(1) Prove that if a and b are rational numbers $b \neq 0$, and r is an irrational number, then $a + br$ is irrational.

Let $a, b \in \mathbb{Q}$ and $r \in \mathbb{I}$. There are integers m, n where $n \neq 0$, $a = \frac{m}{n}$.Then $b = \frac{s}{t}$, $t \neq 0$, for some intgers s and t. We know that $a + br$ is rational, where $a + br = \frac{p}{q}$, for some integers p and q, $q \neq 0$.
We can consider

$$\begin{aligned} a + br &= \frac{p}{q} \\ \frac{m}{n} + \left(\frac{s}{t}\right) \cdot r &= \frac{p}{q} \\ r &= (\frac{p}{q} - \frac{m}{n}) \cdot \frac{t}{s} \\ &= \frac{(t(np - mq))}{qns}. \end{aligned}$$

Where $n \neq 0$, $q \neq 0$, $b \neq 0$ therefore, $qns \neq 0$ using the zero product property. The integer r is over a non integer, proving that r is a rational number, contradicting the fact that r is irrational and also proving $a + br$ is irrational.

(2) Prove that for every integer a, if a^3 is even then a is even.

Let a be an odd integer. Then $a = 2k + 1$ for some $k \in \mathbb{Z}$.

$$\begin{aligned} a^3 &= (2k + 1)^3 \\ &= 8k^3 + 12k^2 + 6k + 1 \\ &= 2(4k^3 + 6k^2 + 3k) + 1. \end{aligned}$$

Since $k \in \mathbb{Z}$, $4k^3 + 6k^2 + 3k \in \mathbb{Z}$. Thus a^3 is odd.

(3) Prove that $\sqrt[3]{2}$ is irrational.
We can do a proof by contradiction. To start, we can assume $\sqrt[3]{2}$ is rational. Let $\sqrt[3]{2} = p/q$ where p and q are integers with no common divisors

$$\begin{aligned} 2 &= (p/q)^3 \\ p^3 &= 2q^3 \end{aligned}$$

Here, p^3 is even by problem 2. Let $p = 2k$ for some integer k. Then,we subsitute $p = 2k$ in $(p/q)^3 = 2$.

$$\begin{aligned} (p/q)^3 &= 2 \\ p^3 &= 2q^3 \\ (2k)^3 &= 2q^3 \\ 4k^3 &= q^3 \end{aligned}$$

Hence, q^3 is even, therefore q is even by number 2. Then p and q are divisible by 2 and hence, p and q have the common factor of 2. This is a contradiction. Therefore, $a = \sqrt[3]{2}$ is an irrational number.

(4) Consider the following statement:

$$(\forall m, n \in \mathbb{Z}) \text{ if } 2m+n \text{ is odd then } m \text{ and } n \text{ are both odd.}$$

(a)

$2m+n$ is odd implies that an even number multiplied by $2m$ then added to 1 equals an odd number. If we assume n is an odd number, $2m+n$ will still result to be an odd number. For example, we can use 4 as m and 3 as n.

$$\begin{aligned} 2m+n &= 8+3 \\ &= 11, \end{aligned}$$

which is odd.

(b)

The negation of the statement is :

$$\exists \text{ integers } m \text{ and } n, 2m+n \text{ is odd and both } m \text{ and } n \text{ are not odd}$$

(5) Determine, with proof, whether this statement is true or false: The product of any two irrational numbers is irrational.

We can have $9+2\sqrt{3}$ be an irrational number and $9-2\sqrt{3}$ be another irrational number.If we were to multiply $9+2\sqrt{3}$ and $9-2\sqrt{3}$ we then get our answer:

$$\begin{aligned} (9+2\sqrt{3})(9-2\sqrt{3}) &= 81-4(3) \\ &= 81-12 \\ &= 69, \end{aligned}$$

Which is rational. This is a contradiction to the statement 'The product of any two irrational numbers is irrational.' Therefore the statement is false.