Asymptotic properties of tandem service and control operation in a class of cyclic algorithms with prolongation

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Operations research is a field of mathematics which deals with finding optimal way of *operation* on different objects. The nature of these operations might contain very complex stochastic grain. Once the objects are customers or demands and the operation is service and control, queuing theory methods can be used. Queuing theory approaches can be divided in two groups. The first group of methods is classical and assume homogeneity of all customers as well as that of all operations on the customers. For the first time such constraints were considered in the early XX century by F. Johanssen, A.K. Erlang, A.Ya. Khinchine, F. Pollaczek, C. Palm, D. Kendall. During the second half of the XX century this sort of customers and operations on them have been investigated by A.N. Kolmogorov, B.V. Gnedenko, T.D. Saati, Yu.V. Prokhorov, E.S. Ventsel et al.

Though there are a lot of cases where homogeneity assumptions do not hold: information flows processing in local-area computer networks and telecommunication networks, control of conflicting flows of aircrafts at passage of intersecting air lanes, conflicting transport flows control at intersections with complicated crossing geometry etc. The second group of queuing theory methods deal with nonhomogeneous customers and nonhomogeneous operations on these customers. This is done via solving following fundamental problems: 1) classification of nonhomogeneous customers and description of flows of nonhomogeneous customers; 2) formation and development of control algorithms of conflicting flows of nonhomogeneous customers. In this paper we describe one of the problems that can be solved with second group of queuing theory methods.

Consider a real-life system of tandem of two consecutive crossroads (Fig. 1). The input flows are flows of vehicles. The flows Π_1 and Π_5 at the first crossroad are conflicting; Π_2 and Π_3 at the second crossroad are also conflicting. Every vehicle from the flow Π_1 after passing the first road intersection joints the flow Π_4 and enters the queue O_4. After some random time interval the vehicle arrives to the next road intersection. Such a pair of crossroads is an instance of a more general queuing model described below.

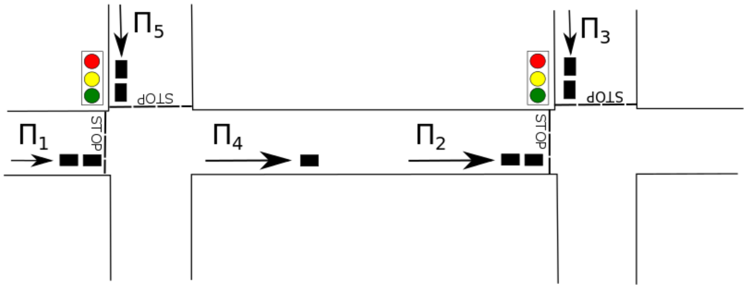

Fig. 1. Tandem of crossroads

Consider a queuing system with four input flows of customers Π_1, Π_2, Π_3, and Π_4 entering the single server queueing system (Π_5 flow has no effect on system behaviour and is omitted in remaining discussion). Customers in the input flow Π_j, $j \in \{1,2,3,4\}$ join a queue O_j with an unlimited capacity. For $j \in \{1,2,3\}$ the discipline of the queue O_j is FIFO (First In First Out). Discipline of the queue O_4 will be described later. The input flows Π_1 and Π_3 are generated by an external environment, which has only one state. Each of these flows is a nonordinary Poisson flow. Denote by λ_1 and λ_3 the intensities of bulk arrivals for the flows Π_1 and Π_3 respectively. The probability generating function of number of customers in a bulk in the flow Π_j is $f_j(z) = \sum_{\nu=1}^{\infty} p_\nu^{(j)} z^\nu, \quad j \in \{1,3\}$. We assume that $f_j(z)$ converges for any $z \in \mathbb{C}$ such that $|z| < (1+\varepsilon)$, $\varepsilon > 0$. Here $p_\nu^{(j)}$ is the probability of a bulk size in flow Π_j being exactly $\nu = 1, 2, \ldots$. Having been serviced the customers from O_1 come back to the system as the Π_4 customers. The Π_4 customers in turn after service enter the system as the Π_2 ones. The flows Π_2 and Π_3 are conflicting in the sense that their customers can't be serviced simultaneously. This implies that the problem can't be reduced to a problem with fewer input flows by merging the flows together.

In order to describe the server behavior positive integers d, $n_0, n_1, \ldots$, n_d are fixed and a finite set $\Gamma = \{\Gamma^{(k,r)} \colon k = 0, 1, \ldots, d; r = 1, 2, \ldots n_k\}$ of states server can reside in is introduced. At the state $\Gamma^{(k,r)}$ the server stays during a constant time $T^{(k,r)}$. We will assume, that for each fixed k^* cycle subset $\{\Gamma^{(k^*,r)} \colon r = 1, 2, \ldots n_k^*\} = C_{k^*}^{\mathrm{N}} \cup C_{k^*}^{\mathrm{O}} \cup C_{k^*}^{\mathrm{I}}$, that is consists of three disjoint sets called neutral, output and input sets of states. In more details server is described in [1].

In general, service durations of different customers can be dependent and may have different laws of probability distributions. So, saturation flows will be used to define the service process. The saturation flow Π_j^{sat}, $j \in \{1,2,3,4\}$, is defined as a virtual output flow under the maximum usage of the server and unlimited number of customer in the queue O_j. The saturation flow Π_j^{sat}, $j \in \{1,2,3\}$ contains a non-random number $\ell(k,r,j) \geqslant 0$ of customers in the server state $\Gamma^{(k,r)}$.

The queuing system under investigation can be regarded as a cybernetic control system, it helps to rigorously construct a formal stochastic model [2]. There are following blocks present in the system: 1) the external environment with one state; 2) input poles of the first type — the input flows Π_1, Π_2, Π_3, and Π_4; 3) input poles of the second type — the saturation flows Π_1^{sat}, Π_2^{sat}, Π_3^{sat}, and Π_4^{sat}; 4) an external memory — the queues O_1, O_2, O_3, and O_4; 5) an information processing device for the external memory — the queue discipline units δ_1, δ_2, δ_3, and δ_4; 6) an internal memory — the server (OY); 7) an information processing device for internal memory — the graph of server state transitions; 8) output poles — the output flows Π_1^{out}, Π_2^{out}, Π_3^{out}, and Π_4^{out}.

Let us introduce the following variables and elements along with their value ranges. To fix a discrete time scale consider the epochs $\tau_0 = 0$, τ_1, τ_2, ... when the server changes its state. Let $\Gamma_i \in \Gamma$ be the server state during the interval $(\tau_{i-1};\tau_i]$, $\varkappa_{j,i} \in \mathbb{Z}_+$ be the number of customers in the queue O_j at the instant τ_i, $\eta_{j,i} \in \mathbb{Z}_+$ be the number of customers arrived into the queue O_j from the flow Π_j during the interval $(\tau_i;\tau_{i+1}]$, $\xi_{j,i} \in \mathbb{Z}_+$ be the number of customers in the saturation flow Π_j^{sat} during the interval $(\tau_i;\tau_{i+1}]$, $\overline{\xi}_{j,i} \in \mathbb{Z}_+$ be the actual number of serviced customers from the queue O_j during the interval $(\tau_i;\tau_{i+1}]$, $j \in \{1,2,3,4\}$. The server changes its state according to the following rule: $\Gamma_{i+1} = h(\Gamma_i, \varkappa_{3,i})$ where the mapping $h(\cdot,\cdot)$ is defined in paper [1]. Lets define function $\psi(\cdot,\cdot,\cdot)$: $\psi(k;y,u) = C_y^k u^k (1-u)^{y-k}$. $\psi(k;y,u)$ is probability of arrival of k Π_2-customers given O_4 has y customers and server is in state $\Gamma^{(k,r)}$, that is $u = p_{k,r}$. If $0 \leqslant k \leqslant y$ does no hold we put $\psi(k;y,u)$ equal 0. Mathematical model in more details can be found in work [3].

We now present several results regarding asymptotic behaviour of described system. Consider stochastic sequences:

$$\{(\Gamma_i(\omega), \varkappa_{3,i}(\omega)); i = 0,1,\ldots\}, \tag{1}$$

$$\{(\Gamma_i(\omega), \varkappa_{1,i}(\omega), \varkappa_{3,i}(\omega)); i = 0,1,\ldots\}, \tag{2}$$

$$\{(\Gamma_i(\omega), \varkappa_{1,i}(\omega), \varkappa_{3,i}(\omega), \varkappa_{4,i}(\omega)); i = 0,1,\ldots\}, \tag{3}$$

which include number of customers $\varkappa_{1,i}(\omega)$, $\varkappa_{3,i}(\omega)$ and $\varkappa_{4,i}(\omega)$ in the queues O_1, O_3 and O_4 respectfully.

Theorem 1. Let $\Gamma_0 = \Gamma^{(k,r)} \in \Gamma$ and $\varkappa_{3,0} = x_{3,0} \in \mathbb{Z}_+$ be fixed. Then the sequence (1) is Markov chain.

Theorem 2. Let $\Gamma_0 = \Gamma^{(k,r)} \in \Gamma$ and $(\varkappa_{1,0}, \varkappa_{3,0}) = (x_{1,0}, x_{3,0}) \in \mathbb{Z}_+^2$ be fixed. Then the sequence (2) is Markov chain.

Theorem 3. For Markov chain (1) to have a stationary distribution it is sufficient to satisfy the following inequalitiy

$$\min_{k=\overline{1,d}} \frac{\sum_{r=1}^{n_k} \ell(k,r,3)}{\lambda_3 f_3'(1) \sum_{r=1}^{n_k} T^{(k,r)}} > 1.$$

Theorem 4. For Markov chain (2) to have a stationary distribution it is sufficient to satisfy the following inequalities

$$\min_{k=\overline{0,d}} \frac{\sum_{r=1}^{n_k} \ell(k,r,1)}{\lambda_1 f_1'(1) \sum_{r=1}^{n_k} T^{(k,r)}} > 1, \quad \min_{k=\overline{1,d}} \frac{\sum_{r=1}^{n_k} \ell(k,r,3)}{\lambda_3 f_3'(1) \sum_{r=1}^{n_k} T^{(k,r)}} > 1.$$

Theorem 1 and *Theorem 3* concern low-priority queue which is also described in [1,3,4]. *Theorem 2* and *Theorem 4* concern primary input flow queues which are referenced in [5].

Theorem 5. For queue sizes in stochastic sequence (3) to be bounded it is sufficient to satisfy assumptions of *Theorem 4* and following inequality:

$$\min_{k=\overline{0,d}, r=\overline{1,n_k}} \{p_{k,r}\} > 0.$$

Proof Let $(\gamma, x_3) \in \Gamma \times \mathbb{Z}_+$ and $\Gamma^{(\tilde{k},\tilde{r})} = h(\gamma, x_3)$. Put $A_i(w_1, w_3, w_4, \gamma) = \{\omega\colon \varkappa_{1,i} = w_1, \varkappa_{3,i} = w_3, \varkappa_{4,i} = w_4, \Gamma_i = \gamma\}$. Considering recurrent expression for $\varkappa_{4,i+1}$ one has:

$$\begin{aligned}
&E[\varkappa_{4,i+1} | \varkappa_{1,i} = w_1, \varkappa_{3,i} = w_3, \varkappa_{4,i} = w_4, \Gamma_i = \gamma] = \\
&\qquad = E[w_4 - \eta_{2,i} + \min\{\xi_{1,i}, w_1 + \eta_{1,i}\} | A_i(w_1, w_3, w_4, \gamma)] \leqslant \\
&\qquad\qquad \leqslant E[w_4 - \eta_{2,i} + \xi_{1,i} | A_i(w_1, w_3, w_4, \gamma)] = \\
&\qquad\qquad = E[w_4 - \eta_{2,i} + \ell(\tilde{k}, \tilde{r}, 1) | A_i(w_1, w_3, w_4, \gamma)] = \\
&\qquad\qquad\qquad = w_4 + \ell(\tilde{k}, \tilde{r}, 1) - E[\eta_{2,i} | A_i(w_1, w_3, w_4, \gamma)].
\end{aligned}$$

From definition of $\psi(\cdot; \cdot, \cdot)$ we get:

$$\begin{aligned}
&E[\eta_{2,i} | A_i(w_1, w_3, w_4, \gamma)] = \\
&\qquad = \sum_{a=0}^{w_4} a \psi(a; w_4, p_{\tilde{k},\tilde{r}}) = \sum_{a=0}^{w_4} a \binom{w_4}{a} p_{\tilde{k},\tilde{r}}^a (1 - p_{\tilde{k},\tilde{r}})^{w_4 - a} = w_4 p_{\tilde{k},\tilde{r}}.
\end{aligned}$$

Hence it is true that $E[\varkappa_{4,i+1}|A_i(w_1,w_3,w_4,\gamma)] \leqslant w_4(1-p_{\tilde{k},\tilde{r}})+\ell(\tilde{k},\tilde{r},1)$. Using law of total expectation one can get:

$$E[\varkappa_{4,i+1}] = \sum_{w_1=0}^{\infty}\sum_{w_3=0}^{\infty}\sum_{w_4=0}^{\infty}\sum_{\gamma\in\Gamma} E[\varkappa_{4,i+1}|A_i(w_1,w_3,w_4,\gamma)]\times$$

$$\times \mathbf{P}(A_i(w_1,w_3,w_4,\gamma)) \leqslant \sum_{w_3=0}^{\infty}\sum_{w_4=0}^{\infty}\sum_{\gamma\in\Gamma}(w_4(1-p_{\tilde{k},\tilde{r}})+\ell(\tilde{k},\tilde{r},1))\times$$

$$\times \mathbf{P}(A_i(w_1,w_3,w_4,\gamma)) \leqslant (1-\min\{p_{\tilde{k},\tilde{r}}\})\times$$

$$\times \sum_{w_4=0}^{\infty} w_4\mathbf{P}(\varkappa_{4,i}=w_4) + \max\{\ell(\tilde{k},\tilde{r},1)\}) \sum_{w_4=0}^{\infty}\mathbf{P}(\varkappa_{4,i}=w_4) =$$

$$= (1-\min\{p_{\tilde{k},\tilde{r}}\})E[\varkappa_{4,i}] + \max\{\ell(\tilde{k},\tilde{r},1)\}).$$

The sequence $M_0 = E[\varkappa_{4,0}]$, $M_{i+1} = (1-\min\{p_{\tilde{k},\tilde{r}}\})M_i+\max\{\ell(\tilde{k},\tilde{r},1)\})$ bounds from above sequence $E[\varkappa_{4,i+1}]$ and under theorem assumptions is limited. It implies O_4 queue size $\varkappa_{4,i}$ to be limited. Since conditions of the *Theorem* 4 are satisfied then Markov chain (2) has stationary distribution and queue sizes $\varkappa_{1,i}$ and $\varkappa_{3,i}$ are limited as well. We are done.

References

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