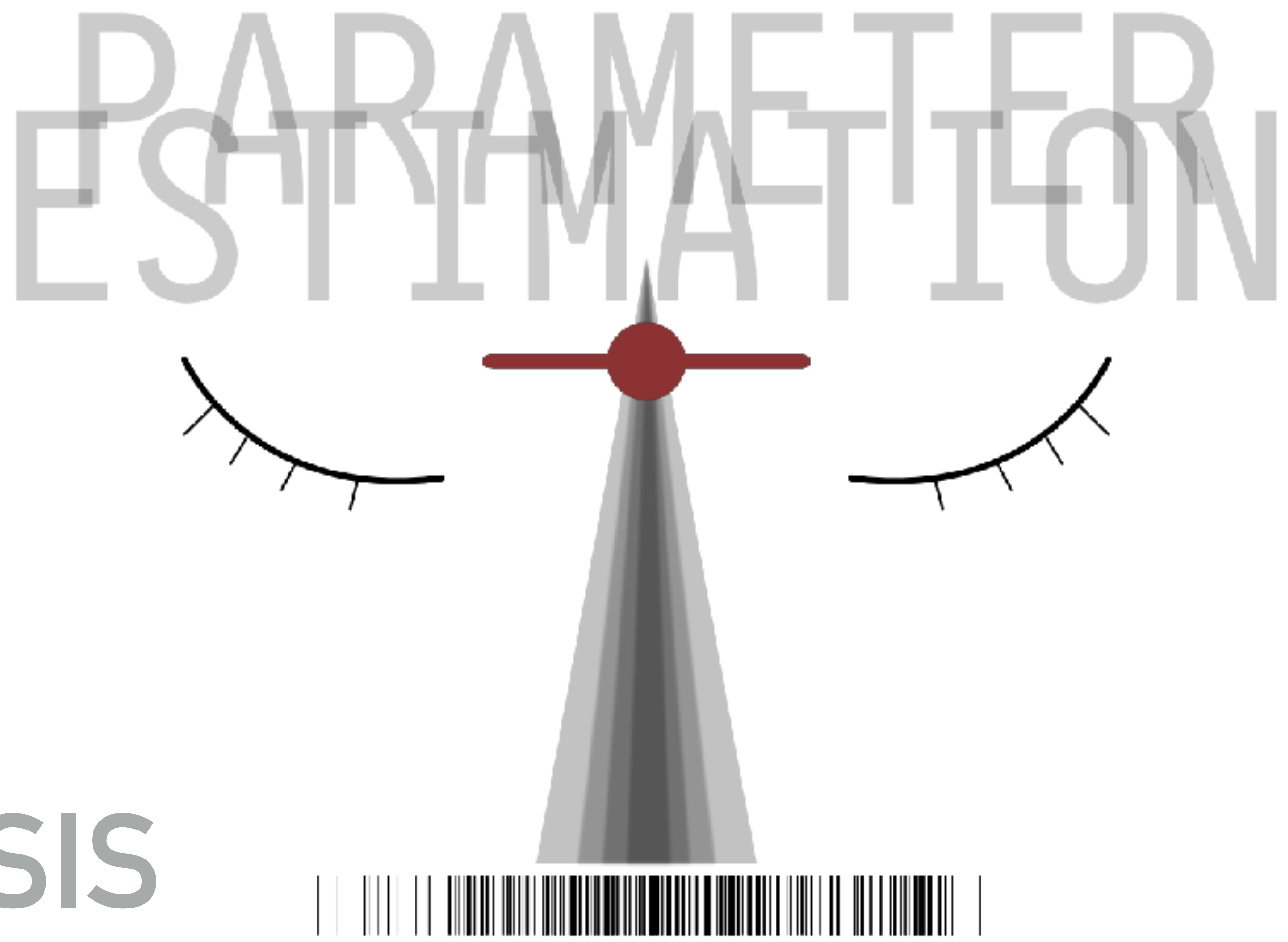
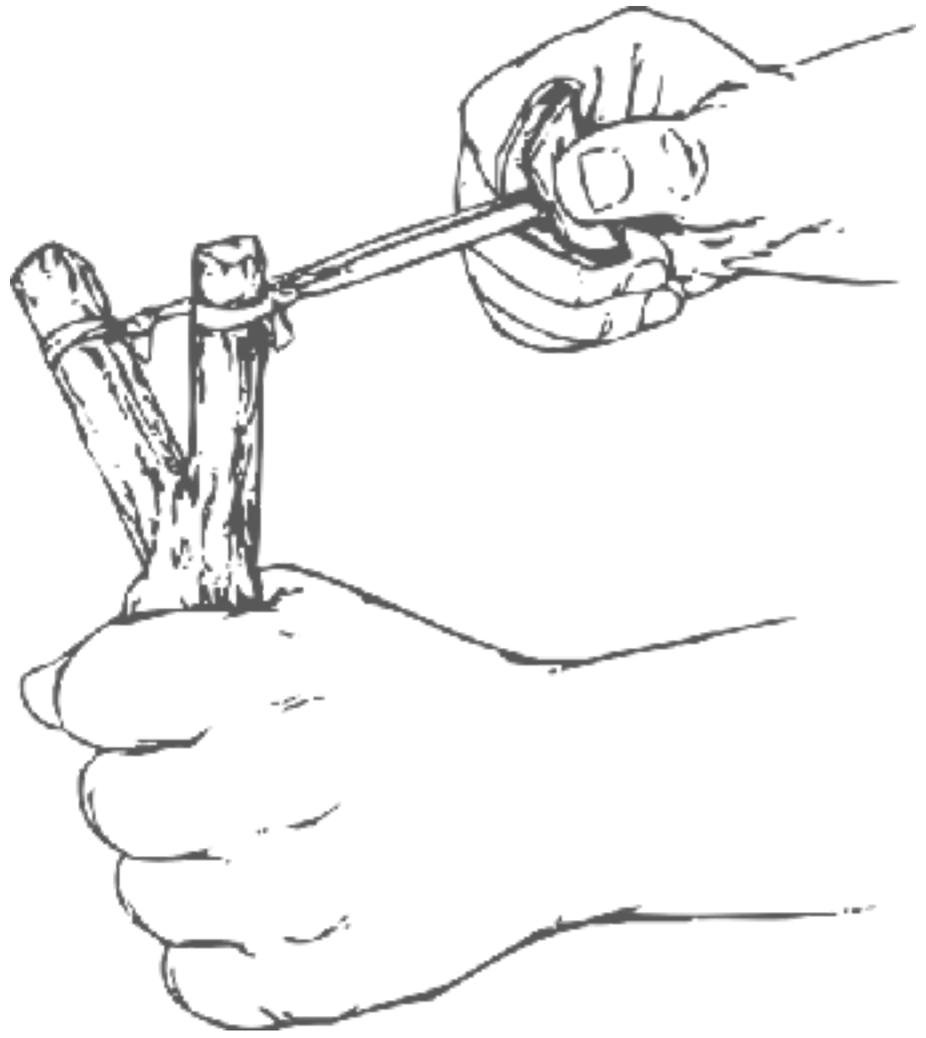


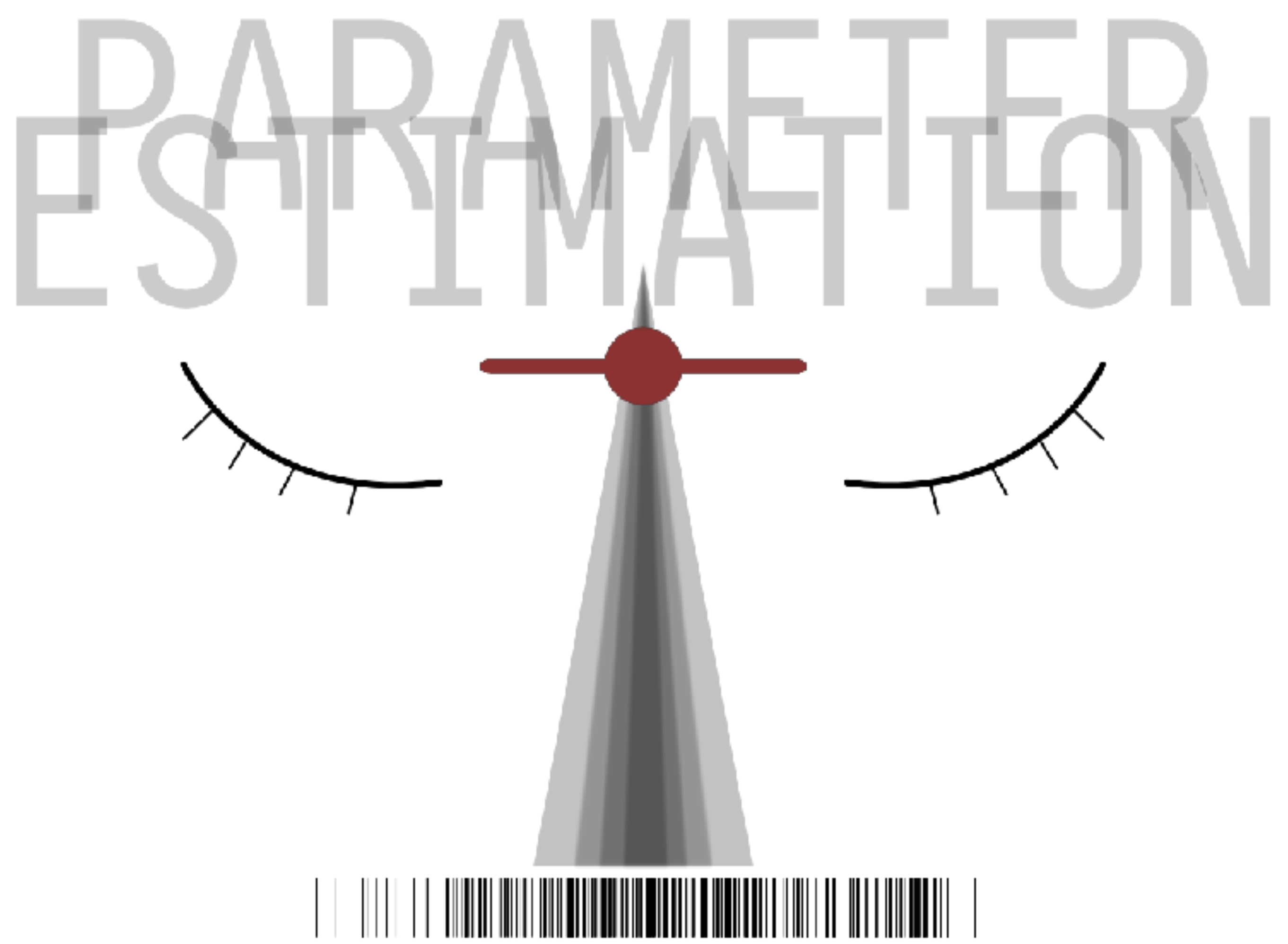
INTRODUCTION TO DATA ANALYSIS

PARAMETER ESTIMATION

LEARNING GOALS



- understand Bayes rule for parameter estimation
    - (conjugate) priors, likelihood
- point-valued & interval-based estimators
    - frequentist: MLE, confidence intervals
    - Bayes: mean of posterior, credible intervals
- implement probabilistic models in `greta`
- compute with posterior samples



ESTIMATES

- point-valued: single “best” values
- interval-range: “good” values (around “best” value)

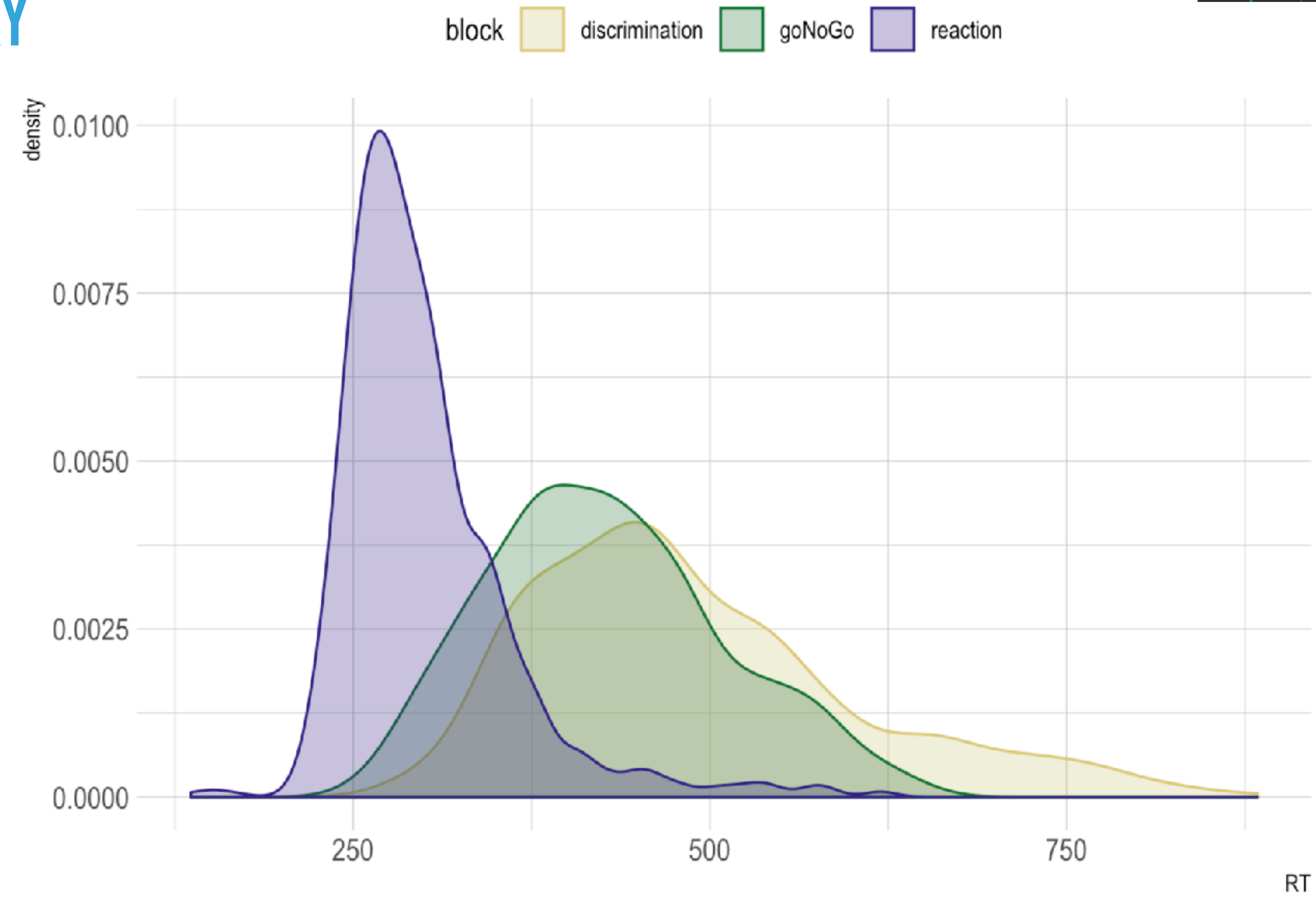
estimate	Bayesian	frequentist
best value	mean of posterior posterior	maximum likelihood estimate
interval range	credible interval (HDI)	confidence interval

model-based hypothesis testing

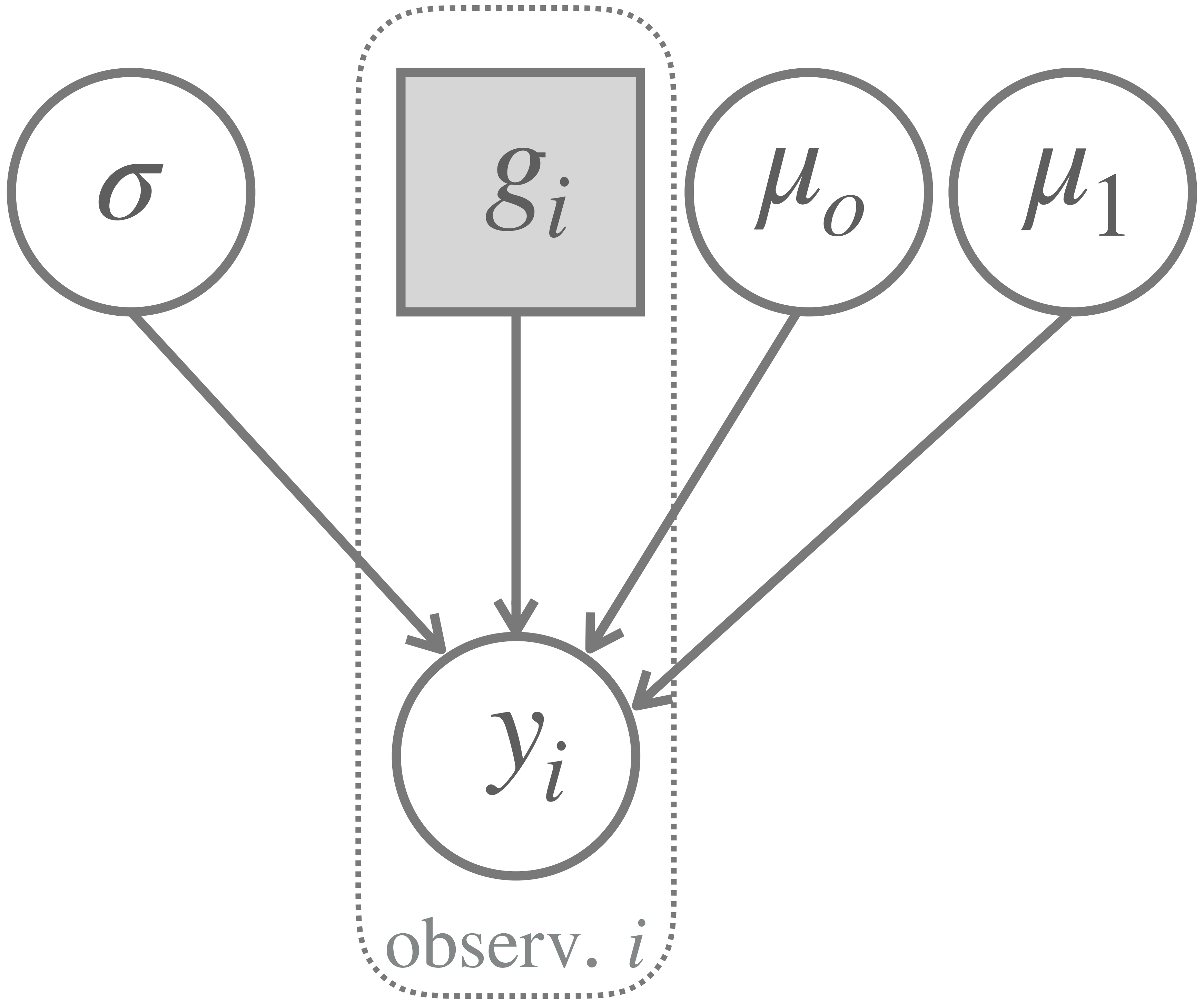
MENTAL CHRONOMETRY

- N=50 participants recruited via Prolific
- three blocks / conditions
  - reaction press button when a shape appears
  - go/no-go press button for shape 1; don't press for shape 2
  - discrimination press one button for shape 1, another for shape 2

MENTAL CHRONOMETRY

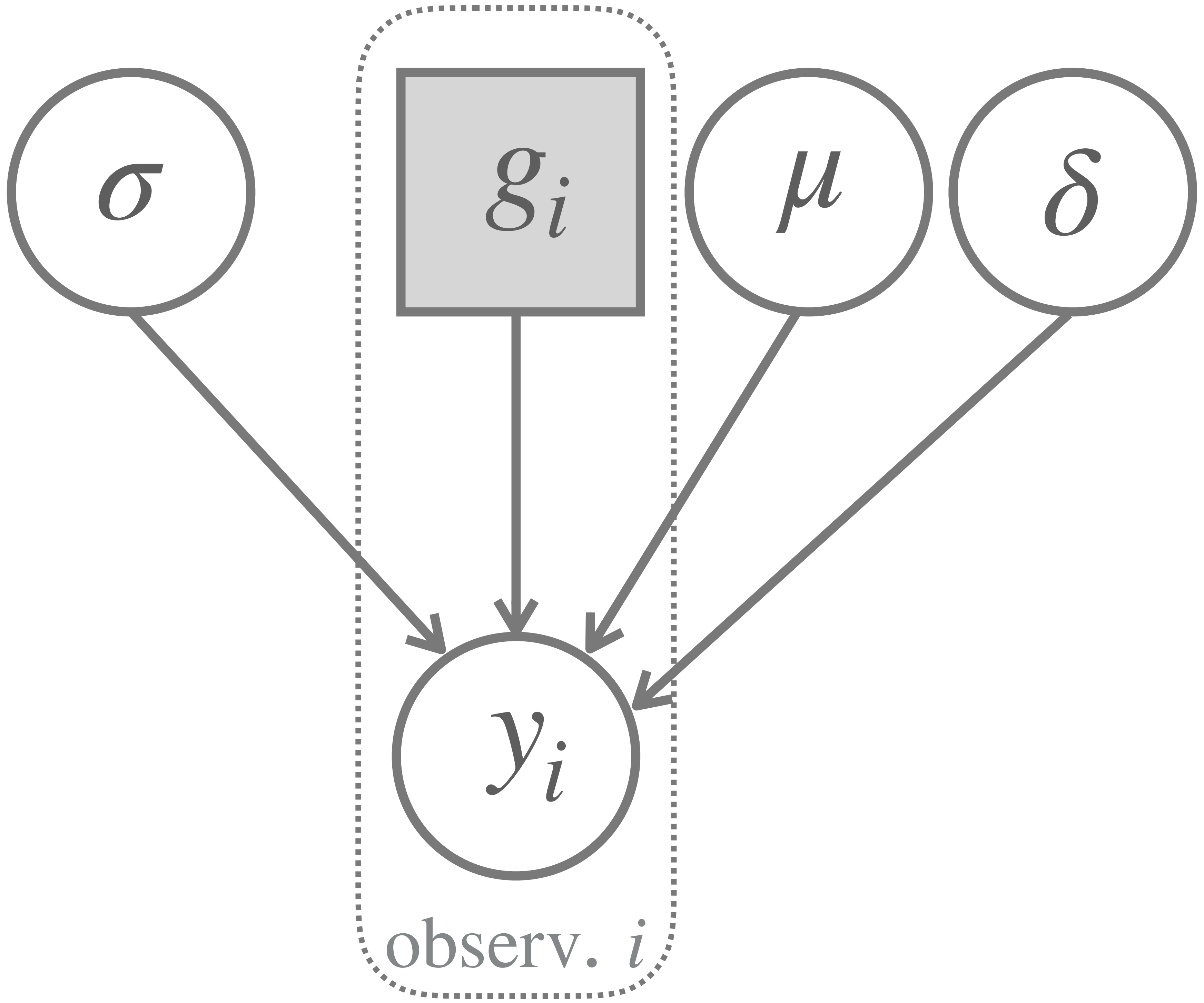


T-TEST MODEL [TWO UNCOUPLED MEANS]



$$\sigma \sim \text{Trunc-Norm}(\ldots, \text{lower} = 0)$$
$$\mu_0 \sim \text{Normal}(\ldots)$$
$$\mu_1 \sim \text{Normal}(\ldots)$$
$$y_i \sim \text{Normal}(\mu_{g_i}, \sigma)$$

T-TEST MODEL [WITH DIFFERENCE BETWEEN MEANS]



$$\sigma \sim \text{Trunc-Norm}(\ldots, \text{lower} = 0)$$

$$\mu \sim \text{Normal}(\ldots)$$

$$\delta \sim \text{Normal}(0, \ldots)$$

$$y_i \sim \begin{cases} \text{Normal}(\mu, \sigma) & \text{if } g_i = 0 \\ \text{Normal}(\mu + \delta, \sigma) & \text{if } g_i = 1 \end{cases}$$

HYPOTHESES & PARAMETER VALUES

- point-valued null hypothesis: $\delta = 0$
- observe data D
- three ways of testing [recall three pillars of DA]:
  - estimation: is 0 among the parameters estimated from D?
  - prediction: is D among the data predicted by a model with $\delta = 0$?
  - comparison: take two models: one with $\delta = 0$, one where δ takes on different values, too; which one explains D better?

Bayes rule for parameter estimation

BAYES RULE FOR PARAMETER ESTIMATION

$$\underbrace{P(\theta \mid D)}_{\text{posterior}} = \frac{\underbrace{P(D \mid \theta)}_{\text{likelihood}} \; \underbrace{P(\theta)}_{\text{prior}}}{\underbrace{P(D)}_{\text{marginal likelihood}}}$$

$$\underbrace{P(D)}_{\text{marginal likelihood}} = \int P(D \mid \theta) \; P(\theta) \; \mathrm{d}\theta$$

REMARKS ON NOTATION

- if there is only one model M, we leave out the model index, writing $P(\theta)$ instead of $P_M(\theta)$
- we write $P(\theta \mid D)$ instead of $P(\Theta = \theta \mid \mathcal{D} = D)$
- short-hand with non-normalized probabilities (implicit normalizing constant):

$$\underbrace{P(\theta \mid D)}_{posterior} \propto \underbrace{P(\theta)}_{prior} \ \underbrace{P(D \mid \theta)}_{likelihood}$$

EXAMPLE

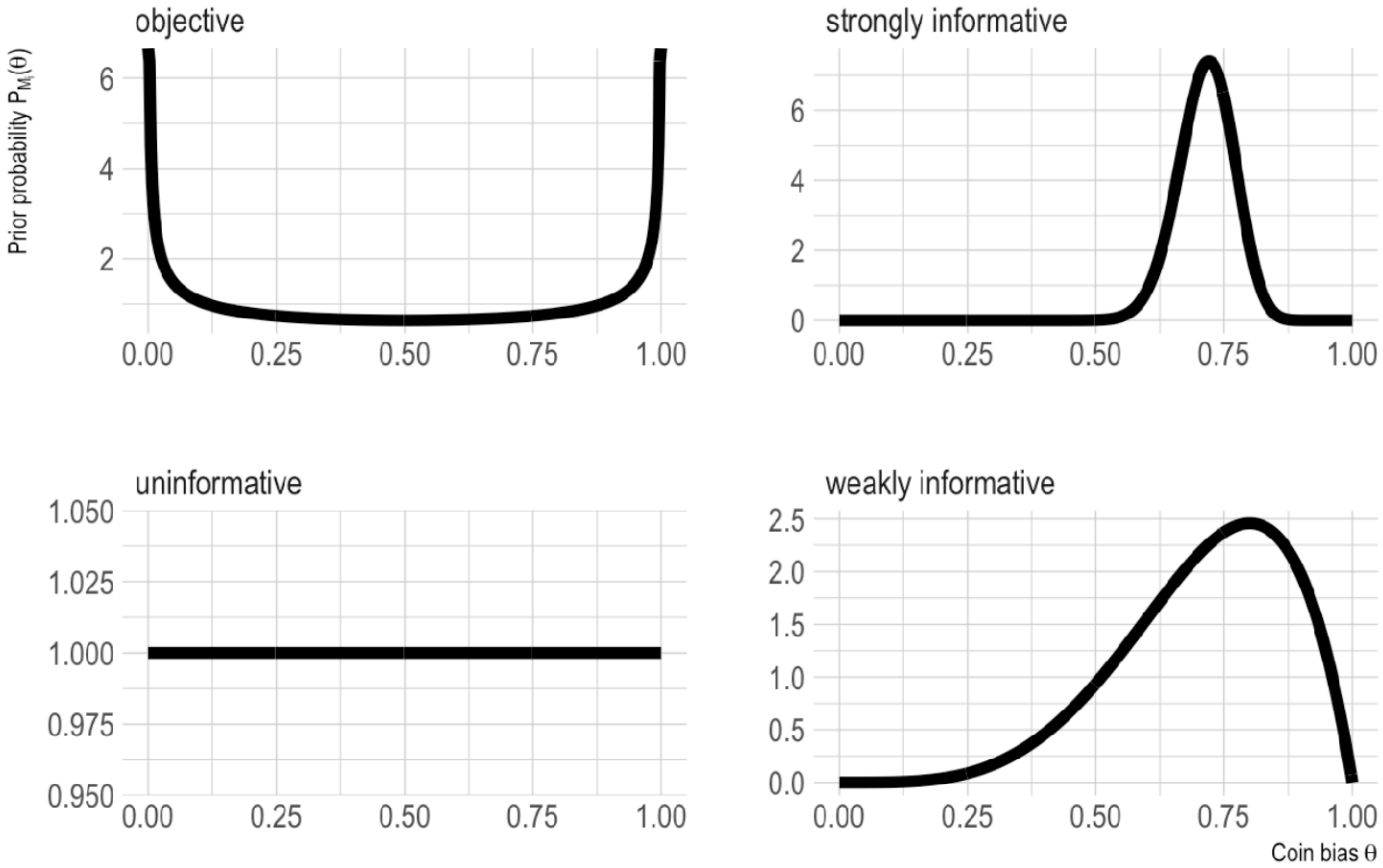
- model:

$$k \sim \text{Binomial}(N, \theta)$$
$$\theta \sim \text{Beta}(\alpha, \beta)$$

- data:
  - "24/7" $k = 7$ $N = 24$
  - "KoF" $k = 109$ $N = 311$

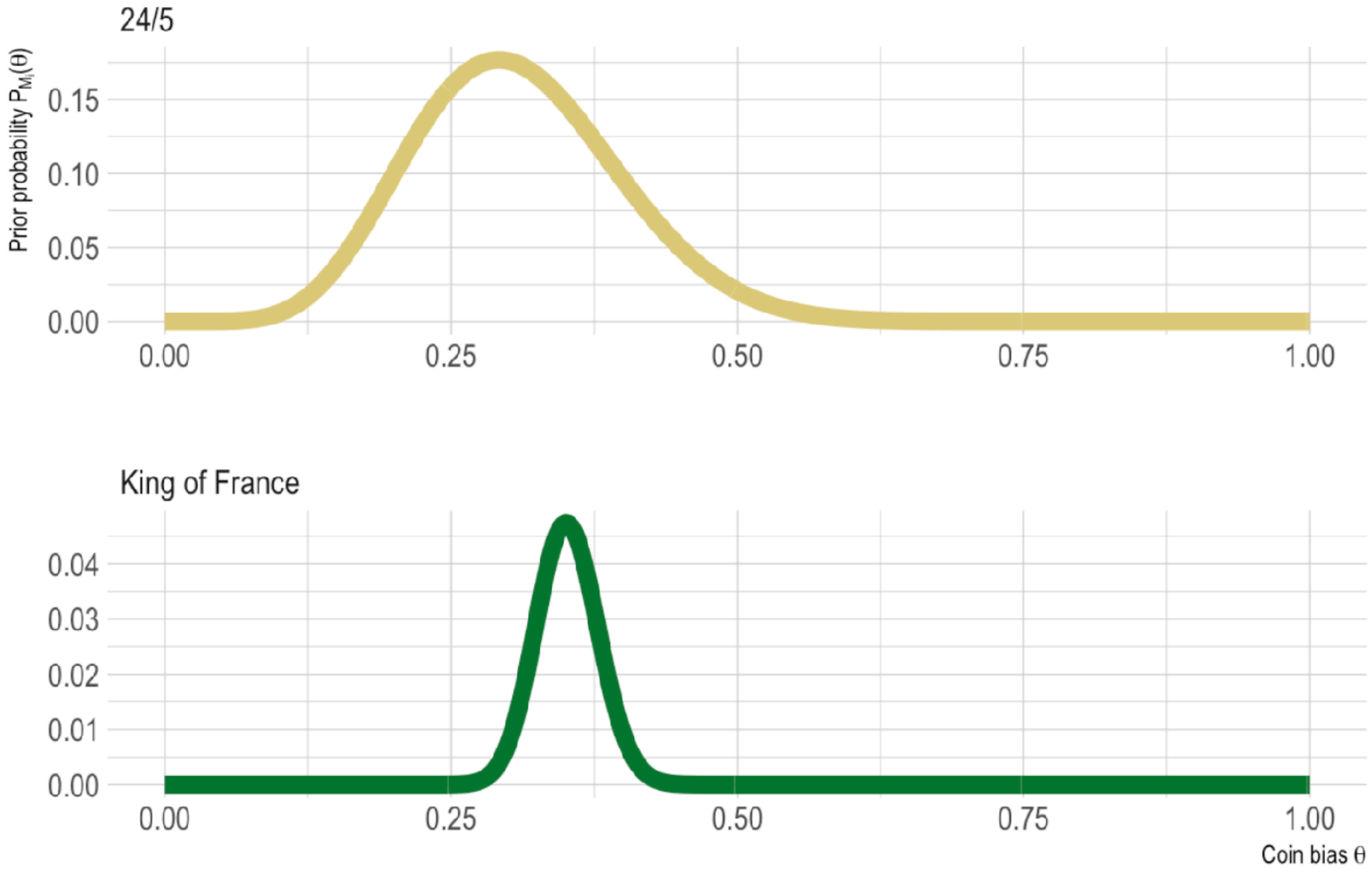
  [number of "true" responses to all sentences with a false presupposition]

PRIOR



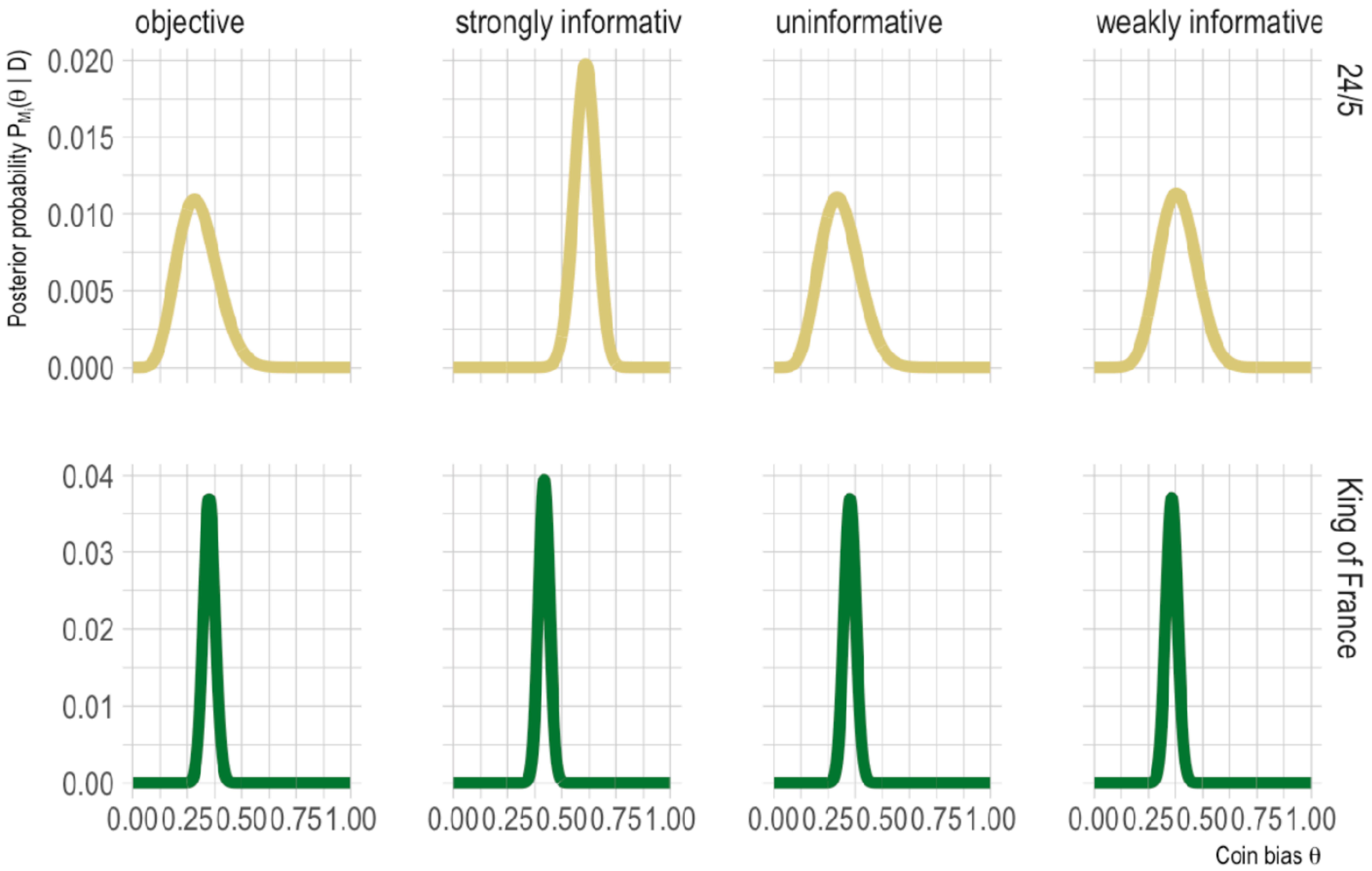
objective
strongly informative
uninformative
weakly informative
Prior probability $P_{M_i}(\theta)$
Coin bias θ
0.00
0.25
0.50
0.75
1.00
2
4
6
0
1.050
1.025
1.000
0.975
0.950
2.5
2.0
1.5
1.0
0.5
0.0

LIKELIHOOD



24/5
Prior probability $P_{M_i}(\theta)$
0.15
0.10
0.05
0.00
0.00
0.25
0.50
0.75
1.00
King of France
0.04
0.03
0.02
0.01
0.00
0.00
0.25
0.50
0.75
1.00
Coin bias θ

POSTERIOR

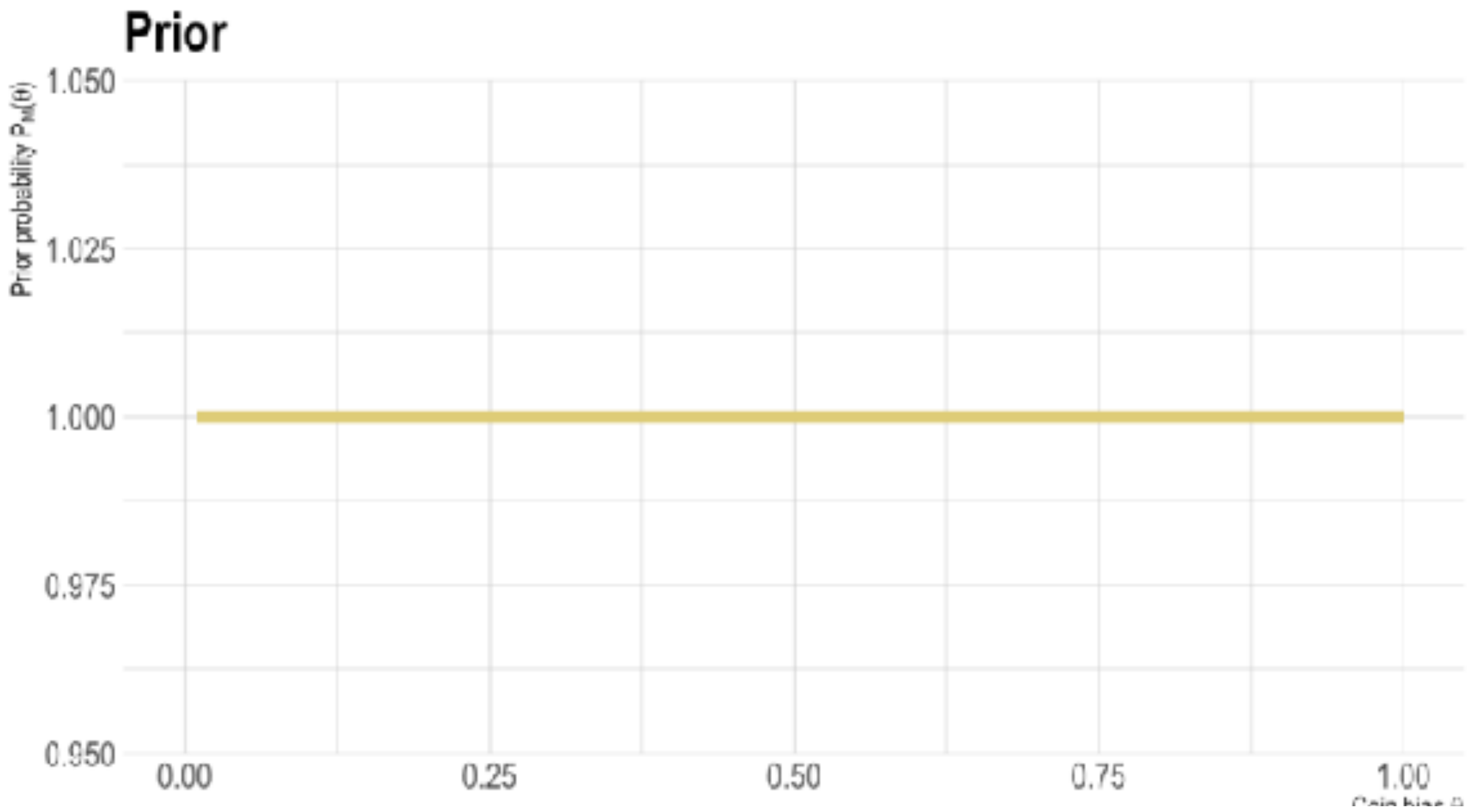


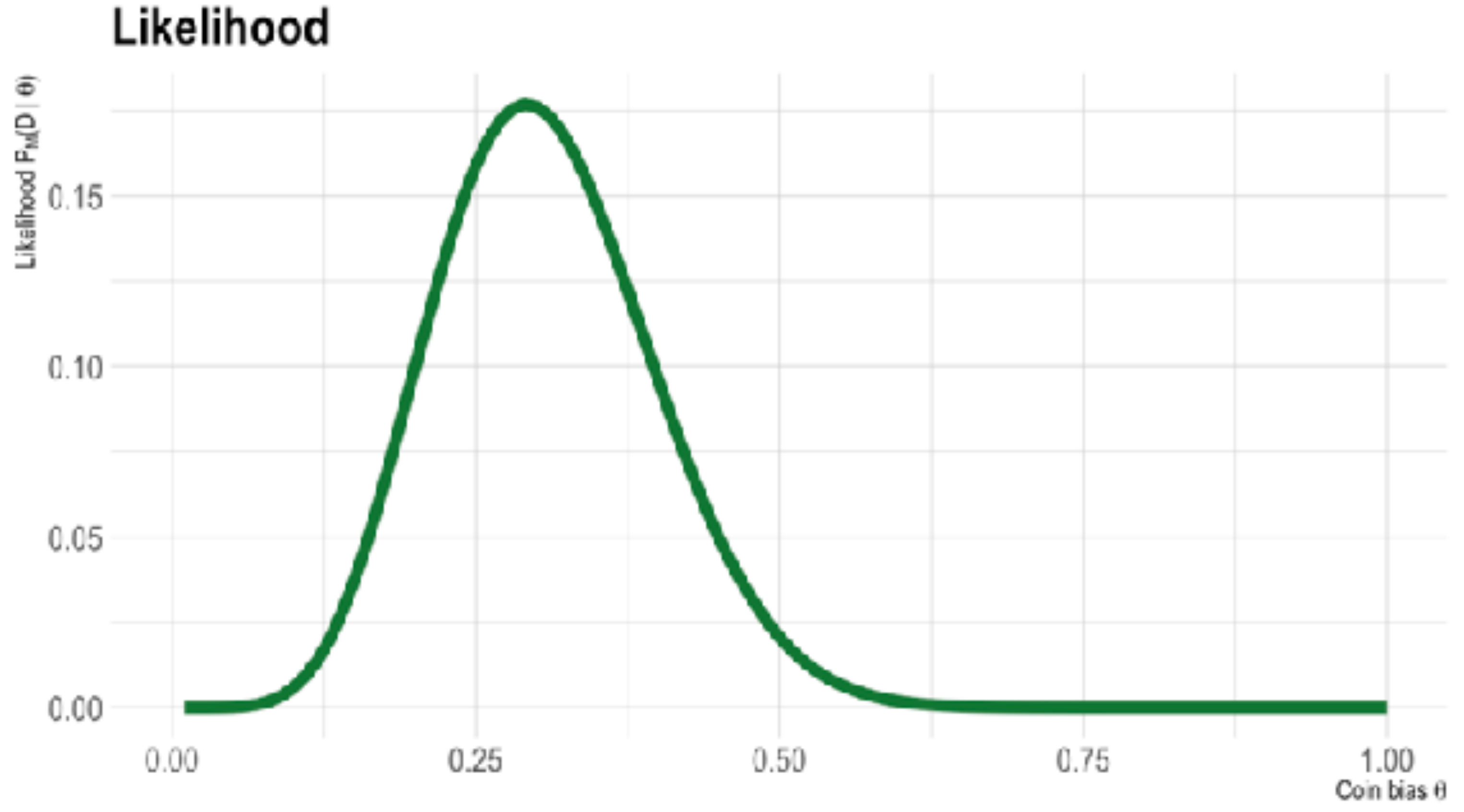
objective
strongly informativ
uninformative
weakly informative
24/5
King of France
Posterior probability $P_{M_i}(\theta \mid D)$
0.020
0.015
0.010
0.005
0.000
0.04
0.03
0.02
0.01
0.00
0.00 0.25 0.50 0.75 1.00
Coin bias θ

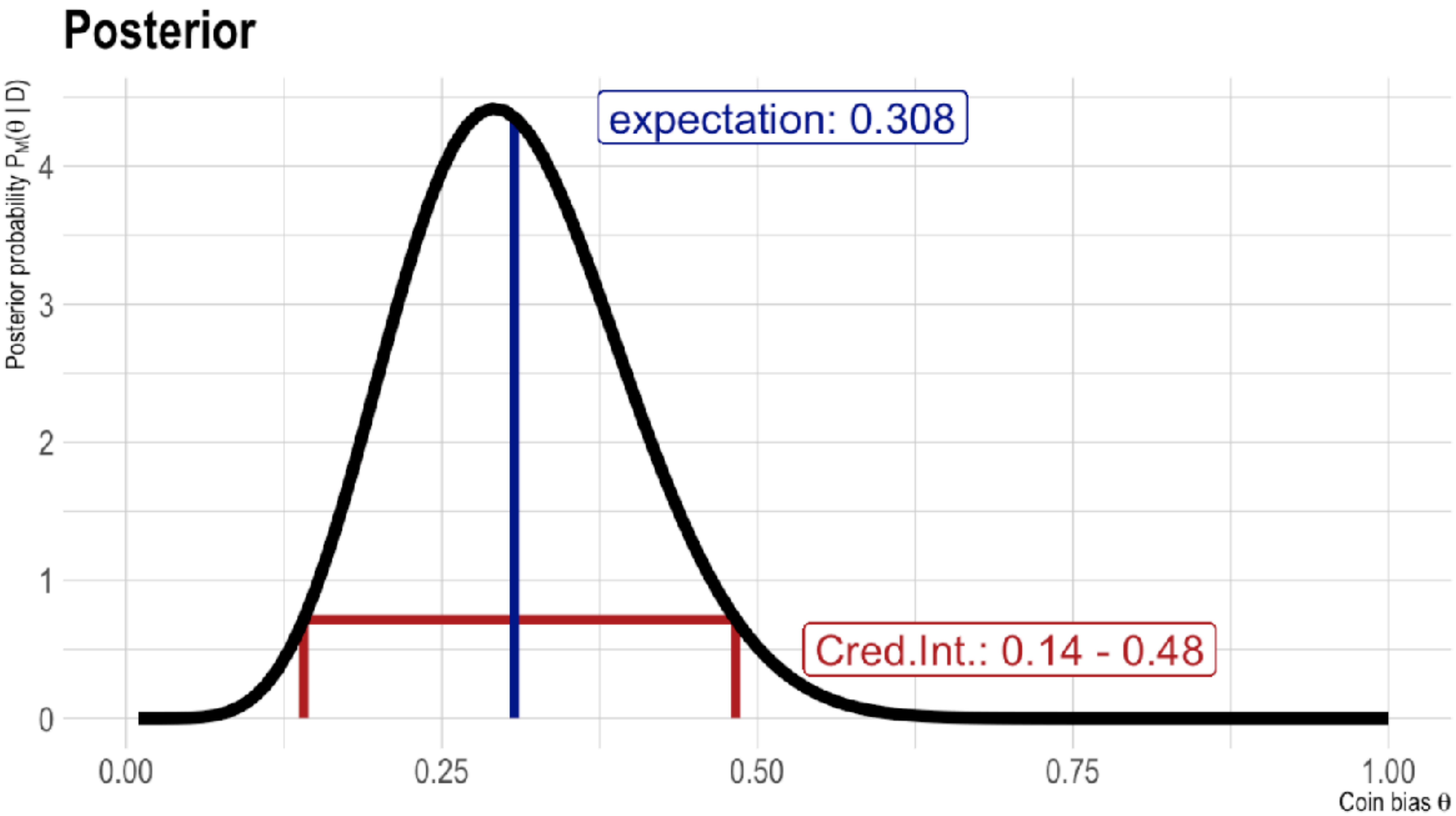
Bayesian point- & interval-estimates

EXAMPLE

- model: $k \sim \text{Binomial}(N, \theta), \theta \sim \text{Beta}(1,1)$
- data: $k = 7,\ N = 24$







POSTERIOR MEAN & MAP

- posterior mean:

$$\mathbb{E}_{P(\theta|D)} = \int \theta \; P(\theta \mid D) \; \mathrm{d}\theta$$

- maximum a posteriori:

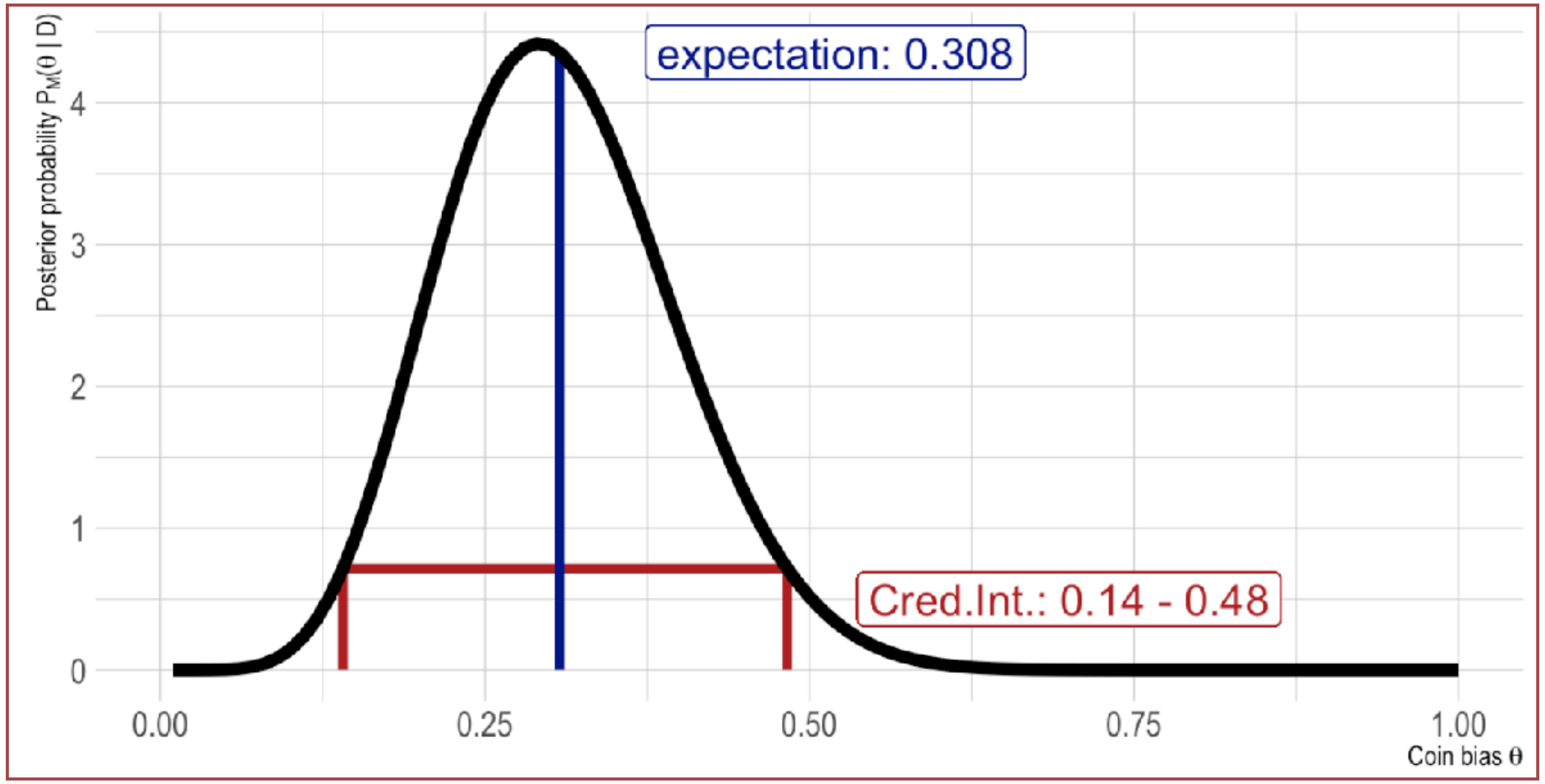
$$\text{MAP}(P(\theta \mid D)) = \arg\max_{\theta} P(\theta \mid D)$$

- posterior mean is proper Bayesian measure, because it is holistic = influenced by whole distribution
- MAP is local, not influenced by whole distribution
- estimation of posterior mean is (usually) less error-prone than estimation of MAP

CREDIBLE INTERVAL

- interval $[l; u]$ is a $\gamma\,\%$ credible interval for a random variable X if
  - (I) $P(l \leq X \leq u) = \dfrac{\gamma}{100}$, and
  - (II) for every $x \in [l; u]$ and $x' \notin [l; u]$ we have $P(X = x) > P(X = x')$
- "range of values too probable to properly ignore"

[see David Lewis on "Elusive Knowledge"]



posteriors from conjugacy

BAYES RULE FOR PARAMETER ESTIMATION

$$P(\theta \mid D) = \frac{P(D \mid \theta)\, P(\theta)}{\int P(D \mid \theta)\, P(\theta)\, \mathrm{d}\theta}$$

✓fast & easy

✓fast & easy

❌☠ possibly intractable ☠❌

CONJUGACY

- prior $P(\theta)$ is a conjugate prior for likelihood $P(D \mid \theta)$ iff prior $P(\theta)$ and posterior $P(\theta \mid D)$ are of the same kind of probability distribution (possibly with different parameter values)
- e.g., prior and posterior are both normal distributions, but have different means and standard deviations

CONJUGACY OF BETA & BINOMIAL

- claim: beta & binomial are conjugate
- proof:

$$P(\theta \mid k, N) \propto \text{Binomial}(k; N, \theta)\ \text{Beta}(\theta \mid a, b)$$

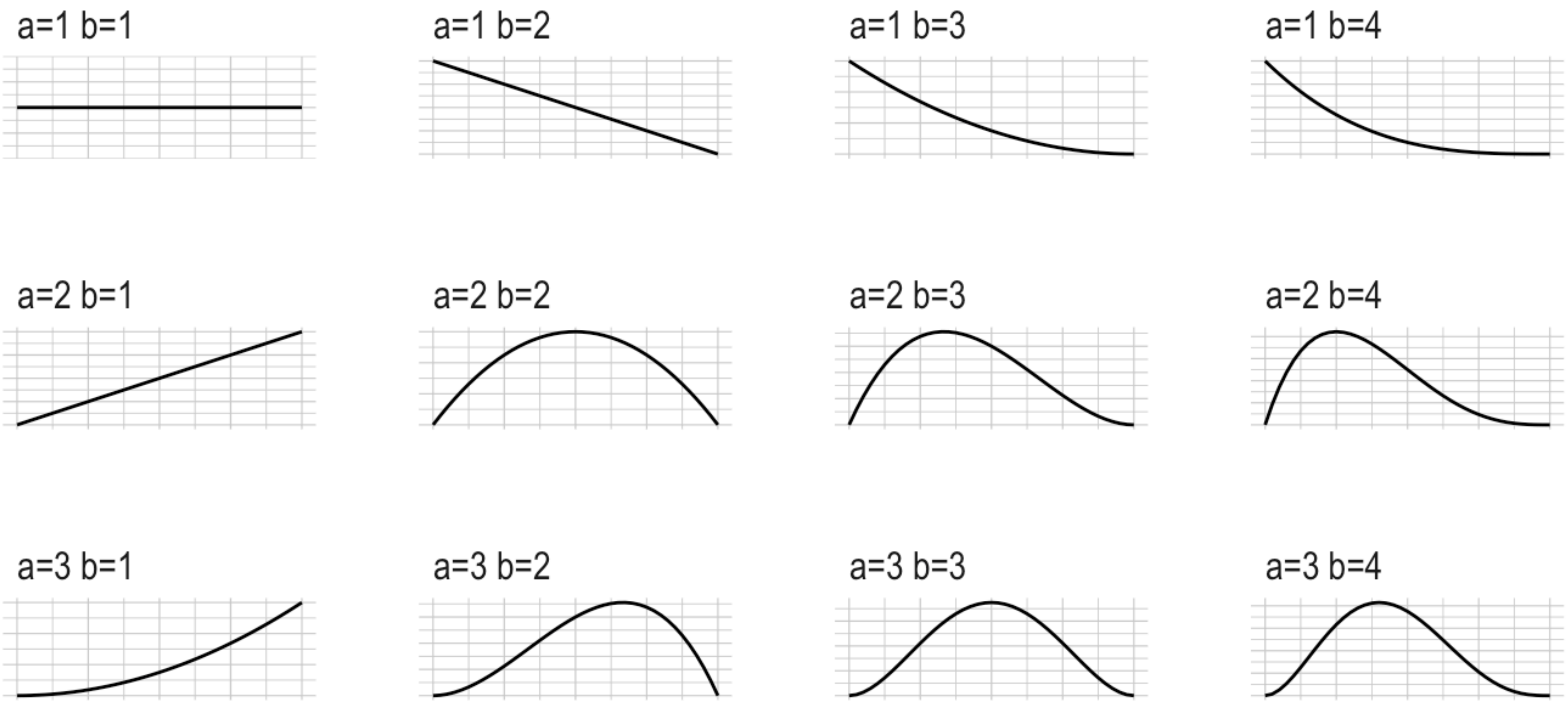
$$P(\theta \mid k, N) \propto \theta^{k}\,(1-\theta)^{N-k}\,\theta^{a-1}\,(1-\theta)^{b-1}$$

$$P(\theta \mid k, N) \propto \theta^{k+a-1}\,(1-\theta)^{N-k+b-1}$$

$$P(\theta \mid k, N) = \text{Beta}(\theta \mid k+a, N-k+b)$$

sequential updating

SEQUENTIAL UPDATING IN THE BETA-BINOMIAL MODEL



SEQUENTIAL UPDATING IN GENERAL

- claim: if D_1 and D_2 are disjoint and $D_1 \cup D_2 = D$, $P(\theta \mid D) \propto P(\theta \mid D_1)\ P(D_2 \mid \theta)$
- proof: $P(\theta \mid D) = \dfrac{P(\theta)\ P(D \mid \theta)}{\int P(\theta')\ P(D \mid \theta') \mathrm{d}\theta'}$

$$= \frac{P(\theta)\ P(D_1 \mid \theta)\ P(D_2 \mid \theta)}{\int P(\theta')\ P(D_1 \mid \theta')\ P(D_2 \mid \theta') \mathrm{d}\theta'}$$

[from multiplicativity of likelihood]

$$= \frac{P(\theta)\ P(D_1 \mid \theta)\ P(D_2 \mid \theta)}{\frac{k}{k} \int P(\theta')\ P(D_1 \mid \theta')\ P(D_2 \mid \theta') \mathrm{d}\theta'}$$

[for random positive k]

$$= \frac{\frac{P(\theta)\ P(D_1 \mid \theta)}{k}\ P(D_2 \mid \theta)}{\int \frac{P(\theta')\ P(D_1 \mid \theta')}{k}\ P(D_2 \mid \theta') \mathrm{d}\theta'}$$

[rules of integration; basic calculus]

$$= \frac{P(\theta \mid D_1)\ P(D_2 \mid \theta)}{\int P(\theta' \mid D_1)\ P(D_2 \mid \theta') \mathrm{d}\theta'}$$

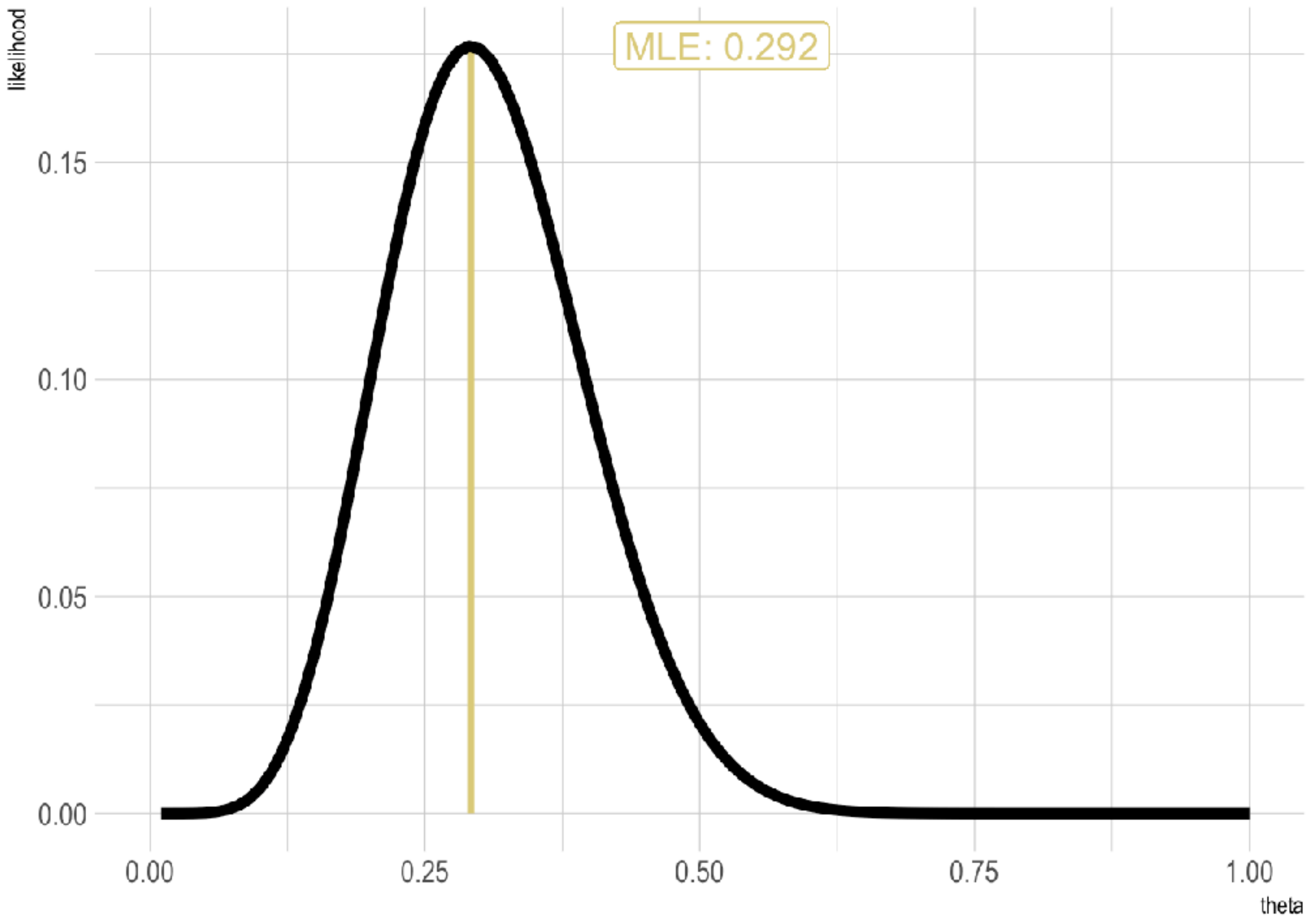
[Bayes rule with $k = \int P(\theta) P(D_1 \mid \theta) \mathrm{d}\theta$]

frequentist estimation

MAXIMUM LIKELIHOOD ESTIMATE

- maximum likelihood estimate:

$$\hat{\theta} = \arg\max_{\theta} P(d \mid \theta)$$



CONFIDENCE INTERVAL [MATHEMATICALLY]

- let $\mathscr{D}$ be the random variable describing the probability of data
- X_l and X_u are random variables derived from $\mathscr{D}$ via functions g_l and g_u so that $g_{l,u} \colon D \mapsto \mathbb{R}$
- a $\gamma\,\%$ confidence interval for observed data D_{obs} is the interval:

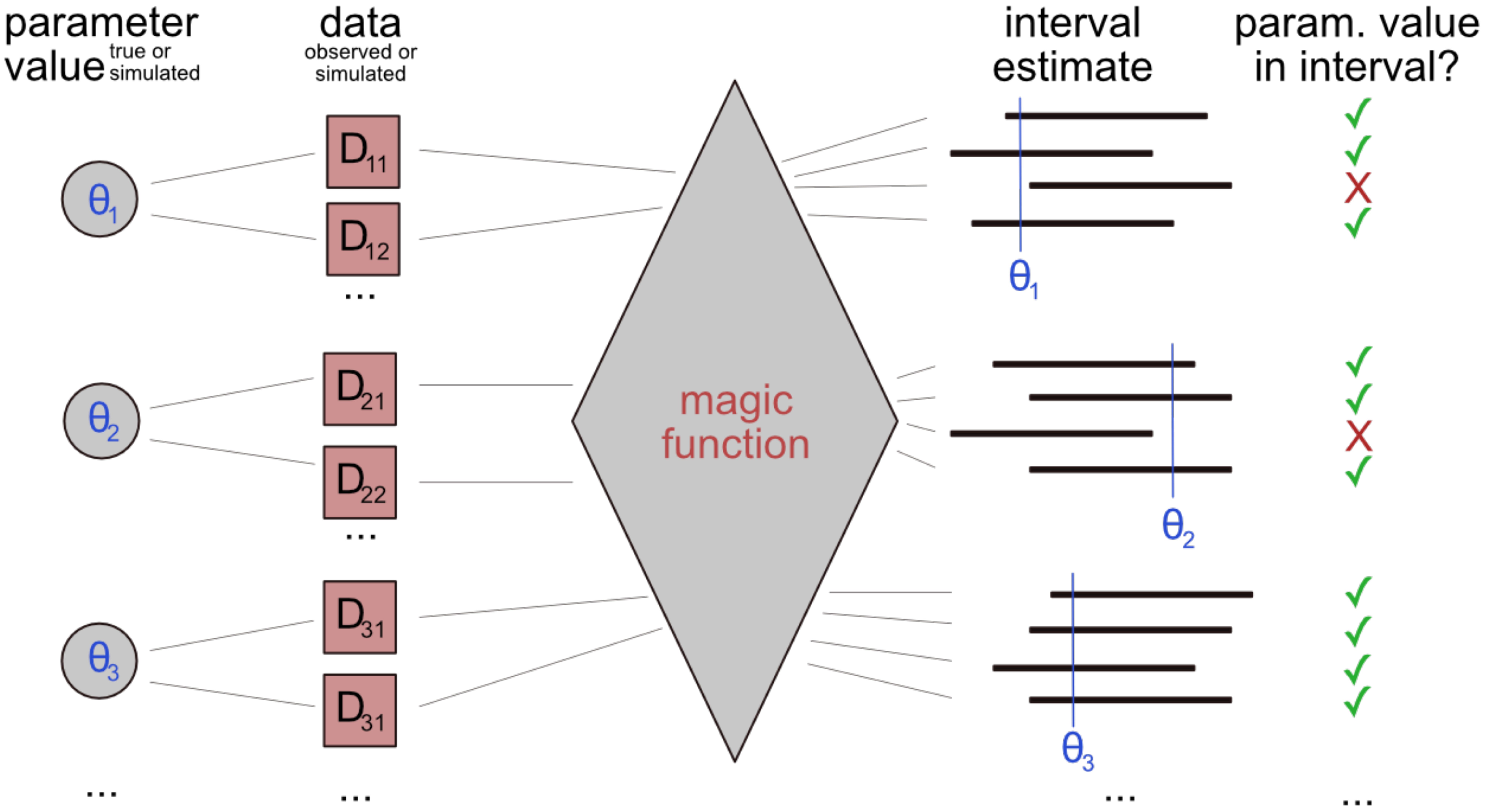
$$\left[g_l(D_{\text{obs}}), g_u(D_{\text{obs}})\right]$$

- where functions $g_{l,u}$ are constructed so that:

$$P(X_l \leq \theta_{\text{true}} \leq X_u) = \frac{\gamma}{100}$$

- and where θ_{true} is the true value

CONFIDENCE INTERVAL [ALGORITHMICALLY]



CONFIDENCE INTERVAL [ALGORITHMICALLY]

- fix number of coin flips N (not really necessary, but easier)
- suppose the true coin bias is θ_{true} (but we don't know it)
- we have a magic function $MF: k \mapsto [u_k; l_k]$
- we now sample repeatedly $k \sim \text{Binomial}(N, \theta_{\text{true}})$
- for each sample k, compute $MF(k) = [u_k; l_k]$
- MF gives us a $\gamma\,\%$ confidence interval if θ_{true} is inside of $MF(k) = [u_k; l_k]$ in $\gamma\,\%$ of the sampled ks

addressing point-valued hypotheses with estimation

ADDRESSING POINT-VALUED HYPOTHESES [FREQUENTIST]

- $\Theta_i = \theta_i^*$ is out point-valued hypothesis
- we do not consider a ROPE
- for a frequentist credible interval $[l; u]$ for Θ_i, we:
  - reject the point-valued hypothesis iff $\theta_i^* \notin [l; u]$; and
  - withhold judgement otherwise.

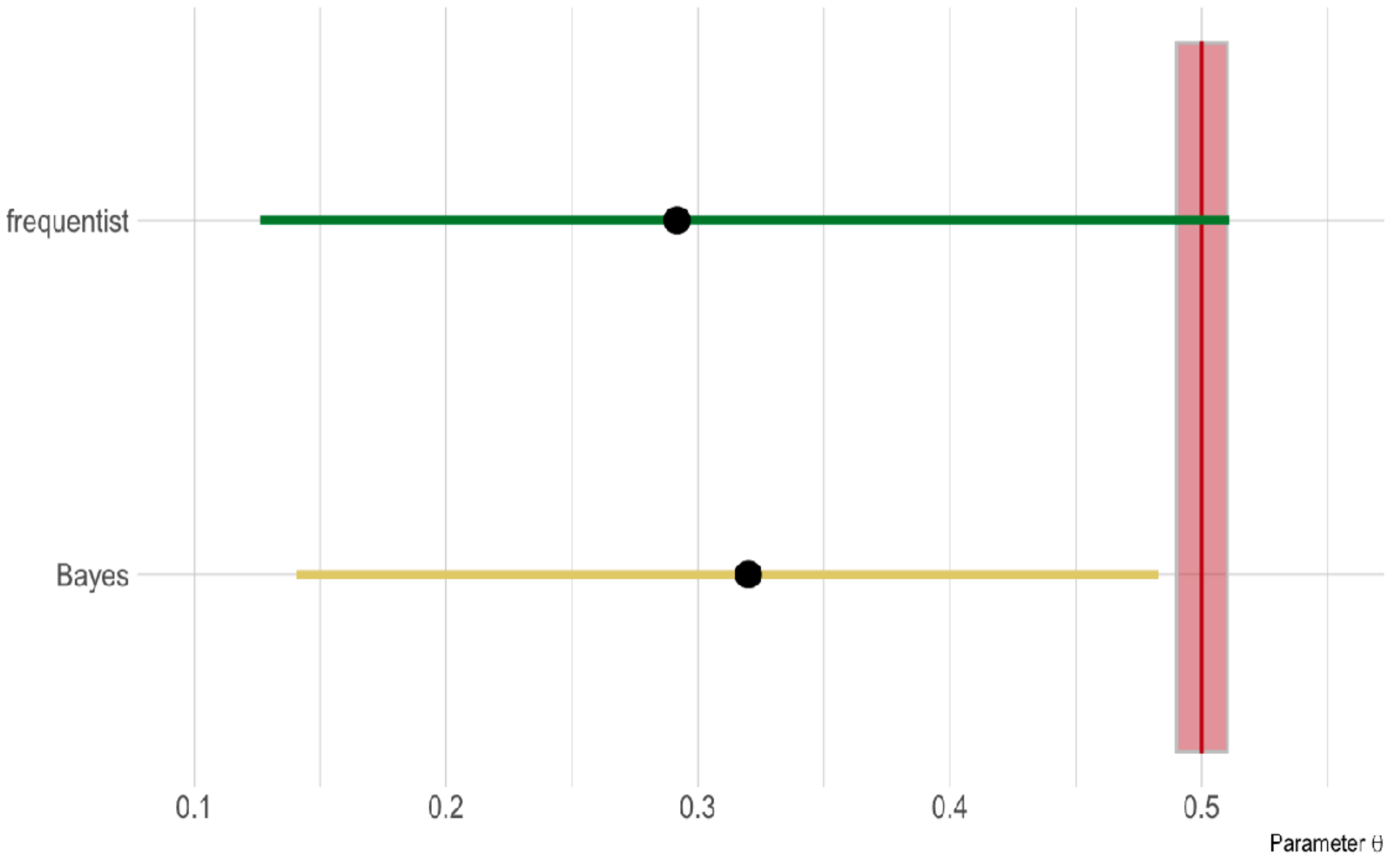
ADDRESSING POINT-VALUED HYPOTHESES [BAYES]

- $\Theta_i = \theta_i^*$ is out point-valued hypothesis
- a region of practical equivalence [ROPE] is an ϵ-region around θ_i^*:
  $\mathrm{ROPE}(\theta_i^*) = [\theta_i^* - \epsilon, \theta_i^* + \epsilon]$
- for a Bayesian credible interval $[l; u]$ for Θ_i, we:
  - accept the point-valued hypothesis iff $[l; u]$ is contained entirely in $\mathrm{ROPE}(\theta_i^*)$;
  - reject the point-valued hypothesis iff $[l; u]$ and $\mathrm{ROPE}(\theta_i^*)$ have no overlap;
  - withhold judgement otherwise.

EXAMPLE

- 24/7 example, uninformative priors for Bayesian model
- point- and interval estimates:

```
## # A tibble: 2 x 4
##   approach    lower point upper
##   <chr>       <dbl> <dbl> <dbl>
## 1 Bayes       0.141 0.32  0.483
## 2 frequentist 0.126 0.292 0.511
```



comparison

BAYESIAN VS FREQUENTIST ESTIMATES

- for Bayesianism the full posterior is the primary object of concern; point- and interval-estimates are essentially just summary statistics for the full posterior
- for frequentists the point- and interval-estimates are the primary object of concern
- MLEs are much easier to compute but might not exist
- posteriors can be very hard to compute (long run time)

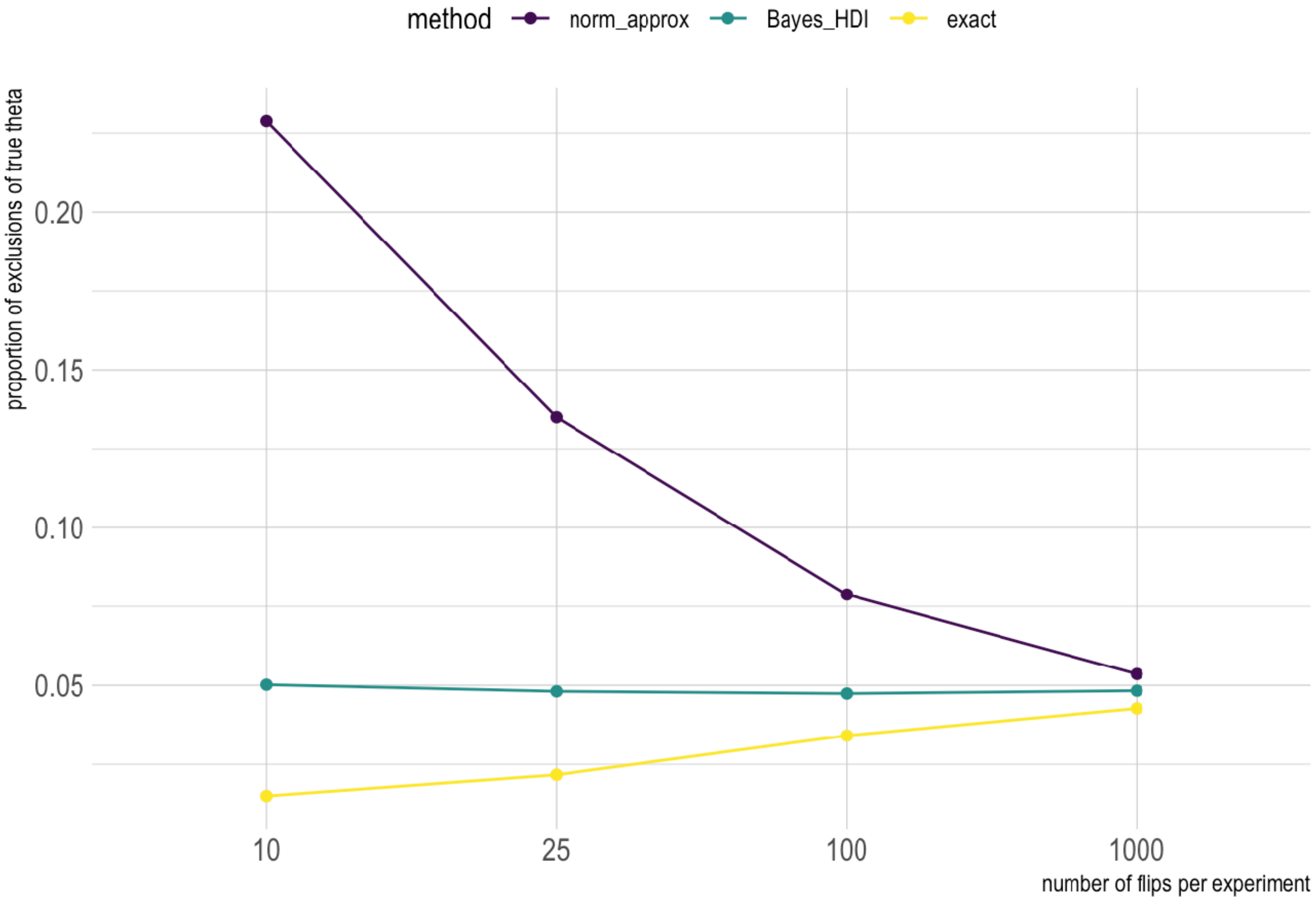
A PUZZLE ABOUT POINT-ESTIMATES

- flip a coin of unknown bias once
- suppose you see heads
- what's your best estimate of the bias?
  - MLE = 1
  - posterior mean (uninformative priors) = $^2/_3$

SIMULATION-BASED COMPARISON OF INTERVAL-ESTIMATES

- fix $N \in \{10{,}25{,}100{,}1000\}$
- repeatedly do:
  - sample $\theta_{\text{true}} \sim \text{Beta}(1{,}1)$
  - sample $k \sim \text{Binomial}(\theta_{\text{true}}, N)$
  - compute intervals for k and N
    - HDI, exact CI, approximate CI
- look at percentage that θ_{true} is included in each interval construction

RESULTS


method
norm_approx
Bayes_HDI
exact
proportion of exclusions of true theta
0.20
0.15
0.10
0.05
10
25
100
1000
number of flips per experiment

computing estimates

OPTIMIZING FUNCTIONS

```
# function for the negative log-likelihood of the given
# data and fixed parameter values
nll = function(y, x, beta_0, beta_1, sd) {
  # negative sigma is logically impossible
  if (sd <= 0) {return( Inf )}
  # predicted values
  yPred = beta_0 + x * beta_1
  # negative log-likelihood of each data point
  nll = -dnorm(y, mean=yPred, sd=sd, log = T)
  # sum over all observations
  sum(nll)
}
```

```
fit_lh = optim(
  # initial parameter values
  par = c(1.5, 0, 0.5),
  # function to optimize
  fn = function(par) {
    with(avocado_data,
         nll(average_price, total_volume_sold,
             par[1], par[2], par[3])
    )
  }
)
fit_lh$par
```

```
## [1]  1.425080e+00 -2.247373e-08  3.950978e-01
```

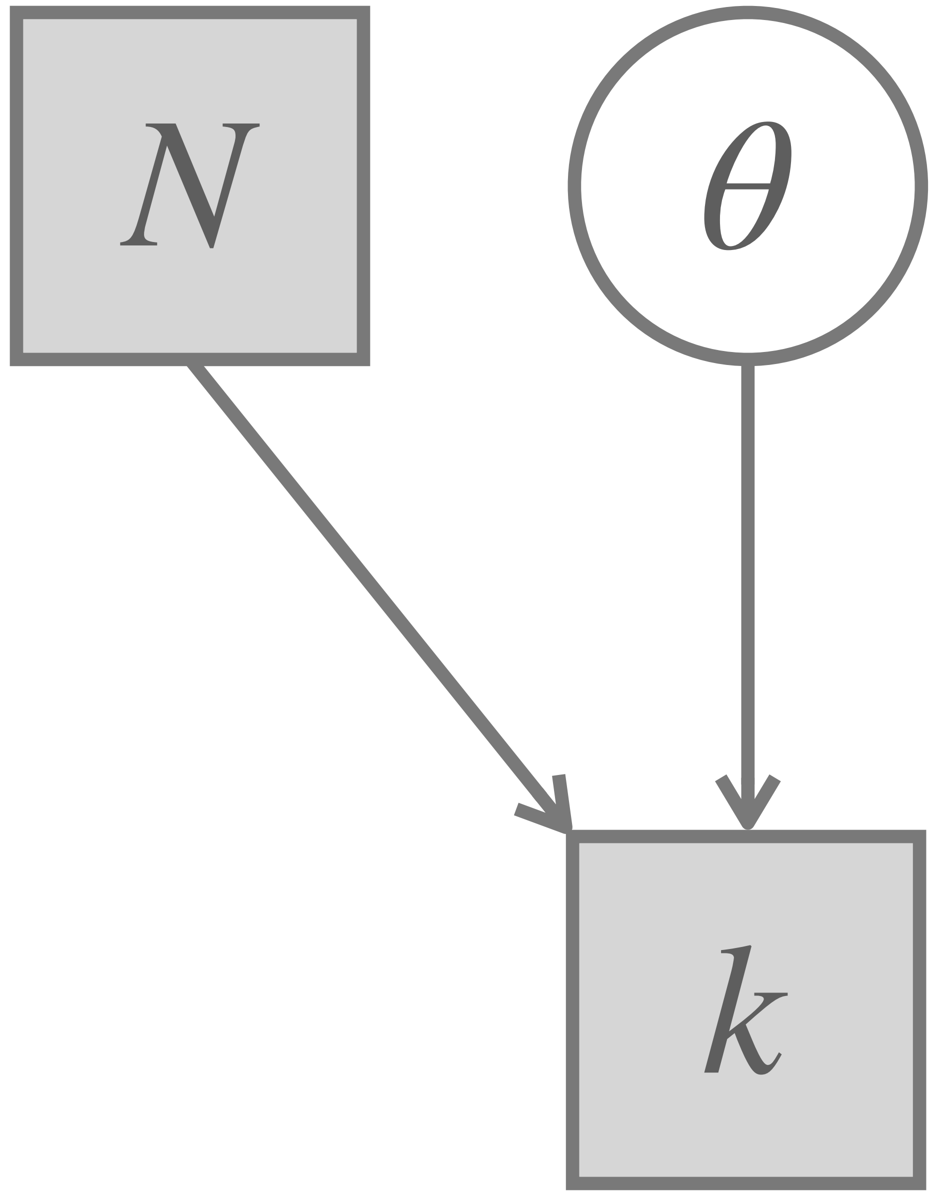
```
lm(average_price ~ total_volume_sold, avocado_data)$coef
```

```
##       (Intercept) total_volume_sold
##      1.425096e+00     -2.247455e-08
```

MARKOV CHAIN MONTE CARLO

probabilistic models with greta

BINOMIAL MODEL



$$\theta \sim \text{Beta}(1,1)$$
$$k \sim \text{Binomial}(\theta, N)$$

```
# greta data
k <- as_data(109)
N <- as_data(311)
```

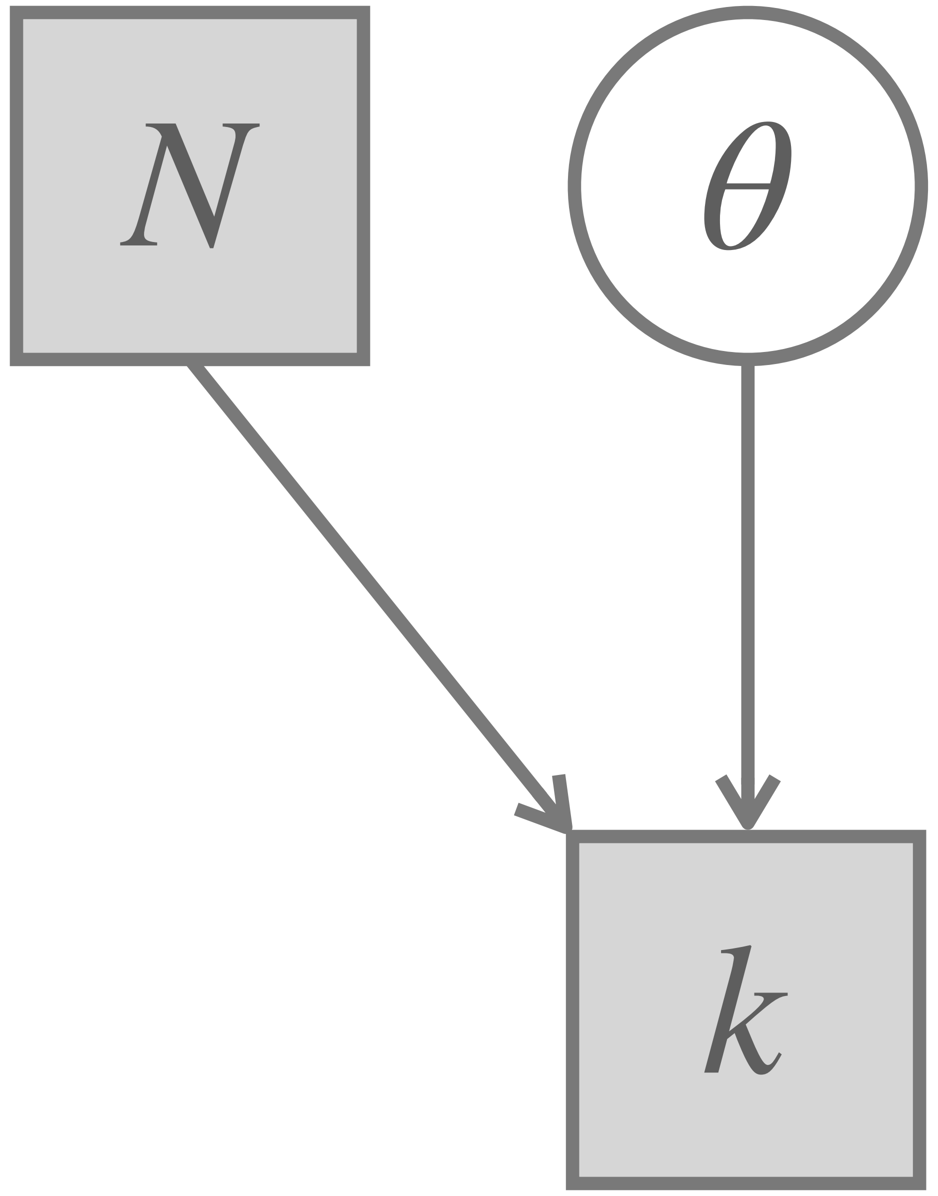
```
# coin bias & prior (here: uninformative)
theta    <- beta(1,1)
```

```
# likelihood of data given theta
distribution(k) <- binomial(N, theta)
```

```
# declare the greta model
m <- model(theta)
```

```
# take 4 chains of 1000 samples
draws <- greta::mcmc(
  model = m,
  n_samples = 1000,
  warmup = 1000,
  chains = 4
)
```

BINOMIAL MODEL

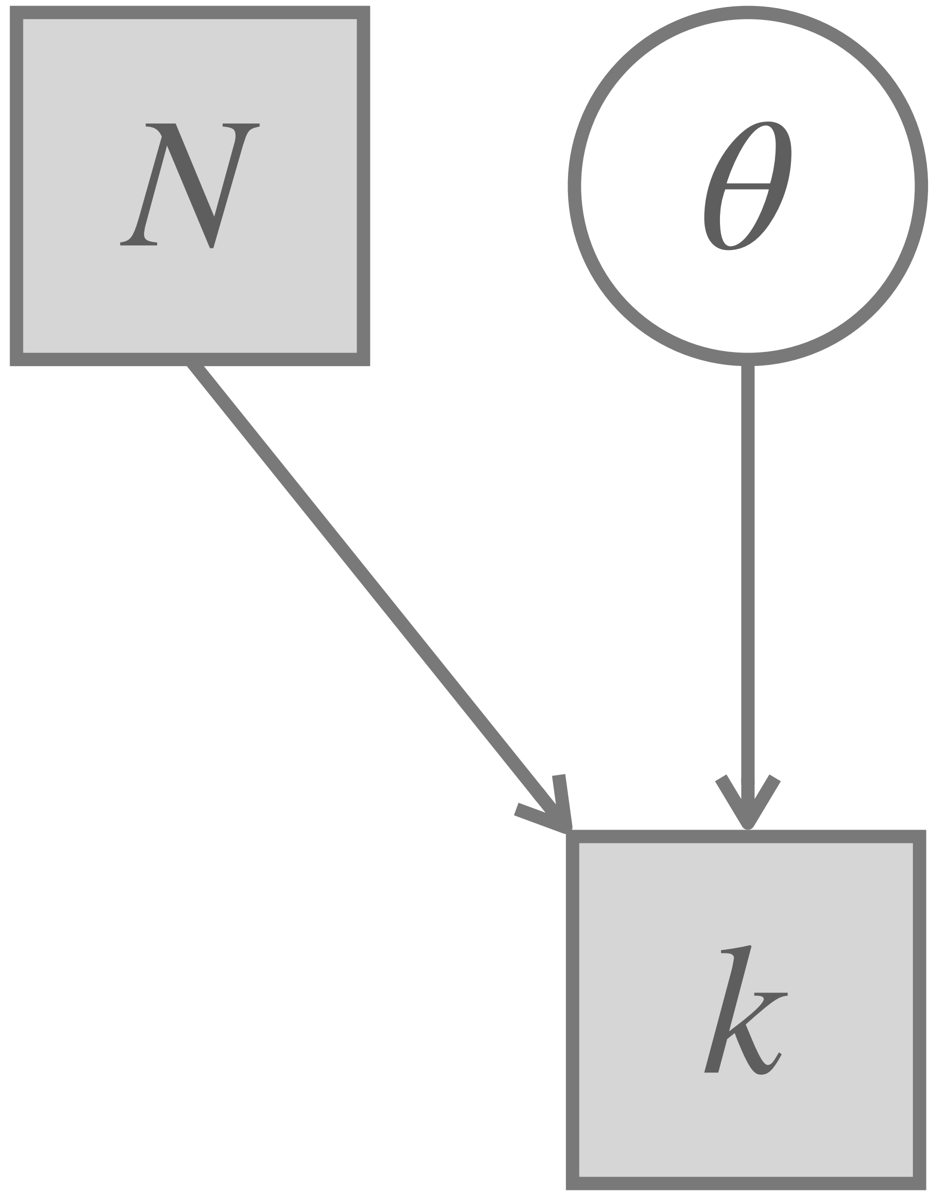


$$\theta \sim \text{Beta}(1,1)$$
$$k \sim \text{Binomial}(\theta, N)$$

```r
# cast results (type 'mcmc.list') into tidy tibble
tidy_draws = ggmcmc::ggs(draws)
tidy_draws
```

```
## # A tibble: 4,000 x 4
##    Iteration Chain Parameter value
##        <int> <int> <fct>     <dbl>
##  1         1     1 theta     0.343
##  2         2     1 theta     0.323
##  3         3     1 theta     0.352
##  4         4     1 theta     0.356
##  5         5     1 theta     0.356
##  6         6     1 theta     0.398
##  7         7     1 theta     0.398
##  8         8     1 theta     0.346
##  9         9     1 theta     0.405
## 10        10     1 theta     0.308
## # ... with 3,990 more rows
```

BINOMIAL MODEL



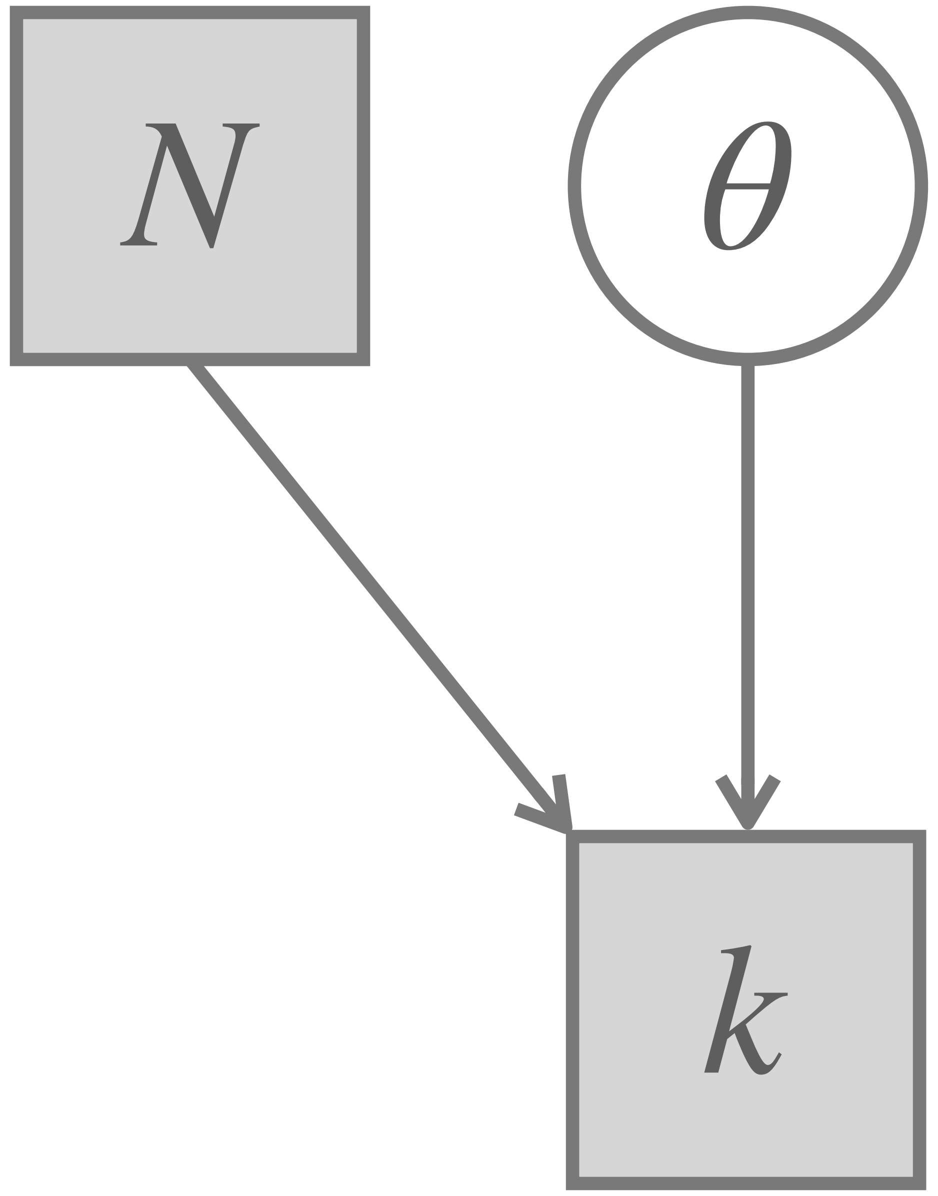
$$\theta \sim \text{Beta}(1,1)$$

$$k \sim \text{Binomial}(\theta, N)$$

```r
# obtain Bayesian point and interval estimates
Bayes_estimates <- tidy_draws %>%
  group_by(Parameter) %>%
  summarise(
    '|95%' = HDInterval::hdi(value)[1],
    mean = mean(value),
    '95|%' = HDInterval::hdi(value)[2]
  )
Bayes_estimates
```

```
## # A tibble: 1 x 4
##   Parameter `|95%`  mean `95|%`
##   <fct>      <dbl> <dbl>  <dbl>
## 1 theta      0.300 0.350  0.403
```

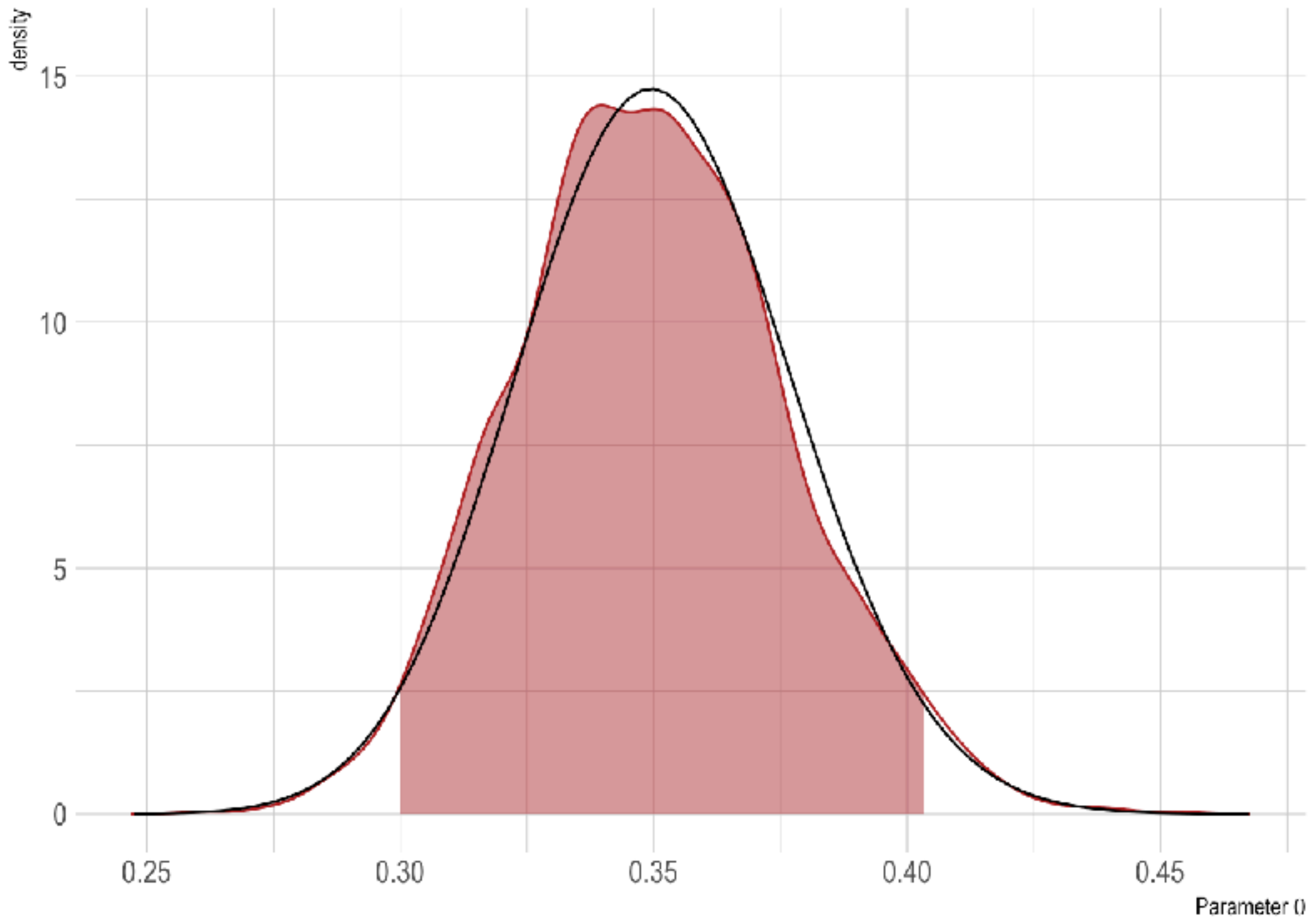
BINOMIAL MODEL



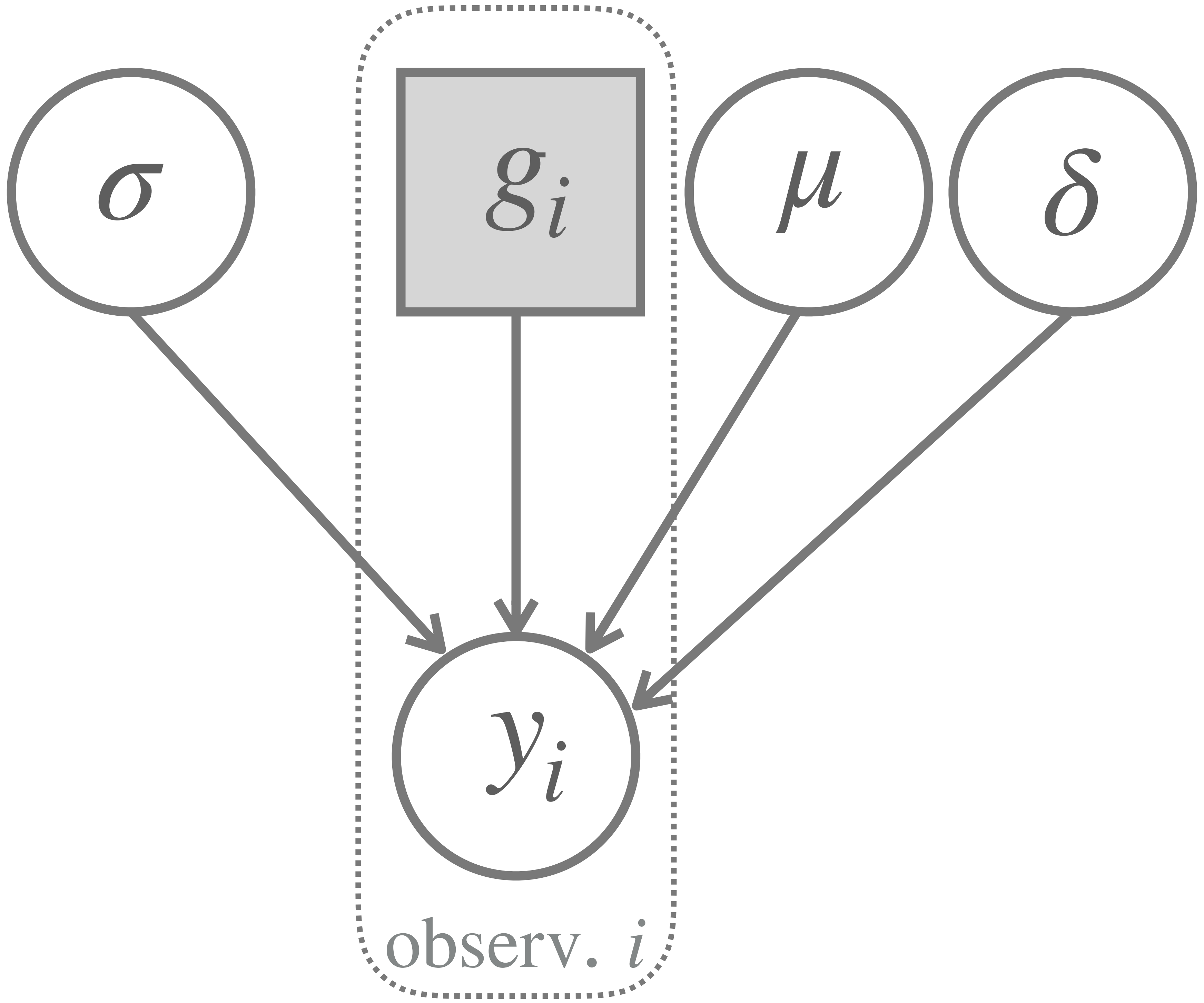
$$\theta \sim \text{Beta}(1,1)$$

$$k \sim \text{Binomial}(\theta, N)$$

```
## # A tibble: 1 x 4
##   Parameter `|95%`  mean `95|%`
##   <fct>       <dbl> <dbl>  <dbl>
## 1 theta       0.300 0.350  0.403
```



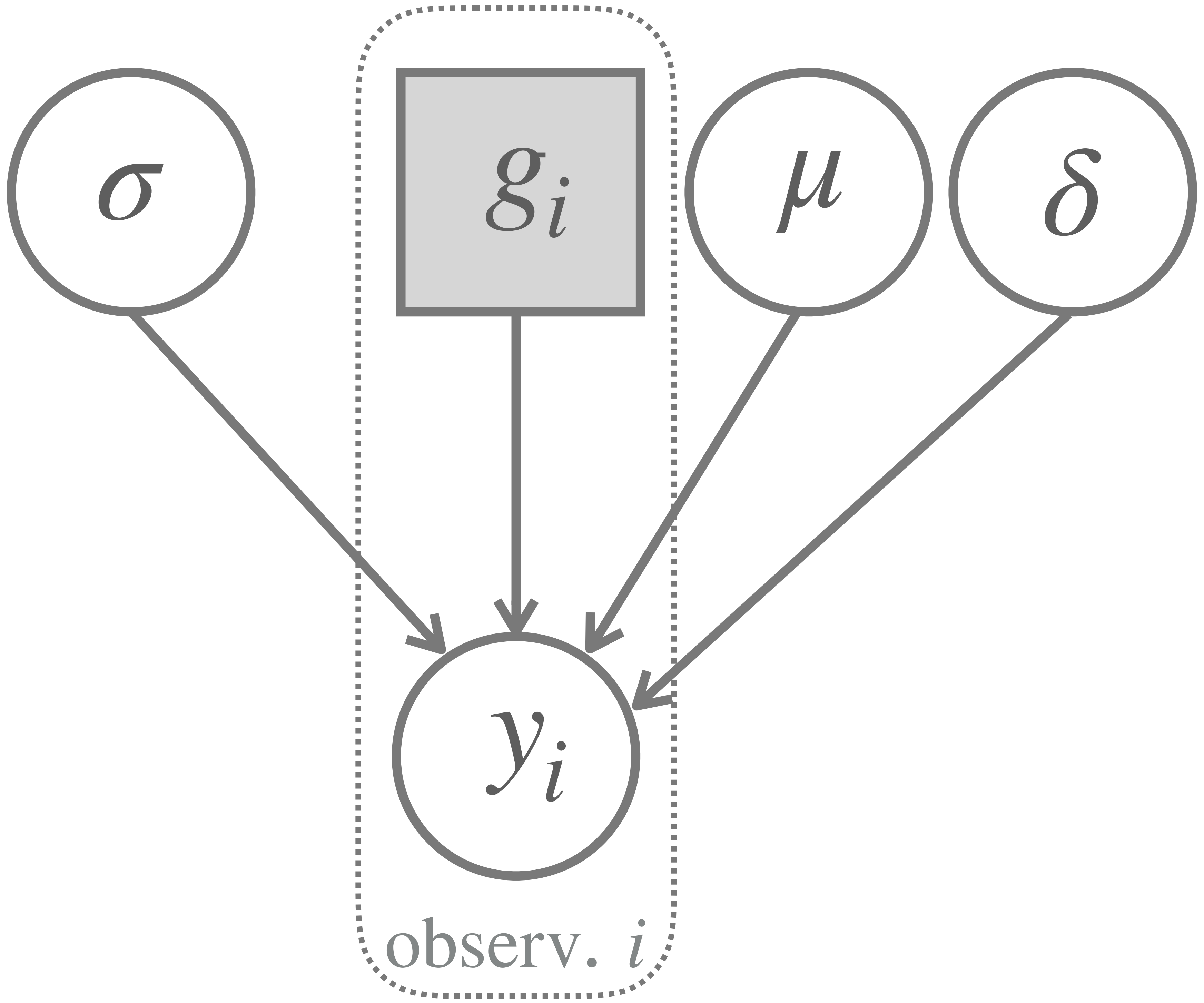
T-TEST MODEL [WITH DELTA]




```r
# isolate data vectors
RT_goNoGo <- mc_data_cleaned %>% filter(block == "goNoGo") %>% pull(RT)
RT_discrm <- mc_data_cleaned %>% filter(block == "discrimination") %>% pull(RT)
# declare as greta data arrays
y0 <- as_data(RT_goNoGo)
y1 <- as_data(RT_discrm)
```

```r
# priors
mean_0   <- normal(430, 50)
delta    <- normal(0, 100)
sigma    <- normal(100, 10, truncation = c(0, Inf))
# derived prameters
mean_1   <- mean_0 + delta
# likelihood
distribution(y0) <- normal(mean_0, sigma)
distribution(y1) <- normal(mean_1, sigma)
# model
m <- model(mean_0, mean_1, delta, sigma)## --- sampling ---
draws <- greta::mcmc(m, warmup = 4000, n_samples = 6000, thin = 2)
```

T-TEST MODEL [WITH DELTA]




```
## # A tibble: 4 x 4
##   Parameter `|95%`  mean `95|%`
##   <fct>      <dbl> <dbl>  <dbl>
## 1 delta       49.6  60.1   71.2
## 2 mean_0     419.  427.   436.
## 3 mean_1     481.  488.   494.
## 4 sigma      101.  105.   109.
```