Hybrid Model (CP–LP) via Logic-Based Benders Decomposition (LBBD)

Patricia Silva

January 2026

1 Hybrid CP–LP model as LBBD / CP–LP decomposition

Our hybrid model follows a *Logic-Based Benders Decomposition (LBBD)* (also known as a CP–LP decomposition):

- **Master (CP-SAT)** decides the **discrete/boolean variables** (runway assignment and ordering decisions for uncertain pairs) and maintains a cost estimate using "proxy" time/deviation variables and an estimator variable θ.
- **Subproblem (LP/GLOP)** receives the Master decisions fixed, solves the **continuous** timing problem (landing times x_i, deviations α_i, β_i), and returns either an optimal cost or infeasibility.
- **Iteratively** we add cuts: *no-good* cuts (infeasibility) and *optimality* cuts.

The following formulation is aligned with the implementation, including the *strengthened master* using proxy variables x^m, α^m, β^m.

2 Notation

Sets

$$P = \{1, \ldots, \texttt{num\_planes}\}, \qquad R = \{1, \ldots, \texttt{num\_runways}\}$$
$$W, V, U \subseteq P \times P, \qquad i \neq j$$

where W, V, U are computed as in the implementation.

Parameters

- Time windows: E_i, L_i, and target time T_i.
- Penalties: g_i (early), h_i (late).
- Separation times:
  - S_{ij} if aircraft i lands before j on the **same runway**;
  - s_{ij} if aircraft i lands before j on **different runways**.

3 Master Problem (CP): Strengthened Master

3.1 Decision Variables (discrete + proxies)

Runway assignment

$$r_i \in \{1, \ldots, R\} \qquad \forall i \in P$$

Ordering for uncertain pairs

$$b_{ij} \in \{0,1\} \qquad \forall (i,j) \in U$$

where

$$b_{ij} = 1 \iff i \text{ lands before } j.$$

Proxy variables (integer timing and deviations)

$$x_i^m \in [E_i, L_i] \cap \mathbb{Z}, \qquad \alpha_i^m \in \mathbb{Z}_+, \qquad \beta_i^m \in \mathbb{Z}_+ \qquad \forall i \in P$$

Benders estimator

$$\theta \in \mathbb{Z}_+$$

3.2 Master Constraints

(M1) Exclusivity of order in uncertain pairs

$$b_{ij} + b_{ji} = 1 \qquad \forall \{i,j\} : (i,j) \in U$$

(In the code this is enforced for all pairs in U, ensuring consistency.)

(M2) Proxy deviation definition

$$x_i^m + \alpha_i^m - \beta_i^m = T_i \qquad \forall i \in P$$

(with $x_i^m \in [E_i, L_i]$ embedded in its domain).

(M3) Separation constraints for certain-order pairs V

For $(i,j) \in V$, i must land before j. Separation depends on whether they share the same runway:

$$x_j^m \geq x_i^m + \begin{cases} S_{ij}, & \text{if } r_i = r_j, \\ s_{ij}, & \text{if } r_i \neq r_j, \end{cases} \qquad \forall (i,j) \in V.$$

(In CP this is modeled via a reified boolean `same_ij` and two `OnlyEnforceIf`.)

(M4) Separation constraints for uncertain pairs U

If $b_{ij} = 1$ (i before j), then:

$$x_j^m \geq x_i^m + \begin{cases} S_{ij}, & \text{if } r_i = r_j, \\ s_{ij}, & \text{if } r_i \neq r_j, \end{cases} \qquad \forall (i,j) \in U.$$

Since $b_{ji} = 1$ in the opposite case, the symmetric constraint applies when the reverse pair is considered.

Note. The set W is not explicitly used in the Master, because for W the separation is automatically satisfied by the time windows (hence redundant).

(M5) Lower bound (proxy cost) and link to θ

Define the proxy cost:

$$C^m(r, b, x^m, \alpha^m, \beta^m) = \sum_{i \in P} \left(g_i \alpha_i^m + h_i \beta_i^m \right).$$

(In the code g_i, h_i are treated as integers, or scaled to be integers.) We impose:

$$\theta \geq C^m(\cdot).$$

3.3 Master Objective

$$\min\ \theta.$$

4 Subproblem (LP) given (r, b) fixed

The subproblem receives:

- fixed runways $\bar{r}_i$,
- fixed order decisions $\bar{b}_{ij}$ (for U),
- and sets W, V, U.

4.1 Continuous Variables

$$x_i \in [E_i, L_i], \qquad \alpha_i \geq 0, \qquad \beta_i \geq 0 \qquad \forall i \in P.$$

4.2 Subproblem Constraints

(SP1) Exact deviation definition

$$x_i + \alpha_i - \beta_i = T_i \qquad \forall i \in P.$$

(This matches the LP in the implementation and avoids using $\max(\cdot)$ in the linear solver.)

(SP2) Separation constraints induced by the fixed order

For every pair (i, j) for which the subproblem considers "i precedes j" (in the code: all $(i, j) \in V \cup W$ and all $(i, j) \in U$ with $\bar{b}_{ij} = 1$), define the effective separation:

$$\Delta_{ij}(\bar{r}) = \begin{cases} S_{ij}, & \text{if } \bar{r}_i = \bar{r}_j, \\ s_{ij}, & \text{if } \bar{r}_i \neq \bar{r}_j. \end{cases}$$

Then impose:

$$x_j \geq x_i + \Delta_{ij}(\bar{r}) \qquad \forall (i, j) \in \mathcal{P}(\bar{b}),$$

where the enforced precedence set is:

$$\mathcal{P}(\bar{b}) = V \cup W \cup \{(i, j) \in U : \bar{b}_{ij} = 1\}.$$

4.3 Subproblem Objective

$$\min \sum_{i \in P} (g_i \alpha_i + h_i \beta_i).$$

Let $z(\bar{r}, \bar{b})$ denote the LP optimum if feasible.

5 LBBD / Benders Cuts used in the implementation

We use a combination of *no-good cuts* (infeasibility) and *pattern-based optimality cuts* (common in LBBD with CP).

5.1 Infeasibility cut (no-good)

If the subproblem is infeasible for a Master solution $(\bar{r}, \bar{b})$, then that exact combination must not reappear.

Define literals representing a "match" with the current solution:

- for each aircraft i: the literal $[r_i = \bar{r}_i]$,
- for each $(i, j) \in U$: the literal $[b_{ij} = \bar{b}_{ij}]$.

The no-good cut is:

$$\neg \left(\bigwedge_i (r_i = \bar{r}_i) \ \wedge \bigwedge_{(i,j)\in U} (b_{ij} = \bar{b}_{ij}) \right).$$

This is equivalent to a disjunction of negated literals (as in `AddBoolOr` on the negations).

5.2 Optimality cut (when LP cost is larger than θ)

If the Master proposes $(\bar{r}, \bar{b})$ and the LP returns cost $z(\bar{r}, \bar{b})$ (in the code rounded up to an integer), then whenever the Master chooses the same combination again, θ must be at least that cost:

$$\left(\bigwedge_i (r_i = \bar{r}_i) \ \wedge \bigwedge_{(i,j)\in U} (b_{ij} = \bar{b}_{ij}) \right) \Rightarrow \theta \geq \lceil z(\bar{r}, \bar{b}) \rceil.$$

In CP this is implemented via a boolean `is_same` encoding whether all decisions match, and then enforcing $\theta \geq \lceil z(\bar{r}, \bar{b}) \rceil$ with `OnlyEnforceIf(is_same)`.