LOW-PRIORITY QUEUE FLUCTUATIONS IN TANDEM OF QUEUING SYSTEMS UNDER CYCLIC CONTROL WITH PROLONGATIONS

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Abstract

A tandem of queuing systems is considered. Each system has a high-priority input flow and a low-priority input flow which are conflicting. In the first system, the customers are serviced in the class of cyclic algorithms. The serviced high-priority customers are transferred from the first system to the second one with random delays and become the high-priority input flow of the second system. In the second system, customers are serviced in the class of cyclic algorithms with prolongations. Low-priority customers are serviced when their number exceeds a threshold. A mathematical model is constructed in form of a multidimensional denumerable discrete-time Markov chain. The recurrent relations for partial probability generating functions for the low-priority queue in the second system are found.

Keywords: tandem of controlling queuing systems, cyclic algorithm with prolongations, conflict flows, multidimensional denumerable discrete-time Markov chain

1. Introduction

Conflict traffic flows control at a crossroad is one of the classical problem of queuing theory. The problem has been solved for different classes of algorithms: the class of algorithms with a cyclic fixed rhythm, with renewals (”with a loop”) with dynamic priorities, etc. However, several (two in our case) consecutive crossroads are of great interest, because in a real life situation after a car passes one highway intersection it finds itself at another one. In other words, output flow of the first intersection forms input flow of the second intersection. Hence, the second input flow no longer has simple probabilistic structure known a priori (for example, non-ordinary Poison process) and knowledge about service algorithm should be taken into account to deduce formation properties of the first output flow.

In [1] the problem of a tandem of two crossroads was carried out and rigorous probabilistic model was built. Description of the system is the following. At each of the road intersections, in addition to the high priority flow, there is the traffic flow ”in the perpendicular direction” with lower priority. The service of the flows on the first intersection is supposed to be in the class of cyclic algorithms. In contrast the second intersection has cyclic algorithm with prolongations service type. Over and above, vehicles could not turn from one conflict direction to another. In this paper we continue the study of described problem and deepen in low priority flow of the second intersection investigation.

2. The setting of the problem

This section contains statement of the problem according to [1]. First of all the tandem is represented as queuing system to give generic formulation. In the end of the section system of two consecutive crossroads is given as simple example. Detailed mathematical model is built in next section.

Consider queuing system of the following type (see Fig. 1). There are four input

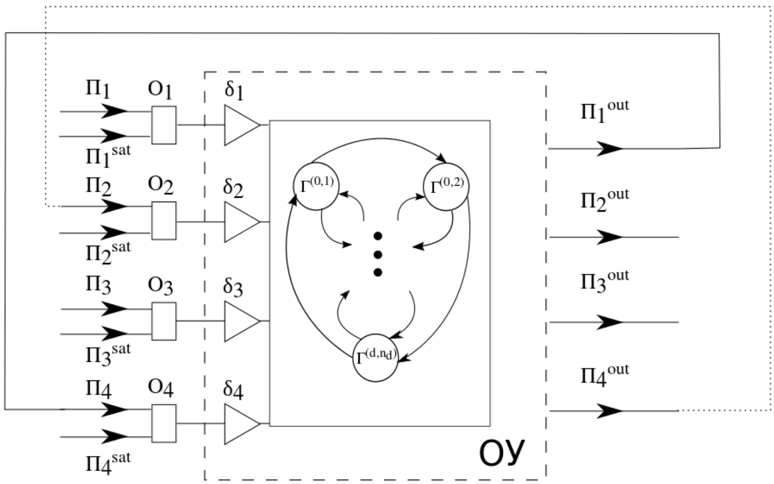

Figure 1: Structure scheme of queuing system

flows of customers coming into the system with one server. Customers from input flow Π_j arrive to corresponding queue with an unbounded capacity, $j \in \{1, 2, 3, 4\}$. For $j \in \{1, 2, 3\}$ discipline of the queue O_j is FIFO (First In First Out). In other words, the earlier customer come to system the earlier it is serviced. Disciplie of the queue O_4 is described below. Input flows Π_1 and Π_3 are generated by external environment, which has only one state. Each of these flows is a nonordinary Poisson process. Denote λ_1 and λ_3 the intensity for the bulk arrival for the flows Π_1 and Π_3 respectively. Generating function of customer number in the bulk of flow Π_j is

$$f_j(z) = \sum_{\nu=1}^{\infty} p_\nu^{(j)} z^\nu, \quad j \in \{1, 3\}, \tag{1}$$

assuming it is finite for any $z \in \mathbb{C}$ such as $|z| < (1+\varepsilon)$, $\varepsilon > 0$. Quantity $p_{\nu}^{(j)}$ is probability of event that customers number in a bulk of flow Π_j is exactly ν. Being serviced Π_1 customers come back to the system as Π_4 customers. Π_4 customers in turn after service come back to the system as Π_2 ones. Flows Π_2 and Π_3 are conflicting in the sense that customers arriving from different sources can't be serviced simultaneously in the same queuing system. This means that the problem can't be reduced to a corresponding problem with fewer input flows by merging the flows together.

Given positive integers d, n_0, n_1, …, n_d, we introduce a finite set $\Gamma = \{\Gamma^{(k,r)}\colon k = 0, 1, \dots, d; r = 1, 2, \dots n_k\}$ of states in which server can be. At each state $\Gamma^{(k,r)}$ sever stays during time $T^{(k,r)}$. Define sets Γ^{I}, Γ^{II}, Γ^{III} and Γ^{IV} as follows. In state $\gamma \in \Gamma^{\mathrm{I}}$ only customers of queues O_1, O_2 and O_4 are serviced. In state $\gamma \in \Gamma^{\mathrm{II}}$ only customers of queues O_2 and O_4 are serviced. In state $\gamma \in \Gamma^{\mathrm{III}}$ only customers of queuesO_1, O_3 and O_4 are serviced. In state $\gamma \in \Gamma^{\mathrm{IV}}$ only customers of queues O_3 and O_4 are serviced. The set Γ is defined as union $\Gamma = \Gamma^{\mathrm{I}} \cup \Gamma^{\mathrm{II}} \cup \Gamma^{\mathrm{III}} \cup \Gamma^{\mathrm{IV}}$ disjoint sets. We will also need following sets in the future ${}^{1}\Gamma = \Gamma^{\mathrm{I}} \cup \Gamma^{\mathrm{III}}$, ${}^{2}\Gamma = \Gamma^{\mathrm{I}} \cup \Gamma^{\mathrm{II}}$, ${}^{3}\Gamma = \Gamma^{\mathrm{III}} \cup \Gamma^{\mathrm{IV}}$.

Server changes its state according to the following rule. We will call the set $C_k = \{\Gamma^{(k,r)}\colon r = 1, 2, \dots n_k\}$ k-th cycle, $k = 1, 2, \dots, d$. When $k = 0$ state of the form $\Gamma^{(0,r)}$ is called prolongation state, $r = 1, 2, \dots, n_0$. Denote $r \oplus_k 1 = r + 1$ for $r < n_k$ and $r \oplus_k 1 = 1$ if $r = n_k$, $k = 0, 1, \dots, d$. In cycle C_k we underline subsets C_k^{O} input, C_k^{I} output and $C_k^{\mathrm{N}} = C_k \setminus (C_k^{\mathrm{O}} \cup C_k^{\mathrm{I}})$ neutral states. Then after server was in state $\Gamma^{(k,r)} \in C_k \setminus C_k^{\mathrm{O}}$ it switches to the state $\Gamma^{(k,r\oplus_k 1)}$ of the same cycle C_k. After state $\Gamma^{(k,r)}$ in C_k^{O} server switches to the state $\Gamma^{(k,r\oplus_k 1)}$, if number of customers in the queue O_3 in switching moment greater than predetermined threshold L. In other case, that is when number of customers in the queue O_3 lesser or equal than L, new state is prolongation one $\Gamma^{(0,r_1)}$, where $r_1 = h_1(\Gamma^{(k,r)})$ and $h_1(\cdot)$ — given mapping of the set $\bigcup\limits_{k=1}^{d} C_k^{\mathrm{O}}$ to $\{1, 2, \dots, n_0\}$. After state $\Gamma^{(0,r)}$ the state of the same type $\Gamma^{(0,r_2)}$ is chosen, if number of customers in queue O_3 is lesser or equal than L, where $r_2 = h_2(r)$ and $h_2(\cdot)$ — given mapping of the set $\{1, 2, \dots, n_0\}$ to itself; in other case state of the form $\Gamma^{(k,r_3)} \in C_k^{\mathrm{I}}$ is on, where $\Gamma^{(k,r_3)} = h_3(r)$ and $h_3(\cdot)$ — given mapping $\{1, 2, \dots, n_0\}$ to the set $\bigcup\limits_{k=1}^{d} C_k^{\mathrm{I}}$. We assume that all prolongation states $\Gamma^{(0,r)}$ belong to the set ${}^{2}\Gamma$, and relations $C_k^{\mathrm{O}} \subset {}^{2}\Gamma$ and $C_k^{\mathrm{I}} \subset {}^{3}\Gamma$ are hold. We also assume that all the cycles have exactly one input and output state. And last assumption is that all the prolongation vertices form one cycle, that is we can put $h_2(r) = r \oplus_0 1$.

More formally server changes its states according to the following rule:

$$h(\Gamma^{(k,r)}, y) = \begin{cases} \Gamma^{(k,r\oplus_k 1)}, & \text{if } (\Gamma^{(k,r)} \in C_k \setminus C_k^{\mathrm{O}}) \text{ or } (\Gamma^{(k,r)} \in C_k^{\mathrm{O}} \ \&\ y > L); \\ \Gamma^{(0,h_1(\Gamma^{(k,r)}))}, & \text{if } \Gamma^{(k,r)} \in C_k^{\mathrm{O}} \text{ and } y \leqslant L; \\ \Gamma^{(0,r\oplus_0 1)}, & \text{if } k = 0 \text{ and } y \leqslant L; \\ h_3(r), & \text{if } k = 0 \text{ and } y > L. \end{cases} \tag{2}$$

In general, service durations of different customers can be dependent and have different distributions. That is why instead of classic approach which specifies distributions of every particular customer, saturation flows will be used. Such a flow Π_j^{sat}, $j \in \{1, 2, 3, 4\}$, is defined as virtual output flow given maximum loading of the server, and for $j \in \{1, 2, 3\}$ given also maximum loading of the corresponding queue. Saturation flow Π_j^{sat}, $j \in \{1, 2, 3\}$, contains non-random number of customers $\ell_{k,r,j}$, serviced during time $T^{(k,r)}$, if server's state is $\Gamma^{(k,r)}$. Let $\mathbb{Z}_+$ be the set of non-negative integer numbers. Then given the fact, that queue O_4 contains $x \in \mathbb{Z}_+$ customers, saturation flow Π_4^{sat} is determined to contain all the x customers. Finally in the state $\Gamma^{(k,r)}$ every customer from queue O_4 with probability $p_{k,r}$ and independently of others ends servicing and come to queue O_2 of the flow Π_2. The customer of queue O_4 stay on up to the next round with probability $1 - p_{k,r}$. On next round the picture is the same.

As we mentioned earlier in the end of the section we introduce real-life example of defined queuing system: tandem of two consecutive crossroads (Fig. 2). Flows of

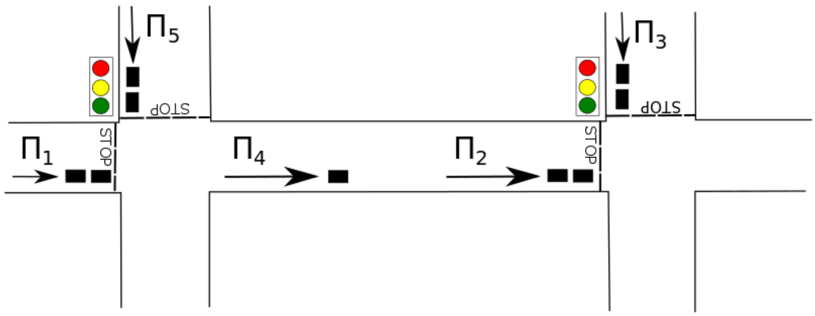

Figure 2: Example: crossroads tandem

arriving cars play role of customer flows formed by external environment: conflicting flows Π_1 and Π_5 on the first crossroad; flow Π_3 on the second. Every car from flow Π_1, after passing first road intersection, comes to the queue O_4 of the flow Π_4. Whereupon the car arrives to the next road intersection with some probability ($p_{k,r}$ for the state $\Gamma^{(k,r)}$) or does not have time to do so and stays for another round. In case it has time it comes in queue O_2 and waits it's turn to pass it. Such system of two crossroads is an instance of more general queuing model described above.

3. Mathematical model

To build rigorous and complete probabilistic model of queuing system described in previous section cybernetic approach was used (see [2]). Scheme of the controlling system was introduced on Fig. 1. There are following blocks presented on figure: 1) external environment with one state; 2) input poles of the first type — input flows Π_1, Π_2, Π_3 and Π_4; 3) input poles of the second type — saturation flows Π_1^{sat}, Π_2^{sat}, Π_3^{sat} and Π_4^{sat}; 4) external memory — queues O_1, O_2, O_3 and O_4; 5) device for external memory processing — queue disciplines δ_1, δ_2, δ_3 and δ_4; 6) internal memory — server; 7) device for internal memory processing — state changing graph; 8) output poles Π_1^{out}, Π_2^{out}, Π_3^{out} and Π_4^{out}. The coordinate of the block is its number on scheme.

To specify blocks information we introduce following variables and elements, along with their value ranges. To fix a discrete time scale consider the moments $\tau_0 = 0$, $\tau_1, \tau_2, \ldots$ when server changes its state. Denote Γ_i from the set Γ server's state during $(\tau_{i-1}; \tau_i]$, $\varkappa_{j,i} \in \mathbb{Z}_+$ number of customers in queue O_j at the time τ_i, $\eta_{j,i} \in \mathbb{Z}_+$ number of customers, came to the queue O_j from the flow Π_j during $(\tau_i; \tau_{i+1}]$, $\xi_{j,i} \in \mathbb{Z}_+$ number of customers in the saturation flow Π_j^{sat} during $(\tau_i; \tau_{i+1}]$, $\overline{\xi}_{j,i} \in \mathbb{Z}_+$ number of actually serviced customers in the flow Π_j during $(\tau_i; \tau_{i+1}]$, $j \in \{1, 2, 3, 4\}$.

Server changes it's state according to the following rule:

$$\Gamma_{i+1} = h(\Gamma_i, \varkappa_{3,i}), \tag{3}$$

where mapping $h(\cdot, \cdot)$ is determined by (2). To define a duration T_{i+1} of the server state for time interval $(\tau_i; \tau_{i+1}]$ it's handy to use mapping $h_T(\cdot, \cdot)$:

$$T_{i+1} = h_T(\Gamma_i, \varkappa_{3,i}) = T^{(k,r)}, \quad \text{where } \Gamma^{(k,r)} = \Gamma_{i+1} = h(\Gamma_i, \varkappa_{3,i}).$$

Functional relation

$$\overline{\xi}_{j,i} = \min\{\varkappa_{j,i} + \eta_{j,i}, \xi_{j,i}\}, \quad j \in \{1, 2, 3\}, \tag{4}$$

between $\overline{\xi}_{j,i}$ and $\varkappa_{j,i}$, $\eta_{j,i}$, $\xi_{j,i}$ implements service strategy. Further since

$$\varkappa_{j,i+1} = \varkappa_{j,i} + \eta_{j,i} - \overline{\xi}_{j,i}, \quad j \in \{1, 2, 3\},$$

and due to (4) it follows that

$$\varkappa_{j,i+1} = \max\{0, \varkappa_{j,i} + \eta_{j,i} - \xi_{j,i}\}, \quad j \in \{1, 2, 3\}. \tag{5}$$

The setting of the problem (see also structure scheme on Fig. 1) implies these relations for flow Π_4:

$$\eta_{4,i} = \min\{\xi_{1,i}, \varkappa_{1,i} + \eta_{1,i}\}, \quad \varkappa_{4,i+1} = \varkappa_{4,i} + \eta_{4,i} - \eta_{2,i}, \quad \xi_{4,i} = \varkappa_{4,i}. \tag{6}$$

Non-local description of the input and saturation flows involves specification of some particular features of conditional distributions of highlighted discrete components $\eta_i = (\eta_{1,i}, \eta_{2,i}, \eta_{3,i}, \eta_{4,i})$ and $\xi_i = (\xi_{1,i}, \xi_{2,i}, \xi_{3,i}, \xi_{4,i})$ of marked point processes $\{(\tau_i, \nu_i, \eta_i); i \geqslant 0\}$ and $\{(\tau_i, \nu_i, \xi_i); i \geqslant 0\}$ given fixed label value $\nu_i = (\Gamma_i; \varkappa_i)$, where $\varkappa_i = (\varkappa_{1,i}, \varkappa_{2,i}, \varkappa_{3,i}, \varkappa_{4,i})$. Let $\varphi_1(\cdot, \cdot)$ and $\varphi_3(\cdot, \cdot)$ be defined from decompositions

$$\sum_{\nu=0}^{\infty} z^{\nu} \varphi_j(\nu, t) = \exp\{\lambda_j t (f_j(z) - 1)\},$$

where $f_j(z)$ are defined in (1), $j \in \{1, 3\}$. Function $\varphi_j(\nu, t)$ defines probability of coming $\nu = 0, 1, \ldots$ customers of the flow Π_j during time $t \geqslant 0$. If $\nu < 0$ value of $\varphi_j(\nu, t)$ is set to zero. Function $\psi(\cdot, \cdot, \cdot)$ is defined by formula

$$\psi(k; y, u) = C_y^k u^k (1-u)^{y-k}.$$

Semantically $\psi(k; y, u)$ is a probability of coming k customers from flow Π_2 given y customers in queue O_4 and $\Gamma^{(k,r)}$ as a state of the server, such that $u = p_{k,r}$. If condition $0 \leqslant k \leqslant y$ does not hold $\psi(k; y, u)$ is set to zero.

Let $a = (a_1, a_2, a_3, a_4) \in \mathbb{Z}_+^4$ and $x = (x_1, x_2, x_3, x_4) \in \mathbb{Z}_+^4$. Then the setting of the problem implies that given fixed label value $\nu_i = (\Gamma^{(k,r)}; x)$ probability $\varphi(a, k, r, x)$ of following equations simultaneous holding $\eta_{1,i} = a_1$, $\eta_{2,i} = a_2$, $\eta_{3,i} = a_3$, $\eta_{4,i} = a_4$ is

$$\varphi_1(a_1, h_T(\Gamma^{(k,r)}, x_3)) \times \psi(a_2, x_4, p_{\tilde{k},\tilde{r}}) \times \varphi_3(a_3, h_T(\Gamma^{(k,r)}, x_3)) \times \delta_{a_4, \min\{\ell(\tilde{k},\tilde{r},1), x_1+a_1\}}, \quad (7)$$

where $\Gamma^{(\tilde{k},\tilde{r})} = h(\Gamma^{(k,r)}, x_3)$ and $\delta_{i,j}$ is Kroneker's delta

$$\delta_{i,j} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j, \end{cases} \quad .$$

Let $b = (b_1, b_2, b_3, b_4) \in \mathbb{Z}_+^4$. The setting of the problem also implies that probability $\zeta(b, k, r, x)$ of following equations simultaneous holding $\xi_{1,i} = b_1$, $\xi_{2,i} = b_2$, $\xi_{3,i} = b_3$, $\xi_{4,i} = b_4$ given fixed label value $\nu_i = (\Gamma^{(k,r)}; x)$ is

$$\delta_{b_1,\ell(\tilde{k},\tilde{r},1)} \times \delta_{b_2,\ell(\tilde{k},\tilde{r},2)} \times \delta_{b_3,\ell(\tilde{k},\tilde{r},3)} \times \delta_{b_4,x_4}. \quad (8)$$

To gather all the assumptions made so far, we introduce below a theorem, which is described in more details in [1]. Specifically, theorem constructs probabilistic space on which all the mentioned random variables/elements and properties are consistent and therefore make mathematical sense.

Theorem 1. *Let $\gamma_0 = \Gamma^{(k_0,r_0)} \in \Gamma$ and $x^0 = (x_{1,0}, x_{2,0}, x_{3,0}, x_{4,0}) \in \mathbb{Z}_+^4$ be fixed. Then there exists probabilistic space $(\Omega, \mathcal{F}, \mathbf{P}(\cdot))$ and random variables $\eta_{j,i} = \eta_{j,i}(\omega)$, $\xi_{j,i} = \xi_{j,i}(\omega)$, $\varkappa_{j,i} = \varkappa_{j,i}(\omega)$ and random elements $\Gamma_i = \Gamma_i(\omega)$, $i \geqslant 0$, $j \in \{1, 2, 3, 4\}$, which are defined on this space, such that 1) equations $\Gamma_0(\omega) = \gamma_0$ and $\varkappa_0(\omega) = x^0$ hold; 2) propositions* (3), (5), (6) *hold; 3) for any a, b, $x^t = (x_{1,t}, x_{2,t}, x_{3,t}, x_{4,t}) \in \mathbb{Z}_+^4$, $\Gamma^{(k_t,r_t)} \in \Gamma$, $t = 1, 2, \ldots$, conditional distribution of vectors η_i, and ξ_i has the form*

$$\mathbf{P}\left(\{\omega\colon \eta_i = a, \xi_i = b\} \,\middle|\, \bigcap_{t=0}^{i} \{\omega\colon \Gamma_t = \Gamma^{(k_t,r_t)}, \varkappa_t = x^t\}\right) = \varphi(a, k_i, r_i, x^i) \times \zeta(b, k_i, r_i, x^i), \quad (9)$$

where mappings $\varphi(\cdot,\cdot,\cdot,\cdot)$ and $\zeta(\cdot,\cdot,\cdot,\cdot)$ are defined by (7) *and* (8) *respectively, $i \geqslant 0$.*

4. Low-priority queue analysis

The ultimate goal of the study is to analyse random sequence

$$\{(\Gamma_i, \varkappa_{1,i}, \varkappa_{2,i}, \varkappa_{3,i}, \varkappa_{4,i}); i \geqslant 0\} \quad (10)$$

That is researcher is interested about it's Markove'ness, making classification of it's states, finding necessary and sufficient conditions on stationary distribution existence and finding important characteristics of such distribution. To come up with the solution of such a big problem it is easier to reduce problem into small ones. After the solutions for these issues is found researcher builds from them the solution for the main problem.

One of such small problems is analyses of sequence

$$\{(\Gamma_i, \varkappa_{3,i}); i \geqslant 0\} \tag{11}$$

which includes low priority queue size $\varkappa_{3,i}$ (that is, size of queue O_3). This is a subsequence of $\{(\Gamma_i, \varkappa_i); i \geqslant 0\}$ that is why some of it's properties will be likely generalized on the entire sequence. Below we introduce several results which have been already obtained on the way of solving this sub-problem.

First result concerns Markov'ness of the subsequence.

Theorem 2. *Let* $\Gamma_0 = \Gamma^{(k,r)} \in \Gamma$ *and* $\varkappa_{3,0} = x_{3,0} \in \mathbb{Z}_+$ *be fixed. Then sequence* $\{(\Gamma_i, \varkappa_{3,i}); i \geqslant 0\}$ *is homogeneous denumerable Markov chain.*

To continue we need following function. For any $y_0, y, \tilde{y} \in \mathbb{Z}_+$ and $t \in \mathbb{R}$, $t \geqslant 0$

$$\widetilde{\varphi}_3(k, r, t, y, \tilde{y}) = (1 - \delta_{\tilde{y},0})\varphi_3(\tilde{y} + \ell(k, r, 3) - y, t) + \delta_{\tilde{y},0} \sum_{a=0}^{\ell(k,r,3)-y} \varphi_3(a, t), \tag{12}$$

where k and r such that $\Gamma^{(k,r)} \in \Gamma$.

In these notations following proposition is true.

Theorem 3. *Let* x_3, $\tilde{x}_3 \in \mathbb{Z}_+$ *and* $\Gamma^{(k,r)}$, $\Gamma^{(\tilde{k},\tilde{r})} = h(\Gamma^{(k,r)}, x_3) \in \Gamma$. *Then transition probabilities of homogeneous denumerable Markov chain* $\{(\Gamma_i, \varkappa_{3,i}); i \geqslant 0\}$ *can be found from expressions*

$$\mathbf{P}(\Gamma_{i+1} = \Gamma^{(\tilde{k},\tilde{r})}, \varkappa_{3,i+1} = \tilde{x}|\Gamma_i = \Gamma^{(k,r)}, \varkappa_{3,i} = x) = \widetilde{\varphi}_3(\tilde{k}, \tilde{r}, h_T(\Gamma^{(k,r)}, x_3), x_3, \tilde{x}_3), \tag{13}$$

Last theorem gives insight on what states of the chain $\{(\Gamma_i, \varkappa_{3,i}); i \geqslant 0\}$ are important. To make complete classification, we introduce sets

$$S^3_{0,r} = \{(\Gamma^{(0,r)}, x_3)\colon x_3 \in Z_+, L \geqslant x_3 > L - \max_{k=1,2,\dots,d}\{\sum_{t=0}^{n_k} \ell_{k,t,3}\}\}, \quad 1 \leqslant r \leqslant n_0,$$

$$S^3_{k,r} = \{(\Gamma^{(k,r)}, x_3)\colon x_3 \in Z_+, x_3 > L - \sum_{t=0}^{r-1} \ell_{k,t,3}\}\}, \quad 1 \leqslant k \leqslant d, \quad 1 \leqslant r \leqslant n_k.$$

Theorem 4. *The set of important states of the Markov chain* $\{(\Gamma_i, \varkappa_{3,i}); i \geqslant 0\}$ *consists of sets* $\bigcup\limits_{1 \leqslant r \leqslant n_0} S^3_{0,r}$ *and* $\bigcup\limits_{\substack{1 \leqslant k \leqslant d \\ 1 \leqslant r \leqslant n_k}} S^3_{k,r}$.

At the end of this article we give result which is principle in finding necessary and sufficient conditions on stationary distribution existence of chain $\{(\Gamma_i, \varkappa_{3,i}); i \geqslant 0\}$.

Denote for $\gamma \in \Gamma$ and $x_3 \in Z_+$

$$Q_{3,i}(\gamma, x) = \mathbf{P}(\Gamma_i = \gamma, \varkappa_{3,i} = x_3). \tag{14}$$

Suppose k and r such that $\Gamma^{(k,r)} \in \Gamma$. Partial generation functions are defined as follows:

$$\mathfrak{M}^{(i)}(k,r,v) = \sum_{w=0}^{\infty} Q_{3,i}(\Gamma^{(k,r)}, w)v^w,$$

$$\Phi^{(i)}(k,r,v) = q_{k,r}(v) \sum_{x_3=0}^{\infty} \sum_{\gamma\in H_{-1}(\Gamma^{(k,r)},x_3)} Q_{3,i}(\gamma, x_3)v^{x_3},$$

where

$$q_{k,r}(v) = v^{-\ell(k,r,3)} \sum_{w=0}^{\infty} \varphi_3(w, T^{(k,r)})v^w.$$

Theorem 5. *Let $\tilde{\gamma} = \Gamma^{(\tilde{k},\tilde{r})} \in \Gamma$. The following recurrent by $i \geqslant 0$ relations take place for partial generation functions of Markov chain $\{(\Gamma_i, \varkappa_{3,i}); i \geqslant 0\}$:*

$$\mathfrak{M}^{(i+1)}(\tilde{k},\tilde{r},v) = \Phi^{(i)}(\tilde{k},\tilde{r},v) + \sum_{x_3=0}^{\ell(\tilde{k},\tilde{r},3)} \sum_{\gamma\in H_{-1}(\tilde{\gamma},x_3)} Q_{3,i}(\gamma, x_3) \sum_{a=0}^{\ell(\tilde{k},\tilde{r},3)-x_3} \varphi_3(a, T^{(\tilde{k},\tilde{r})}) -$$
$$- \sum_{x_3=0}^{\ell(\tilde{k},\tilde{r},3)} \sum_{\gamma\in H_{-1}(\tilde{\gamma},x_3)} Q_{3,i}(\gamma, x_3)v^{x_3-\ell(\tilde{k},\tilde{r},3)} \sum_{w=0}^{\ell(\tilde{k},\tilde{r},3)+1-x_3} \varphi_3(w, T^{(\tilde{k},\tilde{r})})v^w. \quad (15)$$

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