No. Tarea #1

Date. 15 Feb 2024

Tarea Vectores

Sección 1.5.7

2a. Demostrar $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$ donde

$\phi = \phi(r) = \phi(x,y,z)$ ⋆ $r = (x\hat{i} + y\hat{j} + z\hat{k}) = x^i\hat{i}_i$

$\psi = \psi(r) = \psi(x,y,z)$

$$\nabla(\phi\psi) = \phi\left[\left(\frac{\partial\psi}{\partial x}\hat{i} + \frac{\partial\psi}{\partial y}\hat{j} + \frac{\partial\psi}{\partial z}\hat{k}\right)\right] + \psi\left[\left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)\right]$$

$$\nabla(\phi\psi) = \phi\frac{\partial\psi}{\partial x}\hat{i} + \phi\frac{\partial\psi}{\partial y}\hat{j} + \phi\frac{\partial\psi}{\partial z}\hat{k} + \psi\frac{\partial\phi}{\partial x}\hat{i} + \psi\frac{\partial\phi}{\partial y}\hat{j} + \psi\frac{\partial\phi}{\partial z}\hat{k}$$

$$\nabla(\phi\psi) = \left(\phi\frac{\partial\psi}{\partial x} + \psi\frac{\partial\phi}{\partial x}\right)\hat{i} + \left(\phi\frac{\partial\psi}{\partial y} + \psi\frac{\partial\phi}{\partial y}\right)\hat{j} + \left(\phi\frac{\partial\psi}{\partial z} + \psi\frac{\partial\phi}{\partial z}\right)$$

$$\nabla(\phi\psi) = \frac{\partial(\phi\psi)}{\partial x}\hat{i} + \frac{\partial(\phi\psi)}{\partial y}\hat{j} + \frac{\partial(\phi\psi)}{\partial z}\hat{k} \quad ∴$$

Notación de índices

$$\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi = \phi(\partial^i\psi(x^i)e_i) + \psi(\partial^i\phi(x^i)e_i) = \partial^i\phi\psi(x^i)e_i$$

2d. Demostrar $\nabla\cdot(\nabla\times a)$ donde a es un campo vectorial

⋆ $a = a(r) = a(x,y,z) = a^i(x,y,z)\hat{i}_i$

$\nabla\times a = \varepsilon^{ijk}\partial_j a_k \hat{i}_i$

rotacional

La divergencia de un campo vectorial mide como el campo converge hacia cierto punto, en cambio el rotacional mide como el campo rota alrededor de dado punto. No puede haber rotacional asociado a ese flujo de campo por lo cual

$\nabla\cdot(\nabla\times a) = \nabla\times(\nabla\cdot a) = \phi$ → Demostración

$$\nabla\cdot a = \frac{\partial a_x}{\partial x}\hat{i} + \frac{\partial a_y}{\partial y}\hat{j} + \frac{\partial a_z}{\partial z}\hat{k}$$

$$\nabla\times(\nabla\cdot a) = \nabla\times\left(\frac{\partial a_x}{\partial x}\hat{i} + \frac{\partial a_y}{\partial y}\hat{j} + \frac{\partial a_z}{\partial z}\hat{k}\right)$$

$$= \left(\frac{\partial}{\partial y}\left(\frac{\partial a_z}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial a_y}{\partial y}\right)\right)\hat{i}, \left(\frac{\partial}{\partial z}\left(\frac{\partial a_x}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial a_z}{\partial z}\right)\right)\hat{j}, \left(\frac{\partial}{\partial x}\left(\frac{\partial a_y}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial a_x}{\partial x}\right)\right)\hat{k}$$

Cada término con las derivadas cruzadas, se anula

2f. Demostrar $\nabla\times(\nabla\times a) = \nabla(\nabla\cdot a) - \nabla^2 a$

$$(\nabla\times a)_i = \varepsilon^{ijk}\frac{\partial a_k}{\partial x_j}$$

$$\nabla\times(\nabla\times a)_i = \varepsilon^{ijk}\frac{\partial}{\partial x_j}(\nabla\times a)_k = \varepsilon^{ijk}\frac{\partial}{\partial x_j}\left(\varepsilon^{klm}\frac{\partial a_m}{\partial x_l}\right)$$

$$\nabla\times(\nabla\times a)_i = \varepsilon^{ijk}\varepsilon^{klm}\frac{\partial^2 a_m}{\partial x_l\partial x_j}$$

⋆ Aplicando Kronecker $\varepsilon^{ijk}\varepsilon^{klm} = \delta^i_l\delta^j_m - \delta^i_m\delta^j_l$

$$\nabla\times(\nabla\times a)_i = (\delta^i_l\delta^j_m - \delta^i_m\delta^j_l)\frac{\partial^2 a_m}{\partial x_l\partial x_j}$$

$$\nabla\times(\nabla\times a)_i = \frac{\partial^2 a_i}{\partial x_j\partial x_j} - \frac{\partial^2 a_j}{\partial x_j\partial x_i}$$

⋆ Usando $\nabla^2 = \frac{\partial^2}{\partial x_j\partial x_j}$

$$\nabla\times(\nabla\times a)_i = \frac{\partial^2 a_i}{\partial x_j\partial x_j} - (\nabla^2 a)_i$$

$$\nabla\times(\nabla\times a)_i = \nabla(\nabla\cdot a)_i - (\nabla^2 a)_i$$

Sección 1.6.6

2a. Demostrar $\cos(3\alpha) = \cos^3(\alpha) - 3\cos(\alpha)\operatorname{sen}^2(\alpha)$

$z = |z|(\cos\alpha + i\operatorname{sen}\alpha)$, supongamos $|z| = 1$, $z = \cos\alpha + i\operatorname{sen}\alpha$

$z^3 = (\cos\alpha + i\operatorname{sen}\alpha)^3 = \cos^3\alpha - 3\cos\alpha\operatorname{sen}^2\alpha + 3i\operatorname{sen}\alpha\cos^2\alpha - i\operatorname{sen}^3\alpha$

$z^3 = \cos^3\alpha - 3\cos\alpha\operatorname{sen}^2\alpha + i(3\operatorname{sen}\alpha\cos^2\alpha - \operatorname{sen}^3\alpha)$

⋆ A partir de la fórmula de Moivre tenemos que

$z^n = |z|^n(\cos(n\alpha) + i\operatorname{sen}(n\alpha))$, $z^3 = \cos(3\alpha) + i\operatorname{sen}(3\alpha)$

Si hacemos $z^3 = z^3$ e igualamos la parte real nos queda que $\cos(3\alpha) = \cos^3\alpha - 3\cos\alpha\operatorname{sen}^2\alpha$ ∴

2b. Demostrar que $\operatorname{sen}(3\alpha) = 3\cos^2\alpha\operatorname{sen}\alpha - \operatorname{sen}^3\alpha$

Con el desarrollo anterior, volvemos a igualar $z^3 = z^3$ pero esta vez su parte imaginaria

$$i\operatorname{sen}3\alpha = i(3\operatorname{sen}\alpha\cos^2\alpha - \operatorname{sen}^3\alpha)$$

$$\operatorname{sen}3\alpha = 3\operatorname{sen}\alpha\cos^2\alpha - \operatorname{sen}^3\alpha \quad ∴$$

5a. Encontrar las raíces de $(2i)^{1/2}$

$z = 2i \rightarrow \sqrt{z} = \sqrt{2i} \rightarrow z^{1/2} = (2i)^{1/2}$; $|z| =$ $n = 2$, $k = 0, 1$, $\theta = \frac{\pi}{2}$

$z^{1/2}$ para $k = 0 \rightarrow \sqrt{2}\left[\cos\left(\frac{\pi}{4}\right) + i\operatorname{sen}\left(\frac{\pi}{4}\right)\right] = 1 + i$

$z^{1/2}$ para $k = 1 \rightarrow \sqrt{2}\left[\cos\left(\frac{\frac{\pi}{2} + 2\pi}{2}\right) + i\operatorname{sen}\left(\frac{\frac{\pi}{2} + 2\pi}{2}\right)\right]$

$= -1 - i$

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5b. Encontrar las raícez de $(1-\sqrt{3}\,i)^{1/2}$

$z = 1-\sqrt{3}i$, $\sqrt{z} = \sqrt{1-\sqrt{3}i}$; $n = 2$; $k = 0, 1$; $\theta = -\frac{\pi}{3}$; $|z| = 2$

$z^{1/2}$ para $k=0 \rightarrow \sqrt{2}\left[\cos\left(\frac{\pi}{6}\right) - i\,\text{sen}\left(\frac{\pi}{6}\right)\right] = \frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}i$

$z^{1/2}$ para $k=1 \rightarrow \sqrt{2}\left[\cos\left(\frac{2\pi - \frac{\pi}{3}}{2}\right) - i\,\text{sen}\left(\frac{2\pi - \frac{\pi}{3}}{2}\right)\right]$

$= -\frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}i$

5c. Encontrar las raícez de $(-1)^{1/3}$

$z = -1$, $z^{1/3} = (-1)^{1/3}$; $n = 3$; $k = 0, 1, 2$; $\theta = 0$; $|z| = 1$

$z^{1/3}$ para $k=0 \rightarrow (-\cos(0) + i\,\cancel{\text{sen}(0)}^{0}) = -1$

$z^{1/3}$ para $k=1 \rightarrow \left(-\cos\left(\frac{2\pi}{3}\right) + i\,\cancel{\text{sen}\left(\frac{2\pi}{3}\right)}^{0}\right) = +1/2 + \sqrt{3}i/2$

$z^{1/3}$ para $k=2 \rightarrow \left(-\cos\left(\frac{4\pi}{3}\right) + i\,\cancel{\text{sen}\left(\frac{4\pi}{3}\right)}\right) = \frac{+1}{2} - \sqrt{3}i/2$

5d. Encontrar las raícez de $8^{1/6}$

$z = 8$, $z^{1/6} = (8)^{1/6}$; $n = 6$; $k = 0, 1, 2, 3, 4, 5$; $\theta = 0$; $|z| = 8$

$z^{1/6}$ para $k=0 \rightarrow 8^{1/6}(\cos(0) + i\,\text{sen}(0)) = 8^{1/6}$

$z^{1/6}$ para $k=1 \rightarrow 8^{1/6}\left(\cos\left(\frac{2\pi}{6}\right) + i\,\text{sen}\left(\frac{2\pi}{6}\right)\right) =$

$z^{1/6}$ para $k=2 \rightarrow 8^{1/6}$

$z^{1/6}$ para $k=3 \rightarrow 8^{1/6}$

$z^{1/6}$ para $k=4 \rightarrow 8^{1/6}$

$z^{1/6}$ para $k=5 \rightarrow 8^{1/6}$