Computer Vision 6.S058 Problem Set 1

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1 Problem 1

2 Problem 2

We are in effect given the equation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \alpha \cdot P \cdot R_x(\theta) \cdot \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix},$$

where $R_x(\theta)$ represents a rotation amtrix around the X-axis by an angle θ, P is a projection matrix that reduces 3D world coordinates to 2D image coordinates, α is a scaling factor to account for the camera sensor size, and (x_0, y_0) represents the image coordinates of the origin of the camera coordinate system.

In this case, the rotation matrix around the X axis by an angle θ is

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

and the projection matrix is

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \end{bmatrix}.$$

Therefore, this equation becomes

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= \alpha \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \\ &= \alpha \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos^2\theta & -\sin^2\theta & -2\cos\theta\sin\theta \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \\ &= \begin{bmatrix} \alpha(X + x_0) \\ \alpha(\cos\theta Y - \sin\theta Z + y_0) \end{bmatrix}. \end{aligned}$$

We know that $(0, 0, 0)$ projects to $(0, 0)$, i.e. when $X = 0, Y = 0$, and $z = 0$, we get $x = 0$ and $y = 0$. Therefore, the equation becomes

$$\begin{aligned} x &= \alpha(0) + x_0 = 0 \implies x_0 = 0 \\ y &= \alpha(\cos\theta(0) - \sin\theta(0)) + y_0 = 0 \implies y_0 = 0. \end{aligned}$$

So, $x_0 = 0$ and $y_0 = 0$. Now, applying the second condition (that $(1, 0, 0)$ projects to $(3, 0)$), we get that

$$\begin{aligned} x &= \alpha(1) + x_0 = 3 \implies \alpha = 3 \\ y &= \alpha(\cos\theta(1) - \sin\theta(0)) + y_0. \end{aligned}$$

So, $\alpha = 3$, $x_0 = 0$, and $y_0 = 0$.

3 Problem 3

We know that

$$Z(x,y) = \frac{Y(x,y)\cos\theta}{\sin\theta} - \frac{y}{\sin\theta}.$$

3.1 Constraint Along Vertical Edges

We know that

$$\begin{aligned}
\frac{\partial Z}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{Y\cos\theta}{\sin\theta} - \frac{y}{\sin\theta}\right) \\
&= \frac{\cos\theta}{\sin\theta}\cdot\frac{\partial Y}{\partial y} - \frac{1}{\sin\theta} \\
&= \frac{\cos\theta}{\sin\theta}\cdot\frac{1}{\cos\theta} - \frac{1}{\sin\theta} \\
&= 0.
\end{aligned}$$

3.2 Constraint Along Horizontal Edges

For the constraint along horizontal edges,

$$\begin{aligned}
\frac{\partial Z}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{Y\cos\theta}{\sin\theta} - \frac{y}{\sin\theta}\right) \\
&= \frac{\cos\theta}{\sin\theta}\cdot\frac{\delta Y}{\delta t} - \frac{1}{\sin\theta}\cdot\frac{\delta y}{\delta t}.
\end{aligned}$$

Since $\frac{\delta Y}{\delta t} = 0$ (from the horizontal edge constraint) and $\frac{\delta y}{\delta t} = n_x$ (because $t = (-n_y, n_x)$),

$$\frac{\partial Z}{\partial t} = 0 - \frac{n_x}{\sin\theta} = -\frac{n_x}{\sin\theta}.$$

3.3 Constraint on Flat Surfaces

For flat surfaces, the second derivative of $Y(x,y)$ is zero, so

$$\frac{\partial^2 Y}{\partial x^2} = \frac{\partial^2 Y}{\partial y^2} = \frac{\partial^2 Y}{\partial x \partial y} = 0.$$

Some algebra shows that

$$\begin{aligned}
\frac{\partial^2 Z}{\partial x^2} &= \frac{\partial^2}{\partial x^2}\left(\frac{Y\cos\theta}{\sin\theta} - \frac{y}{\sin\theta}\right) = 0 \\
\frac{\partial^2 Z}{\partial x^2} &= \frac{\partial^2}{\partial y^2}\left(\frac{Y\cos\theta}{\sin\theta} - \frac{y}{\sin\theta}\right) = 0 \\
\frac{\partial^2 Z}{\partial x^2} &= \frac{\partial^2}{\partial x \partial y}\left(\frac{Y\cos\theta}{\sin\theta} - \frac{y}{\sin\theta}\right) = 0.
\end{aligned}$$

So,

$$\frac{\partial^2 Z}{\partial x^2} = \frac{\partial^2 Z}{\partial y^2} = \frac{\partial^2 Z}{\partial x \partial y} = 0.$$