Taller 2.

Punto 1.3: Encuentre la función de densidad espectral (transformada de Fourier) para las siguientes señales (sin aplicar propiedades):

a. $e^{-a|t|}$, $a \in \mathbb{R}^+$ $|t| \rightarrow$ se puede partir en $-t=(-\infty,0)$ y $t=(0,\infty)$

$$F\{e^{-a|t|}\} = \int_{-\infty}^{\infty} e^{-a|t|} e^{-jwt}\,dt = \int_{-\infty}^{0} e^{(a-jw)t}\,dt + \int_{0}^{\infty} e^{-(a+jw)t}\,dt.$$

$$= \frac{1}{a-jw} e^{(a-jw)t}\Big|_{-\infty}^{0} + \frac{-1}{a+jw} e^{-(a+jw)t}\Big|_{0}^{\infty} \rightarrow e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0.$$

$$= \frac{1}{a-jw}\left(\cancelto{1}{e^{0}} - \cancelto{0}{e^{-\infty}}\right) + \frac{-1}{a+jw}\left(\cancelto{0}{e^{-\infty}} - \cancelto{1}{e^{0}}\right) = \frac{1}{a-jw} + \frac{1}{a+jw}$$

$$= \frac{(a+jw)+(a-jw)}{(a-jw)(a+jw)} = \frac{2a}{a^2 + \cancel{ajw} - \cancel{ajw} - \cancelto{-1}{(jw)^2}} = \frac{2a}{a^2+w^2} = X(w)$$

b. $\cos(w_c t)$, $w_c \in \mathbb{R}$ Usando la propiedad $\rightarrow \cos(w_c t) = (e^{jw_c t} + e^{-jw_c t})/2$

$$F\{\cos(w_c t)\} = \int_{-\infty}^{\infty} \cos(w_c t)\, e^{-jwt}\,dt = \int_{-\infty}^{\infty} \frac{e^{jw_c t} + e^{-jw_c t}}{2}\, e^{-jwt}\,dt.$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} e^{jw_c t} e^{-jwt}\,dt + \int_{-\infty}^{\infty} e^{-jw_c t} e^{-jwt}\,dt\right.$$

$$= \frac{1}{2}\left[\underbrace{\int_{-\infty}^{\infty} e^{-(w-w_c)jt}\,dt}_{I_1} + \int_{-\infty}^{\infty} e^{-(w+w_c)jt}\,dt\right.$$

Resolviendo $I_1 = \int_{-T}^{T} e^{-\overbrace{(w-w_c)}^{\alpha} jt}\,dt = \frac{e^{-j\alpha T} - e^{j\alpha T}}{-j\alpha} = \frac{(e^{j\alpha T} - e^{-j\alpha T})}{j\alpha}\left(\frac{2}{2}\right)$

$= \frac{2\,\text{sen}(\alpha T)}{\alpha}$ Si $\lim_{T\to\infty} \frac{\text{sen}(\alpha T)}{\pi\alpha} = \delta(\alpha) \rightarrow \lim_{T\to\infty} \frac{2\,\text{sen}(\alpha T)}{\alpha} = 2\pi\,\delta(\alpha)$

Finalmente $\rightarrow \int_{-\infty}^{\infty} e^{-(w-w_c)jt}\,dt = 2\pi\,\delta(w-w_c)$

$\rightarrow$ Así $\frac{1}{\cancel{2}}\cancel{2}\pi\left[\delta(w-w_c) + \delta(w+w_c)\right] = \pi\left[\delta(w-w_c) + \delta(w+w_c)\right] = X(w)$

c. sen$(w_s t)$; $w_s \in \mathbb{R}$ Usando la propiedad → sen$(w_s t) = (e^{jw_s t} - e^{-jw_s t})/2j$

$$F\{sen(w_s t)\} = \int_{-\infty}^{\infty} sen(w_s t)\, e^{-jwt}\, dt = \int_{-\infty}^{\infty} \frac{e^{jw_s t} - e^{-jw_s t}}{2j}\, e^{-jwt}\, dt$$

$$= \frac{1}{2j}\left[\int_{-\infty}^{\infty} e^{jw_s t} e^{-jwt}\, dt - \int_{-\infty}^{\infty} e^{-jw_s t} e^{-jwt}\, dt\right]$$

$$= \frac{1}{2j}\left[\int_{-\infty}^{\infty} e^{-(w-w_s)jt}\, dt - \int_{-\infty}^{\infty} e^{-(w+w_s)jt}\, dt\right]$$

En el ejercicio b demostramos que $\int_{-\infty}^{\infty} e^{-\alpha jt}\, dt = 2\pi\, \delta(\alpha)$ Así:

$$= \frac{1}{2j} 2\pi \left[\delta(w-w_s) - \delta(w+w_s)\right] = \frac{\pi}{j}\left[\delta(w-w_s) - \delta(w+w_s)\right]\left(\frac{j}{j}\right)$$

$$= \frac{\pi j}{-1}\left[\delta(w-w_s) - \delta(w+w_s)\right] = -\pi j\left[\delta(w-w_s) - \delta(w+w_s)\right]$$

$$= \pi j\left[\delta(w+w_s) - \delta(w-w_s)\right] = X(w)$$

d. $f(t)\cos(w_c t)$, $w_c \in \mathbb{R}$, $f(t) \in \mathbb{R}, \mathbb{C}$.

$$F\{f(t)\cos(w_c t)\} = \int_{-\infty}^{\infty} f(t)\cos(w_c t)\, e^{-jwt}\, dt$$

$$= \int_{-\infty}^{\infty} f(t)\, \frac{e^{jw_c t} + e^{-jw_c t}}{2}\, e^{-jwt}\, dt = \frac{1}{2}\int_{-\infty}^{\infty} f(t)\left(e^{jw_c t} + e^{-jw_c t}\right) e^{-jwt}\, dt$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} f(t)\, e^{jw_c t} e^{-jwt}\, dt + \int_{-\infty}^{\infty} f(t)\, e^{-jw_c t} e^{-jwt}\, dt\right]$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} f(t)\, e^{-(w-w_c)jt}\, dt + \int_{-\infty}^{\infty} f(t)\, e^{-(w+w_c)jt}\, dt\right]$$

Tenemos que $\int_{-\infty}^{\infty} f(t)\, e^{-j(w-w_c)t}\, dt = F(w-w_c)$ → Transformada desplazada.

$$= \frac{1}{2}\left[F(w-w_c) + F(w+w_c)\right] = X(w).$$

e. $e^{-a|t|^2}$, $a \in \mathbb{R}^+$ t → siempre es positivo

$$F\{e^{-at^2}\} = \int_{-\infty}^{\infty} e^{-at^2} e^{-j\omega t}\, dt$$

Operando exponentes → $-at^2 - j\omega t = -a\left(t^2 + \frac{j\omega}{a}t\right) = -a\left[\left(t + \frac{j\omega}{2a}\right)^2 - \left(\frac{j\omega}{2a}\right)^2\right]$

$$= -a\left[\left(t + \frac{j\omega}{2a}\right)^2 - \frac{j^2\omega^2}{2^2a^2}\right] = -a\left[\left(t + \frac{j\omega}{2a}\right)^2 + \frac{\omega^2}{4a^2}\right]$$

$$= -a\left(t + \frac{j\omega}{2a}\right)^2 - \frac{\cancel{a}\omega^2}{4a^{\cancel{2}}} = -a\left(t + \frac{j\omega}{2a}\right)^2 - \frac{\omega^2}{4a} \quad \text{Así:}$$

$$= \int_{-\infty}^{\infty} e^{-\omega^2/4a} e^{-a(t+j\omega/2a)^2}\, dt = e^{-\omega^2/4a} \overbrace{\int_{-\infty}^{\infty} e^{-a(t+j\omega/2a)^2}\, dt}^{I_1}$$

$I_1 \rightarrow u = t + \frac{j\omega}{2a}$ $du = dt$ → $I_1 = \int_{-\infty}^{\infty} e^{-au^2}\, du$

$$I_1^2 = \left(\int_{-\infty}^{\infty} e^{-ax^2}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-ay^2}\, dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-a(x^2+y^2)}\, dx\, dy$$

Haciendo uso de coordenadas polares → $x = r\cos\theta$, $y = r\,\text{sen}\,\theta$ $dx\,dy = r\,dr\,d\theta$, $r \in [0, \infty)$, $\theta \in [0, 2\pi)$

$$I_1^2 = \int_0^{2\pi}\int_0^{\infty} e^{-ar^2} r\, dr\, d\theta = 2\pi \int_0^{\infty} r e^{-ar^2}\, dr \rightarrow \left.\begin{matrix} s = ar^2 \\ ds = 2ar\,dr \end{matrix}\right\} r\,dr = \frac{ds}{2a}$$

$$I_1^2 = 2\pi \int_0^{\infty} e^{-s} \frac{ds}{2a} = \frac{\cancel{2}\pi}{\cancel{2}a}\int_0^{\infty} e^{-s}\, ds = \frac{\pi}{a}\left(-\frac{1}{e^{\infty}} - (-1)\right) = \frac{\pi}{a} \rightarrow I = \sqrt{\frac{\pi}{a}}$$

$$= e^{-\omega^2/4a}\sqrt{\frac{\pi}{a}} = X(\omega)$$

f. $A\,\text{rect}_d(t)$; $A, d \in \mathbb{R}$ $\text{rect}_d(t) = \begin{cases} 1, & |t| \leq d/2 \\ 0, & |t| > d/2 \end{cases}$

$$F\{A\,\text{rect}_d(t)\} = \int_{-\infty}^{\infty} A\,\text{rect}_d(t) e^{-j\omega t}\, dt = \int_{-d/2}^{d/2} A e^{-j\omega t}\, dt$$

$$= A\left(\frac{e^{-j\omega t}}{-j\omega}\right)\Bigg|_{-d/2}^{d/2} = -\frac{A}{j\omega}\left(e^{-j\omega d/2} - e^{(-j\omega)(-d/2)}\right)$$

$$= \frac{2A}{\omega}\left(\frac{e^{-j\omega d/2} - e^{(-j\omega)(-d/2)}}{2j}\right) = \frac{2A}{\omega}\,\text{sen}\left(\frac{\omega d}{2}\right) = X(\omega)$$

Punto 1.4: Aplique las propiedades de la transformada de Fourier para resolver:

a. $F\{e^{-jw_1t}\cos(w_ct)\}$, $w_1, w_c \in \mathbb{R}$

$$e^{-jw_1t}\cos(w_ct) = \frac{e^{jw_ct}+e^{-jw_ct}}{2}\,e^{-jw_1t}\,dt$$

$$= \frac{1}{2}\left(e^{jw_ct}e^{-jw_1t}+e^{-jw_ct}e^{-jw_1t}\right) = \frac{1}{2}\left(e^{-jt(w_1-w_c)}+e^{-jt(w_1+w_c)}\right)$$

Sabemos que $F\{e^{jw_0t}\} = 2\pi\delta(w-w_0)$

$$\to = \frac{1}{2}2\pi\left(\delta[w-(-w_1+w_c)]+\delta[w-(-w_1-w_c)]\right)$$

$$= \pi[\delta(w+w_1-w_c)+\delta(w+w_1+w_c)] = X(w)$$

b. $F\{u(t)\cos^2(w_ct)\}$ $w_c \in \mathbb{R}^+$ Si $u(t)=1$ de $[0,\infty)$

$$\cos^2(w_ct) = \frac{1+\cos(w_ct)}{2}\ ;\ u(t)\cos^2(w_ct) = \frac{1}{2}\left[u(t)+u(t)\cos(2w_ct)\right]$$

$$F\{u(t)\cos^2(w_ct)\} = \frac{1}{2}\left(F\{u(t)\}+F\{u(t)\cos(2w_ct)\}\right)$$

Si $F\{u(t)\} = F\{sgn\}+F\{1/2\} = \frac{1}{jw}+\pi\delta(w)$

y $F\{u(t)\cos(w_0t)\} = \frac{\pi}{2}\left(\delta(w-w_0)+\delta(w+w_0)\right)+\frac{jw}{w_0^2-w^2}$

$$\to = \frac{1}{2}\left[\frac{1}{jw}+\pi\delta(w)+\frac{\pi}{2}\left(\delta(w-2w_c)+\delta(w+2w_c)\right)+\frac{jw}{(2w_c)^2-w^2}\right] = X(w)$$

c. $F^{-1}\left\{\frac{7}{w^2+6w+45} * \frac{10}{(8+jw/3)^2}\right\}$

Producto en frecuencia = Convolución en t

$$F^{-1}\{G(w)H(w)\}(t) = \frac{1}{2\pi}(g*h)(t)$$

Donde $g(t) = F^{-1}\{G(w)\}$ y $h(t) = F^{-1}\{H(w)\}$

$$G(\omega) = \frac{7}{\omega^2 + 6\omega + 45} \rightarrow \omega^2 + 6\omega + 45 = (\omega+3)^2 + 36 = (\omega+3)^2 + 6^2$$

$$\text{si } e^{-a|t|} \rightarrow \frac{2a}{a^2+\omega^2} \; ; \; \frac{1}{(\omega-\omega_0)^2 + a^2} \rightarrow \frac{1}{2a} e^{-a|t|} e^{j\omega_0 t} \qquad a = 6, \; \omega_0 = -3$$

$$\rightarrow g(t) = F^{-1}\left\{ \frac{7}{(\omega+3)^2 + 6^2} \right\} = \frac{7}{2(6)} e^{-6|t|} e^{-j3t} = \frac{7}{12} e^{-6|t|} e^{-j3t}$$

$$H(\omega) = \frac{10}{(8 + j\omega/3)^2} \rightarrow (8 + j\omega/3)^2 = (1/3\,(24 + j\omega))^2 = 1/9\,(24 + j\omega)^2$$

$$\text{si } \quad F\{te^{-at}u(t)\} = \frac{1}{(a+j\omega)^2} \quad \text{para } a > 0 \qquad a = 24$$

$$\rightarrow = F^{-1}\left\{ 90 \frac{1}{(24 + j\omega)} \right\} = 90te^{-24t}u(t) = h(t)$$

$$x(t) = F^{-1}\{G \cdot H\} = \frac{1}{2\pi}(g * h) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\tau)h(t-\tau)\,d\tau$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{7}{12} e^{-6|\tau|} e^{-j3\tau}\, 90(t-\tau)e^{-24(t-\tau)}u(t-\tau)\,d\tau$$

$$x(t) = \frac{1}{2\pi}\frac{7(90)}{12}\int_{-\infty}^{\infty} e^{-6|\tau|}e^{-j3\tau}(t-\tau)e^{-24(t-\tau)}u(t-\tau)\,d\tau$$

$$x(t) = \frac{105}{4\pi}\int_{-\infty}^{\infty} e^{-6|\tau|}e^{-j3\tau}(t-\tau)e^{-24(t-\tau)}\underline{u(t-\tau)}\,d\tau$$

$$x(t) = \frac{105}{4\pi}\int_{-\infty}^{\infty} e^{-6|\tau|}e^{-j3\tau}(t-\tau)e^{-24(t-\tau)}\,d\tau \rightarrow \text{sí } u(t-\tau) = 1 \text{ en } (-\infty, \tau]$$

$$x(t) = \frac{105}{4\pi}\left(\int_{-\infty}^{0} e^{6\tau}e^{-j3\tau}(t-\tau)e^{-24(t-\tau)}\,d\tau + \int_{0}^{t} e^{-6\tau}e^{-j3\tau}(t-\tau)e^{-24(t-\tau)}\,d\tau\right)$$

$$x(t) = \frac{105}{4\pi}e^{-24t}\left(\int_{-\infty}^{0} e^{6\tau}e^{-j3\tau}(t-\tau)e^{24\tau}\,d\tau + \int_{0}^{t} e^{-6\tau}e^{-j3\tau}(t-\tau)e^{24\tau}\,d\tau\right)$$

$$x(t) = \frac{105}{4\pi}e^{-24t}\left(\underbrace{\int_{-\infty}^{0}(t-\tau)e^{\tau(30-j3)}\,d\tau}_{I_1} + \underbrace{\int_{0}^{t}(t-\tau)e^{\tau(-30-j3)}\,d\tau}_{I_2}\right)$$

$$I_1 = \int_{-\infty}^{0} t e^{(30-j3)\tau}\, d\tau - \int_{-\infty}^{0} \tau e^{\tau(-30-j3)\tau}\, d\tau$$

$$= t\cdot \frac{e^{(30-j3)\tau}}{30-j3}\Big|_{-\infty}^{0} - \left(\frac{\tau e^{(30-j3)\tau}}{30-j3} - \frac{e^{(30-j3)\tau}}{(30-j3)^2}\right)\Big|_{-\infty}^{0}$$

$$= \frac{t}{30-j3}\left(\cancelto{1}{e^{0}} - \cancelto{0}{\frac{1}{e^{\infty}}}\right) - \left[\cancelto{0}{\frac{0e^{0}}{30-j3}} - \frac{\cancelto{1}{e^{0}}}{(30-j3)^2} - \left(\cancelto{0}{\frac{1}{(30-j3)e^{\infty}}} - \cancelto{0}{\frac{1}{(30-j3)^2 e^{\infty}}}\right)\right]$$

$$= \frac{t}{30-j3} + \frac{1}{(30-j3)^2}$$

$$I_2 = \int_{0}^{t} (t-\tau) e^{(-30-j3)\tau}\, d\tau = \int_{0}^{t} t e^{(-30-j3)\tau}\, d\tau - \int_{0}^{t} \tau e^{(-30-j3)\tau}\, d\tau$$

$$= t\cdot \frac{e^{(-30-j3)\tau}}{-30-j3}\Big|_{0}^{t} - \left(\frac{\tau e^{(-30-j3)\tau}}{-30-j3} - \frac{e^{(-30-j3)\tau}}{(-30-j3)^2}\right)\Big|_{0}^{t}$$

$$= \frac{t}{-30-j3}\left(e^{(-30-j3)t} - \cancelto{1}{e^{0}}\right) - \left[\frac{e^{(-30-j3)t}}{-30-j3} - \frac{e^{(-30-j3)t}}{(-30-j3)^2} - \left(\cancelto{0}{\frac{0e^{0}}{30-j3}} - \frac{\cancelto{1}{e^{0}}}{(-30-j3)^2}\right)\right]$$

$$= \frac{t}{-30-j3}\left(e^{(-30-j3)t} - 1\right) - \left(\frac{te^{(-30-j3)t}}{-30-j3} - \frac{e^{(-30-j3)t}}{(-30-j3)^2} + \frac{1}{(-30-j3)^2}\right)$$

$$= \cancel{\frac{te^{(-30-j3)t}}{-30-j3}} - \frac{t}{-30-j3} - \cancel{\frac{te^{(-30-j3)t}}{-30-j3}} + \frac{e^{(-30-j3)t}}{(-30-j3)^2} - \frac{1}{(-30-j3)^2}$$

$$= \frac{e^{(-30-j3)t} - 1}{-(30+j3)^2} + \frac{t}{30+j3}$$

$$x(t) = \frac{105}{4\pi} e^{-24t}\left(\frac{t}{30-j3} + \frac{1}{(30+j3)^2} + \frac{e^{(-30-j3)t} - 1}{(30-j3)^2} + \frac{t}{30+j3}\right)$$

$$x(t) = \frac{105}{4\pi} e^{-24t}\left[t\left(\frac{1}{30-j3} + \frac{1}{30+j3}\right) + \left(\frac{1}{(30-j3)^2} - \frac{1}{(30+j3)^2}\right) + \frac{e^{(-30-j3)t}}{(30-j3)^2}\right]$$

$$x(t) = \frac{105}{4\pi} e^{-24t}\left(t\,\frac{60}{30^2+3^2} + \frac{360j}{909^2} + \frac{e^{(-30-j3)t}}{(30+j3)^2}\right)$$

d. $F\{3t^3\} \rightarrow$ si $F\{t^n x(t)\} = j^n \dfrac{d^n}{dw^n} X(w)$, $x(t) = 1 \rightarrow F\{1\} = 2\pi\delta(w)$.

$$F\{3t^3\} = 3j^3 \frac{d^3}{dw^3}\left(2\pi\delta(w)\right) = j^2 j\, 6\pi\delta^{(3)}(w) = -j\,6\pi\delta^{(3)}(w).$$

e. $\dfrac{B}{T}\displaystyle\sum_{n=-\infty}^{+\infty}\left(\frac{1}{a^2+(w-nw_0)^2} + \frac{1}{a+j(w-nw_0)}\right)$ donde $n\in\{0,\pm1,\pm2,\dots\}$ $w_0 = 2\pi/T$ y $B,T\in\mathbb{R}^+$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{B}{T}\sum_{n=-\infty}^{+\infty}\left(\frac{1}{a^2+(w-nw_0)^2} + \frac{1}{a+j(w-nw_0)}\right)e^{jwt}\,dw$$

$v = w - nw_0$
$dv = dw$
$w = v + nw_0$

$$x(t) = \frac{B}{T2\pi}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1}{a^2(w-nw_0)^2} + \frac{1}{a+j(w-nw_0)}\right)e^{jwt}\,dw$$

$$x(t) = \frac{B}{T2\pi}\sum_{n-\infty}^{+\infty}\left(\int_{-\infty}^{\infty}\frac{1}{a^2+v^2}e^{j(v+nw_0)t}\,dv + \int_{-\infty}^{\infty}\frac{1}{a+jv}e^{j(v+nw_0)t}\,dv\right)$$

$$x(t) = \frac{B}{T2\pi}\sum_{n=-\infty}^{\infty}\left(e^{jnw_0t}\left[\int_{-\infty}^{\infty}\frac{1}{a^2+v^2}e^{jvt}\,dv + \int_{-\infty}^{\infty}\frac{1}{a+jv}e^{jvt}\,dv\right)\right.$$

Si $A(v) = \dfrac{1}{a^2+v^2} \rightarrow a(t) = F^{-1}\{A(v)\} = \dfrac{1}{2a}e^{-a|t|}$ para $a>0$.

$B(v) = \dfrac{1}{a+jv} \rightarrow b(t) = F^{-1}\{B(v)\} = e^{-at}u(t)$ para $a>0$

$$x(t) = \frac{B}{T2\pi}\sum_{n=-\infty}^{+\infty}e^{jnw_0t}\left(2\pi\frac{1}{2a}e^{-a|t|} + 2\pi e^{-at}u(t)\right)$$

$$x(t) = \frac{B\cancel{2\pi}}{T\cancel{2\pi}}\left(\frac{1}{2a}e^{-a|t|} + e^{-at}u(t)\right)\boxed{\sum_{n=-\infty}^{\infty}e^{jnw_0t}} = 2\pi\sum_{k=-\infty}^{+\infty}\delta(w_0t - 2\pi k)$$

$$= 2\pi\sum_{k=-\infty}^{\infty}\frac{1}{w_0}\delta\left(t - \frac{2\pi k}{w_0}\right) \quad w_0 = 2\pi/T$$

$$x(t) = \frac{B}{\cancel{T}}\left(\frac{1}{2a}e^{-a|t|} + e^{-at}u(t)\right)\cancel{T}\sum_{k=-\infty}^{\infty}\delta(t-Tk)$$

$$= \frac{2\pi}{w_0}\sum_{k=-\infty}^{\infty}\delta(t-Tk) = T\sum_{k=-\infty}^{\infty}\delta(t-Tk)$$

$$x(t) = B\left(\frac{1}{2a}e^{-a|t|} + e^{-at}u(t)\right)\sum_{k=-\infty}^{\infty}\delta(t-Tk)$$

Punto 2.2 • Compruebe la solución $h(t)$ de la EDO cuando $x(t) = \delta(t)$ de manera manual. Tener en cuenta que $\epsilon'(t) = u'(t) = \delta(t)$.

sistema $\rightarrow y'(t) + y(t) = x(t)$ para cualquier entrada $x(t)$
sabemos que $h(t) = y(t)$ cuando $x(t) = \delta(t)$

Reemplazando $\rightarrow y'(t) + y(t) = \delta(t)$, $[-\varepsilon, \varepsilon] \rightarrow$ intervalo muy pequeño $\varepsilon \rightarrow 0$.

$$\int_{-\varepsilon}^{\varepsilon} (y'(t) + y(t))\, dt = \int_{-\varepsilon}^{\varepsilon} \delta(t)\, dt$$

$\delta(t) \rightarrow$ se enciende solo en $t=0$ y su área $= 1$.

$$\rightarrow y(\varepsilon) - y(-\varepsilon) + \int_{-\varepsilon}^{\varepsilon} y(t)\, dt = 1$$

si $\varepsilon \rightarrow 0$ la integral también $\rightarrow 0$.

$\rightarrow y(0^+) - y(0^-) = 1$ se asume un sistema causal $y(0^-) = 0$

$\rightarrow y(0^+) = 1$ así para $t>0$ la EDO queda $\rightarrow y'(t) + y(t) = 0$ homogénea.

la solución general es $y(t) = Ce^{-t}$, $t>0$

con la condición $y(0^+) = 1 \rightarrow Ce^0 = 1 \rightarrow C = 1$.

$$u(t) = \begin{cases} 0, & t<0 \\ 1, & t>0 \end{cases}$$

Finalmente $h(t) = e^{-t}u(t)$ $\rightarrow u(t)$ para apagar todo en $y(0^-)$

• Comprobar la solución de la integral de convolución de manera manual. Tener en cuenta las funciones Heaviside.

$h(t) = e^{-t}u(t)$, $x(t) = e^{-2t}u(t)$ hallar $y(t) = h(t) * x(t)$

Definición de convolución $\rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau$ como $h(t)$ y $x(t)$ tienen $u(t) \rightarrow t<0 = 0$.

$$y(t) = \int_0^t x(\tau)h(t-\tau)\, d\tau = \int_0^t e^{-2\tau} e^{-(t-\tau)}\, d\tau = \int_0^t e^{-2\tau - t + \tau}\, d\tau$$

$$= \int_0^t e^{-t-\tau}\, d\tau = e^{-t}\int_0^t e^{-\tau}\, d\tau = e^{-t}\left[-e^{-\tau}\right]_0^t = e^{-t}(-e^{-t} + \overset{1}{e^0})$$

$$= e^{-t} - e^{-t-t} = e^{-t} - e^{-2t}, \quad t \geq 0 \rightarrow y(t) = (e^{-t} - e^{-2t})\, u(t).$$

Punto 2.3: Demuestre sí los siguientes sistemas de la forma $y = H\{x\}$, son sistemas lineales e invariantes en el tiempo (SLIT). Simule los sistemas en Python:

- $y[n] = x[n]/3 + 2x[n-1] - y[n-1]$

① $x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$
$y[n] = (1/3)(\alpha_1 x_1[n] + \alpha_2 x_2[n]) + 2(\alpha_1 x_1[n-1] + \alpha_2 x_2[n-1]) - (\alpha_1 y_1[n-1] + \alpha_2 y_2[n-1])$

② $\alpha_1 x_1(t) + \alpha_2 x_2(t)$
$\alpha_1(x_1[n]/3 + 2x_1[n-1] - y_1[n-1]) + \alpha_2(x_2[n]/3 + 2x_2[n-1] - y_2[n-1])$
$\tilde{y}[n] = \alpha_1 x_1[n]/3 + 2\alpha_1 x_1[n-1] - \alpha_1 y_1[n-1] + \alpha_2 x_2[n]/3 + 2\alpha_2 x_2[n-1] - \alpha_2 y_2[n-1]$
$\tilde{y}[n] = (1/3)(\alpha_1 x_1[n] + \alpha_2 x_2[n]) + 2(\alpha_1 x_1[n-1] + \alpha_2 x_2[n-1] - (\alpha_1 y_1[n-1] + \alpha_2 y_2[n-1])$

Es lineal ① = ②

① $x(t) = x(t - t_0)$
$y[n] = (1/3)\,x[n-n_0] + 2x[n-1-n_0] - y[n-1]$

② $\tilde{y}(t - t_0) = x(t - t_0)$
$\tilde{y}[n-n_0] = (1/3)\,x[n-n_0] + 2x[n-1-n_0] - y[n-1-n_0]$

Solo es invariante en el tiempo sí $y[n-1] = y[n-1-n_0] \rightarrow n_0 = 0$

En conclusión es SLIT ya que cumple linealidad y es invariante en el tiempo con $n_0 = 0$.

- $y[n] = \sum_{k=0}^{n} x^2[k]$

① $x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$
$y[n] = \sum_{k=0}^{n} (\alpha_1 x_1[k] + \alpha_2 x_2[k])^2$
$y[n] = \sum_{k=0}^{n} (\alpha_1^2 x_1^2[k] + 2\alpha_1\alpha_2 x_1[k]x_2[k] + \alpha_2^2 x_2^2[k])$

② $\alpha_1 x_1(t) + \alpha_2 x_2(t)$
$\tilde{y}[n] = \sum_{k=0}^{n} \alpha_1(x_1^2[k]) + \alpha_2(x_2^2[k])$

NO se cumple linealidad. ① ≠ ②

① $x(t) = x(t - t_0)$
$y[n] = \sum_{k=0}^{n} x^2[k - n_0]$

② $\tilde{y}(t - t_0) = x(t - t_0)$
$\tilde{y}[n - n_0] = \sum_{k=0}^{n-n_0} x^2[k]$

NO es invariante en el tiempo ya que los límites ① ≠ límites ②

En conclusión NO es SLIT ya que no es lineal ni invariante en el tiempo.

- $y[n] = \text{median}(x[n])$; donde median es la función mediana sobre una ventana de tamaño 3. $y[n] = \text{median}(x[n-1], x[n], x[n+1])$

Propiedades median( )
- $\text{median}(a+b) \neq \text{median}(a) + \text{median}(b)$
- $\text{median}(\alpha x) = \alpha\,\text{median}(x)$

① $x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$
$y[n] = \text{median}(\alpha_1 x_1[n-1] + \alpha_2 x_2[n-1],\ \alpha_1 x_1[n] + \alpha_2 x_2[n],\ \alpha_1 x_1[n+1] + \alpha_2 x_2[n+1])$

② $\alpha_1 x_1(t) + \alpha_2 x_2(t)$
$\tilde{y}[n] = \alpha_1 \text{median}(x_1[n-1], x_1[n], x_1[n+1]) + \alpha_2 \text{median}(x_2[n-1], x_2[n], x_2[n+1])$
$\tilde{y}[n] = \text{median}(\alpha_1 x_1[n-1], \alpha_1 x_1[n], \alpha_1 x_1[n+1]) + \text{median}(\alpha_2 x_2[n-1] + \alpha_2 x_2[n] + \alpha_2 x_2[n+1])$

Como $\text{median}(a+b) \neq \text{median}(a) + \text{median}(b)$
→ NO se cumple linealidad ① ≠ ②

① $x(t) = x(t-t_0)$
$y[n] = \text{median}(x[n-1-n_0], x[n-n_0], x[n+1-n_0])$

② $\tilde{y}(t-t_0) = x(t-t_0)$
$\tilde{y}[n-n_0] = \text{median}(x[n-1-n_0], x[n-n_0], x[n+1-n_0])$

Es invariante en el tiempo ① = ②

En conclusión NO es SLIT ya que aunque es invariante en el tiempo no es lineal.

- $y(t) = Ax(t) + B$; $A, B \in \mathbb{R}$

① $v(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$
$y(t) = A(\alpha_1 x_1(t) + \alpha_2 x_2(t)) + B = A\alpha_1 x_1(t) + A\alpha_2 x_2(t) + B$

$\neq$

② $\alpha_1 x_1(t) + \alpha_2 x_2(t)$
$y(t) = \alpha_1(Ax_1(t) + B) + \alpha_2(Ax_2(t) + B) = A\alpha_1 x_1(t) + A\alpha_2 x_2(t) + \alpha_1 B + \alpha_2 B$

Solo cumple linealidad sí $B = \alpha_1 B + \alpha_2 B \rightarrow \alpha_1 + \alpha_2 = 1 \rightarrow \alpha_1 = \alpha_2 = 1/2$.

① $v(t) = x(t-t_0)$
$y(t) = Ax(t-t_0) + B$

② $y(t-t_0) = x(t-t_0)$
$y(t-t_0) = Ax(t-t_0) + B$

Es invariante en el tiempo ① = ②

En conclusión NO es SLIT, ya que aunque es invariante en el tiempo no es lineal

Punto 2.5: Sea la señal Gaussiana $x(t) = e^{-at^2}$ con $a \in \mathbb{R}^+$, el sistema A con relación entrada-salida $y_A(t) = x^2(t)$, y el sistema lineal e invariante con el tiempo B con respuesta al impulso $h_B(t) = Be^{-bt^2}$:

a. Encuentre la salida del sistema en serie $x(t) \xrightarrow{B} h_B(t) \xrightarrow{A} y_A(t) \rightarrow y(t)$

$$x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

$$y(t) = x(t) * h(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

↳ Respuesta al impulso desplazado

$h_B = Be^{-bt^2}$ $\quad x(t) = e^{-at^2}$ $\quad y_A(t) = x^2(t)$ → Info que tenemos

$x(t) \xrightarrow{h_B} v(t) \xrightarrow{A} y(t)$ → Pide (a)

Se resuelve aparte.

$$v(t) = x(t) * h_B = \int_{-\infty}^{\infty} e^{-a\tau^2} Be^{-b(t-\tau)^2}\,d\tau = B\int_{-\infty}^{\infty} e^{-(a\tau^2 + b(t-\tau)^2)}\,d\tau$$

$$\rightarrow a\tau^2 + b(t-\tau)^2 = a\tau^2 + b(t^2 - 2t\tau + \tau^2) = a\tau^2 + bt^2 - 2bt\tau + b\tau^2$$
$$= (a+b)\tau^2 - 2bt\tau + bt^2$$

$$\rightarrow (a+b)\left(\tau^2 - \frac{2bt}{a+b}\tau\right) = (a+b)\left[\left(\tau - \frac{bt}{a+b}\right)^2 - \left(\frac{bt}{a+b}\right)^2\right]$$

$$= a\tau^2 + b(t-\tau)^2 = (a+b)\left(\tau - \frac{bt}{a+b}\right)^2 bt^2 - (a+b)\left(\frac{bt}{a+b}\right)^2$$

$$\rightarrow bt^2 - (a+b)\frac{bt^2}{(a+b)^2} = bt^2 - \frac{b^2t^2}{a+b} = \frac{b(a+b) - b^2}{a+b}t^2 = \frac{ab}{a+b}t^2$$

$$= a\tau^2 + b(t-\tau)^2 = (a+b)\left(\tau - \frac{bt}{a+b}\right)^2 + \frac{ab}{a+b}t^2$$

Integral Gaussiana

$$B\int_{-\infty}^{\infty} e^{-\left((a+b)\left(\tau - \frac{bt}{a+b}\right)^2 + \frac{ab}{a+b}t^2\right)}\,d\tau = Be^{-\frac{ab}{a+b}t^2}\int_{-\infty}^{\infty} e^{-\left((a+b)\left(\tau - \frac{bt}{a+b}\right)^2\right)}\,d\tau$$

$$= Be^{-\frac{ab}{a+b}}\sqrt{\frac{\pi}{a+b}}$$ → usando la propiedad

$$y(t) = x^2(t) = \left(Be^{-\frac{ab}{a+b}t^2}\sqrt{\frac{\pi}{a+b}}\right)^2 = \frac{\pi}{a+b}B^2 e^{-\frac{2ab}{a+b}t^2} = \frac{B^2\pi}{a+b}e^{\frac{-2ab}{a+b}t^2}$$

b. Encuentre la salida del sistema en serie $x(t) \to y_A(t) \xrightarrow{A} h_B(t) \xrightarrow{B} y(t)$

$$x(t) \xrightarrow{A} u(t) = y_A(t) \xrightarrow{h_B} y(t)$$

$$u(t) = x^2(t) = e^{-2at^2}$$

se resuelve aparte

$$y(t) = u(t) * h_B(t) = \int_{-\infty}^{\infty} e^{-2a\tau^2} Be^{-b(t-\tau)^2} d\tau = B\int_{-\infty}^{\infty} e^{-\overline{(2a\tau^2 + b(t-\tau)^2)}} d\tau$$

$$\to 2a\tau^2 + b(t-\tau)^2 = (2a+b)\tau^2 - 2bt + bt^2$$

$$= (2a+b)\left(\tau - \frac{bt}{2a+b}\right) bt^2 - (2a+b)\left(\frac{bt}{2a+b}\right)^2$$

$$\to bt^2 - \frac{b^2t^2}{2a+b} = \frac{b(2a+b) - b^2}{2a+b} t^2 = \frac{2ab}{2a+b} t^2$$

$$B\int_{-\infty}^{\infty} e^{-\left((2a+b)\left(\tau - \frac{bt}{2a+b}\right)^2 + \frac{2ab}{2a+b}t^2\right)} d\tau = Be^{-\frac{2ab}{2a+b}} \int_{-\infty}^{\infty} e^{-(2a+b)\left(\tau - \frac{bt}{2a+b}\right)^2} d\tau$$

$$= Be^{-\frac{2ab}{2a+b}t^2} \sqrt{\frac{\pi}{2a+b}} \to \text{usando la propiedad}$$

$$y(t) = B\sqrt{\frac{\pi}{2a+b}}\, e^{-\frac{2ab}{2a+b}t^2}$$

Punto 2.10: • Obtener la transformada inversa de laplace de $X(s)$ manualmente.

Dada $X(s) = \dfrac{2s^2+14s+124}{s^3+8s^2+46s+68}$ con Región de convergencia $\rightarrow R\{s\} = -2$.

Comprobar sí $s=-2$ el raíz del denominador $\rightarrow (-2)^3+8(-2)^2+46(-2)+68$
$= -8+32-92+68 = 0$ $\rightarrow$ se divide entre $(s+2)$

División sintética $\rightarrow$

s^3	s^2	s^1	s^0	
1	8	46	68	$s=-2$
1	6	34	0	$\rightarrow$ si es raíz.

$s^3+8s^2+46s+68 = s^2+6s+34\,(s+2)$.

Fracciones parciales $\rightarrow \dfrac{2s^2+14s+124}{(s+2)(s^2+6s+34)} = \dfrac{A}{s+2} + \dfrac{Bs+C}{s^2+6s+34}$ Multiplicamos por $(s+2)(s^2+6s+34)$.

$\rightarrow 2s^2+14s+124 = A(s^2+6s+34)+(Bs+C)(s+2)$
$= 2s^2+14s+124 = As^2+6As+34A+Bs^2+2Bs+Cs+2C$
$2s^2+14s+124 = (A+B)s^2+(6A+2B+C)s+(34A+2C)$

Igualando factores $\rightarrow A+B=2$
$B = 2-A.$

$6A+2B+C = 14$
$6A+2(2-A)+C = 14$
$6A+4-2A+C = 14.$
$4A+C = 10. \rightarrow C = 10-4A.$

$34A+2C = 124.$
$34A+2(10-4A) = 124.$
$34A+20-8A = 124.$
$26A = 104. \rightarrow A = 104/26 = 4$

$B = 2-A = 2-4 = -2$
$C = 10-4(4) = 10-16 = -6$

Se obtiene $\rightarrow A=4,\ B=-2,\ C=-6.$

$X(s) = \dfrac{4}{s+2} + \dfrac{-2s-6}{s^2+6s+34}$ $\rightarrow$ $s^2+6s+34$ Denominador

se completa el cuadrado
$= (s^2+6s+(6/2)^2)-(6/2)^2+34.$
$\underbrace{(s^2+6s+9)}_{(s+3)^2}(-9+34) = 25.$
$(s+3)^2+25.$

Numerador $\rightarrow -2s-6 = -2(s+3)$

$X(s) = \dfrac{4}{s+2} + \dfrac{-2(s+3)}{(s+3)^2+25}$ Usamos transformadas conocidas:

$\mathcal{L}^{-1}\left\{\dfrac{1}{s+a}\right\} = e^{-at}u(t) \rightarrow \mathcal{L}^{-1}\left\{\dfrac{4}{s+2}\right\} = 4e^{-2t}u(t)$

$\mathcal{L}^{-1}\left\{\dfrac{s+a}{(s+a)^2+w^2}\right\} = e^{-at}\cos(wt)u(t) \rightarrow \mathcal{L}^{-1}\left\{\dfrac{-2(s+3)}{(s+3)^2+5^2}\right\} = -2e^{-3t}\cos(5t)u(t)$

$x(t) = (4e^{-2t}-2e^{-3t}\cos(5t))\,u(t)$

• Obtener la transformada inversa de Laplace manualmente de:

$$X(s) = \frac{1}{(s+1)(s+2)^2} \quad \text{para } \mathcal{R}\{s\} > -1.$$

Fracciones parciales $\rightarrow \frac{1}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$ Multiplicar por $(s+1)(s+2)^2$

Se eliminan denominadores $\rightarrow 1 = A(s+2)^2 + B(s+1)(s+2) + C(s+1)$

Al evaluar en $s=-1 \rightarrow 1 = A(-1+2)^2 + \cancel{B(-1+1)(-1+2)}^{0} + \cancel{C(-1+1)}^{0}$

$1 = A(1)^2 = A$

Al evaluar en $s=-2 \rightarrow 1 = \cancel{A(-2+2)^2}^{0} + \cancel{B(-2+1)(-2+2)}^{0} + C(-2+1)$

$1 = C(-1) = -C \rightarrow C = -1$

Al evaluar en $s=0 \rightarrow 1 = A(0+2)^2 + B(0+1)(0+2) + C(0+1)$

$1 = 4 + 2B - 1 \rightarrow B = -2/2 \rightarrow B = -1$

Sustituye $A = 1$, $C = -1$

Sustituimos $\rightarrow \frac{1}{(s+1)(s+2)^2} = \frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{(s+2)^2}$ Usando transformadas conocidas:

$\mathcal{L}^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}u(t) \rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}u(t)$, $\mathcal{L}^{-1}\left\{\frac{-1}{s+2}\right\} = -e^{-2t}u(t)$

$\mathcal{L}^{-1}\left\{\frac{1}{(s+a)^n}\right\} = \frac{t^{n-1}}{n-1!}e^{-at} \rightarrow \mathcal{L}^{-1}\left\{\frac{-1}{(s+2)^2}\right\} = -te^{-2t}u(t)$

$x(t) = \left(e^{-t} - e^{-2t} - te^{-2t}\right)u(t)$

$= \left(e^{t} - 1 - t\right)e^{-2t}u(t)$