1 Part 1: probability review

Exercise 1.1

Let X be a random variable taking values between 0 and π with pdf given by $f(x) = c\sin(x), x \in [0, \pi]$. What is the value of c?

Solution: Since the integral of the pdf is always 1 (by definition),

$$1 = \int_0^\pi f(x)\,\mathrm{d}x = \int_0^\pi c\sin x\,\mathrm{d}x = -c\Big|_0^\pi \cos x = -c(\cos\pi - \cos 0) = -c(-2) = 2c.$$

And so, $2c = 1$ and $c = \frac{1}{2}$. $\boxed{B}$. ∎

Exercise 1.2

What is $\mathbb{E}[X]$?

Solution: By the definition of expectation,

$$\mathbb{E}[X] = \int_0^\pi xf(x)\,\mathrm{d}x = \int_0^\pi cx\sin x\,\mathrm{d}x = c\Big|_0^\pi (\sin x - x\cos x) = c(-\pi\cos\pi + \sin\pi - 0\cos 0 + \sin 0) = \pi c.$$

We know from the previous problem that $c = \frac{1}{2}$, so $\mathbb{E}[X] = \frac{\pi}{2}$. $\boxed{A}$. ∎

Exercise 1.3

Let X be a Gaussian random variable with mean $\mu > 0$ and variance μ^2. What is $\mathbb{E}[X]$?

Solution: The mean, that is $\mathbb{E}[X]$, is μ by definition. $\boxed{B}$. ∎

Exercise 1.4

What is $\mathbb{E}[X^2]$?

Solution: By definition of variance,

$$\mu^2 = \mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mathbb{E}[X^2] - \mu^2,$$

so $\mathbb{E}[X^2] = 2\mu^2$. $\boxed{C}$. ∎

Exercise 1.5

What is $\mathbb{E}[X^3]$?

Solution: Using the binomial theorem,

$$\begin{aligned}\mathbb{E}[X^3] &= \mathbb{E}[((X-\mu)+\mu)^3] \\ &= \mathbb{E}[(X-\mu)^3] + 3\mathbb{E}[(X-\mu)^2\mu] + 3\mathbb{E}[(X-\mu)\mu^2] + \mathbb{E}[\mu^3] \\ &= 3\mu\mathbb{E}[(X-\mu)^2] + \mu^3.\end{aligned}$$

Since $X - \mu$ is a Gaussian random variable with mean 0 and variance μ^2, $\mathbb{E}[(X-\mu)^2] = \mu^2$,

$$\mathbb{E}[X^3] = 3\mu^3 + \mu^{=}4\mu^3.3$$

$\boxed{C}$. ∎

Exercise 1.6

What is $\mathrm{Var}[X^2]$?

Solution: For a normal random variable with mean μ and standard deviation σ, $\mathbb{E}[X^4] = 3\sigma^4 + 6\mu^2\sigma^2 + \mu^4$. Since our mean and standard deviation are both μ in this case, $\mathbb{E}[X^4] = 10\mu^4$. Using this,

$$\begin{aligned}\mathrm{Var}[X^2] &= \mathbb{E}[X^4] - \mathbb{E}[X^2]^2 \\ &= 10\mu^4 - (2\mu^2)^2 \\ &= 6\mu^4.\end{aligned}$$

$\boxed{B}$. ■

Exercise 1.7

What is $\mathbb{P}[X > 0]$ in terms of the CDF Φ of the standard Gaussian distribution?

Solution: By definition of the cdf,

$$\mathbb{P}[X > 0] = \mathbb{P}\left[\frac{X-\mu}{\sigma} > -\frac{\mu}{\sigma}\right] = \mathbb{P}\left[\frac{X-\mu}{\sigma} > -1\right] = \mathbb{P}\left[\frac{X-\mu}{\sigma} <\right] = \Phi(1).$$

$\boxed{B}$. ■

Exercise 1.8

Let X be a random variable such that

$$X = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } 1-p \end{cases}$$

for some $p \in [0, 1]$. What is $\mathbb{E}[X]$?

Solution: Routine algebra shows that $\mathbb{E}[X] = 1 \cdot p + (-1) \cdot (1-p) = -1 + 2p$. $\boxed{D}$. ■

Exercise 1.9

What is $\mathrm{Var}[X]$?

Solution: The variance of X is $\mathbb{E}[X^2] - \mathbb{E}[X]^2 = 1 - (1-2p)^2 = 4p - 4p^2 = 4p(1-p)$. $\boxed{C}$. ■

Exercise 1.10

For what p is $\mathrm{Var}[X]$ maximized?

Solution: We know from the previous problem that $\mathrm{Var}[X] = 4p - 4p^2$, which has derivative $4 - 8p$. The variance is maximized when that derivative is 0, so when $4 - 8p = 0 \Longrightarrow p = \frac{1}{2}$. $\boxed{C}$. ■

Exercise 1.11

What is $\mathbb{E}[X^k]$?

Solution: The expected value is $\mathbb{E}[X^k] = 1^k \cdot p + (-1)^k \cdot (1-p) = p + (-1)^k \cdot (1-p)$. $\boxed{D}$. ■

Exercise 1.12

Let X and Y be two independent standard Gaussian random variables. What is $\mathbb{E}[X^2Y]$?

Solution: Since X and Y are independent, $\mathbb{E}[X^2Y] = \mathbb{E}[X^2] \cdot \mathbb{E}[Y] = 0$. $\boxed{A}$. ∎

Exercise 1.13

What is $\mathrm{Var}(X + Y)$?

Solution: Since X and Y are independent, $\mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] = 1 + 1 = 2$. $\boxed{C}$. ∎

Exercise 1.14

What is $\mathrm{Var}[XY]$?

Solution: The variance is $\mathrm{Var}[XY] = \mathbb{E}[(XY)^2] - \mathbb{E}[XY]^2 = \mathbb{E}[X^2]\mathbb{E}[Y^2] = 1$. $\boxed{B}$. ∎

Exercise 1.15

What is $\mathrm{Cov}[X, X + Y]$?

Solution: The covariance is $\mathrm{Cov}[X, X + Y] = \mathrm{Cov}[X, X] + \mathrm{Cov}[X, Y] = 1 + 0 = 1$. $\boxed{B}$. ∎

Exercise 1.16

What is $\mathrm{Cov}[X, XY]$?

Solution: The covariance is $\mathrm{Cov}[X, XY] = \mathbb{E}[X^2Y] - \mathbb{E}[X]\mathbb{E}[XY] = 0$. $\boxed{A}$. ∎

Exercise 1.17

Exercise 1.18

Exercise 1.19

Let$X_1, \dots X_n$ be i.i.d. with mean μ and variance σ^2. What is $\mathbb{E}\left[\sum_{i=1}^n X_i\right]$?

Solution: By linearity of expectation,

$$\mathbb{E}\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n \mathbb{E}[X_i] = n\mu.$$

$\boxed{C}$. ∎

Exercise 1.20

What is $\mathrm{Var}[\sum_{i=1}^n X_i]$?

Solution: Since each of the X_is are independent,

$$\text{Var}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \text{Var}[X_i] = n\sigma^2.$$

$\boxed{B}$. ■

Exercise 1.21

What is $\mathbb{E}[(\sum_{i=1}^{n} X_i)^2]$?

Solution: Let $Y = \sum_{i=1}^{n} X_i$. Then,

$$n\sigma^2 = \text{Var}[Y] = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = \mathbb{E}[Y^2] - (n\mu)^2,$$

so $\mathbb{E}[Y^2] = n^2\mu^2 + n\sigma^2$. $\boxed{D}$. ■

Exercise 1.22

What is $\text{Var}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right]$?

Solution: Since $\text{Var}[aX] = a^2\text{Var}[X]$,

$$\text{Var}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n^2}\text{Var}\left[\sum_{i=1}^{n} X_i\right] = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}.$$

$\boxed{B}$. ■

2 New Concepts

Exercise 2.1

Let $X_n \sim \text{Unif}(-1/n, 1/n)$ and let X be a random variable such that $\mathbb{P}[X = 0] = 1$.

1. Compute and draw the CDF $F_n(x)$ and $F(x)$ of X_n and X respectively.
2. Does $X_n \xrightarrow{P} X$? (prove or disprove)
3. Does $X_n \rightsquigarrow X$ (prove or disprove)

Solution:

1.
2. We know that for all ϵ,

$$\mathbb{P}[|X_n - X| > \epsilon] = \mathbb{P}[|X_n| > \epsilon] = 1 - \mathbb{P}[|X_n| \leq \epsilon] = 1 - \min\left(\frac{2\epsilon}{\frac{2}{n}}, 1\right) = 1 - \min(n\epsilon, 1).$$

Therefore,

$$\lim_{n\to\infty} \mathbb{P}[|X_n - X| > \epsilon] = \lim_{n\to\infty} 1 - \min(n\epsilon, 1) = 0,$$

so X_n does converge to X probabilistically.

3. The CDF of X is

$$F(t) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

and the CDF of X_n is

$$F_n(t) = \begin{cases} 0 & x < -\frac{1}{n} \\ \frac{nt+1}{2} & -\frac{1}{n} \leq x \leq \frac{1}{n} \\ 1 & x > \frac{1}{n} \end{cases}.$$

As n approaches ∞, $F_n(t)$ approaches

$$F_n(t) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = \frac{1}{2} \\ 1 & x = 1 \end{cases}.$$

Although $F_n(t)$ and $F(t)$ are not the same functions, they have equal values at all points for which F is continous. Thus, X_n converges to X in distribution.

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