

Taller 2.

Punto 1.3: Encuentre la función de densidad espectral (transformada de Fourier) para las siguientes señales (sin aplicar propiedades):

a. $e^{-a|t|}$, $a \in \mathbb{R}^+$ $|t| \rightarrow$ se puede partir en $-t = (-\infty, 0)$ y $t = (0, \infty)$

$$F\{e^{-a|t|}\} = \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t} dt = \int_{-\infty}^0 e^{(a-j\omega)t} dt + \int_0^{\infty} e^{-(a+j\omega)t} dt$$

$$= \frac{1}{a-j\omega} e^{(a-j\omega)t} \Big|_{-\infty}^0 + \frac{-1}{a+j\omega} e^{-(a+j\omega)t} \Big|_0^{\infty} \rightarrow e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$$

$$= \frac{1}{a-j\omega} (e^0 - e^{-\infty}) + \frac{-1}{a+j\omega} (e^{-\infty} - e^0) = \frac{1}{a-j\omega} + \frac{1}{a+j\omega}$$

$$= \frac{(a+j\omega) + (a-j\omega)}{(a-j\omega)(a+j\omega)} = \frac{2a}{a^2 + j\omega a - j\omega a - (j\omega)^2} = \frac{2a}{a^2 + \omega^2} = X(\omega)$$

b. $\cos(\omega_c t)$, $\omega_c \in \mathbb{R}$ Usando la propiedad $\rightarrow \cos(\omega_c t) = (e^{j\omega_c t} + e^{-j\omega_c t})/2$

$$F\{\cos(\omega_c t)\} = \int_{-\infty}^{\infty} \cos(\omega_c t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \frac{e^{j\omega_c t} + e^{-j\omega_c t}}{2} e^{-j\omega t} dt$$

$$= \frac{1}{2} \left[\int_{-\infty}^{\infty} e^{j\omega_c t} e^{-j\omega t} dt + \int_{-\infty}^{\infty} e^{-j\omega_c t} e^{-j\omega t} dt \right]$$

$$= \frac{1}{2} \left[\int_{-\infty}^{\infty} e^{-(\omega - \omega_c)t} dt + \int_{-\infty}^{\infty} e^{-(\omega + \omega_c)t} dt \right]$$

$$\text{Resolviendo } I_1 = \int_{-T}^T e^{-(\omega - \omega_c)t} dt = \frac{e^{-j\alpha T} - e^{j\alpha T}}{-j\alpha} = \frac{(e^{j\alpha T} - e^{-j\alpha T})}{j\alpha} \left(\frac{2}{2} \right)$$

$$= \frac{2 \sin(\alpha T)}{\alpha} \quad \text{Si } \lim_{T \rightarrow \infty} \frac{\sin(\alpha T)}{\pi \alpha} = \delta(\alpha) \rightarrow \lim_{T \rightarrow \infty} \frac{2 \sin(\alpha T)}{\alpha} = 2\pi \delta(\alpha)$$

$$\text{Finalmente } \rightarrow \int_{-\infty}^{\infty} e^{-(\omega - \omega_c)t} dt = 2\pi \delta(\omega - \omega_c)$$

$$\rightarrow \text{Así } \frac{1}{2} 2\pi [\delta(\omega - \omega_c) + \delta(\omega + \omega_c)] = \pi [\delta(\omega - \omega_c) + \delta(\omega + \omega_c)] = X(\omega)$$

c. $\sin(\omega t)$; $\omega_s \in \mathbb{R}$ usando la propiedad $\rightarrow \sin(\omega t) = (e^{j\omega t} - e^{-j\omega t})/2j$

$$F\{\sin(\omega t)\} = \int_{-\infty}^{\infty} \sin(\omega t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \frac{e^{j\omega t} - e^{-j\omega t}}{2j} e^{-j\omega t} dt$$

$$= \frac{1}{2j} \left[\int_{-\infty}^{\infty} e^{j\omega t} e^{-j\omega t} dt - \int_{-\infty}^{\infty} e^{-j\omega t} e^{-j\omega t} dt \right]$$

$$= \frac{1}{2j} \left[\int_{-\infty}^{\infty} e^{-(\omega - \omega_s)t} dt - \int_{-\infty}^{\infty} e^{-(\omega + \omega_s)t} dt \right]$$

En el ejercicio b demostramos que $\int_{-\infty}^{\infty} e^{-\alpha t} dt = 2\pi \delta(\alpha)$ Así:

$$\rightarrow = \frac{1}{2j} 2\pi [\delta(\omega - \omega_s) - \delta(\omega + \omega_s)] = \frac{\pi}{j} [\delta(\omega - \omega_s) - \delta(\omega + \omega_s)]$$

$$= \frac{\pi j}{-1} [\delta(\omega - \omega_s) - \delta(\omega + \omega_s)] = -\pi j [\delta(\omega - \omega_s) - \delta(\omega + \omega_s)]$$

$$= \pi j [\delta(\omega + \omega_s) - \delta(\omega - \omega_s)] = X(\omega)$$

d. $f(t) \cos(\omega_c t)$, $\omega_c \in \mathbb{R}$, $f(t) \in \mathbb{R}$, \mathbb{C} .

$$F\{f(t) \cos(\omega_c t)\} = \int_{-\infty}^{\infty} f(t) \cos(\omega_c t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} f(t) \frac{e^{j\omega_c t} + e^{-j\omega_c t}}{2} e^{-j\omega t} dt = \frac{1}{2} \int_{-\infty}^{\infty} f(t) (e^{j\omega_c t} + e^{-j\omega_c t}) e^{-j\omega t} dt$$

$$= \frac{1}{2} \left[\int_{-\infty}^{\infty} f(t) e^{j\omega_c t} e^{-j\omega t} dt + \int_{-\infty}^{\infty} f(t) e^{-j\omega_c t} e^{-j\omega t} dt \right]$$

$$= \frac{1}{2} \left[\int_{-\infty}^{\infty} f(t) e^{-j(\omega - \omega_c)t} dt + \int_{-\infty}^{\infty} f(t) e^{-j(\omega + \omega_c)t} dt \right]$$

Tenemos que $\int_{-\infty}^{\infty} f(t) e^{-j(\omega - \omega_c)t} dt = F(\omega - \omega_c) \rightarrow$ Transformada t desplazada.

$$\rightarrow = \frac{1}{2} [F(\omega - \omega_c) + F(\omega + \omega_c)] = X(\omega)$$

e^{-at^2} , $a \in \mathbb{R}^+$ $t \rightarrow$ siempre es positivo.

$$F\{e^{-at^2}\} = \int_{-\infty}^{\infty} e^{-at^2} e^{-j\omega t} dt$$

Operando exponentes $\rightarrow -at^2 - j\omega t = -a\left(t^2 + \frac{j\omega t}{a}\right) = -a\left[\left(t + \frac{j\omega}{2a}\right)^2 - \left(\frac{j\omega}{2a}\right)^2\right]$

$$= -a\left[\left(t + \frac{j\omega}{2a}\right)^2 - \frac{j^2\omega^2}{2^2 a^2}\right] = -a\left[\left(t + \frac{j\omega}{2a}\right)^2 + \frac{\omega^2}{4a^2}\right]$$

$$= -a\left(t + \frac{j\omega}{2a}\right)^2 - \frac{\omega^2}{4a} \quad \text{Así:}$$

$$\rightarrow \int_{-\infty}^{\infty} e^{-\omega^2/4a} e^{-a(t+j\omega/2a)^2} dt = e^{-\omega^2/4a} \int_{-\infty}^{\infty} e^{-a(t+j\omega/2a)^2} dt \quad I_1$$

$$I_1 \rightarrow u = t + \frac{j\omega}{2a} \quad du = dt \rightarrow I_1 = \int_{-\infty}^{\infty} e^{-au^2} du$$

$$I_1^2 = \left(\int_{-\infty}^{\infty} e^{-ax^2} dx\right) \left(\int_{-\infty}^{\infty} e^{-ay^2} dy\right) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-a(x^2+y^2)} dx dy$$

Haciendo uso de coordenadas polares $\rightarrow \begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \quad \begin{cases} dx dy = r dr d\theta \\ r \in [0, \infty), \theta \in [0, 2\pi) \end{cases}$

$$I_1^2 = \int_0^{2\pi} \int_0^{\infty} e^{-ar^2} r dr d\theta = 2\pi \int_0^{\infty} r e^{-ar^2} dr \rightarrow \begin{cases} s = ar^2 \\ ds = 2a r dr \end{cases} \quad r dr = \frac{ds}{2a}$$

$$I_1^2 = 2\pi \int_0^{\infty} e^{-s} \frac{ds}{2a} = \frac{2\pi}{2a} \int_0^{\infty} e^{-s} ds = \frac{\pi}{a} \left(-\frac{1}{e^{\infty}} - (-1)\right) = \frac{\pi}{a} \rightarrow I = \sqrt{\frac{\pi}{a}}$$

$$\rightarrow e^{-\omega^2/4a} \sqrt{\frac{\pi}{a}} = \chi(\omega)$$

f. $\text{rect}_d(t)$; $A, d \in \mathbb{R}$ $\text{rect}_d(t) = \begin{cases} 1 & |t| \leq d/2 \\ 0 & |t| > d/2 \end{cases}$

$$F\{A \text{rect}_d(t)\} = \int_{-\infty}^{\infty} A \text{rect}_d(t) e^{-j\omega t} dt = \int_{-d/2}^{d/2} A e^{-j\omega t} dt$$

$$= A \left(\frac{e^{-j\omega t}}{-j\omega} \right) \Big|_{-d/2}^{d/2} = -\frac{A}{j\omega} (e^{-j\omega d/2} - e^{(+j\omega)(-d/2)})$$

$$= \frac{2A}{\omega} \left(\frac{e^{-j\omega d/2} - e^{(+j\omega)(-d/2)}}{2j} \right) = \frac{2A}{\omega} \sin\left(\frac{\omega d}{2}\right) = \chi(\omega)$$

Punto 1.4: Aplique las propiedades de la transformada de Fourier para resolver:

a. $F\{e^{-j\omega_1 t} \cos(\omega_c t)\}$, $\omega_1, \omega_c \in \mathbb{R}$

$$e^{-j\omega_1 t} \cos(\omega_c t) = \frac{e^{j\omega_c t} + e^{-j\omega_c t}}{2} e^{-j\omega_1 t}$$

$$= \frac{1}{2} (e^{j\omega_c t} e^{-j\omega_1 t} + e^{-j\omega_c t} e^{-j\omega_1 t}) = \frac{1}{2} (e^{-jt(\omega_1 - \omega_c)} + e^{-jt(\omega_1 + \omega_c)})$$

Sabemos que $F\{e^{j\omega_0 t}\} = 2\pi \delta(\omega - \omega_0)$

$$\rightarrow = \frac{1}{2} 2\pi (\delta[\omega - (-\omega_1 + \omega_c)] + \delta[\omega - (-\omega_1 - \omega_c)])$$

$$= \pi [\delta(\omega + \omega_1 - \omega_c) + \delta(\omega + \omega_1 + \omega_c)] = X(\omega)$$

b. $F\{u(t) \cos^2(\omega_c t)\}$ $\omega_c \in \mathbb{R}^+$ Si $u(t) = 1$ de $(0, \infty)$

$$\cos^2(\omega_c t) = \frac{1 + \cos(2\omega_c t)}{2}; \quad u(t) \cos^2(\omega_c t) = \frac{1}{2} [u(t) + u(t) \cos(2\omega_c t)]$$

$$F\{u(t) \cos^2(\omega_c t)\} = \frac{1}{2} (F\{u(t)\} + F\{u(t) \cos(2\omega_c t)\})$$

Si $F\{u(t)\} = F\{\text{sgn}\} + F\{1/2\} = \frac{1}{j\omega} + \pi \delta(\omega)$

$$\text{y } F\{u(t) \cos(\omega_c t)\} = \frac{\pi}{2} (\delta(\omega - \omega_c) + \delta(\omega + \omega_c)) + \frac{j\omega}{\omega_c^2 - \omega^2}$$

$$\rightarrow = \frac{1}{2} \left[\frac{1}{j\omega} + \pi \delta(\omega) + \frac{\pi}{2} (\delta(\omega - 2\omega_c) + \delta(\omega + 2\omega_c)) + \frac{j\omega}{(2\omega_c)^2 - \omega^2} \right] = X(\omega)$$

c. $F^{-1} \left\{ \frac{7}{\omega^2 + 6\omega + 45} * \frac{10}{(8 + j\omega/3)^2} \right\}$

Producto en frecuencia = $F^{-1}\{G(\omega)H(\omega)\}(t) = \frac{1}{2\pi} (g * h)(t)$
 Convolución en t

Donde $g(t) = F^{-1}\{G(\omega)\}$ y $h(t) = F^{-1}\{H(\omega)\}$

$$G(\omega) = \frac{7}{\omega^2 + 6\omega + 45} \rightarrow \omega^2 + 6\omega + 45 = (\omega + 3)^2 + 36 = (\omega + 3)^2 + 6^2$$

$$\text{Si } e^{-at} \rightarrow \frac{2a}{a^2 + \omega^2}; \frac{1}{(\omega - \omega_0)^2 + a^2} \rightarrow \frac{1}{2a} e^{-a|t|} e^{j\omega_0 t} \quad a=6, \omega_0=-3$$

$$\rightarrow g(t) = F^{-1} \left\{ \frac{7}{(\omega + 3)^2 + 6^2} \right\} = \frac{7}{2(6)} e^{-6|t|} e^{-j3t} = \frac{7}{12} e^{-6|t|} e^{-j3t}$$

$$H(\omega) = \frac{10}{(8 + j\omega/3)^2} \rightarrow (8 + j\omega/3)^2 = (1/3 (24 + j\omega))^2 = 1/9 (24 + j\omega)^2$$

$$\text{Si } F\{te^{-at}u(t)\} = \frac{1}{(a + j\omega)^2} \text{ para } a > 0 \quad a=24$$

$$\rightarrow = F^{-1} \left\{ 90 \frac{1}{(24 + j\omega)} \right\} = 90te^{-24t}u(t) = h(t)$$

$$x(t) = F^{-1}\{G \cdot H\} = \frac{1}{2\pi} (g * h) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\tau) h(t-\tau) d\tau$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{7}{12} e^{-6|\tau|} e^{-j3\tau} 90(t-\tau) e^{-24(t-\tau)} u(t-\tau) d\tau$$

$$x(t) = \frac{1}{2\pi} \frac{7(90)}{12} \int_{-\infty}^{\infty} e^{-6|\tau|} e^{-j3\tau} (t-\tau) e^{-24(t-\tau)} u(t-\tau) d\tau$$

$$x(t) = \frac{105}{4\pi} \int_{-\infty}^{\infty} e^{-6|\tau|} e^{-j3\tau} (t-\tau) e^{-24(t-\tau)} u(t-\tau) d\tau$$

$$x(t) = \frac{105}{4\pi} \int_{-\infty}^{\infty} e^{-6|\tau|} e^{-j3\tau} (t-\tau) e^{-24(t-\tau)} d\tau \rightarrow \text{Si } u(t-\tau) = 1 \text{ en } (-\infty, t]$$

$$x(t) = \frac{105}{4\pi} \left(\int_{-\infty}^0 e^{6\tau} e^{-j3\tau} (t-\tau) e^{-24(t-\tau)} d\tau + \int_0^t e^{-6\tau} e^{-j3\tau} (t-\tau) e^{-24(t-\tau)} d\tau \right)$$

$$x(t) = \frac{105}{4\pi} e^{-24t} \left(\int_{-\infty}^0 e^{6\tau} e^{-j3\tau} (t-\tau) e^{24\tau} d\tau + \int_0^t e^{-6\tau} e^{-j3\tau} (t-\tau) e^{24\tau} d\tau \right)$$

$$x(t) = \frac{105}{4\pi} e^{-24t} \left(\underbrace{\int_{-\infty}^0 (t-\tau) e^{\tau(30-j3)} d\tau}_{I_1} + \underbrace{\int_0^t (t-\tau) e^{\tau(-30-j3)} d\tau}_{I_2} \right)$$

$$I_1 = \int_{-\infty}^0 t e^{(30-j3)t} d\tau - \int_{-\infty}^0 \tau e^{\tau(-30-j3)} d\tau$$

$$= t \cdot \frac{e^{(30-j3)t}}{30-j3} \Big|_{-\infty}^0 - \left(\frac{\tau e^{(30-j3)\tau}}{30-j3} - \frac{e^{(30-j3)\tau}}{(30-j3)^2} \right) \Big|_{-\infty}^0$$

$$= \frac{t}{30-j3} \left(\cancel{e^0} - \cancel{\frac{1}{e^\infty}} \right) - \left[\cancel{\frac{0e^0}{30-j3}} - \frac{e^0}{(30-j3)^2} - \left(\cancel{\frac{1}{(30-j3)e^\infty}} - \cancel{\frac{1}{(30-j3)^2 e^\infty}} \right) \right]$$

$$= \frac{t}{30-j3} + \frac{1}{(30-j3)^2}$$

$$I_2 = \int_0^t (t-\tau) e^{(-30-j3)\tau} d\tau = \int_0^t t e^{(-30-j3)\tau} d\tau - \int_0^t \tau e^{(-30-j3)\tau} d\tau$$

$$= t \cdot \frac{e^{(-30-j3)\tau}}{-30-j3} \Big|_0^t - \left(\frac{\tau e^{(-30-j3)\tau}}{-30-j3} - \frac{e^{(-30-j3)\tau}}{(-30-j3)^2} \right) \Big|_0^t$$

$$= \frac{t}{-30-j3} \left(e^{(-30-j3)t} - \cancel{e^0} \right) - \left[\frac{e^{(-30-j3)t}}{-30-j3} - \frac{e^{(-30-j3)t}}{(-30-j3)^2} - \left(\cancel{\frac{0e^0}{-30-j3}} - \frac{e^0}{(-30-j3)^2} \right) \right]$$

$$= \frac{t}{-30-j3} \left(e^{(-30-j3)t} - 1 \right) - \left(\frac{t e^{(-30-j3)t}}{-30-j3} - \frac{e^{(-30-j3)t}}{(-30-j3)^2} + \frac{1}{(-30-j3)^2} \right)$$

$$= \cancel{\frac{t e^{(-30-j3)t}}{-30-j3}} - \frac{t}{-30-j3} - \cancel{\frac{t e^{(-30-j3)t}}{-30-j3}} + \frac{e^{(-30-j3)t}}{(-30-j3)^2} - \frac{1}{(-30-j3)^2}$$

$$= \frac{e^{(-30-j3)t} - 1}{(-30-j3)^2} + \frac{t}{30+j3}$$

$$x(t) = \frac{105}{4\pi} e^{-24t} \left(\frac{t}{30-j3} + \frac{1}{(30+j3)^2} + \frac{e^{(-30-j3)t} - 1}{(30-j3)^2} + \frac{t}{30+j3} \right)$$

$$x(t) = \frac{105}{4\pi} e^{-24t} \left[t \left(\frac{1}{30-j3} + \frac{1}{30+j3} \right) + \left(\frac{1}{(30-j3)^2} - \frac{1}{(30+j3)^2} \right) + \frac{e^{(-30-j3)t}}{(30-j3)^2} \right]$$

$$x(t) = \frac{105}{4\pi} e^{-24t} \left(t \frac{60}{30^2 + 3^2} + \frac{360j}{4092} + \frac{e^{(-30-j3)t}}{(30+j3)^2} \right)$$

d. $F\{3t^3\} \rightarrow$ si $F\{t^n x(t)\} = j^n \frac{d^n}{d\omega^n} X(\omega)$; $x(t) = 1 \rightarrow F\{1\} = 2\pi \delta(\omega)$.

$$F\{3t^3\} = 3j^3 \frac{d^3}{d\omega^3} (2\pi \delta(\omega)) = j^2 j 6\pi \delta^{(3)}(\omega) = -j 6\pi \delta^{(3)}(\omega)$$

e. $\frac{B}{T} \sum_{n=-\infty}^{+\infty} \left(\frac{1}{a^2 + (\omega - n\omega_0)^2} + \frac{1}{a + j(\omega - n\omega_0)} \right)$ donde $n \in \{0, \pm 1, \pm 2, \dots\}$
 $\omega_0 = 2\pi/T \rightarrow B, T \in \mathbb{R}^+$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{B}{T} \sum_{n=-\infty}^{+\infty} \left(\frac{1}{a^2 + (\omega - n\omega_0)^2} + \frac{1}{a + j(\omega - n\omega_0)} \right) e^{j\omega t} d\omega$$

$$x(t) = \frac{B}{T 2\pi} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{1}{a^2 + (\omega - n\omega_0)^2} + \frac{1}{a + j(\omega - n\omega_0)} \right) e^{j\omega t} d\omega$$

$\begin{matrix} v = \omega - n\omega_0 \\ dv = d\omega \\ \omega = v + n\omega_0 \end{matrix}$

$$x(t) = \frac{B}{T 2\pi} \sum_{n=-\infty}^{+\infty} \left(\int_{-\infty}^{\infty} \frac{1}{a^2 + v^2} e^{j(v+n\omega_0)t} dv + \int_{-\infty}^{\infty} \frac{1}{a + jv} e^{j(v+n\omega_0)t} dv \right)$$

$$x(t) = \frac{B}{T 2\pi} \sum_{n=-\infty}^{\infty} \left(e^{jn\omega_0 t} \left[\int_{-\infty}^{\infty} \frac{1}{a^2 + v^2} e^{jvt} dv + \int_{-\infty}^{\infty} \frac{1}{a + jv} e^{jvt} dv \right] \right)$$

Si $A(v) = \frac{1}{a^2 + v^2} \rightarrow a(t) = F^{-1}\{A(v)\} = \frac{1}{2a} e^{-a|t|}$ para $a > 0$.

$B(v) = \frac{1}{a + jv} \rightarrow b(t) = F^{-1}\{B(v)\} = e^{-at} u(t)$ para $a > 0$.

$$x(t) = \frac{B}{T 2\pi} \sum_{n=-\infty}^{+\infty} e^{jn\omega_0 t} \left(2\pi \frac{1}{2a} e^{-a|t|} + 2\pi e^{-at} u(t) \right)$$

$$x(t) = \frac{B 2\pi}{T 2\pi} \left(\frac{1}{2a} e^{-a|t|} + e^{-at} u(t) \right) \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t} = 2\pi \sum_{k=-\infty}^{+\infty} \delta(\omega_0 t - 2\pi k)$$

$$x(t) = \frac{B}{T} \left(\frac{1}{2a} e^{-a|t|} + e^{-at} u(t) \right) \sum_{k=-\infty}^{\infty} \delta(t - T_k)$$

$= 2\pi \sum_{k=-\infty}^{\infty} \frac{1}{\omega_0} \delta\left(t - \frac{2\pi k}{\omega_0}\right) \quad \omega_0 = 2\pi/T$
 $= \frac{2\pi}{\omega_0} \sum_{k=-\infty}^{\infty} \delta(t - T_k) = T \sum_{k=-\infty}^{\infty} \delta(t - T_k)$

$$x(t) = B \left(\frac{1}{2a} e^{-a|t|} + e^{-at} u(t) \right) \sum_{k=-\infty}^{\infty} \delta(t - T_k)$$

Punto 2.2: • Compruebe la solución $h(t)$ de la EDO cuando $x(t) = \delta(t)$ de manera manual. Tener en cuenta que $E'(t) = u'(t) = \delta(t)$.

Sistema $\rightarrow y'(t) + y(t) = x(t)$ para cualquier entrada $x(t)$
 Sabemos que $h(t) = y(t)$ cuando $x(t) = \delta(t)$

Reemplazando $\rightarrow y'(t) + y(t) = \delta(t)$, $[-\epsilon, \epsilon] \rightarrow$ intervalo muy pequeño $\epsilon \rightarrow 0$.

$$\int_{-\epsilon}^{\epsilon} (y'(t) + y(t)) dt = \int_{-\epsilon}^{\epsilon} \delta(t) dt \quad \delta(t) \rightarrow \text{se enciende solo en } t=0 \text{ y su área} = 1$$

$$\rightarrow y(\epsilon) - y(-\epsilon) + \int_{-\epsilon}^{\epsilon} y(t) dt = 1 \quad \text{Si } \epsilon \rightarrow 0 \text{ la integral también } \rightarrow 0$$

$$\rightarrow y(0^+) - y(0^-) = 1 \quad \text{se asume un sistema causal } y(0^-) = 0$$

$$\rightarrow y(0^+) = 1 \quad \text{así para } t > 0 \text{ la EDO queda } \rightarrow y'(t) + y(t) = 0 \text{ homogénea.}$$

la solución general es $y(t) = Ce^{-t}$, $t > 0$

$$\text{con la condición } y(0^+) = 1 \rightarrow Ce^0 = 1 \rightarrow C = 1$$

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

Finalmente $h(t) = e^{-t}u(t) \rightarrow u(t)$ para apagar todo en $y(0^-)$

• Comprobar la solución de la integral de convolución de manera manual. Tener en cuenta las funciones Heaviside.

$$h(t) = e^{-t}u(t), \quad x(t) = e^{-2t}u(t) \quad \text{hallar } y(t) = h(t) * x(t)$$

$$\text{Definición de convolución } \rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \quad \text{como } h(t) \text{ y } x(t) \text{ tienen } u(t) \rightarrow t < 0 = 0$$

$$y(t) = \int_0^t x(\tau) h(t-\tau) d\tau = \int_0^t e^{-2\tau} e^{-(t-\tau)} d\tau = \int_0^t e^{-2\tau - t + \tau} d\tau$$

$$= \int_0^t e^{-t-\tau} d\tau = e^{-t} \int_0^t e^{-\tau} d\tau = e^{-t} [-e^{-\tau}]_0^t = e^{-t} (-e^{-t} + e^0)$$

$$= e^{-t} - e^{-t-t} = e^{-t} - e^{-2t}, \quad t \geq 0 \rightarrow y(t) = (e^{-t} - e^{-2t}) u(t)$$

Punto 2.3: demuestre si los siguientes sistemas de la forma $y = H\{x\}$, son sistemas lineales e invariantes en el tiempo (SLIT). Simule los sistemas en Python:

• $y[n] = x[n]/3 + 2x[n-1] - y[n-1]$

① $x(t) = d_1 x_1(t) + d_2 x_2(t)$
 $y[n] = (1/3)(d_1 x_1[n] + d_2 x_2[n]) + 2(d_1 x_1[n-1] + d_2 x_2[n-1]) - (d_1 y_1[n-1] + d_2 y_2[n-1])$

② $d_1 x_1(t) + d_2 x_2(t)$
 $d_1 (x_1[n]/3 + 2x_1[n-1] - y_1[n-1]) + d_2 (x_2[n]/3 + 2x_2[n-1] - y_2[n-1])$
 $\tilde{y}[n] = d_1 x_1[n]/3 + 2d_1 x_1[n-1] - d_1 y_1[n-1] + d_2 x_2[n]/3 + 2d_2 x_2[n-1] - d_2 y_2[n-1]$
 $\tilde{y}[n] = (1/3)(d_1 x_1[n] + d_2 x_2[n]) + 2(d_1 x_1[n-1] + d_2 x_2[n-1]) - (d_1 y_1[n-1] + d_2 y_2[n-1])$

Es lineal ① = ②

① $x(t) = x(t - t_0)$
 $y[n] = (1/3)x[n - n_0] + 2x[n-1 - n_0] - y[n-1 - n_0]$

② $\tilde{y}(t - t_0) = x(t - t_0)$
 $\tilde{y}[n - n_0] = (1/3)x[n - n_0] + 2x[n-1 - n_0] - y[n-1 - n_0]$

Solo es invariante en el tiempo si $y[n-1] = y[n-1 - n_0] \rightarrow n_0 = 0$.

En conclusión es SLIT ya que cumple linealidad y es invariante en el tiempo con $n_0 = 0$.

• $y[n] = \sum_{k=-\infty}^n x^2[k]$

① $x(t) = d_1 x_1(t) + d_2 x_2(t)$
 $y[n] = \sum_{k=-\infty}^n (d_1 x_1[k] + d_2 x_2[k])^2$
 $y[n] = \sum_{k=-\infty}^n (d_1^2 x_1^2[k] + 2d_1 d_2 x_1[k] x_2[k] + d_2^2 x_2^2[k])$

② $d_1 x_1(t) + d_2 x_2(t)$
 $\tilde{y}[n] = \sum_{k=-\infty}^n d_1 (x_1^2[k]) + d_2 (x_2^2[k])$

NO se cumple linealidad ① \neq ②

① $x(t) = x(t - t_0)$
 $y[n] = \sum_{k=-\infty}^n x^2[k - n_0]$

② $\tilde{y}(t - t_0) = x(t - t_0)$
 $\tilde{y}[n - n_0] = \sum_{k=-\infty}^{n-n_0} x^2[k]$

Para ①: $m = k - n_0$
 $k = -\infty \rightarrow m = -\infty$
 $k = n \rightarrow m = n - n_0$

Es invariante en el tiempo ① = ②

En conclusión NO es SLIT ya que aunque es invariante en el tiempo no es lineal.

- $y[n] = \text{median}(x[n])$; donde median es la función mediana sobre una ventana de tamaño 3. $y[n] = \text{median}(x[n-1], x[n], x[n+1])$

Propiedad del median()

- $\text{median}(a+b) \neq \text{median}(a) + \text{median}(b)$
- $\text{median}(d \cdot x) = d \cdot \text{median}(x)$

① $x(t) = d_1 x_1(t) + d_2 x_2(t)$

$$y[n] = \text{median}(d_1 x_1[n-1] + d_2 x_2[n-1], d_1 x_1[n] + d_2 x_2[n], d_1 x_1[n+1] + d_2 x_2[n+1])$$

② $d_1 x_1(t) + d_2 x_2(t)$

$$\tilde{y}[n] = d_1 \text{median}(x_1[n-1], x_1[n], x_1[n+1]) + d_2 \text{median}(x_2[n-1], x_2[n], x_2[n+1])$$

$$\tilde{y}[n] = \text{median}(d_1 x_1[n-1], d_1 x_1[n], d_1 x_1[n+1]) + \text{median}(d_2 x_2[n-1], d_2 x_2[n], d_2 x_2[n+1])$$

Como $\text{median}(a+b) \neq \text{median}(a) + \text{median}(b)$

→ NO se cumple linealidad. ① \neq ②

① $x(t) = x(t-t_0)$

$$y[n] = \text{median}(x[n-1-n_0], x[n-n_0], x[n+1-n_0])$$

② $\tilde{y}(t-t_0) = x(t-t_0)$

$$\tilde{y}[n-n_0] = \text{median}(x[n-1-n_0], x[n-n_0], x[n+1-n_0])$$

Es invariante en el tiempo ① = ②

En conclusión NO es SLIT ya que aunque es invariante en el tiempo no es lineal.

- $y(t) = Ax(t) + B$; $A, B \in \mathbb{R}$

① $x(t) = d_1 x_1(t) + d_2 x_2(t)$

$$y(t) = A(d_1 x_1(t) + d_2 x_2(t)) + B = Ad_1 x_1(t) + Ad_2 x_2(t) + \underbrace{B}_{\neq d_1 B + d_2 B}$$

② $d_1 x_1(t) + d_2 x_2(t)$

$$y(t) = d_1 (Ax_1(t) + B) + d_2 (Ax_2(t) + B) = Ad_1 x_1(t) + Ad_2 x_2(t) + d_1 B + d_2 B$$

Solo cumple linealidad si $B = d_1 B + d_2 B \rightarrow d_1 + d_2 = 1 \rightarrow d_1 = d_2 = 1/2$.

① $x(t) = x(t-t_0)$

$$y(t) = A x(t-t_0) + B$$

② $\tilde{y}(t-t_0) = x(t-t_0)$

$$\tilde{y}(t-t_0) = A x(t-t_0) + B$$

Es invariante en el tiempo ① = ②

En conclusión NO es SLIT, ya que aunque es invariante en el tiempo no es lineal.

Ejemplo 2.5: Sea la señal gaussiana $x(t) = e^{-at^2}$ con $a \in \mathbb{R}^+$, el sistema A con relación entrada-salida $y_A(t) = x^2(t)$, y el sistema lineal e invariante con el tiempo B con respuesta al impulso $h_B(t) = Be^{-bt^2}$:

a. Encuentre la salida del sistema en serie $x(t) \xrightarrow{B} h_B(t) \xrightarrow{A} y_A(t) \rightarrow y(t)$

$$y(t) * d(t) = \int_{-\infty}^{\infty} x(\tau) d(t-\tau) d\tau$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

↳ respuesta al impulso desplazado

$$y(t) = x(t) * h(t)$$

$$h_B = Be^{-bt^2}$$

$$x(t) = e^{-at^2}$$

$$y_A(t) = x^2(t)$$

→ Info que tenemos

$$x(t) \xrightarrow{h_B} v(t) \xrightarrow{A} y(t) \rightarrow \text{Pide } \odot$$

se resuelve aparte.

$$v(t) = x(t) * h_B = \int_{-\infty}^{\infty} e^{-a\tau^2} Be^{-b(t-\tau)^2} d\tau = B \int_{-\infty}^{\infty} e^{-(a\tau^2 + b(t-\tau)^2)} d\tau$$

$$\rightarrow a\tau^2 + b(t-\tau)^2 = a\tau^2 + b(t^2 - 2t\tau + \tau^2) = a\tau^2 + b\tau^2 - 2bt\tau + bt^2 = (a+b)\tau^2 - 2bt\tau + bt^2$$

$$\rightarrow (a+b) \left(\tau^2 - \frac{2bt\tau}{a+b} \right) = (a+b) \left[\left(\tau - \frac{bt}{a+b} \right)^2 - \left(\frac{bt}{a+b} \right)^2 \right]$$

$$= a\tau^2 + b(t-\tau)^2 = (a+b) \left(\tau - \frac{bt}{a+b} \right)^2 - (a+b) \left(\frac{bt}{a+b} \right)^2$$

$$\rightarrow bt^2 - (a+b) \frac{bt^2}{(a+b)^2} = bt^2 - \frac{b^2 t^2}{a+b} = \frac{b(a+b) - b^2}{a+b} t^2 = \frac{ab}{a+b} t^2$$

$$= a\tau^2 + b(t-\tau)^2 = (a+b) \left(\tau - \frac{bt}{a+b} \right)^2 + \frac{ab}{a+b} t^2$$

Integral Gaussiana

$$\rightarrow B \int_{-\infty}^{\infty} e^{-((a+b) \left(\tau - \frac{bt}{a+b} \right)^2 + \frac{ab}{a+b} t^2)} d\tau = Be^{-\frac{ab}{a+b} t^2} \int_{-\infty}^{\infty} e^{-((a+b) \left(\tau - \frac{bt}{a+b} \right)^2)} d\tau$$

$$= Be^{-\frac{ab}{a+b} t^2} \sqrt{\frac{\pi}{a+b}} \rightarrow \text{usando la propiedad}$$

$$y(t) = x^2(t) = \left(Be^{-\frac{ab}{a+b} t^2} \sqrt{\frac{\pi}{a+b}} \right)^2 = \frac{\pi}{a+b} B^2 e^{-\frac{2ab}{a+b} t^2} = \frac{B^2 \pi}{a+b} e^{-\frac{2ab}{a+b} t^2}$$

b. Encuentre la salida del sistema en serie $x(t) \xrightarrow{A} u(t) \xrightarrow{B} y(t)$

$$x(t) \xrightarrow{A} u(t) = y_A(t) \xrightarrow{B} y(t)$$

$$u(t) = x^2(t) = e^{-2at^2}$$

$$y(t) = u(t) * h_B(t) = \int_{-\infty}^{\infty} e^{-2a\tau^2} B e^{-b(t-\tau)^2} d\tau = B \int_{-\infty}^{\infty} e^{-\overbrace{(2a\tau + b(t-\tau))^2}^{\text{se resuelve aparte}}} d\tau$$

$$\begin{aligned} \rightarrow 2a\tau^2 + b(t-\tau)^2 &= (2a+b)\tau^2 - 2bt + bt^2 \\ &= (2a+b) \left(\tau - \frac{bt}{2a+b} \right)^2 - (2a+b) \left(\frac{bt}{2a+b} \right)^2 \end{aligned}$$

$$\rightarrow bt^2 - \frac{b^2 t^2}{2a+b} = \frac{b(2a+b) - b^2}{2a+b} t^2 = \frac{2ab}{2a+b} t^2$$

$$\begin{aligned} B \int_{-\infty}^{\infty} e^{-\left((2a+b) \left(\tau - \frac{bt}{2a+b} \right)^2 + \frac{2ab}{2a+b} t^2 \right)} d\tau &= B e^{-\frac{2ab}{2a+b} t^2} \int_{-\infty}^{\infty} e^{-(2a+b) \left(\tau - \frac{bt}{2a+b} \right)^2} d\tau \\ &= B e^{-\frac{2ab}{2a+b} t^2} \sqrt{\frac{\pi}{2a+b}} \rightarrow \text{usando la propiedad.} \end{aligned}$$

$$y(t) = B \sqrt{\frac{\pi}{2a+b}} e^{-\frac{2ab}{2a+b} t^2}$$

Ejercicio 2.10: • Obtener la transformada inversa de Laplace de $X(s)$ manualmente.

Dada $X(s) = \frac{2s^2 + 14s + 124}{s^3 + 8s^2 + 46s + 68}$ con Región de convergencia $\rightarrow R(s) = -2$.

Comprobar si $s = -2$ es raíz del denominador $\rightarrow (-2)^3 + 8(-2)^2 + 46(-2) + 68 = -8 + 32 - 92 + 68 = 0$ \rightarrow se divide entre $(s+2)$

División sintética \rightarrow

s^3	s^2	s^1	s^0	
1	8	46	68	$s = -2$
1	6	34	0	\rightarrow sí es raíz.

$s^3 + 8s^2 + 46s + 68 = (s+2)(s^2 + 6s + 34)$

Fración parcial $\rightarrow \frac{2s^2 + 14s + 124}{(s+2)(s^2 + 6s + 34)} = \frac{A}{s+2} + \frac{Bs+C}{s^2 + 6s + 34}$ Multiplicamos por $(s+2)(s^2 + 6s + 34)$

$\rightarrow 2s^2 + 14s + 124 = A(s^2 + 6s + 34) + (Bs+C)(s+2)$
 $= 2s^2 + 14s + 124 = As^2 + 6As + 34A + Bs^2 + 2Bs + Cs + 2C$
 $2s^2 + 14s + 124 = (A+B)s^2 + (6A+2B+C)s + (34A+2C)$

Iguando factores $\rightarrow A+B=2$ $6A+2B+C=14$
 $B=2-A$ $6A+2(2-A)+C=14$
 $6A+4-2A+C=14$
 $4A+C=10 \rightarrow C=10-4A$
 $34A+2C=124$
 $34A+2(10-4A)=124$
 $34A+20-8A=124$
 $26A=104 \rightarrow A=104/26=4$ $C=10-4(4)=10-16=-6$
 $B=2-A=2-4=-2$

Se obtiene $\rightarrow A=4, B=-2, C=-6$

se completa el cuadrado

$X(s) = \frac{4}{s+2} + \frac{-2s-6}{s^2+6s+34}$ $\rightarrow s^2+6s+34 = (s^2+6s+(6/2)^2) - (6/2)^2 + 34$
 Denominador $\rightarrow (s^2+6s+9) - 9 + 34 = 25$
 $(s+3)^2 + 25$

Numerador $\rightarrow -2s-6 = -2(s+3)$

$X(s) = \frac{4}{s+2} + \frac{-2(s+3)}{(s+3)^2 + 25}$ Usamos transformadas conocidas:

$\mathcal{L}^{-1} \left\{ \frac{1}{s+a} \right\} = e^{-at} u(t) \rightarrow \mathcal{L}^{-1} \left\{ \frac{4}{s+2} \right\} = 4e^{-2t} u(t)$

$\mathcal{L}^{-1} \left\{ \frac{s+a}{(s+a)^2 + w^2} \right\} = e^{-at} \cos(wt) u(t) \rightarrow \mathcal{L}^{-1} \left\{ \frac{-2(s+3)}{(s+3)^2 + 5^2} \right\} = -2e^{-3t} \cos(5t) u(t)$

$x(t) = (4e^{-2t} - 2e^{-3t} \cos(5t)) u(t)$

• obtener la transformada inversa de Laplace manualmente de:

$$X(s) = \frac{1}{(s+1)(s+2)^2} \text{ para } \operatorname{Re}\{s\} > -1.$$

Fraciones parciales $\rightarrow \frac{1}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$ Multiplicar por $(s+1)(s+2)^2$

Se eliminan denominadores $\rightarrow 1 = A(s+2)^2 + B(s+1)(s+2) + C(s+1)$

Al evaluar en $s = -1 \rightarrow 1 = A(-1+2)^2 + \cancel{B(-1+1)(-1+2)} + \cancel{C(-1+1)}$
 $1 = A(1)^2 = A$

Al evaluar en $s = -2 \rightarrow 1 = \cancel{A(-2+2)^2} + \cancel{B(-2+1)(-2+2)} + C(-2+1)$
 $1 = C(-1) = -C \rightarrow C = -1$

Al evaluar en $s = 0 \rightarrow 1 = A(0+2)^2 + B(0+1)(0+2) + C(0+1)$
 $1 = 4 + 2B - 1 \rightarrow B = -2/2 \rightarrow B = -1$

Substitute

$$A = 1$$

$$C = -1$$

Substituímos $\rightarrow \frac{1}{(s+1)(s+2)^2} = \frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{(s+2)^2}$ Usando transformadas conocidas:

$$\mathcal{L}^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}u(t) \rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}u(t), \mathcal{L}^{-1}\left\{\frac{-1}{s+2}\right\} = -e^{-2t}u(t).$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+a)^n}\right\} = \frac{t^{n-1}}{(n-1)!} e^{-at} \rightarrow \mathcal{L}^{-1}\left\{\frac{-1}{(s+2)^2}\right\} = -te^{-2t}u(t).$$

$$x(t) = (e^{-t} - e^{-2t} - te^{-2t})u(t). \\ = (e^t - 1 - t)e^{-2t}u(t).$$