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Parcial 1: Señales y sistemas.

1. La distancia media entre 2 señales periódicas $x_1(t) \in \mathbb{R}, \mathbb{C}$ y $x_2(t) \in \mathbb{R}, \mathbb{C}$; se puede expresar a partir de la potencia media de la diferencia entre ellas:

$$d^2(x_1, x_2) = \bar{P}_{x_1 - x_2} = \lim_{T \to \infty} \frac{1}{T} \int_T |x_1(t) - x_2(t)|^2 \, dt$$

Sea $x_1(t)$ y $x_2(t)$ 2 señales definidas como:

$x_1(t) = Ae^{-jnw_0t}$ $\quad$ $x_2(t) = Be^{jmw_0t}$

con $w_0 = 2\pi/T$; $T, A, B \in \mathbb{R}^+$ y $n, m \in \mathbb{Z}$. Determine la distancia entre las 2 señales. Compruebe sus resultados con Python.

$$\bar{P}_{x_1 - x_2} = \frac{1}{T} \int_T |x_1(t) - x_2(t)|^2 \, dt = \frac{1}{T} \left(\int_T (x_1(t) - x_2(t))(x_1(t) - x_2(t))^* \right) dt$$

$$= \frac{1}{T} \left(\int_T (x_1(t) - x_2(t))(x_1^*(t) - x_2^*(t)) \right) dt$$

$$= \frac{1}{T} \left(\int_T x_1(t) x_1^*(t) \, dt - \int_T x_1(t) x_2^*(t) \, dt - \int_T x_2(t) x_1^*(t) \, dt + \int_T x_2(t) x_2^*(t) \, dt \right)$$

$$= \underbrace{\frac{1}{T} \int_T |x_1(t)|^2 \, dt}_{\bar{P}_{x_1}} - \underbrace{\frac{2}{T} \int_T x_1(t) x_2^*(t) \, dt}_{G_{12}} + \underbrace{\frac{1}{T} \int_T |x_2(t)|^2 \, dt}_{\bar{P}_{x_2}}.$$

$$\bar{P}_{x_1} = \frac{1}{T} \int_T x_1(t) x_1^*(t) \, dt = \frac{1}{T} \int_0^T Ae^{-jnw_0t} (Ae^{jnw_0t}) \, dt$$

$$= \frac{A^2}{T} \int_0^T \underbrace{e^{j\cancel{(n-n)}^{0} w_0 t}}_{1} \, dt = \frac{A^2}{T} \int_0^T dt = \frac{A^2}{T} t \Big|_0^T = \frac{A^2}{\cancel{T}} (\cancel{T}) = A^2$$

$$\bar{P}_{x_2} = \frac{1}{T} \int_T x_2(t) x_2^*(t) \, dt = \frac{1}{T} \int_0^T Be^{jmw_0t} (Be^{-jmw_0t}) \, dt$$

$$= \frac{B^2}{T} \int_0^T \underbrace{e^{j\cancel{(m-m)}^{0} w_0 t}}_{1} \, dt = \frac{B^2}{T} \int_0^T dt = \frac{B^2}{T} t \Big|_0^T = \frac{B^2}{\cancel{T}} (\cancel{T}) = B^2$$

$$C_{12} = -\frac{2}{T}\int_T x_1(t)\, x_2^*(t)\, dt = -\frac{2}{T}\int_0^T (Ae^{-jn\omega_0 t})(Be^{-jm\omega_0 t})\, dt$$

$$= -\frac{2AB}{T}\int_0^T e^{-jn\omega_0 t - jm\omega_0 t}\, dt = -\frac{2AB}{T}\int_0^T e^{-j\overbrace{(n+m)}^{k}\omega_0 t}\, dt$$

$\longrightarrow$ Sí $n+m=0$, $n=-m$:

$$C_{12} = -\frac{2AB}{T}\int_0^T \underbrace{e^{-j(0)\omega_0 t}}_{1}\, dt = -\frac{2AB}{T}\int_0^T dt = -\frac{2AB}{T}\, t\Big|_0^T = -\frac{2AB}{\cancel{T}}\cancel{T}$$

$\longrightarrow$ Sí $n+m \neq 0$, $n \neq -m$; $n+m = k \in \mathbb{Z}$:

$$C_{12} = -\frac{2AB}{T}\int_0^T e^{-jk\omega_0 t}\, dt = -\frac{2AB}{T}\left(\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right)\Big|_0^T$$

$$= \frac{2AB}{jk\omega_0 T}\left(e^{-jk\omega_0 T} - \underbrace{e^{-jk\omega_0 (0)}}_{1}\right)$$

$$= \cos(k\omega_0 T) - j\,\text{sen}(k\omega_0 T) = \underbrace{\cos(k2\pi)}_{1} - j\,\underbrace{\text{sen}(k2\pi)}_{0} = 1$$

$$= \frac{2AB}{jk\omega_0 T}(1-1) = 0$$

$$\overline{P}_{x_1-x_2} = \begin{cases} A^2 - 2AB + B^2 & ; n = m \\ A^2 + B^2 & ; n \neq m \end{cases} \qquad \overline{P}_{x_1-x_2} = \overline{P}_{x_1} + C_{12} + \overline{P}_{x_2}$$

2. Encuentre la señal en tiempo discreto al utilizar un conversor análogo digital con frecuencia de muestreo de 5KHz y 4 bits de capacidad de representación, aplicado a la señal continua:

$$x(t) = 3\cos(1000\pi t) + 5\,\text{sen}(3000\pi t) + 10\cos(11000\pi t)$$

Realizar la simulación del proceso de discretización (incluyendo al menos 3 períodos de $x(t)$). En caso de que la discretización no sea apropiada, diseñe e implemente un conversor adecuado para la señal estudiada.

$F_s = 5KHz$ Estados $= 2^{\#\text{bits}} = 2^4 = 16$

$t = nT_s \rightarrow F_s = \frac{1}{T_s}$; $F = \frac{1}{T}$; $T = \frac{2\pi}{\omega}$

$A\cos(\Omega n) = A\cos(2\pi f n) \rightarrow \Omega = 2\pi f \longrightarrow f = \frac{T_s}{T} = \frac{F}{F_s}$

Igual para sen, reemplaza cos por sen.

Para $x_1(t) = 3\cos(1000\pi t)$

$\omega_1 = 1000\pi \longrightarrow T_1 = \frac{2\pi}{\omega_1} = \frac{\cancel{2\pi}}{1000\cancel{\pi}} = \frac{1}{500} \longrightarrow F_1 = \frac{1}{T_1} = \frac{1}{1/500} = 500\ Hz$

$x_1[n] = A\cos\left(2\pi n \frac{F_1}{F_s}\right) = 3\cos\left(2\pi n \frac{500}{5000}\right) = 3\cos\left(\frac{n\pi}{5}\right)$

Para $x_2(t) \to 5\,sen(3000\pi t)$

$\omega_2 = 3000\pi \longrightarrow T_2 = \frac{2\pi}{\omega_2} = \frac{\cancel{2\pi}}{3000\cancel{\pi}} = \frac{1}{1500} \longrightarrow F_2 = \frac{1}{T_2} = \frac{1}{1/1500} = 1500\ Hz$

$x_2[n] = A\,sen\left(2\pi n \frac{F_2}{F_s}\right) = 5\,sen\left(2\pi n \frac{1500}{5000}\right) = 3\,sen\left(\frac{3n\pi}{5}\right)$

Para $x_3(t) \to 10\cos(11000\pi t)$

$\omega_3 = 11000\pi \longrightarrow T_3 = \frac{2\pi}{11000\pi} = \frac{1}{5500} \longrightarrow F_3 = \frac{1}{T_3} = \frac{1}{1/5500} = 5500\ Hz$

$x_3[n] = A\cos\left(2\pi n \frac{F_3}{F_s}\right) = 10\cos\left(2\pi n \frac{5500}{5000}\right) = 10\cos\left(\frac{11n\pi}{5}\right)$

$x[n] = 3\cos\left(\frac{n\pi}{5}\right) + 3\,sen\left(\frac{3n\pi}{5}\right) + 10\cos\left(\frac{11n\pi}{5}\right)$ $\quad MCM = 1/5.$

$\frac{\omega_1}{\omega_2} = \frac{1000\pi}{3000\pi} = \frac{1}{3} \in \mathbb{Q} \qquad \frac{\omega_2}{\omega_3} = \frac{3000\pi}{11000\pi} = \frac{3}{11} \in \mathbb{Q}$

$\frac{\omega_1}{\omega_3} = \frac{1000\pi}{11000\pi} = \frac{1}{11} \in \mathbb{Q}$ $\quad \therefore$ la señal es cuasiperiódica

NO cumple Nyquist.

Si $F_s \geq 2F_{max} \longrightarrow 2F_{max} = 2(5500\ Hz) = 11000\ Hz$ pero $F_s = 5000\ Hz$

Comparando con las originales $\longrightarrow -\pi \leq \Omega < \pi$

Copia o aliasing

$\Omega_1 = \frac{1}{5}\pi$ cumple, $\Omega_2 = \frac{3}{5}\pi$ cumple, $\Omega_3 = \frac{11}{5}\pi$ no cumple $> \pi$

$\Omega_3\ original = \frac{11}{5}\pi - 2\pi = \frac{1}{5}\pi$ ahora sí cumple.

Se supone una frecuencia de muestreo más grande

$F_s = 3F_{max} = 3(5500\,Hz) = 16500\,Hz \rightarrow T_s = \frac{1}{F_s} = \frac{1}{16500}$

Para $x_1(t) = 3\cos(1000\pi t)$

$x_1[n] = 3\cos\left(2\pi n \frac{500}{16500}\right) = 3\cos\left(\frac{2}{33}n\pi\right) \quad \Omega_1 = \frac{2\pi}{33} \approx 0{,}06\pi$

Para $x_2(t) = 5\,sen(3000\pi t)$

$x_2[n] = 5\,sen\left(2\pi n \frac{1500}{16500}\right) = 5\,sen\left(\frac{2}{11}n\pi\right) \quad \Omega_2 = \frac{2\pi}{11} \approx 0{,}18\pi$

Para $x_3(t) = 10\cos(11000\pi t)$

$x_3[n] = 10\cos\left(2\pi n \frac{5500}{16500}\right) = 10\cos\left(\frac{2n\pi}{3}\right) \quad \Omega_3 = \frac{2\pi}{3} \approx 0{,}67\pi$

$-\pi \leq \Omega_1, \Omega_2, \Omega_3 \leq \pi \rightarrow$ las 3 frecuencias digitales están en las originales.

$x[n] = 3\cos\left(\frac{2n\pi}{33}\right) + 5\,sen\left(\frac{2n\pi}{11}\right) + 10\cos\left(\frac{2n\pi}{3}\right)$

3. Sea $x''(t)$ la segunda derivada de la señal $x(t)$, donde $t \in [t_i, t_f]$. Demuestre que los coeficientes de la serie exponencial de Fourier se pueden calcular según:

$$C_n = \frac{1}{(t_i - t_f)n^2 w_0^2} \int_{t_i}^{t_f} x''(t)\, e^{-jnw_0 t}\, dt \;;\; n \in \mathbb{Z}.$$

¿Cómo se pueden calcular los coeficientes a_n y b_n desde $x''(t)$ en la serie trigonométrica de Fourier?

Sabemos que $x(t) = \sum_n C_n e^{jnw_0 t}$

$\rightarrow x'(t) = \frac{d}{dt}\sum_n C_n e^{jnw_0 t} = \sum_n C_n \boxed{\frac{d}{dt} e^{jnw_0 t}} \longrightarrow \frac{d}{dt} e^{jnw_0 t} = jnw_0\, e^{jnw_0 t}$

$\rightarrow x''(t) = \frac{d}{dt}x'(t) = \frac{d}{dt}\sum_n C_n \frac{d}{dt} e^{jnw_0 t} = \sum_n C_n \boxed{\frac{d^2}{dt^2} e^{jnw_0 t}}$

$\frac{d^2}{dt^2} e^{jnw_0 t} = \frac{d}{dt} jnw_0\, e^{jnw_0 t} = (jnw_0)^2 e^{jnw_0 t} = j^2 n^2 w_0^2 e^{jnw_0 t}$
$= -n^2 w_0^2 e^{jnw_0 t}$

$$x''(t) = \sum_n \underbrace{-C_n n^2 \omega_0^2}_{\tilde{C}_n} e^{jn\omega_0 t} = \sum_n \tilde{C}_n e^{jn\omega_0 t} \quad ; \text{ con } \tilde{C}_n = -C_n n^2 \omega_0^2$$

Si $C_n = \frac{1}{T}\int_T x(t) e^{-jn\omega_0 t}\, dt$ y $\tilde{C}_n = \frac{1}{T}\int_T x''(t) e^{-jn\omega_0 t}\, dt$

Se reemplaza el $\tilde{C}_n \to C_n n^2 \omega_0^2 = \frac{1}{T}\int_T x''(t) e^{-jn\omega_0 t}\, dt$.

$$C_n = \frac{1}{-(t_f - t_i) n^2 \omega_0^2}\int_T x''(t) e^{-jn\omega_0 t}\, dt = \boxed{\frac{1}{(t_i - t_f) n^2 \omega_0^2}\int_T x''(t) e^{-jn\omega_0 t}\, dt}$$

$$C_n = \frac{1}{(t_i - t_f) n^2 \omega_0^2}\int_T x''(t) e^{-jn\omega_0 t}\, dt = \frac{1}{(t_i - t_f) n^2 \omega_0^2}\int x''(t)\left[\cos(n\omega_0 t) - j\,\text{sen}(n\omega_0 t)\right] dt$$

$$= \frac{1}{(t_i - t_f) n^2 \omega_0^2}\int_T x''(t)\cos(n\omega_0 t)\, dt - j\,\frac{1}{(t_i - t_f) n^2 \omega_0^2}\int_T x''(t)\,\text{sen}(n\omega_0 t)\, dt$$

Sabemos que $a_n = \frac{2}{T}\int_T x(t)\cos(n\omega_0 t)\, dt$; $b_n = \frac{2}{T}\int_T x(t)\,\text{sen}(n\omega_0 t)\, dt$.

y que $a_n = 2\,\text{Re}\{C_n\}$; $b_n = -2\,\text{Im}\{C_n\}$

$$\to a_n = 2\,\text{Re}\{C_n\} = 2\,\frac{1}{(t_i - t_f) n^2 \omega_0^2}\int_T x''\cos(n\omega_0 t)\, dt$$

$$\to b_n = \cancel{-}2\,\text{Im}\{C_n\} = \cancel{-}2\,\frac{1}{(t_i - t_f) n^2 \omega_0^2}\int_T x''\,\text{sen}(n\omega_0 t)\, dt$$

O también con $x(t) = \sum_n a_n \cos(n\omega_0 t) + b_n\,\text{sen}(n\omega_0 t)$

$$x(t) = \sum_n a_n \cos(n\omega_0 t) + b_n\,\text{sen}(n\omega_0 t)$$

$$x'(t) = \sum_n -a_n\,\text{sen}(n\omega_0 t)\, n\omega_0 + b_n \cos(n\omega_0 t)\, n\omega_0$$

$$x''(t) = \sum_n -a_n \cos(n\omega_0 t)(n\omega_0)^2 - b_n\,\text{sen}(n\omega_0 t)(n\omega_0)^2$$

$$= \sum_n \underbrace{-a_n n^2 \omega_0^2}_{\tilde{a}_n} \cos(n\omega_0 t) \underbrace{- b_n n^2 \omega_0^2}_{\tilde{b}_n} \text{sen}(n\omega_0 t)$$

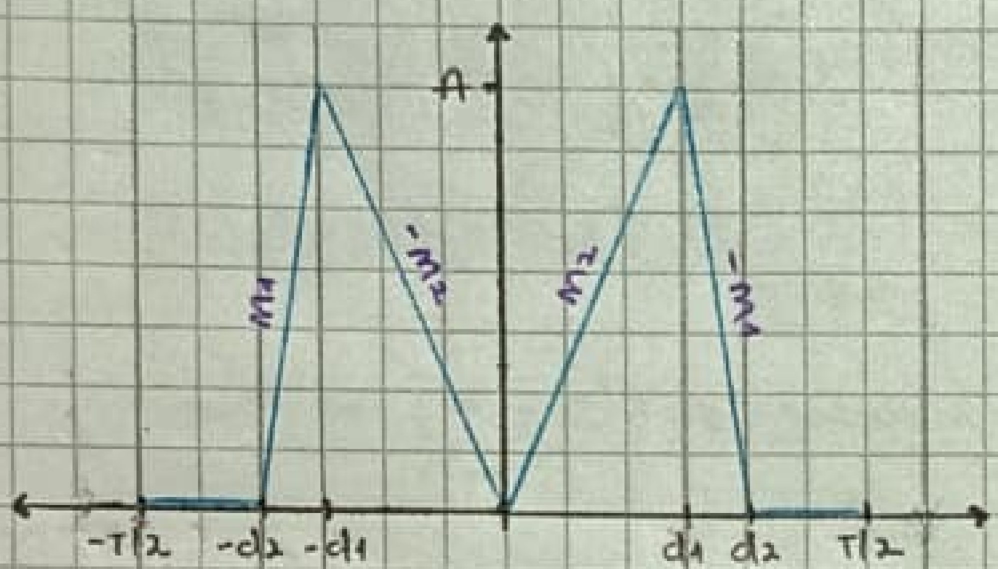
$$a_n = \frac{2}{T}\int_T x(t)\cos(n\omega_0 t)\, dt \longrightarrow \tilde{a}_n = \frac{2}{T}\int_T x''(t)\cos(n\omega_0 t)\, dt.$$

$$-a_n n^2 \omega_0^2 = \frac{2}{T}\int_T x''(t)\cos(n\omega_0 t)\, dt \longrightarrow \boxed{a_n = \frac{2}{\underbrace{-(t_f - t_i)}_{t_i - t_f} n^2 \omega_0^2}\int x''(t)\cos(n\omega_0 t)\, dt.}$$

$$b_n = \frac{2}{T}\int_T x(t)\,\text{sen}(n\omega_0 t)\,dt \to \vec{b_n} = \frac{2}{T}\int_T x''(t)\,\text{sen}(n\omega_0 t)\,dt.$$

$$-b_n n^2 \omega_0^2 = \frac{2}{T}\int_T x''(t)\,\text{sen}(n\omega_0 t)\,dt \to \boxed{b_n = \frac{2}{\frac{-(t_f-t_i)\,n^2\omega_0^2}{t_i-t_f}}\int_T x''(t)\,\text{sen}(n\omega_0 t)\,dt}$$

4. Encuentre el espectro de Fourier, su parte real, imaginaria, magnitud, fase y el error relativo para $n \in \{0, \pm1, \pm2, \pm3, \pm4, \pm5\}$, a partir de $x''(t)$ para la señal $x(t)$ en la figura. Compruebe el espectro obtenido con la estimación a partir de $x(t)$. Presente las simulaciones de Python respectivas.



$$m_1 = \frac{y_{max} - y_{min}}{x_{max} - x_{min}} = \frac{A - 0}{-d_1 - (-d_2)} = \frac{A}{d_2 - d_1}$$

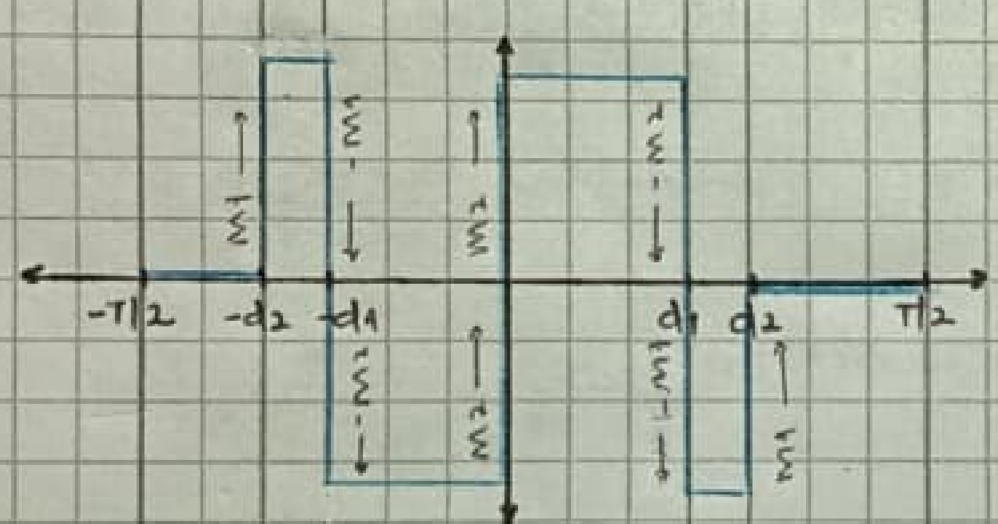
$$m_2 = \frac{y_{max} - y_{min}}{x_{max} - x_{min}} = \frac{A - 0}{d_1 - 0} = \frac{A}{d_1}$$

$x_1(t) = m_1 x_1 + b_1$ $\quad$ $x_2(t) = m_2 x_2 + b_2$

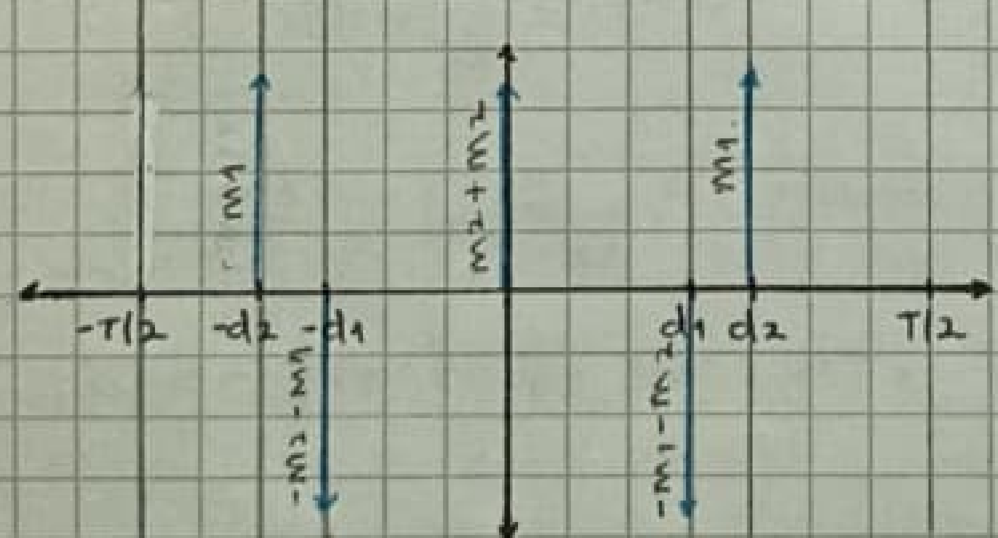
$\to b_1 = x_1(t) - m_1 x_1$ $\quad$ $\to b_2 = x_2(t) - m_2 x_2$

$b_1 = 0 - \frac{A}{d_2 - d_1}(-d_2)$ $\quad$ $b_2 = 0 - \frac{A}{d_1}(0)$

$b_1 = \frac{A d_2}{d_2 - d_1}$ $\quad$ $b_2 = 0.$



$$x(t) = \begin{cases} 0, & d_2 \le |t| \le T/2 \\ \frac{At}{d_2 - d_1} + \frac{A d_2}{d_2 - d_1} = At + Ad_2 & d_2 \le |t| \le d_1 \\ \frac{At}{d_1} & 0 \le |t| \le d_1 \end{cases}$$



$x''(t) = 0$ en cada tramo y en los puntos donde hay cambios tiende a $\pm\infty$ y estos cambios los podemos representar como $m_k\delta(t + t_k)$ donde m_k es la diferencia de la derivada por la derecha y la derivada por la izquierda y t_k donde ocurre el salto (cambio instantáneo)

$$x''(t) = \overbrace{m_{-d_2}}^{m_1}\delta(t+d_2) + \overbrace{m_{-d_1}}^{-m_2-m_1}\delta(t+d_1) + \overbrace{m_0}^{2m_2}\delta(t+0) + \overbrace{m_{d_1}}^{-m_1-m_2}\delta(t-d_1) + \overbrace{m_{d_2}}^{m_1}\delta(t-d_2)$$

$$x''(t) = m_1[\delta(t+d_2) - \delta(t-d_2)] - (m_1+m_2)[\delta(t+d_1) + \delta(t-d_1)] + 2m^2\delta(t+0)$$

Sabemos que $\int_{-\infty}^{\infty}\delta(t+T) = 1$

$$C_n = \frac{1}{\frac{(t_i - t_f)}{T} n^2 \omega_0^2} \int_T x''(t) e^{-jn\omega_0 t}\, dt \qquad T = t_f - t_i = \frac{T}{2} - \left(-\frac{T}{2}\right) = T \; ; \; \omega_0 = \frac{2\pi}{T}$$

$$= \frac{1}{T n^2 \omega_0^2} \int_{-T/2}^{T/2} \left[\frac{A}{d_2 - d_1} (\delta(t + d_2) + \delta(t - d_2)) - \left(\frac{A}{d_1} + \frac{A}{d_2 - d_1}\right)(\delta(t + d_1) + \delta(t - d_1)) + \right.$$

$$\left. + \frac{2A}{d_1} \delta(t) \right] e^{-jn\omega_0 t}\, dt \longrightarrow \text{como } x(t) \text{ con simetría par } \operatorname{sen}(\theta) = 0$$

$$= \frac{1}{T n^2 \omega_0^2} \left[\int_{-T/2}^{T/2} \frac{A}{d_2 - d_1} \delta(t + d_2) \cos(n\omega_0 t)\, dt + \int_{-T/2}^{T/2} \frac{A}{d_2 - d_1} \delta(t - d_2) \cos(n\omega_0 t)\, dt \right.$$

$$- \int_{-T/2}^{T/2} A \left(\frac{1}{d_1} + \frac{1}{d_2 - d_1}\right) \delta(t + d_1) \cos(n\omega_0 t)\, dt - \int_{-T/2}^{T/2} A \left(\frac{1}{d_1} + \frac{1}{d_2 - d_1}\right)$$

$$\left. \delta(t - d_1) \cos(n\omega_0 t)\, dt + \int_{-T/2}^{T/2} \frac{2A}{d_1} \delta(t) \cos(n\omega_0 t)\, dt \right]$$

usamos $\rightarrow \int_{-\infty}^{\infty} x(t) \delta(t \pm t_0)\, dt = x(\mp t_0)$ $\qquad \boxed{\cos(x) = \cos(-x)}$

$$C_n = \frac{A}{T n^2 \omega_0^2} \left(\frac{1}{d_2 - d_1} \cos(n\omega_0(-d_2)) + \frac{1}{d_2 - d_1} \cos(n\omega_0 d_2) - \frac{d_2 \cancel{- d_1} \cancel{+ d_1}}{d_1(d_2 - d_1)} \cos(n\omega_0(-d_1)) \right.$$

$$\left. - \frac{d_2 \cancel{- d_1} \cancel{+ d_1}}{d_1(d_2 - d_1)} \cos(n\omega_0 d_1) + \frac{2}{d_1} \overbrace{\cancel{\cos(n\omega_0 \cdot 0)}}^{1} \right)$$

$$= \frac{A}{(t_i - t_f) n^2 \omega_0^2} \left(\frac{2}{d_2 - d_1} \cos(n\omega_0 d_2) - \frac{2 d_2}{d_1(d_2 - d_1)} \cos(n\omega_0 d_1) + \frac{2}{d_1} \right)$$

$$= \frac{\cancel{2}A}{\left(\cancel{\frac{T}{2}} - \frac{T}{2}\right) \frac{n^2 2^2 \pi^2}{T^2}} \left(\frac{1}{d_2 - d_1} \cos(n\omega_0 d_2) - \frac{d_2}{d_1(d_2 - d_1)} \cos(n\omega_0 d_1) + \frac{1}{d_1} \right)$$

$$= \frac{-AT}{2n^2\pi^2} \left(\frac{1}{d_2 - d_1} \cos(n\omega_0 d_2) - \frac{1}{d_1(d_2 - d_1)} \cos(n\omega_0 d_1) + \frac{1}{d_1} \right) \rightarrow Re\{C_n\}$$

$$|C_n| = \sqrt{Re^2\{C_n\} + 0} = Re\{C_n\}$$

$$\theta_{C_n} = \tan^{-1}\left(\frac{Im\{C_n\}}{Re\{C_n\}}\right) = \tan^{-1}\left(\frac{0}{Re\{C_n\}}\right) = 0$$

$$e_r[\%] = \left(1 - \sum |C_n|^2 \frac{P_n}{P_x}\right) \cdot 100\, [\%] \quad , \quad P_n = \frac{1}{T} E_n = \frac{1}{T} \int_T \overline{|e^{-jn\omega_0 t}|^2\, dT}^{\,T} = 1$$

Señal par → Se integra de 0 a d_2 → $x(t) = x(-t)$

$$P_x = \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\,dt = \frac{2}{T}\int_0^{T/2} |x(t)|^2\,dt$$ → De d_2 a $T/2$ = 0

$$= \frac{2}{T}\int_0^{d_2} |x(t)|^2\,dt = \frac{2}{T}\left[\int_0^{d_1}\left(\frac{A}{d_1}t\right)^2 dt + \int_{d_1}^{d_2}\left(-\frac{A}{d_2-d_1}(t+d_2)\right)^2 dt\right]$$

$$= \frac{2}{T}\left[\frac{A^2}{d_1^2}\int_0^{d_2} t^2\,dt - \frac{A^2}{(d_2-d_1)^2}\int_{d_1}^{d_2}(t+d_2)^2\,dt\right]$$

$$= \frac{2}{T}\left[\frac{A^2}{d_1^2}\frac{t^3}{3}\Big|_0^{d_1} - \frac{A}{(d_2-d_1)^2}\int_{d_1}^{d_2}(t^2+2td_2+d_2^2)\,dt\right]$$

$$= \frac{2}{T}\left[\frac{A^2}{d_1^2}\left(\frac{d_1^3}{3}-0\right) - \frac{A^2}{(d_2-d_1)^2}\left(\frac{t^3}{3}+\frac{2t^2d_2}{2}+d_2^2t\right)\Big|_{d_1}^{d_2}\right]$$

$$= \frac{2A^2}{T}\left[\frac{d_1}{3} - \frac{1}{(d_2-d_1)^2}\left(\frac{d_2^3}{3}+\cancel{\frac{2d_2^2d_2}{2}}+d_2^2d_2-\frac{d_1^3}{3}-\cancel{\frac{2d_1^2d_2}{2}}-d_2^2d_1\right)\right]$$

$$= \frac{2A^2}{T}\left[\frac{d_1}{3} - \frac{1}{(d_2-d_1)^2}\left(\frac{d_2^3}{3}+\frac{2d_2^2d_2}{2}+d_2^2d_2-\frac{d_1^3}{3}-\frac{2d_1^2d_2}{2}-d_2d_1\right)\right]$$

$$= \frac{2A^2}{T}\left[\frac{d_1}{3} - \frac{A^2}{(d_2-d_1)^2}\left(\frac{d_2^3}{3}+2d_2^3-\frac{d_1^3}{3}-d_1^3d_2-d_2^2d_1\right)\right]$$

$$|C_n|^2 = \left(\sqrt{\left(-\frac{AT}{2n^2\pi^2}\left(\frac{1}{(d_2-d_1)}\cos(n\omega_0 d_2)-\left(\frac{1}{d_1}+\frac{1}{d_2-d_1}\right)\cos(n\omega_0 d_1)+\frac{1}{d_1}\right)\right)^2}\right)^2$$

$$= \left(-\frac{AT}{2n^2\pi^2}\left(\frac{1}{d_2-d_1}\cos(n\omega_0 d_2)-\left(\frac{1}{d_1}+\frac{1}{d_2-d_1}\right)\cos(n\omega_0 d_1)+\frac{1}{d_1}\right)\right)^2$$

$$e_r(\%) = \left(1-\frac{\sum_n\left(-\frac{AT}{2n^2\pi^2}\left(\frac{1}{d_2-d_1}\cos(n\omega_0 d_2)-\left(\frac{1}{d_1}+\frac{1}{d_2-d_1}\right)\cos(n\omega_0 d_1)+\frac{1}{d_1}\right)\right)^2}{\frac{2}{T}\left(\frac{A^2d_1}{3}-\frac{A^2}{(d_2-d_1)^2}\left(\frac{d_2^3}{3}+2d_2^3-\frac{d_1^3}{3}-d_1^2d_2-d_2^2d_1\right)\right)}\right)$$