(1) Use the Quotient Remainder Theorem with $d = 3$ to prove that the product of any two consecutive integers has the form $3k$ or $3k + 2$.

Let $n \in \mathbb{Z}$, The Quotient Remainder Theorem tells us that every integer is one of the form $3m$, $3m+1$, or $3m+2$.

[Case 1] $n = 3m$, then the consecutive integers are of the form $3m$, $3m+1$.The product should be $n(n+1) = 3m(3m+1)$, this then equals $3k$ for some integer $k = (3m+1)$.

[Case 2] n=$3m$+1 If the consecutive integers are $3m$+1 and $3m$+2 then the product is

$$\begin{aligned} n(n+1) &= (3m+1)(3m+2) \\ &= 3(3m^2+3m)+2. \end{aligned}$$

[Case 3] $n = 3m+2$ if the consecutive integers are of the form $3m$+2 and $3m$+3, then the product is of the form

$$\begin{aligned} n(n+1) &= (3m+2)(3m+3) \\ &= 3(3m^2+5m+2). \end{aligned}$$

From these three cases we can confirm that the product of any two integers is of the form $3k$ or $3k$+2.

(2)

(a) The standard factored form of a^3 would be the standard form of a cubed.

$$\begin{aligned} a^3 &= (p_1^{e_1} \cdot p_2^{e_2} ... p_k^{e_k})^3, \\ a^3 &= p_1^{3e_1} \cdot p_2^{3e_2} ... p_k^{3e_k}. \end{aligned}$$

(b) To start, we must find the lowest positive integer k, such that $2^4 \cdot 3^5 \cdot 7 \cdot 11^2 \cdot k$ is a perfect cube, or in other words the integer must equal the third power. Let's take the least positive integer k = 2^4·3^5·7 · 11^2.

$$\begin{aligned} 2^4 \cdot 3^5 \cdot 7 \cdot 11^2 &= 2^4 \cdot 3^5 \cdot 7 \cdot 11^2 \cdot 2^2 \cdot 3^1 \cdot 7^2 \cdot 11 \\ &= 2^6 \cdot 3^6 \cdot 11^3 \cdot 7^3 \\ &= (2^2)^3 \cdot (3^2)^3 \cdot 11^3 \cdot 7^3 \\ &= (2^2 \cdot 3^2 \cdot 11 \cdot 7)^3 \\ &= (2772)^3. \end{aligned}$$

By definition of a perfect cube,

$$\begin{aligned} k &= 2^2 \cdot 3^1 \cdot 7^2 \cdot 11 \\ &= 4 \cdot 3 \cdot 49 \cdot 11. \end{aligned}$$

Finally, the least positive integer $k = 6468$