Taller 2

Punto 1.B: Encuentre la función de densidad espectral (transformada de Fourier) para las siguientes señales (sin aplicar propiedades):

a. $e^{-a|t|}$, $a \in \mathbb{R}^+$ $\quad |t| \to$ se puede partir en $t = (-\infty, 0)$ y $t = (0, \infty)$

$$F\{e^{-a|t|}\} = \int_{-\infty}^{\infty} e^{-a|t|} e^{-jwt}\, dt = \int_{-\infty}^{0} e^{(a-jw)t}\, dt + \int_{0}^{\infty} e^{-(a+jw)t}\, dt$$

$$= \frac{1}{a-jw} e^{(a-jw)t} \Big|_{-\infty}^{0} + \frac{-1}{a+jw} e^{-(a+jw)t} \Big|_{0}^{\infty} \to e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$$

$$= \frac{1}{a-jw}\left(\cancel{e^{0}}^{1} - \cancel{e^{-\infty}}^{0}\right) + \frac{-1}{a+jw}\left(\cancel{e^{-\infty}}^{0} - \cancel{e^{0}}^{1}\right) = \frac{1}{a-jw} + \frac{1}{a+jw}$$

$$= \frac{(a+jw)+(a-jw)}{(a-jw)(a+jw)} = \frac{2a}{a^2 + \cancel{ajw} - \cancel{ajw} - (jw)^2} = \frac{2a}{a^2+w^2} = X(w)$$

b. $\cos(w_c t)$, $w_c \in \mathbb{R}$ $\quad$ Usando la propiedad $\to \cos(w_c t) = (e^{jw_c t} + e^{-jw_c t})/2$

$$F\{\cos(w_c t)\} = \int_{-\infty}^{\infty} \cos(w_c t) e^{-jwt}\, dt = \int_{-\infty}^{\infty} \frac{e^{jw_c t} + e^{-jw_c t}}{2} e^{-jwt}\, dt$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} e^{jw_c t} e^{-jwt}\, dt + \int_{-\infty}^{\infty} e^{-jw_c t} e^{-jwt}\, dt\right]$$

$$= \frac{1}{2}\left[\underbrace{\int_{-\infty}^{\infty} e^{-(w-w_c)jt}\, dt}_{I_1} + \int_{-\infty}^{\infty} e^{-(w+w_c)jt}\, dt\right]$$

Resolviendo $I_1 = \int_{-T}^{T} e^{-\overbrace{(w-w_c)}^{\alpha}jt}\, dt = \frac{e^{-j\alpha T} - e^{j\alpha T}}{-j\alpha} = \frac{(e^{j\alpha T} - e^{-j\alpha T})}{j\alpha}\left(\frac{2}{2}\right)$

$$= \frac{2\,\mathrm{sen}(\alpha T)}{\alpha} \quad \text{Si } \lim_{T\to\infty} \frac{\mathrm{sen}(\alpha T)}{\pi\alpha} = \delta(\alpha) \to \lim_{T\to\infty} \frac{2\,\mathrm{sen}(\alpha T)}{\alpha} = 2\pi\,\delta(\alpha)$$

Finalmente $\to \int_{-\infty}^{\infty} e^{-(w-w_c)jt}\, dt = 2\pi\,\delta(w-w_c)$

Así $\frac{1}{\cancel{2}} \cancel{2}\pi\left[\delta(w-w_c) + \delta(w+w_c)\right] = \pi\left[\delta(w-w_c) + \delta(w+w_c)\right] = X(w)$

c. $sen(w_s t)$; $w_s \in \mathbb{R}$ Usando la propiedad → $sen(w_s t) = (e^{jw_s t} - e^{-jw_s t})/2j$

$$F\{sen(w_s t)\} = \int_{-\infty}^{\infty} sen(w_s t) e^{-jwt} dt = \int_{-\infty}^{\infty} \frac{e^{jw_s t} - e^{-jw_s t}}{2j} e^{-jwt} dt$$

$$= \frac{1}{2j}\left[\int_{-\infty}^{\infty} e^{jw_s t} e^{-jwt} dt - \int_{-\infty}^{\infty} e^{-jw_s t} e^{-jwt} dt\right]$$

$$= \frac{1}{2j}\left[\int_{-\infty}^{\infty} e^{-(w-w_s)jt} dt - \int_{-\infty}^{\infty} e^{-(w+w_s)jt} dt\right.$$

En el ejercicio b demostramos que $\int_{-\infty}^{\infty} e^{-\alpha jt} dt = 2\pi\delta(\alpha)$ Así:

$$= \frac{1}{2j} 2\pi\left[\delta(w-w_s) - \delta(w+w_s)\right] = \frac{\pi}{j}\left[\delta(w-w_s) - \delta(w+w_s)\right]\left(\frac{j}{j}\right)$$

$$= \frac{\pi j}{-1}\left[\delta(w-w_s) - \delta(w+w_s)\right] = -\pi j\left[\delta(w-w_s) - \delta(w+w_s)\right]$$

$$= \pi j\left[\delta(w+w_s) - \delta(w-w_s)\right] = X(w)$$

d. $f(t)\cos(w_c t)$, $w_c \in \mathbb{R}$, $f(t) \in \mathbb{R}, \mathbb{C}$.

$$F\{f(t)\cos(w_c t)\} = \int_{-\infty}^{\infty} f(t)\cos(w_c t) e^{-jwt} dt$$

$$= \int_{-\infty}^{\infty} f(t) \frac{e^{jw_c t} + e^{-jw_c t}}{2} e^{-jwt} dt = \frac{1}{2}\int_{-\infty}^{\infty} f(t)(e^{jw_c t} + e^{-jw_c t}) e^{-jwt} dt$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} f(t) e^{jw_c t} e^{-jwt} dt + \int_{-\infty}^{\infty} f(t) e^{-jw_c t} e^{-jwt} dt\right]$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} f(t) e^{-(w-w_c)jt} dt + \int_{-\infty}^{\infty} f(t) e^{-(w+w_c)jt} dt\right]$$

Tenemos que $\int_{-\infty}^{\infty} f(t) e^{-j(w-w_c)t} dt = F(w-w_c)$ → Transformada t desplazada.

$$= \frac{1}{2}\left[F(w-w_c) + F(w+w_c)\right] = X(w).$$

e. $e^{-a|t|^2}$, $a \in \mathbb{R}^+$ t siempre es positivo

$F\{e^{-at^2}\} = \int_{-\infty}^{\infty} e^{-at^2} e^{-j\omega t}\, dt$

Operando exponentes $\rightarrow$ $-at^2 - j\omega t = -a\left(t^2 + \frac{j\omega}{a}t\right) = -a\left[\left(t + \frac{j\omega}{2a}\right)^2 - \left(\frac{j\omega}{2a}\right)^2\right]$

$= -a\left[\left(t + \frac{j\omega}{2a}\right)^2 - \frac{j^2\omega^2}{2^2 a^2}\right] = -a\left[\left(t + \frac{j\omega}{2a}\right)^2 + \frac{\omega^2}{4a^2}\right]$

$= -a\left(t + \frac{j\omega}{2a}\right)^2 - \frac{a\omega^2}{4a^2} = -a\left(t + \frac{j\omega}{2a}\right)^2 - \frac{\omega^2}{4a}$ Así:

$= \int_{-\infty}^{\infty} e^{-\omega^2/4a} e^{-a(t + j\omega/2a)^2}\, dt = e^{-\omega^2/4a} \overbrace{\int_{-\infty}^{\infty} e^{-a(t + j\omega/2a)^2}\, dt}^{I_1}$

$I_1 \rightarrow u = t + \frac{j\omega}{2a}$ $du = dt$ $\rightarrow I_1 = \int_{-\infty}^{\infty} e^{-au^2}\, du.$

$I_1^2 = \left(\int_{-\infty}^{\infty} e^{-ax^2}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-ay^2}\, dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-a(x^2+y^2)}\, dx\, dy$

Haciendo uso de coordenadas polares $\rightarrow$ $x = r\cos\theta$, $y = r\,\text{sen}\,\theta$ $dx\,dy = r\,dr\,d\theta$. $r \in [0, \infty)$, $\theta \in [0, 2\pi]$

$I_1^2 = \int_0^{2\pi}\int_0^{\infty} e^{-ar^2}\, r\, dr\, d\theta = 2\pi \int_0^{\infty} r e^{-ar^2}\, dr$ $\rightarrow$ $\left.\begin{matrix} s = ar^2 \\ ds = 2ar\,dr \end{matrix}\right\}$ $r\,dr = \frac{ds}{2a}$

$I_1^2 = 2\pi \int_0^{\infty} e^{-s} \frac{ds}{2a} = \frac{\cancel{2}\pi}{\cancel{2}a}\int_0^{\infty} e^{-s}\, ds = \frac{\pi}{a}\left(-\frac{1}{e^{\infty}} - (-1)\right) = \frac{\pi}{a}$ $\rightarrow I = \sqrt{\frac{\pi}{a}}$

$= e^{-\omega^2/4a}\sqrt{\frac{\pi}{a}} = X(\omega)$

f. $A\,\text{rect}_d(t)$; $A, d \in \mathbb{R}$ $\text{rect}_d(t) = \begin{cases} 1 & |t| \le d/2 \\ 0 & |t| > d/2 \end{cases}$

$F\{A\,\text{rect}_d(t)\} = \int_{-\infty}^{\infty} A\,\text{rect}_d(t)\, e^{-j\omega t}\, dt = \int_{-d/2}^{d/2} A e^{-j\omega t}\, dt.$

$= A\left(\frac{e^{-j\omega t}}{-j\omega}\right)\Big|_{-d/2}^{d/2} = -\frac{A}{j\omega}\left(e^{-j\omega d/2} - e^{(-j\omega)(-d/2)}\right)$

$= \frac{2A}{\omega}\left(\frac{e^{-j\omega d/2} - e^{(-j\omega)(-d/2)}}{2j}\right) = \frac{2A}{\omega}\,\text{sen}\left(\frac{\omega d}{2}\right) = X(\omega).$