ such that:
$$\begin{cases} H \circ \tilde{\Delta} = H(x^+, p^+) = H(x^-, p^-) \\ p^+ = p^- + \epsilon dh \end{cases}$$

The bouncing ball with dissipation



- e.o.m. $\begin{cases} \dot{x} = \frac{1}{m} y \\ \dot{y} = -mg + u \end{cases}$
- guard: $S = x = 0 = h^{-1}(0)$, $h(x, y) = x$.
- reset: $\Delta(x, y) = (x, -c^2 y)$, $0 < c < 1$.
- cost: $J = \int_0^T \frac{u^2}{2} \, dt$.

Optimal control solution

$$\hat{H}(x, y, p_x, p_y) = \min_u \left\{ \frac{1}{2}u^2 + \frac{1}{m} y p_x + (-mg + u) p_y \right\} \implies u = -p_y$$

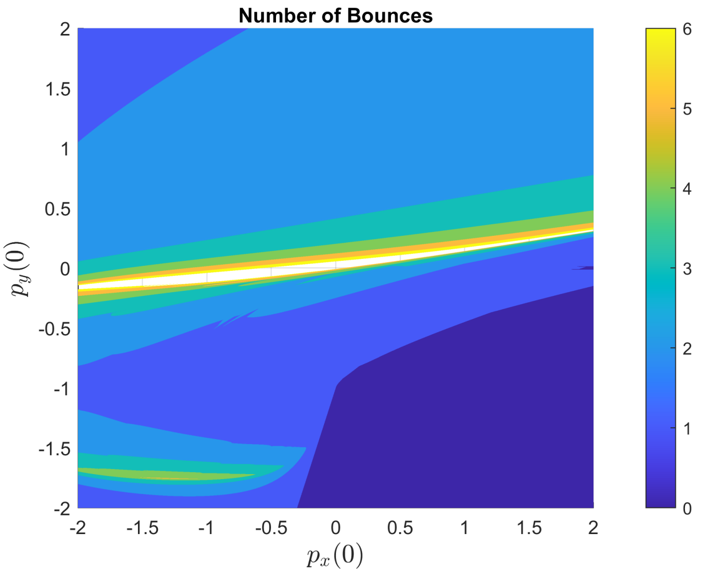
Continuous dynamics:

$$\begin{aligned} \dot{x} &= \frac{y}{m} \\ \dot{y} &= -mg - p_y \\ \dot{p_x} &= 0 \\ \dot{p_y} &= -\frac{p_x}{m} \end{aligned}$$

Discrete dynamics:

$$\begin{aligned} x^+ &= x^- \\ y^+ &= -c^2 y^- \\ p_x^+ &= -\frac{1}{c^2} p_y^- + \frac{m}{2c^2} \frac{(p_y^-)^2}{y} (1 - c^{-4}) \\ &\quad + \frac{m^2 g}{c^2} \frac{p_y}{y} (1 + c^{-2}) \\ p_y^+ &= -\frac{1}{c^2} p_y^- \end{aligned}$$

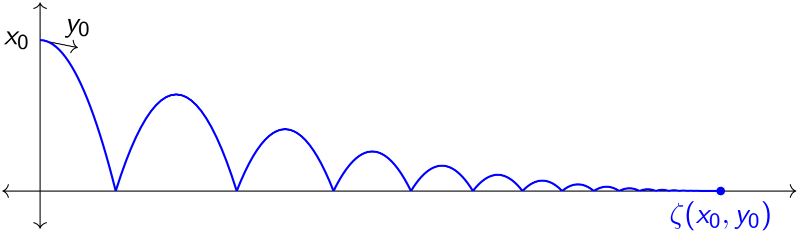
$y = 0?$


Number of Bounces
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
$p_y(0)$
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2
$p_x(0)$
6
5
4
3
2
1
0

Zeno for the bouncing ball

No controls $\implies$ every trajectory is Zeno.

$$\zeta(x_0, y_0) = \frac{1}{mg} y_0 + \frac{3}{g(1-c^2)} \sqrt{\frac{y_0^2}{m^2} + 2gx_0}$$



Zeno can be optimal

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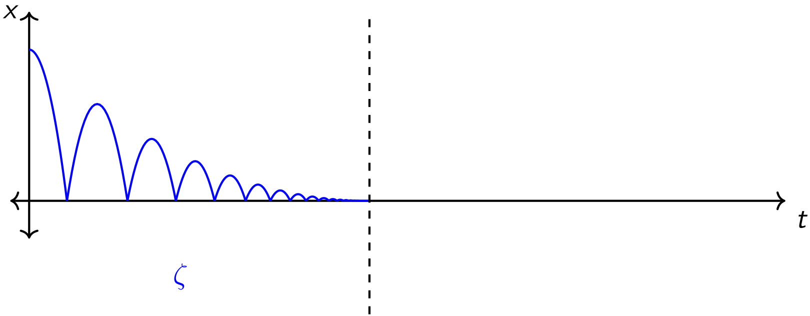
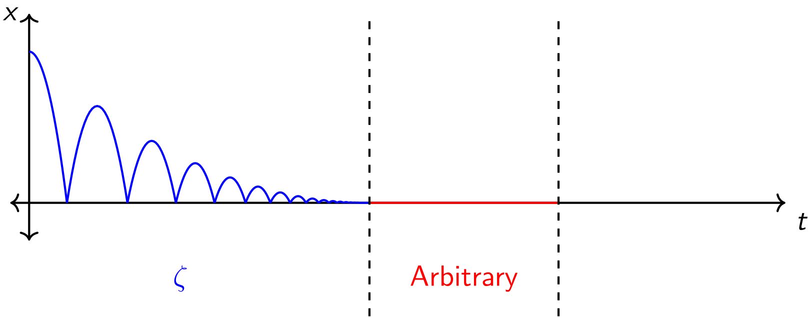
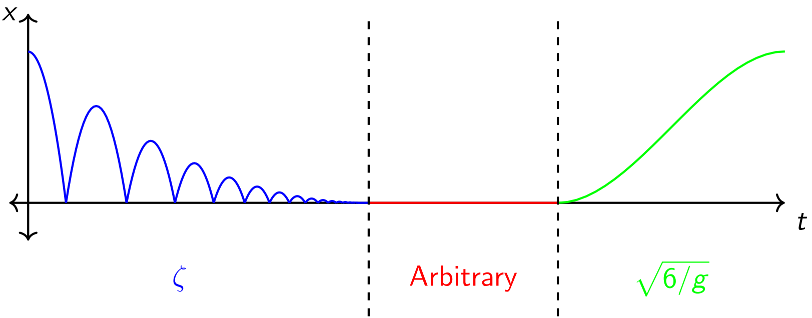


Zeno can be optimal



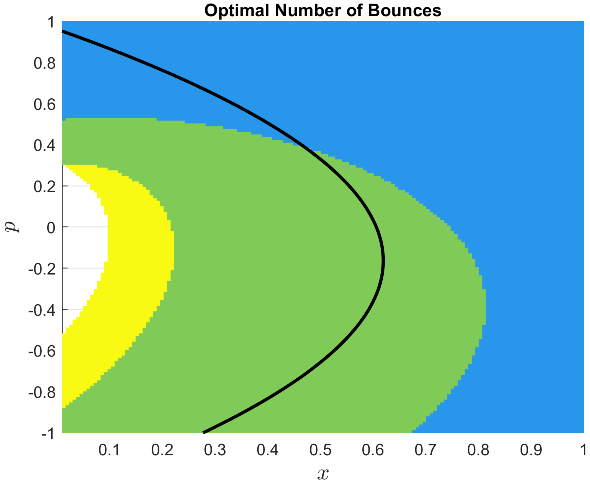
Zeno can be optimal



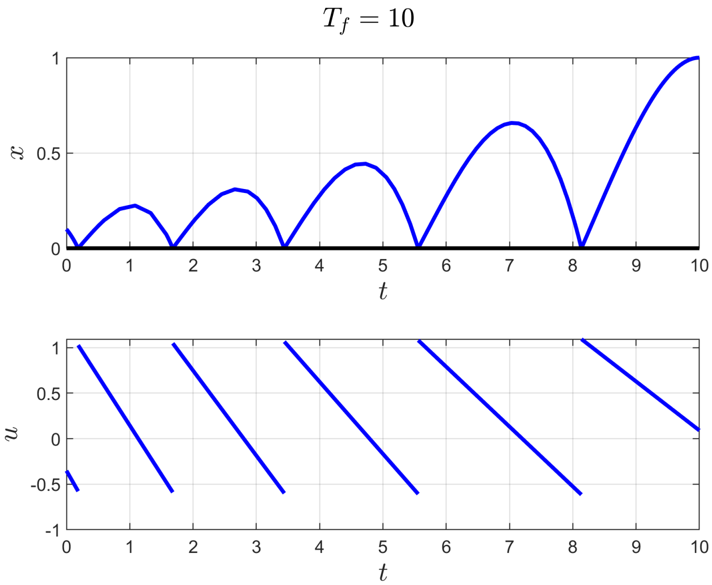
Zeno becomes locally optimal if $T > \zeta + \sqrt{\frac{6}{g}}$ with cost $J_{Zeno} = \frac{2}{3}g\sqrt{6g}$

Numerical observations

$$J_{optimal} = \min\{J_{shoot}, J_{Zeno}\}$$

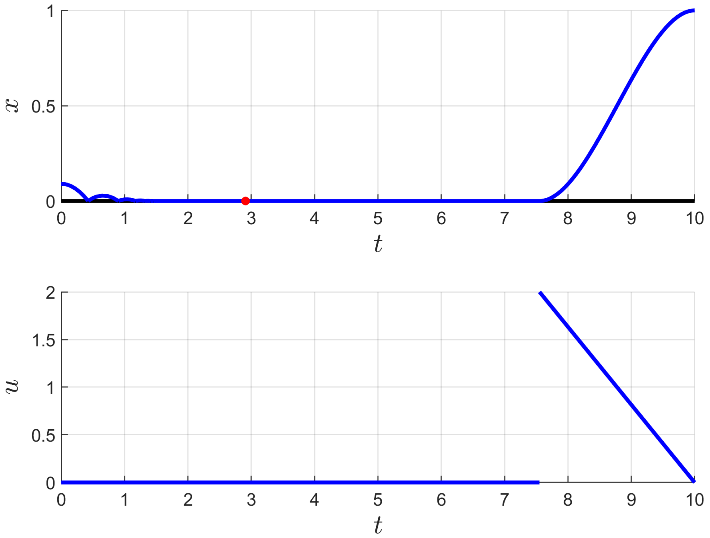


Maximum principle trajectories

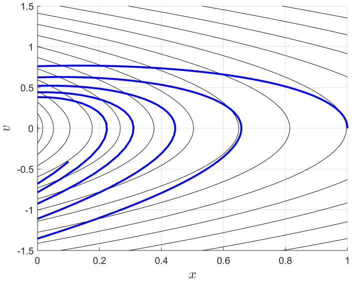


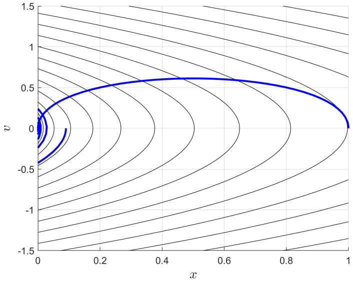
Zeno trajectories

$T_f = 10$



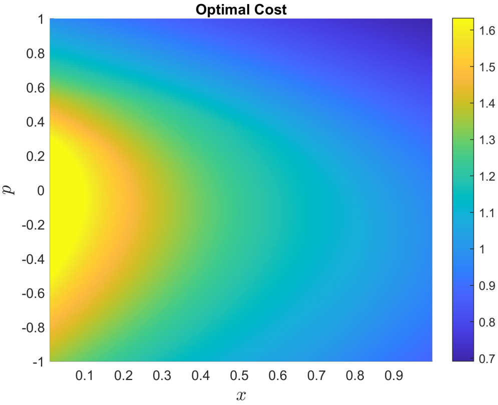
Phase space comparison





Numerical observations

- Zeno doesn't satisfy the extended reset map but still optimal $\implies$ Maximum principle is not neccesary for optimality!
- The value function is constant in the Zeno area! What is the boundary?



Questions?

Paper: arXiv:2210.01056

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