18.650 Homework 1

Maria Chrysafis

February 9, 2025

1 Expectation

Exercise 1.1

Suppose we play a game where we start with c dollars. On each play of the game, you either double or halve your money, with equal probability. What is your expected fortune after n trials?

Solution: Let X_i denote the amount of money you have after playing the game i times. When $i = 0$, by definition, $\mathbb{P}[X_0 = c] = 1$, and so, $\mathbb{E}[X_0] = c$. When $i > 0$,

$$\mathbb{E}[X_i] = \mathbb{E}\left[\frac{1}{2} \cdot (2 \cdot X_{i-1}) + \frac{1}{2} \cdot \left(\frac{1}{2} \cdot X_i\right)\right] = \mathbb{E}\left[\frac{5}{4} X_{i-1}\right] = \frac{5}{4}\,\mathbb{E}\left[X_{i-1}\right].$$

It immediately follows that $\mathbb{E}[X_i] = \left(\frac{5}{4}\right)^i \cdot c$. Thus, after n trials, your expected fortune is $c \cdot \left(\frac{5}{4}\right)^n$. ∎

Exercise 1.2

Show that $\mathrm{Var}[X] = 0$ if and only if there is a constant c such that $\mathbb{P}[X = c] = 1$.

Solution: We first prove the easier direction, namely that if $\mathbb{P}[X = c] = 1$, then $[X] = 0$. In this case, $\mathbb{E}[X^2] = c^2$ and $\mathbb{E}[X]^2 = c^2$ too, so $[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 0$, as desired.
As for the other direction, ∎