1 probability review

Exercise 1.1

Let X be a random variable taking values between 0 and π with pdf given by $f(x) = c\sin(x), x \in [0, \pi]$. What is the value of c?

Solution: Since the integral of the pdf is always 1 (by definition),

$$1 = \int_0^\pi f(x)\,\mathrm{d}x = \int_0^\pi c\sin x\,\mathrm{d}x = -c\Big|_0^\pi \cos x = -c(\cos\pi - \cos 0) = -c(-2) = 2c.$$

And so, $2c = 1$ and $c = \frac{1}{2}$. $\boxed{B}$. ■

Exercise 1.2

What is $\mathbb{E}[X]$?

Solution: By the definition of expectation,

$$\mathbb{E}[X] = \int_0^\pi xf(x)\,\mathrm{d}x = \int_0^\pi cx\sin x\,\mathrm{d}x = c\Big|_0^\pi (\sin x - x\cos x) = c(-\pi\cos\pi + \sin\pi - 0\cos 0 + \sin 0) = \pi c.$$

We know from the previous problem that $c = \frac{1}{2}$, so $\mathbb{E}[X] = \frac{\pi}{2}$. $\boxed{A}$. ■

Exercise 1.3

Let X be a Gaussian random variable with mean $\mu > 0$ and variance μ^2. What is $\mathbb{E}[X]$?

Solution: The mean, that is $\mathbb{E}[X]$, is μ by definition. $\boxed{B}$. ■

Exercise 1.4

What is $\mathbb{E}[X^2]$?

Solution: By definition of variance,

$$\mu^2 = \mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mathbb{E}[X^2] - \mu^2,$$

so $\mathbb{E}[X^2] = 2\mu^2$. $\boxed{C}$. ■

Exercise 1.5

What is $\mathbb{E}[X^3]$?

Solution: Using the binomial theorem,

$$\begin{aligned}\mathbb{E}[X^3] &= \mathbb{E}[((X-\mu)+\mu)^3] \\ &= \mathbb{E}[(X-\mu)^3] + 3\mathbb{E}[(X-\mu)^2\mu] + 3\mathbb{E}[(X-\mu)\mu^2] + \mathbb{E}[\mu^3] \\ &= 3\mu\mathbb{E}[(X-\mu)^2] + \mu^3.\end{aligned}$$

Since $X - \mu$ is a Gaussian random variable with mean 0 and variance μ^2, $\mathbb{E}[(X-\mu)^2] = \mu^2$,

$$\mathbb{E}[X^3] = 3\mu^3 + \mu^{=}4\mu^3.3$$

$\boxed{C}$. ■

Exercise 1.6

What is $\mathrm{Var}[X^2]$?

Solution: For a normal random variable with mean μ and standard deviation σ, $\mathbb{E}[X^4] = 3\sigma^4 + 6\mu^2\sigma^2 + \mu^4$. Since our mean and standard deviation are both μ in this case, $\mathbb{E}[X^4] = 10\mu^4$. Using this,

$$\begin{aligned}\mathrm{Var}[X^2] &= \mathbb{E}[X^4] - \mathbb{E}[X^2]^2 \\ &= 10\mu^4 - (2\mu^2)^2 \\ &= 6\mu^4.\end{aligned}$$

$\boxed{B}$. ■

Exercise 1.7

What is $\mathbb{P}[X > 0]$ in terms of the CDF Φ of the standard Gaussian distribution?

Solution: By definition of the cdf,

$$\mathbb{P}[X > 0] = \mathbb{P}\left[\frac{X-\mu}{\sigma} > -\frac{\mu}{\sigma}\right] = \mathbb{P}\left[\frac{X-\mu}{\sigma} > -1\right] = \mathbb{P}\left[\frac{X-\mu}{\sigma} <\right] = \Phi(1).$$

$\boxed{B}$. ■

Exercise 1.8

Let X be a random variable such that

$$X = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } 1-p \end{cases}$$

for some $p \in [0, 1]$. What is $\mathbb{E}[X]$?

Solution: Routine algebra shows that $\mathbb{E}[X] = 1 \cdot p + (-1) \cdot (1-p) = -1 + 2p$. $\boxed{D}$. ■

Exercise 1.9

What is $\mathrm{Var}[X]$?

Solution: The variance of X is $\mathbb{E}[X^2] - \mathbb{E}[X]^2 = 1 - (1-2p)^2 = 4p - 4p^2 = 4p(1-p)$. $\boxed{C}$. ■

Exercise 1.10

For what p is $\mathrm{Var}[X]$ maximized?

Solution: We know from the previous problem that $\mathrm{Var}[X] = 4p - 4p^2$, which has derivative $4 - 8p$. The variance is maximized when that derivative is 0, so when $4 - 8p = 0 \implies p = \frac{1}{2}$. $\boxed{C}$. ■

Exercise 1.11

What is $\mathbb{E}[X^k]$?

Solution: The expected value is $\mathbb{E}[X^k] = 1^k \cdot p + (-1)^k \cdot (1-p) = p + (-1)^k \cdot (1-p)$. $\boxed{D}$. ■

Exercise 1.12

Let X and Y be two independent standard Gaussian random variables. What is $\mathbb{E}[X^2Y]$?

Solution: Since X and Y are independent, $\mathbb{E}[X^2Y] = \mathbb{E}[X^2] \cdot \mathbb{E}[Y] = 0$. $\boxed{A}$. ■

Exercise 1.13

What is $\text{Var}(X + Y)$?

Solution: Since X and Y are independent, $\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y] = 1 + 1 = 2$. $\boxed{C}$. ■

Exercise 1.14

What is $\text{Var}[XY]$?

Solution: The variance is $\text{Var}[XY] = \mathbb{E}[(XY)^2] - \mathbb{E}[XY]^2 = \mathbb{E}[X^2]\mathbb{E}[Y^2] = 1$. $\boxed{B}$. ■

Exercise 1.15

What is $\text{Cov}[X, X + Y]$?

Solution: The covariance is $\text{Cov}[X, X + Y] = \text{Cov}[X, X] + \text{Cov}[X, Y] = 1 + 0 = 1$. $\boxed{B}$. ■

Exercise 1.16

What is $\text{Cov}[X, XY]$?

Solution: The covariance is $\text{Cov}[X, XY] = \mathbb{E}[X^2Y] - \mathbb{E}[X]\mathbb{E}[XY] = 0$. $\boxed{A}$. ■

Exercise 1.17

Let X be an exponential random variable with parameter $\frac{1}{2}$ that model sthe lifetime (in years) of a lightbulb. What is (approximately) the probability that the lightbulb will last at least two years?

Solution: The exponential random variable has CDF of $1 - e^{-\lambda x}$, so the probability that it lasts two or fewer years is $1 - e^{-2 \cdot 1} = 1 - e^{-1}$. The probability that it lasts more than two years is therefore e^{-1}, which is closest to $\boxed{C}$. ■

Exercise 1.18

Given that the lightbulb has lasted for at least 3 years, what is approximately the probability that it will last for at least two more years?

Solution: Exponential distribution is memoryless, so $\boxed{C}$ (same as previous question). ■

Exercise 1.19

Let$X_1, \ldots X_n$ be i.i.d. with mean μ and variance σ^2. What is $\mathbb{E}\left[\sum_{i=1}^{n} X_i\right]$?

Solution: By linearity of expectation,

$$\mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{E}[X_i] = n\mu.$$

$\boxed{C}$. ∎

Exercise 1.20

What is $\mathrm{Var}[\sum_{i=1}^{n} X_i]$?

Solution: Since each of the X_is are independent,

$$\mathrm{Var}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathrm{Var}[X_i] = n\sigma^2.$$

$\boxed{B}$. ∎

Exercise 1.21

What is $\mathbb{E}[(\sum_{i=1}^{n} X_i)^2]$?

Solution: Let $Y = \sum_{i=1}^{n} X_i$. Then,

$$n\sigma^2 = \mathrm{Var}[Y] = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = \mathbb{E}[Y^2] - (n\mu)^2,$$

so $\mathbb{E}[Y^2] = n^2\mu^2 + n\sigma^2$. $\boxed{D}$. ∎

Exercise 1.22

What is $\mathrm{Var}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right]$?

Solution: Since $\mathrm{Var}[aX] = a^2\mathrm{Var}[X]$,

$$\mathrm{Var}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n^2}\mathrm{Var}\left[\sum_{i=1}^{n} X_i\right] = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}.$$

$\boxed{B}$. ∎

2 new concepts

Exercise 2.1

Let $X_n \sim \mathrm{Unif}(-1/n, 1/n)$ and let X be a random variable such that $\mathbb{P}[X = 0] = 1$.

1. Compute and draw the CDF $F_n(x)$ and $F(x)$ of X_n and X respectively.
2. Does $X_n \xrightarrow{P} X$? (prove or disprove)
3. Does $X_n \rightsquigarrow X$ (prove or disprove)

Solution:

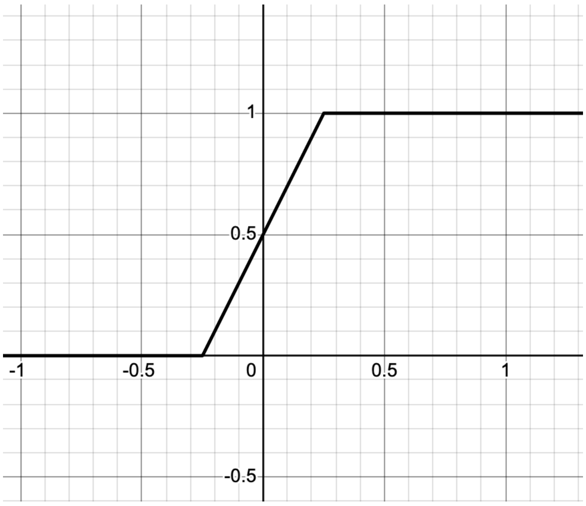

Figure 1: This is the cdf for X_n, when $n = 4$.

Figure 2: This is the cdf for X.

1. The CDF of X is

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

and the CDF of X_n is

$$F_n(x) = \begin{cases} 0 & x < -\frac{1}{n} \\ \frac{nt+1}{2} & -\frac{1}{n} \leq \frac{1}{n} \\ 1 & x > \frac{1}{n} \end{cases}.$$

2. We know that for all ϵ,

$$\mathbb{P}[|X_n - X| > \epsilon] = \mathbb{P}[|X_n| > \epsilon] = 1 - \mathbb{P}[|X_n| \leq \epsilon] = 1 - \min\left(\frac{2\epsilon}{\frac{2}{n}}, 1\right) = 1 - \min(n\epsilon, 1).$$

Therefore,

$$\lim_{n\to\infty} \mathbb{P}[|X_n - X| > \epsilon] = \lim_{n\to\infty} 1 - \min(n\epsilon, 1) = 0,$$

so X_n does converge to X probabilistically.

3. The CDF of X is

$$F(t) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

and the CDF of X_n is

$$F_n(t) = \begin{cases} 0 & x < -\frac{1}{n} \\ \frac{nt+1}{2} & -\frac{1}{n} \leq x \leq \frac{1}{n} \\ 1 & x > \frac{1}{n} \end{cases}.$$

As n approaches ∞, $F_n(t)$ approaches

$$F_n(t) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = \frac{1}{2} \\ 1 & x = 1 \end{cases}.$$

Although $F_n(t)$ and $F(t)$ are not the same functions, they have equal values at all points for which F is continous. Thus, X_n converges to X in distribution.

■

Exercise 2.2

Let $X \sim \mathcal{N}(1, 2.25)$. Compute the following probabilities (show your work):

1. $\mathbb{P}[X > 1]$
2. $\mathbb{P}[|X - 2| \leq 1]$
3. $\mathbb{P}[|X| < 1]$
4. $\mathbb{P}[X^2 - 2X - 1 > 0]$.

Solution:

1. Since X is normally distributed around 1, by symmetry, $\mathbb{P}[X > 1] = 0.5$.
2. We know that $\mathbb{P}[|X - 2| \leq 1] = \mathbb{P}[1 \leq X \leq 3]$. If we let Y be $\frac{X-\mu}{\sigma} = \frac{X-1}{1.5}$, then we can rewrite this as $\mathbb{P}\left[0 \leq Y \leq \frac{4}{3}\right]$. Since Y is standard normal, we can now reference the standard normal table, which tells us that $\mathbb{P}\left[Y \leq \frac{4}{3}\right] \approx 0.9082$ and $\mathbb{P}[Y \leq 0] = 0.5$, so $\mathbb{P}\left[0 \leq Y \leq \frac{4}{3}\right] \approx 0.4082$.
3. Again, we can use a similar technique as the previous part and let $Y = \frac{X-\mu}{\sigma} = \frac{X-1}{1.5}$. Then, it becomes clear that $\mathbb{P}[|X| < 1] = \mathbb{P}[-1 \leq X \leq 1] = \mathbb{P}\left[-\frac{4}{3} \leq Y \leq 0\right]$, which by symmetry means that $\mathbb{P}[|X| \leq 1] = \mathbb{P}\left[0 \leq Y \leq \frac{4}{3}\right] \approx 0.4082$.
4. Since $(X-1)^2 - 2 = X^2 - 2X - 1$, $\mathbb{P}[X^2 - 2X - 1 > 0] = \mathbb{P}[(X-1)^2 - 2 > 0] = \mathbb{P}[(X-1)^2 > 2] = \mathbb{P}[1 - \sqrt{2} \leq X \leq 1 + \sqrt{2}]$. Letting $Y = \frac{X-\mu}{\sigma}$, we reduce this to $\mathbb{P}\left[-\frac{\sqrt{2}}{1.5} < X < \frac{\sqrt{2}}{1.5}\right]$. Using a standard normal table, we can compute this to be around 0.3472.

■

Exercise 2.3

Let

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\right).$$

Compute the following quantities:

1. $\mathrm{Var}[X]$

2. $\mathbb{E}[Y^2 + X]$

3. $\mathbb{E}[(X - Y)^2]$

4. $\mathrm{Var}[X + 2Y]$

5. Find $\alpha > 0$ such that $\alpha X = Y$ with probability 1 or prove that no such α exists.

Solution:

1. By the covariance matrix, we know that $\mathrm{Cov}[X, X] = 1$, so $\mathrm{Var}[X] = 1$.

2. Taking a look at the covariance matrix, we know that $\mathrm{Cov}[Y, Y] = 2$, so $2 = \mathrm{Var}[Y] = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = \mathbb{E}[Y^2]$. Thus, $\mathbb{E}[Y^2 + X] = \mathbb{E}[Y^2] + \mathbb{E}[X] = 2 + 1 = 3$.

3. Some computation reveals that

$$\begin{aligned}
\mathbb{E}[(X - Y)^2] &= \mathbb{E}[X^2] - 2\mathbb{E}[XY] + \mathbb{E}[Y^2] \\
&= \mathrm{Var}[X] + \mathbb{E}[X]^2 - 2\left(\mathrm{cov}[X, Y] + \mathbb{E}[X]\mathbb{E}[Y]\right) + \mathbb{E}[Y^2] \\
&= 1 + 1 - 2 + 2 \\
&= 2
\end{aligned}$$

4. With some algebra, we see that

$$\begin{aligned}
\mathrm{Var}[X + 2Y] &= \mathbb{E}[(X + 2Y)^2] - \mathbb{E}[X + 2Y]^2 \\
&= \mathbb{E}[X^2] + 4\mathbb{E}[XY] + 4\mathbb{E}[Y^2] - \left(\mathbb{E}[X] + 2\mathbb{E}[Y]\right)^2 \\
&= 2 + 4 \cdot 1 + 4 \cdot 2 - (1 + 2 \cdot 0)^2 \\
&= 13.
\end{aligned}$$

5. Let us assume that such an α existed, by sake of contradiction. Then,

$$\mathrm{Cov}[X, Y] = \mathrm{Cov}[X, \alpha X] = \mathbb{E}[\alpha X^2] - \mathbb{E}[X]\mathbb{E}[\alpha X] = \alpha \mathrm{Var}[X].$$

Since $\mathrm{Cov}[X, Y] = 1$ and $\mathrm{Var}[X] = 1$, this would mean that $\alpha = 1$. But, α cannot be 1, since Y has a different variance than X.

■

Exercise 2.4

Let $X_1, \ldots, X_n \sim \mathrm{Ber}(1/2)$ and $Y_1, \ldots, Y_n \sim \mathrm{Exp}(1)$. Assume further that all the random variables are mutually independent. Write a central limit theorem for each of the following quantities in the form $\sqrt{n}(Z_n - \mu) \rightsquigarrow \mathcal{N}(0, \sigma^2)$ if Z_n is a random variable, and $\sqrt{n}(Z_n - \mu) \rightsquigarrow \mathcal{N}(0, \Sigma)$ if Z_n is a random vector. Show your work.

1. $Z_n = \begin{pmatrix} \overline{X}_n \\ \overline{Y}_n \end{pmatrix}$

2. $Z_n = \left(\overline{X}_n - \overline{Y}_n\right)$

3. $Z_n = \frac{1}{n}\sum_{i=1}^{n} X_i Y_i$

4. $\overline{X}_n^2 / \overline{Y}_n$

Solution: By properties of the Bernoulli distribution, $E[X_i] = \frac{1}{2}$ and $\mathrm{Var}(X_i) = \frac{1}{4}$; by properties of the Exponential distribution, $E[Y_i] = 1$ and $\mathrm{Var}[Y_i] = 1$.

1. We know that the variance of the Bernoulli distribution with parameter p is is $p \cdot (1-p)$; since $p = \frac{1}{2}$, its variance is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. $\frac{1}{4}$. The variance of the exponential distribution is $\frac{1}{\lambda^2}$, and since $\lambda = 1$, this means that we have variance 1. Further, since all variables are assumed to be mutually independent, the covariance matrix must be

$$\Sigma = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{pmatrix}.$$

2. Since $\text{Var}[X_i - Y_i] = \text{Var}[X_i] + \text{Var}[Y_i] = 1 + \frac{1}{4} = \frac{5}{4}$,

$$\sqrt{n}(Z_n - \mu) = \sqrt{n}\left(\overline{X}_n - \overline{Y}_n + \frac{1}{2}\right) \rightsquigarrow \mathcal{N}(0, \text{Var}(X_i - Y_i)) = \mathcal{N}\left(0, \frac{5}{4}\right).$$

3. Since X and Y are independent,

$$\begin{aligned} \text{Var}[X_i Y_i] &= \mathbb{E}[X_i^2 Y_i^2] - \mathbb{E}[X_i Y_i]^2 \\ &= \mathbb{E}[X_i]^2 \mathbb{E}[Y_i]^2 - \frac{1}{4} \\ &= \left(\text{Var}[X_i] + \mathbb{E}[X_i]^2\right)\left(\text{Var}[Y_i] + \mathbb{E}[Y_i]^2\right) - \frac{1}{4} \\ &= \frac{1}{2} \cdot 2 - \frac{1}{4} = \frac{3}{4}. \end{aligned}$$

   So, $\sigma^2 = \frac{3}{4}$.

4. We can use the delta method. We can let $g(x, y) = \frac{x^2}{y}$; some elementary calculus shows that

$$\nabla g(x, y) = \begin{bmatrix} \frac{2x}{y} \\ -\frac{x^2}{y^2} \end{bmatrix},$$

   and if we evaluate it at $(\mu_X, \mu_Y) = (1/2, 1)$, we get

$$\nabla g(1/2, 1) = \begin{bmatrix} 1 \\ -\frac{1}{4} \end{bmatrix}.$$

   The asymptotic variance is therefore

$$\begin{bmatrix} 1 & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{1}{4} \end{bmatrix} = \frac{5}{16}.$$

■