

## 1 Part 1: probability review

### Exercise 1.1

Let  $X$  be a random variable taking values between 0 and  $\pi$  with pdf given by  $f(x) = c \sin(x)$ ,  $x \in [0, \pi]$ . What is the value of  $c$ ?

**Solution:** Since the integral of the pdf is always 1 (by definition),

$$1 = \int_0^\pi f(x) dx = \int_0^\pi c \sin x dx = -c \Big|_0^\pi \cos x = -c(\cos \pi - \cos 0) = -c(-2) = 2c.$$

And so,  $2c = 1$  and  $c = \frac{1}{2}$ .  $\boxed{B}$ . ■

### Exercise 1.2

What is  $\mathbb{E}[X]$ ?

**Solution:** By the definition of expectation,

$$\mathbb{E}[X] = \int_0^\pi x f(x) dx = \int_0^\pi cx \sin x dx = c \Big|_0^\pi (\sin x - x \cos x) = c(-\pi \cos \pi + \sin \pi - 0 \cos 0 + \sin 0) = \pi c.$$

We know from the previous problem that  $c = \frac{1}{2}$ , so  $\mathbb{E}[X] = \frac{\pi}{2}$ .  $\boxed{A}$ . ■

### Exercise 1.3

Let  $X$  be a Gaussian random variable with mean  $\mu > 0$  and variance  $\mu^2$ . What is  $\mathbb{E}[X]$ ?

**Solution:** The mean, that is  $\mathbb{E}[X]$ , is  $\mu$  by definition.  $\boxed{B}$ . ■

### Exercise 1.4

What is  $\mathbb{E}[X^2]$ ?

**Solution:** By definition of variance,

$$\mu^2 = \text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mathbb{E}[X^2] - \mu^2,$$

so  $\mathbb{E}[X^2] = 2\mu^2$ .  $\boxed{C}$ . ■

### Exercise 1.5

What is  $\mathbb{E}[X^3]$ ?

**Solution:** Using the binomial theorem,

$$\begin{aligned} \mathbb{E}[X^3] &= \mathbb{E}[(X - \mu) + \mu]^3 \\ &= \mathbb{E}[(X - \mu)^3] + 3\mathbb{E}[(X - \mu)^2\mu] + 3\mathbb{E}[(X - \mu)\mu^2] + \mathbb{E}[\mu^3] \\ &= 3\mu\mathbb{E}[(X - \mu)^2] + \mu^3. \end{aligned}$$

Since  $X - \mu$  is a Gaussian random variable with mean 0 and variance  $\mu^2$ ,  $\mathbb{E}[(X - \mu)^2] = \mu^2$ ,

$$\mathbb{E}[X^3] = 3\mu^3 + \mu^3 = 4\mu^3.$$

$\boxed{C}$ . ■

**Exercise 1.6**

What is  $\text{Var}[X^2]$ ?

**Solution:** For a normal random variable with mean  $\mu$  and standard deviation  $\sigma$ ,  $\mathbb{E}[X^4] = 3\sigma^4 + 6\mu^2\sigma^2 + \mu^4$ . Since our mean and standard deviation are both  $\mu$  in this case,  $\mathbb{E}[X^4] = 10\mu^4$ . Using this,

$$\begin{aligned}\text{Var}[X^2] &= \mathbb{E}[X^4] - \mathbb{E}[X^2]^2 \\ &= 10\mu^4 - (2\mu^2)^2 \\ &= 6\mu^4.\end{aligned}$$

$\boxed{B}$ . ■

**Exercise 1.7**

What is  $\mathbb{P}[X > 0]$  in terms of the CDF  $\Phi$  of the standard Gaussian distribution?

**Solution:** By definition of the cdf,

$$\mathbb{P}[X > 0] = \mathbb{P}\left[\frac{X - \mu}{\sigma} > -\frac{\mu}{\sigma}\right] = \mathbb{P}\left[\frac{X - \mu}{\sigma} > -1\right] = \mathbb{P}\left[\frac{X - \mu}{\sigma} < 1\right] = \Phi(1).$$

$\boxed{B}$ . ■

**Exercise 1.8**

Let  $X$  be a random variable such that

$$X = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } 1 - p \end{cases}$$

for some  $p \in [0, 1]$ . What is  $\mathbb{E}[X]$ ?

**Solution:** Routine algebra shows that  $\mathbb{E}[X] = 1 \cdot p + (-1) \cdot (1 - p) = -1 + 2p$ .  $\boxed{D}$ . ■

**Exercise 1.9**

What is  $\text{Var}[X]$ ?

**Solution:** The variance of  $X$  is  $\mathbb{E}[X^2] - \mathbb{E}[X]^2 = 1 - (1 - 2p)^2 = 4p - 4p^2 = 4p(1 - p)$ .  $\boxed{C}$ . ■

**Exercise 1.10**

For what  $p$  is  $\text{Var}[X]$  maximized?

**Solution:** We know from the previous problem that  $\text{Var}[X] = 4p - 4p^2$ , which has derivative  $4 - 8p$ . The variance is maximized when that derivative is 0, so when  $4 - 8p = 0 \implies p = \frac{1}{2}$ .  $\boxed{C}$ . ■

**Exercise 1.11**

What is  $\mathbb{E}[X^k]$ ?

**Solution:** The expected value is  $\mathbb{E}[X^k] = 1^k \cdot p + (-1)^k \cdot (1 - p) = p + (-1)^k \cdot (1 - p)$ .  $\boxed{D}$ . ■

**Exercise 1.12**

Let  $X$  and  $Y$  be two independent standard Gaussian random variables. What is  $\mathbb{E}[X^2Y]$ ?

**Solution:** Since  $X$  and  $Y$  are independent,  $\mathbb{E}[X^2Y] = \mathbb{E}[X^2] \cdot \mathbb{E}[Y] = 0$ . A. ■

**Exercise 1.13**

What is  $\text{Var}(X + Y)$ ?

**Solution:** Since  $X$  and  $Y$  are independent,  $\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y] = 1 + 1 = 2$ . C. ■

**Exercise 1.14**

What is  $\text{Var}[XY]$ ?

**Solution:** The variance is  $\text{Var}[XY] = \mathbb{E}[(XY)^2] - \mathbb{E}[XY]^2 = \mathbb{E}[X^2]\mathbb{E}[Y^2] = 1$ . B. ■

**Exercise 1.15**

What is  $\text{Cov}[X, X + Y]$ ?

**Solution:** The covariance is  $\text{Cov}[X, X + Y] = \text{Cov}[X, X] + \text{Cov}[X, Y] = 1 + 0 = 1$ . B. ■

**Exercise 1.16**

What is  $\text{Cov}[X, XY]$ ?

**Solution:** The covariance is  $\text{Cov}[X, XY] = \mathbb{E}[X^2Y] - \mathbb{E}[X]\mathbb{E}[XY] = 0$ . A. ■

**Exercise 1.17****Exercise 1.18****Exercise 1.19**

Let  $X_1, \dots, X_n$  be i.i.d. with mean  $\mu$  and variance  $\sigma^2$ . What is  $\mathbb{E}[\sum_{i=1}^n X_i]$ ?

**Solution:** By linearity of expectation,

$$\mathbb{E}\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n \mathbb{E}[X_i] = n\mu.$$

C. ■

**Exercise 1.20**

What is  $\text{Var}[\sum_{i=1}^n X_i]$ ?

**Solution:** Since each of the  $X_i$ s are independent,

$$\text{Var} \left[ \sum_{i=1}^n X_i \right] = \sum_{i=1}^n \text{Var}[X_i] = n\sigma^2.$$

$\square$ . ■

### Exercise 1.21

What is  $\mathbb{E}[(\sum_{i=1}^n X_i)^2]$ ?

**Solution:** Let  $Y = \sum_{i=1}^n X_i$ . Then,

$$n\sigma^2 = \text{Var}[Y] = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = \mathbb{E}[Y^2] - (n\mu)^2,$$

so  $\mathbb{E}[Y^2] = n^2\mu^2 + n\sigma^2$ .  $\square$ . ■

### Exercise 1.22

What is  $\text{Var} \left[ \frac{1}{n} \sum_{i=1}^n X_i \right]$ ?

**Solution:** Since  $\text{Var}[aX] = a^2\text{Var}[X]$ ,

$$\text{Var} \left[ \frac{1}{n} \sum_{i=1}^n X_i \right] = \frac{1}{n^2} \text{Var} \left[ \sum_{i=1}^n X_i \right] = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}.$$

$\square$ . ■

## 2 New Concepts

### Exercise 2.1

Let  $X_n \sim \text{Unif}(-1/n, 1/n)$  and let  $X$  be a random variable such that  $\mathbb{P}[X = 0] = 1$ .

1. Compute and draw the CDF  $F_n(x)$  and  $F(x)$  of  $X_n$  and  $X$  respectively.
2. Does  $X_n \xrightarrow{P} X$ ? (prove or disprove)
3. Does  $X_n \rightsquigarrow X$  (prove or disprove)

**Solution:**

1. The CDF of  $X$  is

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

and the CDF of  $X_n$  is

$$F_n(x) = \begin{cases} 0 & x < -\frac{1}{n} \\ \frac{nt+1}{2} & -\frac{1}{n} \leq \frac{1}{n} \\ 1 & x > \frac{1}{n} \end{cases}.$$

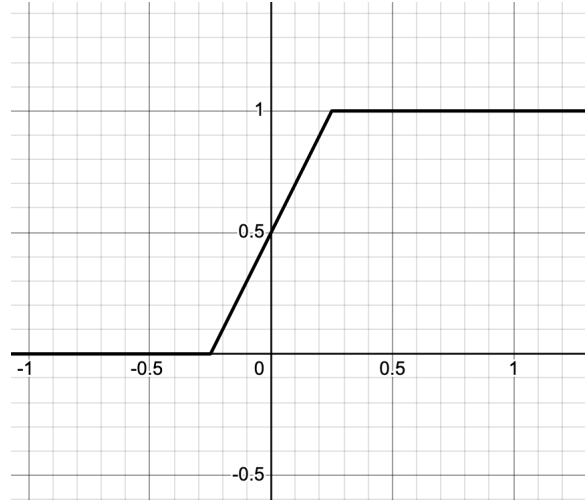


Figure 1: This is the cdf for  $X_n$ , when  $n = 4$ .

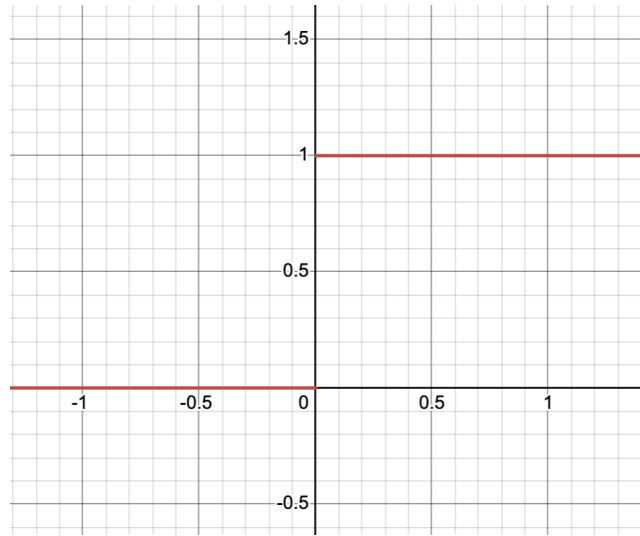


Figure 2: This is the cdf for  $X$ .

2. We know that for all  $\epsilon$ ,

$$\mathbb{P}[|X_n - X| > \epsilon] = \mathbb{P}[|X_n| > \epsilon] = 1 - \mathbb{P}[|X_n| \leq \epsilon] = 1 - \min\left(\frac{2\epsilon}{\frac{2}{n}}, 1\right) = 1 - \min(n\epsilon, 1).$$

Therefore,

$$\lim_{n \rightarrow \infty} \mathbb{P}[|X_n - X| > \epsilon] = \lim_{n \rightarrow \infty} 1 - \min(n\epsilon, 1) = 0,$$

so  $X_n$  does converge to  $X$  probabilistically.

3. The CDF of  $X$  is

$$F(t) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

and the CDF of  $X_n$  is

$$F_n(t) = \begin{cases} 0 & x < -\frac{1}{n} \\ \frac{nt+1}{2} & -\frac{1}{n} \leq x \leq \frac{1}{n} \\ 1 & x > \frac{1}{n} \end{cases}.$$

As  $n$  approaches  $\infty$ ,  $F_n(t)$  approaches

$$F_n(t) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = \frac{1}{2} \\ 1 & x = 1 \end{cases}.$$

Although  $F_n(t)$  and  $F(t)$  are not the same functions, they have equal values at all points for which  $F$  is continuous. Thus,  $X_n$  converges to  $X$  in distribution.

■