All of Statistics Chapter 3

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February 9, 2025

1 Expectation

Exercise 1.1

Suppose we play a game where we start with c dollars. On each play of the game, you either double or halve your money, with equal probability. What is your expected fortune after n trials?

Solution: Let X_i denote the amount of money you have after playing the game i times. When $i = 0$, by definition, $\mathbb{P}[X_0 = c] = 1$, and so, $\mathbb{E}[X_0] = c$. When $i > 0$,

$$\mathbb{E}[X_i] = \mathbb{E}\left[\frac{1}{2} \cdot (2 \cdot X_{i-1}) + \frac{1}{2} \cdot \left(\frac{1}{2} \cdot X_i\right)\right] = \mathbb{E}\left[\frac{5}{4} X_{i-1}\right] = \frac{5}{4}\mathbb{E}\left[X_{i-1}\right].$$

It immediately follows that $\mathbb{E}[X_i] = \left(\frac{5}{4}\right)^i \cdot c$. Thus, after n trials, your expected fortune is $c \cdot \left(\frac{5}{4}\right)^n$. ■

Exercise 1.2

Show that $\text{Var}[X] = 0$ if and only if there is a constant c such that $\mathbb{P}[X = c] = 1$.

Solution: We first prove the easier direction, namely that if $\mathbb{P}[X = c] = 1$, then $\text{Var}[X] = 0$. In this case, $\mathbb{E}[X^2] = c^2$ and $\mathbb{E}[X]^2 = c^2$ too, so $\text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 0$, as desired.
As for the other direction, by Jensen's inequality,

$$\mathbb{E}[X^2] \geq \mathbb{E}[X]^2 \iff \text{Var}[X] \geq 0,$$

with equality holding iff X is constant, i.e. $\mathbb{P}[X = c] = 1$ for some c. ■

Exercise 1.3

Let $X_1, \ldots, X_n \sim \text{Uniform}[0, 1]$ and let $Y_n = \max\left(X_1, \ldots X_n\right)$. Find $\mathbb{E}[Y_n]$.

Solution: Consider the CDF of Y_n:

$$F_{Y_n}(x) = \mathbb{P}[Y_n \leq x] = \mathbb{P}[X_1, X_2, \ldots, X_n \leq x] = \mathbb{P}[X_1 \leq x]^n = x^n.$$

The PDF of Y_n is therefore $f_{Y_n}(x)\frac{\text{d}}{\text{d}x}\left[x^n\right] = nx^{n-1}$, and so

$$\mathbb{E}[Y_n] = \int f_{Y_n}(x) \cdot x\text{d}x = \int_0^1 nx^{n-1} \cdot x\text{d}x = \int_0^1 nx^n\text{d}x = \frac{n}{n+1}.$$

■

Exercise 1.4

A particle starts at the origin of the real line and moves along the line in jumps of one unit. For each jump the probability is p that the particle will jump one unit to the left and the probability is $1 - p$ that the particle will jump one unit to the right. Let X_n be theposition of the particle after n units. Find $\mathbb{E}[X_n]$ and $\operatorname{Var}[X_n]$.

Solution: When $n = 0$, $\mathbb{E}[X_n] = 0$, and when $n > 0$,

$$\begin{aligned}\mathbb{E}[X_n] &= \mathbb{E}[p \cdot (X_{n-1} - 1)] + \mathbb{E}[(1-p) \cdot (X_{n-1} + 1)] \\ &= p\,\mathbb{E}[X_{n-1}] - p + (1-p)\,\mathbb{E}[X_{n-1}] + 1 - p \\ &= \mathbb{E}[X_{n-1}] + 1 - 2p.\end{aligned}$$

Therefore, by induction $\mathbb{E}[X_n] = (1-2p) \cdot n$.
As for the variance, let Y_i denote whether or not you move left (-1) or right $(+1)$ on the ith jump.Then,

$$\begin{aligned}\operatorname{Var}[X_n] &= \operatorname{Var}[Y_1 + Y_2 + \ldots + Y_n] \\ &= n\operatorname{Var}[Y_1] \\ &= n(4p^2 - 4p).\end{aligned}$$

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Exercise 1.5

A fair coin is tossed until a head is obtained. What is the expected number of tosses that will be required?

Solution: If we let X_i denote the event that the first time we get heads is on the ith toss, then $\mathbb{P}[X_i] = \frac{1}{2^i}$, so

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} \frac{i}{2^i} = 2,$$

where X denotes the number of tosses required. ■

Exercise 1.6

Prove theorem 3.6 for discrete random variables.

Solution: For discrete random variables, the expected value of X is

$$\sum_i \mathbb{P}[X = x_i]x_i,$$

so the expected value of $r(X)$ is

$$\sum_i \mathbb{P}[r(X) = r_i]r_i = \sum_i \mathbb{P}[r(X) = r(x_i)]x_i.$$

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Exercise 1.7

Let X be a continous random variable with CDF F. Suppose that $\mathbb{P}[X > 0] = 1$ and that $\mathbb{E}[X]$ exists. Show that $\mathbb{E}[X] = \int_0^\infty \mathbb{P}[X > x]\mathrm{d}x$.

Solution: We know that $\mathbb{P}[X > x] = \int_x^\infty f_X(y)\mathrm{d}y$, so

$$\begin{aligned}\int_0^\infty \mathbb{P}[X > x]\mathrm{d}x &= \int_0^\infty \int_x^\infty f_X(y)\mathrm{d}y\mathrm{d}x \\ &= \int_0^\infty \int_0^y f_X(y)\mathrm{d}x\mathrm{d}y \\ &= \int_0^\infty f_X(y)y\mathrm{d}y \\ &= \mathbb{E}[X],\end{aligned}$$

from which the desired follows. ■

Exercise 1.8

Prove theorem 3.17.

Solution: By linearity of expectation

$$\mathbb{E}[\overline{X}_n] = \mathbb{E}\left[\frac{X_1 + X_2 + \ldots + X_n}{n}\right] = \frac{\mathbb{E}[X_1] + \mathbb{E}[X_2] + \ldots + \mathbb{E}[X_n]}{n} = \frac{n\mu}{n} = \mu.$$

Additionally,

$$\operatorname{Var}[\overline{X}_n] = \operatorname{Var}\left[\frac{X_1 + \ldots + X_n}{n}\right] = \frac{1}{n^2}\operatorname{Var}[X_1 + \ldots + X_n] = \frac{1}{n^2}\left(\operatorname{Var}[X_1] + \ldots + \operatorname{Var}[X_n]\right) = \frac{\sigma^2}{n^2}.$$

The expected value of the sample variance is

$$\begin{aligned} S_n &= \mathbb{E}\left[\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2\right] \\ &= \frac{n}{n-1}\cdot\mathbb{E}\left[\left(X_1 - \overline{X}_n\right)^2\right] \\ &= \frac{1}{n(n-1)}\cdot\mathbb{E}\left[((n-1)X_1 - X_2 - X_3\ldots - X_n)^2\right]. \end{aligned}$$

If we let $Y = (n-1)X_1 - X_2 - \ldots - X_n$, then

$$\operatorname{Var}[Y] = \operatorname{Var}[(n-1)X_1] + \operatorname{Var}[X_2] + \ldots + \operatorname{Var}[X_n] = (n-1)^2\sigma + (n-1)\sigma = n(n-1)\sigma,$$

so $\mathbb{E}[Y^2] = \operatorname{Var}[Y] + \mathbb{E}[Y]^2 = n(n-1)\sigma^2$. Plugging this into the prior equation, $\mathbb{E}[S_n^2] = \frac{1}{n(n-1)}\cdot n(n-1)\sigma^2 = \sigma^2$. ■

Exercise 1.9

Let $X_1, X_2, \ldots X_n$ be $\mathrm{N}[0,1]$ random variables and let $\overline{X}_n = n^{-1}\sum_{i=1}^{n} X_i$. Plot $\overline{X}_n$ verus n for $n = 1, \ldots, 10,000$. Repeat for $X_1, X_2, \ldots X_n \sim$ Cauchy. Explain why there is such a difference.

Solution: See the `.ipynb` file. Expectedly, the sample mean for the normal distribution appears to converge to 1, but the sample mean of the Cauchy sequence does not converge (since the Cauchy sequence is so fat-tailed, it does not have a well-defined mean). ■

Exercise 1.10

Let $X \sim \mathrm{N}[0,1]$ and let $Y = e^X$. Find $\mathbb{E}[Y]$ and $\operatorname{Var}[Y]$.

Solution: The MGF of X is $\psi_X(t) = \mathbb{E}[e^{Xt}] = e^{\frac{1}{2}t^2}$, so $\mathbb{E}[Y] = \sqrt{e}$ and $\mathbb{E}[Y^2] = e^2$. It follows that $\operatorname{Var}[Y] = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = e^2 - e$. ■

Exercise 1.11

Let $Y_1, Y_2, \ldots$ be independent random variables such that $\mathbb{P}[Y_i = 1] = \mathbb{P}[Y_i = -1] = \frac{1}{2}$. Let $X_n = \sum_{i=1}^{n} Y_i$. Think of Y_i as "the stock price increased by one dollar", $Y_i = -1$ as "the stock price decreased by one dollar", and X_n as the value of the stock on day n.

(a) Find $\mathbb{E}[X_n]$ and $\operatorname{Var}[X_n]$.

(b) Stumlate X_n and plot X_n versus n for $n = 1, 2, \ldots 10,000$. Repeat the whole stimulation several times. Notice two things. First, it is easy to see patterns in the sequence even though it is random.

Second, you will find that the four runs look very different even though they were generated the same way. How do the calcualtions in (a) explain the second observation?

Solution:

(a)

$$\mathbb{E}[X_n] = \mathbb{E}\left[\sum_{i=1}^{n} Y_i\right] = \sum_{i=1}^{n} \mathbb{E}[Y_i] = 0$$

$$\mathrm{Var}[X_n] = \mathrm{Var}\left[\sum_{i=1}^{n} Y_i\right] = \sum_{i=1}^{n} \mathrm{Var}[Y_i] = \sum_{i=1}^{n} 1 = n.$$

(b) See the `.ipynb` file. There is expectedly, high variance, since the variance is a linear factor of n.

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Exercise 1.12

Prove the formulas given in the table at the beginning of Section 3.4 for the Bernoulli, Poisson, Uniform, Exponential, Gamma, and Beta.

Solution:

- **Bernoulli**: For the Bernoulli distribution, the mean is clearly p, since with probability p, the event happens $(+1)$ and with probability $1-p$, the event does not happen (-1). The variance can be computed by noting that $\mathbb{E}[X^2] = p$ and $\mathbb{E}[X]^2 = p^2$, so $\mathrm{Var}[X] = p(1-p)$.
- **Poisson**: The moment generating function of the Poisson distribution is
- **Exponential**: The moment generating function of the exponential distribution is

$$M_X(t) = \mathbb{E}[e^{Xt}] = \int_0^\infty e^{tx}\lambda e^{-\lambda x}\mathrm{d}x = \lambda \int_0^\infty e^{(t-\lambda)x}\mathrm{d}x = \lambda \cdot \frac{e^{(t-\lambda)x}}{t-\lambda}\Big|_0^\infty = \frac{\lambda}{\lambda - t}.$$

  Some computation yields that $M_X'(t) = \lambda(\lambda-t)^{-2}$ and $M_X''(t) = 2\lambda(\lambda-t)^{-3}$, so $\mathbb{E}[X] = M_x'(0) = \frac{1}{\lambda} = \beta$ and $\mathbb{E}[X^2] = M_X''(0) = \frac{2}{\lambda^2}$. The variance is therefore $\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2} = \beta^2$.
- **Gamma**:
- **Beta**:

■