All of Statistics Chapter 3

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1 Expectation

> **Exercise 1.1**
>
> Suppose we play a game where we start with c dollars. On each play of the game, you either double or halve your money, with equal probability. What is your expected fortune after n trials?

Solution: Let X_i denote the amount of money you have after playing the game i times. When $i = 0$, by definition, $\mathbb{P}[X_0 = c] = 1$, and so, $\mathbb{E}[X_0] = c$. When $i > 0$,

$$\mathbb{E}[X_i] = \mathbb{E}\left[\frac{1}{2} \cdot (2 \cdot X_{i-1}) + \frac{1}{2} \cdot \left(\frac{1}{2} \cdot X_i\right)\right] = \mathbb{E}\left[\frac{5}{4}X_{i-1}\right] = \frac{5}{4}\mathbb{E}\left[X_{i-1}\right].$$

It immediately follows that $\mathbb{E}[X_i] = \left(\frac{5}{4}\right)^i \cdot c$. Thus, after n trials, your expected fortune is $c \cdot \left(\frac{5}{4}\right)^n$. ∎

> **Exercise 1.2**
>
> Show that $\text{Var}[X] = 0$ if and only if there is a constant c such that $\mathbb{P}[X = c] = 1$.

Solution: We first prove the easier direction, namely that if $\mathbb{P}[X = c] = 1$, then $\text{Var}[X] = 0$. In this case, $\mathbb{E}[X^2] = c^2$ and $\mathbb{E}[X]^2 = c^2$ too, so $\text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 0$, as desired.
As for the other direction, by Jensen's inequality,

$$\mathbb{E}[X^2] \geq \mathbb{E}[X]^2 \iff \text{Var}[X] \geq 0,$$

with equality holding iff X is constant, i.e. $\mathbb{P}[X = c] = 1$ for some c. ∎

> **Exercise 1.3**
>
> Let $X_1, \ldots, X_n \sim \text{Uniform}[0, 1]$ and let $Y_n = \max(X_1, \ldots X_n)$. Find $\mathbb{E}[Y_n]$.

Solution: Consider the CDF of Y_n:

$$F_{Y_n}(x) = \mathbb{P}[Y_n \leq x] = \mathbb{P}[X_1, X_2, \ldots, X_n \leq x] = \mathbb{P}[X_1 \leq x]^n = x^n.$$

The PDF of Y_n is therefore $f_{Y_n}(x)\frac{\mathrm{d}}{\mathrm{d}x}[x^n] = nx^{n-1}$, and so

$$\mathbb{E}[Y_n] = \int f_{Y_n}(x) \cdot x \mathrm{d}x = \int_0^1 nx^{n-1} \cdot x \mathrm{d}x = \int_0^1 nx^n \mathrm{d}x = \frac{n}{n+1}.$$

∎

> **Exercise 1.4**
>
> A particle starts at the origin of the real line and moves along the line in jumps of one unit. For each jump the probability is p that the particle will jump one unit to the left and the probability is $1 - p$ that

the particle will jump one unit to the right. Let X_n be theposition of the particle after n units. Find $\mathbb{E}[X_n]$ and $\mathrm{Var}[X_n]$.

Solution: When $n = 0$, $\mathbb{E}[X_n] = 0$, and when $n > 0$,

$$\begin{aligned}\mathbb{E}[X_n] &= \mathbb{E}[p \cdot (X_{n-1} - 1)] + \mathbb{E}[(1-p) \cdot (X_{n-1} + 1)] \\ &= p\,\mathbb{E}[X_{n-1}] - p + (1-p)\,\mathbb{E}[X_{n-1}] + 1 - p \\ &= \mathbb{E}[X_{n-1}] + 1 - 2p.\end{aligned}$$

Therefore, by induction $\mathbb{E}[X_n] = (1 - 2p) \cdot n$.
As for the variance, let Y_i denote whether or not you move left (-1) or right $(+1)$ on the ith jump.Then,

$$\begin{aligned}\mathrm{Var}[X_n] &= \mathrm{Var}[Y_1 + Y_2 + \ldots + Y_n] \\ &= n\,\mathrm{Var}[Y_1] \\ &= n(4p^2 - 4p).\end{aligned}$$

∎

Exercise 1.5

A fair coin is tossed until a head is obtained. What is the expected number of tosses that will be required?

Solution: If we let X_i denote the event that the first time we get heads is on the ith toss, then $\mathbb{P}[X_i] = \frac{1}{2^i}$, so

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} \frac{i}{2^i} = 2,$$

where X denotes the number of tosses required.

∎

Exercise 1.6

Prove theorem 3.6 for discrete random variables.

Solution: For discrete random variables, the expected value of X is

$$\sum_i \mathbb{P}[X = x_i] x_i,$$

so the expected value of $r(X)$ is

$$\sum_i \mathbb{P}[r(X) = r_i] r_i = \sum_i \mathbb{P}[r(X) = r(x_i)] x_i.$$

∎

Exercise 1.7

Let X be a continous random variable with CDF F. Suppose that $\mathbb{P}[X > 0] = 1$ and that $\mathbb{E}[X]$ exists. Show that $\mathbb{E}[X] = \int_0^{\infty} \mathbb{P}[X > x]\mathrm{d}x$.

Solution: We know that $\mathbb{P}[X > x] = \int_x^{\infty} f_X(y)\mathrm{d}y$, so

$$\begin{aligned}\int_0^{\infty} \mathbb{P}[X > x]\mathrm{d}x &= \int_0^{\infty} \int_x^{\infty} f_X(y)\mathrm{d}y\mathrm{d}x \\ &= \int_0^{\infty} \int_0^{y} f_X(y)\mathrm{d}x\mathrm{d}y \\ &= \int_0^{\infty} f_X(y)y\mathrm{d}y \\ &= \mathbb{E}[X],\end{aligned}$$

from which the desired follows. ∎

Solution: By linearity of expectation

$$\mathbb{E}[\overline{X}_n] = \mathbb{E}\left[\frac{X_1 + X_2 + \ldots + X_n}{n}\right] = \frac{\mathbb{E}[X_1] + \mathbb{E}[X_2] + \ldots + \mathbb{E}[X_n]}{n} = \frac{n\mu}{n} = \mu.$$

Additionally,

$$\text{Var}[\overline{X}_n] = \text{Var}\left[\frac{X_1 + \ldots + X_n}{n}\right] = \frac{1}{n^2}\text{Var}[X_1 + \ldots + X_n] = \frac{1}{n^2}\left(\text{Var}[X_1] + \ldots + \text{Var}[X_n]\right) = \frac{\sigma^2}{n^2}.$$

The expected value of the sample variance is

$$\begin{aligned}
S_n &= \mathbb{E}\left[\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2\right] \\
&= \frac{n}{n-1}\cdot\mathbb{E}\left[\left(X_1 - \overline{X}_n\right)^2\right] \\
&= \frac{1}{n(n-1)}\cdot\mathbb{E}\left[((n-1)X_1 - X_2 - X_3 \ldots - X_n)^2\right].
\end{aligned}$$

If we let $Y = (n-1)X_1 - X_2 - \ldots - X_n$, then

$$\text{Var}[Y] = \text{Var}[(n-1)X_1] + \text{Var}[X_2] + \ldots + \text{Var}[X_n] = (n-1)^2\sigma + (n-1)\sigma = n(n-1)\sigma,$$

so $\mathbb{E}[Y^2] = \text{Var}[Y] + \mathbb{E}[Y]^2 = n(n-1)\sigma^2$. Plugging this into the prior equation, $\mathbb{E}[S_n^2] = \frac{1}{n(n-1)}\cdot n(n-1)\sigma^2 = \sigma^2$.
∎

Exercise 1.9

Let $X_1, X_2, \ldots X_n$ be N[0,1] random variables and let $\overline{X}_n = n^{-1}\sum_{i=1}^{n}X_i$. Plot $\overline{X}_n$ verus n for $n = 1, \ldots, 10,000$. Repeat for $X_1, X_2, \ldots X_n \sim$ Cauchy. Explain why there is such a difference.

Solution: See the corresponding `.ipynb` file for graphs. Expectedly, the sample mean for the normal distribution appears to converge to 1, but the sample mean of the Cauchy sequence does not converge (since the Cauchy sequence is so fat-tailed, it does not have a well-defined mean). ∎

Exercise 1.10

Let $X \sim$ N[0,1] and let $Y = e^X$. Find $\mathbb{E}[Y]$ and $\text{Var}[Y]$.

Solution: The MGF of X is $\psi_X(t) = \mathbb{E}[e^{Xt}] = e^{\frac{1}{2}t^2}$, so $\mathbb{E}[Y] = \sqrt{e}$ and $\mathbb{E}[Y^2] = e^2$. It follows that $\text{Var}[Y] = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = e^2 - e$. ∎