18.650 Homework 1

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1 Expectation

Exercise 1.1

Suppose we play a game where we start with c dollars. On each play of the game, you either double or halve your money, with equal probability. What is your expected fortune after n trials?

Solution: Let X_i denote the amount of money you have after playing the game i times. When $i = 0$, by definition, $\mathbb{P}[X_0 = c] = 1$, and so, $\mathbb{E}[X_0] = c$. When $i > 0$,

$$\mathbb{E}[X_i] = \mathbb{E}\left[\frac{1}{2} \cdot (2 \cdot X_{i-1}) + \frac{1}{2} \cdot \left(\frac{1}{2} \cdot X_i\right)\right] = \mathbb{E}\left[\frac{5}{4} X_{i-1}\right] = \frac{5}{4}\,\mathbb{E}\left[X_{i-1}\right].$$

It immediately follows that $\mathbb{E}[X_i] = \left(\frac{5}{4}\right)^i \cdot c$. Thus, after n trials, your expected fortune is $c \cdot \left(\frac{5}{4}\right)^n$. ■

Exercise 1.2

Show that $\mathrm{Var}[X] = 0$ if and only if there is a constant c such that $\mathbb{P}[X = c] = 1$.

Solution: We first prove the easier direction, namely that if $\mathbb{P}[X = c] = 1$, then $\mathrm{Var}[X] = 0$. In this case, $\mathbb{E}[X^2] = c^2$ and $\mathbb{E}[X]^2 = c^2$ too, so $\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 0$, as desired.
As for the other direction, by Jensen's inequality,

$$\mathbb{E}[X^2] \geq \mathbb{E}[X]^2 \iff \mathrm{Var}[X] \geq 0,$$

with equality holding iff X is constant, i.e. $\mathbb{P}[X = c] = 1$ for some c. ■

Exercise 1.3

Let $X_1, \ldots, X_n \sim \mathrm{Uniform}[0, 1]$ and let $Y_n = \max\left(X_1, \ldots X_n\right)$. Find $\mathbb{E}[Y_n]$.

Solution: Consider the CDF of Y_n:

$$F_{Y_n}(x) = \mathbb{P}[Y_n \leq x] = \mathbb{P}[X_1, X_2, \ldots, X_n \leq x] = \mathbb{P}[X_1 \leq x]^n = x^n.$$

The PDF of Y_n is therefore $f_{Y_n}(x) \frac{\mathrm{d}}{\mathrm{d}x}\left[x^n\right] = nx^{n-1}$, and so

$$\mathbb{E}[Y_n] = \int f_{Y_n}(x) \cdot x\mathrm{d}x = \int_0^1 nx^{n-1} \cdot x\mathrm{d}x = \int_0^1 nx^n \mathrm{d}x = \frac{n}{n+1}.$$

■

Exercise 1.4

A particle starts at the origin of the real line and moves along the line in jumps of one unit. For each jump the probability is p that the particle will jump one unit to the left and the probability is $1 - p$ that the particle will jump one unit to the right. Let X_n be theposition of the particle after n units. Find $\mathbb{E}[X_n]$ and $\mathrm{Var}[X_n]$.

Solution: When $n = 0$, $\mathbb{E}[X_n] = 0$, and when $n > 0$,

$$\begin{aligned}\mathbb{E}[X_n] &= \mathbb{E}[p \cdot (X_{n-1} - 1)] + \mathbb{E}[(1-p) \cdot (X_{n-1} + 1)] \\ &= p\,\mathbb{E}[X_{n-1}] - p + (1-p)\,\mathbb{E}[X_{n-1}] + 1 - p \\ &= \mathbb{E}[X_{n-1}] + 1 - 2p.\end{aligned}$$

Therefore, by induction $\mathbb{E}[X_n] = (1-2p) \cdot n$.
As for the variance, let Y_i denote whether or not you move left (-1) or right $(+1)$ on the ith jump.Then,

$$\begin{aligned}\mathrm{Var}[X_n] &= \mathrm{Var}[Y_1 + Y_2 + \ldots + Y_n] \\ &= n\,\mathrm{Var}[Y_1] \\ &= n(4p^2 - 4p).\end{aligned}$$

■

Exercise 1.5

A fair coin is tossed until a head is obtained. What is the expected number of tosses that will be required?

Solution: If we let X_i denote the event that the first time we get heads is on the ith toss, then $\mathbb{P}[X_i] = \frac{1}{2^i}$, so

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} \frac{i}{2^i} = 2,$$

where X denotes the number of tosses required. ■