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as a manuscript

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Minimax pricing method for Exotic and American options in an incomplete market with a finite horizon (the case of discrete-time)

DISSERTATION SUMMARY

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Calculation problem for options in an incomplete market. Suppose time is discrete. There are assets of two types. Prices of the assets of the first type evolve in a way allowing description by a predictable random sequence. Such assets are called risk-free assets. The assets of the second type are such, that evolution of its prices can be described by adapted random sequence only, they say these assets are risky. The market is a finite set of risk-free and risky assets[1]. There in the dissertation, it is assumed that: (1) it is free to exchange assets, i.e. there are no transaction costs, (2) there are no trading restrictions.

An option is a contract between the Buyer and the Seller implying: (1) the Seller'sconveying the right to make a transaction in the future on the settlement terms fixed in the contract when the option was sold, and the Buyer's paying the Seller a fee (the option's value, the option's premium); (2)the transaction is being fulfilled, conditionally the Buyer exercises the option, on some date in the future according to the settlement terms. The deal could be either (a) buying or selling of risky assets by the Seller of the option to the Buyer of the option (option assuming delivery of the actual assets) or (b) tendering the equivalent cash amount of difference between assets' prices, at least one of which is a risky one. The Seller forms a set of risky and risk-free assets (portfolio) with premium and managers it to make sure that the aggregate cost of the assets in the portfolio (capital of portfolio) will enable him to fulfill his obligations according to the contract reliably. A portfolio satisfying the last requirement is called the hedge[1].

Based on the approach used to determine exercise moment they classify options into European, Exotic and American style options. European style options might be exercised at the date settled in the contract. The exercise moment of the Exotic style option could depend on the occurrence of a random event described in the contract. For example, the Barrier option might be exercised if a risky asset's price has passed a certain level or has left the corridor fixed in the contract[2]. Both for European and Exotic options contract parties can not influence the coming of the exercise moment. The Buyer of an American option is empowered to make a transaction settled by the option at any moment he or she chooses: (a) before or on the fixed date (option with finite horizon); (b) with no time restrictions (the case of the infinite horizon). There in the dissertation, the case of the finite horizon only is considered.

When modeling the European option's pricing in an incomplete market it is required to determine the price of the option and hedge («to calculate an option»). For the American option, one has to determine the exercise moment also[1].

As known, the market might be wholly specified by the distribution of the assets' prices. Generally, they assume that the distribution is known up to equivalence. Besides, it is standard to

[1] Shiryaev A.N. Essentials of Stochastic Finance. Volume 2. Theory. *FAZIS*, Moscow, 1998[in Russian].

assume that the market is arbitrage-free. Arbitrage-free markets might be characterized by the existence of equivalent martingale (risk-neutral) measure[1]. Among arbitrage-free markets they distinguish «ideal» markets: where any payoff (1) has the only arbitrage-free price and (2) can be replicated, i.e. there is such portfolio that at the execution moment its capital equals the payoff. These markets are called perfect or complete. As known[1], a market is complete if and only if an equivalent martingale measure is unique. In complete markets, the arbitrage-free price of European option equals its expected payoff with respect to the unique martingale measure, and calculation of American option is reduced to an optimal stopping problem of dynamic payoff, where the expectation is to be calculated with respect to the said measure[1].

The complete market is an ideal model. Generally, arbitrage-free markets are incomplete, i.e. equivalent martingale measure is not unique, and, generally speaking, different option's prices and portfolios correspond to equivalent martingale measures (option's prices form a continual set)[2]. It is required to describe and to draw the basis for parties' approaches to the choice of arbitrage-free price and portfolio.

The degree of the problem' development. There in the literature, they have studied well two approaches to the stated problem of choice: (a) expected utility maximization for a market party and (b) superhedging. Within the approaches, they consider static and dynamic problems with discrete and continuous time.

The static expected utility maximization problem for a market party is to find a probability measure maximizing the expected terminal value of utility function for the party over a set of measures. The problem has been considered in publications by such authors as Schachermayer W., Kramkov D.O., Gushchin A.A., Rokhlin D.B., Delbaen F., Grandits P. and Summer Ch., Biagini S. andFrittelli M., Khasanov R.V. and others. For different market settings it was proved, that the problem is the «dual» one for a superhedging problem[3], they obtained existence conditions for the problems' solutions, but it is quite a problem to verify them (for example, see [4,5]). Dynamic expected utility maximization problem was considered by Quenez M.C. and coauthors, where for different market settings in dynamic utility maximization problems they obtained existence conditions and some properties of solutions.

The second approach is to construct a portfolio (call superhedge) with capital allowing guaranteed execution of the option with respect to any equivalent martingale measure (see

[2] Föllmer H., Schied A. Stochastic Finance. An Introduction in Discrete Time. *MTsNMO,* Moscow, 2008 [in Russian].

[3] Cvitanic J., Schachermayer W., Wang H. Utility Maximization in Incomplete Markets with Random Endowment // Finance and Stochastics. 2001. Vol. 5. Issue 2. P. 259-272.

[4] Delbaen F., Schachermayer W. The Fundamental Theorem of Asset Pricing for Unbounded Stochastic Processes // Mathematische Annalen. 1998. Vol. 312. Issue 2. P. 215-260.

[5] Biagini S., Frittelli M. A unified framework for utility maximization problems: an Orlicz space approach // Annals of Applied Probability. 2008. Vol. 18. No. 3. P. 929-966.

publications by Shiryaev A.N., Kramkov D.O., Kabanov Y.M., Föllmer H., El Karoui N., Quenez M.C., Delbaen F., Schachermayer W., Berkaoui A., Neufeld A., Nutz M. and others). It is known that[2]: (1) upper hedging price for European (American) option is representable as the upper value of expected payoff over the set of equivalent (local) martingale measures (and stopping moments); (2) construction of superhedge is based on optional decomposition. They found existence conditions of the representation (and, consequently, existence conditions of superhedge) for supermartingales with respect to the set of equivalent martingale measures[2] and with respect to an abundant enough set of absolutely continuous measures stable to the bifurcation[6]. Alas, any methods to construct optional decomposition and superhedge (and optimal stopping moment) do not follow from these results.

In articles by Khametov V.M., Zverev O.V. and Silaev A.A. they described the game-theoretic approach (minimax and maximin) to option's calculation in an incomplete market. Recurrent relations of Bellman type for the game value have been derived, sufficient existence conditions for the solution of the minimax problem, and some of its properties have been established.

For the American option's calculation problem in an incomplete market, Riedel F.[7] has proposed a game-theoretic maximin approach, representing the Buyer's point of view. For the case, when it is possible to specify market by a weakly compact time-consistent subset of equivalent probability measures set, he obtained: 1) recurrent relation for the maximin expected payoff; 2) existence conditions for probability measure delivering minimum to option's expected payoff and for optimal stopping moment. Example calculations of American options were made. The proposed by Riedel F. approach does not include the construction of a hedge. Also, neither sets of equivalent probability measures, no sets of martingale equivalent probability measures, with which they commonly specify market, are weakly compact.

Actuality. So, it is still an actual issue to construct a solution for the calculation problem options in an incomplete market, specified by a set of equivalent probability measures. There in the dissertation, the minimax game-theoretic approach is described and used to solve this problem for Exotic and American options. It is a further development of said publications by Khametov V.M. and Zverev O.V. This justifies the importance of the theme of the dissertation.

Goals and objectives of the research. The goal of the dissertation is to: (1) justify usage of the minimax method for dynamic calculation of Exotic and American options in an incomplete market; (2) construct and study a minimax pricing model for Exotic and American

[6] Berkaoui A. On a generalized optional decomposition theorem [Electronic source] // An International Journal of Probability and Stochastic Processes. 2014. Vol.86. Issue 6.

[7] Riedel F. Optimal stopping with multiple priors // Econometrica. 2009. Vol.77. No.3. P.857-908.

options in an incomplete market without transaction costs and trading restrictions, with the finite horizon and a discrete time, representing the Seller's point of view. To do so the following objectives were set: (1) to describe and to give an economic interpretation for the minimax method of calculating for Exotic and American options in an incomplete market without transaction costs and trading restrictions; (2)based on this description to formulate optimization problems for Exotic and American options' calculation, when time is discrete, there is only one risk-free asset with constant price, they observe finite-dimensional adapted random sequences representing the evolution of risky assets' prices, the horizon is finite, the dynamic payoff is a uniformly bounded adapted random sequence, the Seller's risk function is exponential; (3) to justify stochastic version of dynamic programming method as a method to solve the problems; (4) to find existence conditions for the solution of the problems; (5) to study properties of solution for minimax problems; (6) to design method allowing solutions construction for minimax problems.

Research methodology. There in the dissertation, pricing models for Exotic and American options in an incomplete market are constructed through the minimax game-theoretic method, representing the Seller's point of view. According to the method calculation problem for the Exotic (American) option is interpreted as a game of two (three) parties: the Seller, market (and the Buyer). The Seller managers portfolio. Equivalent probability measures are the market's strategies. (The Buyer of American option chooses execution moment for the contract.) In the dissertation, the minimax method is applied in a way to represent the Seller's point of view. In respect to the Seller it is assumed, that: (1) he or she is rational; (2) his or her risk function is exponential, depends on the deficit of the portfolio's capital. The Seller's rationality means the following. In a situation when the distribution of risky-assets prices is known up to equivalence only (and the Buyer of an American option is empowered to choose any moment for the contract to be executed), the Seller calculates the option's price with respect to the worst situation for him, that is when the distribution of risky-assets prices (and execution moment for American option) maximizes his or her expected risk. Meanwhile, the Seller chooses a portfolio to minimize the «worst» value of expected risk. Hence the Seller has to solve the stochastic optimization problem (Seller's minimax problem): the Seller's expected risk is an objective functional, inner supremum is to be found over the set of equivalent probability measures (and stopping moments), outer infimum is to be found over the set of portfolios. The solution of the problem is defined as a set consisting of: (1) the «worst-case» probability measure (and optimal stopping moment) delivering inner supremum; (2) the Seller's optimal portfolio delivering outer infimum, and (3) the Seller's minimax expected risk. There in the dissertation, the study of the existence

conditions for minimax problems solutions and its properties is based on methods of random processes theory and optimal stochastic control, stochastic and functional analysis, game theory.

Principal results to be defended.

- For an incomplete market with the finite horizon without transaction costs, when risky assets' prices evolve as adapted sequences, it is proved, that minimax expected exponential risk of Exotic and American options' Seller evolve according to recurrent relations of Bellman type (Theorems 2, 32- 34).
- Existence criterion and sufficient conditions of uniqueness for an optimal portfolio of Exotic and American options' Seller with exponential risk function in an incomplete market without transaction costs and trading restrictions, the Seller's optimal portfolio coincidence with superhedge of minimal capital (Theorems 7, 13, 15, 18, 36 and Corollaries 16-17, 39-41).
- Existence conditions for a solution of minimax problems for the Seller of Exotic and American options (sufficient conditions – by the combination of statements 7, 20, 32-33, criteria – Theorems 7 and 22 together, 23, 28).
- The «worst-case» measure properties: it is martingale (Theorems 21, 33), discrete (Theorems 26, 33), the initial incomplete market is complete with respect to the «worst-case» measure (Theorem 26, Corollaries 27 and 42).
- Statements linking minimax problems for Exotic and American options (Theorems 32-33, Corollary 44), minimax problems and superhedging problems (Corollaries 16-17, 39-40) and, for American option, minimax problem and optimal stopping problem with respect to the «worst-case» measure (Corollary 43).
- Stepwise algorithm to construct solutions for minimax problems (Algorithms 1 and 2, Theorems 28, 32-33, Corollary 29).

Usage of algorithms has allowed construction in the dissertation of some new examples: (a) of analytical calculation for of Exotic and American options in a one-dimensional incomplete market with risky asset price evolution described by geometric random motion: Binary, Barrier options, European option on a maximum of the risky asset price, vanilla American call and put, Russian option have been calculated; (b) calculation if American call with discount in a binomial market in symbolic computing system Maple 14.

Scientific novelty. All results to be defended are new; they imply some known statements of theory for options calculation in an incomplete market.

General conclusions. There in the dissertation, the minimax calculation method has been used for the first time for Exotic and American options in incomplete marker without transaction costs and trading restrictions, with the finite horizon and a discrete time. The method allowed: (1) to formulate minimax optimization problems for the Seller of an option with exponential risk

function; (2) to describe the evolution of the Seller's minimax expected exponential risk; (3) to prove criteria for solution existence for the minimax problems; (4) to find such important properties of a solution as martingality and discreteness of the «worst-case» measure; (5) to offer a method of construction for the solutions; (6) to prove coincidence between solutions of the minimax problems and superhedging problems for Exotic and American options. Some new examples were constructed for options calculation in an incomplete market.

Contribution by the author. All results to be defended were obtained by the author personally. Contributions of other authors of published articles on the theme of the dissertation are as follows: Khametov V.M. – research problem statement and general supervision, Vasiliev G.A. – overview of Choquet theory and Monge-Kantorovich Problem, Jasonov E.V. – translation of proposed by the author algorithm into the language of computer algebra Maple 14.

List of published articles that reflect the main scientific results of the dissertation.

The main results of the dissertation research are presented in the following publications.

Articles published in peer-reviewed scientific journals indexed by the international citation databases Web of Science and Scopus.

1. Khametov V. M., Shelemekh E. A. On the Uniqueness of the Optional Decomposition of Semimartingales // Mathematical Notes. - 2019. - Vol. 105. - Iss. 3-4. - Pp. 478-482.

2. Khametov V. M., Shelemekh E. A. Upper and Lower Bounds of Optimal Stopping for a Random Sequence: The Case of Finite Horizon // Automation and Remote Control. - 2019. - Vol. 80. - Iss. 3. - Pp. 513-530.

3. Khametov V. M., Shelemekh E. A. Extremal Measures and Hedging in American Options // Automation and Remote Control. - 2016. - Vol. 77. - Iss. 6. - Pp. 1041-1059.

4. Khametov V.M., Shelemekh E.A. Superhedging of American options on an incomplete market with discrete time and finite horizon // Automation and Remote Control. –2015. -Vol. 76. - Iss. 9. - Pp. 1616-1634.

5. Vasil'ev G. A., Khametov V. M., Shelemekh E. A. Conditions for the Discreteness of Extremal Probability Measures (the Finite-Dimensional Case) // Mathematical Notes. - 2013. - Vol. 94. - No. 6. - Pp. 963-967.

Articles published in peer-reviewed scientific journals included in the list of high-rating journals by NRU HSE.

1. Shelemekh E.A. Exotic options calculation in incomplete market // *Economica i matematicheskie metody*. – 2017. – Vol. 53. - № 3. – Pp. 78-92 [in Russian].

2. Khametov V.M., Shelemekh E.A., Jusonov E.V. Algorithm to solve optimal stopping problem with finite horizon // *Upravlenie bolshimi sistemami: sbornik trudov*. - 2014. - Iss. 52. - Pp. 6 -22 [in Russian].

The work has been approbated at the following international and national conferences:

- IX Moscow International Conference on Operations Research (ORM 2018), Moscow, October 22–27, 2018, the theme of the report – «Example calculation of exotic options in incomplete {1,S}-market»;
- Fourth national conference «Molodaya economica: economicheskaya nayka glazami molodih uchenih», Moscow, December 01, 2017, the theme of the report – «Fair value of an option with convex payoff function in incomplete {1,S}-market» [in Russian];
- Sixth international conference «Modern methods, problems and applications of operator theory and harmonic analysis- VI», Rostov-on-Don, April 24 -29, 2016, the theme of the report – «Calculation of exotic option in incomplete {1,S}-market with discrete measure (a finite number of states)»;
- VIII Moscow International Conference on Operations Research (ORM2016). Moscow, October 17-22, 2016, the theme of the report – «Superhedging of American options in an incomplete {1,S}-markets (discrete-time, final horizon)»;
- V International Youth Research and Practice Conference «Mathematical and Computer Modelling in Economics, Insurance and Risk Management», Saratov, November 9-12, 2016, themes of the report – « Calculation of Exotic options in incomplete market specified by Markov chain with finite states number» [in Russian], «Solving Optimal Stopping Problem by Using Computer Algebra Systems»;
- Third national conference «Molodaya economica: economicheskaya nayka glazami molodih uchenih», Moscow, December 07, 2016, the theme of the report – «Example superhedging of options on a maximum of the risky asset price in incomplete {1,S}-market» [in Russian];
- Third Russian Economic Congress (REC-2016), Moscow, December 19-23, 2016, the theme of the report – «Calculation of Exotic options in incomplete marker with discrete time and final horizon» [in Russian];
- Second Russian Economic Congress (REC-2013), Suzdal, February 18-22, 2013, the theme of the report – «Existence of solution for American option calculation problem in incomplete market with discrete time and finite horizon» [in Russian].