Low-priority queue and server's steady-state existence in a tandem under prolongable cyclic service

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Abstract. A mathematical model of a tandem of queuing systems is considered. Each system has a high-priority input flow and a low-priority input flow which are conflicting. In the first system, the customers are serviced in the class of cyclic algorithms. The serviced high-priority customers are transferred from the first system to the second one with random delays and become the high-priority input flow of the second system. In the second system, customers are serviced in the class of cyclic algorithms with prolongations. Low-priority customers are serviced when their number exceeds a threshold. A mathematical model is constructed in form of a multidimensional denumerable discrete-time Markov chain. Conditions of low-priority queue stationary distribution existence were found.

Keywords: tandem of controlling queuing systems, cyclic algorithm with prolongations, conflicting flows, multidimensional denumerable discrete-time Markov chain

1 Introduction

An enormous amount of work has been done on the problem of conflicting traffic flows control at crossroad by the moment. In the queuing theory literature one can find following algorithms investigated: fixed duration cyclic algorithm, cyclic algorithm with a loop, cyclic algorithm with changing regimes, etc [1–6]. However, in a real-life situations cars pass several consecutive crossroads on their way rather then only one. In other words, an output flow of cars from the first intersection forms an input flow of cars of the next intersection. Hence, the second input flow no longer has an *a priori* known simple probabilistic structure (for example, a non-ordinary Poison flow), and knowledge about the service algorithm should be taken into account to deduce formation conditions of the first output flow.

One can find several works about tandems of intersections. In [7] a computer-aided simulation of adjacent intersections was carried out. In [8] a mathematical model of two intersections in tandem governed by cyclic algorithms was investigated and stability conditions were found. In this paper we assume that the first intersection is governed by a cyclic algorithm while the second intersection is governed by a cyclic algorithm with prolongations. The low-priority queue on the second intersection and necessary conditions of its stationary state existence take central place of this paper. This work continues studying in paper [10].

2 The problem settings

Consider a queuing system with a scheme shown in Fig. 1. There are four in-

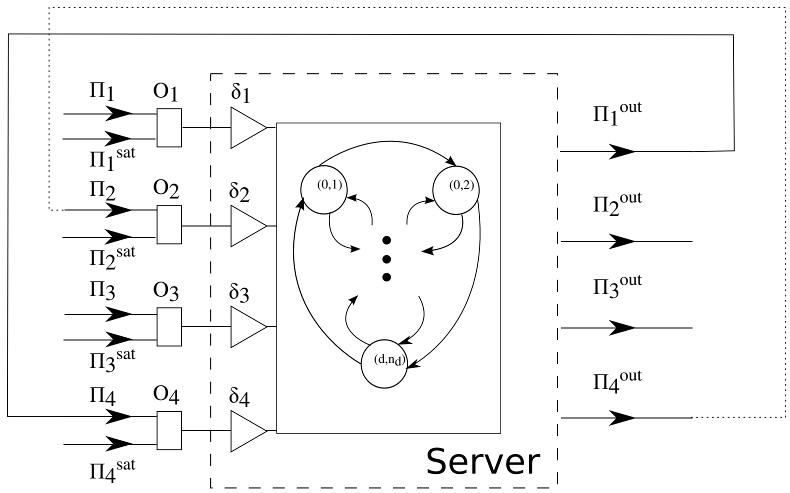

Fig. 1. Scheme of the queuing system as a cybernetic control system

put flows of customers Π_1, Π_2, Π_3, and Π_4 entering the single server queueing system. Customers in the input flow Π_j, $j \in \{1, 2, 3, 4\}$ join a queue O_j with an unlimited capacity. For $j \in \{1, 2, 3\}$ the discipline of the queue O_j is FIFO (First In First Out). Discipline of the queue O_4 will be described later. The input flows Π_1 and Π_3 are generated by an external environment, which has only one state. Each of these flows is a nonordinary Poisson flow. Denote by λ_1 and λ_3 the intensities of bulk arrivals for the flows Π_1 and Π_3 respectively. The probability generating function of number of customers in a bulk in the flow Π_j is

$$f_j(z) = \sum_{\nu=1}^{\infty} p_\nu^{(j)} z^\nu, \quad j \in \{1, 3\}. \tag{1}$$

We assume that $f_j(z)$ converges for any $z \in \mathbb{C}$ such that $|z| < (1 + \varepsilon)$, $\varepsilon > 0$. Here $p_\nu^{(j)}$ is the probability of a bulk size in flow Π_j being exactly $\nu = 1, 2, \ldots$. Having been serviced the customers from O_1 come back to the system as the Π_4

customers. The Π_4 customers in turn after service enter the system as the Π_2 ones. The flows Π_2 and Π_3 are conflicting in the sense that their customers can't be serviced simultaneously. This implies that the problem can't be reduced to a problem with fewer input flows by merging the flows together.

In order to describe the server behavior we fix positive integers d, n_0, n_1, …, n_d and we introduce a finite set $\Gamma = \{\Gamma^{(k,r)}\colon k = 0, 1, \ldots, d; r = 1, 2, \ldots n_k\}$ of states server can reside in. At the state $\Gamma^{(k,r)}$ the server stays during a constant time $T^{(k,r)}$. Define disjoint subsets Γ^{I}, Γ^{II}, Γ^{III}, and Γ^{IV} of Γ as follows. In the state $\gamma \in \Gamma^{\mathrm{I}}$ only customers from the queues O_1, O_2 and O_4 are serviced. In the state $\gamma \in \Gamma^{\mathrm{II}}$ only customers from the queues O_2 and O_4 are serviced. In the state $\gamma \in \Gamma^{\mathrm{III}}$ only customers from queues O_1, O_3, and O_4 are serviced. In the state $\gamma \in \Gamma^{\mathrm{IV}}$ only customers from queues O_3 and O_4 are serviced. We assume that $\Gamma = \Gamma^{\mathrm{I}} \cup \Gamma^{\mathrm{II}} \cup \Gamma^{\mathrm{III}} \cup \Gamma^{\mathrm{IV}}$. Set also ${}^1\Gamma = \Gamma^{\mathrm{I}} \cup \Gamma^{\mathrm{III}}$, ${}^2\Gamma = \Gamma^{\mathrm{I}} \cup \Gamma^{\mathrm{II}}$, ${}^3\Gamma = \Gamma^{\mathrm{III}} \cup \Gamma^{\mathrm{IV}}$.

The server changes its state according to the following rules. We call a set $C_k = \{\Gamma^{(k,r)}\colon r = 1, 2, \ldots n_k\}$ the k-th cycle, $k = 1$, 2, …, d. For $k = 0$ the state $\Gamma^{(0,r)}$ with $r = 1$, 2, …, n_0 is called a prolongation state. Put $r \oplus_k 1 = r + 1$ for $r < n_k$, and $r \oplus_k 1 = 1$ for $r = n_k$ ($k = 0$, 1, …, d). In the cycle C_k we select a subset C_k^{O} of input states, a subset C_k^{I} of output states, and a subset $C_k^{\mathrm{N}} = C_k \setminus (C_k^{\mathrm{O}} \cup C_k^{\mathrm{I}})$ of neutral states. After the state $\Gamma^{(k,r)} \in C_k \setminus C_k^{\mathrm{O}}$ the server switches to the state $\Gamma^{(k,r\oplus_k 1)}$ within the same cycle C_k. After the state $\Gamma^{(k,r)}$ in C_k^{O} the server switches to the state $\Gamma^{(k,r\oplus_k 1)}$ if number of customers in the queue O_3 at switching instant is greater than a predetermined threshold L. Otherwise, if the number of customers in the queue O_3 is less than or equal to L then the new state is the prolongation one $\Gamma^{(0,r_1)}$ where $r_1 = h_1(\Gamma^{(k,r)})$ and $h_1(\cdot)$ is a given mapping of $\bigcup_{k=1}^{d} C_k^{\mathrm{O}}$ into $\{1, 2, \ldots, n_0\}$. After the state $\Gamma^{(0,r)}$ if the number of customers in O_3 is not above L the state of the same type $\Gamma^{(0,r_2)}$ is chosen where $r_2 = h_2(r)$ and $h_2(\cdot)$ is a given mapping of the set $\{1, 2, \ldots, n_0\}$ into itself; in the other case the new state is $\Gamma^{(k,r_3)} \in C_k^{\mathrm{I}}$ where $\Gamma^{(k,r_3)} = h_3(r)$ and $h_3(\cdot)$ is a given mapping of $\{1, 2, \ldots, n_0\}$ to $\bigcup_{k=1}^{d} C_k^{\mathrm{I}}$. We assume that each prolongation state $\Gamma^{(0,r)}$ belongs to the set ${}^2\Gamma$ and that relations $C_k^{\mathrm{O}} \subset {}^2\Gamma$ and $C_k^{\mathrm{I}} \subset {}^3\Gamma$ hold. We also assume that all the cycles have exactly one input and output state. Finally, we assume that all the prolongation states make a cycle, that is $h_2(r) = r \oplus_0 1$. Putting all together, we introduce a function which formalizes the server state changes:

$$h(\Gamma^{(k,r)}, y) = \begin{cases} \Gamma^{(k,r\oplus_k 1)} & \text{if } \Gamma^{(k,r)} \in C_k \setminus C_k^{\mathrm{O}} \text{ or} \\ & \quad (\Gamma^{(k,r)} \in C_k^{\mathrm{O}}) \wedge (y > L); \\ \Gamma^{(0,h_1(\Gamma^{(k,r)}))} & \text{if } \Gamma^{(k,r)} \in C_k^{\mathrm{O}} \text{ and } y \leqslant L; \\ \Gamma^{(0,r\oplus_0 1)} & \text{if } k = 0 \text{ and } y \leqslant L; \\ h_3(r) & \text{if } k = 0 \text{ and } y > L. \end{cases} \tag{2}$$

In general, service durations of different customers can be dependent and may have different laws of probability distributions. So, saturation flows will be used to define the service process. A saturation flow Π_j^{sat}, $j \in \{1, 2, 3, 4\}$, is defined as a virtual output flow under the maximum usage of the server and unlimited number of customer in the queue O_j. The saturation flow Π_j^{sat}, $j \in \{1, 2, 3\}$ contains a non-random number $\ell(k, r, j) \geqslant 0$ of customers in the server state $\Gamma^{(k,r)}$. In particular, $\ell(k, r, j) \geqslant 1$ for $\Gamma^{(k,r)} \in {}^j\Gamma$ and $\ell(k, r, j) = 0$ for $\Gamma^{(k,r)} \notin {}^j\Gamma$. Let $\mathbb{Z}_+$ be the set of non-negative integer numbers. If the queue O_4 contains $x \in \mathbb{Z}_+$ customers the saturation flow Π_4^{sat} also contains the x customers. Finally, in the state $\Gamma^{(k,r)}$ every customer from queue O_4 with probability $p_{k,r}$ and independently of others ends servicing and joins Π_2 to go to O_2. With the complementary probability $1 - p_{k,r}$ the customer stays in O_4 until the next time slot. In the next time slot it repeats its attempt to join Π_2 with a proper probability.

A real-life example of just described queuing system is a tandem of two consecutive crossroads (Fig. 2). The input flows are flows of vehicles. The flows

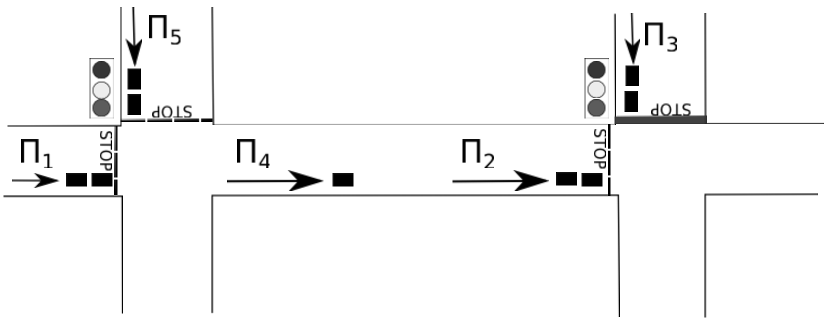

Fig. 2. A tandem of crossroads, the physical interpretation of the queuing system under study

Π_1 and Π_5 at the first crossroad are conflicting; Π_2 and Π_3 at the second crossroad are also conflicting. Every vehicle from the flow Π_1 after passing first road intersection joint the flow Π_4 and enters the queue O_4. After some random time interval the vehicle arrives to the next road intersection. Such a pair of crossroads is an instance of a more general queuing model described above.

3 Mathematical model

The queuing system under investigation can be regarded as a cybernetic control system that helps to rigorously construct a formal stochastic model [8]. The scheme of the control system is shown in Fig. 1. There are following blocks present in the scheme: 1) the external environment with one state; 2) input poles of the first type — the input flows Π_1, Π_2, Π_3, and Π_4; 3) input poles of the second type — the saturation flows Π_1^{sat}, Π_2^{sat}, Π_3^{sat}, and Π_4^{sat}; 4) an external memory — the queues O_1, O_2, O_3, and O_4; 5) an information processing device for the external memory — the queue discipline units δ_1, δ_2, δ_3, and δ_4; 6) an internal memory — the server (OY); 7) an information processing device for internal memory — the graph of server state transitions; 8) output poles — the output flows Π_1^{out}, Π_2^{out}, Π_3^{out}, and Π_4^{out}. The coordinate of a block is its number on the scheme.

Let us introduce the following variables and elements along with their value ranges. To fix a discrete time scale consider the epochs $\tau_0 = 0$, τ_1, τ_2, ... when the server changes its state. Let $\Gamma_i \in \Gamma$ be the server state during the interval $(\tau_{i-1}; \tau_i]$, $\varkappa_{j,i} \in \mathbb{Z}_+$ be the number of customers in the queue O_j at the instant τ_i, $\eta_{j,i} \in \mathbb{Z}_+$ be the number of customers arrived into the queue O_j from the flow Π_j during the interval $(\tau_i; \tau_{i+1}]$, $\xi_{j,i} \in \mathbb{Z}_+$ be the number of customers in the saturation flow Π_j^{sat} during the interval $(\tau_i; \tau_{i+1}]$, $\overline{\xi}_{j,i} \in \mathbb{Z}_+$ be the actual number of serviced customers from the queue O_j during the interval $(\tau_i; \tau_{i+1}]$, $j \in \{1, 2, 3, 4\}$.

The server changes its state according to the following rule:

$$\Gamma_{i+1} = h(\Gamma_i, \varkappa_{3,i}) \tag{3}$$

where the mapping $h(\cdot, \cdot)$ is defined by Formula (2).

Let $\varphi_1(\cdot, \cdot)$ and $\varphi_3(\cdot, \cdot)$ be defined by series expansions

$$\sum_{\nu=0}^{\infty} z^{\nu} \varphi_j(\nu, t) = \exp\{\lambda_j t (f_j(z) - 1)\}$$

with functions $f_j(z)$ defined by (1), $j \in \{1, 3\}$. The function $\varphi_j(\nu, t)$ equals the probability of $\nu = 0$, 1, ... arrivals in the flow Π_j during time $t \geqslant 0$. If $\nu < 0$ the value of $\varphi_j(\nu, t)$ is set to zero.

Mathematical model in more details can be found in work [10]. From now on we focus on low-priority customers in the queue O_3.

4 The low-priority queue

Here we will consider the stochastic sequence

$$\{(\Gamma_i(\omega), \varkappa_{3,i}(\omega)); i = 0, 1, \ldots\}, \tag{4}$$

which includes the number of low-priority customers $\varkappa_{3,i}(\omega)$ in the queue O_3. In this section we will report several results concerning this stochastic sequence.

Let $\Gamma^{(k,r)} \in \Gamma$ and $x_3 \in Z_+$. Denote by $H_{-1}(\Gamma^{(k,r)}, x_3)$ the set of all server states γ such that $h(\gamma, x_3) = \Gamma^{(k,r)}$ and put $r \ominus_k 1 = r - 1$ for $n_k \geqslant r > 1$, and $r \ominus_k 1 = n_k$ for $r = 1$ ($k = 0$, 1, ..., d). Then formula (2) makes it possible to define the mapping $H_{-1}(\varGamma^{(k,r)}, x_3)$ explicitly:

$$H_{-1}(\varGamma^{(k,r)}, x_3) = \begin{cases} \left\{\varGamma^{(k_1,r_1)}, \varGamma^{(0,r\ominus_0 1)}\right\} & \text{if } (k=0) \wedge (x_3 \leqslant L), \\ \left\{\varGamma^{(k,r\ominus_k 1)}, \varGamma^{(0,r_2)}\right\} & \text{if } (\varGamma^{(k,r)} \in C_k^{\mathrm{I}}) \wedge (x_3 > L), \\ \left\{\varGamma^{(k,r\ominus_k 1)}\right\} & \text{if } (\varGamma^{(k,r)} \in C_k^{\mathrm{O}}) \vee (\varGamma^{(k,r)} \in C_k^{\mathrm{N}}); \\ \varnothing & \text{if } (k=0) \wedge (x_3 > L) \\ & \quad \text{or } (\varGamma^{(k,r)} \in C_k^{\mathrm{I}}) \wedge (x_3 \leqslant L) \end{cases} \tag{5}$$

where $h_1(\varGamma^{(k_1,r_1)}) = r$ and $h_3(r_2) = \varGamma^{(k,r)}$.

Let's define for $\gamma \in \varGamma$ and $x_3 \in Z_+$ values

$$Q_{3,i}(\gamma, x) = \mathbf{P}(\{\omega\colon \varGamma_i(\omega) = \gamma, \varkappa_{3,i}(\omega) = x_3\}).$$

Theorem 1 concerns generating functions and corrects ones in paper [10]. Suppose k and r are such that $\varGamma^{(k,r)} \in \varGamma$. Let's define partial probability generating functions

$$\mathfrak{M}^{(3,i)}(k,r,v) = \sum_{w=0}^{\infty} Q_{3,i}(\varGamma^{(k,r)}, w) v^w,$$

$$q_{k,r}(v) = v^{-\ell(k,r,3)} \sum_{w=0}^{\infty} \varphi_3(w, T^{(k,r)}) v^w.$$

and auxilary functions

$$\tilde{\alpha}_i(k,r,v) = \sum_{x_3=0}^{\ell(k,r,3)} \sum_{\gamma \in H_{-1}(\tilde{\gamma},x_3)} Q_{3,i}(\gamma, x_3) \sum_{a=0}^{\ell(k,r,3)-x_3} \varphi_3(a, T^{(k,r)}) - \\ - \sum_{x_3=0}^{\ell(k,r,3)} \sum_{\gamma \in H_{-1}(\tilde{\gamma},x_3)} Q_{3,i}(\gamma, x_3) v^{x_3 - \ell(k,r,3)} \sum_{w=0}^{\ell(k,r,3)-x_3} \varphi_3(w, T^{(k,r)}) v^w,$$

$$\alpha_i(0,r,v) = \tilde{\alpha}_i(0,r,v) + q_{0,r}(v) \times \sum_{x_3=0}^{L} \Big[Q_{3,i}(\varGamma^{(k_1,r_1)}, x_3) + \\ + Q_{3,i}(\varGamma^{(0,r\ominus_0 1)}, x_3) \Big] v^{x_3}, \quad \varGamma^{(0,r)} \in \varGamma,$$

$$\alpha_i(k,r,v) = \tilde{\alpha}_i(k,r,v) - q_{k,r}(v) \sum_{x_3=0}^{L} \Big[Q_{3,i}(\varGamma^{(k,r\ominus_k 1)}, x_3) + \\ + Q_{3,i}(\varGamma^{(0,r_2)}, x_3) \Big] v^{x_3} + q_{k,r}(v) \sum_{x_3 \geqslant 0} Q_{3,i}(\varGamma^{(0,r)}, x_3) v^{x_3}, \quad \varGamma^{(k,r)} \in C_k^{\mathrm{I}},$$

$$\alpha_i(k,r,v) = \tilde{\alpha}_i(k,r,v), \quad \varGamma^{(k,r)} \in C_k^{\mathrm{O}} \cup C_k^{\mathrm{N}}.$$

Theorem 1. *Following recurrent w.r.t. $i \geqslant 0$ relations take place for the partial probability generating functions:*

1. $\Gamma^{(0,r)} \in \Gamma,\ r = \overline{1, n_0}$
$$\mathfrak{M}^{(3,i+1)}(0, r, v) = \alpha_i(0, r, v);$$
2. $\Gamma^{(k,r)} \in \Gamma,\ k = \overline{1, d},\ r = \overline{1, n_k}$
$$\mathfrak{M}^{(3,i+1)}(k, r, v) = q_{k,r}(v) \times \mathfrak{M}^{(3,i)}(k, r \ominus_k 1, v) + \alpha_i(k, r, v);$$

The last result (theorem 2) is new and concerns low-priority queue and server's steady-state existence.

Theorem 2. *For Markov chain (4) to have stationary distribution $Q_3(\gamma, x)$, $(\gamma, x) \in \Gamma \times \mathbb{Z}_+$ it is necessary that following inequality takes place*
$$\min_{k=\overline{1,d}} \frac{\lambda_3 f_3'(1) \sum_{r=1}^{n_k} T^{(k,r)}}{\sum_{r=1}^{n_k} \ell(k, r, 3)} < 1.$$

Proof. Let's assume that stationary distribution $Q_3(\gamma, x)$, $(\gamma, x) \in \Gamma \times \mathbb{Z}_+$, exists. Then choosing this distribution as the initial one imposes existence of following limits:
$$\lim_{i \to \infty} Q_{3,i}(\gamma, w) = Q_3(\gamma, w),$$
which are equal to stationary probabilities of corresponding states.

After defining generating functions
$$\mathfrak{M}^{(3)}(k, r, v) = \sum_{w=0}^{\infty} Q_3(\gamma, w) v^w,$$
for the stationary distribution similar relations can be derived as in Theorem 1:

1. $\Gamma^{(0,r)} \in \Gamma,\ r = \overline{1, n_0}$
$$\mathfrak{M}^{(3)}(0, r, v) = \alpha(0, r, v); \tag{6}$$
2. $\Gamma^{(k,r)} \in \Gamma,\ k = \overline{1, d},\ r = \overline{1, n_k}$
$$\mathfrak{M}^{(3)}(k, r, v) = q_{k,r}(v) \times \mathfrak{M}^{(3)}(k, r \ominus_k 1, v) + \alpha(k, r, v); \tag{7}$$

where
$$\tilde{\alpha}(k, r, v) = \sum_{x_3=0}^{\ell(k,r,3)} \sum_{\gamma \in \mathbb{H}_{-1}(\tilde{\gamma}, x_3)} Q_3(\gamma, x_3) \sum_{a=0}^{\ell(k,r,3)-x_3} \varphi_3(a, T^{(k,r)}) -$$
$$- \sum_{x_3=0}^{\ell(k,r,3)} \sum_{\gamma \in \mathbb{H}_{-1}(\tilde{\gamma}, x_3)} Q_3(\gamma, x_3) v^{x_3 - \ell(k,r,3)} \sum_{w=0}^{\ell(k,r,3)-x_3} \varphi_3(w, T^{(k,r)}) v^w,$$

$$\alpha(0,r,v) = \tilde{\alpha}(0,r,v) + q_{0,r}(v) \times \sum_{x_3=0}^{L} \Big[Q_3(\mathit{\Gamma}^{(k_1,r_1)}, x_3) + \\ + Q_3(\mathit{\Gamma}^{(0,r\ominus_0 1)}, x_3) \Big] v^{x_3}, \quad \mathit{\Gamma}^{(0,r)} \in \mathit{\Gamma},$$

$$\alpha(k,r,v) = \tilde{\alpha}(k,r,v) - q_{k,r}(v) \sum_{x_3=0}^{L} \Big[Q_3(\mathit{\Gamma}^{(k,r\ominus_k 1)}, x_3) + \\ + Q_3(\mathit{\Gamma}^{(0,r_2)}, x_3) \Big] v^{x_3} + q_{k,r}(v) \sum_{x_3 \geqslant 0} Q_3(\mathit{\Gamma}^{(0,r)}, x_3) v^{x_3}, \quad \mathit{\Gamma}^{(k,r)} \in C_k^{\mathrm{I}},$$

$$\alpha(k,r,v) = \tilde{\alpha}(k,r,v), \quad \mathit{\Gamma}^{(k,r)} \in C_k^{\mathrm{O}} \cup C_k^{\mathrm{N}}$$

Taylor expansion of $q_{k,r}(v)$ gives

$$q_{k,r}(v) = v^{-\ell(k,r,3)} \exp\left(\lambda_3 T^{(k,r)}(f_3(v) - 1)\right) = \\ = 1 + (\lambda_3 T^{(k,r)} f_3'(1) - \ell(k,r,3))(v-1) + O((v-1)^2)).$$

Summing all the relations (6) and (7) one can find

$$\sum_{k=0}^{d} \sum_{r=1}^{n_k} \mathfrak{M}^{(3)}(k,r,v) = \\ = \sum_{r=1}^{n_0} \alpha(0,r,v) + \sum_{k=1}^{d} \sum_{r=1}^{n_k} \big[q_{k,r}(v) \mathfrak{M}^{(3)}(k, r \ominus_k 1, v) + \alpha(k,r,v) \big] = \\ = \sum_{k=1}^{d} \sum_{r=1}^{n_k} q_{k,r}(v) \mathfrak{M}^{(3)}(k, r \ominus_k 1, v) + \sum_{k=1}^{d} \sum_{r=1}^{n_k} \alpha(k,r,v) + \sum_{r=1}^{n_0} \alpha(0,r,v). \quad (8)$$

Similarly lets expand $\sum_{k=1}^{d} \sum_{r=1}^{n_k} \alpha(k,r,v)$ and $\sum_{r=1}^{n_0} \alpha(0,r,v)$. First of all

$$\tilde{\alpha}(k,r,v) = \\ = \sum_{x_3=0}^{\ell(k,r,3)} \sum_{\gamma \in \mathbb{H}_{-1}(\tilde{\gamma}, x_3)} Q_3(\gamma, x_3) \sum_{w=0}^{\ell(k,r,3)-x_3} \varphi_3(w, T^{(k,r)})(1 - v^{w-(\ell(k,r,3)-x_3)}) = \\ = -(v-1) \sum_{x_3=0}^{\ell(k,r,3)} \sum_{\gamma \in \mathbb{H}_{-1}(\tilde{\gamma}, x_3)} Q_3(\gamma, x_3) \times \\ \times \sum_{w=0}^{\ell(k,r,3)-x_3} \varphi_3(w, T^{(k,r)})(w - (\ell(k,r,3) - x_3)) + O((v-1)^2).$$

In particular, $\ell(k,r,3)$ equals to zero for $k=0$, that implies $\tilde{\alpha}(0,r,v) = O((v - - 1)^2)$.

And now we are ready to expand further:

$$\sum_{r=1}^{n_0} \alpha(0,r,v) =$$

$$= \sum_{r=1}^{n_0} q_{0,r}(v) \times \sum_{x_3=0}^{L} \big[Q_3(\Gamma^{(k_1,r_1)},x_3) + Q_3(\Gamma^{(0,r\ominus_0 1)},x_3)\big] v^{x_3} + O((v-1)^2) =$$

$$= \sum_{r=1}^{n_0} (1 + (\lambda_3 T^{(0,r)} f_3'(1) - \ell(0,r,3))(v-1)) \times$$

$$\times \sum_{x_3=0}^{L} \Big[Q_3(\Gamma^{(k_1,r_1)},x_3) + Q_3(\Gamma^{(0,r\ominus_0 1)},x_3)\Big] v^{x_3} + O((v-1)^2),$$

$$\sum_{k,r\colon \Gamma^{(k,r)} \in C_k^{\mathrm{I}}} \alpha(k,r,v) = \sum_{k,r\colon \Gamma^{(k,r)} \in C_k^{\mathrm{I}}} q_{k,r}(v) \Bigg[\mathfrak{M}^{(3)}(0,r_2,v) -$$

$$- \sum_{x_3=0}^{L} \big(Q_3(\Gamma^{(k,r\ominus_k 1)},x_3) + Q_3(\Gamma^{(0,r_2)},x_3)\big) v^{x_3}\Bigg] +$$

$$+ \sum_{k,r\colon \Gamma^{(k,r)} \in C_k^{\mathrm{I}}} \tilde{\alpha}(k,r,v) = \sum_{k,r\colon \Gamma^{(k,r)} \in C_k^{\mathrm{I}}} (1 + (\lambda_3 T^{(k,r)} f_3'(1) - \ell(k,r,3))(v-1)) \times$$

$$\times \Bigg[\mathfrak{M}^{(3)}(0,r_2,v) - \sum_{x_3=0}^{L} \big(Q_3(\Gamma^{(k,r\ominus_k 1)},x_3) + Q_3(\Gamma^{(0,r_2)},x_3)\big) v^{x_3}\Bigg] -$$

$$- (v-1) \sum_{k,r\colon \Gamma^{(k,r)} \in C_k^{\mathrm{I}}} \sum_{x_3=0}^{\ell(k,r,3)} \sum_{\gamma \in \mathbb{H}_{-1}(\tilde{\gamma},x_3)} Q_3(\gamma,x_3) \times$$

$$\times \sum_{w=0}^{\ell(k,r,3)-x_3} \varphi_3(w,T^{(k,r)})(w - (\ell(k,r,3) - x_3)) + O((v-1)^2),$$

$$\sum_{k,r\colon \Gamma^{(k,r)} \in C_k^{\mathrm{O}} \cup C_k^{\mathrm{N}}} \alpha(k,r,v) = \sum_{k,r\colon \Gamma^{(k,r)} \in C_k^{\mathrm{O}} \cup C_k^{\mathrm{N}}} \tilde{\alpha}(k,r,v) =$$

$$= -(v-1) \sum_{k,r\colon \Gamma^{(k,r)} \in C_k^{\mathrm{O}} \cup C_k^{\mathrm{N}}} \sum_{x_3=0}^{\ell(k,r,3)} \sum_{\gamma \in \mathbb{H}_{-1}(\tilde{\gamma},x_3)} Q_3(\gamma,x_3) \times$$

$$\times \sum_{w=0}^{\ell(k,r,3)-x_3} \varphi_3(w,T^{(k,r)})(w - (\ell(k,r,3) - x_3)) + O((v-1)^2).$$

It's important to mention, that any input system state corresponds to one and only one prolongation system state. That is why substitution of calculated expressions into (8) gives:

$$
\begin{aligned}
0 = O((v-1)^2) + (v-1)\sum_{k=1}^{d}\sum_{r=1}^{n_k}(\lambda_3 T^{(k,r)} f_3'(1) - \ell(k,r,3))\mathfrak{M}^{(3)}(k, r\ominus_k 1, v) + \\
+ (v-1)\sum_{r=1}^{n_0}(\lambda_3 T^{(0,r)} f_3'(1) - \ell(0,r,3)) \times \sum_{x_3=0}^{L}\Big[Q_3(\mathit{\Gamma}^{(k_1,r_1)}, x_3) + \\
+ Q_3(\mathit{\Gamma}^{(0,r\ominus_0 1)}, x_3)\Big] v^{x_3} + (v-1)\sum_{k,r\colon \mathit{\Gamma}^{(k,r)}\in C_k^{\mathrm{I}}}(\lambda_3 T^{(k,r)} f_3'(1) - \ell(k,r,3)) \times \\
\times \Big[\mathfrak{M}^{(3)}(0, r_2, v) - \sum_{x_3=0}^{L}\big(Q_3(\mathit{\Gamma}^{(k,r\ominus_k 1)}, x_3) + Q_3(\mathit{\Gamma}^{(0,r_2)}, x_3)\big)v^{x_3}\Big] - \\
- (v-1)\sum_{k,r\colon \mathit{\Gamma}^{(k,r)}\in C_k^{\mathrm{O}}\cup C_k^{\mathrm{N}}}\ \sum_{x_3=0}^{\ell(k,r,3)}\ \sum_{\gamma\in\mathbb{H}_{-1}(\tilde{\gamma},x_3)}\Bigg(Q_3(\gamma, x_3) \times \\
\times \sum_{w=0}^{\ell(k,r,3)-x_3}\varphi_3(w, T^{(k,r)})(w - (\ell(k,r,3) - x_3))\Bigg), \quad (9)
\end{aligned}
$$

dividing by $(v-1)$ and then sending v to 1 from the left one continues

$$
\begin{aligned}
0 = \sum_{k,r\colon \mathit{\Gamma}^{(k,r)}\in C_k^{\mathrm{I}}}(\lambda_3 T^{(k,r)} f_3'(1) - \ell(k,r,3)) \times \Big[\mathfrak{M}^{(3)}(k, r\ominus_k 1, 1) - \\
- \sum_{x_3=0}^{L} Q_3(\mathit{\Gamma}^{(k,r\ominus_k 1)}, x_3) + \mathfrak{M}^{(3)}(0, r_2, 1) - \sum_{x_3=0}^{L} Q_3(\mathit{\Gamma}^{(0,r_2)}, x_3)\Big] + \\
+ \sum_{k,r\colon \mathit{\Gamma}^{(k,r)}\in C_k^{\mathrm{O}}\cup C_k^{\mathrm{N}}}(\lambda_3 T^{(k,r)} f_3'(1) - \ell(k,r,3))\mathfrak{M}^{(3)}(k, r\ominus_k 1, 1) + \\
+ \sum_{r=1}^{n_0}\lambda_3 T^{(0,r)} f_3'(1) \times \sum_{x_3=0}^{L}\Big[Q_3(\mathit{\Gamma}^{(k_1,r_1)}, x_3) + Q_3(\mathit{\Gamma}^{(0,r\ominus_0 1)}, x_3)\Big] + \\
+ \sum_{k=1}^{d}\sum_{r=1}^{n_k}\sum_{x_3=0}^{\ell(k,r,3)}\ \sum_{\gamma\in\mathbb{H}_{-1}(\tilde{\gamma},x_3)} Q_3(\gamma, x_3)\sum_{w=0}^{\ell(k,r,3)-x_3}\varphi_3(w, T^{(k,r)})(\ell(k,r,3) - x_3 - w).
\end{aligned}
\quad (10)
$$

Fixing $v = 1$ in corresponding generating functions relations one can get

$$
\begin{aligned}
\mathfrak{M}^{(3)}(0, r, 1) = \sum_{x_3=0}^{L}\Big(Q_3(\mathit{\Gamma}^{(k_1,r_1)}, x_3) + Q_3(\mathit{\Gamma}^{(0,r\ominus_0 1)}, x_3)\Big), \\
\mathfrak{M}^{(3)}(k, r, 1) = \mathfrak{M}^{(3)}(k, r\ominus_k 1, 1) + \mathfrak{M}^{(3)}(0, r_2, 1) - \\
- \sum_{x_3=0}^{L}\Big(Q_3(\mathit{\Gamma}^{(k,r\ominus_k 1)}, x_3) + Q_3(\mathit{\Gamma}^{(0,r_2)}, x_3)\Big), \quad \mathit{\Gamma}^{(k,r)}\in C_k^{\mathrm{I}},
\end{aligned}
$$

$\mathfrak{M}^{(3)}(k,r,1) = \mathfrak{M}^{(3)}(k, r \ominus_k 1, 1), \quad \Gamma^{(k,r)} \in C_k^{\mathrm{O}} \bigcup C_k^{\mathrm{N}}.$

The last equation gives $\mathfrak{M}^{(3)}(k, n_k, 1) = \mathfrak{M}^{(3)}(k, n_k \ominus_k 1, 1) = \ldots = \mathfrak{M}^{(3)}(k,1,1) = M_k$. Using these facts, let's simplify expression (10)

$$0 = \sum_{k,r\colon \Gamma^{(k,r)} \in C_k^{\mathrm{I}}} (\lambda_3 T^{(k,r)} f_3'(1) - \ell(k,r,3)) \times M_k +$$
$$+ \sum_{k,r\colon \Gamma^{(k,r)} \in C_k^{\mathrm{O}} \cup C_k^{\mathrm{N}}} (\lambda_3 T^{(k,r)} f_3'(1) - \ell(k,r,3)) M_k +$$
$$+ \sum_{r=1}^{n_0} \lambda_3 T^{(0,r)} f_3'(1) \times \sum_{x_3=0}^{L} \Big[Q_3(\Gamma^{(k_1,r_1)}, x_3) + Q_3(\Gamma^{(0, r \ominus_0 1)}, x_3) \Big] +$$
$$+ \sum_{\substack{k,r\colon \Gamma^{(k,r)} \in \\ \in C_k^{\mathrm{O}} \cup C_k^{\mathrm{N}}}} \sum_{x_3=0}^{\ell(k,r,3)} \sum_{\gamma \in \mathbb{H}_{-1}(\tilde{\gamma}, x_3)} Q_3(\gamma, x_3) \sum_{w=0}^{\ell(k,r,3)-x_3} \varphi_3(w, T^{(k,r)}) (\ell(k,r,3) - x_3 - w).$$

After grouping terms we get:

$$0 = \sum_{k=1}^{d} \Bigg(M_k \times \sum_{r=1}^{n_k} (\lambda_3 T^{(k,r)} f_3'(1) - \ell(k,r,3)) \Bigg) +$$
$$+ \sum_{r=1}^{n_0} \lambda_3 T^{(0,r)} f_3'(1) \times \sum_{x_3=0}^{L} \Big[Q_3(\Gamma^{(k_1,r_1)}, x_3) + Q_3(\Gamma^{(0, r \ominus_0 1)}, x_3) \Big] +$$
$$+ \sum_{\substack{k,r\colon \Gamma^{(k,r)} \in \\ \in C_k^{\mathrm{O}} \cup C_k^{\mathrm{N}}}} \sum_{x_3=0}^{\ell(k,r,3)} \sum_{\gamma \in \mathbb{H}_{-1}(\tilde{\gamma}, x_3)} Q_3(\gamma, x_3) \sum_{w=0}^{\ell(k,r,3)-x_3} \varphi_3(w, T^{(k,r)}) (\ell(k,r,3) - x_3 - w).$$

Assumption that expression $\sum_{r=1}^{n_k} \ell(k,r,3) \big/ \lambda_3 f_3'(1) \sum_{r=1}^{n_k} T^{(k,r)}$ for any $k = \overline{1;d}$ is less than or equal to 1, leads to impossible conclusion $Q_3(\Gamma^{(0,1)}, 0) = 0$. Theorem is proved.

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