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Primary input flows in a tandem under prolongable cyclic service

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Abstract. A tandem of queuing systems is under investigation. There are high and low-priority input flows in each system. Customers of the first system are serviced in the class of cyclic algorithms. After service high-priority customers of the first system are transferred to the second one. In the second system, customers are serviced in the class of cyclic algorithms with prolongations. This means low-priority customers are serviced once their number exceeds predefined threshold. A mathematical model is constructed in form of a multidimensional denumerable discrete-time Markov chain. The queues of primary input flows are of special interest in this paper. Partial generating functions of two-dimensional Markov chain are found. Sufficient condition for steady state existence is also given.

1. Introduction

There is a lot of literature dedicated to conflicting traffic flows control at crossroad. One can find following types of algorithms investigated there: cyclic algorithm with a loop, cyclic algorithm with changing regimes, fixed duration cyclic algorithm, etc. See [1–6]. Though this is almost always the case, when after vehicle passes one crossroad, it finds itself on another. That is in a real life there are more than one crossroad on the way. In other words, an output flow of cars from the first intersection forms an input flow of cars of the next intersection. This imposes that the second input flow no longer has an *a priori* known simple probabilistic structure (for example, a non-ordinary Poison flow), and knowledge about the service algorithm should be taken into account to deduce formation conditions of the first output flow.

Tandems of crossroads are studied for example in the following papers. In [7] a mathematical model of two intersections in tandem governed by cyclic algorithms was investigated and stability conditions were found. In [8] a computer-aided simulation of adjacent intersections was carried out. In this paper we assume that the first intersection is governed by a cyclic algorithm while the second intersection is governed by a cyclic algorithm with prolongations. The high priority input flow of the first system and low priority input flow of the second system we will call primary input flows. This is because they are both generated by the environment and not by the system as the secondary input flows. The queues of primary input flows take central place in this paper. This work continues studying in [10] and [11].

2. The problem settings

Consider a queuing system with a scheme shown in Fig. 1. There are four input flows of customers Π_1, Π_2, Π_3, and Π_4 entering the single server queueing system. Customers in the input flow Π_j, $j \in \{1, 2, 3, 4\}$ join a queue O_j with an unlimited capacity. For $j \in \{1, 2, 3\}$ the discipline of the queue O_j is FIFO (First In First Out). Discipline of the queue O_4 will be described later. The input flows Π_1 and Π_3 are generated by an external environment, which has only one state. Each of these flows is a nonordinary Poisson flow. Denote by λ_1 and λ_3 the intensities of bulk arrivals for the flows Π_1 and Π_3 respectively. The probability generating function of number of customers in a bulk in the flow Π_j is $f_j(z) = \sum_{\nu=1}^{\infty} p_\nu^{(j)} z^\nu, \quad j \in \{1, 3\}$.

We assume that $f_j(z)$ converges for any $z \in \mathbb{C}$ such that $|z| < (1 + \varepsilon)$, $\varepsilon > 0$. Here $p_\nu^{(j)}$ is the probability of a bulk size in flow Π_j being exactly $\nu = 1, 2, \ldots$. Having been serviced the customers from O_1 come back to the system as the Π_4 customers. The Π_4 customers in turn after service enter the system as the Π_2 ones. The flows Π_2 and Π_3 are conflicting in the sense that their customers can't be serviced simultaneously. This implies

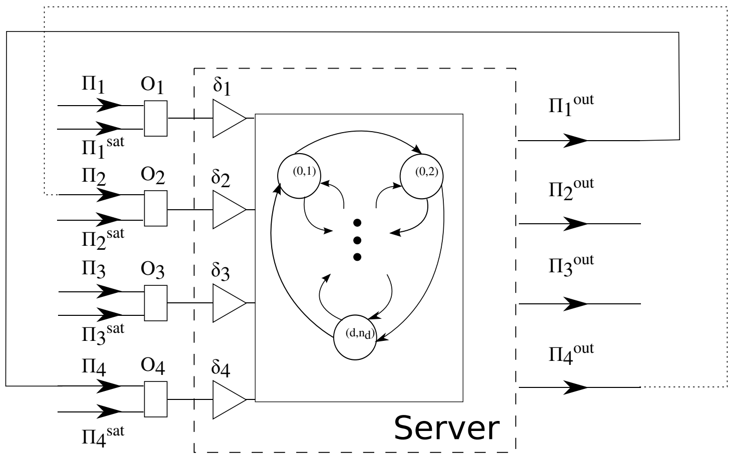

Figure 1. Scheme of the queuing system as a cybernetic control system

that the problem can't be reduced to a problem with fewer input flows by merging the flows together.

In order to describe the server behavior positive integers d, n_0, n_1, ..., n_d are fixed and a finite set $\Gamma = \{\Gamma^{(k,r)} \colon k = 0, 1, \ldots, d; r = 1, 2, \ldots n_k\}$ of states server can reside in is introduced. At the state $\Gamma^{(k,r)}$ the server stays during a constant time $T^{(k,r)}$. We will assume, that for each fixed k^* cycle subset $\{\Gamma^{(k^*,r)} \colon r = 1, 2, \ldots n_k^*\} = C_{k^*}^{\mathrm{N}} \cup C_{k^*}^{\mathrm{O}} \cup C_{k^*}^{\mathrm{I}}$, that is consists of three disjoint sets called neutral, output and input sets of states. In more details server is described in [11].

In general, service durations of different customers can be dependent and may have different laws of probability distributions. So, saturation flows will be used to define the service process. A saturation flow Π_j^{sat}, $j \in \{1, 2, 3, 4\}$, is defined as a virtual output flow under the maximum usage of the server and unlimited number of customer in the queue O_j. The saturation flow Π_j^{sat}, $j \in \{1, 2, 3\}$ contains a non-random number $\ell(k, r, j) \geqslant 0$ of customers in the server state $\Gamma^{(k,r)}$.

A real-life example of just described queuing system is a tandem of two consecutive crossroads (Fig. 2). The input flows are flows of vehicles. The flows Π_1 and Π_5 at the first crossroad are conflicting; Π_2 and Π_3 at the second crossroad are also conflicting. Every vehicle from the flow Π_1 after passing first road intersection joint the flow Π_4 and enters the queue O_4. After some random time interval the vehicle arrives to the next road intersection. Such a pair of crossroads is an instance of a more general queuing model described above.

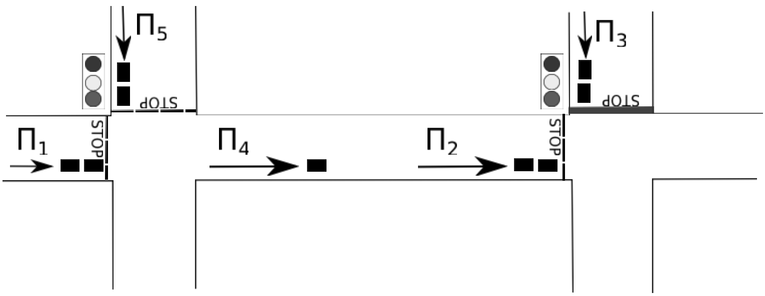

Figure 2. A tandem of crossroads, the physical interpretation of the queuing system under study

3. Mathematical model

The queuing system under investigation can be regarded as a cybernetic control system that helps to rigorously construct a formal stochastic model [7]. The scheme of the control system is shown in Fig. 1. There are following blocks present in the scheme: 1) the external environment with one state; 2) input poles of the first type — the input flows Π_1, Π_2, Π_3, and Π_4; 3) input poles of the second type — the saturation flows Π_1^{sat}, Π_2^{sat}, Π_3^{sat}, and Π_4^{sat}; 4) an external memory — the queues O_1, O_2, O_3, and O_4; 5) an information processing device for the external memory — the queue discipline units δ_1, δ_2, δ_3, and δ_4; 6) an internal memory — the server (OY); 7) an information processing device for internal memory — the graph of server state transitions; 8) output poles — the output flows Π_1^{out}, Π_2^{out}, Π_3^{out}, and Π_4^{out}. The coordinate of a block is its number on the scheme.

Let us introduce the following variables and elements along with their value ranges. To fix a discrete time scale consider the epochs $\tau_0 = 0$, τ_1, τ_2, ... when the server changes its state. Let $\Gamma_i \in \Gamma$ be the server state during the interval $(\tau_{i-1}; \tau_i]$, $\varkappa_{j,i} \in \mathbb{Z}_+$ be the number of customers in the queue O_j at the instant τ_i, $\eta_{j,i} \in \mathbb{Z}_+$ be the number of customers arrived into the queue O_j from the flow Π_j during the interval $(\tau_i; \tau_{i+1}]$, $\xi_{j,i} \in \mathbb{Z}_+$ be the number of customers in the saturation flow Π_j^{sat} during the interval $(\tau_i; \tau_{i+1}]$, $\overline{\xi}_{j,i} \in \mathbb{Z}_+$ be the actual number of serviced customers from the queue O_j during the interval $(\tau_i; \tau_{i+1}]$, $j \in \{1, 2, 3, 4\}$.

The server changes its state according to the following rule: $\Gamma_{i+1} = h(\Gamma_i, \varkappa_{3,i})$ where the mapping $h(\cdot, \cdot)$ is defined in paper [11].

Let $\varphi_1(\cdot, \cdot)$ and $\varphi_3(\cdot, \cdot)$ be defined by series expansions

$$\sum_{\nu=0}^{\infty} z^{\nu} \varphi_j(\nu, t) = \exp\{\lambda_j t (f_j(z) - 1)\},$$

$j \in \{1, 3\}$. The function $\varphi_j(\nu, t)$ equals the probability of $\nu = 0$, 1, ... arrivals in the flow Π_j during time $t \geqslant 0$. If $\nu < 0$ the value of $\varphi_j(\nu, t)$ is set to zero.

Mathematical model in more details can be found in work [10]. From now on we focus on primary input flows customers in the queues O_1 and O_3.

4. The queues of primary input flows

Here we will consider the stochastic sequence

$$\{(\Gamma_i(\omega), \varkappa_{1,i}(\omega), \varkappa_{3,i}(\omega)); i = 0, 1, \ldots\}, \tag{1}$$

which includes the number of customers $\varkappa_{1,i}(\omega)$ and $\varkappa_{3,i}(\omega)$ in the queues O_1 and O_3 respectfully. In this section we will report several results concerning this stochastic sequence.

Let's define for $\gamma \in \Gamma$ and $(x_1, x_3) \in Z_+^2$ values

$$Q_{1,i}(\gamma, x_1, x_3) = \mathbf{P}(\{\omega\colon \Gamma_i(\omega) = \gamma, \varkappa_{1,i}(\omega) = x_1, \varkappa_{3,i}(\omega) = x_3\}).$$

Suppose k and r are such that $\Gamma^{(k,r)} \in \Gamma$. Let's define partial probability generating functions

$$\mathfrak{M}^{(1,i)}(k, r, v_1, v_3) = \sum_{w_1=0}^{\infty} \sum_{w_3=0}^{\infty} Q_{1,i}(\Gamma^{(k,r)}, w_1, w_3) v_1^{w_1} v_3^{w_3},$$

and auxilary functions

$$q^{(1,i)}(k, r, v_1) = v_1^{-\ell(k,r,1)} \sum_{w=0}^{\infty} \varphi_1(w, T^{(k,r)}) v_1^{w};$$

$$q^{(3,i)}(k, r, v_3) = q_{k,r}(v_3) = v_3^{-\ell(k,r,3)} \sum_{w=0}^{\infty} \varphi_3(w, T^{(k,r)}) v_3^{w},$$

and

$$g^{(1,i)}(k, r, v_1, v_3) = \sum_{x_3=L+1}^{\infty} \sum_{x_1=\ell(k,r,1)+1}^{\infty} [Q_{1,i}(\Gamma^{(k, r\ominus_k 1)}, x_1, x_3)+$$
$$+Q_{1,i}(\Gamma^{(0,r_2)}, x_1, x_3)] v_1^{x_1} v_3^{x_3}, \quad \Gamma^{(k,r)} \in C_k^{\mathrm{I}}$$
$$g^{(1,i)}(k, r, v_1, v_3) =$$
$$= \sum_{x_3=\ell(k,r,3)+1}^{\infty} \sum_{x_1=\ell(k,r,1)+1}^{\infty} Q_{1,i}(\Gamma^{(k, r\ominus_k 1)}, x_1, x_3) v_1^{x_1} v_3^{x_3}, \ \Gamma^{(k,r)} \in C_k^{\mathrm{O}} \cup C_k^{\mathrm{N}}$$

$$\alpha^{(1,i)}(k, r, v_1, v_3) = \sum_{x_3=0}^{\ell(k,r,3)} \sum_{w_3=1}^{\infty} \sum_{w_1=1}^{\infty} \sum_{x_1=0}^{w_1+\ell(k,r,1)} [Q_{1,i}(\Gamma^{(k_1,r_1)}, x_1, x_3)+$$
$$+Q_{1,i}(\Gamma^{(0,r\ominus_0 1)}, x_1, x_3)] \times \varphi_3(w_3 + \ell(k,r,3) - x_3, T^{(k,r)})\times$$
$$\times\varphi_1(w_1 + \ell(k,r,1) - x_1, T^{(k,r)}) \times v_1^{w_1} v_3^{w_3} + q^{(3,i)}(k, r, v_3)\times$$
$$\times \sum_{x_1=0}^{\ell(k,r,1)} \sum_{x_3=\ell(k,r,3)+1}^{L} v_3^{x_3} \sum_{w_1=1}^{\infty} [Q_{1,i}(\Gamma^{(k_1,r_1)}, x_1, x_3)+$$
$$+Q_{1,i}(\Gamma^{(0,r\ominus_0 1)}, x_1, x_3)] \varphi_1(w_1 + \ell(k,r,1) - x_1, T^{(k,r)}) v_1^{w_1}+$$

$$+q^{(1,i)}(k,r,v_1)q^{(3,i)}(k,r,v_3)\sum_{x_3=\ell(k,r,3)+1}^{L}\sum_{x_1=\ell(k,r,1)+1}^{\infty}[Q_{1,i}(\Gamma^{(k_1,r_1)},x_1,x_3)+$$

$$+Q_{1,i}(\Gamma^{(0,r\ominus_0 1)},x_1,x_3)]v_1^{x_1}v_3^{x_3}+\sum_{x_1=0}^{\ell(k,r,1)}\sum_{x_3=0}^{\ell(k,r,3)}[Q_{1,i}(\Gamma^{(k_1,r_1)},x_1,x_3)+$$

$$+Q_{1,i}(\Gamma^{(0,r\ominus_0 1)},x_1,x_3)]\times\sum_{a=0}^{\ell(k,r,3)-x_3}\varphi_3(a,T^{(k,r)})\times\sum_{a=0}^{\ell(k,r,1)-x_1}\varphi_1(a,T^{(k,r)})+$$

$$+\sum_{w_3=1}^{\infty}\sum_{x_1=0}^{\ell(k,r,1)}\sum_{x_3=0}^{\min\{L,w_3+\ell(k,r,3)\}}[Q_{1,i}(\Gamma^{(k_1,r_1)},x_1,x_3)+$$

$$+Q_{1,i}(\Gamma^{(0,r\ominus_0 1)},x_1,x_3)]\times\varphi_3(w_3+\ell(k,r,3)-x_3,T^{(k,r)})\times$$

$$\times\sum_{a=0}^{\ell(k,r,1)-x_1}\varphi_1(a,T^{(k,r)})v_3^{w_3}+\sum_{w_1=1}^{\infty}\sum_{x_1=0}^{w_1+\ell(k,r,1)}\sum_{x_3=0}^{\ell(k,r,3)}[Q_{1,i}(\Gamma^{(k_1,r_1)},x_1,x_3)+$$

$$+Q_{1,i}(\Gamma^{(0,r\ominus_0 1)},x_1,x_3)]\times\sum_{a=0}^{\ell(k,r,3)-x_3}\varphi_3(a,T^{(k,r)})\times$$

$$\times\varphi_1(w_1+\ell(k,r,1)-x_1,T^{(k,r)})v_1^{w_1},\quad k=0$$

$$\alpha^{(1,i)}(k,r,v_1,v_3)=q^{(3,i)}(k,r,v_3)\times$$

$$\times\sum_{x_1=0}^{\ell(k,r,1)}\sum_{x_3=L}^{\infty}v_3^{x_3}\sum_{w_1=1}^{\infty}[Q_{1,i}(\Gamma^{(0,r_2)},x_1,x_3)+Q_{1,i}(\Gamma^{(k,r\ominus_k 1)},x_1,x_3)]\times$$

$$\times\varphi_1(w_1+\ell(k,r,1)-x_1,T^{(k,r)})v_1^{w_1}+$$

$$+\sum_{w_3=L-\ell(k,r,3)+1}^{\infty}\sum_{x_1=0}^{\ell(k,r,1)}\sum_{x_3=L+1}^{w_3+\ell(k,r,3)}[Q_{1,i}(\Gamma^{(0,r_2)},x_1,x_3)+Q_{1,i}(\Gamma^{(k,r\ominus_k 1)},x_1,x_3)]\times$$

$$\times\varphi_3(w_3+\ell(k,r,3)-x_3,T^{(k,r)})\times\sum_{a=0}^{\ell(k,r,1)-x_1}\varphi_1(a,T^{(k,r)})v_3^{w_3},\ \Gamma^{(k,r)}\in C_k^{\mathrm{I}}$$

$$\alpha^{(1,i)}(k,r,v_1,v_3)=\sum_{x_3=0}^{\ell(k,r,3)}\sum_{w_3=1}^{\infty}\sum_{w_1=1}^{\infty}\sum_{x_1=0}^{w_1+\ell(k,r,1)}Q_{1,i}(\Gamma^{(k,r\ominus_k 1)},x_1,x_3)\times$$

$$\times\varphi_3(w_3+\ell(k,r,3)-x_3,T^{(k,r)})\varphi_1(w_1+\ell(k,r,1)-x_1,T^{(k,r)})v_1^{w_1}v_3^{w_3}+$$

$$+q^{(3,i)}(k,r,v_3)\times\sum_{x_1=0}^{\ell(k,r,1)}\sum_{x_3=\ell(k,r,3)+1}^{\infty}v_3^{x_3}\sum_{w_1=1}^{\infty}Q_{1,i}(\Gamma^{(k,r\ominus_k 1)},x_1,x_3)\times$$

$$\times\varphi_1(w_1+\ell(k,r,1)-x_1,T^{(k,r)})v_1^{w_1}+\sum_{x_1=0}^{\ell(k,r,1)}\sum_{x_3=0}^{\ell(k,r,3)}Q_{1,i}(\Gamma^{(k,r\ominus_k 1)},x_1,x_3)\times$$

$$\times\sum_{a=0}^{\ell(k,r,3)-x_3}\varphi_3(a,T^{(k,r)})\times\sum_{a=0}^{\ell(k,r,1)-x_1}\varphi_1(a,T^{(k,r)})+$$

$$+\sum_{w_3=1}^{\infty}\sum_{x_1=0}^{\ell(k,r,1)}\sum_{x_3=0}^{w_3+\ell(k,r,3)}Q_{1,i}(\Gamma^{(k,r\ominus_k 1)},x_1,x_3)\times$$

$$\times\varphi_3(w_3+\ell(k,r,3)-x_3,T^{(k,r)})\times\sum_{a=0}^{\ell(k,r,1)-x_1}\varphi_1(a,T^{(k,r)})v_3^{w_3}+$$

$$+\sum_{w_1=1}^{\infty}\sum_{x_1=0}^{w_1+\ell(k,r,1)}\sum_{x_3=0}^{\ell(k,r,3)}Q_{1,i}(\Gamma^{(k,r\ominus_k 1)},x_1,x_3)\times$$

$$\times\sum_{a=0}^{\ell(k,r,3)-x_3}\varphi_3(a,T^{(k,r)})\varphi_1(w_1+\ell(k,r,1)-x_1,T^{(k,r)})v_1^{w_1},\ \Gamma^{(k,r)}\in C_k^{\mathrm{O}}\cup C_k^{\mathrm{N}}$$

Theorem 4.1. *Following recurrent w.r.t. $i\geqslant 0$ relations take place for the partial probability generating functions:*

1. $\Gamma^{(0,\tilde r)}\in\Gamma$, $\tilde r=\overline{1,n_0}$

$$\mathfrak{M}^{(1,i+1)}(0,\tilde r,v)=\alpha^{(1,i)}(0,\tilde r,v_1,v_3);$$

2. $\Gamma^{(\tilde k,\tilde r)}\in C_{\tilde k}^{\mathrm{I}}\cup C_{\tilde k}^{\mathrm{O}}\cup C_{\tilde k}^{\mathrm{N}}$

$$\mathfrak{M}^{(1,i+1)}(\tilde k,\tilde r,v)=q^{(1,i)}(\tilde k,\tilde r,v_1)q^{(3,i)}(\tilde k,\tilde r,v_3)\times g^{(1,i)}(\tilde k,\tilde r,v_1,v_3)+ +\alpha^{(1,i)}(\tilde k,\tilde r,v_1,v_3);$$

Theorem 4.2. *For Markov chain* (1) *to have a stationary distribution $Q_1(\gamma,x)$, $(\gamma,x)\in\Gamma\times\mathbb{Z}_+$ it is sufficient to satisfy the following inequality*

$$\min_{\substack{k=\overline{1,d}\\ j=1,3}}\frac{\sum_{r=1}^{n_k}\ell(k,r,j)}{\lambda_j f_j'(1)\sum_{r=1}^{n_k}T^{(k,r)}}>1.$$

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