

Taller 2.

Punto 1.3: Encuentre la función de densidad espectral (transformada de Fourier) para las siguientes señales (sin aplicar propiedades del)

a. $e^{-at}u(t)$, $a \in \mathbb{R}^+$ $|t| \rightarrow$ se puede partition en $-t = (-\infty, 0)$ y $t = (0, \infty)$

$$\begin{aligned} F\{e^{-at}u(t)\} &= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t} dt = \int_{-\infty}^0 e^{(a-j\omega)t} dt + \int_0^{\infty} e^{-(a+j\omega)t} dt \\ &= \frac{1}{a-j\omega} e^{(a-j\omega)t} \Big|_{-\infty}^0 + \frac{-1}{a+j\omega} e^{-(a+j\omega)t} \Big|_0^{\infty} \rightarrow e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0. \\ &= \frac{1}{a-j\omega} (e^0 - e^{-\infty}) + \frac{-1}{a+j\omega} (e^{-\infty} - e^0) = \frac{1}{a-j\omega} + \frac{1}{a+j\omega} \\ &= \frac{(a+j\omega) + (a-j\omega)}{(a-j\omega)(a+j\omega)} = \frac{2a}{a^2 + j\omega a - a^2 - (j\omega)^2} = \frac{2a}{a^2 + \omega^2} = X(\omega) \end{aligned}$$

b. $\cos(\omega_0 t)$, $\omega_0 \in \mathbb{R}$. Usando la propiedad $\cos(\omega_0 t) = (e^{j\omega_0 t} + e^{-j\omega_0 t})/2$

$$\begin{aligned} F\{\cos(\omega_0 t)\} &= \int_{-\infty}^{\infty} \cos(\omega_0 t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} e^{-j\omega t} dt \\ &\equiv \frac{1}{2} \left[\int_{-\infty}^{\infty} e^{j\omega_0 t} e^{-j\omega t} dt + \int_{-\infty}^{\infty} e^{-j\omega_0 t} e^{-j\omega t} dt \right] \\ &= \frac{1}{2} \left[\int_{-\infty}^{\infty} e^{-(\omega-\omega_0)t} dt + \int_{-\infty}^{\infty} e^{-(\omega+\omega_0)t} dt \right] \end{aligned}$$

$$\text{Resolviendo } I_1 = \int_{-\infty}^{\infty} e^{-(\omega-\omega_0)t} dt = \frac{e^{-j\omega_0 T} - e^{j\omega_0 T}}{-j\omega_0} = \frac{(e^{j\omega_0 T} - e^{-j\omega_0 T})}{j\omega_0} \left(\frac{2}{2} \right)$$

$$= \frac{2 \sin(\omega_0 T)}{\omega_0} \quad \text{Si } \lim_{T \rightarrow \infty} \frac{\sin(\omega_0 T)}{\pi \omega_0} = \delta(\omega_0) \rightarrow \lim_{T \rightarrow \infty} \frac{2 \sin(\omega_0 T)}{\omega_0} = 2\pi \delta(\omega_0)$$

$$\text{Finalmente } \rightarrow \int_{-\infty}^{\infty} e^{-(\omega-\omega_0)t} dt = 2\pi \delta(\omega - \omega_0)$$

$$\rightarrow \text{Así } \frac{1}{2} 2\pi [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] = \pi [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] = X(\omega)$$

c. $\operatorname{sen}(wst)$; $w_s \in \mathbb{R}$ Usando la propiedad $\rightarrow \operatorname{sen}(wst) = (e^{jwst} - e^{-jwst})/2j$

$$\begin{aligned} F\{\operatorname{sen}(wst)\} &= \int_{-\infty}^{\infty} \operatorname{sen}(wst) e^{-jwt} dt = \int_{-\infty}^{\infty} \frac{e^{jwst} - e^{-jwst}}{2j} e^{-jwt} dt \\ &= \frac{1}{2j} \left[\int_{-\infty}^{\infty} e^{jwst} e^{-jwt} dt - \int_{-\infty}^{\infty} e^{-jwst} e^{-jwt} dt \right] \\ &= \frac{1}{2j} \left[\int_{-\infty}^{\infty} e^{-(w-w_s)jt} dt - \int_{-\infty}^{\infty} e^{-(w+w_s)jt} dt \right] \end{aligned}$$

En el ejercicio b demostramos que $\int_{-\infty}^{\infty} e^{-xit} dt = 2\pi \delta(x)$ Así:

$$\begin{aligned} &\hookrightarrow = \frac{1}{2j} 2\pi \left[\delta(w-w_s) - \delta(w+w_s) \right] = \frac{\pi}{j} \left[\delta(w-w_s) - \delta(w+w_s) \right] \left(\frac{j}{j} \right) \\ &= \frac{\pi j}{-1} \left[\delta(w-w_s) - \delta(w+w_s) \right] = -\pi j \left[\delta(w-w_s) - \delta(w+w_s) \right] \\ &= \pi j \left[\delta(w+w_s) - \delta(w-w_s) \right] = X(w) \end{aligned}$$

d. $f(t) \cos(w_c t)$, $w_c \in \mathbb{R}$, $f(t) \in \mathbb{R}, \mathbb{C}$.

$$\begin{aligned} F\{f(t) \cos(w_c t)\} &= \int_{-\infty}^{\infty} f(t) \cos(w_c t) e^{-jwt} dt \\ &= \int_{-\infty}^{\infty} f(t) \frac{e^{jw_c t} + e^{-jw_c t}}{2} e^{-jwt} dt = \frac{1}{2} \int_{-\infty}^{\infty} f(t) (e^{jw_c t} + e^{-jw_c t}) e^{-jwt} dt \\ &= \frac{1}{2} \left[\int_{-\infty}^{\infty} f(t) e^{jw_c t} e^{-jwt} dt + \int_{-\infty}^{\infty} f(t) e^{-jw_c t} e^{-jwt} dt \right] \\ &\quad - \frac{1}{2} \left[\int_{-\infty}^{\infty} f(t) e^{-(w-w_c)t} dt + \int_{-\infty}^{\infty} f(t) e^{(w+w_c)t} dt \right] \end{aligned}$$

Tenemos que $\int_{-\infty}^{\infty} f(t) e^{-j(w-w_c)t} dt = F(w-w_c) \rightarrow$ Transformada t desplazada.

$$\hookrightarrow = \frac{1}{2} [F(w-w_c) + F(w+w_c)] = X(w).$$

e. e^{-at^2} , $a \in \mathbb{R}^+$ $t \rightarrow$ siempre es positivo.

$$F\{e^{-at^2}\} = \int_{-\infty}^{\infty} e^{-at^2} e^{-jw t} dt$$

$$\text{Operando exponentes} \rightarrow -at^2 - jw t = -a(t^2 + \frac{jw t}{a}) = -a \left[\left(t + \frac{jw}{2a} \right)^2 - \left(\frac{jw}{2a} \right)^2 \right]$$

$$= -a \left[\left(t + \frac{jw}{2a} \right)^2 - \frac{j^2 w^2}{2^2 a^2} \right] = -a \left[\left(t + \frac{jw}{2a} \right)^2 + \frac{w^2}{4a^2} \right]$$

$$= -a \left(t + \frac{jw}{2a} \right)^2 - \frac{aw^2}{4a^2} = -a \left(t + \frac{jw}{2a} \right)^2 - \frac{w^2}{4a} \quad \text{Así:}$$

$$\boxed{I_1 = \int_{-\infty}^{\infty} e^{-w^2/4a} e^{-a(t+jw/2a)^2} dt = e^{-w^2/4a} \int_{-\infty}^{\infty} e^{-a(t+jw/2a)^2} dt}$$

$$I_1 \rightarrow u = t + \frac{jw}{2a} \quad du = dt \rightarrow I_1 = \int_{-\infty}^{\infty} e^{-au^2} du.$$

$$I_1^2 = \left(\int_{-\infty}^{\infty} e^{-ax^2} dx \right) \left(\int_{-\infty}^{\infty} e^{-ay^2} dy \right) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-a(x^2+y^2)} dx dy$$

Haciendo uso de coordenadas polares $\rightarrow x = r \cos \theta \quad dx dy = r dr d\theta$
 $y = r \sin \theta \quad r \in [0, \infty), \theta \in [0, 2\pi)$

$$I_1^2 = \int_0^{2\pi} \int_0^{\infty} e^{-ar^2} r dr d\theta = 2\pi \int_0^{\infty} r e^{-ar^2} dr \rightarrow s = ar^2 \quad \begin{cases} r dr = ds \\ ds = 2a r dr \end{cases}$$

$$I_1^2 = 2\pi \int_0^{\infty} e^{-s} \frac{ds}{2a} = \frac{2\pi}{2a} \int_0^{\infty} e^{-s} ds = \frac{\pi}{a} \left(-\frac{1}{e^a} - (-1) \right) = \frac{\pi}{a} \rightarrow I = \sqrt{\frac{\pi}{a}}$$

$$\rightarrow = e^{-w^2/4a} \sqrt{\frac{\pi}{a}} = \chi(w)$$

f. $\text{Arecta}(t)$; $A, d \in \mathbb{R}$ $\text{recta}(t) = \begin{cases} 1, & |t| \leq d/2 \\ 0, & |t| > d/2 \end{cases}$

$$F\{\text{Arecta}(t)\} = \int_{-\infty}^{\infty} \text{Arecta}(t) e^{-jw t} dt = \int_{-d/2}^{d/2} A e^{-jw t} dt.$$

$$= A \left(\frac{e^{-jw t}}{-jw} \right) \Big|_{-d/2}^{d/2} = -\frac{A}{jw} (e^{-jwd/2} - e^{(jw)(-d/2)})$$

$$= \frac{2A}{w} \left(\frac{e^{-jwd/2} - e^{(jw)(-d/2)}}{2j} \right) = \frac{2A}{w} \sin \left(\frac{wd}{2} \right) = \chi(w).$$

Punto 1.4: Aplique las propiedades de la transformada de Fourier para resolver:

a. $F\{e^{-jw_1 t} \cos(w_0 t)\}$, $w_1, w_0 \in \mathbb{R}$

$$e^{-jw_1 t} \cos(w_0 t) = \frac{e^{jw_0 t} + e^{-jw_0 t}}{2} e^{-jw_1 t} dt$$

$$= \frac{1}{2} (e^{jw_0 t} e^{-jw_1 t} + e^{-jw_0 t} e^{-jw_1 t}) = \frac{1}{2} (e^{-jt(w_0 - w_1)} + e^{-jt(w_0 + w_1)})$$

sabemos que $F\{e^{jw_0 t}\} = 2\pi \delta(w - w_0)$

$$\Rightarrow = \frac{1}{2} 2\pi (\delta(w - (-w_1 + w_0)) + \delta(w - (w_1 + w_0)))$$

$$= \pi [\delta(w + w_1 - w_0) + \delta(w + w_1 + w_0)] = X(w)$$

b. $F\{u(t) \cos^2(w_0 t)\}$ $w_0 \in \mathbb{R}^+$ Si $u(t) = 1$ de $(0, \infty)$

$$\cos^2(w_0 t) = \frac{1 + \cos(2w_0 t)}{2}; u(t) \cos^2(w_0 t) = \frac{1}{2} [u(t) + u(t) \cos(2w_0 t)]$$

$$F\{u(t) \cos^2(w_0 t)\} = \frac{1}{2} (F\{u(t)\} + F\{u(t) \cos(2w_0 t)\})$$

si $F\{u(t)\} = F\{\text{sgn}\} + F\{1|2\} = \frac{1}{jw} + \pi \delta(w)$

$$\text{y } F\{u(t) \cos(2w_0 t)\} = \frac{\pi}{2} (\delta(w - w_0) + \delta(w + w_0)) + \frac{jw}{w_0^2 - w^2}$$

$$\Rightarrow = \frac{1}{2} \left[\frac{1}{jw} + \pi \delta(w) + \frac{\pi}{2} (\delta(w - 2w_0) + \delta(w + 2w_0)) + \frac{jw}{(2w_0)^2 - w^2} \right] = X(w)$$

c. $F^{-1} \left\{ \frac{7}{w^2 + 6w + 45} * \frac{10}{(8 + jw/3)^2} \right\}$

Producto en frecuencia = $F^{-1}\{G(w)H(w)\}(t) = \frac{1}{2\pi} (g * h)(t)$
 Convolución en t

Donde $g(t) = F^{-1}\{G(w)\}$ y $h(t) = F^{-1}\{H(w)\}$

$$G(w) = \frac{7}{w^2 + 6w + 45} \rightarrow w^2 + 6w + 45 = (w+3)^2 + 36 = (w+3)^2 + 6^2$$

$$\text{Si } e^{-at} \rightarrow \frac{2a}{a^2 + w^2}; \frac{1}{(w - w_0)^2 + a^2} \rightarrow \frac{1}{2a} e^{-at} e^{jwst} \quad a=6 \quad w_0=-3$$

$$\rightarrow g(t) = F^{-1} \left\{ \frac{7}{(w+3)^2 + 6^2} \right\} = \frac{7}{2(6)} e^{-6t} e^{-j3t} = \frac{7}{12} e^{-6t} e^{-j3t}$$

$$H(w) = \frac{10}{(8+jw/3)^2} \rightarrow (8+jw/3)^2 = (1/3(24+jw))^2 = 1/9 (24+jw)^2$$

$$\text{Si } F\{te^{-at}u(t)\} = \frac{1}{(a+jw)^2} \text{ para } a > 0 \quad a=24$$

$$\rightarrow = F^{-1} \left\{ \frac{90}{(24+jw)} \right\} = 90te^{-24t}u(t) = h(t)$$

$$x(t) = F^{-1}\{G \cdot H\} = \frac{1}{2\pi} (g+h) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\tau) h(t-\tau) d\tau$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{7}{12} e^{-6|\tau|} e^{-j3\tau} 90(t-\tau) e^{-24(t-\tau)} u(t-\tau) d\tau$$

$$x(t) = \frac{1}{2\pi} \frac{7(90)}{12} \int_{-\infty}^{\infty} e^{-6|\tau|} e^{-j3\tau} (t-\tau) e^{-24(t-\tau)} u(t-\tau) d\tau$$

$$x(t) = \frac{105}{4\pi} \int_{-\infty}^{\infty} e^{-6|\tau|} e^{-j3\tau} (t-\tau) e^{-24(t-\tau)} u(t-\tau) d\tau$$

$$x(t) = \frac{105}{4\pi} \int_{-\infty}^{\infty} e^{-6|\tau|} e^{-j3\tau} (t-\tau) e^{-24(t-\tau)} d\tau \rightarrow \text{si } u(t-\tau) = 1 \text{ en } (-\infty, t]$$

$$x(t) = \frac{105}{4\pi} \left(\int_{-\infty}^0 e^{6\tau} e^{-j3\tau} (t-\tau) e^{-24(t-\tau)} d\tau + \int_0^t e^{-6\tau} e^{-j3\tau} (t-\tau) e^{-24(t-\tau)} d\tau \right)$$

$$x(t) = \frac{105}{4\pi} e^{-24t} \left(\int_{-\infty}^0 e^{6\tau} e^{-j3\tau} (t-\tau) e^{24\tau} d\tau + \int_0^t e^{-6\tau} e^{-j3\tau} (t-\tau) e^{24\tau} d\tau \right)$$

$$x(t) = \frac{105}{4\pi} e^{-24t} \left(\int_{-\infty}^0 (t-\tau) e^{\tau(30-j3)} d\tau + \int_0^t (t-\tau) e^{\tau(-30-j3)} d\tau \right)$$

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$$\begin{aligned}
 I_1 &= \int_{-\infty}^0 t e^{(30-j3)t} dt - \int_{-\infty}^0 t e^{t(-30+j3)t} dt \\
 &= t \frac{e^{(30-j3)t}}{30-j3} \Big|_{-\infty}^0 - \left(t \frac{e^{(30-j3)t}}{30-j3} - \frac{e^{(30-j3)t}}{(30-j3)^2} \right) \Big|_{-\infty}^0 \\
 &= \frac{t}{30-j3} \left(e^{(30-j3)t} - \cancel{\frac{1}{e^{(30-j3)t}}} \right) - \left[\cancel{\frac{te^0}{30-j3}} - \frac{e^{(30-j3)t}}{(30-j3)^2} \right] \left(\cancel{\frac{1}{(30-j3)e^0}} - \cancel{\frac{1}{(30-j3)^2 e^0}} \right) \\
 &= \frac{t}{30-j3} + \frac{1}{(30-j3)^2}
 \end{aligned}$$

$$\begin{aligned}
 I_2 &= \int_0^t (t-t) e^{(-30+j3)t} dt = \int_0^t t e^{(-30+j3)t} dt - \int_0^t t e^{t(-30+j3)t} dt \\
 &= t \frac{e^{(-30+j3)t}}{-30+j3} \Big|_0^t - \left(t \frac{e^{(-30+j3)t}}{-30+j3} - \frac{e^{(-30+j3)t}}{(-30+j3)^2} \right) \Big|_0^t \\
 &= \frac{t}{-30+j3} \left(e^{(-30+j3)t} - \cancel{\frac{1}{e^{(-30+j3)t}}} \right) - \left[\cancel{\frac{e^{(-30+j3)t}}{-30+j3}} - \frac{e^{(-30+j3)t}}{(-30+j3)^2} - \left(\cancel{\frac{0e^0}{-30+j3}} - \cancel{\frac{e^0}{(-30+j3)^2}} \right) \right] \\
 &= \frac{t}{-30+j3} \left(e^{(-30+j3)t} - 1 \right) - \left(\frac{t e^{(-30+j3)t}}{-30+j3} - \frac{e^{(-30+j3)t}}{(-30+j3)^2} + \frac{1}{(-30+j3)^2} \right) \\
 &= \frac{t e^{(-30+j3)t}}{-30+j3} - \frac{t}{-30+j3} - \frac{t e^{(-30+j3)t}}{-30+j3} + \frac{e^{(-30+j3)t}}{(-30+j3)^2} + \frac{1}{(-30+j3)^2} \\
 &= \frac{e^{(-30+j3)t} - 1}{(-30+j3)^2} + \frac{t}{30+j3}
 \end{aligned}$$

$$x(t) = \frac{105}{4\pi} e^{-24t} \left(\frac{t}{30-j3} + \frac{1}{(30+j3)^2} + \frac{e^{(-30+j3)t} - 1}{(30-j3)^2} + \frac{t}{30+j3} \right)$$

$$x(t) = \frac{105}{4\pi} e^{-24t} \left[t \left(\frac{1}{30-j3} + \frac{1}{30+j3} \right) + \left(\frac{1}{(30-j3)^2} - \frac{1}{(30+j3)^2} \right) + \frac{e^{(-30+j3)t}}{(30-j3)^2} \right]$$

$$x(t) = \frac{105}{4\pi} e^{-24t} \left(t \frac{60}{30^2+3^2} + \frac{360j}{909^2} + \frac{e^{(-30+j3)t}}{(30+j3)^2} \right)$$

d. $F\{3t^3\} \rightarrow$ si $F\{t^n x(t)\} = j^n \frac{d^n}{dw^n} X(w)$, $x(t) = 1 \rightarrow F\{1\} = 2\pi \delta(w)$.

$$F\{3t^3\} = 3j^3 \frac{d^3}{dw^3} (2\pi \delta(w)) = j^2 j 6\pi \delta^{(3)}(w) = -j 6\pi \delta^{(3)}(w).$$

e. $\frac{B}{T} \sum_{n=-\infty}^{+\infty} \left(\frac{1}{a^2 + (w - nw_0)^2} + \frac{1}{a + j(w - nw_0)} \right)$ donde $n \in \{0, \pm 1, \pm 2, \dots\}$
 $w_0 = 2\pi/T \quad B, T \in \mathbb{R}^+$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{B}{T} \sum_{n=-\infty}^{+\infty} \left(\frac{1}{a^2 + (w - nw_0)^2} + \frac{1}{a + j(w - nw_0)} \right) e^{jwt} dw$$

$v = w - nw_0$
 $dv = dw$

$$x(t) = \frac{B}{T 2\pi} \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{\infty} \left(\frac{1}{a^2 + (v + nw_0)^2} + \frac{1}{a + j(v + nw_0)} \right) e^{jvt} dv$$

$w = v + nw_0$

$$x(t) = \frac{B}{T 2\pi} \sum_{n=-\infty}^{+\infty} \left(\int_{-\infty}^{\infty} \frac{1}{a^2 + v^2} e^{j(v + nw_0)t} dv + \int_{-\infty}^{\infty} \frac{1}{a + jv} e^{j(v + nw_0)t} dv \right)$$

$$-x(t) = \frac{B}{T 2\pi} \sum_{n=-\infty}^{+\infty} \left(e^{jnwt} \left[\int_{-\infty}^{\infty} \frac{1}{a^2 + v^2} e^{jvt} dv + \int_{-\infty}^{\infty} \frac{1}{a + jv} e^{jvt} dv \right] \right)$$

Si $A(v) = \frac{1}{a^2 + v^2} \rightarrow a(t) = F^{-1}\{A(v)\} = \frac{1}{2a} e^{-|at|}$ para $a > 0$.

$B(v) = \frac{1}{a + jv} \rightarrow b(t) = F^{-1}\{B(v)\} = e^{-at} u(t)$ para $a > 0$.

$$\hookrightarrow x(t) = \frac{B}{T 2\pi} \sum_{n=-\infty}^{+\infty} e^{jnwt} \left(2\pi \frac{1}{2a} e^{-|at|} + 2\pi e^{-at} u(t) \right)$$

$$x(t) = \frac{B 2\pi}{T 2\pi} \left(\frac{1}{2a} e^{-|at|} + e^{-at} u(t) \right) \sum_{n=-\infty}^{+\infty} e^{jnwt} = 2\pi \sum_{k=-\infty}^{+\infty} \delta(wt - 2\pi k)$$

$$x(t) = \frac{B}{T} \left(\frac{1}{2a} e^{-|at|} + e^{-at} u(t) \right) T \sum_{k=-\infty}^{+\infty} \delta(t - Tk) = 2\pi \sum_{k=-\infty}^{+\infty} \frac{1}{w_0} \delta(t - \frac{2\pi k}{w_0}) \quad (w_0 = 2\pi/T)$$

$$x(t) = B \left(\frac{1}{2a} e^{-|at|} + e^{-at} u(t) \right) \sum_{k=-\infty}^{+\infty} \delta(t - Tk) = \frac{2\pi}{w_0} \sum_{k=-\infty}^{+\infty} \delta(t - Tk) = T \sum_{k=-\infty}^{+\infty} \delta(t - Tk)$$

Punto 2.2: • Compruébese la solución $h(t)$ de la EDO cuando $x(t) = f(t)$ de manera manual. Tener en cuenta que $E'(t) = u'(t) = f(t)$.

Sistema $\rightarrow y'(t) + y(t) = x(t)$ para cualquier entrada $x(t)$
sabemos que $h(t) = y(t)$ cuando $x(t) = f(t)$

Reemplazando $\rightarrow y'(t) + y(t) = f(t)$, $[-\varepsilon, \varepsilon] \rightarrow$ intervalo muy pequeño $\varepsilon \rightarrow 0$.

$$\int_{-\varepsilon}^{\varepsilon} (y'(t) + y(t)) dt = \int_{-\varepsilon}^{\varepsilon} f(t) dt \quad f(t) \rightarrow \text{se cumple solo en } t=0 \\ \text{y su área} = 1$$

$$\rightarrow y(\varepsilon) - y(-\varepsilon) + \int_{-\varepsilon}^{\varepsilon} y(t) dt = 1 \quad \text{si } \varepsilon \rightarrow 0 \text{ la integral también} \rightarrow 0.$$

$$\rightarrow y(0^+) - y(0^-) = 1 \quad \text{se asume un sistema causal } y(0^-) = 0$$

$$\rightarrow y(0^+) = 1 \quad \text{así para } t > 0 \text{ la EDO queda} \rightarrow y'(t) + y(t) = 0 \text{ homogénea.}$$

La solución general es $y(t) = Ce^{-t}$, $t > 0$

$$(con la condición y(0^+) = 1 \rightarrow Ce^0 = 1 \rightarrow C = 1.)$$

$$\text{Finalmente } h(t) = e^{-t} u(t) \rightarrow u(t) \text{ para apagar todo en } y(0^-)$$

• Comprobá la solución de la integral de convolución de manera manual. Tener en cuenta las funciones Heaviside.

$$h(t) = e^{-t} u(t), \quad x(t) = e^{-2t} u(t) \quad \text{hallar } y(t) = h(t) * x(t)$$

Definición de convolución $\rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau \quad (\text{como } h(t) \text{ y } x(t) \text{ tienen } u(t) \rightarrow t < 0 = 0)$

$$y(t) = \int_0^t x(\tau) h(t - \tau) d\tau = \int_0^t e^{-2\tau} e^{-(t-\tau)} d\tau = \int_0^t e^{-2\tau-t+\tau} d\tau$$

$$= \int_0^t e^{-t-\tau} d\tau = e^{-t} \int_0^t e^{-\tau} d\tau = e^{-t} [-e^{-\tau}]_0^t = e^{-t} (-e^{-t} + e^0)$$

$$= e^{-t} - e^{-t-t} = e^{-t} - e^{-2t}, \quad t > 0 \rightarrow y(t) = (e^{-t} - e^{-2t}) u(t).$$

PUNTO 2.3: Demuestra si los siguientes sistemas de la forma $y = h[x]$, son sistemas lineales e invariantes en el tiempo (SLIT). Simula los sistemas en Python:

- $y[n] = x[n]/3 + 2x[n-1] - y[n-1]$

$$\textcircled{1} \quad x(t) = d_1 x_1(t) + d_2 x_2(t)$$

$$y[n] = (1/3)(d_1 x_1[n] + d_2 x_2[n]) + 2(d_1 x_1[n-1] + d_2 x_2[n-1]) - (d_1 y_1[n-1] + d_2 y_2[n-1])$$

$$\textcircled{2} \quad d_1 x_1(t) + d_2 x_2(t)$$

$$d_1(x_1[n]/3 + 2x_1[n-1] - y_1[n-1]) + d_2(x_2[n]/3 + 2x_2[n-1] - y_2[n-1])$$

$$\tilde{y}[n] = d_1 x_1[n]/3 + 2d_1 x_1[n-1] - d_1 y_1[n-1] + d_2 x_2[n]/3 + 2d_2 x_2[n-1] - d_2 y_2[n-1]$$

$$\tilde{y}[n] = (1/3)(d_1 x_1[n] + d_2 x_2[n]) + 2(d_1 x_1[n-1] + d_2 x_2[n-1]) - (d_1 y_1[n-1] + d_2 y_2[n-1])$$

Es lineal! $\textcircled{1} = \textcircled{2}$

$$\textcircled{1} \quad x(t) = x(t-t_0)$$

$$y[n] = (1/3)x[n-n_0] + 2x[n-1-n_0] - y[n-1]$$

$$\textcircled{2} \quad \tilde{y}(t-t_0) = x(t-t_0)$$

$$\tilde{y}[n-n_0] = (1/3)x[n-n_0] + 2x[n-1-n_0] - y[n-1-n_0]$$

solo es invariante en el tiempo si $y[n-1] = y[n-1-n_0] \rightarrow n_0 = 0$.

En conclusión es SLIT ya que cumple linealidad y el invariante en el tiempo con $n_0 = 0$.

- $y[n] = \sum_{k=-\infty}^n x^2[k]$.

$$\textcircled{1} \quad x(t) = d_1 x_1(t) + d_2 x_2(t)$$

$$y[n] = \sum_{k=-\infty}^n (d_1 x_1[k] + d_2 x_2[k])^2$$

$$y[n] = \sum_{k=-\infty}^n (d_1^2 x_1^2[k] + 2d_1 d_2 x_1[k] x_2[k] + d_2^2 x_2^2[k])$$

$$\textcircled{2} \quad d_1 x_1(t) + d_2 x_2(t)$$

$$\tilde{y}[n] = \sum_{k=-\infty}^n d_1(x_1^2[k]) + d_2(x_2^2[k])$$

NO se cumple linealidad. $\textcircled{1} \neq \textcircled{2}$

$$\textcircled{1} \quad x(t) = x(t-t_0)$$

$$y[n] = \sum_{k=-\infty}^n x^2[k-n_0]$$

$$\textcircled{2} \quad \tilde{y}(t-t_0) = x(t-t_0)$$

$$\tilde{y}[n-n_0] = \sum_{k=-\infty}^{n-n_0} x^2[k]$$

Para $\textcircled{1}$: $m = k - n_0$

$k = -\infty \rightarrow m = -\infty$

$k = n \rightarrow m = n - n_0$

Es invariante en el tiempo $\textcircled{1} = \textcircled{2}$

En conclusión NO es SLIT ya que aunque es invariante en el tiempo no es lineal.

- $y[n] = \text{median}(x[n])$; donde median es la función mediana sobre una ventana de tamaño 3. $y[n] = \text{median}(x[n-1], x[n], x[n+1])$

Propiedad del median()

- $\text{median}(a+b) \neq \text{median}(a) + \text{median}(b)$
- $\text{median}(ax) = a \text{median}(x)$

$$\textcircled{1} \quad x(t) = d_1 x_1(t) + d_2 x_2(t).$$

$$y[n] = \text{median}(d_1 x_1[n-1] + d_2 x_2[n-1], d_1 x_1[n] + d_2 x_2[n], d_1 x_1[n+1] + d_2 x_2[n+1])$$

$$\textcircled{2} \quad d_1 x_1(t) + d_2 x_2(t).$$

$$\tilde{y}[n] = d_1 \text{median}(x_1[n-1], x_1[n], x_1[n+1]) + d_2 \text{median}(x_2[n-1], x_2[n], x_2[n+1])$$

$$\tilde{y}[n] = \text{median}(d_1 x_1[n-1], d_1 x_1[n], d_1 x_1[n+1]) + \text{median}(d_2 x_2[n-1] + d_2 x_2[n] + d_2 x_2[n+1])$$

(Como $\text{median}(a+b) \neq \text{median}(a) + \text{median}(b)$)

→ NO se cumple linealidad. $\textcircled{1} \neq \textcircled{2}$

$$\textcircled{1} \quad x(t) = x(t-t_0)$$

$$y[n] = \text{median}(x[n-1-n_0], x[n-n_0], x[n+1-n_0])$$

$$\textcircled{2} \quad \tilde{y}(t-t_0) = x(t-t_0)$$

$$\tilde{y}[n-n_0] = \text{median}(x[n-1-n_0], x[n-n_0], x[n+1-n_0])$$

Es invariante en el tiempo $\textcircled{1} = \textcircled{2}$

En conclusión NO es SIT ya que aunque es invariante en el tiempo no es

- $y(t) = Ax(t) + B; A, B \in \mathbb{R}$

$$\textcircled{1} \quad v(t) = d_1 x_1(t) + d_2 x_2(t)$$

$$v(t) = A(d_1 x_1(t) + d_2 x_2(t)) + B = Ad_1 x_1(t) + Ad_2 x_2(t) + \underbrace{B}_{\neq}$$

$$\textcircled{2} \quad d_1 x_1(t) + d_2 x_2(t)$$

$$v(t) = d_1(Ax_1(t) + B) + d_2(Ax_2(t) + B) = Ad_1 x_1(t) + Ad_2 x_2(t) + \underbrace{d_1 B + d_2 B}_{=}$$

Solo cumple linealidad si $B = d_1 B + d_2 B \rightarrow d_1 + d_2 = 1 \rightarrow d_1 = d_2 = 1/2$.

$$\textcircled{1} \quad v(t) = x(t-t_0)$$

$$v(t) = A(x(t-t_0) + B)$$

$$\textcircled{2} \quad \tilde{v}(t-t_0) = x(t-t_0)$$

$$\tilde{v}(t-t_0) = Ax(t-t_0) + B$$

Es invariante en el tiempo $\textcircled{1} = \textcircled{2}$

En conclusión NO es SIT, ya que aunque es invariante en el tiempo no es

lineal.

Punto 2.5: sea la señal gaussiana $x(t) = e^{-at^2}$ con $a \in \mathbb{R}^+$, el sistema A con relación entrada-salida $y_A(t) = x^2(t)$, y el sistema lineal e invariante con el tiempo B con respuesta al impulso $h_B(t) = Be^{-bt^2}$:

a. Encuentre la salida del sistema en serie $x(t) \xrightarrow{\text{A}} h_B(t) \xrightarrow{*} y_A(t) \rightarrow y(t)$

$$v(t) + d(t) = \int_{-\infty}^{\infty} x(\tau) d(t-\tau) d\tau$$

$$v(t) = x(t) * h(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \quad \rightarrow \text{Respuesta al impulso desplazado}$$

$$h_B = Be^{-bt^2} \quad x(t) = e^{-at^2} \quad y_A(t) = x^2(t) \quad \rightarrow \text{Info que tenemos}$$

$$x(t) \xrightarrow{\text{A}} v(t) \xrightarrow{*} y(t) \quad \rightarrow \text{Pide } \Theta \quad \text{se resuelve aparte.}$$

$$v(t) = x(t) * h_B = \int_{-\infty}^{\infty} e^{-a\tau^2} Be^{-b(t-\tau)^2} d\tau = B \int_{-\infty}^{\infty} e^{-(a\tau^2 + b(t-\tau)^2)} d\tau$$

$$\rightarrow a\tau^2 + b(t-\tau)^2 = a\tau^2 + b(t^2 - 2t\tau + \tau^2) = a\tau^2 + bt^2 - 2bt\tau + b\tau^2 \\ = (a+b)\tau^2 - 2bt\tau + bt^2$$

\rightarrow

$$\rightarrow (a+b)\left(\tau^2 - \frac{2bt\tau}{a+b}\right) = (a+b)\left[\left(\tau - \frac{bt}{a+b}\right)^2 - \left(\frac{bt}{a+b}\right)^2\right]$$

$$= a\tau^2 + b(t-\tau)^2 = (a+b)\left(\tau - \frac{bt}{a+b}\right)^2 bt^2 - (a+b)\left(\frac{bt}{a+b}\right)^2$$

$$\rightarrow bt^2 - (a+b)\frac{bt^2}{(a+b)^2} = bt^2 - \frac{b^2t^2}{a+b} = \frac{b(a+b)-b^2}{a+b} t^2 = \frac{ab}{a+b} t^2$$

$$= a\tau^2 + b(t-\tau)^2 = (a+b)\left(\tau - \frac{bt}{a+b}\right)^2 + \frac{ab}{a+b} t^2.$$

Integral Gaussiana.

$$\rightarrow B \int_{-\infty}^{\infty} e^{-((a+b)(\tau - \frac{bt}{a+b})^2 + \frac{ab}{a+b} t^2)} d\tau = B e^{-\frac{ab}{a+b} t^2} \int_{-\infty}^{\infty} e^{-((a+b)(\tau - \frac{bt}{a+b})^2)} d\tau$$

$$= B e^{-\frac{ab}{a+b} \sqrt{\frac{\pi}{a+b}}} \rightarrow \text{usando la propiedad}$$

$$y(t) = x^2(t) = \left(B e^{-\frac{ab}{a+b} t^2} \sqrt{\frac{\pi}{a+b}}\right)^2 = \frac{\pi}{a+b} B^2 e^{-\frac{2ab}{a+b} t^2} = \frac{B^2 \pi}{a+b} e^{-\frac{2ab}{a+b} t^2}$$

b. Encuentre la salida del sistema en serie $x(t) \rightarrow y_A(t) \xrightarrow{h_B(t)} y(t)$

$$x(t) \xrightarrow{*} u(t) = y_A(t) \xrightarrow{h_B} y(t)$$

$$u(t) = x^2(t) = e^{-2at^2}$$

$$y(t) = u(t) * h_B(t) = \int_{-\infty}^{\infty} e^{-2a\tau^2} B e^{-b(t-\tau)^2} d\tau = B \int_{-\infty}^{\infty} e^{-(2a\tau + b(t-\tau))^2} d\tau$$

se resuelve aparte

$$\begin{aligned} & \rightarrow 2a\tau^2 + b(t-\tau)^2 = (2a+b)\tau^2 - 2bt + bt^2 \\ & = (2a+b)\left(\tau - \frac{bt}{2a+b}\right)bt^2 - (2a+b)\left(\frac{bt}{2a+b}\right)^2 \end{aligned}$$

$$\rightarrow bt^2 - \frac{b^2t^2}{2a+b} = \frac{b(2a+b) - b^2}{2a+b} t^2 = \frac{2ab}{2a+b} t^2$$

$$B \int_{-\infty}^{\infty} e^{-((2a+b)(\tau - \frac{bt}{2a+b})^2 + \frac{2ab}{2a+b} t^2)} d\tau = B e^{-\frac{2ab}{2a+b}} \int_{-\infty}^{\infty} e^{-(2a+b)(\tau - \frac{bt}{2a+b})^2} d\tau$$

$$= B e^{-\frac{2ab}{2a+b} t^2} \sqrt{\frac{\pi}{2a+b}}$$

→ usando la propiedad.

$$y(t) = B \sqrt{\frac{\pi}{2a+b}} e^{-\frac{2ab}{2a+b} t^2}$$

Punto 2.10: • Obtener la transformada inversa de Laplace de $X(s)$ manualmente.

Dada $X(s) = \frac{2s^2 + 14s + 124}{s^3 + 8s^2 + 46s + 68}$ con Región de convergencia $\rightarrow \text{Re}(s) > -2$.

Comprobar si $s = -2$ es raíz del denominador $\rightarrow (-2)^3 + 8(-2)^2 + 46(-2) + 68 = -8 + 32 - 92 + 68 = 0$ \rightarrow se divide entre $(s+2)$

División sintética \rightarrow

s^3	s^2	s^1	s^0	$s = -2$	$s^3 + 8s^2 + 46s + 68$
1	8	46	68		$= s^2 + 6s + 34 (s+2)$
1	6	34	0	→ Si es raíz.	

Fracctional por particiones $\rightarrow \frac{2s^2 + 14s + 124}{(s+2)(s^2 + 6s + 34)} = \frac{A}{s+2} + \frac{Bs + C}{s^2 + 6s + 34}$ Multiplicamos por

$$\begin{aligned} \rightarrow 2s^2 + 14s + 124 &= A(s^2 + 6s + 34) + (Bs + C)(s+2) \\ \rightarrow 2s^2 + 14s + 124 &= As^2 + 6As + 34A + Bs^2 + 2Bs + Cs + 2C \\ \rightarrow 2s^2 + 14s + 124 &= (A+B)s^2 + (6A + 2B + C)s + (34A + 2C) \end{aligned}$$

Igualando factores $\rightarrow A + B = 2$, $6A + 2B + C = 14$
 $B = 2 - A$, $6A + 2(2 - A) + C = 14$
 $6A + 4 - 2A + C = 14$
 $4A + 4 = 10 \rightarrow C = 10 - 4A$.

$$\begin{aligned} 34A + 2C &= 124. \\ 34A + 2(10 - 4A) &= 124. \\ 34A + 20 - 8A &= 124. \\ 26A &= 104. \rightarrow A = 104 / 26 = 4 \quad C = 10 - 4(4) = 10 - 16 = -6. \end{aligned}$$

se obtiene $A = 4$, $B = -2$, $C = -6$.

Se completa el cuadrado

$$X(s) = \frac{4}{s+2} + \frac{-2s - 6}{s^2 + 6s + 34} \rightarrow s^2 + 6s + 34 = (s^2 + 6s + (6/2)^2) - (6/2)^2 + 34. \quad \text{Denominador } \underbrace{(s^2 + 6s + 9)}_{(s+3)^2} \underbrace{(-9 + 34)}_{+25} = 25.$$

Numerador $\rightarrow -2s - 6 = -2(s+3)$

$$X(s) = \frac{4}{s+2} + \frac{-2(s+3)}{(s+3)^2 + 25} \quad \text{Usamos transformadas (inversas):}$$

$$\mathcal{I}^{-1} \left\{ \frac{1}{s+a} \right\} = t^{-at} u(t) \rightarrow \mathcal{I}^{-1} \left\{ \frac{4}{s+2} \right\} = 4e^{-2t} u(t).$$

$$\mathcal{I}^{-1} \left\{ \frac{s+a}{(s+a)^2 + w^2} \right\} = e^{-at} \cos(wt) u(t) \rightarrow \mathcal{I}^{-1} \left\{ \frac{-2(s+3)}{(s+3)^2 + 25} \right\} = -2e^{-3t} \cos(5t) u(t)$$

$$x(t) = (4e^{-2t} - 2e^{-3t} \cos(5t)) u(t).$$

• Obtener la transformada inversa de Laplace manualmente de:

$$X(s) = \frac{1}{(s+1)(s+2)^2} \text{ para } \Re\{s\} > -1.$$

Fracciones parciales $\rightarrow \frac{1}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$ multiplicar por $(s+1)(s+2)^2$

se eliminan denominadores $\rightarrow 1 = A(s+2)^2 + B(s+1)(s+2) + C(s+1)$

Al evaluar en $s = -1 \rightarrow 1 = A(-1+2)^2 + B(-1+1)(-1+2) + C(-1+1)$
 $1 = A(1)^2 = A$

Al evaluar en $s = -2 \rightarrow 1 = A(-2+2)^2 + B(-2+1)(-2+2) + C(-2+1)$
 $1 = C(-1) = -C \rightarrow C = -1$

Al evaluar en $s = 0 \rightarrow 1 = A(0+2)^2 + B(0+1)(0+2) + C(0+1)$. $A = 1$
 $1 = 4 + 2B - 1 \rightarrow B = -2/2 \rightarrow B = -1$ $C = -1$ Sustituir

Sustituimos $\rightarrow \frac{1}{(s+1)(s+2)^2} = \frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{(s+2)^2}$ Usando transformadas conocidas:

$$\mathcal{I}^{-1} \left\{ \frac{1}{s+a} \right\} = e^{-at} u(t) \rightarrow \mathcal{I}^{-1} \left\{ \frac{1}{s+1} \right\} = e^{-t} u(t), \quad \mathcal{I}^{-1} \left\{ \frac{-1}{s+2} \right\} = -e^{-2t} u(t).$$

$$\mathcal{I}^{-1} \left\{ \frac{1}{(s+2)^2} \right\} = \frac{t^{n-1}}{n-1!} e^{-2t} \rightarrow \mathcal{I}^{-1} \left\{ \frac{-1}{(s+2)^2} \right\} = -t e^{-2t} u(t).$$

$$x(t) = (e^{-t} - e^{-2t} - t e^{-2t}) u(t)
= (e^t - 1 - t) e^{-2t} u(t).$$