18.650 Homework 1

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1 Expectation

Exercise 1.1

Suppose we play a game where we start with c dollars. On each play of the game, you either double or halve your money, with equal probability. What is your expected fortune after n trials?

Solution: Let X_i denote the amount of money you have after playing the game i times. When i = 0, by definition, $\mathbb{P}[X_0 = c] = 1$, and so, $\mathbb{E}[X_0] = c$. When i > 0,

$$\mathbb{E}[X_i] = \mathbb{E}\left[\frac{1}{2}\cdot(2\cdot X_{i-1}) + \frac{1}{2}\cdot\left(\frac{1}{2}\cdot X_i\right)\right] = \mathbb{E}\left[\frac{5}{4}X_{i-1}\right] = \frac{5}{4}\,\mathbb{E}\left[X_{i-1}\right].$$

It immediately follows that $\mathbb{E}[X_i] = \left(\frac{5}{4}\right)^i \cdot c$. Thus, after n trials, your expected fortune is $c \cdot \left(\frac{5}{4}\right)^n$.

Exercise 1.2

Show that Var[X] = 0 if and only if there is a constant c such that $\mathbb{P}[X = c] = 1$.

Solution: We first prove the easier direction, namely that if $\mathbb{P}[X=c]=1$, then $\mathrm{Var}[X]=0$. In this case, $\mathbb{E}[X^2]=c^2$ and $\mathbb{E}[X]^2=c^2$ too, so $\mathrm{Var}[X]=\mathbb{E}[X^2]-\mathbb{E}[X]^2=0$, as desired. As for the other direction, by Jensen's inequality,

$$\mathbb{E}[X^2] \ge \mathbb{E}[X]^2 \iff \operatorname{Var}[X] \ge 0,$$

with equality holding iff X is constant, i.e. $\mathbb{P}[X=c]=1$ for some c.

Exercise 1.3

Let $X_1, \ldots, X_n \sim \text{Uniform}[0, 1]$ and let $Y_n = \max(X_1, \ldots, X_n)$. Find $\mathbb{E}[Y_n]$.

Solution: Consider the CDF of Y_n :

$$F_{Y_n}(x) = \mathbb{P}[Y_n \le x] = \mathbb{P}[X_1, X_2, \dots, X_n \le x] = \mathbb{P}[X_1 \le x]^n = x^n.$$

The PDF of Y_n is therefore $f_{Y_n}(x) \frac{\mathrm{d}}{\mathrm{d}x} [x^n] = nx^{n-1}$, and so

$$\mathbb{E}[Y_n] = \int f_{Y_n}(x) \cdot x dx = \int_0^1 nx^{n-1} \cdot x dx = \int_0^1 nx^n dx = \frac{n}{n+1}.$$

Exercise 1.4

A particle starts at the origin of the real line and moves along the line in jumps of one unit. For each jump the probability is p that the particle will jump one unit to the left and the probability is 1-p that

the particle will jump one unit to the right. Let X_n be the position of the particle after n units. Find $\mathbb{E}[X_n]$ and $\mathrm{Var}[X_n]$.

Solution: When n = 0, $\mathbb{E}[X_n] = 0$, and when n > 0,

$$\mathbb{E}[X_n] = \mathbb{E}[p \cdot (X_{n-1} - 1)] + \mathbb{E}[(1 - p) \cdot (X_{n-1} + 1)]$$

= $p \, \mathbb{E}[X_{n-1}] - p + (1 - p) \, \mathbb{E}[X_{n-1}] + 1 - p$
= $\mathbb{E}[X_{n-1}] + 1 - 2p$.

Therefore, by induction $\mathbb{E}[X_n] = (1-2p) \cdot n$.

As for the variance, let Y_i denote whether or not you move left (-1) or right (+1) on the *i*th jump. Then,

$$Var[X_n] = Var[Y_1 + Y_2 + \dots + Y_n]$$
$$= n Var[Y_1]$$
$$= n(4p^2 - 4p).$$

Exercise 1.5

A fair coin is tossed until a head is obtained. What is the expected number of tosses that will be required?

Solution: If we let X_i denote the event that the first time we get heads is on the *i*th toss, then $\mathbb{P}[X_i] = \frac{1}{2^i}$, so

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} \frac{i}{2^i} = 2,$$

where X denotes the number of tosses required.

Exercise 1.6

Exercise 1.7

Exercise 1.8

Prove theorem 3.17.

Solution: By linearity of expectation

$$\mathbb{E}[\overline{X}_n] = \mathbb{E}\left[\frac{X_1 + X_2 + \ldots + X_n}{n}\right] = \frac{\mathbb{E}[X_1] + \mathbb{E}[X_2] + \ldots + \mathbb{E}[X_n]}{n} = \frac{n\mu}{n} = \mu.$$

Additionally,

$$\operatorname{Var}[\overline{X}_n] = \operatorname{Var}\left[\frac{X_1 + \ldots + X_n}{n}\right] = \frac{1}{n^2} \operatorname{Var}[X_1 + \ldots + X_n] = \frac{1}{n^2} \left(\operatorname{Var}[X_1] + \ldots + \operatorname{Var}[X_n]\right) = \frac{\sigma^2}{n^2}.$$

The expected value of the sample variance is

$$S_n = \mathbb{E}\left[\frac{1}{n-1}\sum_{i=1}^n (X_i - \overline{X}_n)^2\right]$$

$$= \frac{n}{n-1} \cdot \mathbb{E}\left[\left(X_1 - \overline{X}_n\right)^2\right]$$

$$= \frac{1}{n(n-1)} \cdot \mathbb{E}\left[\left((n-1)X_1 - X_2 - X_3 \dots - X_n\right)^2\right].$$

If we let $Y = (n-1)X_1 - X_2 - ... - X_n$, then

$$Var[Y] = Var[(n-1)X_1] + Var[X_2] + \dots + Var[X_n] = (n-1)^2 \sigma + (n-1)\sigma = n(n-1)\sigma,$$

so $\mathbb{E}[Y^2] = \text{Var}[Y] + \mathbb{E}[Y]^2 = n(n-1)\sigma^2$. Plugging this into the prior equation, $\mathbb{E}[S_n^2] = \frac{1}{n(n-1)} \cdot n(n-1)\sigma^2 = \sigma^2$.

Exercise 1.9

Exercise 1.10

Let $X \sim N[0,1]$ and let $Y = e^X$. Find $\mathbb{E}[Y]$ and Var[Y].

Solution: The MGF of X is $\psi_X(t) = \mathbb{E}[e^{Xt}] = e^{\frac{1}{2}t^2}$, so $\mathbb{E}[Y] = \sqrt{e}$ and $\mathbb{E}[Y^2] = e^2$. It follows that $\operatorname{Var}[Y] = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = e^2 - e$.