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Exploring Mathematics

Problem-Solving and Proof

Springer

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For Ricarda and Leonard

Preface to the English edition

The classical curriculum for university mathematics education has emphasized calculation and – on a higher level – proofs and systematic development of mathematical theories. In recent years there has been a growing interest in supplementing this with a problem-solving approach. It has become clear that this is not only fun but also very useful as preparation for understanding the mathematical theories, these tremendous advances that mathematics has made over the centuries. These new developments have occurred in Britain, the United States, Germany and certainly other countries as well.

In 2011, the University of Oldenburg introduced a new course into the curriculum: 'Mathematical problem-solving and proving' (Mathematisches Problemlösen und Beweisen). It was aimed at first-year students who have just finished Gymnasium (roughly equivalent to A-levels in England and high school plus epsilon in the US) and are starting a degree in mathematics or mathematics education. This course has been very successful, both in bridging the gap between high school and university, and in providing a fun introduction to the higher mathematics taught at university. Similar courses have sprung up at other universities in Germany, and in Britain and the US, and at other places people are discussing the possibility of introducing them, or at least introducing elements of such an approach into existing courses.

This book arose from my teaching the course twice in Oldenburg, and since many colleagues in English-speaking countries have expressed a great interest in it I decided to translate it into English.

There are many other excellent books on problem-solving. What sets this book apart is that it starts at a very elementary level, but then quite explicitly tries to be a bridge to higher mathematics: the emphasis is on your discovery of mathematics by solving problems, but along the way concepts, notation and terminology of higher mathematics (e.g. sets and mappings) are introduced. In addition, most chapters have a section with additional material (titled *Going* *further*) which provides a preview of where similar ideas are used in advanced mathematical disciplines.

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Introduction

Tell me and I will forget.
Show me and I will remember.
Involve me and I will understand.
(Lao Tse)

With this book I invite you to come on a voyage of discovery. You will discover mathematics from a completely new perspective: not as a collection of formulas and rules, but as a world which you can explore for yourself, in which you can develop your own ideas and unearth hidden treasures. Don't worry: you are well equipped for your journey, and along the way you will add new tools to your equipment that will help you master difficult stages. I will be your guide. You can decide for yourself whether you want to travel alone, accept a few hints, or let yourself be comfortably guided through the terrain.

At the end you will return from your voyage enriched. Not only will you have gained a new view of mathematics, but you will also take home a wealth of experiences.

Three themes will recur throughout our voyage:

1. Mathematical **problems** and **problem-solving strategies**
2. Mathematical **proofs:** why do we need them, how do we find them, how do we write them?
3. **General ideas** of mathematics

Problems and problem-solving strategies

Problems are the soul of mathematics.
The problems of how to calculate areas and predict the movement of objects led to the invention of the calculus in the 17th century. FERMAT's problem, whether the equation $x^n + y^n = z^n$ has positive integer solutions for $n > 2$, has stimulated number theory up to the

present day. The four colour problem, whether four colours suffice to colour every conceivable map assuming that neighbouring countries always get different colours, led to a new mathematical theory in the 19th century: graph theory. These are big mathematical problems. But there are also countless smaller interesting problems which are accessible for everyone, which you can explore for yourself and thus experience what mathematics is about. You will find such problems in this book.

Problem-solving is fun and creative.
Initially you are in the dark. You look at examples, make a sketch, observe patterns, and by and by you realise what matters and what doesn't. Slowly the darkness lifts, you feel your way forward, develop ideas, and suddenly: Eureka – I've found it! Understanding is deeply satisfying. Being creative makes you happy.

Problem-solving can be learned.
During your voyage you will learn many problem-solving strategies. Some of them are so general that they can also be used outside of mathematics. Others are specific to mathematics. With every problem that you solve you develop your creativity and enlarge your pool of experience.

Problem-solving makes you curious about mathematical theories.
Theories give answers. You will really appreciate an answer only if you have asked the question beforehand, and even more if you have tried to answer it yourself and experienced difficulties. If you learn new theories (like algebra or analysis) in this way, then you will be able to use them well, and advance beyond them.

Proofs

Proofs are the heart of mathematics.
Mathematical proof is quite extraordinary: what is proved today is true – today, tomorrow and in a thousand years. This distinguishes mathematics from all other sciences. Many scientific theories were considered valid for centuries but then had to be corrected. Mathematics also advances, new connections of hitherto unconnected worlds are discovered all the time, problems are solved which had seemed intractable for years; but what is proved once will always be true.

Proofs tame infinity.
By experimenting you may discover that you can find more and more prime numbers, no matter how many you have found already. But does this continue for ever? Are there infinitely many primes? Only a proof can give certainty.

Proofs give certainty.
Many a supposed fact turns out to be a fallacy – and if you have experienced this then you will appreciate what proofs do for you. Without a proof you can never be sure.[1]

Proofs help you to understand.
When you have thought in detail about why a mathematical assertion is true, when you have removed any doubt by logical arguments, then you will understand the assertion itself better, and also you will be better at explaining it to others. And you will remember it better.

Finding proofs is problem-solving and therefore creative. And it can be learned.
A proof is a logically complete argument. But how do we find a proof? Here we need to be creative. Finding a proof is to mastering logic as drawing a picture is to knowing about colours, or as composing a symphony is to knowing about musical notes and harmonies. And just like a picture or a symphony, a proof can be beautiful. There is no general recipe for finding proofs, but just as for general problem-solving there are recurring patterns and ideas. Getting to know these will help you get better at finding proofs. You will find many of these patterns in this book.

General ideas of mathematics

Often mathematics is divided into subdisciplines: geometry, algebra, analysis and so on. However, many ideas occur in all disciplines. In this book you will get to know such ideas by simple examples. You will also learn about general methods of scientific work. You may be surprised to notice that many of them will be familiar to you. We merely name them so you can use them systematically.

[1] See Exercises **E 1.10** and **E 5.24** and the beginning of Chapter 7.2 for good examples of this.

Among the scientific methods are the cycle of exploring, making hypotheses (also called conjectures in mathematics) and investigating them systematically (in mathematics: proof or disproof); introducing concepts and notation; and identifying essential aspects of a problem and neglecting inessential ones (abstraction).

Among the general mathematical ideas are the extremal principle, the invariance principle and the principle of counting in two ways. You will learn about these ideas here as problem-solving strategies and use them to find out surprising things. But when you delve deeper into mathematics then you will meet these ideas again and again in different guises. They will guide your way in the vast landscape of mathematics.

Who is this book for?

This book is for everyone who likes mathematics and who likes to think about problems that cannot be solved by simply applying a given method. For all who want to improve their problem-solving skills. For mathematically interested high school or secondary school students. For students in colleges and universities who would like to get a different view of mathematics, supplementing their standard maths courses. For teachers in high or secondary schools, colleges and universities who want to teach problem-solving in a more systematic way than just by giving homework problems, maybe even to offer a course that is method-oriented rather than topic-oriented. For leaders of maths clubs or math circles.

As *prerequisites* you only need to know a bit about numbers, basic geometry, how to rearrange equations. You learn these ideas quite early on at high or secondary school. But even if you know some higher mathematics already you will get your money's worth: at many places in the book you will find material on how the elementary ideas introduced here are used in higher mathematics, including present-day research.

Contents

The core of the book is the problems. Lots of problems, their investigation and solution. In the investigation we approach each problem step by step: What are we looking for? What could be a route of approach? Not every route that we try will lead to a solution; sometimes we need to backtrack and start again. Here you experience how mathematics is born, invented, developed. While discovering mathematics you learn how mathematicians think. Here you learn problem-solving strategies. These are collected in the *Toolbox* sections. In Appendix A you will find an overview of all the strategies introduced in the book.

What kind of problems do you find in this book? Mathematical problems for which no straightforward method of solution offers itself. Problems where you need an idea. Problems that lead to an interesting mathematical concept. Some chapters introduce an area of mathematics, for example graph theory in Chapter 4 or number theory in Chapter 8. These topics are also introduced via a sequence of problems, and at places where it is natural to state a theorem and its proof, the proof is developed from the point of view of a problem-solver.

Here is a more detailed outline. In Chapters 1 and 2 you will find problems where you count things. Counting is the most basic mathematical activity. By learning to advance step by step you will soon have the wonderful experience of discovering mathematics yourself. An important technique for counting is to use recurrence relations, which are discussed in Chapter 2. The basic idea, called recursion, is to reduce a problem to a smaller problem of the same kind. This is also behind the principle of mathematical induction introduced in Chapter 3. Using induction you will prove EULER's famous formula in Chapter 4 and then use it to show a remarkable impossibility result: you cannot connect five points in the plane pairwise so that the connecting curves don't intersect. That we can prove such an assertion rigorously is actually quite amazing: how can we control the infinitely many ways to draw the points and curves? After this excursion into graph theory we return to counting in Chapter 5, this time formulating general systematic counting principles which are often useful.

Chapters 6 and 7 are central: Chapter 6 introduces general problem-solving strategies, that is, strategies which are useful beyond mathematics. Chapter 7 is about logical foundations and the most important general types of proof. These are illustrated by many examples. Chapter 8 is about number theory, that is, prime numbers, divisibility etc.

Here you can use the various types of proof to answer interesting questions about numbers. Number theory also provides useful tools and pretty examples for the following chapters. In Chapter 9 you learn about the pigeonhole principle, a simple idea which spawns amazing consequences when used aptly. Chapters 10 and 11 introduce the extremal principle and the invariance principle. These are versatile tools for solving mathematical problems, and also turn up again and again in all exact sciences. Here you also learn about the fundamental notions of permutations and their signature.

Sections titled *Going further* at the end of Chapters 4, 5, 6, 10 and 11 encourage you to delve deeper into the matters touched upon in these chapters and give you a glimpse of the higher summits of mathematics. Occasionally you may find here an unfamiliar term without detailed explanation, but you should read on, just to get an idea of what else mathematics has to offer. The references at the end of the book guide you to literature that will be useful for further exploration.

The mathematical topics treated in the book are elementary, that is, they can be understood with very few prerequisites. However, we take a higher perspective than you may be used to: often problems are formulated in greater generality than is common at school, we pay attention to logically complete arguments, and although the tone is generally informal, we use modern mathematical terminology, in particular the language of sets and maps. This is explained in Appendix B.

The chapters are mostly independent of each other, so you may read them in a different order. However, later chapters tend to be more demanding than earlier chapters.

Hints for using the book

You learn problem-solving by solving problems, and by understanding other people's solutions. You learn proof by proving, and by carefully reading and understanding proofs. The problems in the text and the exercises at the end of each chapter give you ample opportunity for this. Be an active reader! After reading a problem, first try to solve it yourself. After reading part of the solution, put the book aside and think about how to go on. The symbol

Think about it!

is meant to remind you of this. Whenever you read a statement, ask yourself whether you are convinced that it is true. Keep paper and pencil handy. Ask yourself: why do it like that, couldn't I do it better?

Explain your solution, or the solution you have just read, to a friend. Let him or her play the devil's advocate, who doesn't let you get away with the slightest imprecision, who tries to find gaps in your argument, who asks after each step: "But what if ...". After a while you will learn to be your own devil's advocate.

Sometimes you will have other ideas for solutions than what is given in the text. Every problem has many solutions, and also the same idea can be formulated in many ways. Try to work out whether your solution has the same core idea or whether it is completely different.

The exercises at the end of each chapter are a particularly important part of the book. Work on them. It will pay off. You will find hints for some of the exercises at the end of the book. But you will profit more from an exercise if you don't immediately succumb to the temptation to look them up.

Each exercise has been assigned a level of difficulty. This should be understood as a rough guideline, as there is no objective way to judge difficulty. Whether and when you find a solution depends on many factors. The levels are as follows:

- **1** – easy, often solvable in your head
- **2** – should be doable when you have worked through the chapter
- **3** – requires more commitment and ideas
- **4** – difficult

Most problems in the book are treated in four steps:

1. Understanding the problem
2. Investigating the problem
3. Writing up the solution properly

4. Review

These are similar to the four steps of problem-solving introduced by Pólya in his classic book *How to solve it* (Pólya, 2014). Steps 1. and 2. are closely interwoven and carried out in the sections marked by the symbol 🔍. The solution is marked by the symbol !, the review by the symbol ↻. The end of each of these sections is marked by the same symbol, and the end of a proof is marked q. e. d. (quod erat demonstrandum = which was to be proven).

Notes for the teacher

This book arose from a course whose main aims were to let students see mathematics from a new perspective (not rules to be followed but a world to be discovered), to give them confidence that they can discover mathematics for themselves and to lead them to appreciate proofs.[2] Along the way students acquire tools for problem-solving, get to know the main types of proofs, learn to write solutions and proofs properly, learn about fundamental ideas that occur everywhere in mathematics, and also pick up important concepts like graphs, congruences, permutations and their signatures. These are introduced in informal, often playful contexts, so they can be shared with friends and will be remembered more vividly than if learned in more abstract ways.

If you want to teach such a course my main piece of advice is: *Less is more.* Don't try to 'cover' a certain curriculum. Take time to solve problems together with the students in class, to develop ideas for solutions, to try out different approaches, to suffer the frustration of getting stuck and enjoy the satisfaction of success together. Share with the students your experience of how to approach a mathematical problem, rather than presenting prefabricated solutions. *Always start with an easy problem.* There will be enough challenges for everyone.

I am happy to acknowledge the help and valuable feedback that

[2]This course, titled *Mathematisches Problemlösen und Beweisen* (Mathematical problem-solving and proving), was introduced in the mathematics curriculum at Carl von Ossietzky University of Oldenburg, Germany, in 2011, with the goal of smoothing the difficult transition from high school mathematics to university mathematics.

I got from many students, colleagues and readers of the first German edition. I want to thank especially the student tutors Stefanie Arend, Simone Barz, Karen Johannmeyer, Marlies Händchen, Stefanie Kuhlemann, Roman Rathje, Kathrin Schlarmann, Steffen Smoor and Eric Stachitz, and my colleague Andreas Defant. Sunke Schlüters contributed many excellent ideas for additional problems and helped me with many of the pictures in the book. Sophy Darwin made many valuable suggestions beyond her thorough language editing. My deepest gratitude is due to my wife Ricarda Tomczak for continuous encouragement and innumerable discussions and suggestions.

Now I wish you many joyful hours of solving problems and thinking about proofs, of discovering the beauty of mathematics. If you have any comments or suggestions please write me an email at daniel.grieser@uni-oldenburg.de.

Oldenburg,
February 2018 *Daniel Grieser*