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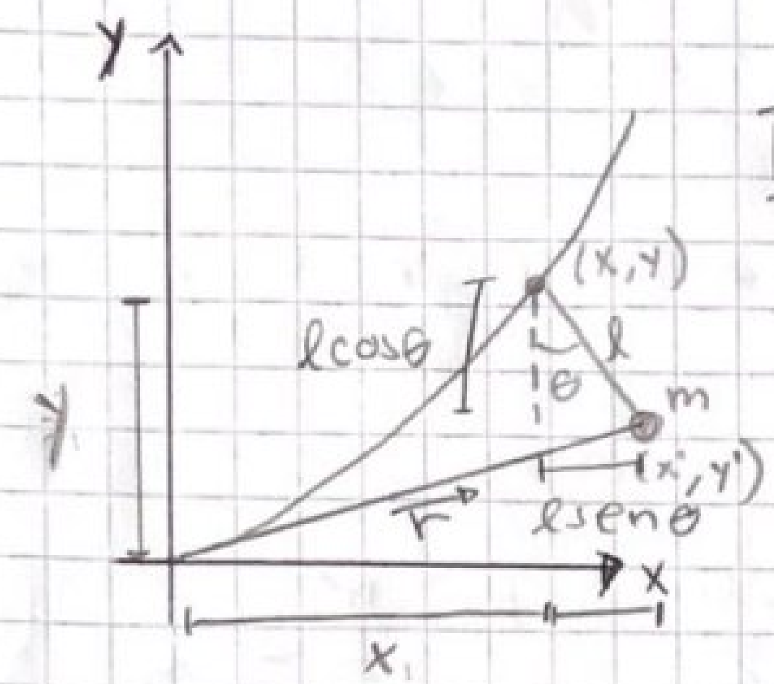
1) Tal y como se muestra en la figura, el punto de suspensión de un pendulo esta obligado a moverse a lo largo de la parabola $y = ax^2$. Encontrar el hamiltoniano.



$\boxed{y_1 = ax^2}$, a; parametro cte

coordenadas generalizadas:

x, θ

$x' = x_1 + l\,sen\theta$

$y' = y_1 - l\cos\theta$

$$\vec{r} = (x_1 + l\,sen\theta)\,\hat{x} + (y_1 - l\cos\theta)\,\hat{y}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = (\dot{x} + l\cos\theta\,\dot{\theta})\,\hat{x} + (a2x\dot{x} + l\,sen\theta\,\dot{\theta})\,\hat{y}$$

$$v^2 = \dot{x}^2 + l^2\cos^2\theta\,\dot{\theta}^2 + 2\dot{x}\,l\cos\theta\,\dot{\theta} + 4a^2x^2\dot{x}^2 + l^2 sen^2\theta\,\dot{\theta}^2$$
$$+ 4ax\dot{x}\,l\,sen\theta\,\dot{\theta}$$

=> Simplificando

$$v^2 = \dot{x}^2(1 + 4a^2x^2) + l^2\dot{\theta}^2 + 2\dot{x}\,l\,\dot{\theta}\,(\cos\theta + 2ax\,sen\theta)$$

=> El lagrangiano

$$\mathcal{L} = T - V = \frac{1}{2}mv^2 - mgy'$$

$$= \frac{1}{2}m\left(\dot{x}^2(1+4a^2x^2) + l^2\dot{\theta}^2 + 2\dot{x}\,l\,\dot{\theta}\,(\cos\theta + 2ax\,sen\theta)\right)$$

$$- mg(ax^2 - l\cos\theta)$$

Para calcular el hamiltoniano

$$P_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = ml^2\dot{\theta} + m\dot{x}\,l\,(\cos\theta + 2ax\,sen\theta)$$

$$P_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}\,(1 + 4a^2x^2) + ml\dot{\theta}(\cos\theta + 2ax\,sen\theta)$$

Tenemos este sistema de ecuaciones

$$\begin{pmatrix} P_x \\ P_\theta \end{pmatrix} = \underbrace{\begin{pmatrix} m(1+4a^2x^2) & ml(\cos\theta + 2ax\,sen\theta) \\ ml(\cos\theta + 2ax\,sen\theta) & ml^2 \end{pmatrix}}_{\text{Matriz A}} \begin{pmatrix} \dot{x} \\ \dot{\theta} \end{pmatrix}$$

Tal que el hamiltoniano puede ser escrito como

$$H = \frac{1}{2} p^T A^{-1} + V_0 \quad ; \quad P_i = \begin{pmatrix} P_x \\ P_\theta \end{pmatrix} \quad ; \quad P_i^T = (P_x , P_\theta)$$

$$H = \frac{1}{2\left(m^2\ell^2(\cancel{x} + 4a^2x^2) - m^2\ell^2(\cos\theta + 2ax\,\mathrm{sen}\theta)^2\right)} (P_x , P_\theta)$$

$$\cdots \begin{pmatrix} m\ell^2 & -m\ell(\cos\theta + 2ax\,\mathrm{sen}\theta) \\ -m\ell(\cos\theta + 2ax\,\mathrm{sen}\theta) & m(1 + 4a^2x^2) \end{pmatrix} \begin{pmatrix} P_x \\ P_\theta \end{pmatrix} + mg(ax^2 - \ell\cos\theta)$$

Simplificando y resolviendo

$$H = \frac{m\ell}{2\left(m^2\ell^2(1 + 4a^2x^2) - m^2\ell^2(\cos\theta + 2ax\,\mathrm{sen}\theta)^2\right)} \left[P_x^2 - 2(\cos\theta + 2ax\,\mathrm{sen}\theta) P_x P_\theta + (1 + 4a^2x^2) P_\theta^2 \right] + mg(ax^2 - \ell\cos\theta)$$

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3 Sea el hamiltoniano $H = \frac{P^2}{2m} - A\left(\frac{P}{m}\cos\gamma t + \gamma q\,\mathrm{sen}\gamma t\right) + \frac{1}{2}kq^2$ donde A, γ y k son ctes

a) Hallar el lagrangeano $\mathcal{L}$

b) Encontrar un lagrangiano equivalente, $\mathcal{L}'$, que no dependa de t

c) Calcular la nueva forma del hamiltoniano asociado a $\mathcal{L}'$ ¿Cual es la relación entre los dos hamiltonianos?

a) Para obtener el lagrangiama, usamos la formula

$$\mathcal{L} = p\dot{q} - H.$$

tal que

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{P}{m} - \frac{A}{m}\cos\gamma t$$

$$P = m\left(\dot{q} + \frac{A}{m}\cos\gamma t\right)$$

remplazando tenemos

$$\mathcal{L} = p\dot{q} - H$$

$$= \dot{q}\left(m\left(\dot{q} + \frac{A}{m}\cos\gamma t\right)\right) - \frac{\left(m\left(\dot{q} + \frac{A}{m}\cos\gamma t\right)\right)^2}{2m}$$

$$+ A\left[\frac{\cancel{m}\left(\dot{q} + \frac{A}{m}\cos\gamma t\right)}{\cancel{m}}\cos\gamma t + \gamma q \operatorname{sen}\gamma t\right] - \frac{1}{2}Kq^2$$

simplificando y realizando los debidos calculos

$$\mathcal{L} = \frac{m\dot{q}^2}{2} + \frac{A^2}{2m}\cos^2\gamma t + A\dot{q}\cos\gamma t + A\gamma q \operatorname{sen}\gamma t - \frac{1}{2}Kq^2$$

b) el lagrangiano se puede escribir como:

$$\mathcal{L} = \mathcal{L}_0 + \frac{dF}{dt}$$

$$= \frac{m\dot{q}^2}{2} - \frac{1}{2}Kq^2 + \frac{d}{dt}\left(\frac{A^2}{4m}\operatorname{sen}(2\gamma t) + Aq\cos(\gamma t)\right)$$

con, $\mathcal{L}_0 = \dfrac{m\dot{q}^2}{2} - \dfrac{1}{2}Kq^2$

$$F(q,t) = \frac{A^2}{4m}\operatorname{sen}(2\gamma t) + Aq\cos(\gamma t)$$

c) el hamiltoniano del nuevo lagrangiano:

$$\mathcal{L}_0 = \frac{m\dot{q}^2}{2} - \frac{1}{2}Kq^2$$

$$H_0 \Rightarrow p_q = \frac{\partial \mathcal{L}}{\partial \dot{q}} = \dot{q}m \rightarrow \dot{q} = \frac{p_q}{m}$$

remplazando

$$H_0 = \frac{p_q^2}{2m} + \frac{1}{2}Kq^2$$

Para encontrar gp relacia tienen

si $\underline{P} = \dot{q}$ y $Q = q$, el hamiltoniano original es

$$H = H_0 - A\left(\frac{p}{m}\cos\gamma t + \gamma q\,\mathrm{sen}\,\gamma t\right) + \frac{1}{2}kq^2$$

remplazando $p = m\left(\dot{q} + \frac{A}{m}\cos\gamma t\right)$

$$= m\left(\underline{P} + \frac{A}{m}\cos\gamma t\right)$$

$$H = H_0 - A\left[\frac{\cancel{m}\left(\underline{P} + \frac{A}{m}\cos\gamma t\right)}{\cancel{m}}\cos\gamma t + \gamma Q\,\mathrm{sen}\,\gamma t\right]$$

$$H = H_0 - A\left[\left(\underline{P} + \frac{A}{m}\cos\gamma t\right)\cos\gamma t + \gamma Q\,\mathrm{sen}\,\gamma t\right]$$

$H = q + te^p$ (1) $\quad Q = q + te^p$ (2) $\wedge$ $P = p$ (3)

usaremos una transformación del tipo F_3

(4) $dF_3 = \frac{\partial F_3}{\partial p} dp + \frac{\partial F_3}{\partial Q} dQ$ donde $q = \frac{-\partial F_3}{\partial p}$ (5) $\wedge$ $P = \frac{-\partial F_3}{\partial Q}$ (6)

(7) $dF_3 = -q\,dp - P\,dQ = -(Q - te^p)dp - p\,dQ = d[p(te^p - Q)]$

(8) $F_3 = p(te^p - Q)$ y $K(Q,P;t) = H(q(Q,P), p(Q,P);t) + \frac{\partial F_3}{\partial t}$ (9)

$K = Q - te^p + te^p + \frac{\partial}{\partial t} p(te^p - Q)$ (10)

(11) $K = Q + Pe^P$ Nuevo hamiltoniano; Kamiltoniano.

(12) $\dot{Q} = \frac{\partial K}{\partial P} = e^P(1 + P)$ $\quad \|$ (13) $\dot{P} = \frac{\partial K}{\partial Q} = 1 \rightarrow \frac{dP}{dt} = 1 \rightarrow \boxed{P = t + C}$ (14)

(14) en (12) $\rightarrow \frac{dQ}{dt} = e^{t+C}(1 + t + C)$ dejamos $C = 0$

$\int dQ = \int e^t(1+t)dt = e^t + \int te^t dt = e^t + (t-1)e^t = \boxed{te^t = Q}$ (15)

$Q = q + te^p$ pero como $p = P = t$ y por (15) $\rightarrow te^t = q + te^t \rightarrow q = 0$

así el Hamiltoniano $H = q + te^p \Rightarrow \boxed{H = te^t}$

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$$H = \frac{1}{2}\left(qp^3 + \frac{q}{p}\right)$$

Ecuaciones de movimiento.

$$\dot{q} = \frac{\partial H}{\partial p} \quad , \quad \dot{p} = -\frac{\partial H}{\partial q}$$

Hallamos $\dot{q}$

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{1}{2}\left(3qp^2 - \frac{q}{p^2}\right)$$

calculamos $\dot{p}$

$$\dot{p} = -\frac{\partial H}{\partial q} = -\frac{1}{2}\left(p^3 + \frac{1}{p}\right)$$