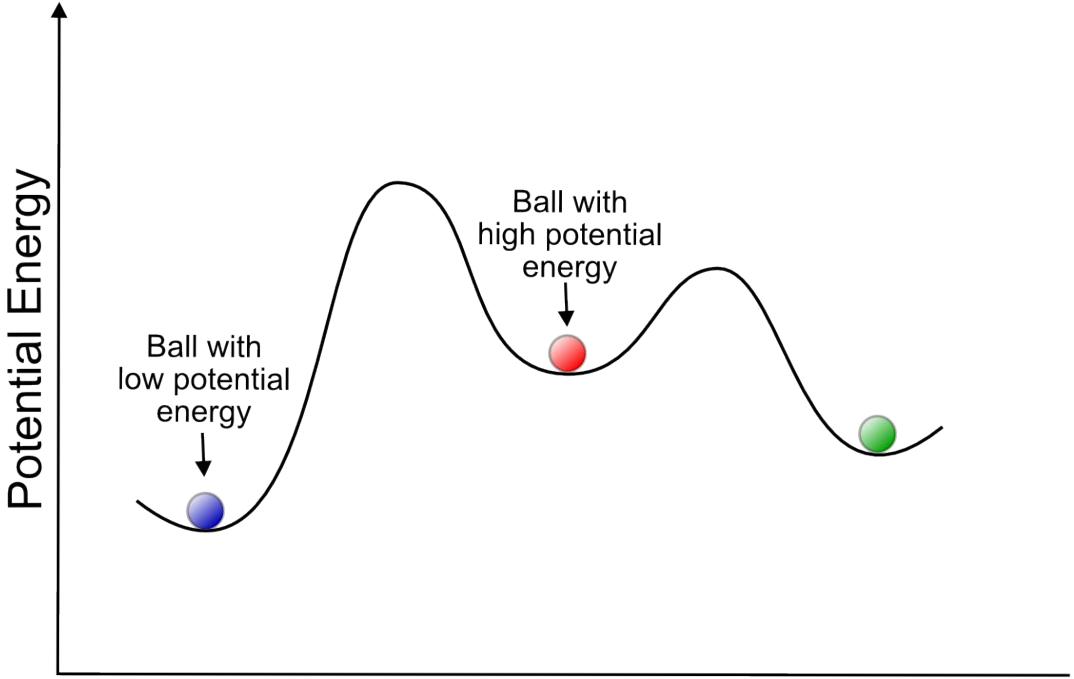
Neutron Stars

Lecture 4: perturbing TOV solutions

Ignacio F. Ranea-Sandoval (Argentina)

Stability analysis basics

Difference between stationary and stable configurations



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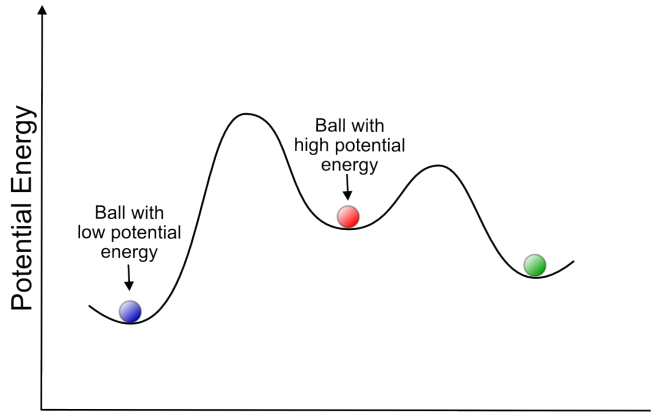

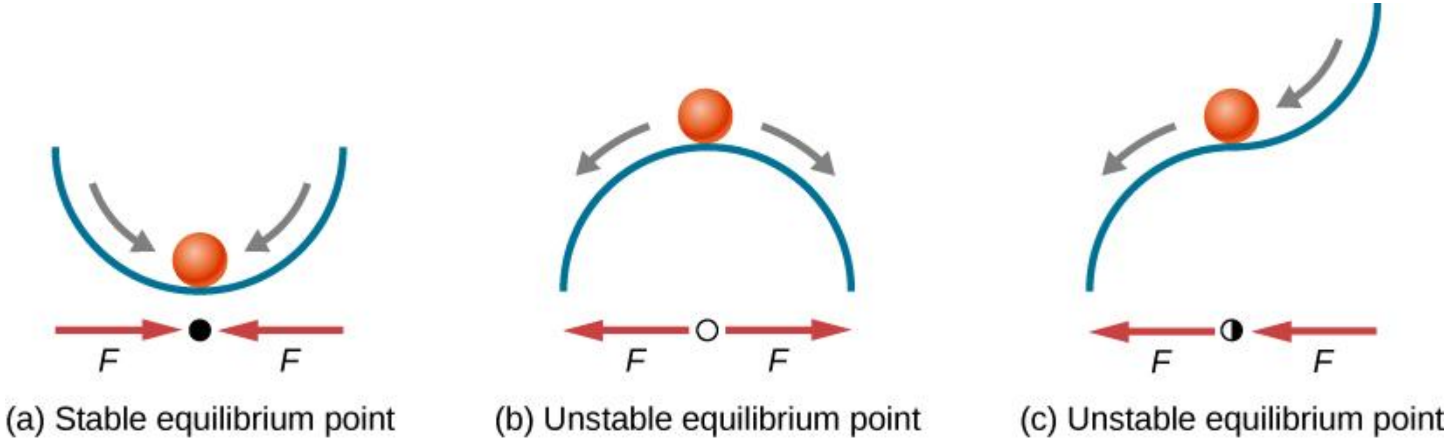

Stability analysis basics

Difference between stationary
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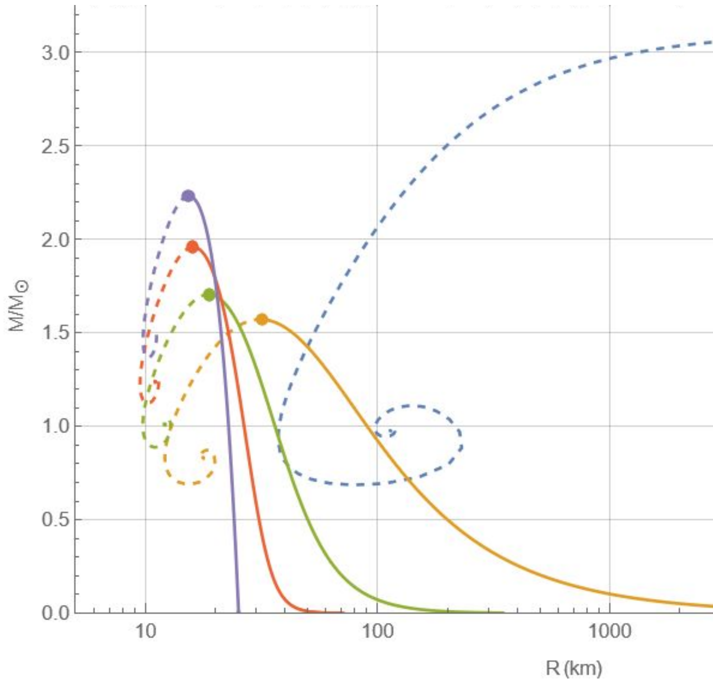
Remember our TOV structure equations?

$$\frac{dm}{dr} = \frac{4\pi}{c^2} r^2 \epsilon \qquad \text{with} \qquad m(0) = 0$$

$$\frac{d\nu}{dr} = \frac{2Gm}{r^2 c^2} \left[1 + \frac{4\pi G p r^3}{mc^2}\right] \left(1 - \frac{2Gm}{rc^2}\right)^{-1}$$

$$\frac{dp}{dr} = -\frac{1}{2}(p + \epsilon) \frac{d\nu}{dr}$$



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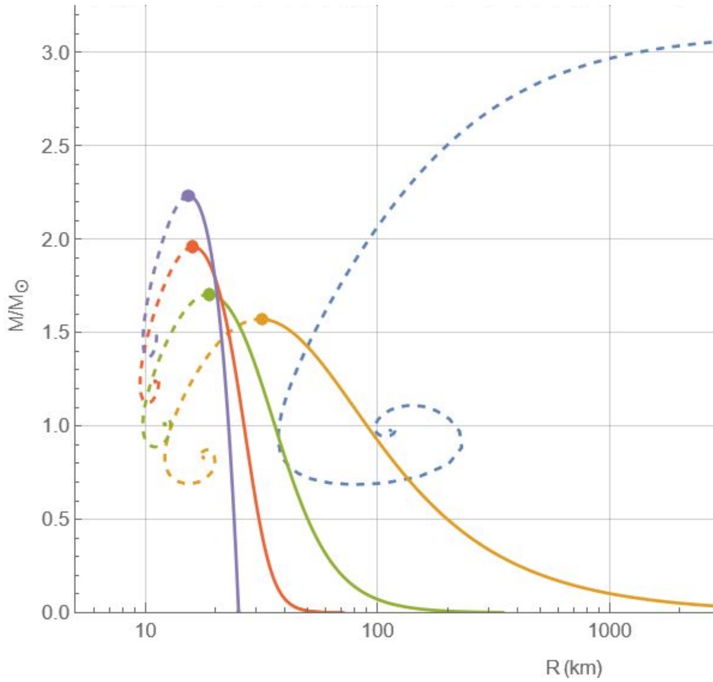
Why the different line styles?



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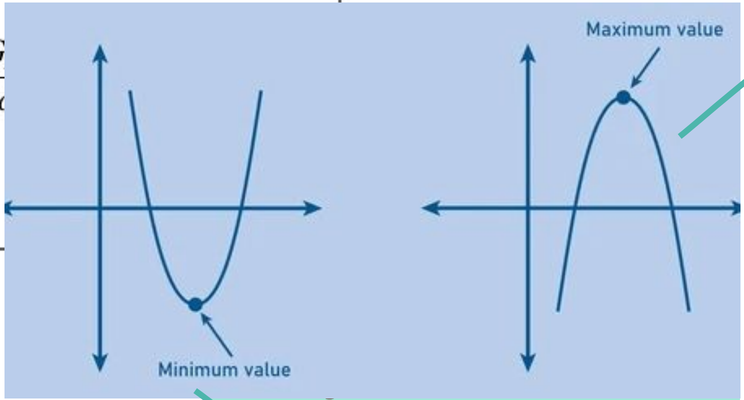
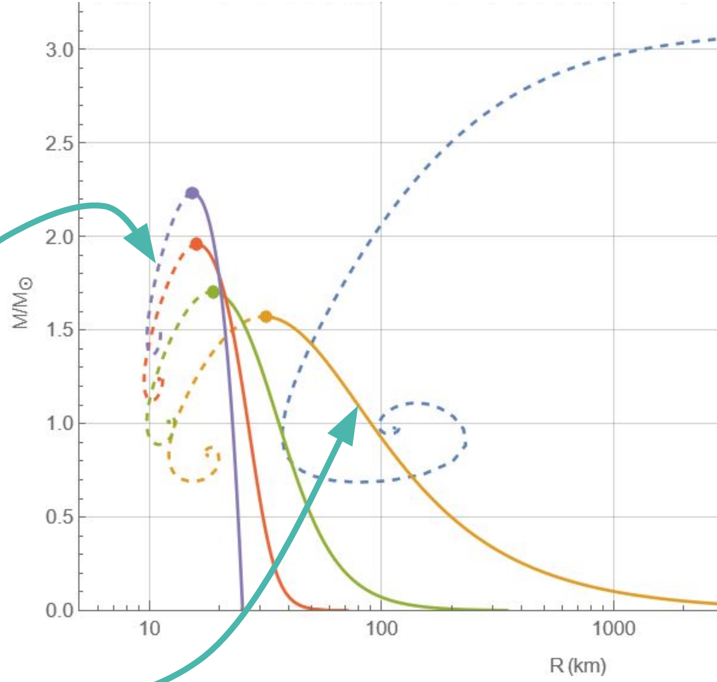


line styles?



Linearized radial perturbations of a TOV solution

A TOV solution is a spherically symmetric solution that is in hydrostatic equilibrium.

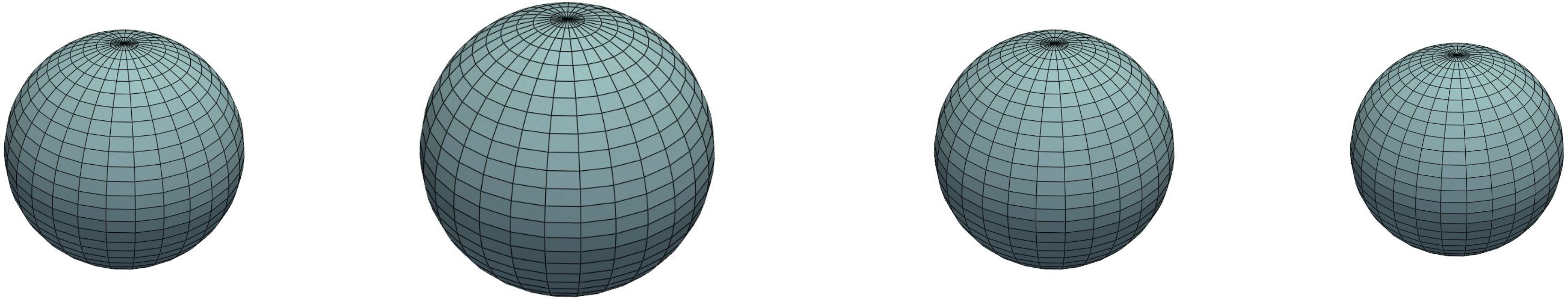
This does not imply stability

Linearized radial perturbations of a TOV solution

A TOV solution is a spherically symmetric solution that is in hydrostatic equilibrium.

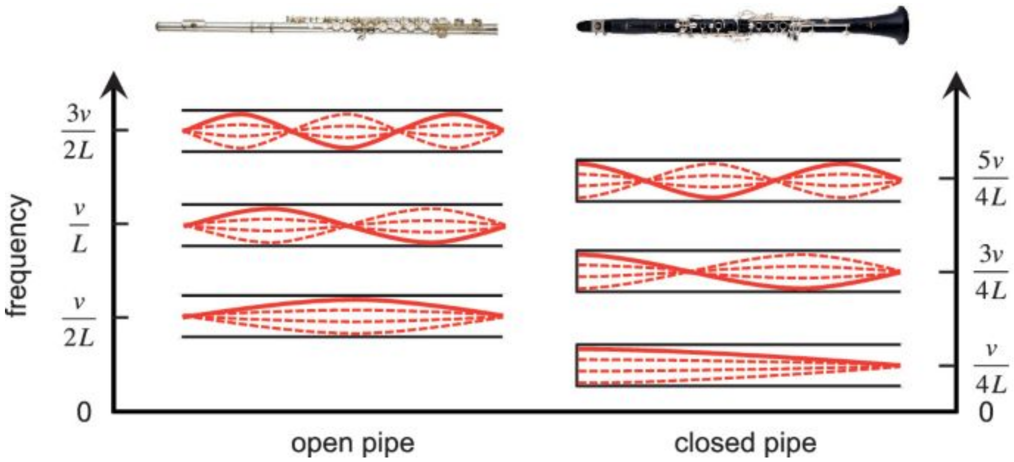
This does not imply stability

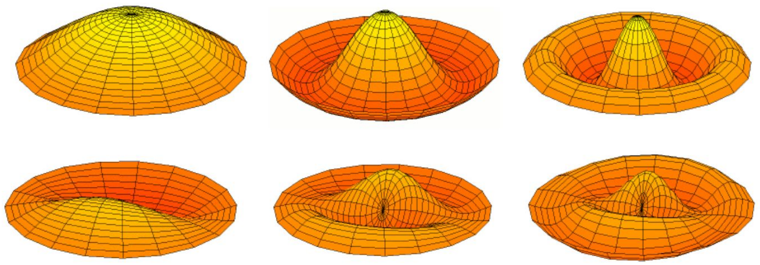
Chandrasekhar in 1964 started the study of stability of such solutions against radial perturbations.



(A short parenthesis)

Eigenmodes or proper modes of a system:





Drumhead Vibration Modes

(A short parenthesis)

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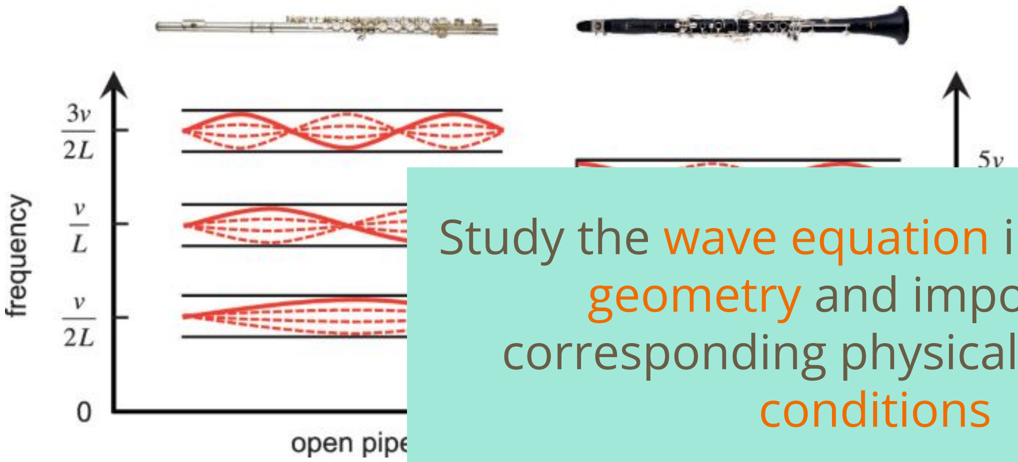
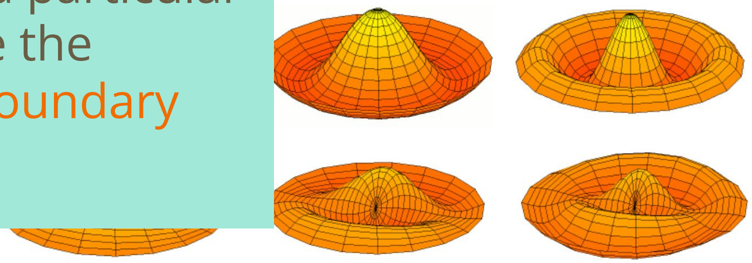


Study the wave equation in a particular geometry and impose the corresponding physical boundary conditions



Drumhead Vibration Modes

Linearized radial perturbations of a TOV solution

Radial perturbations preserve the
spherical symmetry of the TOV solution

How can we study them?

Linearized radial perturbations of a TOV solution

Radial perturbations preserve the spherical symmetry of the TOV solution

How can we study them?

$$G^i_j = \frac{8\pi G}{c^4} T^i_j$$

but now... radial velocity is different from zero!

Linearized radial perturbations of a TOV solution

Radial perturbations preserve the spherical symmetry of the TOV solution

$$G^i_j = \frac{8\pi G}{c^4} T^i_j$$

$$\begin{aligned}
\lambda(r,t) &= \lambda_0(r) + \delta\lambda(r,t),\\
\nu(r,t) &= \nu_0(r) + \delta\nu(r,t),\\
p(r,t) &= p_0(r) + \delta p(r,t),\\
\epsilon(r,t) &= \epsilon_0(r) + \delta\epsilon(r,t).
\end{aligned}$$

TOV solution

To-be-determined-perturbation
note the time dependence!

Linearized radial perturbations of a TOV solution

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A couple of definitions
Radial velocity

$$v = \frac{dx^1}{dx^0} = \frac{dx^1/ds}{dx^0/ds} = \frac{u^1}{u^0} \qquad \Longrightarrow \quad u^1 = v u^0$$

After using normalization of 4-velocity $\quad e^{\nu}(u^0)^2 + (-e^{\lambda})(u^1)^2 = 1$

and keeping 1st-order terms of the perturbations we have that

$$\boxed{u^0 \approx e^{-\nu/2}} \qquad \boxed{u^1 \approx v e^{-\nu/2}}$$

Linearized radial perturbations of a TOV solution

Radial perturbations preserve the spherical symmetry of the TOV solution

The fluid stress-energy tensor $$T^i_j = (p + \epsilon)u^i u_j + p\delta^i_j$$

defining $$p = p_0 + \delta p, \quad \epsilon = \epsilon_0 + \delta\epsilon$$

$$T^1_1 = T^2_2 = T^3_3 = p \qquad T^0_0 = -\epsilon$$

to first order, the non-vanishing components of the stress-energy tensor are

$$T^1_0 = (p_0 + \epsilon_0)u^1 u_0 = -(p_0 + \epsilon_0)\, v$$

$$T^0_1 = (p_0 + \epsilon_0)u^0 u_1 = e^{\lambda_0 - \nu_0}(p_0 + \epsilon_0)\, v$$

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Lagrangian displacement ξ and so $$v = \frac{d\xi}{dx^0}$$

Linearized radial perturbations of a TOV solution

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Lagrangian displacement ξ and so $\quad v = \dfrac{d\xi}{dx^0}$

All together the perturbed stress-energy tensor can be written as

$$T^i_j = \begin{pmatrix} -(\epsilon_0 + \delta\epsilon) & e^{\lambda_0 - \nu_0}(p_0 + \epsilon_0)\,\frac{d\xi}{dx^0} & 0 & 0 \\ -(p_0 + \epsilon_0)\,\frac{d\xi}{dx^0} & p_0 + \delta p & 0 & 0 \\ 0 & 0 & p_0 + \delta p & 0 \\ 0 & 0 & 0 & p_0 + \delta p \end{pmatrix}$$

Linearized radial perturbations of a TOV solution

Radial perturbations preserve the spherical symmetry of the TOV solution

The line element

$$ds^2 = e^{\nu_0+\delta\nu}c^2dt^2 - e^{\lambda_0+\delta\lambda}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

From the Einstein Field Equations at linear level

$$\delta\lambda = -\xi \frac{d}{dr}(\lambda_0+\nu_0).$$

$$\delta\epsilon = -\frac{1}{r^2}\frac{\partial}{\partial r}[r^2(p_0+\epsilon_0)\xi]$$

$$(p_0+\epsilon_0)\frac{\partial}{\partial r}\delta\nu = \left[\delta p - (p_0+\epsilon_0)\left(\frac{d\nu_0}{dr}+\frac{1}{r}\right)\xi\right]\frac{d}{dr}(\lambda_0+\nu_0)$$

$$e^{\lambda_0-\nu_0}(p_0+\epsilon_0)\frac{\partial v}{\partial x^0} + \frac{\partial}{\partial r}\delta p + \tfrac{1}{2}(p_0+\epsilon_0)\frac{\partial}{\partial r}\delta\nu + \tfrac{1}{2}(\delta p+\delta\epsilon)\frac{d\nu_0}{dr} = 0.$$

Linearized radial perturbations of a TOV solution

Radial perturbations preserve the spherical symmetry of the TOV solution

Harmonic time-dependence of the perturbations

$$\delta\lambda(r,t) = \delta\lambda(r)e^{i\omega x^0}$$

$$\delta\nu(r,t) = \delta\nu(r)e^{i\omega x^0}$$

$$\delta p(r,t) = \delta p(r)e^{i\omega x^0}$$

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unknown amplitudes of the perturbations

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unknown characteristic frequencies

unknown amplitudes of the perturbations

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Radial perturbations preserve the
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Harmonic time-de

$\delta\lambda(r,t) = \delta\lambda($

characteristic frequencies

After some algebra
we can obtain

unknown amplitudes of the

$$\omega^2 e^{\lambda_0-\nu_0}(p_0+\epsilon_0)\xi = \frac{4}{r}\frac{dp_0}{dr}\xi - e^{-(\lambda_0+2\nu_0)/2}\frac{d}{dr}\left[e^{(\lambda_0+3\nu_0)/2}\frac{\gamma p_0}{r^2}\frac{d}{dr}(r^2 e^{-\nu_0/2}\xi)\right]$$

$$+\frac{8\pi G}{c^4}e^{\lambda_0}p_0(p_0+\epsilon_0)\xi - \frac{1}{p_0+\epsilon_0}\left(\frac{dp_0}{dr}\right)^2\xi\,.$$

With boundary conditions
$\xi(0) = 0$ and $\delta p(R) = 0$

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Form a Sturm-Liouville problem
(an eigenvalue eigenfunction problem)

Tons and tons of properties are known!

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Form a Sturm-Liouville problem (an eigenvalue eigenfunction problem)
Tons and tons of properties are known!

Those that matter most to us:
All eigenstates are non degenerate.
The square of the frequencies are real and ordered, being ω_0^2 the smallest, one associated to the fundamental mode.

number of nodes of the radial eigenfunction

Linearized radial perturbations of a TOV solution

Radial perturbations preserve the spherical symmetry of the TOV solution

A TOV solution is stable against linearized radial perturbation if the fundamental frequency is real, if not they are unstable.

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Linearized radial perturbations of a TOV solution

Radial perturbations preserve the spherical symmetry of the TOV solution

A TOV solution is stable against linearized radial perturbation if the fundamental frequency is real, if not they are unstable.

Harrison, Thorne, Wakano and Wheeler 1965 prove that for cold matter in chemical equilibrium described with an EOS free of discontinuities that stability can be studied locating critical points of the mass vs central energy density curve

$$\frac{\partial M(\epsilon_c)}{\partial \epsilon_c} = 0$$

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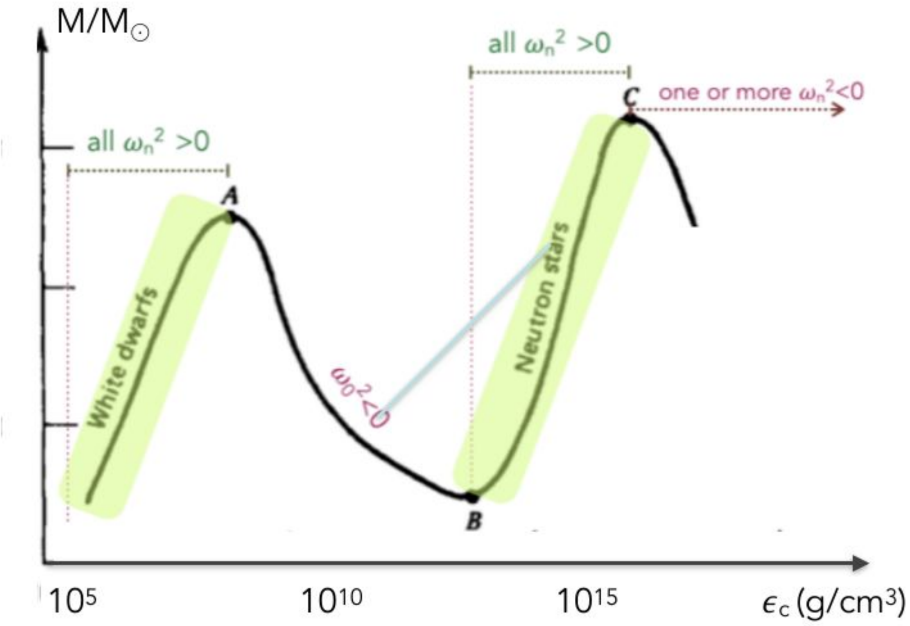
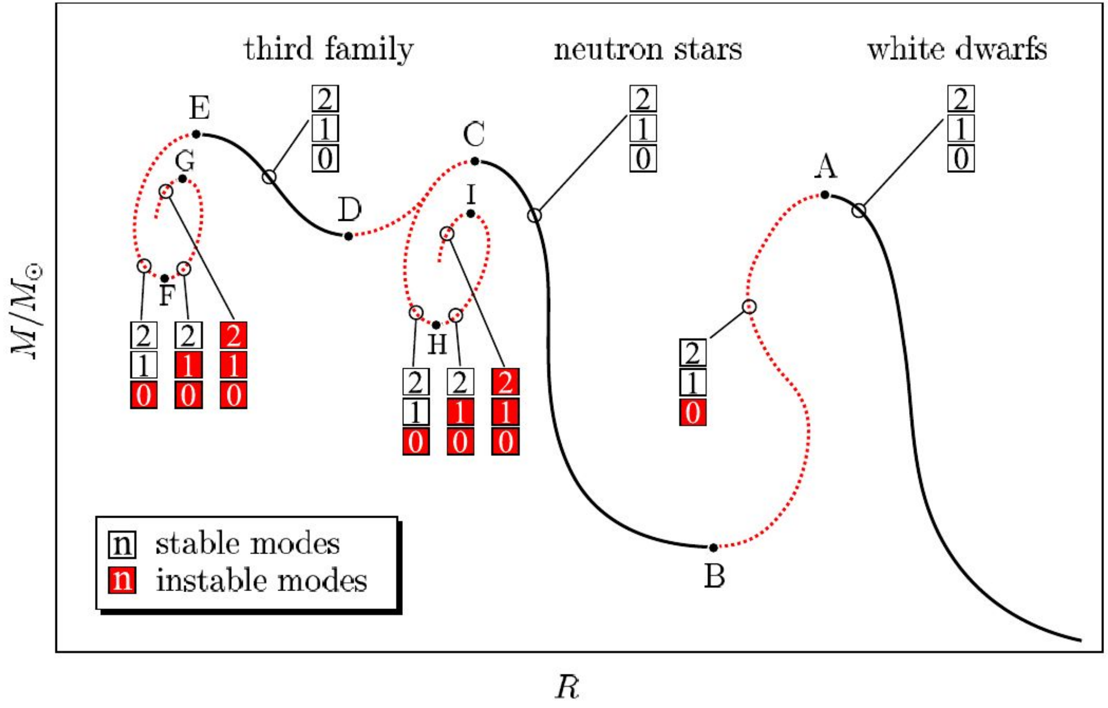


Linearized radial perturbations of a TOV solution

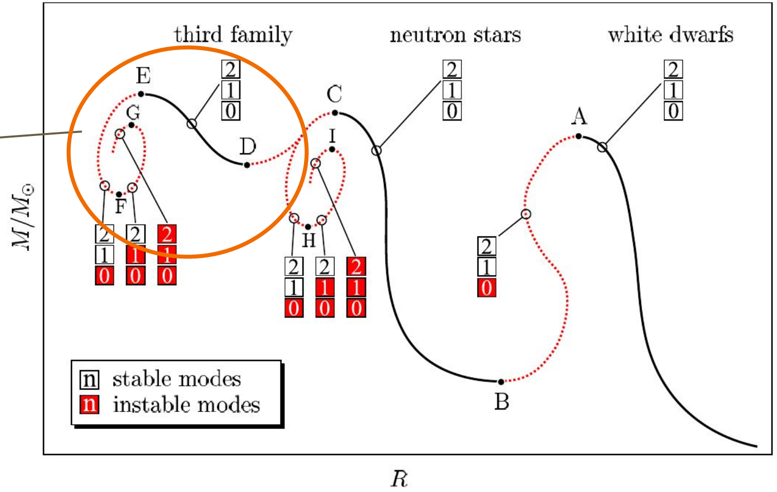
Radial perturbations preserve the spherical symmetry of the TOV solution



Linearized radial perturbations of a TOV solution

Radial perturbations preserve the spherical symmetry of the TOV solution

Typical behaviour of hybrid EOS



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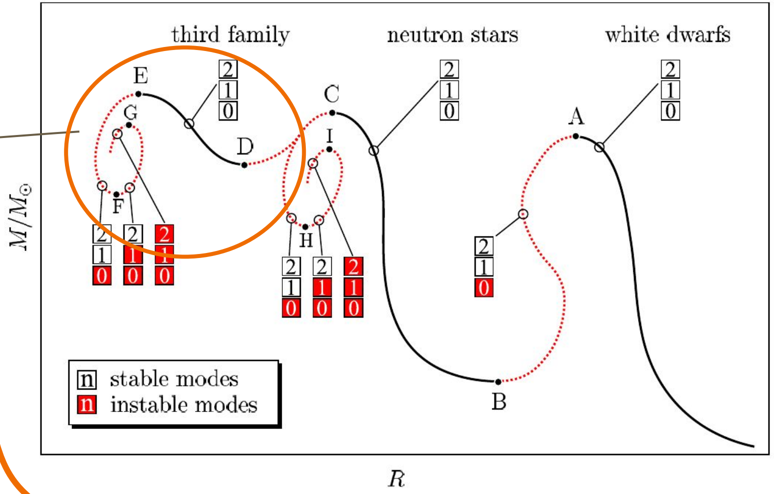

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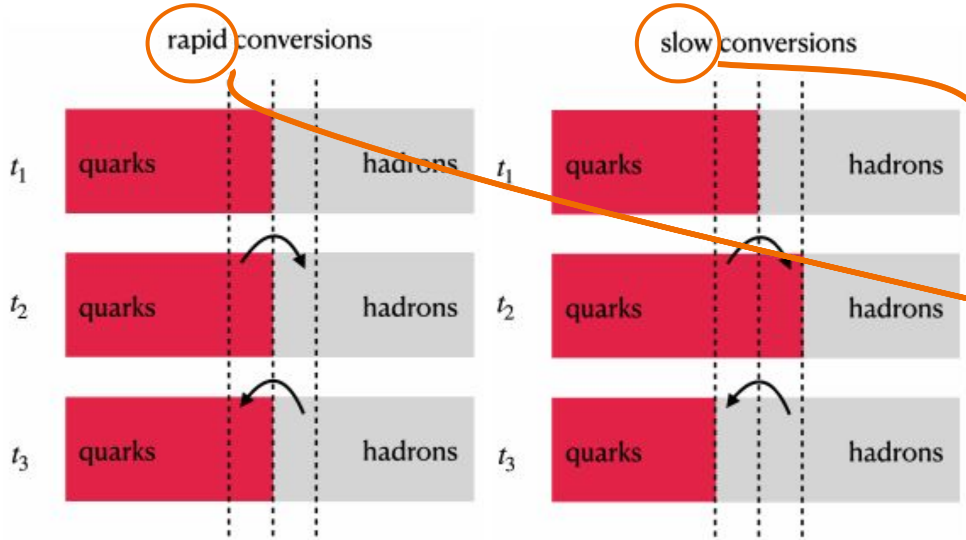
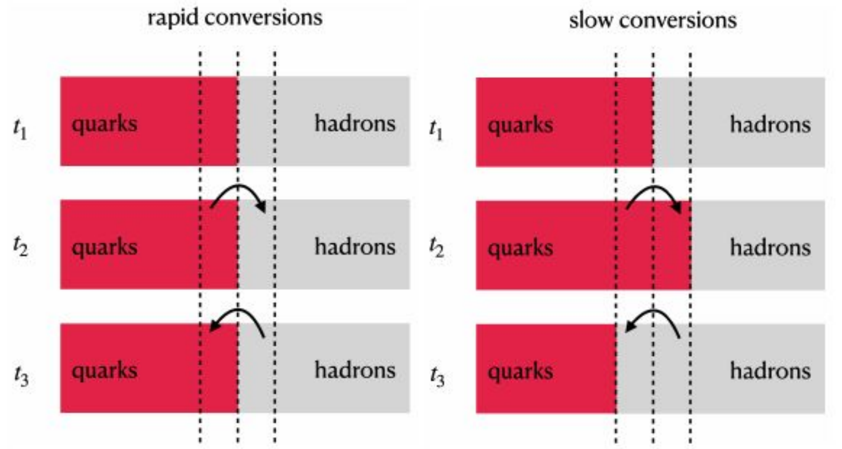


in relation to the characteristic timescale of the perturbation

Linearized radial perturbations of a TOV solution

Radial perturbations preserve the spherical symmetry of the TOV solution



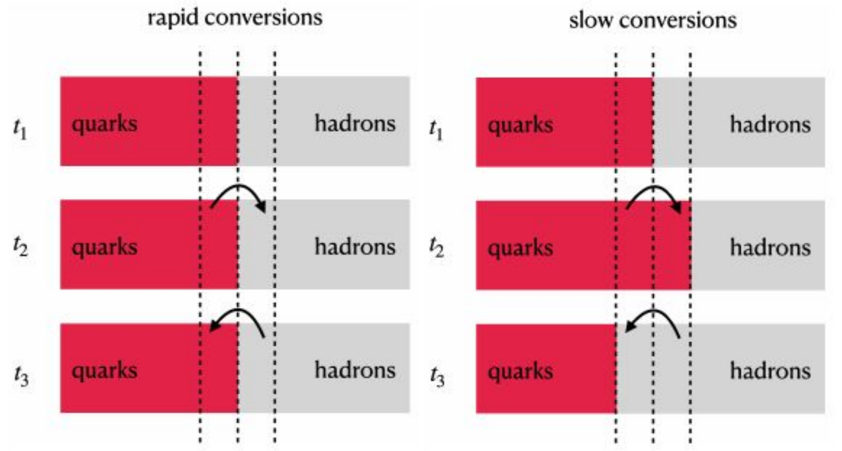
$$[\xi]^+_- \equiv \xi^+ - \xi^- = 0, \quad [\Delta p]^+_- \equiv \Delta p^+ - \Delta p^- = 0$$

$$[\xi^r]^+_- = \Delta p \left[\frac{1}{p'_0}\right]^+_-, \quad [\Delta p]^+_- = 0$$

This boundary conditions have to be taken into account

Linearized radial perturbations of a TOV solution

Radial perturbations preserve the spherical symmetry of the TOV solution



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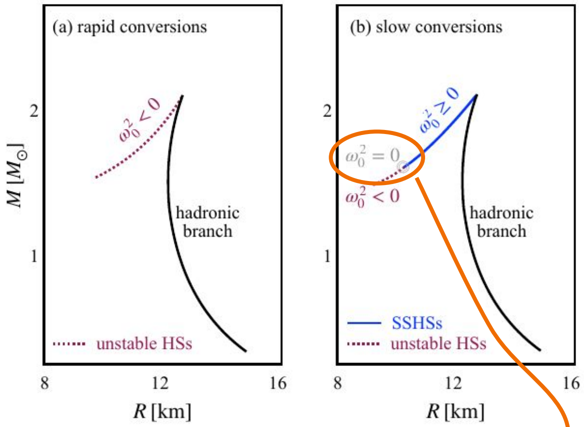
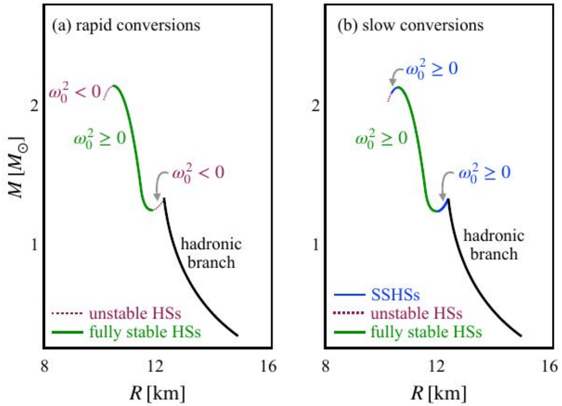
Slow Stable Hybrid Stars are viable astronomical objects

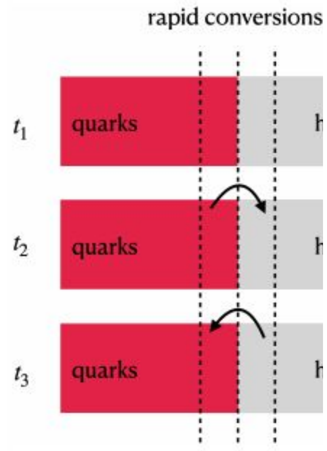




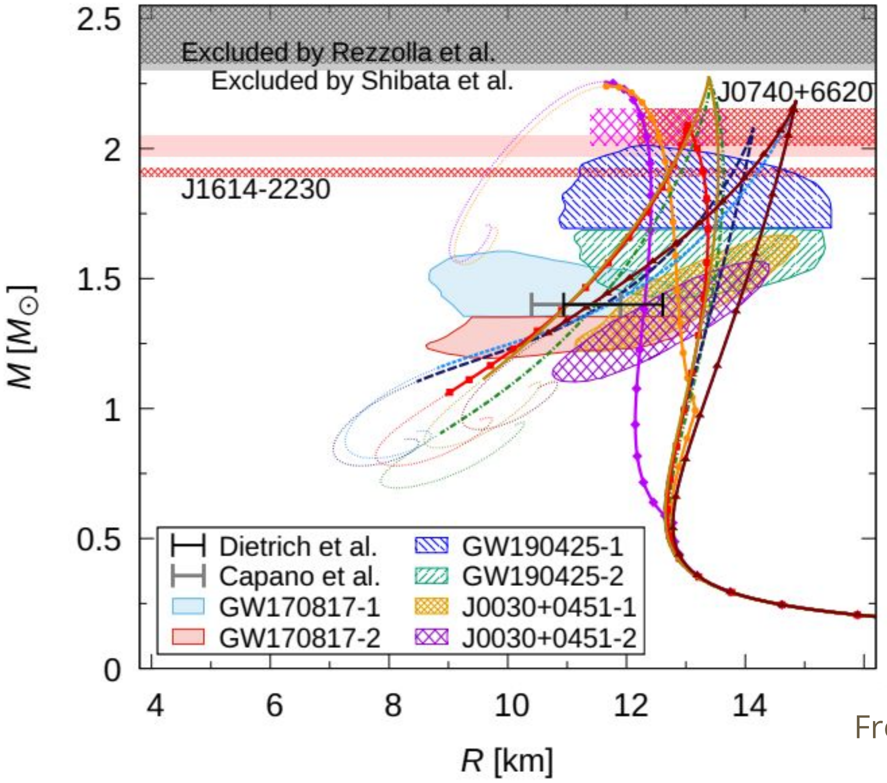
Terminal mass configuration

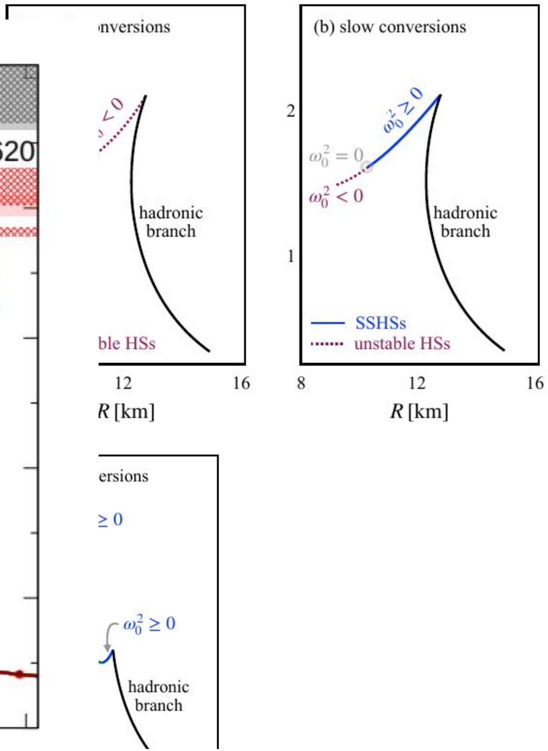
Linearized radial perturbations of a TOV solution

Radial perturb
spherical symm



$[\xi]^{+}_{-} \equiv \xi^{+} - \xi^{-} = 0, \quad [\Delta p]^{+}_{-} \equiv \Delta p^{+}$ -



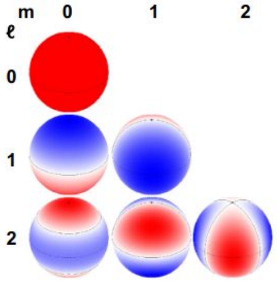


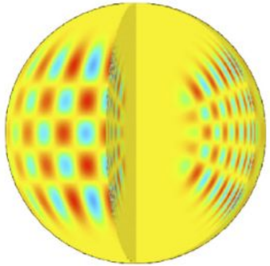
From Lugones et al. 2023

Linearized non-radial perturbations of a TOV solution

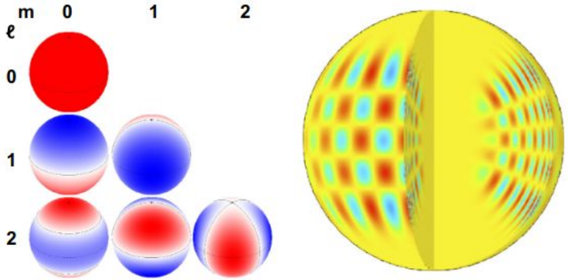
Non-radial perturbations do not preserve the spherical symmetry of the TOV solution

How can we study them?





Linearized non-radial perturbations of a TOV solution

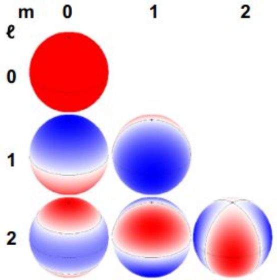


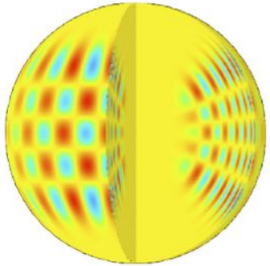
Non-radial perturbations do not preserve the spherical symmetry of the TOV solution

How can we study them?

In 1967, Thorne and Campolattaro set the foundation stone for this long standing topic

Linearized non-radial perturbations of a TOV solution





Non-radial perturbations do not preserve the spherical symmetry of the TOV solution

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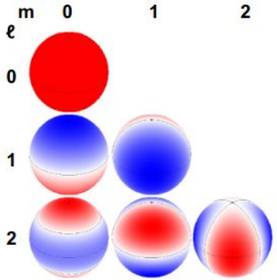
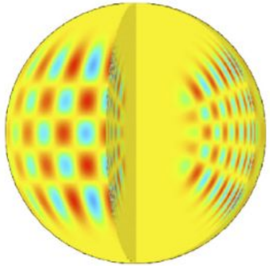
The general idea is the same that we have seen for linearized radial perturbations of a TOV solution, but harder algebra!

Linearized non-radial perturbations of a TOV solution





Non-radial perturbations do not preserve the spherical symmetry of the TOV solution

How can we study them?

Some key points:

1. Time dependence is assumed harmonic with an unknown complex frequency ω
2. Angular dependence of the perturbation can be written in term of spherical harmonics Y_{lm}
3. We focus on modes with l greater or equal that 2 ← They emit GWs
4. Only consider m = 0 as other m's can be obtained by suitable rotations
5. Fluid and axial (purely spacetime) modes

Linearized non-radial perturbations of a TOV solution

Fluid modes, interior perturbation equations

$$H_1' = -r^{-1}[\ell + 1 + 2Me^{\lambda}r^{-1} + 4\pi r^2 e^{\lambda}(p - \epsilon)]H_1 + e^{\lambda}r^{-1}[H_0 + K - 16\pi(\epsilon + p)V],$$

$$K' = r^{-1}H_0 + \frac{\ell(\ell+1)}{2}r^{-1}H_1 - \left[(\ell+1)r^{-1} - \frac{\nu'}{2}\right]K - 8\pi(\epsilon + p)e^{\lambda/2}r^{-1}W,$$

$$W' = -(\ell+1)r^{-1}W + re^{\lambda/2}\left[e^{-\nu/2}\gamma^{-1}p^{-1}X - \ell(\ell+1)r^{-2}V + \frac{1}{2}H_0 + K\right],$$

$$X' = -\ell r^{-1}X + \frac{(\epsilon + p)e^{\nu/2}}{2}\left[\left(r^{-1} + \frac{\nu'}{2}\right)H_0 + \left(r\omega^2 e^{-\nu} + \frac{\ell(\ell+1)}{2}r^{-1}\right)H_1 + \left(\frac{3}{2}\nu' - r^{-1}\right)K - \ell(\ell+1)r^{-2}\nu' V - 2r^{-1}\left(4\pi(\epsilon + p)e^{\lambda/2} + \omega^2 e^{\lambda/2-\nu} - \frac{r^2}{2}(e^{-\lambda/2}r^{-2}\nu')'\right)W\right],$$

Perturbation equations for fluid modes inside the star

V and W describe fluid perturbations

$$X = \omega^2(\epsilon + p)e^{-\nu/2}V - \frac{p'}{r}e^{(\nu-\lambda)/2}W + \tfrac{1}{2}(\epsilon + p)e^{\nu/2}H_0 \qquad a_1 H_0 = a_2 X - a_3 H_1 + a_4 K,$$

The a_i are known

Linearized non-radial perturbations of a TOV solution

Outside the star

Perturbation equations outside the star reduce to the Zerilli equation

$$\frac{d^2 Z}{dr^{*2}} = \left[V_Z(r^*) - \omega^2 \right] Z.$$

Schrödinger-like equation with potential given by

where

$$V_Z(r^*) = \frac{(1-2M/r)}{r^3(nr+3M)^2}\left[2n^2(n+1)r^3 + 6n^2Mr^2 + 18nM^2r + 18M^3\right]$$

$$r^* = r + 2M\log(r/2M - 1)$$

Linearized non-radial perturbations of a TOV solution

Joining both regions

We have to stitch both
solutions at r = R!

$$
\begin{aligned}
H_1' &= -r^{-1}[\ell + 1 + 2Me^{\lambda}r^{-1} + 4\pi r^2 e^{\lambda}(p-\epsilon)]H_1 + e^{\lambda}r^{-1}[H_0 + K - 16\pi(\epsilon + p)V],\\
K' &= r^{-1}H_0 + \frac{\ell(\ell+1)}{2}r^{-1}H_1 - \left[(\ell+1)r^{-1} - \frac{\nu'}{2}\right]K - 8\pi(\epsilon+p)e^{\lambda/2}r^{-1}W,\\
W' &= -(\ell+1)r^{-1}W + re^{\lambda/2}\left[e^{-\nu/2}\gamma^{-1}p^{-1}X - \ell(\ell+1)r^{-2}V + \frac{1}{2}H_0 + K\right],\\
X' &= -\ell r^{-1}X + \frac{(\epsilon+p)e^{\nu/2}}{2}\left[\left(r^{-1} + \frac{\nu'}{2}\right)H_0 + \left(r\omega^2 e^{-\nu} + \frac{\ell(\ell+1)}{2}r^{-1}\right)H_1 + \left(\frac{3}{2}\nu' - r^{-1}\right)K\right.\\
&\quad \left. - \ell(\ell+1)r^{-2}\nu' V - 2r^{-1}\left(4\pi(\epsilon+p)e^{\lambda/2} + \omega^2 e^{\lambda/2-\nu} - \frac{r^2}{2}(e^{-\lambda/2}r^{-2}\nu')'\right)W\right],
\end{aligned}
$$

$$
\frac{d^2 Z}{dr^{*2}} = [V_Z(r^*) - \omega^2]Z.
$$

Linearized non-radial perturbations of a TOV solution

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X' &= -\ell r^{-1}X + \frac{(\epsilon + p)e^{\nu/2}}{2}\left[\left(r^{-1} + \frac{\nu'}{2}\right)H_0 + \left(r\omega^2 e^{-\nu} + \frac{\ell(\ell+1)}{2}r^{-1}\right)H_1 + \left(\frac{3}{2}\nu' - r^{-1}\right)K\right.\\
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\end{aligned}
$$

$$\frac{d^2Z}{dr^{*2}} = \left[V_Z(r^*) - \omega^2\right]Z.$$

And impose proper boundary conditions at r = 0 and at infinity

At the origin, study series expansion

At infinite, no incoming radiation allowed

Linearized non-radial perturbations of a TOV solution

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And impose proper boundary conditions at r = 0 and at infinity

At the origin, study series expansion

At infinite, no incoming radiation allowed

This solutions are called Quasinormal modes and are characterized by a frequency, f, and a damping time, τ

Linearized non-radial perturbations of a TOV solution

The numerical (and theoretical) scheme to solve these equations was developed by Lindblom and Detweiler in two works of 1983 and 1985

Linearized non-radial perturbations of a TOV solution

The numerical (and theoretical) scheme to solve these equations was developed by Lindblom and Detweiler in two works of 1983 and 1985

A simpler way to deal with such problem is using the Relativistic Cowling approximation see, for example, Ranea-Sandoval et al. 2018

Spacetime perturbations are not considered.

Frequencies are real values

$$\frac{\mathrm{d}W(r)}{\mathrm{d}r} = \frac{\mathrm{d}\epsilon}{\mathrm{d}P}\left[\omega^2 r^2 \mathrm{e}^{\Lambda(r)-2\Phi(r)}V(r) + \frac{\mathrm{d}\Phi(r)}{\mathrm{d}r}W(r)\right] - \ell(\ell+1)\mathrm{e}^{\Lambda(r)}V(r),$$

$$\frac{\mathrm{d}V(r)}{\mathrm{d}r} = 2\frac{\mathrm{d}\Phi(r)}{\mathrm{d}r}V(r) - \frac{1}{r^2}\mathrm{e}^{\Lambda(r)}W(r).$$

Families QNMs

QNMs are usually classified using the Cowling scheme of counting nodes of the perturbing functions (see Rodríguez et al. 2024 for different approaches in the context of hot NSs created in a core-collapse supernova event)

Spacetime modes.
No classical counterpart
No fluid motion (hard to excite)

Different formalism than the one presented today!

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Spacetime modes.
No classical counterpart
No fluid motion (hard to excite)

Different formalism than the one presented today!

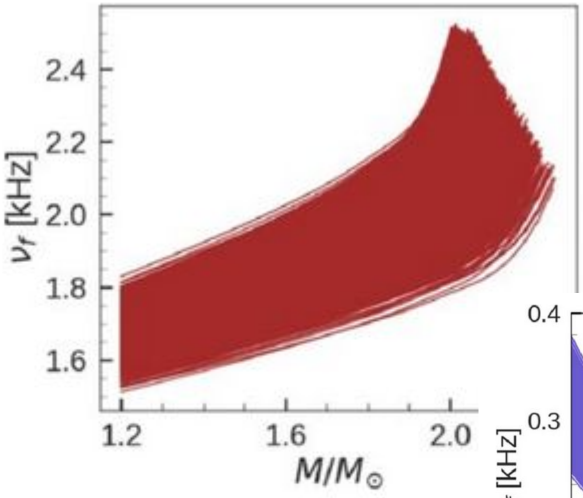
Fluid modes.
Fluid motion is excited
Expected to be related to GW emission

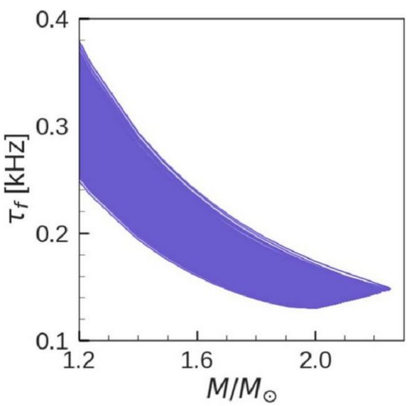
- f-mode
- acoustic p-modes
- gravity g-modes

Expected to be dominant for GW emission

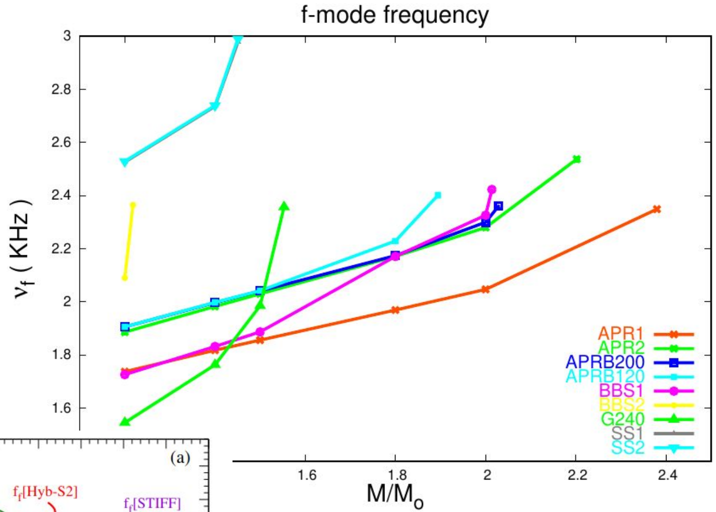
The f-mode of NSs and HSs

Eigenmodes with no nodes inside the star

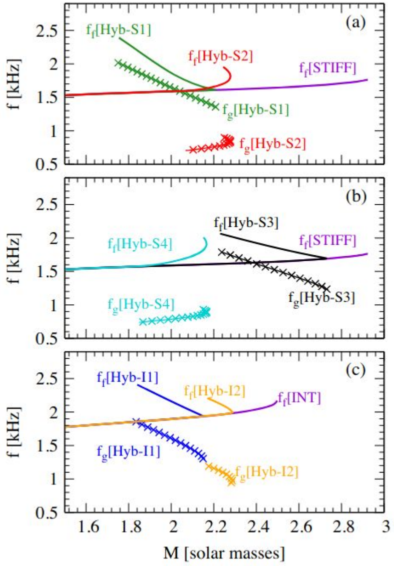




From Roy et al. 2024



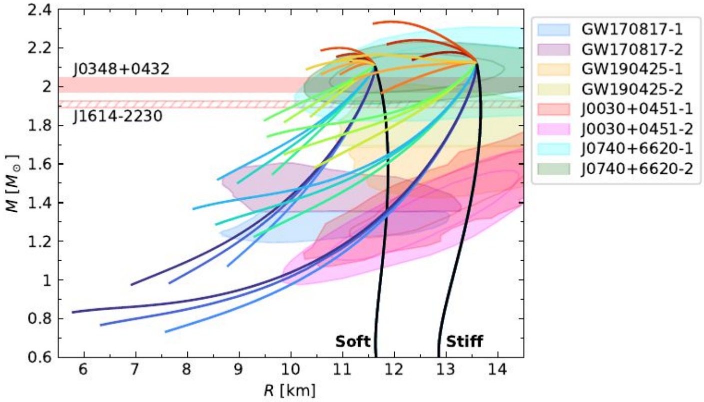
From Benhar et al. 2004



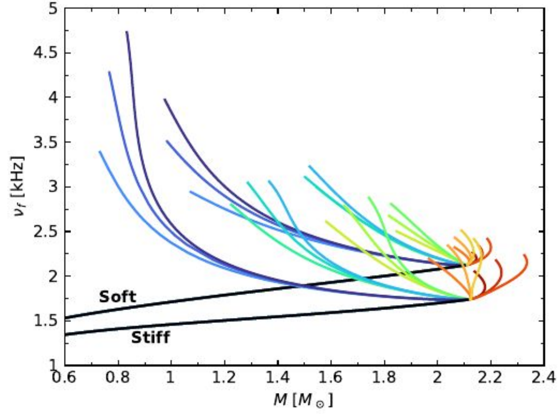
From Tonetto and Lugones 2020

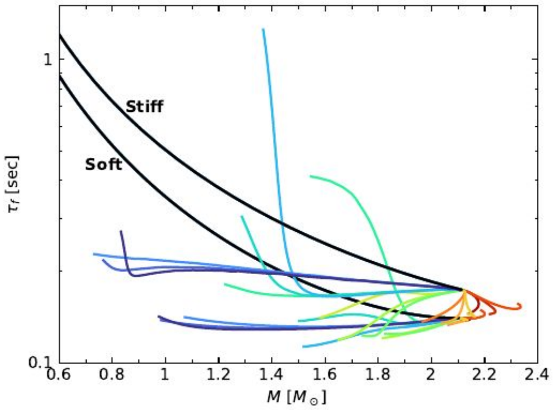
The f-mode of SSHSs

What if we consider SSHSs?



From Ranea-Sandoval et al. 2023





Behaviour is deeply affected!

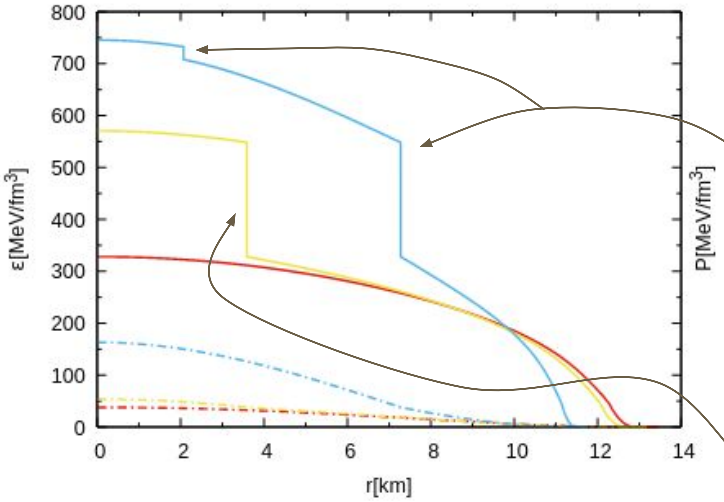
The g-mode of HSs

When sharp hadron-quark phase transitions occur
a characteristic mode shows up

The g-mode of HSs

When sharp hadron-quark phase transitions occur
a characteristic mode shows up

Detection of such mode could be key to understand some mysteries of the behaviour of dense matter



A g-mode associated with each sharp phase transition

The g-mode of HSs

When sharp hadron-quark phase transitions occur a characteristic mode shows up

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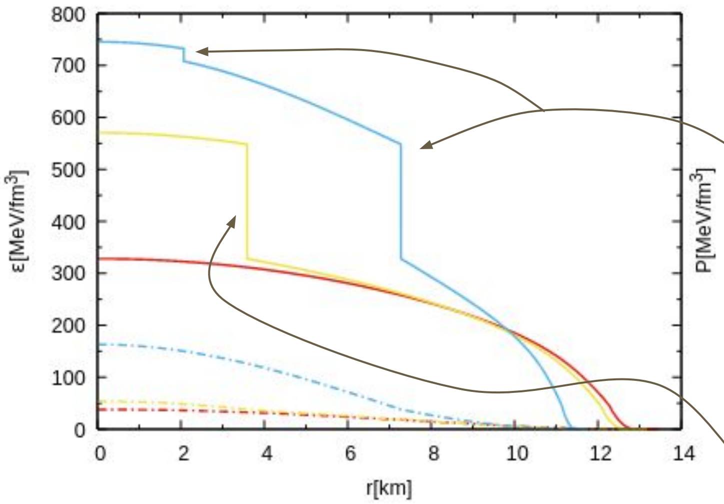


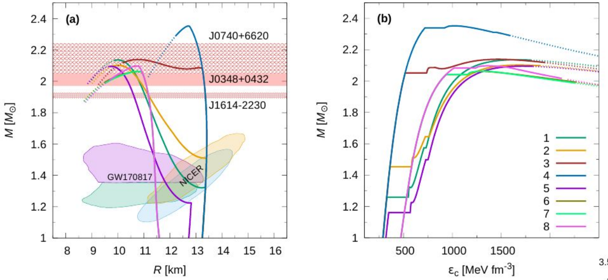
From Rodríguez et al. 2021

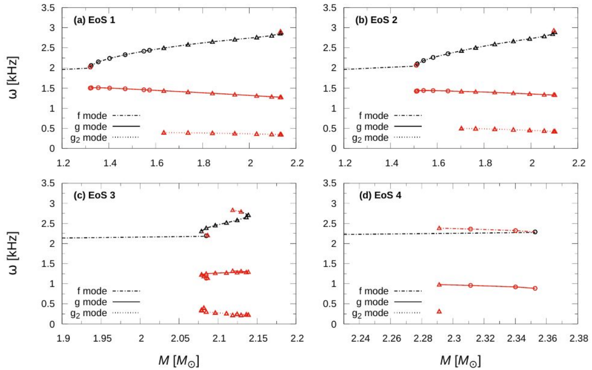
A g-mode associated with each sharp phase transition

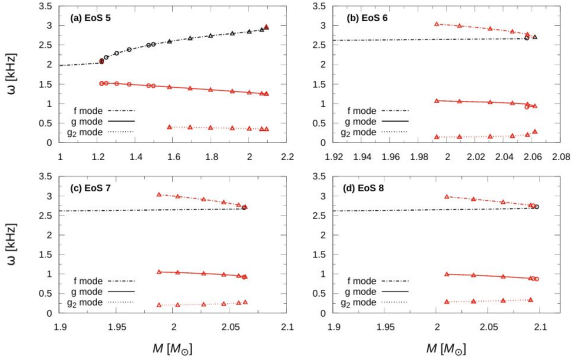
Tonetto and Lugones (2020) showed that they can be excited only if phase transition is slow!

The g-mode of HSs

When sharp hadron-quark phase transitions occur a characteristic mode shows up







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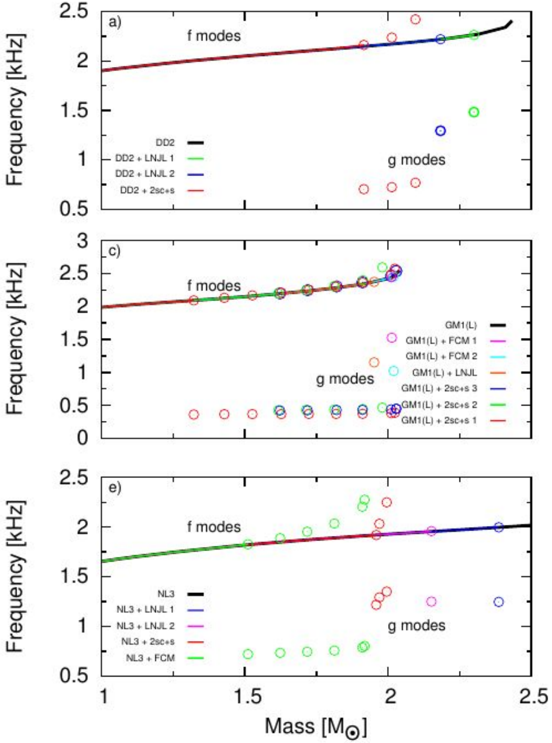

From Ranea-Sandoval et al. 2018

The g-mode of HSs

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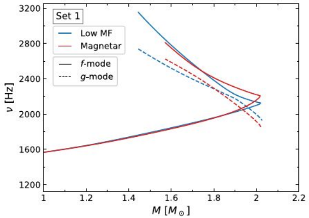
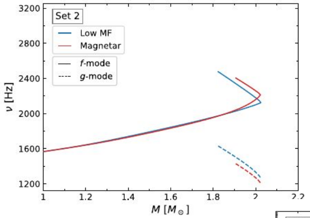
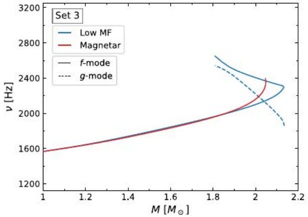
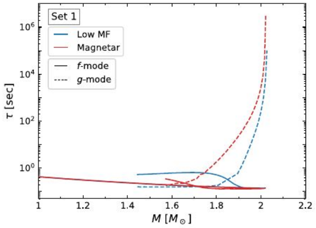
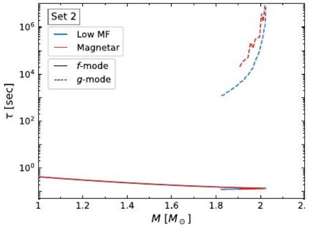
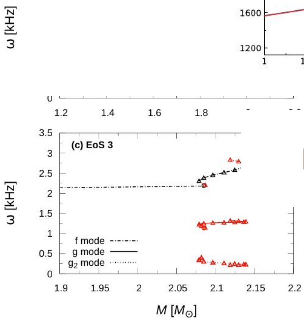


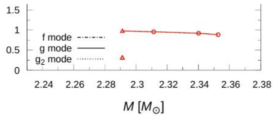




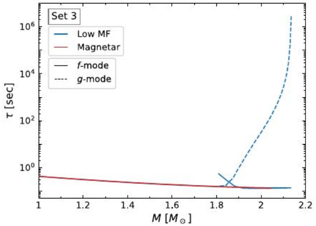


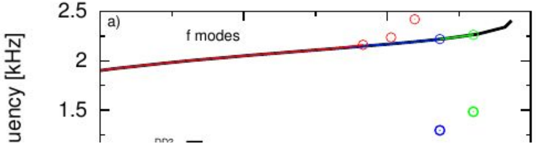


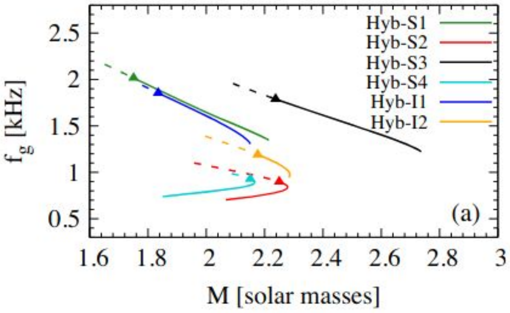


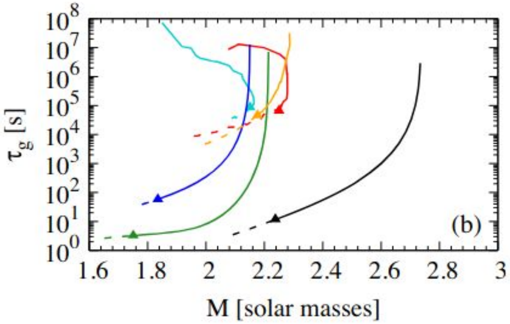


From Mariani et al. 2022









From Tonetto and Lugones 2020

Spacetimes modes

Beside fluid modes, spacetime modes are present

$$\zeta_{,rr} + \left(1 - \frac{2m(r)}{r}\right)^{-1} \left[\left(\frac{2m(r)}{r^2} - \frac{Q(r)}{e^{\lambda(r)}}\right) \zeta_{,r} \right.$$
$$\left. - \left[\frac{6}{r^2}\left(1 - \frac{m(r)}{r}\right)^{-1} + \frac{Q(r)}{re^{\lambda(r)}}\right] \zeta + \omega^2 e^{-\nu(r)} \zeta \right] = 0, \qquad Q(r) = 4\pi r e^{\lambda(r)} (\epsilon(r) - P(r))$$

$$\zeta_{,rr} + \left(1 - \frac{2M}{r}\right)^{-1}$$
$$\times \left[\frac{2M}{r^2}\zeta_{,r} - \frac{6}{r^2}\left(1 - \frac{M}{r}\right)^{-1} \zeta + \omega^2 \frac{\zeta}{1 - \frac{2M}{r}}\right] = 0,$$

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$$\zeta_{,rr} + \left(1 - \frac{2M}{r}\right)^{-1} \times \left[\frac{2M}{r^2} \zeta_{,r} - \frac{6}{r^2} \left(1 - \frac{M}{r}\right)^{-1} \zeta + \omega^2 \frac{\zeta}{1 - \frac{2M}{r}} \right] = 0,$$

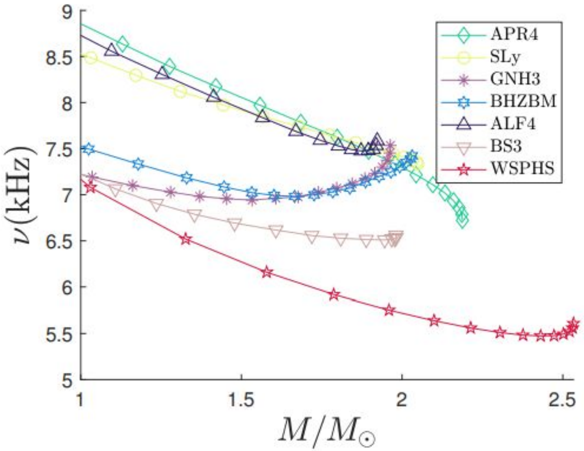
Plus:
Regularity conditions at the origin
Purely outgoing wave at infinity

Eigenvalue problem for the frequencies

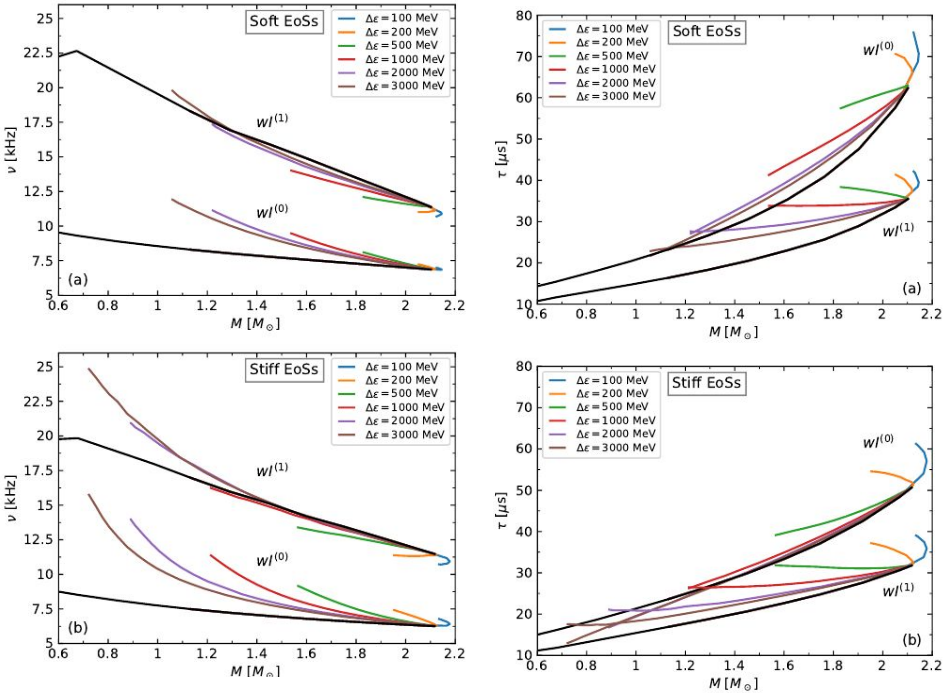
Spacetimes modes

Beside fluid modes, spacetime modes are present

w-modes
first presented by Kokkotas and Schutz, 1992



From Mena-Fernández et al. 2019



From Ranea-Sandoval et al. 2022

Universal relationships and astroseismology

Properly selected relationships including frequency or damping time and macroscopic properties of pulsating objects can produce relations that are (almost) EOS independent

Universal relationships and astroseismology

Properly selected relationships including frequency or damping time and macroscopic properties of pulsating objects can produce relations that are (almost) EOS independent

Universal relationships
Seminal paper Andersson and Kokkotas 1998

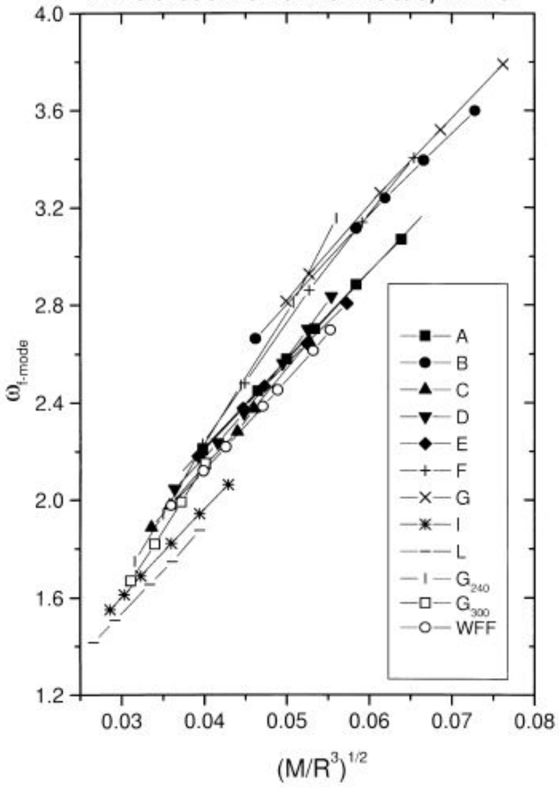
Universal relationships and astroseismology

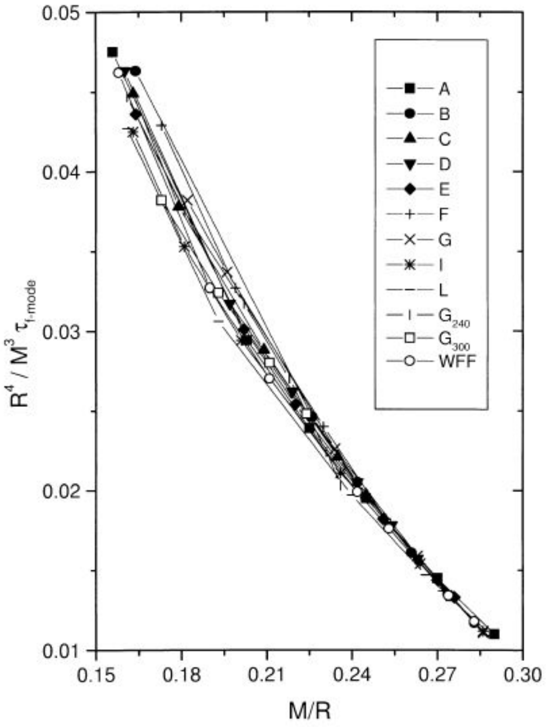
For the f-mode

$$\omega_f(\mathrm{kHz}) \approx 0.78 + 1.635\left(\frac{\bar{M}}{\bar{R}^3}\right)^{1/2}$$

$$\frac{1}{\tau_f(\mathrm{s})} \approx \frac{\bar{M}^3}{\bar{R}^4}\left[22.85 - 14.65\left(\frac{\bar{M}}{\bar{R}}\right)\right].$$

Andersson and Kokkotas, 1998



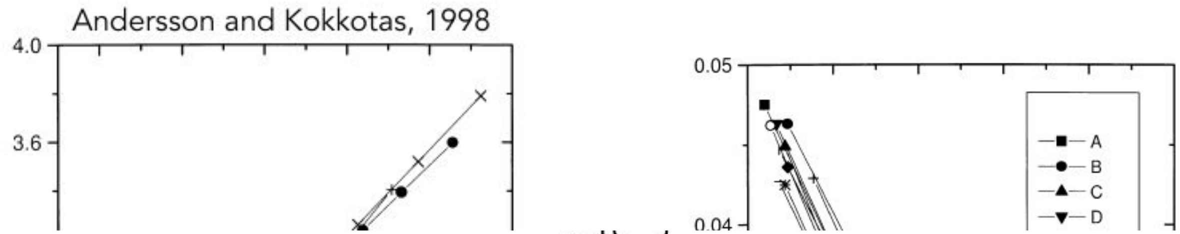


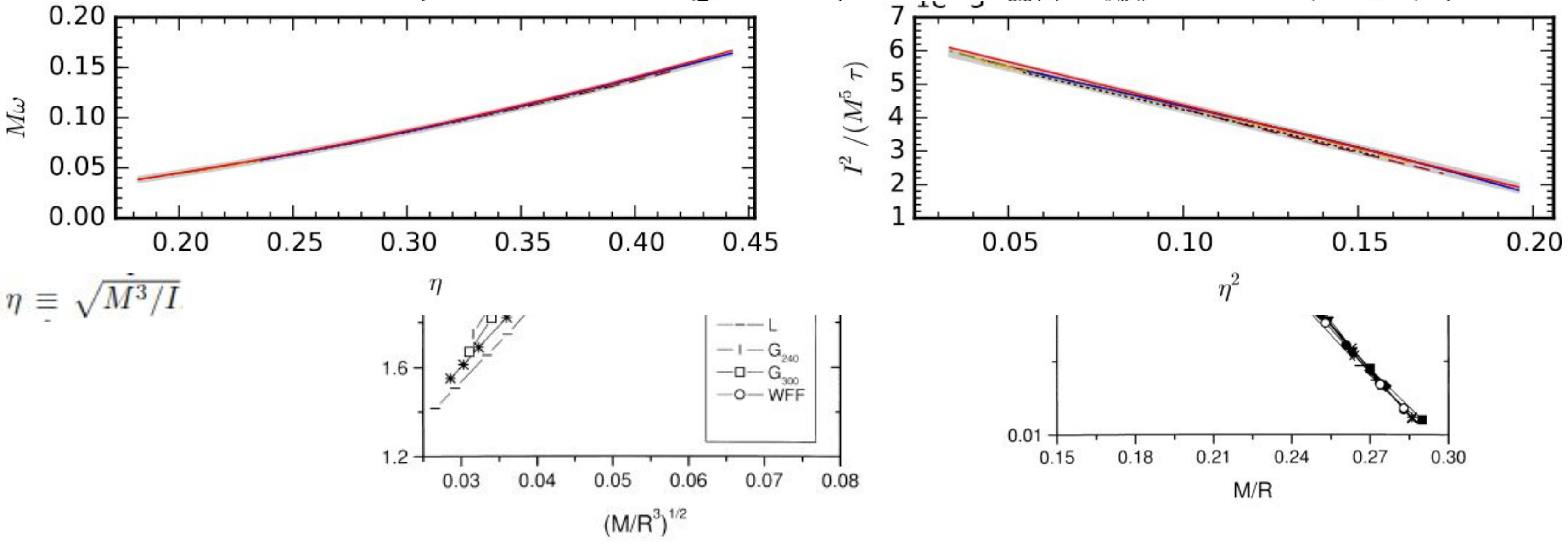
Universal relationships and astroseismology

For the f-mode

From Chirenti et al. 2015

(ideas of Lau et al. 2010)

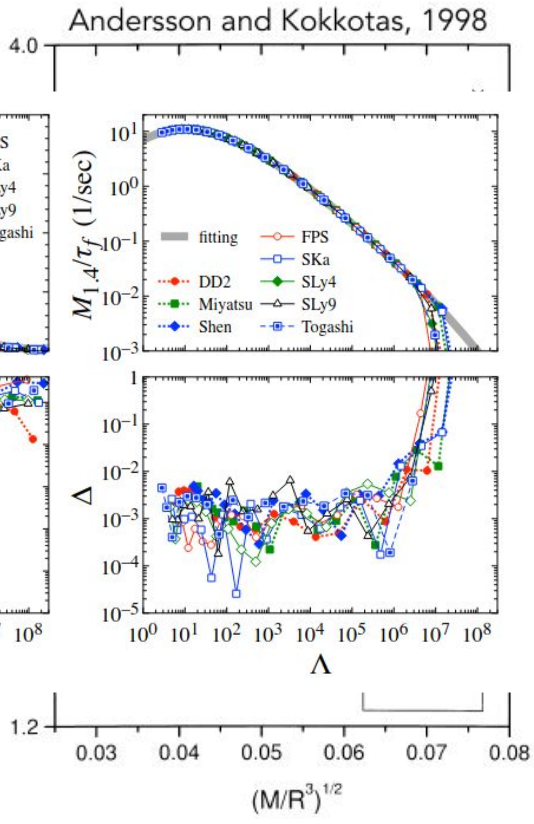


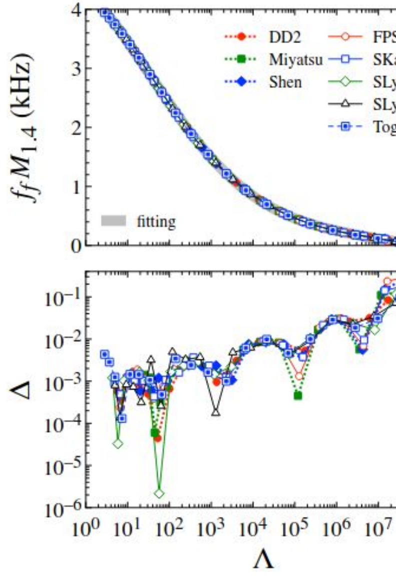


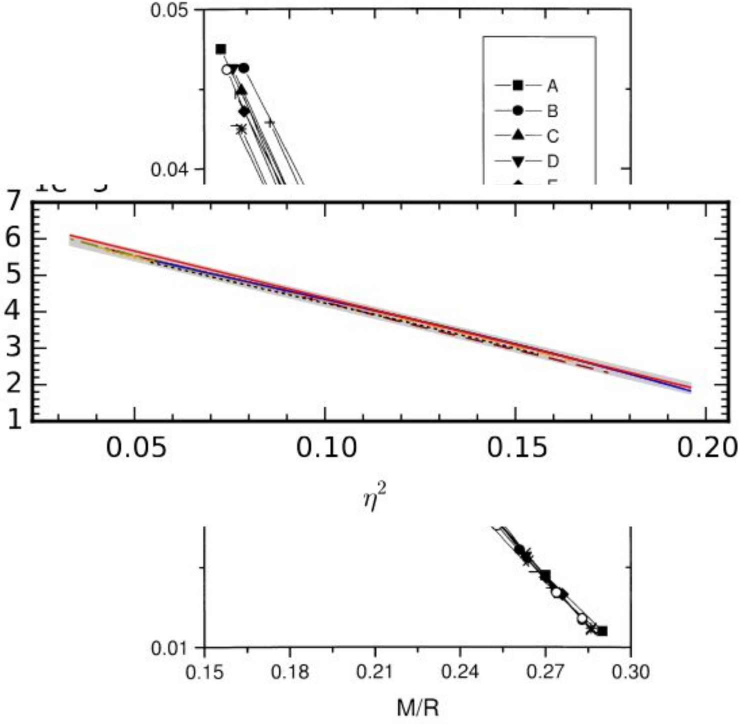
$\eta \equiv \sqrt{M^3/I}$

Universal relationships and astroseismology

For the f-mode







From Sotani and Kumar 2021

Universal relationships and astroseismology

For the f-mode

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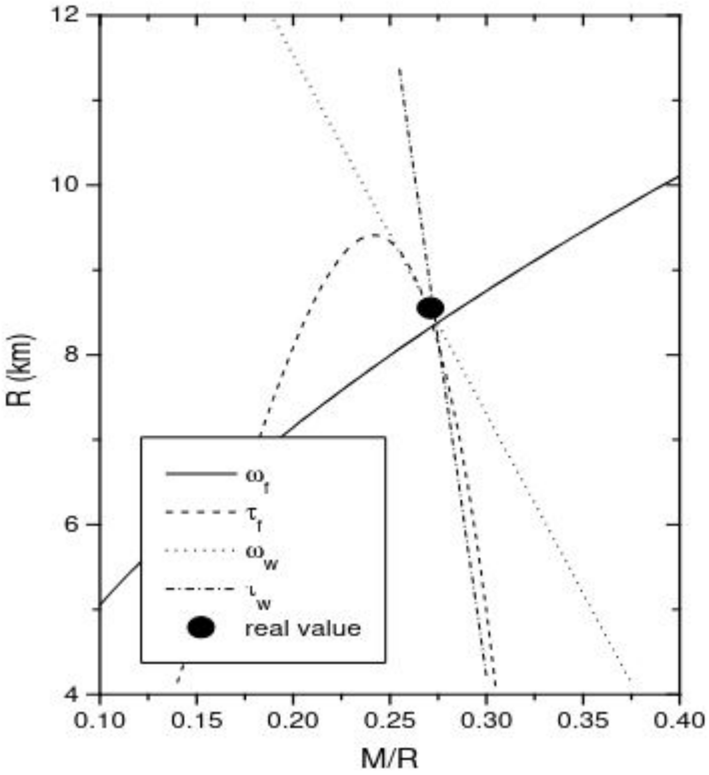
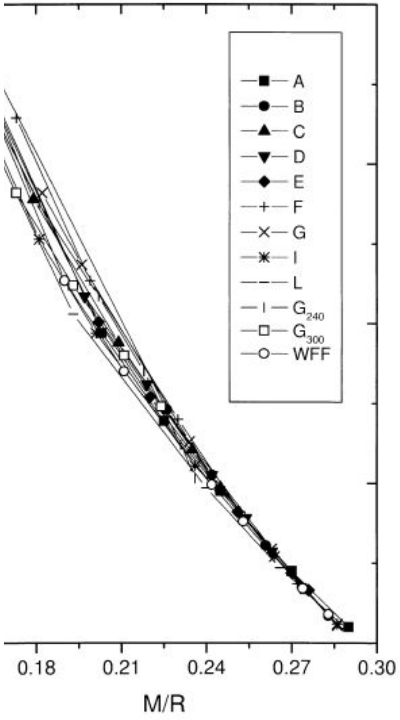
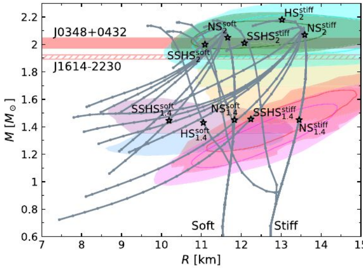
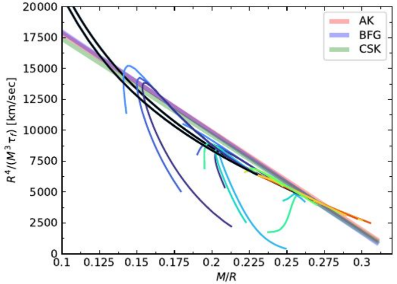


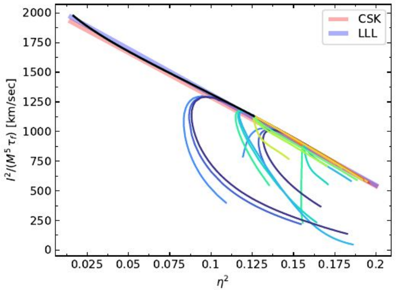


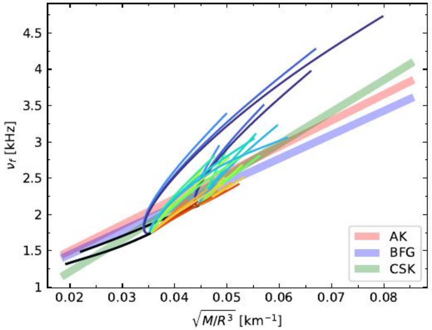
Universal relationships and astroseismology

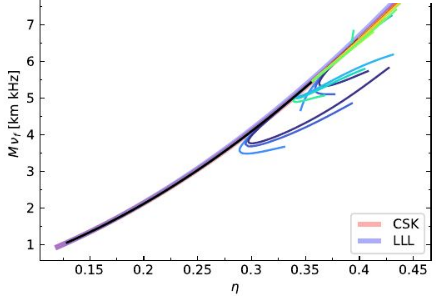
For the f-modes including SSHSs (From Ranea-Sandoval et al. 2023)

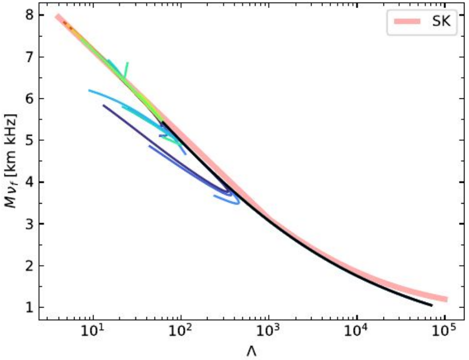


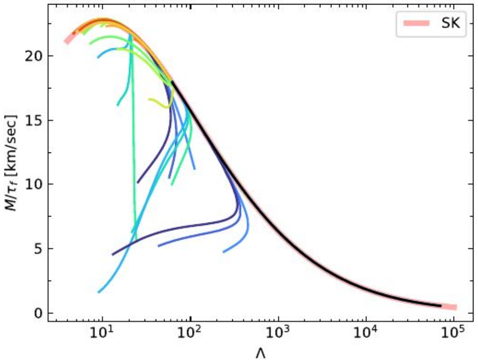










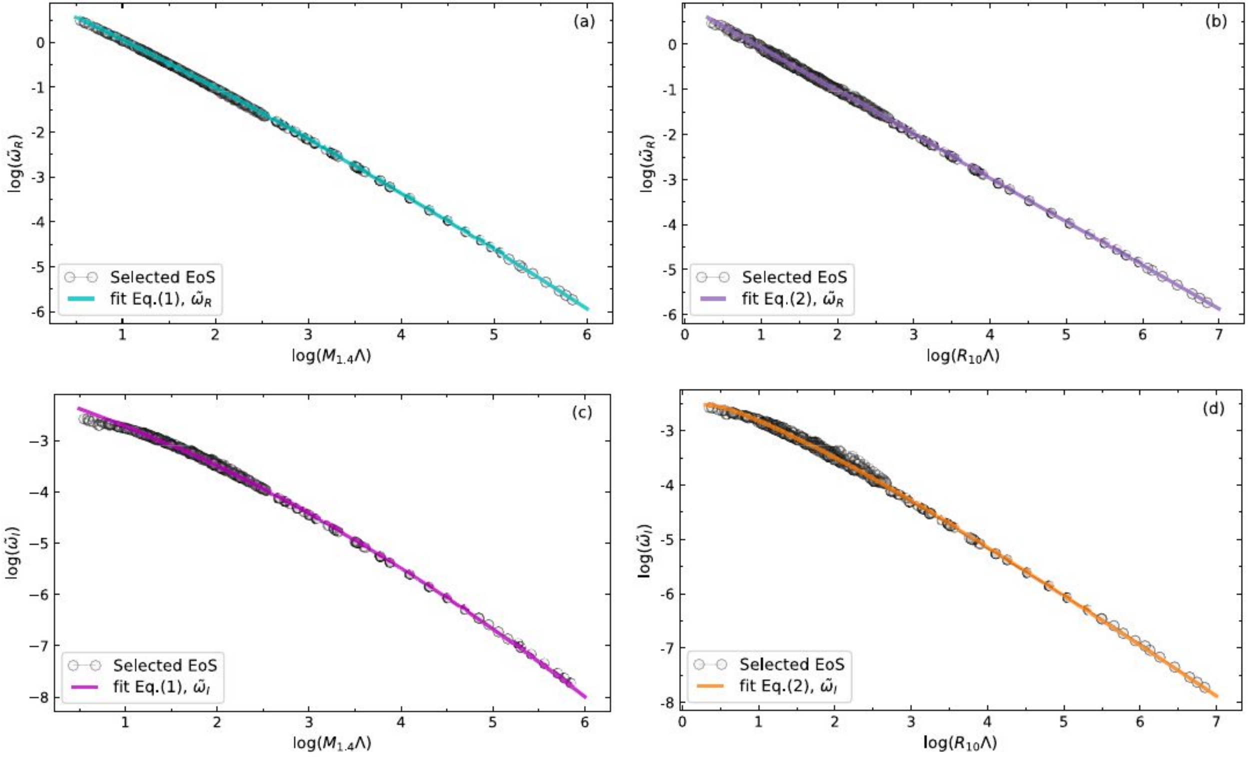


Relationships breakdown

Need to revise
Can test the
SSHS hypothesis

Universal relationships and astroseismology

For the w-modes including SSHSs (From Ranea-Sandoval et al. 2023)



For

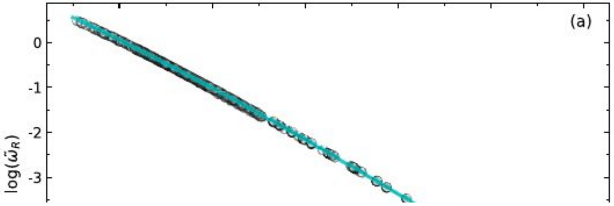
$$\tilde{\omega}_R = \nu/\Lambda$$

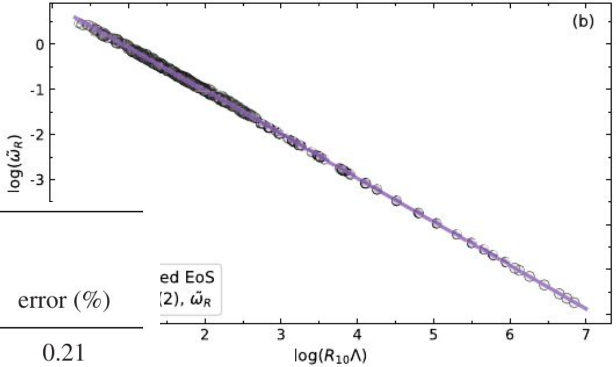
$$\tilde{\omega}_I = 1/(\Lambda\tau)$$

For the fundamental w-mode, these URs include SSHSs

Universal relationships and astroseismology

For the w-modes including SSHSs (From Ranea-Sandoval et al. 2023)





For

$$\tilde{\omega}_R = \nu/\Lambda$$

$$\tilde{\omega}_I = 1/(\Lambda\tau)$$

Universal relationships

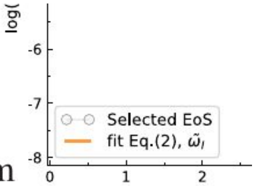
$\log(\tilde{\omega}_{R,I}) = a_{R,I} + b_{R,I}\,x + c_{R,I}\,x^2$

	value ($\pm$)	error (%)
a_R	1.052 ($\pm$ 0.0022)	0.21
b_R	-0.984 ($\pm$ 0.0017)	0.17
c_R	-0.030 ($\pm$ 0.0003)	0.99
a_I	-2.065 ($\pm$ 0.0243)	1.16
b_I	-0.594 ($\pm$ 0.0061)	1.01
c_I	-0.066 ($\pm$ 0.0011)	1.51

$$x = \log(M_{1.4}/\Lambda)$$

$$y = \log(R_{10}/\Lambda)$$

$$M_{1.4} = M/1.4\,\mathrm{M}_\odot \text{ and } R_{10} = R/10\,\mathrm{km}$$



Universal relationships

$\log(\tilde{\omega}_{R,I}) = \alpha_{R,I} + \beta_{R,I}\,y + \gamma_{R,I}\sqrt{y}$

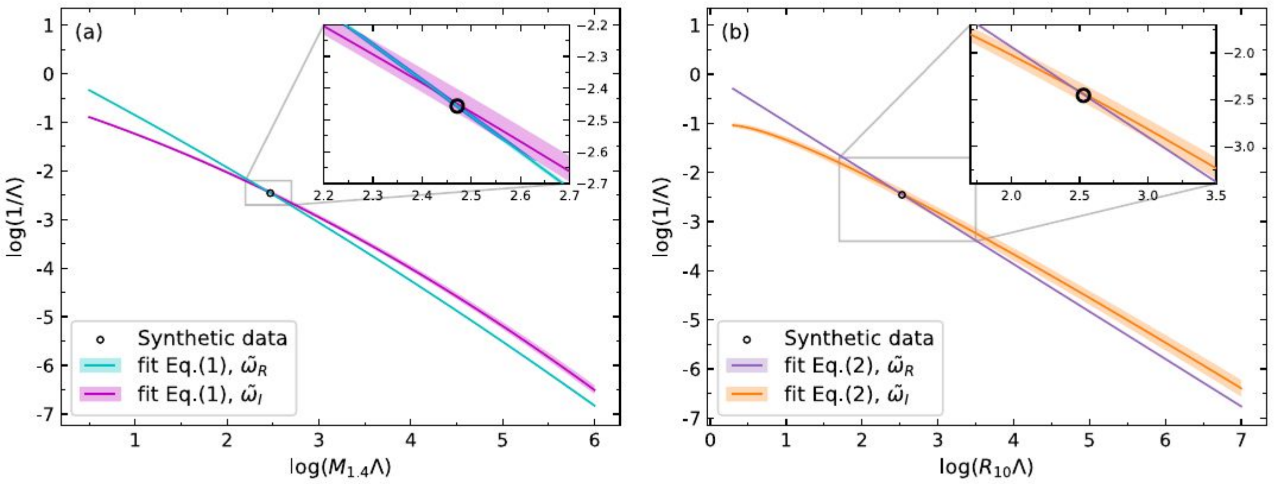
	value ($\pm$)	error (%)
α_R	0.885 ($\pm$ 0.0024)	0.27
β_R	-0.965 ($\pm$ 0.0008)	0.08
γ_R	–	–
α_I	-2.794 ($\pm$ 0.0245)	0.88
β_I	-1.151 ($\pm$ 0.0092)	0.80
γ_I	1.123 ($\pm$ 0.0306)	2.72

log($R_{10}\Lambda$)

Universal relationships and astroseismology

For the w-modes including SSHSs (From Ranea-Sandoval et al. 2023)

They have astroseismologic capabilities!



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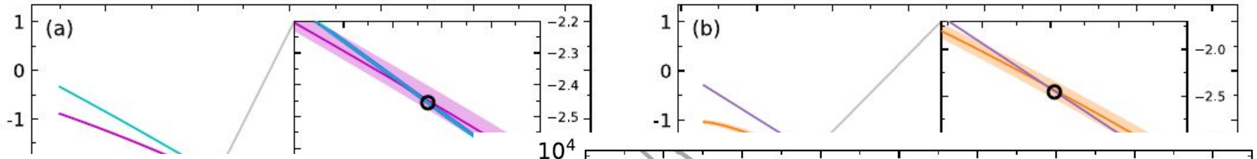

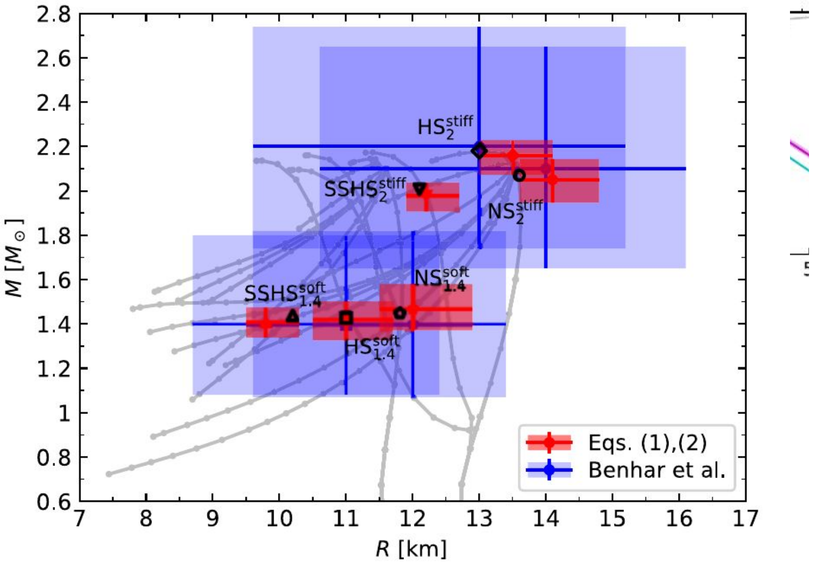

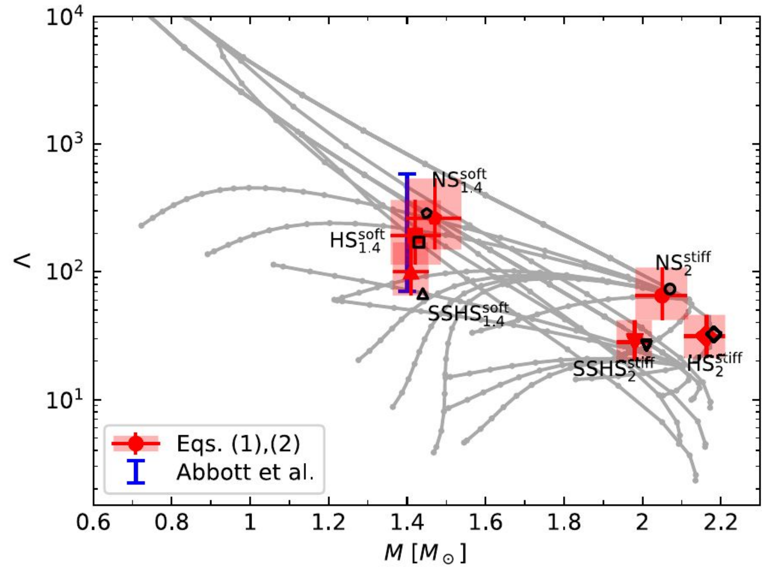

Universal relationships and astroseismology

For the w-modes including SSHSs (From Ranea-Sandoval et al. 2023)

They ha
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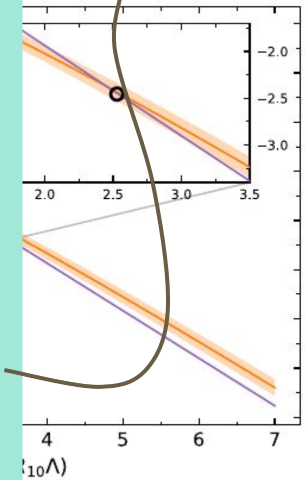
Potential exercise:

	Synthetic data					URs of equations (1) and (2)		
	$M\ [M_\odot]$	R [km]	Λ	ν [kHz]	τ [μsec]	$M\ [M_\odot]$	R [km]	Λ
$NS_{1.4}^{soft}$	1.45	11.8	285.8	7.78	30.56	$1.47^{+0.111}_{-0.099}$	$12.0^{+0.9}_{-0.5}$	263^{+273}_{-114}
$HS_{1.4}^{soft}$	1.43	11.0	170.1	8.39	30.60	$1.42^{+0.083}_{-0.095}$	$11.0^{+0.7}_{-0.5}$	192^{+172}_{-78}
$SSHS_{1.4}^{soft}$	1.44	10.2	67.4	9.13	32.87	$1.41^{+0.066}_{-0.072}$	$9.8^{+0.5}_{-0.3}$	100^{+70}_{-35}
$NS_{1.4}^{stiff}$	1.44	13.4	696.9	7.00	29.78	$1.49^{+0.125}_{-0.141}$	$13.7^{+1.3}_{-0.7}$	489^{+694}_{-238}
$SSHS_{1.4}^{stiff}$	1.45	12.2	261.7	7.77	30.13	$1.45^{+0.106}_{-0.101}$	$12.0^{+0.9}_{-0.6}$	285^{+306}_{-98}
NS_{2}^{soft}	2.05	11.7	23.37	6.94	57.62	$2.04^{+0.062}_{-0.069}$	$12.2^{+0.6}_{-0.5}$	23^{+11}_{-6}
$SSHS_{2}^{soft}$	2.00	11.1	16.3	7.28	61.07	$2.00^{+0.051}_{-0.058}$	$11.6^{+0.4}_{-0.5}$	16^{+7}_{-4}
NS_{2}^{stiff}	2.07	13.6	73.2	6.30	48.40	$2.05^{+0.093}_{-0.104}$	$14.1^{+0.7}_{-0.5}$	65^{+43}_{-23}
HS_{2}^{stiff}	2.18	13.0	32.1	6.41	57.10	$2.16^{+0.069}_{-0.088}$	$13.5^{+0.6}_{-0.5}$	31^{+15}_{-10}
$SSHS_{2}^{stiff}$	2.01	12.1	26.3	7.09	54.17	$1.98^{+0.060}_{-0.074}$	$12.2^{+0.5}_{-0.3}$	28^{+14}_{-8}

Using the URs and the synthetic data from the table. Estimate the values of the mass, radius and dimensionless tidal deformability of a few the objects considered

(never mind the error bars)



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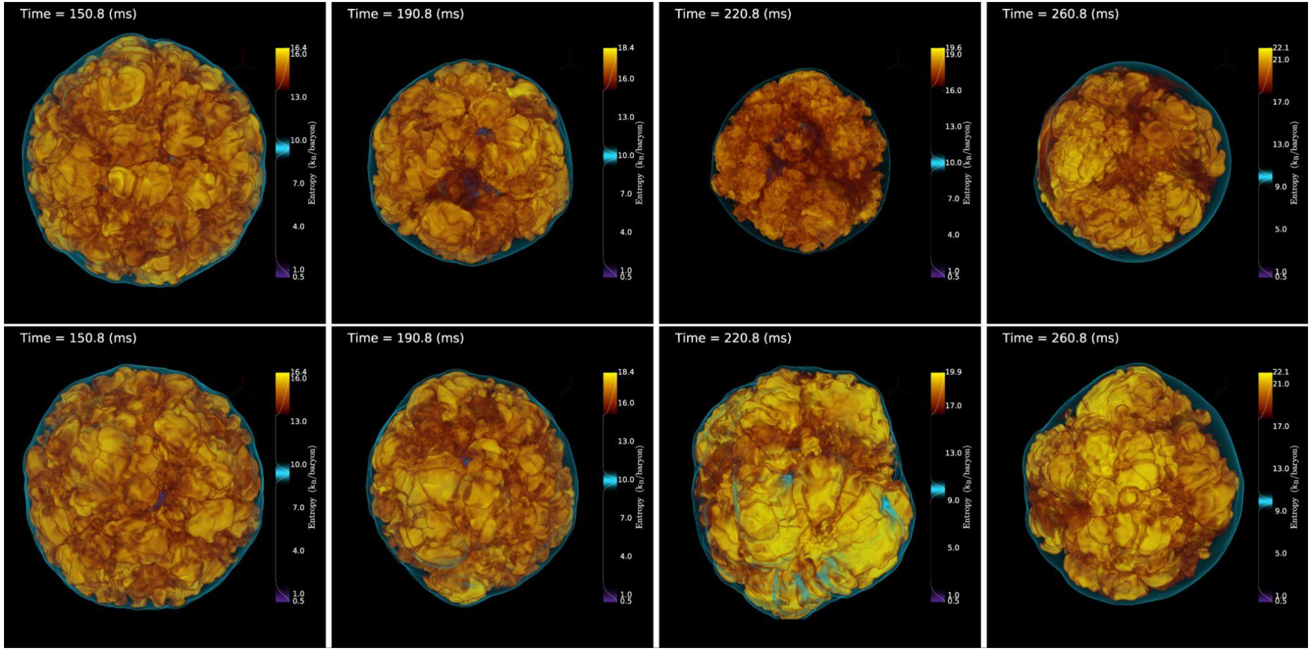
GWs and Core-Collapse Supernova

3D core-collapse supernova simulations (extremely CPU expensive)



from O’Connor and Couch 2018

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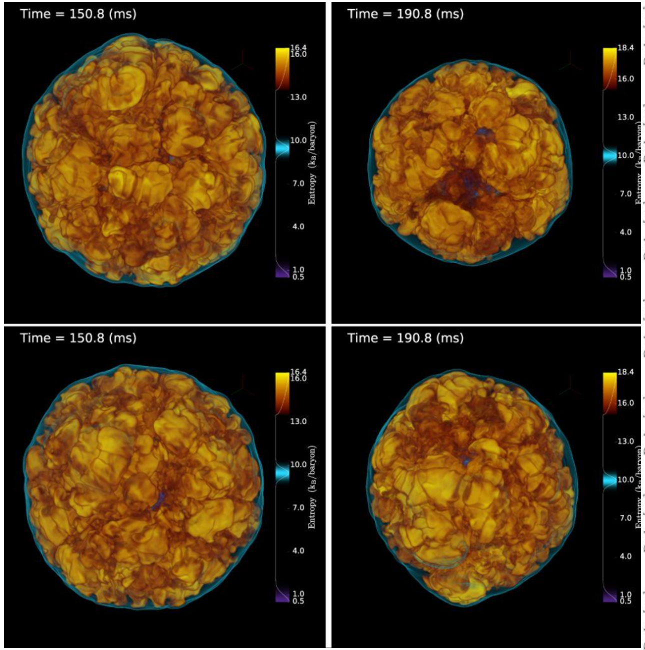

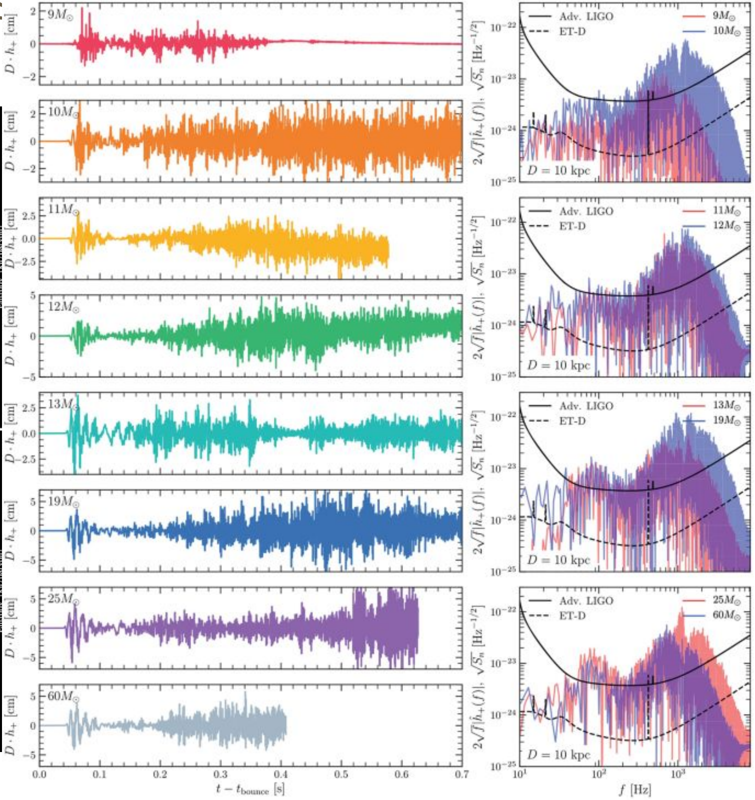

from Abdikamalov et al. 2021

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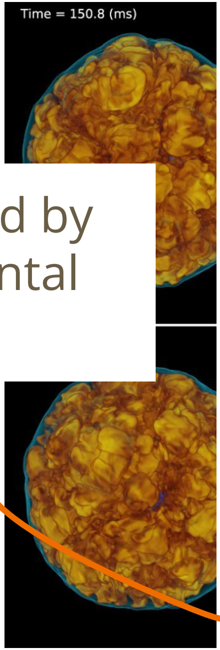
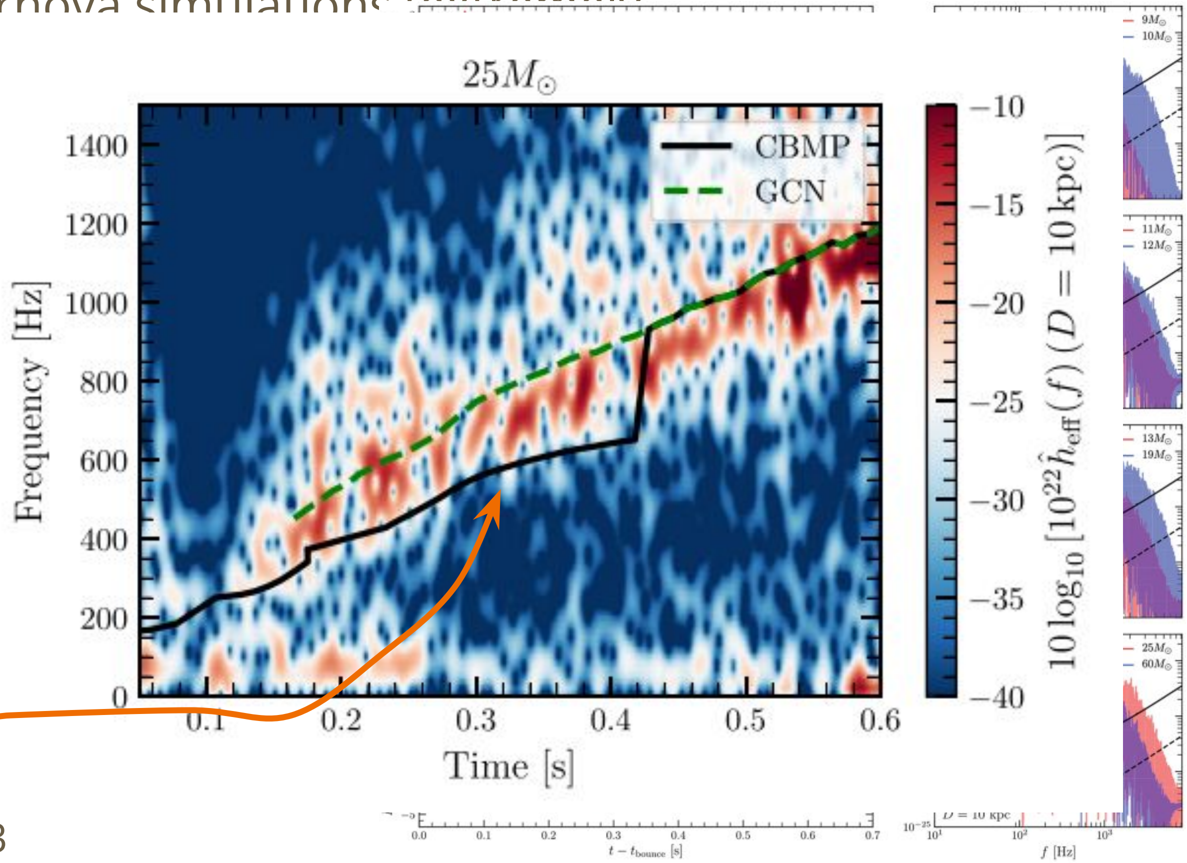


Dominated by fundamental mode



from Rodríguez et al. 2023

GWs and Core-Collapse Supernova

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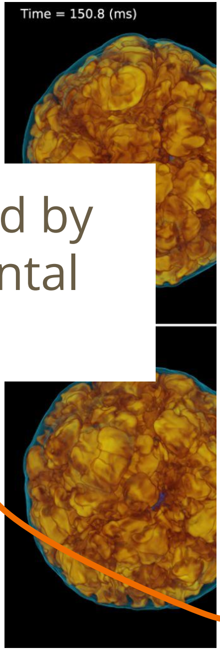

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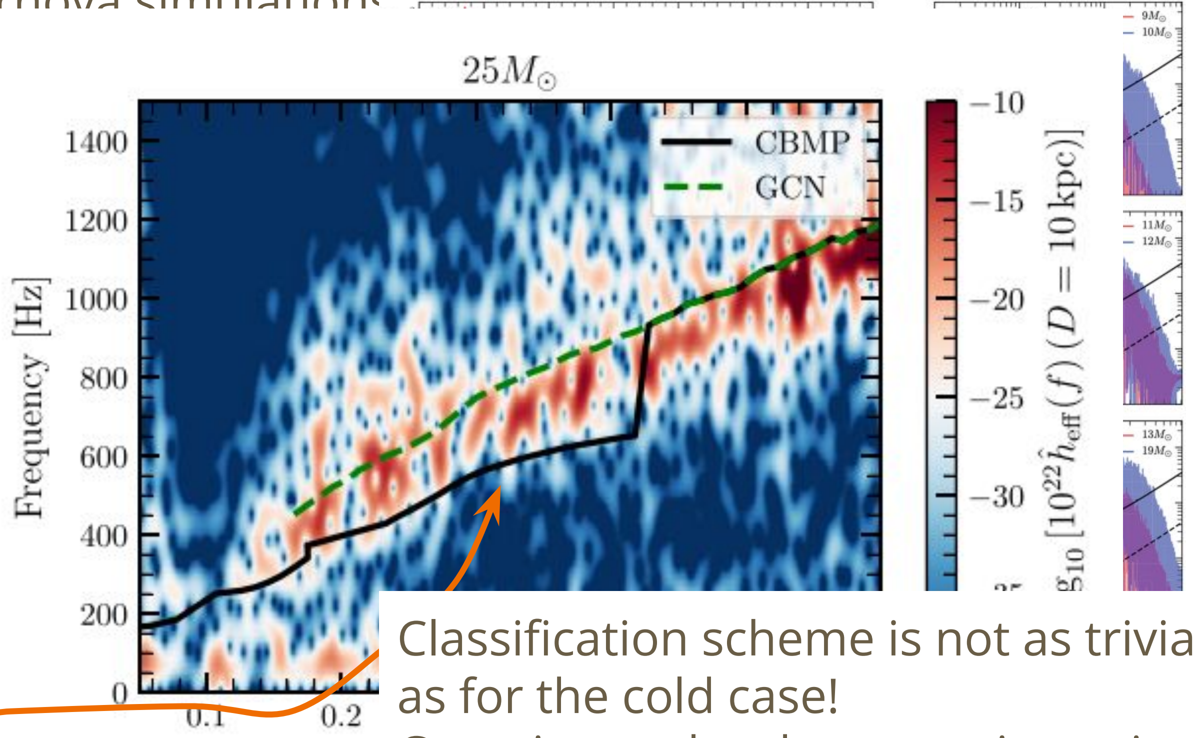

Classification scheme is not as trivial as for the cold case!
Counting nodes does not give unique results...

from Rodríguez et al. 2023

GWs and Core-Collapse Supernova

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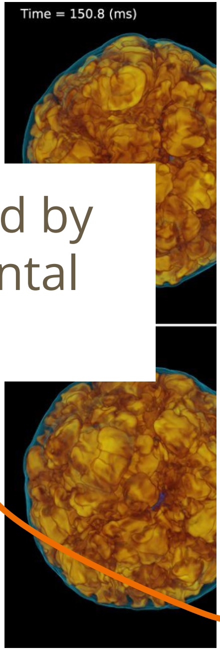

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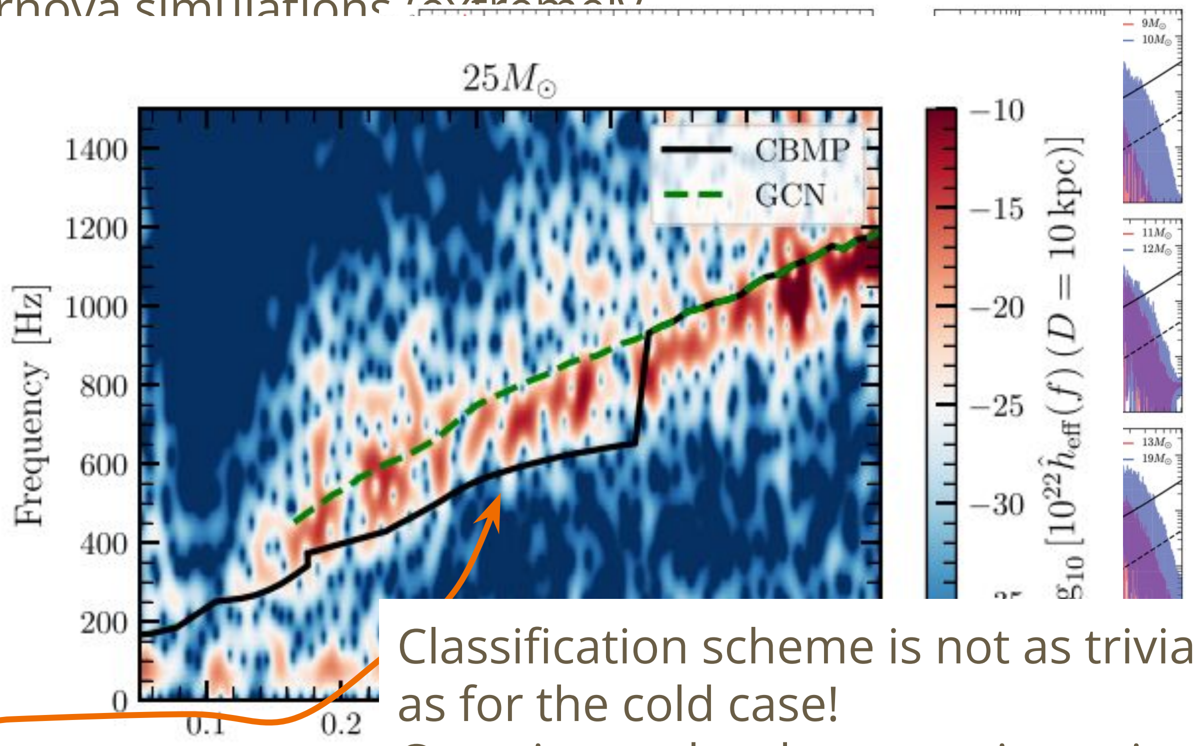

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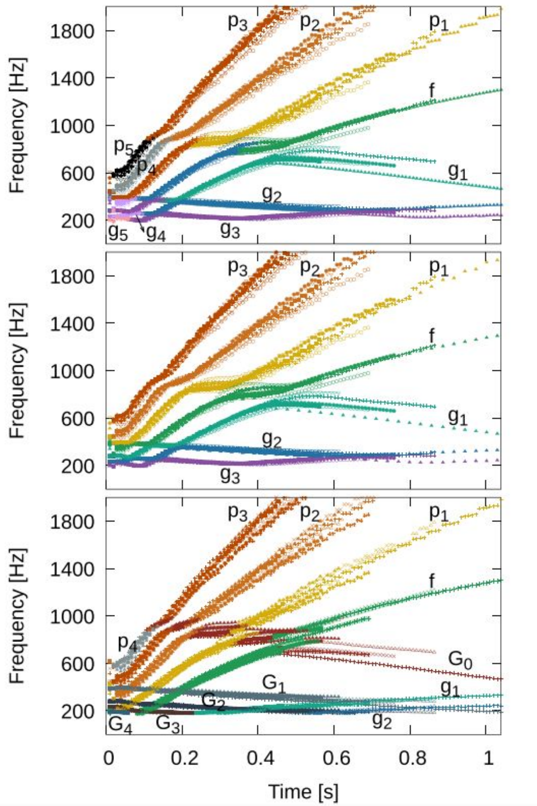

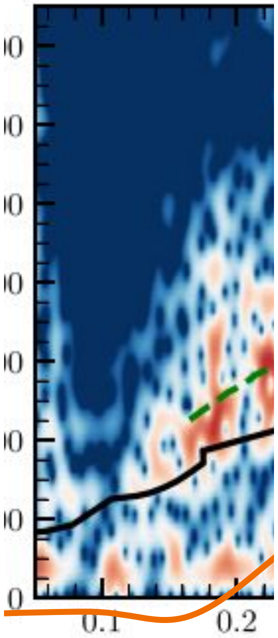

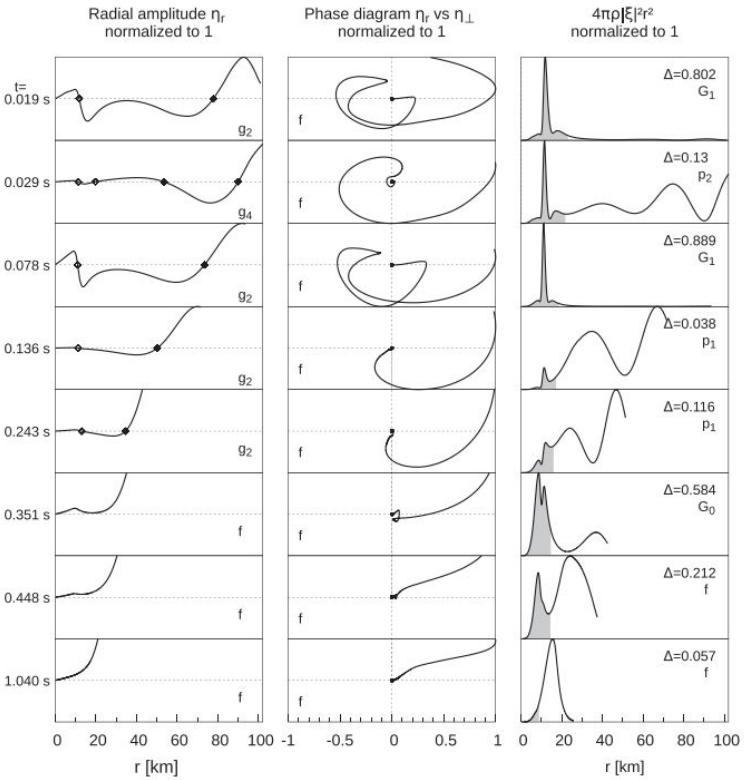

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