Analysis of White Dwarfs and Neutron Stars

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Overview

1. White Dwarfs
2. Pure Neutron Stars
3. Neutron Stars with protons and electrons
4. Neutron Stars with compOSE

Introduction

- Compact stars, i.e. white dwarfs and neutron stars, are the final stages in the evolution of ordinary stars.
- White Dwarfs: stabilized by **electrons**' degeneracy pressure
- Neutron Stars: stabilized by **neutrons**' degeneracy pressure

White Dwarfs

Structure Equations

- Two forces acting on the star: gravitational one and the one related to the degeneracy pressure. The gravitational force is given by:

$$dF = -\frac{Gdm \cdot m(r)}{r^2} \tag{1}$$

with

$$\frac{dm}{dr} = \rho(r) 4\pi r^2 \tag{2}$$

Therefore :

$$\frac{dp}{dr} = -\frac{G\rho(r) \cdot m(r)}{r^2} \tag{3}$$

- Boundary conditions: $m(r = 0) = 0$ and $p(r = 0) = p_c$ with p_c central pressure

White Dwarfs

Equations of State

- One needs to fix the relation between the pressure p and the energy density ϵ.
- Matter in white dwarfs can be treated as an ideal Fermi gas of degenerate electrons: their distributions is described by the Fermi-Dirac statistics

$$f(E) = \frac{1}{exp[(E-\mu)/k_B T] + 1} \tag{4}$$

- number density of the degenerate electron gas:

$$n_e = \int_0^{k_F} \frac{2}{(2\pi\hbar)^3} d^3k = \frac{8\pi}{(2\pi\hbar)^3} \int_0^{k_F} k^2 dk = \frac{k_F^3}{3\pi^2\hbar^3} \tag{5}$$

White Dwarfs

Equations of State

- The star should be electrically neutral: each electron must be neutralised by a proton. To compute the mass density in the white dwarf one can concentrate just on the nucleon mass m_n:

$$\rho = n_e \cdot m_N \cdot \frac{A}{Z} \tag{6}$$

- The energy density can be divided into two components, one term coming from nucleons and the other from electrons:

$$\epsilon \quad = \quad n_e m_N \frac{A}{Z} c^2 + \epsilon_{\rm elec}(k_F) \quad . \tag{7}$$

where $\epsilon_{\rm elec}(k_F)$ is the energy density of electrons.

White Dwarfs

Equations of State

- With the electron energy

$$E(k) = \sqrt{k^2c^2 + m_e^2c^4} \tag{8}$$

one can write

$$\begin{aligned} \epsilon_{\rm elec}(k_F) &= \frac{8\pi}{(2\pi\hbar)^3}\int_0^{k_F} E(k)k^2dk \\ &= \frac{\epsilon_0}{8}\left[(2x^3+x)(1+x^2)^{1/2} - \sinh^{-1}(x)\right] \end{aligned} \tag{9}$$

with

$$\epsilon_0 = \frac{m_e^4c^5}{\pi^2\hbar^3} \tag{10}$$

and

$$x = k_F/m_ec. \tag{11}$$

White Dwarfs

Equations of State

- One can also obtain the electrons' pressure, given by:

$$\begin{aligned} p(k_F) &= \frac{1}{3}\frac{8\pi}{(2\pi\hbar)^3}\int_0^{k_F}\frac{k^2c^2}{E(k)}k^2dk \\ &= \frac{\epsilon_0}{24}\left[(2x^3-3x)(1+x^2)^{1/2}+3\sinh^{-1}(x)\right]. \end{aligned} \quad (12)$$

- The energy density is dominated by the mass density of nucleons while the electrons contribute to most of the pressure.
- To get an equation of the form $p = p(\epsilon)$ one can consider two estreme cases: $x << 1$ (non-relativistic one) and $x >> 1$ (relativistic one).

White Dwarfs

Polytropic Equations of State

- Non relativistic case:

$$p \approx K_{\mathrm{non-rel}} \epsilon^{5/3} \tag{13}$$

with

$$K_{\mathrm{non-rel}} = \frac{\hbar^2}{15\pi^2 m_e} \left(\frac{3\pi^2 Z}{m_N c^2 A} \right)^{5/3} . \tag{14}$$

- Relativistic case:

$$p(\epsilon) \approx K_{\mathrm{rel}}\, \epsilon^{4/3} \tag{15}$$

where

$$K_{\mathrm{rel}} = \frac{\hbar c}{12\pi^2} \left(\frac{3\pi^2 Z}{m_N c^2 A} \right)^{4/3} . \tag{16}$$

White Dwarfs

General Relativity Corrections

- If the stars are very compact one has to take into account effects from General Relativity.
- Using Einstein's equations:

$$G_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu} \quad , \tag{17}$$

one arrives at the Tolman–Oppenheimer–Volkoff (TOV) equation:

$$\frac{dp}{dr} = -\frac{G\epsilon(r)m(r)}{c^2 r^2}\left[1+\frac{p(r)}{\epsilon(r)}\right]\left[1+\frac{4\pi r^3 p(r)}{m(r)c^2}\right]\left[1-\frac{2Gm(r)}{c^2 r}\right]^{-1} \quad . \tag{18}$$

Computational Analysis for White Dwarfs

Polytropic EoS

- Case of a polytropic EoS (for both non-relativistic and relativistic case) in a Newtonian regime:

$$p = K\epsilon^{\gamma} \quad , \tag{19}$$

$$\frac{dm}{dr} = \rho(r)4\pi r^2, \tag{20}$$

$$\frac{dp}{dr} = -\frac{G\rho(r)\cdot m(r)}{r^2}. \tag{21}$$

- One can rewrite these equations in terms of dimensionless quantities.

Computational Analysis for White Dwarfs

Polytropic EoS

- Defining ρ_c as the central matter density,

$$R_0 = \frac{c}{(4\pi G \rho_c)^{1/2}} \tag{22}$$

and

$$M_0 = \frac{4\pi c^3 \rho_c}{(4\pi G \rho_c)^{3/2}} \tag{23}$$

One can introduce these dimensionless quantities:

$$\hat{r} = r/R_0 \tag{24}$$

$$\hat{\rho} = \rho/\rho_c \tag{25}$$

$$\hat{m} = m/M_0 \tag{26}$$

$$\hat{p} = p/(\rho_c c^2) \tag{27}$$

Computational Analysis for White Dwarfs

Polytropic EoS

- From these definitions, one can obtain the differential equations for the dimensionless quantities:

$$\hat{p} = w\hat{\rho}^{\gamma} \tag{28}$$

with

$$w = Kc^{2(\gamma-1)}{\rho_c}^{\gamma-1} \tag{29}$$

and

$$\frac{d\hat{m}}{d\hat{r}} = \hat{r}^2\hat{\rho} \tag{30}$$

$$\frac{d\hat{p}}{d\hat{r}} = -\frac{\hat{m}\hat{\rho}}{\hat{r}^2} \tag{31}$$

Computational Analysis for White Dwarfs

NON RELATIVISTIC Polytropic EoS in Newtonian regime

- Define the system of differential equations:

```
#Define the system of dimensionless differential eqns
def white_dwarf(r, y, w ,z):
    p ,m  = y
    if p <= 0:
        return np.array([0, 0])
    dpdr = -m/(r**2)*(p/w)**z
    dmdr = (r**2)*(p/w)**z
    return np.array([dpdr, dmdr])
```

- Define the central pressure and all the needed parameters for the system of dimensionless differential equations:

```
#Parameters for EoS

#pressure in the centre of the white dwarf
p_c=2.33002e21

#non relativistic constant in the eos p=Knr*e**(5/3)
Knr = h_t**2/(15*np.pi**2*m_e)*(3*np.pi**2*((2*m_n*c**2)**(-1)))**(5/3)

#matter density in the centre of the white dwarf
ro_c=(p_c/Knr)**(3/5)*c**(-2)

#dimensionless param for the dimensionless EoS
w = Knr*c**(4/3)*ro_c**(2/3)

#inverse of gamma
z= 3/5

#initial values for the dimensionless system of differential eqns
p0=p_c/ro_c*c**(-2)
m0 = 0.0
y0 = np.array([p0, m0])
r0, rf, dr = 1e-6, 0.04, 0.001
```

Computational Analysis for White Dwarfs

RK4

- To solve the system of differential equations one can use the RK4 method, which is implemented by this code:

```
#Define RK4 method
def rk4(f, y0, r0, rf, dr, w, z):
    r_values = np.arange(r0, rf, dr)
    y_values = np.zeros((len(r_values), len(y0)))
    y_values[0] = y0

    for i in range(1, len(r_values)):
        r = r_values[i - 1]
        y = y_values[i - 1]
        k1 = dr * f(r, y, w, z)
        k2 = dr * f(r + dr/2, y + k1/2, w, z)
        k3 = dr * f(r + dr/2, y + k2/2, w, z)
        k4 = dr * f(r + dr, y + k3, w, z)

        y_values[i] = y + (k1 + 2*k2 + 2*k3 + k4) / 6

    return r_values, y_values
```

Computational Analysis for White Dwarfs

NON RELATIVISTIC Polytropic EoS in Newtonian regime

- Solve the system of differential equations:

```
# Solve using RK4
r_values, y_values = rk4(white_dwarf, y0, r0, rf, dr, w, z)
```

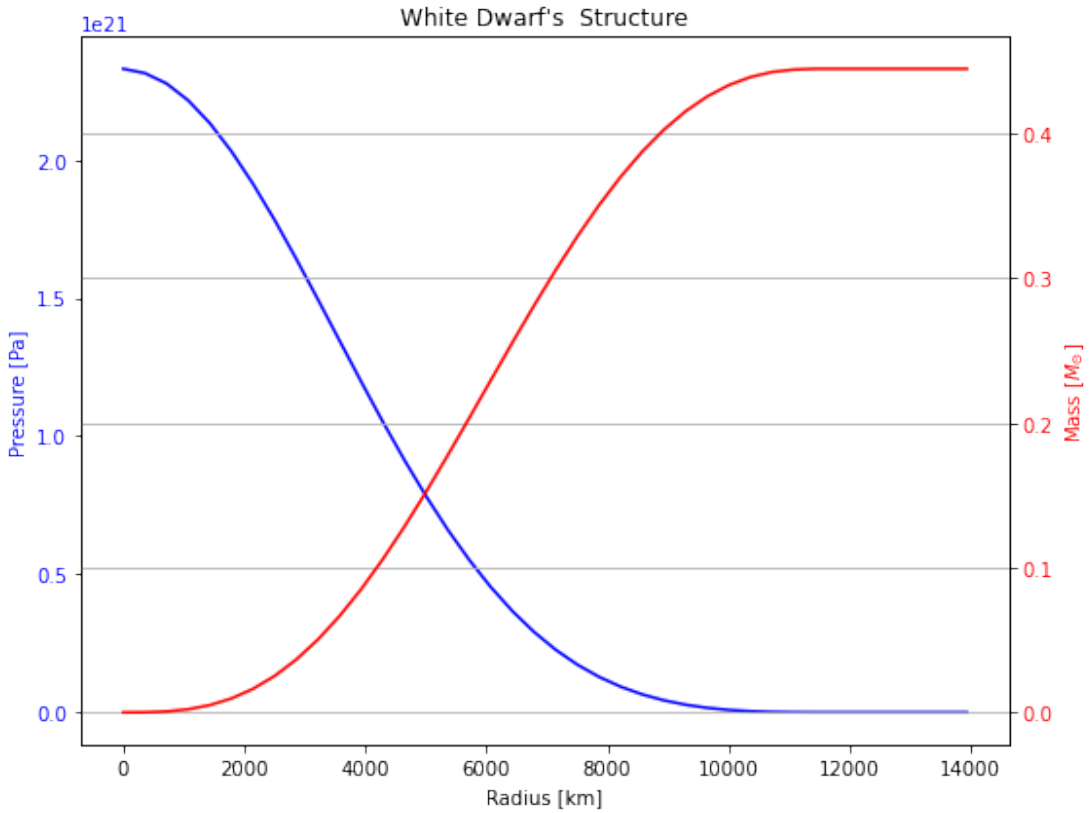
- Return to dimensionful quantities:

```
#Back to dimensionful quantities

m_values= y_values[:, 1]*4*np.pi*ro_c/(ro_c*4*np.pi*G)**(3/2)*c**3/m_sun
p_values= y_values[:, 0]*ro_c*c**2
r_values1=r_values/(ro_c*4*np.pi*G)**(1/2)*c*10**(-3)
```

Computational Analysis for White Dwarfs

NON RELATIVISTIC Polytropic EoS in Newtonian regime

- Plot the behaviour of the pressure and the mass inside the white dwarf:



Computational Analysis for White Dwarfs

NON RELATIVISTIC Polytropic EoS in Newtonian regime

- Find the final radius and mass of the white dwarf:

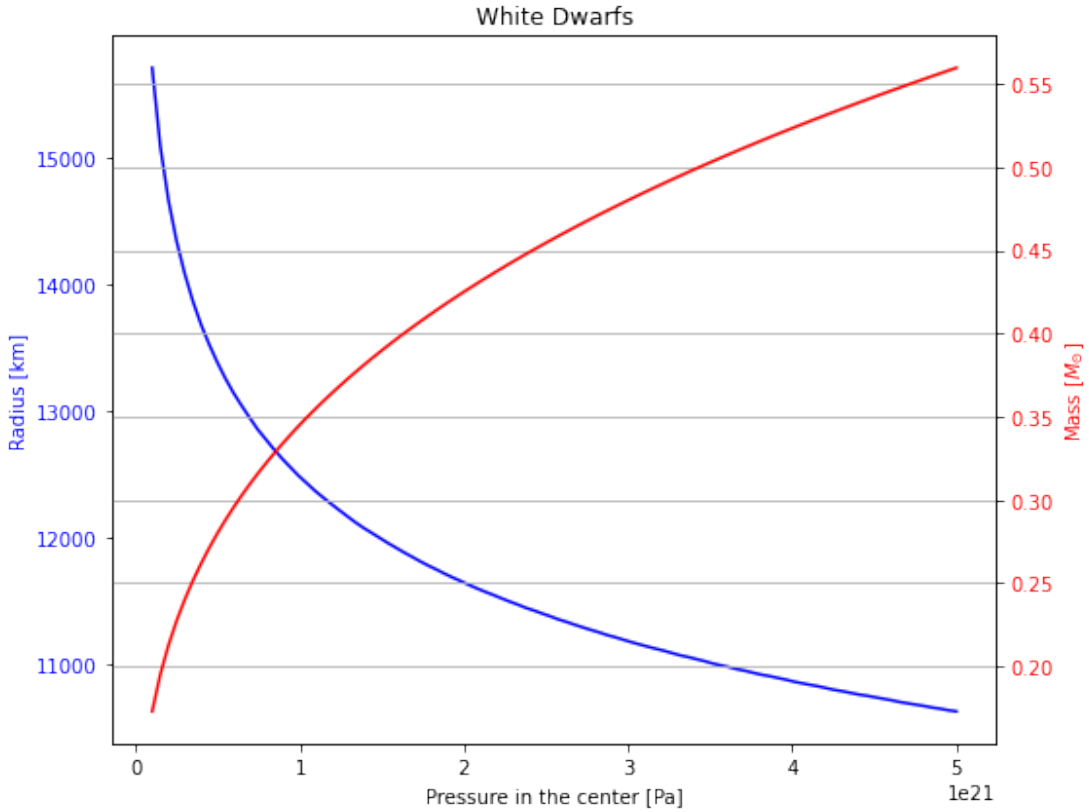
```
#Find mass and radius of the white dwarf(Km and M_sun)
for i in range(len(p_values)):
    if p_values[i] <= 0:
        r_stop = r_values1[i]  # White dwarf's radius
        M_star = m_values[i]  # White dwarf's mass
        break

print(r_stop)
print(M_star)
```

```
11437.953522005835
0.4448101643696684
```

Computational Analysis for White Dwarfs

NON RELATIVISTIC Polytropic EoS in Newtonian regime

- Plot the radii and masses of white dwarfs for various central pressures:



Computational Analysis for White Dwarfs

RELATIVISTIC Polytropic EoS in Newtonian regime

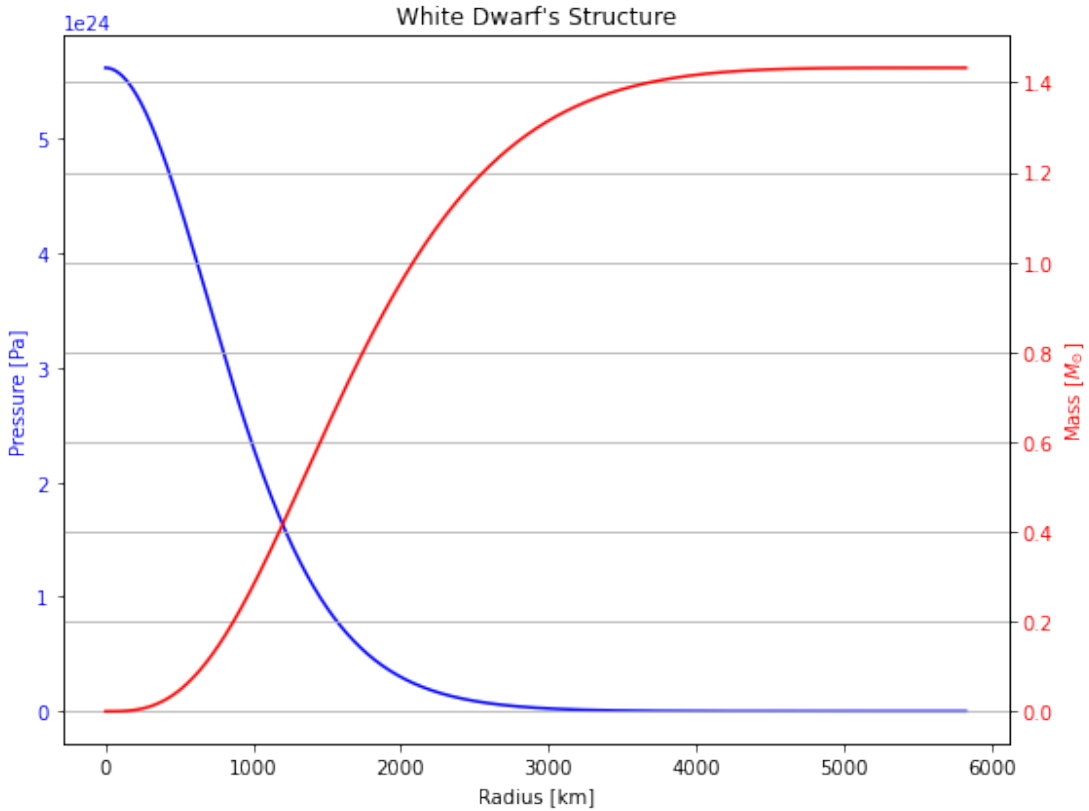
- Same system of differential equations
- Define the central pressure and all the needed parameters for the system of dimensionless differential equations:

```
#Parameters for EoS

#relativistic constant in the eos p=Kr*e**(4/3)
Kr = h_t*c/(12*np.pi**2)*(3*np.pi**2*((2*m_n*c**2)**(-1)))**(4/3)

#pressure in the centre of the white dwarf
p_c=5.62e24

#matter density in the centre of the white dwarf
ro_c=(p_c/Kr)**(3/4)*c**(-2)

#dimensionless param for dimensionless EoS
w = Kr*c**(2/3)*ro_c**(1/3)

#inverse of gamma
z= 3/4

#initial values for the dimensionless system of differential eqns
p0=p_c/ro_c*c**(-2)
m0 = 0.0
y0 = np.array([p0, m0])
r0, rf, dr = 1e-6, 0.25, 0.0001
```

Computational Analysis for White Dwarfs

RELATIVISTIC Polytropic EoS in Newtonian regime

- Solve the system of differential equations
- Return to dimensionful quantities
- Plot the behaviour of pressure and mass inside the white dwarf:



Computational Analysis for White Dwarfs

RELATIVISTIC Polytropic EoS in Newtonian regime

- Find the final radius and mass of the white dwarf:

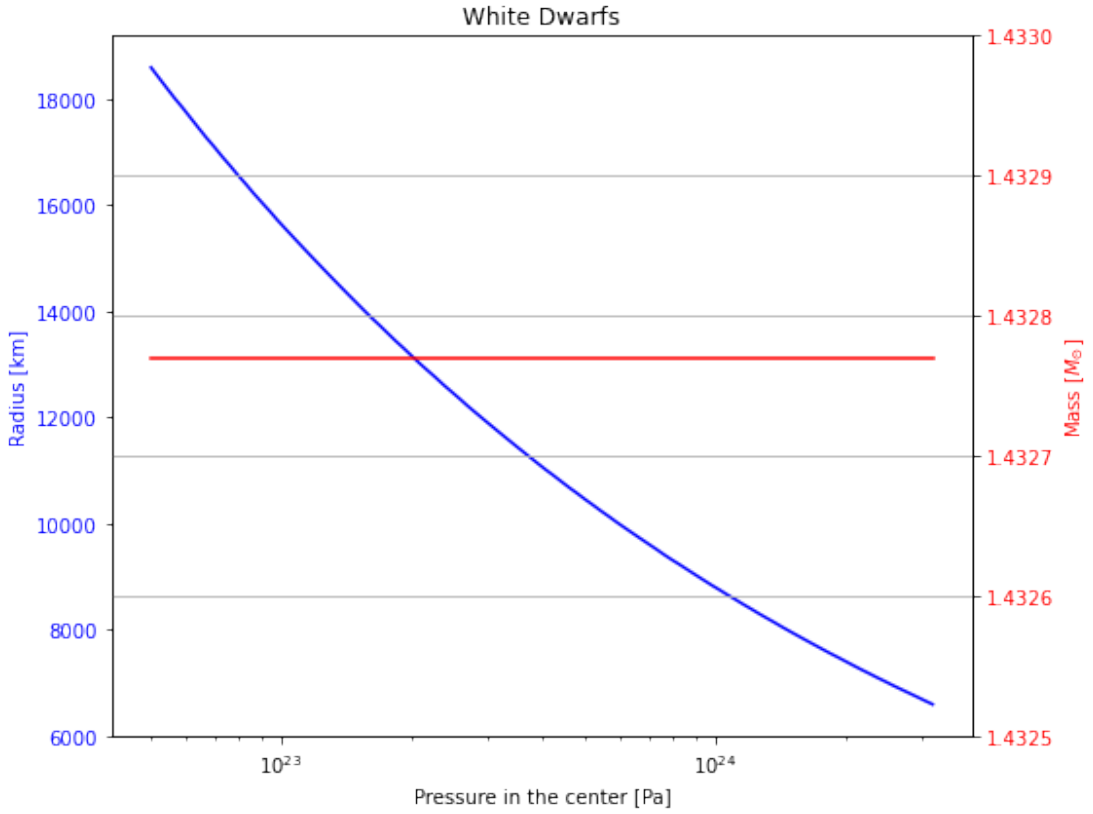
```
#Find mass and radius of the White Dwarf (Km and M_sun)
for i in range(len(p_values)):
    if p_values[i] <= 0:
        r_stop = r_values1[i]  # White Dwarf's radius
        M_star = m_values[i]  # White Dwarf's mass
        break

print(r_stop)
print(M_star)
```

```
5711.1504325725255
1.4327703586476153
```

Computational Analysis for White Dwarfs

RELATIVISTIC Polytropic EoS in Newtonian regime

- Plot the radii and masses of white dwarfs for various central pressures:



Computational Analysis for White Dwarfs

NON RELATIVISTIC Polytropic EoS in GR regime

- Define the system of differential equations and its solver by using the function *solveivp* from *scipy.integrate*

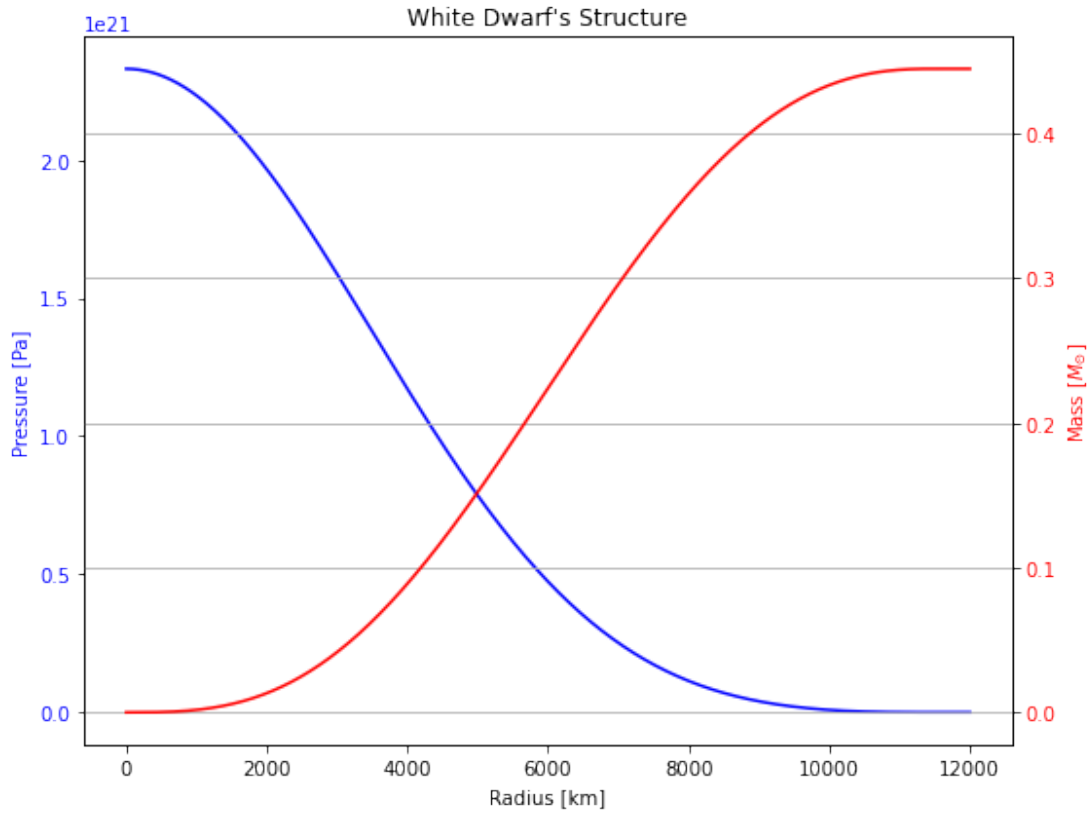
```
def tov_equations(r, y):
    p, m = y
    if p <= 0:
        return np.array([0, 0])
    e = (p / w)**z
    dmdr = 4 * np.pi * r**2 * e / c**2
    dpdr = -G * e * m / (c**2 * r**2) * (1 + p / e) * (1 + 4 * np.pi * r**3 * p / (m * c**2)) / (1 - 2 * G * m / (c**2 * r))
    return [dpdr, dmdr]


def solve_tov(p_c):
    y0 = [p_c, 1e-6]
    r_span = (1e-6, 1.2e7)

    sol = solve_ivp(tov_equations, r_span, y0, method='RK45', max_step=1e3)
    return sol
```

- Define central pressure p_c (same value as the one considered in the NR newtonian regime to compare the two cases)

Computational Analysis for White Dwarfs

NON RELATIVISTIC Polytropic EoS in GR regime

- Plot the behaviour of the pressure and the mass inside the white dwarf:



Computational Analysis for White Dwarfs

NON RELATIVISTIC Polytropic EoS in GR regime

- Find the final radius and mass of the white dwarf:

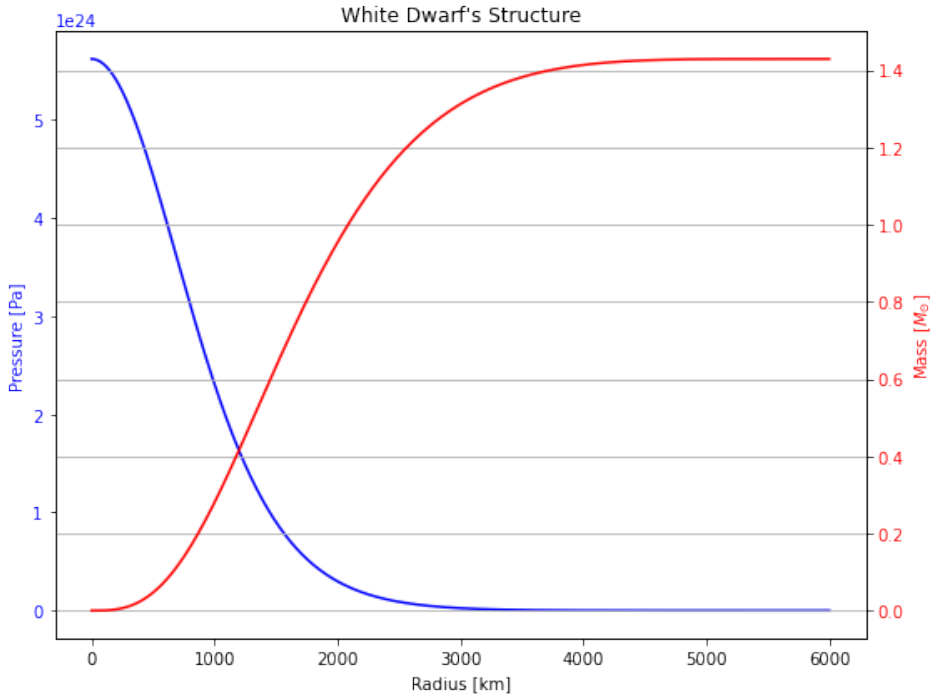
```
#Find radius and mass of the white dwarf (km and M_sun)
for i in range(len(p_values)):
    if p_values[i] <= 0:
        r_stop = r_values1[i]  # White dwarf's radius
        M_star = m_values[i]  # White dwarf's mass
        break

print(r_stop)
print(M_star)
```

```
11462.904188806258
0.4448416920588513
```

Computational Analysis for White Dwarfs

RELATIVISTIC Polytropic EoS in GR regime

- Same system of differential equations and same solving method
- Define central pressure p_c (same value as the one considered in the R newtonian regime to compare the two cases)
- Plot the behaviour of pressure and mass inside the white dwarf:



Computational Analysis for White Dwarfs

RELATIVISTIC Polytropic EoS in GR regime

- Find the final radius and mass of the white dwarf:

```
#Find radius and mass of the white dwarf (km and M_sun)
for i in range(len(p_values)):
    if p_values[i] <= 0:
        r_stop = r_values1[i]  # White dwarf's radius
        M_star = m_values[i]  # White dwarf's mass
        break

print(r_stop)
print(M_star)
```

```
5711.145622209247
1.4292272583081747
```

Computational Analysis for White Dwarfs

Looking for zero-points

- It is necessary to have a relation between the energy density and the pressure to calculate the mass and the radius of the star
- From equations 7-12 one can see that both the pressure and the energy density are functions of the Fermi momentum
- One can use a root-finding routine to find for a given pressure the corresponding Fermi momentum and use it in the energy density's expression.
- In this way one can obtain the energy density ϵ for a given pressure p.
- To get the Fermi-momentum one can apply a root-finding method to the equation

$$p(k_F) - p = 0 \tag{32}$$

where p is given and $p(k_F)$ taken from 12.

Computational Analysis for White Dwarfs

Looking for zero-points: code

- Define the explicit expressions for the pressure and the energy density
- Define a function which, via a root-finding method (*brentq* in this case), gives the energy density associated to a given pressure

```
e0=m_e**4*c**5/(np.pi**2*h_t**3)

#Explicit form of the pressure
def pressure(x, p_g):
    p=e0/24*((2*x**3-3*x)*(1+x**2)**(1/2)+3*np.arcsinh(x))
    p_diff=p-p_g
    return(p_diff)


#Explicit form of the energy density
def energy(x):
    e_elec=e0/8*((2*x**3+x)*(1+x**2)**(1/2)-np.arcsinh(x))
    k=x*m_e*c
    n=k**3/(3*np.pi**2*h_t**3)
    e_tot=2*n*m_n*c**2+e_elec
    return(e_tot)


#Definition of a function which gives me the corresponding e for a given p
def energy_density(p):
    root = brentq(pressure, -100000, 100000, args=(p))
    e=energy(root)
    return e
```

Computational Analysis for White Dwarfs

Looking for zero-points: code

- Use this defined function inside the system of differential equations:

```
#Define TOV equations using the defined function that gives me the energy density for a given pressure

def tov_equations(r, y):
    p, m = y
    if p <= 0:
        return np.array([0, 0])
    e=energy_density(p)
    dmdr = 4 * np.pi * r**2 * e / c**2
    dpdr = -G * e * m / (c**2 * r**2) * (1 + p / e) * (1 + 4 * np.pi * r**3 * p / (m * c**2)) / (1 - 2 * G * m / (c**2 * r))
    return [dpdr, dmdr]
```

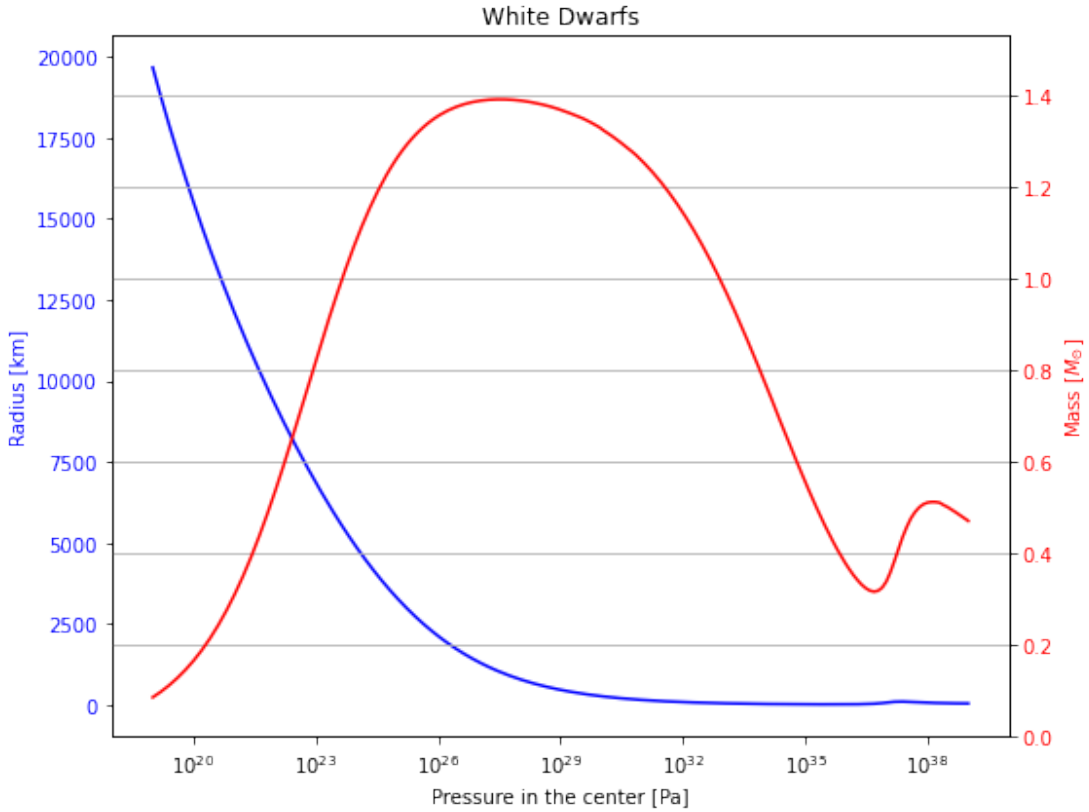
- Define the solver for the differential equations using *solveivp* from *scipy.integrate*

```
def solve_tov(p_c):
    y0 = [p_c, 1e-6]
    r_span = (1e-6, 1e8)

    sol = solve_ivp(tov_equations, r_span, y0, method='LSODA', max_step=1e4)
    return sol
```

Computational Analysis for White Dwarfs

Looking for zero-points: code

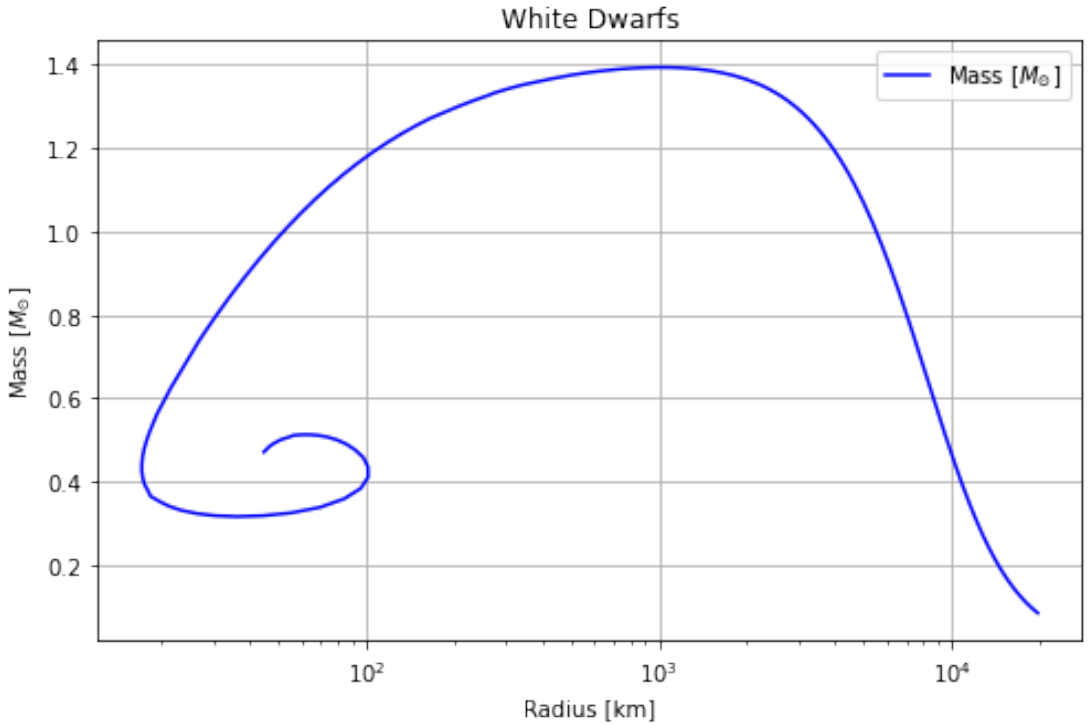
- Solve for many possible central pressures
- Plot the radii and masses of various white dwarfs



Computational Analysis for White Dwarfs

Looking for zero-points: code

- Plot the masses of various white dwarfs with respect to their radii:



Pure Neutron Stars

Structure Equations

- Pure non-relativistic neutron stars are described by an EoS of a Fermi gas of neutrons.
- The neutron number density is given by:

$$n_n = \frac{k_n^3}{3\pi^2\hbar^3}. \tag{33}$$

- In analogy to the Fermi gas for electrons, one has:

$$\epsilon(x) = \frac{\epsilon_0}{8}[(2x^3 + x)(1 + x^2)^{1/2} - \sinh^{-1}(x)] \tag{34}$$

$$p(x) = \frac{\epsilon_0}{24}[(2x^3 - 3x)(1 + x^2)^{1/2} + 3\sinh^{-1}(x)] \tag{35}$$

with

$$x = \frac{k_n}{m_n c}, \quad \epsilon_0 = \frac{m_n^4 c^5}{\pi^2\hbar^3} \quad . \tag{36}$$

Pure Neutron Stars

NON RELATIVISTIC polytropic case

- In the non-relativistic case these equations can be simplified and can lead to a polytropic EoS:

$$p(\epsilon) = K_{nrel}\epsilon^{5/3} \tag{37}$$

with

$$K_{nrel} = \frac{\hbar^2}{15\pi^2 m_n}\left(\frac{3\pi^2}{m_n c^2}\right)^{5/3} \tag{38}$$

- Using the same process (and the same definitions) that was considered to obtain dimensionless differential equations in the case of white dwarfs, one can obtain a system of dimensionless differential equations also for these pure neutron stars.

Computational Analysis for Pure Neutron Stars

NON RELATIVISTIC Polytropic EoS in Newtonian regime

- Define the system of dimensionless differential equations:

```
#Define the dimensionless system of differential eqns

def neutron_star(r, y, w ,z):
    p ,m  = y
    if p <= 0:
        return np.array([0, 0])
    dpdr = -m/(r**2)*(p/w)**z
    dmdr = (r**2)*(p/w)**z
    return np.array([dpdr, dmdr])
```

- Define the central pressure and all the needed parameters for the system of dimensionless differential equations:

```
#Parameters for EoS

#non relativistic constant in the eos p=Knr*e**(5/3)
Knr = h_t**2/(15*np.pi**2*m_n)*(3*np.pi**2/(m_n*c**2))**(5/3)

#pressure in the centre of the neutron star
p_c=1e31

#matter density in the centre of the neutron star
ro_c=(p_c/Knr)**(3/5)*c**(-2)

#dimensionless param for dimensionless eos
w = Knr*c**(4/3)*ro_c**(2/3)

#inverse of gamma
z= 3/5

#initial values for the dimensionless system of differential eqns
p0=p_c/ro_c*c**(-2)
m0 = 0.0
y0 = np.array([p0, m0])
r0, rf, dr = 1e-6, 0.6, 0.0001
```

Computational Analysis for Pure Neutron Stars

NON RELATIVISTIC Polytropic EoS in Newtonian regime

- Solve the system of differential equations:

```
# Solve using RK4
r_values, y_values = rk4(neutron_star, y0, r0, rf, dr, w, z)
```

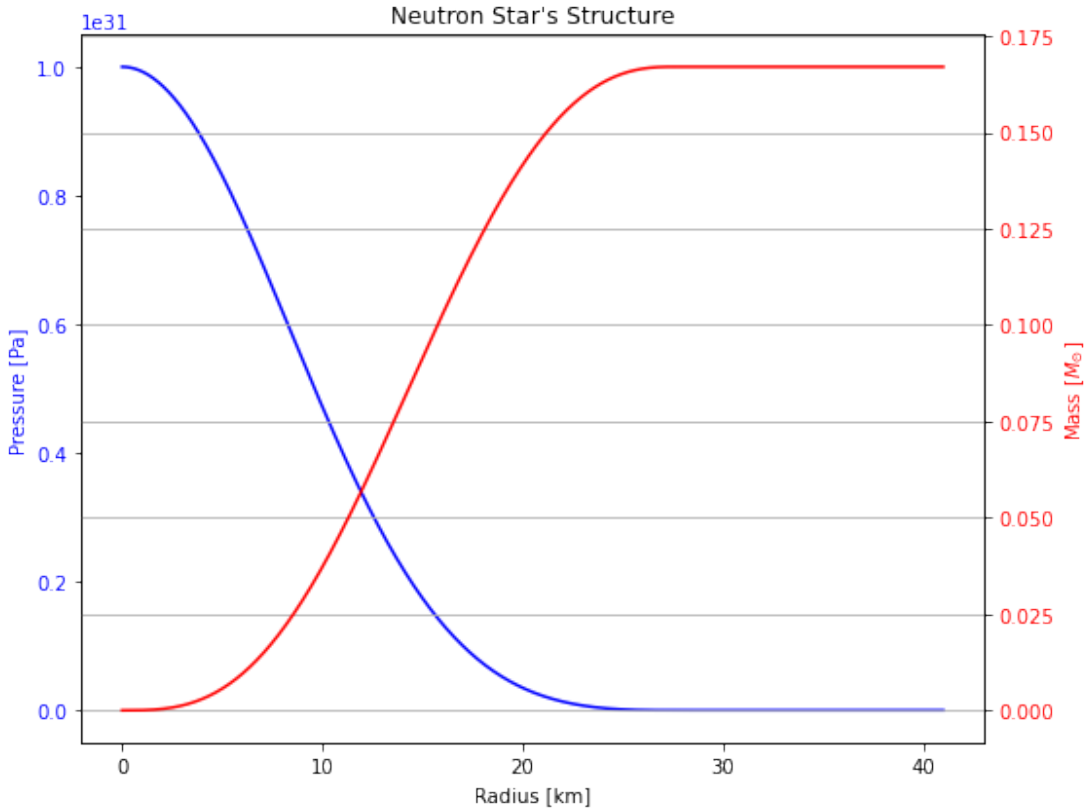
- Return to dimensionful quantities:

```
#Back to the dimensionful quantities

m_values= y_values[:, 1]*4*np.pi*ro_c/(ro_c*4*np.pi*G)**(3/2)*c**3/m_sun
p_values= y_values[:, 0]*ro_c*c**2
r_values1=r_values/(ro_c*4*np.pi*G)**(1/2)*c*10**(-3)
```

Computational Analysis for Pure Neutron Stars

NON RELATIVISTIC Polytropic EoS in Newtonian regime

- Plot the behaviour of the pressure and the mass inside the neutron star:



Computational Analysis for Pure Neutron Stars

NON RELATIVISTIC Polytropic EoS in Newtonian regime

- Find the final radius and mass of the neutron star:

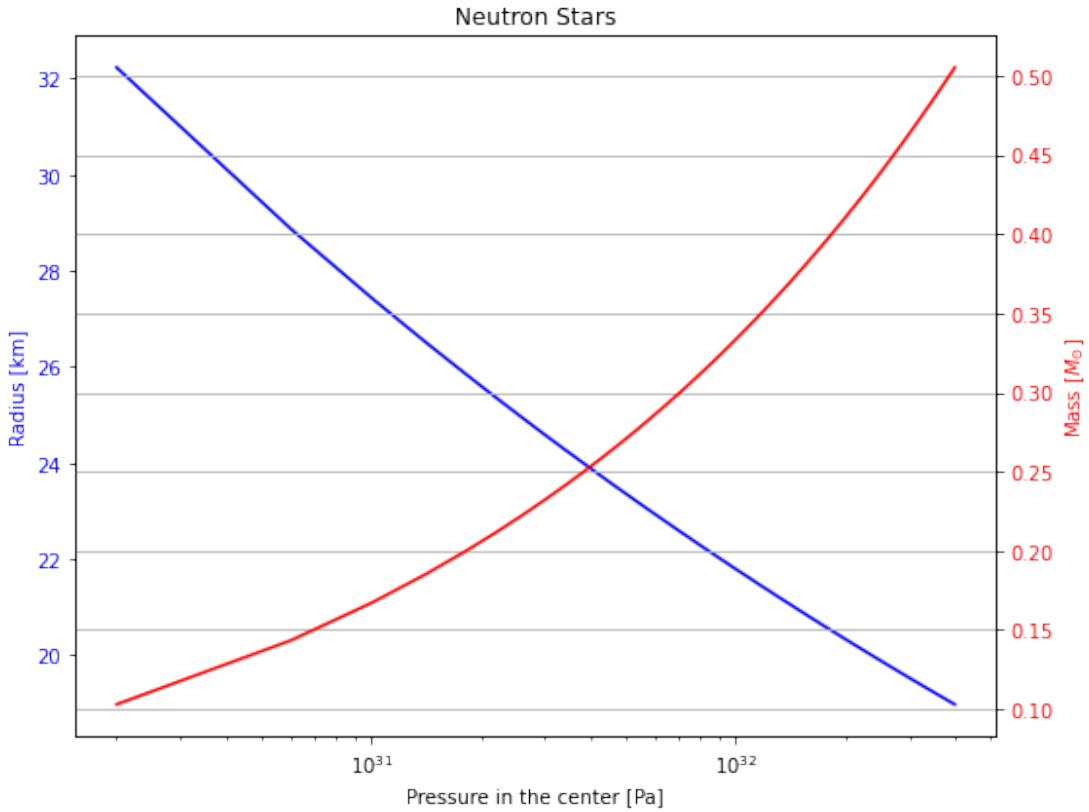
```
#Radius and Mass of the Neutron Star (Km and M_sun)
for i in range(len(p_values)):
    if p_values[i] <= 0:
        r_stop = r_values1[i]
        M_star = m_values[i]
        break

print(r_stop)
print(M_star)
```

```
27.444700095384896
0.16701177616868582
```

Computational Analysis for Pure Neutron Stars

NON RELATIVISTIC Polytropic EoS in Newtonian regime

- Plot the radii and masses of neutron stars for various central pressures:



Computational Analysis for Pure Neutron Stars

NON RELATIVISTIC Polytropic EoS in GR regime

- Define the system of differential equations and its solver by using the function *solveivp* from *scipy.integrate*

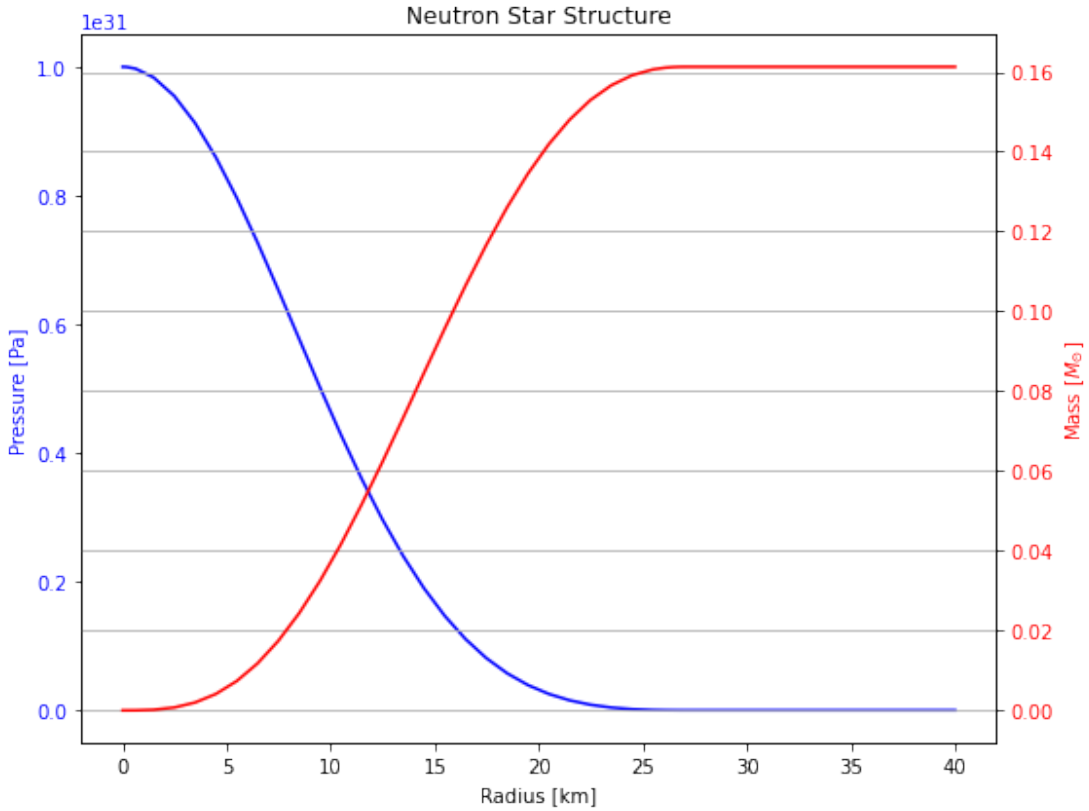
```
def tov_equations(r, y):
    p, m = y
    if p <= 0:
        return np.array([0, 0])

    e = (p / w)**z
    dmdr = 4 * np.pi * r**2 * e / c**2
    dpdr = -G * e * m / (c**2 * r**2) * (1 + p / e) * (1 + 4 * np.pi * r**3 * p / (m * c**2)) / (1 - 2 * G * m / (c**2 * r))
    return [dpdr, dmdr]


def solve_tov(p_c):
    y0 = [p_c, 1e-6]
    r_span = (1e-6, 4e4)

    sol = solve_ivp(tov_equations, r_span, y0, method='RK45', max_step=1e3)
    return sol
```

- Define central pressure p_c (same value as the one considered in the newtonian regime to compare the two cases)

Computational Analysis for Pure Neutron Stars

NON RELATIVISTIC Polytropic EoS in GR regime

- Plot the behaviour of the pressure and the mass inside the neutron star:



Computational Analysis for Pure Neutron Stars

NON RELATIVISTIC Polytropic EoS in GR regime

- Find the final radius and mass of the neutron star:

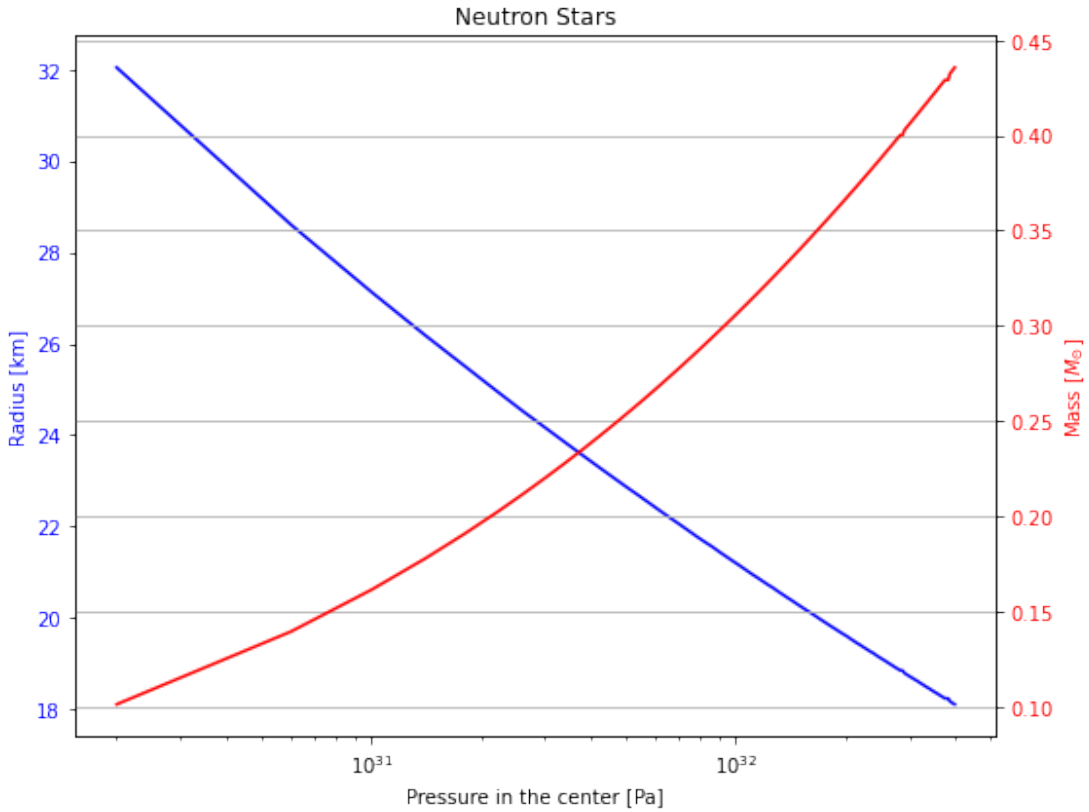
```
#Radius and Mass of the Neutron star (Km and M_sun)
for i in range(len(p_values)):
    if p_values[i] <= 0:
        r_stop = r_values1[i]
        M_star = m_values[i]
        break

print(r_stop)
print(M_star)
```

```
27.135347192907975
0.16126890962034873
```

Computational Analysis for Pure Neutron Stars

NON RELATIVISTIC Polytropic EoS in GR regime

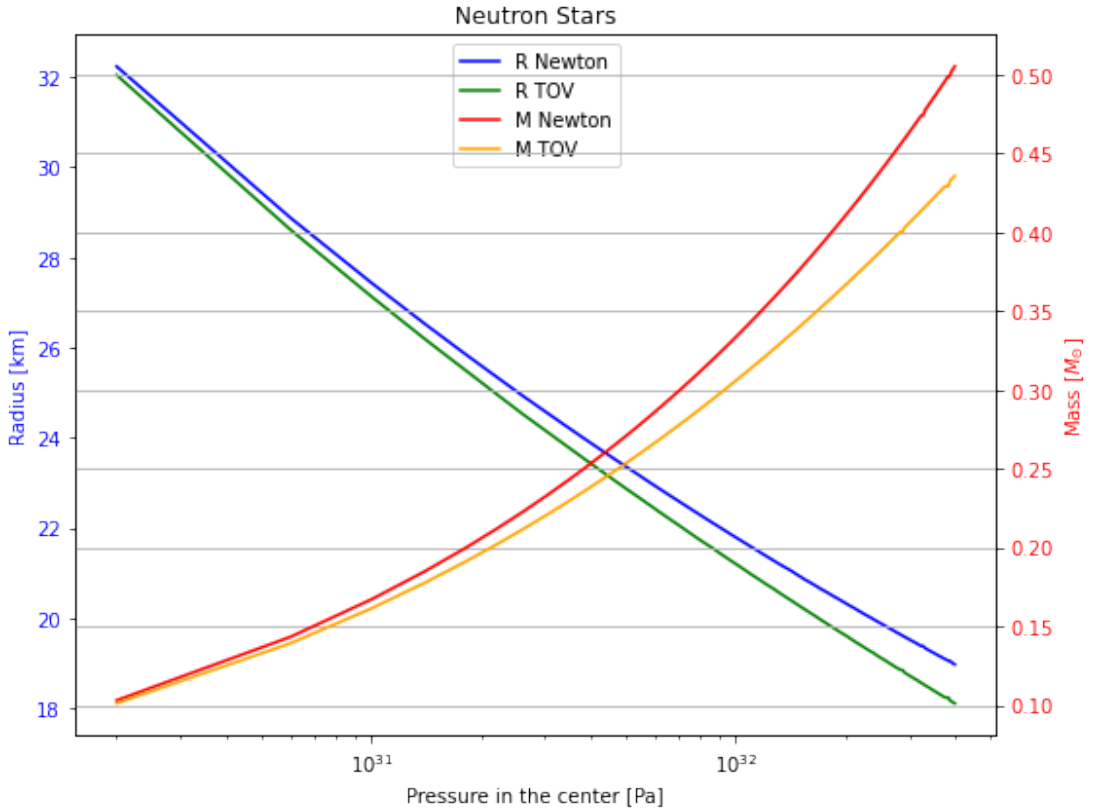
- Plot the radii and masses of neutron stars for various central pressures:



Computational Analysis for Pure Neutron Stars

NON RELATIVISTIC Polytropic EoS: Newtonian Regime VS GR regime

- Compare the results obtained from the Newtonian regime with the ones obtained in the GR one :



Computational Analysis for Pure Neutron Stars

Looking for zero points: code

- Once again one can use a root-finding routine to obtain an equation of state valid over the whole range of pressure and energy density.
- Define the explicit expressions for the pressure and the energy density
- Define a function which, via a root-finding method (brentq in this case), gives the energy density associated to a given pressure

```
e0=m_n**4*c**5/(np.pi**2*h_t**3)

#Explicit form of the pressure

def pressure(x, p_g):
    p=e0/24*((2*x**3-3*x)*(1+x**2)**(1/2)+3*np.arcsinh(x))
    p_diff=p-p_g
    return(p_diff)


#Explicit form of the energy
def energy(x):
    e_n=e0/8*((2*x**3+x)*(1+x**2)**(1/2)-np.arcsinh(x))
    return(e_n)


#Definition of a function which gives me the corresponding e for a given p
def energy_density(p):
    root = brentq(pressure, 0, 10000, args=(p))
    e=energy(root)
    return e
```

Computational Analysis for Pure Neutron Stars

Looking for zero-points: code

- Use this defined function inside the system of differential equations:

```
#Define TOV equations using the defined function that gives me the energy density for a given pressure

def tov_equations(r, y):
    p, m = y
    if p <= 0:
        return np.array([0, 0])
    e=energy_density(p)
    dmdr = 4 * np.pi * r**2 * e / c**2
    dpdr = -G * e * m / (c**2 * r**2) * (1 + p / e) * (1 + 4 * np.pi * r**3 * p / (m * c**2)) / (1 - 2 * G * m / (c**2 * r))
    return [dpdr, dmdr]
```

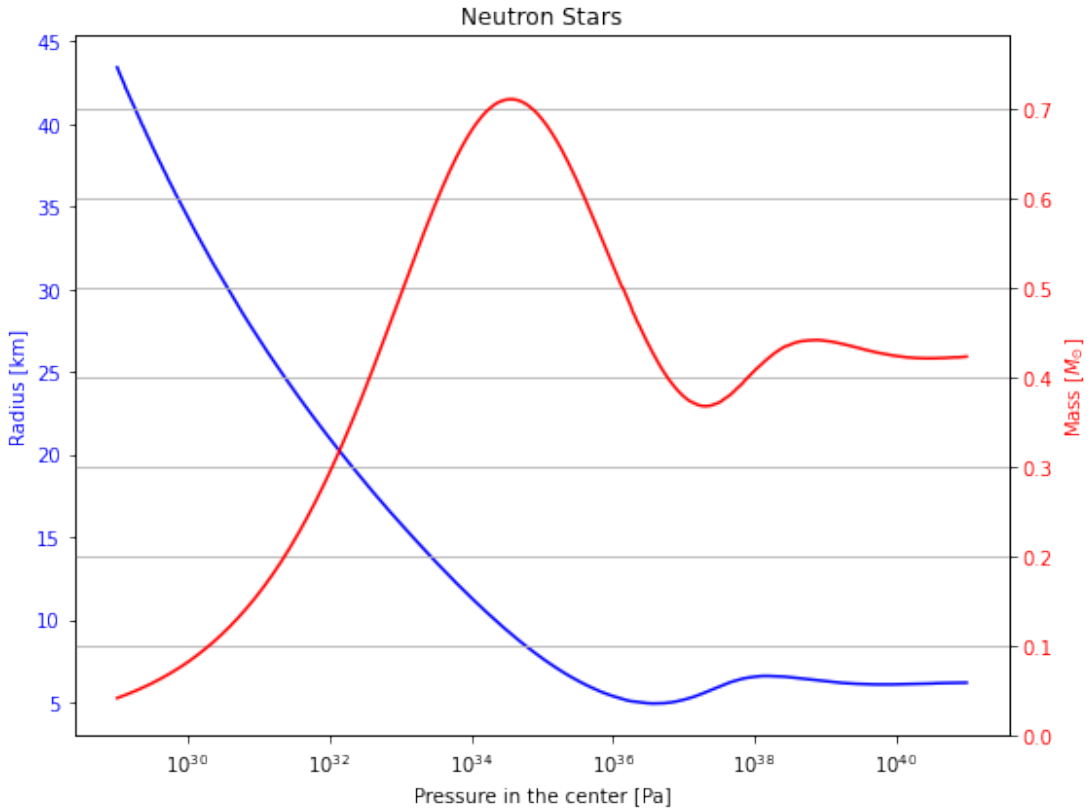
- Define the solver for the differential equations using *solveivp* from *scipy.integrate*

```
def solve_tov(p_c):
    y0 = [p_c, 1e-6]
    r_span = (1e-6, 5e4)

    sol = solve_ivp(tov_equations, r_span, y0, method='LSODA', max_step=1e4)
    return sol
```

Computational Analysis for Pure Neutron Stars

Looking for zero-points: code

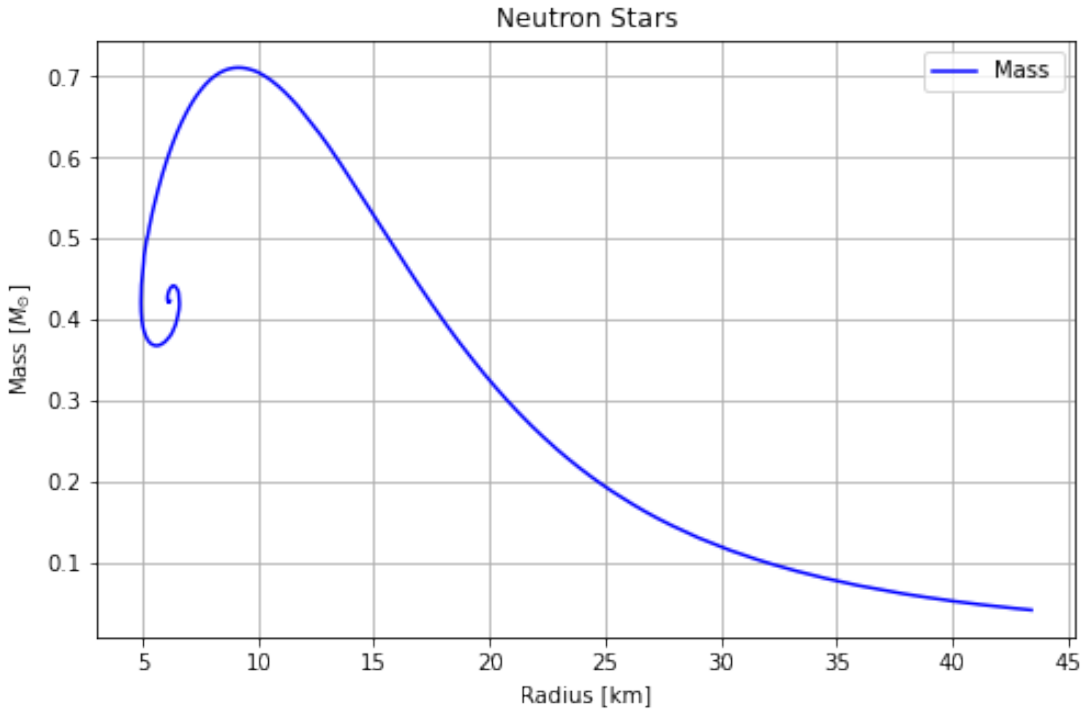
- Solve for many possible central pressures
- Plot the radii and masses of neutron stars for various central pressures:



Computational Analysis for Pure Neutron Stars

Looking for zero-points: code

- Plot the masses of various neutron stars with respect to their radii:



Neutron Stars with protons and electrons

Equations

- Since the neutron is unstable, a neutron star will not consist of neutrons only, but also of protons and electrons
- The electrons and protons are produced by β–decay:

$$n \longrightarrow p + e^- + \bar{\nu}_e. \tag{39}$$

- As all reactions are in chemical equilibrium, for the chemical potentials holds:

$$\mu_n = \mu_p + \mu_e. \tag{40}$$

- Due to stability reasons, it is assumed that the whole star is electrically neutral:

$$n_p = n_e \Leftrightarrow k_p = k_e \tag{41}$$

and the Fermi momenta of protons and electrons must be equal.

Neutron Stars with protons and electrons

Equations

- With the definition of the chemical potential for an ideal Fermi gas with zero temperature,

$$\mu_i(k_i) = \frac{d\epsilon_i}{dn_i} = (k_i^2c^2 + m_i^2c^4)^{1/2}, \quad i = n, p, e, \tag{42}$$

and using eqs. (40) and (41), one obtains:

$$(k_n^2c^2 + m_n^2c^4)^{1/2} - (k_p^2c^2 + m_p^2c^4)^{1/2} - (k_p^2c^2 + m_e^2c^4)^{1/2} = 0. \tag{43}$$

- This equation can be solved for the Fermi momenta of protons k_p as a function of the Fermi momenta of neutrons k_n:

$$\mathrm{k}_p(k_n) = \frac{[(k_n^2c^2+m_n^2c^4-m_e^2c^4)^2-2m_p^2c^4(k_n^2c^2+m_n^2c^4+m_e^2c^4)+m_p^4c^8]^{1/2}}{2c(k_n^2c^2+m_n^2c^4)^{1/2}} \tag{44}$$

Neutron Stars with protons and electrons

Equations

- One should notice that if there are no neutrons present, i.e. $k_n = 0$:

$$k_p(0) = 1.264 \cdot 10^{-3} m_n c. \tag{45}$$

  Therefore equation 44 can only be used if $k_p > 1.264 \cdot 10^{-3} m_n c$, which is fulfilled for $p > p_{\mathrm{crit}} = 3.038 \cdot 10^{23}\ \mathrm{Pa}$.

- Below this pressure, no neutrons are present, so the "neutron star" consists of protons and electrons only.

Neutron Stars with protons and electrons

Equations

- The total energy and the total pressure are the sum of the energy and pressure of electrons, protons and neutrons:

$$\epsilon_{tot} = \sum_{i=n,p,e} \epsilon_i \,, \quad p_{tot} = \sum_{i=n,p,e} p_i \,, \tag{46}$$

where

$$\epsilon_i(k_i) = \frac{8\pi}{(2\pi\hbar)^3} \int_0^{k_i} (k^2c^2 + m_i^2c^4)^{1/2} k^2 dk \tag{47}$$

$$p_i(k_i) = \frac{1}{3}\frac{8\pi}{(2\pi\hbar)^3} \int_0^{k_i} (k^2c^2 + m_i^2c^4)^{-1/2} k^4 dk. \tag{48}$$

Computational Analysis for Neutron Stars with protons and electrons

Equation of State

- One can obtain the equation of state using these previous expressions
- To do this one considers the same root-finding process to get a function which , for a given pressure, returns the associated energy density
- In this case one can define dimensionless quantities such as $x = k_n/m_nc$, $w = k_p/m_pc$ and $z = k_e/m_ec$ that must be related consistently to 41 and 44.
- One should also pay attention to the existence of a critical pressure, under which there are no neutrons.

Computational Analysis for Neutron Stars with protons and electrons

Equation of State

```
#Building the equation of state

e0_p = m_p**4 * c**5 / (np.pi**2 * h_t**3)
e0_n = m_n**4 * c**5 / (np.pi**2 * h_t**3)
e0_e = m_e**4 * c**5 / (np.pi**2 * h_t**3)

#Define dimensionless quantities for the root finding of the momentum

#Dimensionless parameter for the proton w=k_p/(m_p*c) as a function of x=k_n/(m_n*c)
def proton_w(x):
    term1 = (x**2 + 1 - (m_e**2 / m_n**2))**2
    term2 = -2 * (m_p**2 / m_n**2) * (x**2 + 1 + (m_e**2 / m_n**2))
    term3 = (m_p**4 / m_n**4)
    numerator = np.sqrt(m_n**4 * (term1 + term2 + term3))
    denominator = 2 * m_p * m_n * np.sqrt(x**2 + 1)
    return numerator / denominator

#Dimensionless parameter for the electron z=k_e/(m_e*c)=k_p/(m_e*c) as a function of w=k_p/(m_p*c)
def electron_z(w):
    return w * (m_p / m_e)

#Define difference between pressure given by protons, electrons and neutrons and a given pressure
def pressure_tot_func(x, p_g):
    w = proton_w(x)
    z = electron_z(w)
    p_p = e0_p / 24 * ((2 * w**3 - 3 * w) * np.sqrt(1 + w**2) + 3 * np.arcsinh(w))
    p_e = e0_e / 24 * ((2 * z**3 - 3 * z) * np.sqrt(1 + z**2) + 3 * np.arcsinh(z))
    p_n = e0_n / 24 * ((2 * x**3 - 3 * x) * np.sqrt(1 + x**2) + 3 * np.arcsinh(x))
    return p_n + p_e + p_p - p_g

#Define energy density given by protons, electrons and neutrons
def energy_tot_func(x):
    w = proton_w(x)
    z = electron_z(w)
    e_p = e0_p / 8 * ((2 * w**3 + w) * np.sqrt(1 + w**2) - np.arcsinh(w))
    e_e = e0_e / 8 * ((2 * z**3 + z) * np.sqrt(1 + z**2) - np.arcsinh(z))
    e_n = e0_n / 8 * ((2 * x**3 + x) * np.sqrt(1 + x**2) - np.arcsinh(x))
    return e_n + e_e + e_p
```

Computational Analysis for Neutron Stars with protons and electrons

Equation of State

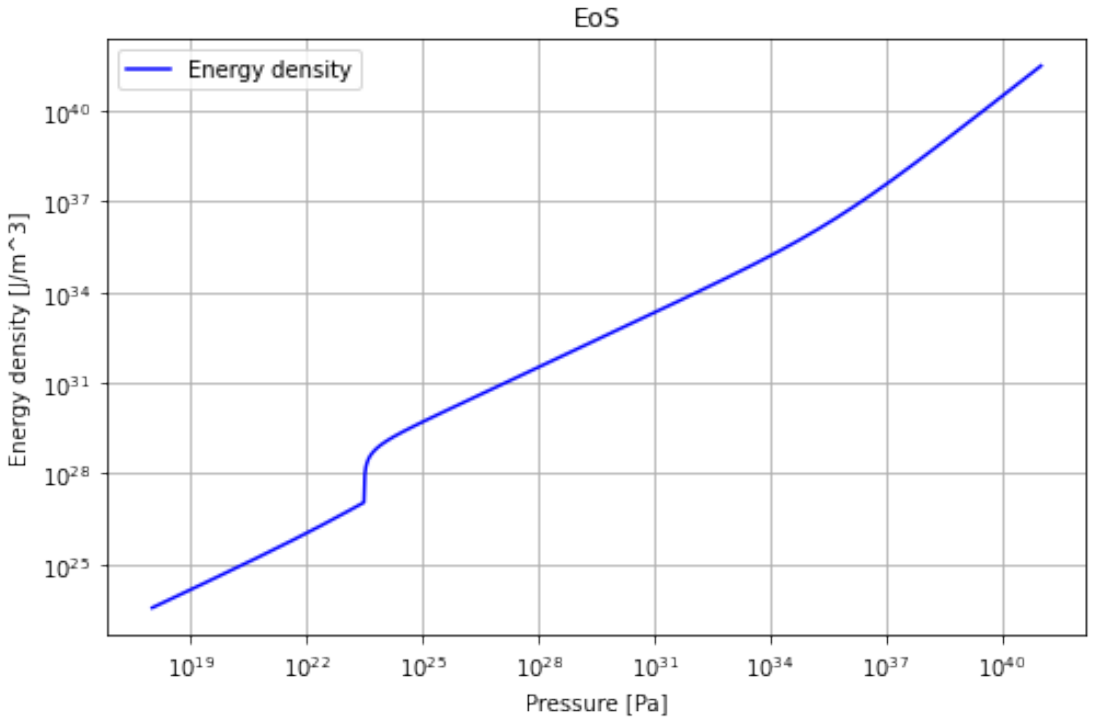
```
#Define difference between pressure given only by protons and electrons and a given pressure (happens under criti
def pressure_1tot_func(u, p_g):
    z = electron_z(u)
    p_p = e0_p / 24 * ((2 * u**3 - 3 * u) * np.sqrt(1 + u**2) + 3 * np.arcsinh(u))
    p_e = e0_e / 24 * ((2 * z**3 - 3 * z) * np.sqrt(1 + z**2) + 3 * np.arcsinh(z))
    return p_p + p_e - p_g

#Define energy density when there are only protons and electrons, which happens under a critical pressure
def energy_1tot_func(u):
    z = electron_z(u)
    e_p = e0_p / 8 * ((2 * u**3 + u) * np.sqrt(1 + u**2) - np.arcsinh(u))
    e_e = e0_e / 8 * ((2 * z**3 + z) * np.sqrt(1 + z**2) - np.arcsinh(z))
    return e_p + e_e
```

```
#Define total energy density: this function gives me the corresponding energy density for a given p
#taking into account that there exists a critical pressure
def energy_density(p):
    p_threshold = 3.054e23

    if p < p_threshold:
        root = brentq(pressure_1tot_func, -1000, 1000, args=(p,))
        return energy_1tot_func(root)
    else:
        root = brentq(pressure_tot_func, -1000, 1000, args=(p,))
        return energy_tot_func(root)
```

Computational Analysis for Neutron Stars with protons and electrons

Equation of State

- One can obtain the plot of the equation of state:



Computational Analysis for Neutron Stars with protons and electrons

Codes

- Use this defined function inside the differential equations:

```
#Define TOV equations using the defined function that gives me the energy density for a given pressure

def tov_equations(r, y):
    p, m = y
    if p <= 0:
        return np.array([0, 0])
    e=energy_density(p)
    dmdr = 4 * np.pi * r**2 * e / c**2
    dpdr = -G * e * m / (c**2 * r**2) * (1 + p / e) * (1 + 4 * np.pi * r**3 * p / (m * c**2)) / (1 - 2 * G * m / (c**2 * r))
    return [dpdr, dmdr]
```

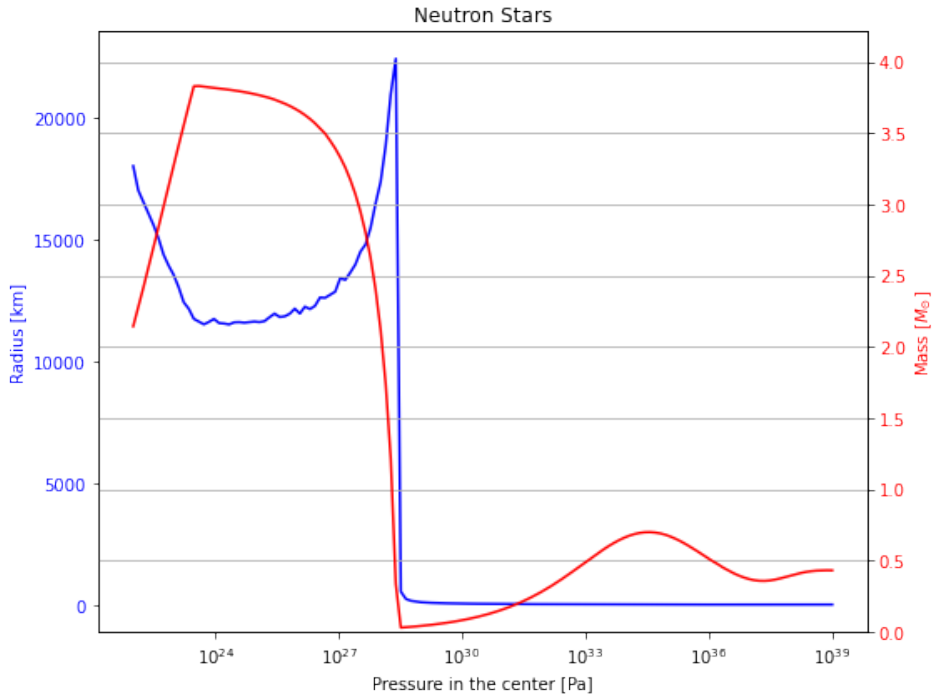
- Define the solver for the differential equations using *solveivp* from *scipy.integrate*

```
def solve_tov(p_c):
    y0 = [p_c, 1e-6]
    r_span = (1e-6, 3e7)

    sol = solve_ivp(tov_equations, r_span, y0, method='LSODA', max_step=5e2)
    return sol
```

Computational Analysis for Neutron Stars with protons and electrons

Codes

- Solve for many possible central pressures
- Plot the radii and masses of the neutron stars for various central pressures



Computational Analysis for Neutron Stars with protons and electrons

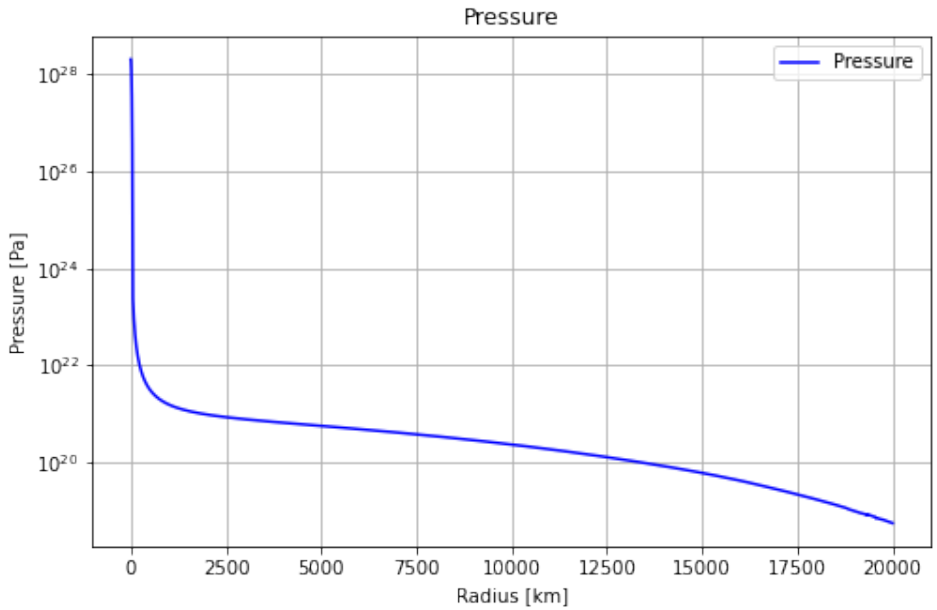
Explanation

- Below the critical pressure, the neutron star consists of protons and electrons. Above this pressure, neutrons appear resulting in a bend of $M(p_c)$ and $R(p_c)$.
- The unusual peak of $R(p_c)$ can be explained by the fact that around $p_c \simeq 10^{28}\ \mathrm{Pa}$ the pressure approaches zero very slowly but never reaches it or falls below.
- Therefore, it is necessary to introduce a cutoff ($p = 10^{18}\ \mathrm{Pa}$), so that the integration terminates as soon as this pressure is reached.
- This kind of neutron star consists of a dense central core and a huge low-density crust.

Computational Analysis for Neutron Stars with protons and electrons

Codes

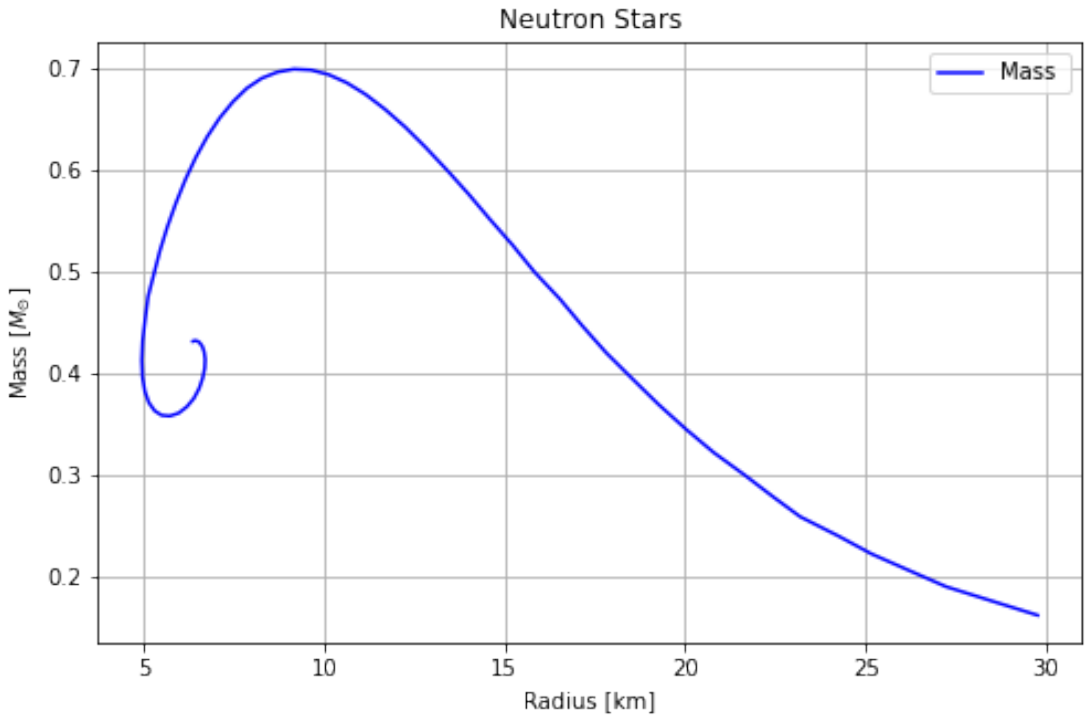
- One can see the pressure's particular behaviour mentioned above by plotting $p(r)$ in an neutron star for a given central pressure
- In this case $p_c = 2 \cdot 10^{28}\ \mathrm{Pa}$ and one considers the same equation of state used before



Computational Analysis for Neutron Stars with protons and electrons

Codes

- Plot the masses of various neutron stars with protons and electrons with respect to their radii:



Neutron Stars compOSE

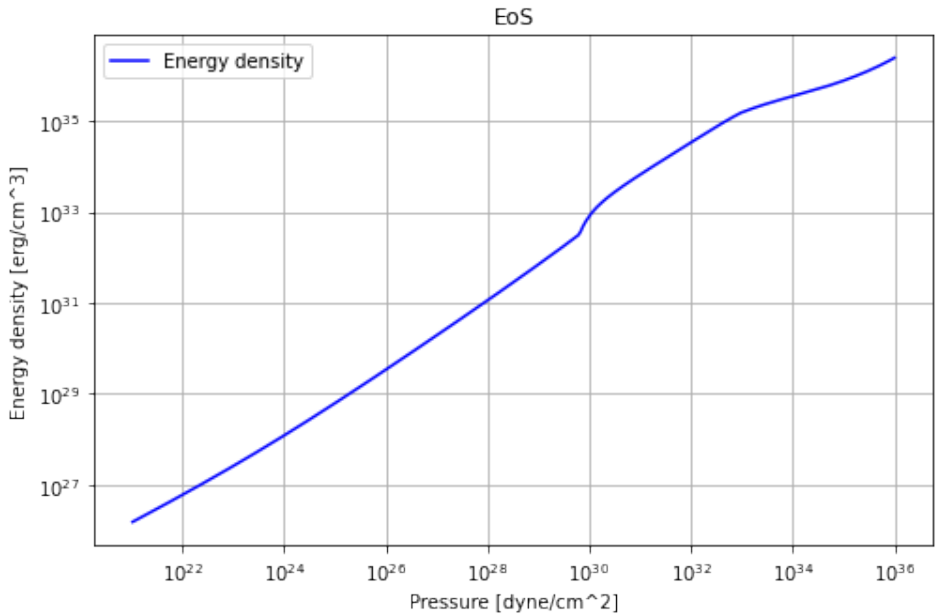
compOSE

- The online service CompOSE provides data tables for different equations of state (EoS).
- One can download their data, plot the equations of state and then study the resulting neutron stars.

Neutron Stars compOSE

RMF1 at T=0

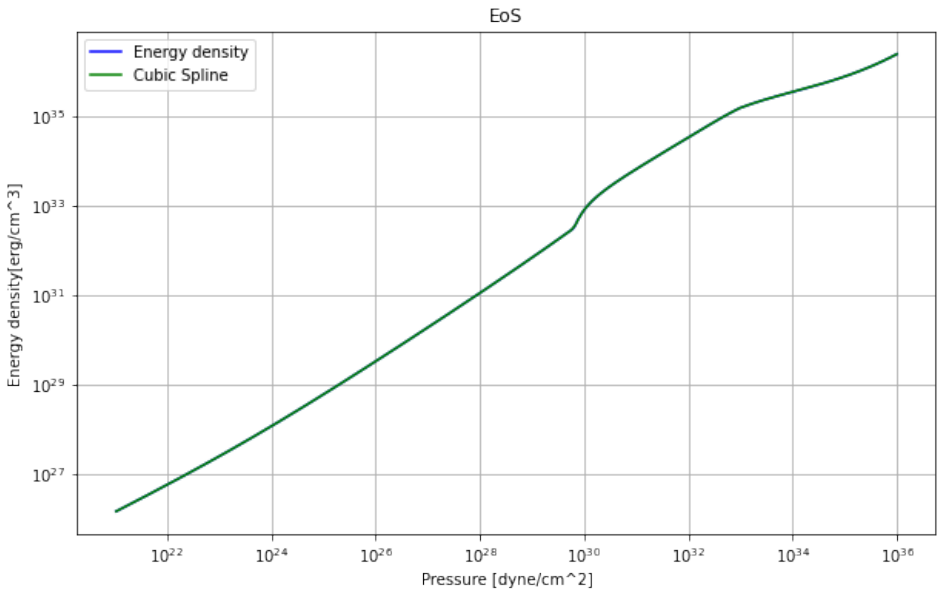
- This EoS describes a neutron star with a core of homogeneous n,p,e matter and a reconstructed crust.
- The EoS can be plotted using the downloaded data.



Neutron Stars compOSE

RMF1 at T=0

- One can use *Cubic spline* to get a function which fits the downloaded data.
- The downloaded data and the obtained function can be plotted together and compared.



Neutron Stars compOSE

RMF1 at T=0

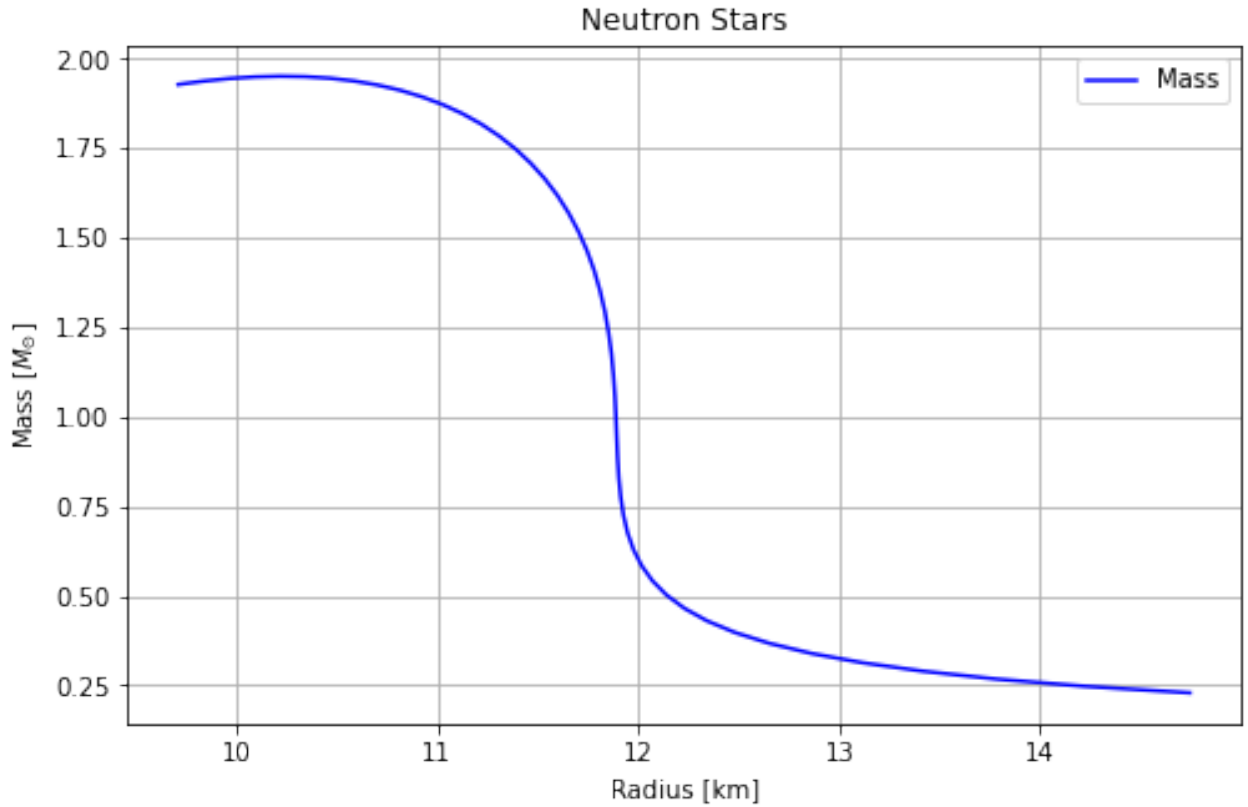
- At this point, one can use this EoS to solve the TOV equations for many possible central pressures.

```
def tov_equations(r, y):
    p, m = y
    e=eos(p)
    dmdr = 4 * np.pi * r**2 * e / c**2
    dpdr = -G * e * m / (c**2 * r**2) * (1 + p / e) * (1 + 4 * np.pi * r**3 * p / (m * c**2)) / (1 - 2 * G * m / (c**2 * r))
    return [dpdr, dmdr]


def solve_tov(p_c):
    y0 = [p_c, 1e-6]
    r_span = (1e-6, 5e8)

    sol = solve_ivp(tov_equations, r_span, y0, method='LSODA', max_step=1e3)

    return sol
```

Neutron Stars compOSE

RMF1 at T=0

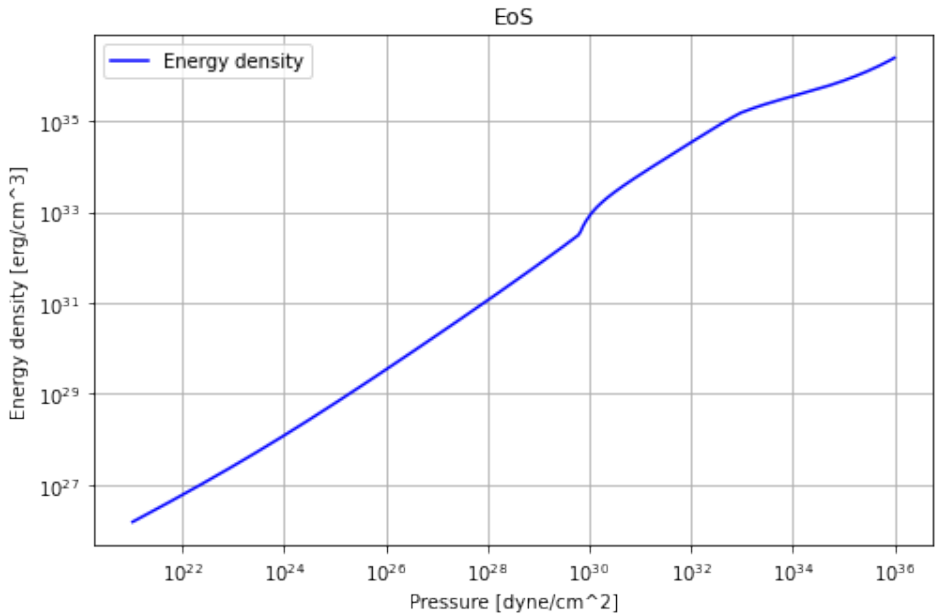
- Plot of the masses of various neutron stars with respect to their radii:



Neutron Stars compOSE

RMF2 at T=0

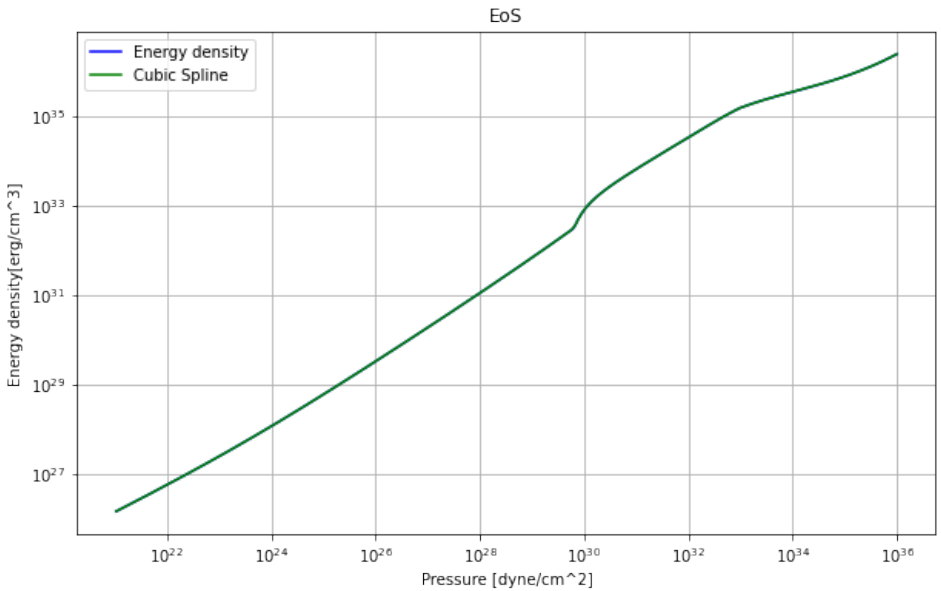
- This EoS is in β-equilibrium at T $= 0$.
- The core is homogeneous n, p, e matter.
- The EoS can be plotted using the downloaded data.



Neutron Stars compOSE

RMF2 at T=0

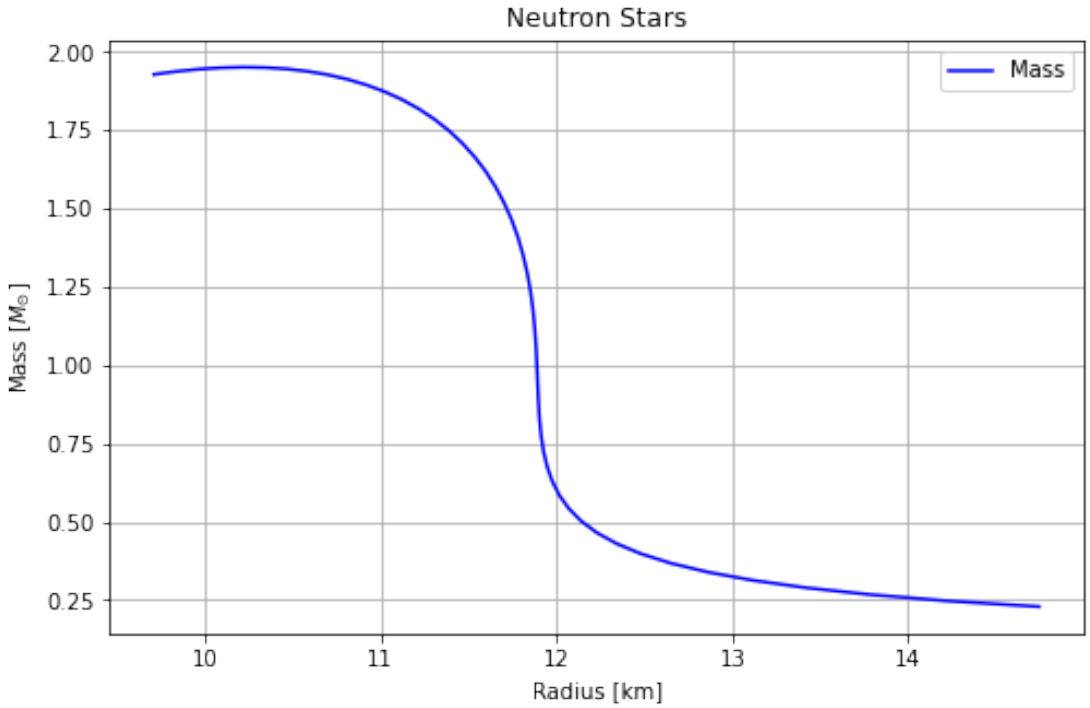
- One can use *Cubic spline* to get a function which fits the downloaded data.
- The downloaded data and the obtained function can be plotted together and compared.



Neutron Stars compOSE

RMF2 at T=0

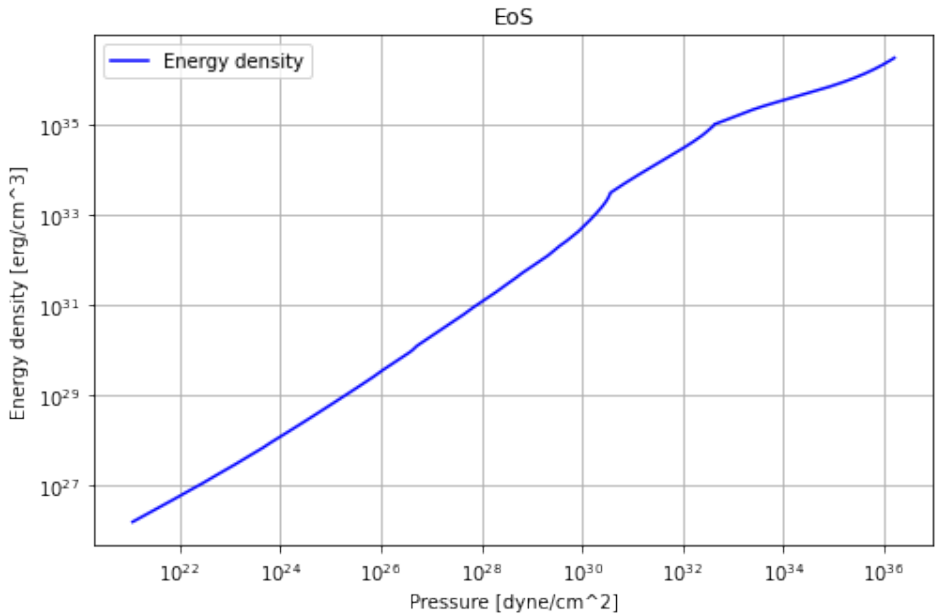
- At this point, one can use this EoS to solve the TOV equations for many possible central pressures.
- Plot of the masses of various neutron stars with respect to their radii:



Neutron Stars compOSE

RMF3 at T=0

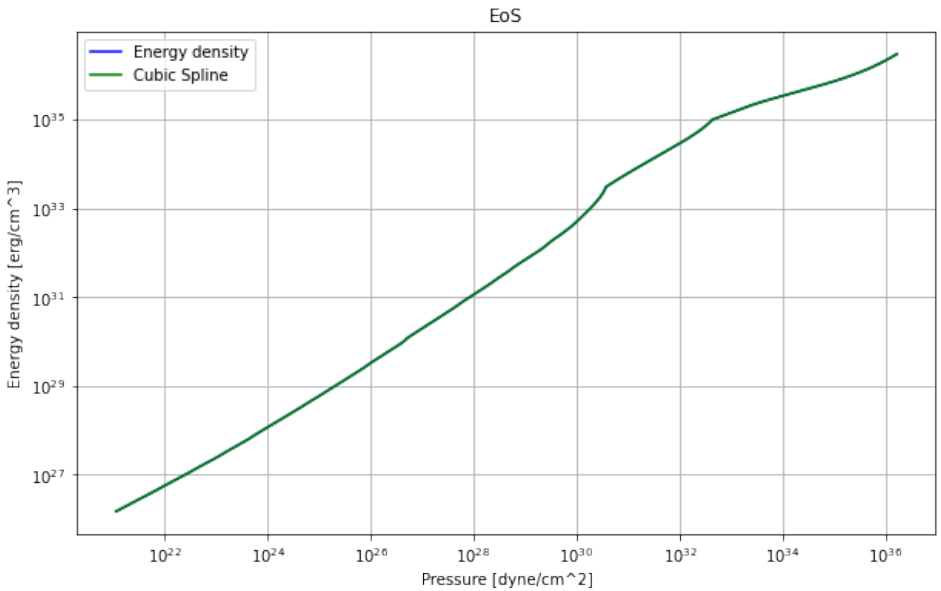
- This EoS is in β-equilibrium at T $= 0$.
- The core is homogeneous n, p, e matter.
- The EoS can be plotted using the downloaded data.



Neutron Stars compOSE

RMF3 at T=0

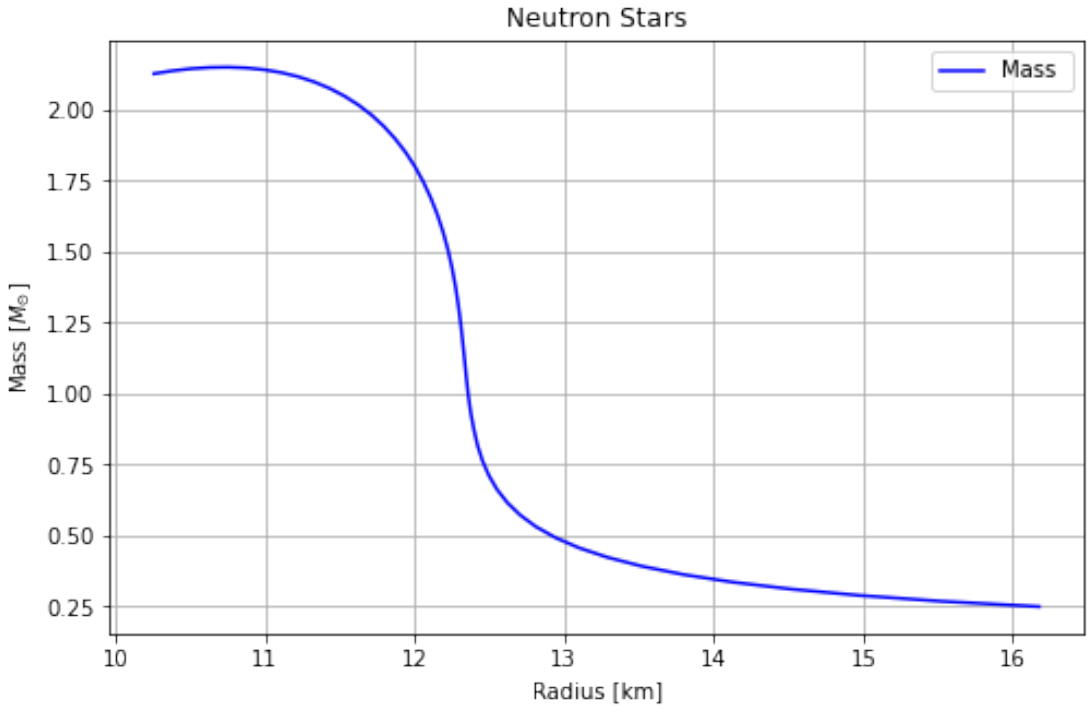
- One can use *Cubic spline* to get a function which fits the downloaded data.
- The downloaded data and the obtained function can be plotted together and compared.



Neutron Stars compOSE

RMF3 at T=0

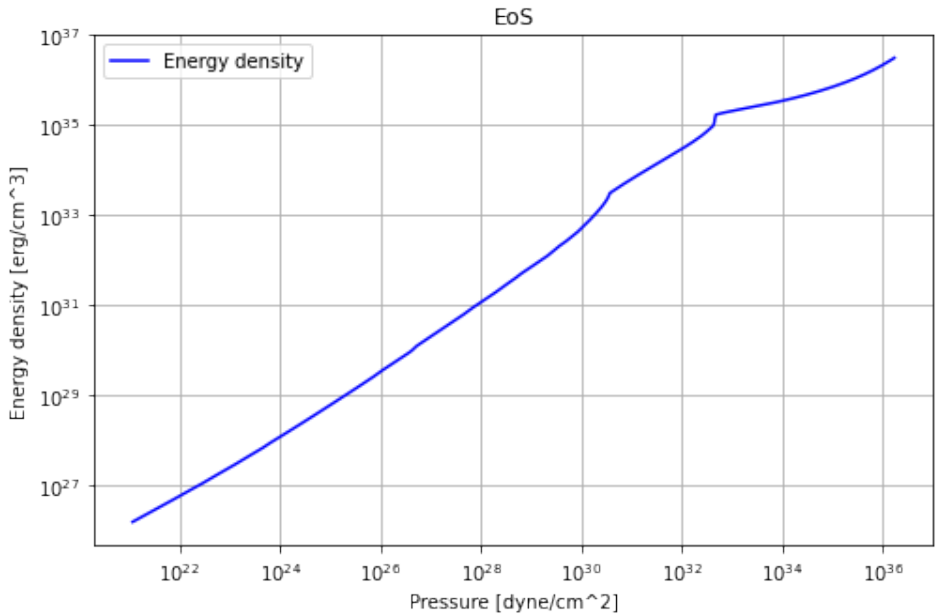
- At this point, one can use this EoS to solve the TOV equations for many possible central pressures.
- Plot of the masses of various neutron stars with respect to their radii:



Neutron Stars compOSE

RMF4 at T=0

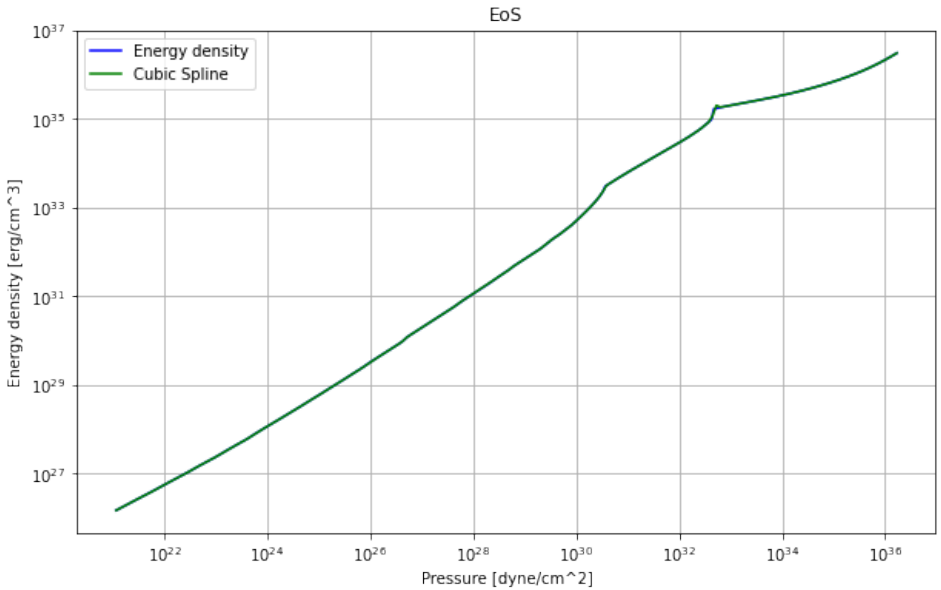
- This EoS is in β-equilibrium at T $= 0$.
- The core is homogeneous n, p, e matter.
- The EoS can be plotted using the downloaded data.



Neutron Stars compOSE

RMF4 at T=0

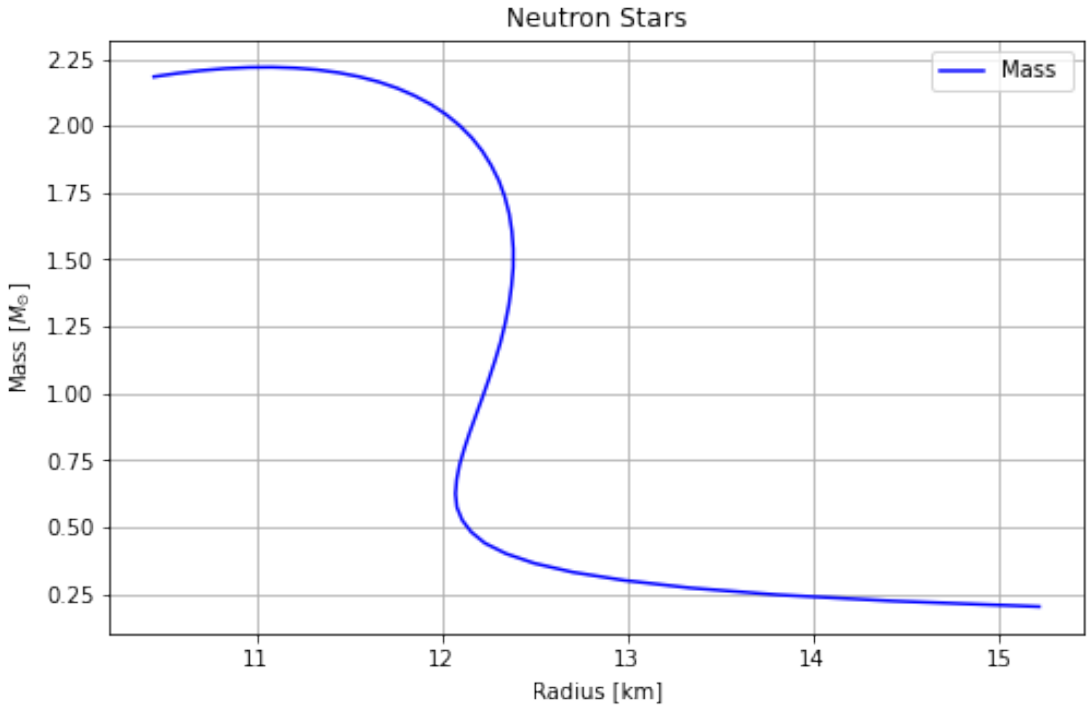
- One can use *Cubic spline* to get a function which fits the downloaded data.
- The downloaded data and the obtained function can be plotted together and compared.



Neutron Stars compOSE

RMF4 at T=0

- At this point, one can use this EoS to solve the TOV equations for many possible central pressures.
- Plot of the masses of various neutron stars with respect to their radii:



References

- *Compact Stars for Undergraduates* , Irina Sagert, Matthias Hempel, Carsten Greiner, and Jurgen Schaffner-Bielich
- *Computational Physics*, *Lecture Notes Fall* 2015, M. Hjorth-Jensen (in particular Part III)

Thank you for the attention!

May 2025