Computer Vision 6.S058 Problem Set 1

Maria Chrysafis

February 12, 2025

1 Problem 1

2 Problem 2

We are in effect given the equation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \alpha \cdot P \cdot R_x(\theta) \cdot \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix},$$

where $R_x(\theta)$ represents a rotation amtrix around the X-axis by an angle θ, P is a projection matrix that reduces 3D world coordinates to 2D image coordinates, α is a scaling factor to account for the camera sensor size, and (x_0, y_0) represents the image coordinates of the origin of the camera coordinate system.

In this case, the rotation matrix around the X axis by an angle θ is

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

and the projection matrix is

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \end{bmatrix}.$$

Therefore, this equation becomes

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= \alpha \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \\ &= \alpha \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos^2\theta & -\sin^2\theta & -2\cos\theta\sin\theta \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \\ &= \begin{bmatrix} \alpha(X + x_0) \\ \alpha(\cos\theta Y - \sin\theta Z + y_0) \end{bmatrix}. \end{aligned}$$

We know that $(0,0,0)$ projects to $(0,0)$, i.e. when $X = 0, Y = 0$, and $z = 0$, we get $x = 0$ and $y = 0$. Therefore, the equation becomes

$$\begin{aligned} x &= \alpha(0) + x_0 = 0 \implies x_0 = 0 \\ y &= \alpha(\cos\theta(0) - \sin\theta(0)) + y_0 = 0 \implies y_0 = 0. \end{aligned}$$

So, $x_0 = 0$ and $y_0 = 0$. Now, applying the second condition (that $(1,0,0)$ projects to $(3,0)$), we get that

$$\begin{aligned} x &= \alpha(1) + x_0 = 3 \implies \alpha = 3 \\ y &= \alpha(\cos\theta(1) - \sin\theta(0)) + y_0. \end{aligned}$$

So, $\alpha = 3$, $x_0 = 0$, and $y_0 = 0$.

3 Problem 3

We know that

$$Z(x,y) = \frac{Y(x,y)\cos\theta}{\sin\theta} - \frac{y}{\sin\theta}.$$

3.1 Constraint Along Vertical Edges

We know that

$$\begin{aligned}
\frac{\partial Z}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{Y\cos\theta}{\sin\theta} - \frac{y}{\sin\theta}\right) \\
&= \frac{\cos\theta}{\sin\theta}\cdot\frac{\partial Y}{\partial y} - \frac{1}{\sin\theta} \\
&= \frac{\cos\theta}{\sin\theta}\cdot\frac{1}{\cos\theta} - \frac{1}{\sin\theta} \\
&= 0.
\end{aligned}$$

3.2 Constraint Along Horizontal Edges

For the constraint along horizontal edges,

$$\begin{aligned}
\frac{\partial Z}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{Y\cos\theta}{\sin\theta} - \frac{y}{\sin\theta}\right) \\
&= \frac{\cos\theta}{\sin\theta}\cdot\frac{\delta Y}{\delta t} - \frac{1}{\sin\theta}\cdot\frac{\delta y}{\delta t}.
\end{aligned}$$

Since $\frac{\delta Y}{\delta t} = 0$ (from the horizontal edge constraint) and $\frac{\delta y}{\delta t} = n_x$ (because $t = (-n_y, n_x)$),

$$\frac{\partial Z}{\partial t} = 0 - \frac{n_x}{\sin\theta} = -\frac{n_x}{\sin\theta}.$$

3.3 Constraint on Flat Surfaces

For flat surfaces, the second derivative of $Y(x,y)$ is zero, so

$$\frac{\partial^2 Y}{\partial x^2} = \frac{\partial^2 Y}{\partial y^2} = \frac{\partial^2 Y}{\partial x \partial y} = 0.$$

Some algebra shows that

$$\begin{aligned}
\frac{\partial^2 Z}{\partial x^2} &= \frac{\partial^2}{\partial x^2}\left(\frac{Y\cos\theta}{\sin\theta} - \frac{y}{\sin\theta}\right) = 0 \\
\frac{\partial^2 Z}{\partial x^2} &= \frac{\partial^2}{\partial y^2}\left(\frac{Y\cos\theta}{\sin\theta} - \frac{y}{\sin\theta}\right) = 0 \\
\frac{\partial^2 Z}{\partial x^2} &= \frac{\partial^2}{\partial x \partial y}\left(\frac{Y\cos\theta}{\sin\theta} - \frac{y}{\sin\theta}\right) = 0.
\end{aligned}$$

So,

$$\frac{\partial^2 Z}{\partial x^2} = \frac{\partial^2 Z}{\partial y^2} = \frac{\partial^2 Z}{\partial x \partial y} = 0.$$

4 Problem 4

5 Problem 5

6 Problem 6

7 Problem 7

Part (a): Equations Relating Angles of Rays Passing Through a Plano-Convex Lens

For a plano-convex lens, the front surface has a radius of curvature R, and the back surface is flat. The equations relating the angles of rays passing through the lens are derived using Snell's Law and the paraxial approximation (small angles).

1. **Snell's Law at the First Surface (Curved Surface):**

   $$n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$$

   Where:

   - n_1 is the refractive index of the medium before the lens (usually air, $n_1 \approx 1$).
   - n_2 is the refractive index of the lens material.
   - θ_1 is the angle of incidence.
   - θ_2 is the angle of refraction.

2. **Angle of Incidence at the First Surface:**

   $$\theta_1 = \alpha + \phi$$

   Where:

   - α is the angle the ray makes with the optical axis before refraction.
   - ϕ is the angle the normal to the surface makes with the optical axis.

3. **Angle of Refraction at the First Surface:**

   $$\theta_2 = \phi + \beta$$

   Where:

   - β is the angle the refracted ray makes with the optical axis.

4. **Snell's Law at the Second Surface (Flat Surface):**

   $$n_2 \sin(\beta) = n_1 \sin(\gamma)$$

   Where:

   - γ is the angle of the ray after exiting the lens.

5. **Paraxial Approximation:**

   $$\sin(\theta) \approx \theta \quad \text{(for small angles)}$$

   This approximation simplifies the trigonometric relationships, making the equations linear and easier to solve.

Part (b): Expression for the Lens Focal Length

The lensmaker's formula for a thin lens is given by:

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

Where:

- f is the focal length of the lens.
- n is the refractive index of the lens material.
- R_1 and R_2 are the radii of curvature of the two lens surfaces.

For a plano-convex lens:

- $R_1 = R$ (radius of curvature of the curved surface).
- $R_2 = \infty$ (since the back surface is flat).

Substituting these into the lensmaker's formula:

$$\frac{1}{f} = (n-1)\left(\frac{1}{R} - \frac{1}{\infty}\right) = (n-1)\left(\frac{1}{R} - 0\right) = \frac{n-1}{R}$$

Therefore, the focal length f of the plano-convex lens is:

$$f = \frac{R}{n-1}$$