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How complex is a random picture?☆

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ABSTRACT

We study the amount of information that is contained in "random pictures", by which we mean the sample sets of a Boolean model. To quantify the notion "amount of information", two closely connected questions are investigated: on the one hand, we study the probability that a large number of balls is needed for a full reconstruction of a Boolean model sample set. On the other hand, we study the quantization error of the Boolean model w.r.t. the Hausdorff distance as a distortion measure.

1. Introduction and results

1.1. Introduction

We are interested in quantifying the amount of information contained in certain random pictures. Let us first fix some notation. We work in dimension $d \geq 1$. Let $(\xi_i)_{i\geq 1}$ be i.i.d. random variables uniformly distributed in $[0, 1]^d$, $(R_i)_{i\geq 1}$ be i.i.d. positive random variables, and let N be a Poisson random variable with parameter λ. Assume that (ξ_i), (R_i), and N are independent. Notice that the set $(\xi_i)_{i\leq N}$ is just a Poisson point field controlled by the measure $\lambda\,\mathrm{vol}_d$, where vol_d stands for the d-dimensional Lebesgue measure.

Define a "random picture" by

$$S := \bigcup_{i=1}^{N} B(\xi_i, R_i) \cap [0, 1]^d.$$

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Here, $B(x, r)$ is the closed ball with centre x and radius r, where for the time being any norm $\|.\|$ on $\mathbb{R}^d$ is fine. The set S is a union of balls in $[0, 1]^d$; imagine the balls are painted 'black', thus we have a black picture over white background.

We are interested in the (lossy) encoding of the picture S by a finite number of bits. This problem, the well-known *quantization problem*, will be described below. It turns out that for the analysis of the quantization problem, the following random variable is crucial. Moreover, we believe that it may be of independent interest. Define the *effective number of balls visible in the picture* as

$$K := \min\{r \geq 1 | \exists i_1, \ldots, i_r \in \{1, \ldots, N\} : S = \bigcup_{s=1}^{r} B(\xi_{i_s}, R_{i_s}) \cap [0, 1]^d\}.$$

In other words, with K balls one can reproduce the black picture S exactly as with the original N balls.

We are interested in the upper tail of K, i.e. $\mathbb{P}[K \geq n]$ when $n \to \infty$. This means we study the probability that one needs many balls in order to reconstruct the picture S. In particular, we would like to understand when one can "save balls" w.r.t. the original Poisson number of balls N. To make this more precise, note that clearly $K \leq N$, and so

$$\mathbb{P}[K \geq n] \leq \mathbb{P}[N \geq n] = \exp(-n \log n \cdot (1 + o(1))), \qquad n \to \infty.$$

We would like to show that the upper tail of K is thinner, i.e. for some $\beta > 1$

$$\mathbb{P}[K \geq n] = \exp\left(-\beta \cdot n \log n \cdot (1 + o(1))\right), \qquad n \to \infty.$$

It turns out that finding such a β is non-trivial and interesting. The value of β depends on the dimension d, on the type of norm used, and on the distribution of the radii $\mathcal{L}(R_1)$.

Boolean models are fundamental objects in stochastic geometry and have a large range of applications, [4,17]. However, to the knowledge of the authors, until recently mostly the *average of observables* of Boolean models is studied. Often this plays a role when estimating parameters of the model in applications. On the contrary, the present paper deals with *rare events*, i.e. with large deviation probabilities.

As mentioned above, the upper tail of the random variable K is an essential ingredient for solving the so-called quantization problem, which we recall now. Let an arbitrary norm $\|.\|$ be fixed on $\mathbb{R}^d$. Let d_H denote the corresponding Hausdorff distance between compact subsets of $\mathbb{R}^d$: $d_H(A, B) := \max(\sup_{a\in A} \inf_{b\in B} \|a - b\|, \sup_{b\in B} \inf_{a\in A} \|a - b\|)$. We define the respective quantization error for pictures by

$$D^{(q)}(r) := \inf_{\#\mathcal{C} \leq e^r} \mathbb{E} \min_{A\in\mathcal{C}} d_H(S, A), \qquad r > 0.$$

Here, the sets $\mathcal{C}$ are called codebooks and the upper index (q) stands for "quantization". The idea is that the "analog" signal S should be encoded by an element $A \in \mathcal{C}$. This incurs an error, $d_H(A, S)$, measured in Hausdorff distance. Loosely speaking, $D^{(q)}$ is then the minimal average error over all codebooks $\mathcal{C}$ of a size not exceeding e^r. We are interested in letting $r \to \infty$, that is, the size of the codebooks grows; and we would like to understand the rate of decay of the corresponding quantization error.

Basic references for quantization problems are [6,11,13]. The analysis of the quantization problem started in the 40ies of the 20th century, and mainly finite-dimensional quantization was the subject of interest until about 2000. Since then, research has shifted to infinite dimensional quantization, e.g. for Brownian motion, fractional Brownian motion, Lévy processes, etc. attaining values in some function spaces, therefore called functional quantization, see e.g. [1,5,7–10,12,14,15] and references therein for a selection, and for their applications to numerical probability, see e.g. [16]. In the present paper, the signal attains values in a more abstract space, namely, in the class of all compact subsets of $[0, 1]^d$.

Results similar to those in this paper are obtained in [2,18]. In [2], certain types of jump processes are studied that resemble (and contain as a special case) compound Poisson processes with values in an abstract space. In the PhD thesis [18], the question we are interested in this paper appears for the first time. The detailed relation to the present results will be described in Section 1.3.

The rest of this paper is structured as follows. In Section 1.2, we summarize our results for the large deviation probabilities of K. In Section 1.3, the results for the quantization error are listed. The proofs are given in Section 2 (lower bounds for the large deviation results), Section 3 (upper bounds for the large deviation results), Section 4 (dimension $d = 1$), and Section 5 (quantization results), respectively.

1.2. Results for the large deviations of K

Dimension $d = 1$. Let us start with the case $d = 1$, which is particularly easy.

In $\mathbb{R}^1$ there is essentially one norm, thus we will work with absolute values. The balls here are just intervals, $B(x, r) = [x - r, x + r]$. The proof of the following result is given in Section 4.

Theorem 1. *Let $d = 1$. Assume that the distribution of R_1 has a Lebesgue density p with $p(z) \approx z^{\alpha-1}$ for $z \to 0$ and some $\alpha > 0$. Then*

$$\mathbb{P}[K \geq n] = \exp(-(1+\alpha)n \log n \cdot (1 + o(1))), \qquad \text{as } n \to \infty. \tag{1}$$

Here and below, the notion $p(z) \approx q(z)$ $(z \to 0)$ stands for the fact that $p(z)/q(z)$ is bounded away from zero and infinity for z small enough. Likewise, we use $p(z) \sim q(z)$ if $\lim p(z)/q(z) = 1$.

For the case of a constant radius, the large deviations turn out to be trivial, which is quite natural. Namely, if $R_1 \equiv c < 1$ is constant, Remark 24 below shows that $\mathbb{P}[K \geq n] = 0$, for $n > 2/c$.

From now on, we assume $d \geq 2$.

Constant radius. Let us now deal with the seemingly simple case of constant radii. It turns out that the rates (and the proofs) are non-trivial and may possibly depend on the geometry of the balls.

Theorem 2. *Assume $R_1 \equiv c < 1$ is constant. Then*

1. *for ℓ_1-balls, we have*
$$\mathbb{P}[K \geq n] = \exp\left(-(1 + \frac{1}{d-1})n \log n \cdot (1 + o(1))\right),$$
2. *for ℓ_2-balls, we have*
$$\exp\left(-(1 + \frac{2}{d-1})n \log n \cdot (1 + o(1))\right)$$
$$\leq \mathbb{P}[K \geq n] \leq \exp\left(-(1 + \frac{1}{d-1})n \log n \cdot (1 + o(1))\right),$$
3. *for ℓ_∞-balls, we have*
$$\exp\left(-(1 + \frac{1}{d-1})n \log n \cdot (1 + o(1))\right) \leq \mathbb{P}[K \geq n].$$

These results are proved in Section 2.1 (lower bounds) and Section 3.2 (upper bounds).

Radius distribution with density. Now we deal with a radius distribution that has a Lebesgue density. The result here is a summary of the results in Sections 2.2, 2.3 (lower bounds) and 3.1 (upper bounds), where slightly more general results are stated and proved.

Theorem 3. *Assume that the radius distribution has a Lebesgue density p with $p(z) \approx z^{\alpha-1}$, for $z \to 0$. Set $\bar{\alpha} := \alpha \wedge 1$. Then for any norm on $\mathbb{R}^d$*

$$\exp\left(-\left(1 + \frac{\alpha}{d}\right) n \log n \cdot (1 + o(1))\right)$$
$$\leq \mathbb{P}[K \geq n] \leq \exp\left(-\left(1 + \frac{\bar{\alpha}}{d}\right) n \log n \cdot (1 + o(1))\right).$$

Furthermore, assume that p is bounded. Then, additionally, for ℓ_1-balls,

$$\exp\left(-(1+\frac{1}{d-1})n\log n\cdot(1+o(1))\right)\le\mathbb{P}[K\ge n],$$

and for ℓ_2-balls,

$$\exp\left(-(1+\frac{2}{d-1})n\log n\cdot(1+o(1))\right)\le\mathbb{P}[K\ge n].$$

1.3. Results for the quantization error

Dimension $d = 1$. Also here we start with dimension $d = 1$. First, we treat the case of a constant radius.

Theorem 4. *Let $d = 1$. Assume that the radius is constant $R_1 \equiv c < 1/2$. Then there are constants $c_1, c_2 > 0$ such that for large enough r we have*

$$c_1 e^{-r/2m} \le D^{(q)}(r) \le c_2 e^{-r/2m}, \tag{2}$$

where $m := \max\{k \in \mathbb{N} : 2kc < 1\}$. For $c \ge 1/2$, the statement holds with $m = 1/2$.

Now we look at $d = 1$ and non-constant radius. Note that Lemma 26 below immediately turns an upper bound for the large deviations of K into an upper bound for the quantization rate. In particular, from the upper bound in Theorem 1 (in particular, one only requires the upper bound for the density p in the assumption) we obtain the following.

Corollary 5. *Let $d = 1$. Assume that the distribution of R_1 has a Lebesgue density p with $p(z) \le cz^{\alpha-1}$ for $z \to 0$ and some $\alpha > 0$. Then*

$$D^{(q)}(r) \le \exp\left(-\sqrt{(1+\alpha)r\log r}\cdot(1+o(1))\right) \qquad \text{as } r\to\infty.$$

Lower bounds could be obtained in a similar manner as for larger dimensions below (cf. [19]), we do not pursue this here to keep the exposition comprehensive. We further mention that Theorem 4.2.1 in [18] treats the case $\alpha = 1$, which is extended to general α here.

For the rest of this section, we deal with $d \ge 2$.

Constant radius. Let us first consider the case of a constant radius.

Theorem 6. *For ℓ_1-balls and constant radius we have*

$$D^{(q)}(r) = \exp\left(-\sqrt{\frac{2}{d-1}\,r\log r}\cdot(1+o(1))\right), \qquad \text{as } r\to\infty.$$

For ℓ_2-balls and constant radius we have

$$\exp\left(-\sqrt{\frac{4(d+1)}{d(d-1)}\,r\log r}\cdot(1+o(1))\right)$$
$$\le D^{(q)}(r) \le \exp\left(-\sqrt{\frac{2}{d-1}\,r\log r}\cdot(1+o(1))\right), \qquad \text{as } r\to\infty.$$

Radius distribution with density. Finally, we deal with a radius distribution with a density.

Theorem 7. *Assume that the distribution of R_1 has a Lebesgue density p with $p(z) \approx z^{\alpha-1}$ for $z \to 0$ for some $\alpha > 0$. Set $\bar{\alpha} := \alpha \wedge 1$. Then*

$$\exp\left(-b\sqrt{r\log r}\cdot(1+o(1))\right) \le D^{(q)}(r) \le \exp\left(-\bar{b}\sqrt{r\log r}\cdot(1+o(1))\right),$$

as $r \to \infty$, where $b := \sqrt{\frac{2(1+\alpha/d)}{d+1}}$ and $\bar{b} := \sqrt{\frac{2(1+\bar{\alpha}/d)}{d+1}}$.

Theorem 4.3.2 and Theorem 4.3.3 in [18] treat the case $\alpha = 1$ and give the right lower bound in that case. Here, we show that this lower bound is sharp and extend both bounds to more general α.

Remark 8. We believe that all results of Sections 1.2 and 1.3 that are stated here for ℓ_2-norms, are also true for ℓ_p-norms with $1 < p < \infty$, and, more generally, for all uniformly convex norms.

2. Lower bounds for the large deviation results

2.1. Constant radius for ℓ_1-, ℓ_2-, ℓ_∞-norms

2.1.1. ℓ_2-norm

Proposition 9. *Assume that the radius is a.s. constant $R_1 \equiv c < 1$ and we work with ℓ_2-balls. If $d \geq 2$, then*

$$\mathbb{P}[K \geq n] \geq \exp(-(1 + \frac{2}{d-1})n \log n(1 + o(1))), \qquad \text{as } n \to \infty.$$

Proof. Consider the following collection of boxes:

$$\left\{ \prod_{m=1}^{d-1} \left[\frac{k_m}{(2n)^{1/(d-1)}} + \frac{1/4}{(2n)^{1/(d-1)}}, \frac{k_m}{(2n)^{1/(d-1)}} + \frac{3/4}{(2n)^{1/(d-1)}} \right] \right\} \times \left[0, \frac{c_1}{n^{2/(d-1)}}\right]$$

with $k_m \in \{0, \dots, \lfloor (2n)^{1/(d-1)} \rfloor - 1\}$ and $c_1 := 2^{-(4+2/(d-1))}$. The number of boxes being of order $2n$, we may choose among them n distinct boxes, say $V_1, \dots, V_n$. Define the following event:

$$E := E_n := \{N = n\} \cap \bigcup_{\pi \text{ permutation of } \{1,\dots,n\}} \{\xi_i \in V_{\pi(i)}, i = 1, \dots, n\}.$$

We will show that – given E – each ball $B(\xi_i, R_1)$ $(i = 1, \dots, n)$ contains a point that is not covered by any other ball $B(\xi_j, R_1)$, $j = 1, \dots, n$, $j \neq i$. Therefore, E implies $K \geq n$. More precisely, the point $x_i := \xi_i + (0, \dots, 0, R_1)$ is obviously in the ball $B(\xi_i, R_1)$ and it is not covered by any other ball: Indeed, for $j \neq i$:

$$\begin{aligned}
\|x_i - \xi_j\|_2^2 &= |\xi_i^{(d)} + R_1 - \xi_j^{(d)}|^2 + \sum_{m=1}^{d-1} |\xi_i^{(m)} - \xi_j^{(m)}|^2 \\
&= |\xi_i^{(d)} - \xi_j^{(d)}|^2 + R_1^2 + 2R_1(\xi_i^{(d)} - \xi_j^{(d)}) + \sum_{m=1}^{d-1} |\xi_i^{(m)} - \xi_j^{(m)}|^2 \\
&\geq R_1^2 - 2R_1 c_1 n^{-2/(d-1)} + \left(\frac{1/2}{(2n)^{1/(d-1)}} \right)^2 > R_1^2,
\end{aligned}$$

by the choice of c_1. Further, note that for large enough n the point x_i is indeed in $[0, 1]^d$, as $R_1 \equiv c < 1$.

Therefore, the event E implies $K \geq n$ and so

$$\begin{aligned}
\mathbb{P}[K \geq n] \geq \mathbb{P}[E] &= \frac{\lambda^n}{n!} e^{-\lambda} \cdot n! \cdot \left(\left(\frac{1/2}{(2n)^{1/(d-1)}} \right)^{d-1} \cdot c_1 n^{-2/(d-1)} \right)^n \\
&= \exp\left(- \left(1 + \frac{2}{d-1} \right) n \log n(1 + o(1)) \right). \quad \square
\end{aligned}$$

2.1.2. ℓ_1-norm

Now we consider the case of a constant radius and ℓ_1-balls.

Proposition 10. *Assume that the radius satisfies $R_1 = c$ a.s. with $c < 1$ and we work with ℓ_1-balls. If $d \geq 2$, then*

$$\mathbb{P}[K \geq n] \geq \exp\left(-\left(1+\frac{1}{d-1}\right) n \log n(1+o(1))\right), \qquad \text{as } n \to \infty.$$

The proof is completely analogous to the ℓ_2-norm case, with the only difference being the possibility to keep the first component in a larger set due to the geometric structure of ℓ_1-balls. This results in the larger bound.

Proof. Consider the following collection of boxes:

$$\left\{\prod_{m=1}^{d-1}\left[\frac{k_m}{(2n)^{1/(d-1)}}+\frac{1/4}{(2n)^{1/(d-1)}}, \frac{k_m}{(2n)^{1/(d-1)}}+\frac{3/4}{(2n)^{1/(d-1)}}\right]\right\} \times \left[0, \frac{c_2}{n^{1/(d-1)}}\right],$$

with $k_m \in \{0, \ldots, \lfloor (2n)^{1/(d-1)} \rfloor - 1\}$. Here, $c_2 := 2^{-(2+1/(d-1))}$. The number of boxes being of order $2n$, we may choose among them n distinct boxes, say $V_1, \ldots, V_n$. Define the following event:

$$E := E_n := \{N = n\} \cap \bigcup_{\pi \text{ permutation of } \{1,\ldots,n\}} \{\xi_i \in V_{\pi(i)}, i = 1, \ldots, n\}.$$

We will show that – given E – each ball $B(\xi_i, R_1)$ $(i = 1, \ldots, n)$ contains point that is not covered by any other ball $B(\xi_j, R_1)$, $j = 1, \ldots, n$, $j \neq i$. Therefore, E implies $K \geq n$. More precisely, the point $x_i := \xi_i + (0, \ldots, 0, R_1)$ is obviously in the ball $B(\xi_i, R_1)$ and it is not covered by any other ball: Indeed, for $j \neq i$:

$$\begin{aligned}
\|x_i - \xi_j\|_1 &= |\xi_i^{(d)} + R_1 - \xi_j^{(d)}| + \sum_{m=1}^{d-1} |\xi_i^{(m)} - \xi_j^{(m)}| \\
&\geq R_1 - |\xi_i^{(d)} - \xi_j^{(d)}| + \sum_{m=1}^{d-1} |\xi_i^{(m)} - \xi_j^{(m)}| \\
&\geq R_1 - c_2 n^{-1/(d-1)} + \frac{1/2}{(2n)^{1/(d-1)}} > R_1,
\end{aligned}$$

by the choice of c_2. Further, note that for large enough n the point x_i is indeed in $[0, 1]^d$, as $R_1 = c < 1$.

Therefore, the event E implies $K \geq n$. Thus,

$$\begin{aligned}
\mathbb{P}[K \geq n] \geq \mathbb{P}[E] &= \frac{\lambda^n}{n!} e^{-\lambda} \cdot n! \cdot \left(\left(\frac{1/2}{(2n)^{1/(d-1)}}\right)^{d-1} \cdot c_2 n^{-1/(d-1)}\right)^n \\
&\geq \exp\left(-\left(1+\frac{1}{d-1}\right) n \log n(1+o(1))\right). \quad \square
\end{aligned} \tag{3}$$

2.1.3. ℓ_∞-norm

Now we consider the case of a constant radius and ℓ_∞-balls. The approach is very similar to the previous ones but the centres of the balls are placed near a "diagonal" hyperplane.

Proposition 11. *Assume that the radius is a.s. constant $R_1 = c < 1$ and we work with ℓ_∞-balls. If $d \geq 2$, then*

$$\mathbb{P}[K \geq n] \geq \exp\left(-\left(1+\frac{1}{d-1}\right) n \log n(1+o(1))\right), \qquad \text{as } n \to \infty.$$

Proof. Let us fix ρ_1, ρ_2 such that $R_1 < \rho_1 < \rho_2 < 1$ and consider the nonempty set (a part of a $(d-1)$-dimensional hyperplane)

$$H := \left\{ x \in [0,1]^d : \sum_{m=1}^{d} x^{(m)} = d\rho_2, \min_{1\le m\le d} x^{(m)} > \rho_1 \right\}.$$

For sufficiently small $c_1 = c_1(d, \rho_1, \rho_2, c)$ we may choose n points $\beta_1, \ldots, \beta_n$ in H such that $\|\beta_i - \beta_j\|_1 > c_1 n^{-1/(d-1)}$ for all $i \neq j$. (We stress that we take ℓ_1-norm here.)

Consider the following collection of boxes:

$$V_i := B(\beta_i, c_2 n^{-1/(d-1)}), \ 1 \le i \le n,$$

with $c_2 < c_1/(4d)$.

Define the following event:

$$E := E_n := \{N = n\} \cap \bigcup_{\pi \text{ permutation of } \{1,\ldots,n\}} \{\xi_i \in V_{\pi(i)}, i = 1, \ldots, n\}.$$

We will show that – given E – each ball $B(\xi_i, R_1)$ $(i = 1, \ldots, n)$ contains a point that is not covered by any other ball $B(\xi_j, R_1), j \neq i$.

Consider the point $x_i := \xi_i - (R_1, \ldots, R_1)$. For $1 \le m \le d$ we have

$$\begin{aligned} x_i^{(m)} &= \xi_i^{(m)} - R_1 \ge \beta_{\pi(i)}^{(m)} - \|\xi_i - \beta_{\pi(i)}\|_\infty - R_1 \\ &\ge \rho_1 - c_2 n^{-1/(d-1)} - R_1 > 0 \end{aligned}$$

for sufficiently large n, which yields $x_i \in [0,1]^d$. It is also obvious that $x_i \in B(\xi_i, R_1)$.

We show now that x_i is not covered by any other ball. Let $x_i' := \beta_{\pi(i)} - (R_1, \ldots, R_1)$. Then

$$\|x_i - x_i'\|_\infty = \|\xi_i - \beta_{\pi(i)}\|_\infty \le c_2 n^{-1/(d-1)}.$$

It follows that for any $j \neq i$ we have

$$\begin{aligned} \|x_i - \xi_j\|_\infty &\ge \|x_i' - \beta_{\pi(j)}\|_\infty - \|x_i - x_i'\|_\infty - \|\xi_j - \beta_{\pi(j)}\|_\infty \\ &\ge \|x_i' - \beta_{\pi(j)}\|_\infty - 2c_2 n^{-1/(d-1)}. \end{aligned} \tag{4}$$

It remains to evaluate $\|x_i' - \beta_{\pi(j)}\|_\infty$. Since $\beta_{\pi(i)}, \beta_{\pi(j)} \in H$, we have

$$\sum_{m=1}^{d} \left(\beta_{\pi(j)}^{(m)} - \beta_{\pi(i)}^{(m)} \right) = \sum_{m=1}^{d} \beta_{\pi(j)}^{(m)} - \sum_{m=1}^{d} \beta_{\pi(i)}^{(m)} = 0. \tag{5}$$

In other words,

$$\sum_{m=1}^{d} \left(\beta_{\pi(j)}^{(m)} - \beta_{\pi(i)}^{(m)} \right)_+ = \sum_{m=1}^{d} \left(\beta_{\pi(j)}^{(m)} - \beta_{\pi(i)}^{(m)} \right)_- .$$

On the other hand, by construction,

$$\sum_{m=1}^{d} \left(\beta_{\pi(j)}^{(m)} - \beta_{\pi(i)}^{(m)} \right)_+ + \sum_{m=1}^{d} \left(\beta_{\pi(j)}^{(m)} - \beta_{\pi(i)}^{(m)} \right)_- = \|\beta_{\pi(j)} - \beta_{\pi(i)}\|_1 > c_1 n^{-1/(d-1)}.$$

It follows that

$$\sum_{m=1}^{d} \left(\beta_{\pi(j)}^{(m)} - \beta_{\pi(i)}^{(m)} \right)_+ > c_1 n^{-1/(d-1)}/2$$

and

$$\begin{aligned}\|x_i' - \beta_{\pi(j)}\|_\infty &\ge \max_{1\le m\le d} \left(\beta_{\pi(j)}^{(m)} - x_i'^{(m)}\right)\\ &= \max_{1\le m\le d} \left(\beta_{\pi(j)}^{(m)} - \beta_{\pi(i)}^{(m)}\right) + R_1.\end{aligned}$$

It follows from (5) that

$$\max_{1\le m\le d} \left(\beta_{\pi(j)}^{(m)} - \beta_{\pi(i)}^{(m)}\right) = \max_{1\le m\le d} \left(\beta_{\pi(j)}^{(m)} - \beta_{\pi(i)}^{(m)}\right)_+ \ge \frac{1}{d}\sum_{m=1}^{d} \left(\beta_{\pi(j)}^{(m)} - \beta_{\pi(i)}^{(m)}\right)_+$$

and we obtain

$$\begin{aligned}\|x_i' - \beta_{\pi(j)}\|_\infty &\ge \frac{1}{d}\sum_{m=1}^{d} \left(\beta_{\pi(j)}^{(m)} - \beta_{\pi(i)}^{(m)}\right)_+ + R_1 &\quad (6)\\ &> c_1 n^{-1/(d-1)}/(2d) + R_1. &\quad (7)\end{aligned}$$

By using the bounds (4), (7), and the definition of c_2, we obtain

$$\|x_i - \xi_j\|_\infty \ge c_1 n^{-1/(d-1)}/(2d) + R_1 - 2c_2 n^{-1/(d-1)} > R_1.$$

This means $x_i \notin B(\xi_i, R_1)$, as claimed. Therefore, the event E implies $K \ge n$ and so

$$\begin{aligned}\mathbb{P}[K \ge n] \ge \mathbb{P}[E] &= \frac{\lambda^n}{n!} e^{-\lambda} \cdot n! \cdot \left(c_2 n^{-1/(d-1)}\right)^{dn}\\ &= \exp\left(-\left(1 + \frac{1}{d-1}\right) n \log n(1 + o(1))\right). \quad \square\end{aligned}$$

2.2. Generic radius: Lower bound via small balls

The following result is valid for arbitrary norms in $\mathbb{R}^d$, $d \ge 1$.

Proposition 12. *Assume that the distribution of R_1 has a Lebesgue density p with $p(z) \ge cz^{\alpha-1}$ for small z and some constants $c > 0$ and $\alpha > 0$. Then*

$$\mathbb{P}[K \ge n] \ge \exp(-(1 + \alpha/d)n \log n(1 + o(1))), \qquad \text{as } n \to \infty.$$

Proof. Let $n \ge (2^{1/d} - 1)^d$. A lower bound is obtained from the following scenario. Consider the following collection of cubic boxes:

$$\prod_{m=1}^{d} \left[\frac{k_m}{(2n)^{1/d}} + \frac{1/4}{(2n)^{1/d}}, \frac{k_m}{(2n)^{1/d}} + \frac{3/4}{(2n)^{1/d}}\right], \qquad k_m \in \{0, \ldots, \lfloor (2n)^{1/d} \rfloor - 1\}.$$

The number of boxes being of order $2n$, we may choose among them n distinct boxes, say $V_1, \ldots, V_n$. Define the following event:

$$E := E_n := \{N = n\} \cap \bigcup_{\pi \text{ permutation of } \{1,\ldots,n\}} E_\pi,$$

where

$$E_\pi := \{\xi_i \in V_{\pi(i)}, R_i \in [c_1 n^{-1/d}, c_2 n^{-1/d}], \forall i = 1, \ldots, n\},$$

with some constants $c_2 > c_1 > 0$. The constant c_2 depending on the norm under consideration can be chosen so small that for distinct i and j the balls $B(\xi_i, R_i)$ and $B(\xi_j, R_j)$ are disjoint. Therefore, the event E implies $K \ge n$.

Finally, note that for large n

$$\mathbb{P}[K \ge n] \ge \mathbb{P}[E]$$

$$
\begin{aligned}
&= n! \cdot \frac{\lambda^n}{n!} e^{-\lambda} \cdot \left(\left(\frac{1/2}{(2n)^{1/d}} \right)^d \cdot \int_{c_1 n^{-1/d}}^{c_2 n^{-1/d}} p(z)\mathrm{d}z \right)^n \\
&\geq \left(\lambda 2^{-(d+1)} \right)^n e^{-\lambda} \cdot n^{-n} \cdot \left(\int_{c_1 n^{-1/d}}^{c_2 n^{-1/d}} c z^{\alpha-1} \mathrm{d}z \right)^n \\
&\geq \left(\lambda 2^{-(d+1)} (c_2^\alpha - c_1^\alpha)/\alpha \right)^n e^{-\lambda} \cdot n^{-n} \cdot n^{-\alpha n/d} \\
&= \exp(-(1+\alpha/d) n \log n \cdot (1+o(1))). \quad \Box
\end{aligned}
$$

2.3. Generic radius: Lower bound via surfaces

Proposition 13. *Assume the radius distribution has a bounded density. Then,*

- *for ℓ_1-norm balls,*
$$\mathbb{P}[K \geq n] \geq \exp\left(-\left(1 + \frac{1}{d-1} \right) n \log n \cdot (1+o(1)) \right);$$
- *for ℓ_2-norm balls,*
$$\mathbb{P}[K \geq n] \geq \exp\left(-\left(1 + \frac{2}{d-1} \right) n \log n \cdot (1+o(1)) \right).$$

We will prepare the proof with the following lemma. It shows that any probability distribution with bounded density has many intervals in its support with the mass proportional to the length of those intervals or larger.

Lemma 14. *Let $R \in [0,1]$ be a random variable having a bounded Lebesgue density p, say*

$$\operatorname*{esssup}_{x\in[0,1]} p(x) \leq c < \infty. \tag{8}$$

Then for any $\beta \in [0,1]$ and any $\delta > 0$ we have

$$\mathrm{vol}_1\{x \in [0,1] : \mathbb{P}[R \in [x, x+\beta]] > \delta\beta\} > \frac{1 - 2c\beta - \delta}{c},$$

where vol_1 denotes the Lebesgue measure on $\mathbb{R}$.

Proof. Set, for ease of notation, $\mathbb{P}[R \in [x, x+\beta]] =: Q_x(\beta)$. First note that we have

$$
\begin{aligned}
\int_0^{1-\beta} Q_x(\beta)\mathrm{d}x &= \int_0^{1-\beta} \int_x^{x+\beta} p(y)\mathrm{d}y\mathrm{d}x = \int_0^1 \int_{0\vee(y-\beta)}^{(1-\beta)\wedge y} \mathrm{d}x p(y)\mathrm{d}y \\
&\geq \int_\beta^{1-\beta} \int_{y-\beta}^{y} \mathrm{d}x p(y)\mathrm{d}y = \beta \int_\beta^{1-\beta} p(y)\mathrm{d}y \\
&\geq \beta \left(\int_0^1 p(y)\mathrm{d}y - 2c\beta \right) = \beta(1 - 2c\beta).
\end{aligned}
$$

On the other hand, note that $Q_x(\beta) \leq c\beta$. Therefore,

$$
\begin{aligned}
\int_0^{1-\beta} Q_x(\beta)\mathrm{d}x &= \int_0^{1-\beta} Q_x(\beta) \mathbb{1}_{Q_x(\beta)\leq\delta\beta}\mathrm{d}x + \int_0^{1-\beta} Q_x(\beta) \mathbb{1}_{Q_x(\beta)>\delta\beta}\mathrm{d}x \\
&\leq \delta\beta + c\beta \int_0^1 \mathbb{1}_{Q_x(\beta)>\delta\beta}\mathrm{d}x.
\end{aligned}
$$

It follows that

$$1 - 2c\,\beta \leq \delta + c\,\mathrm{vol}_1\{x \in [0,1] : \mathbb{P}[R \in [x, x+\beta]] > \delta\beta\}.$$

Rearranging the terms gives the claim. $\Box$

Proof of Proposition 13. We shall proceed in a number of steps: after some preparations, we define a scenario, estimate its probability, and then show that the scenario implies $K \geq n$.

Preparation: $(d-1)$-dimensional boxes. Set $\varepsilon_n := (2n)^{-1/(d-1)}/2$. Let us consider the following collection of boxes:

$$\prod_{m=1}^{d-1} \left[2k_m\varepsilon_n, (2k_m+1)\varepsilon_n\right], \qquad k_m \in \{0, \ldots, \lfloor \frac{\varepsilon_n^{-1}-1}{2} \rfloor\}.$$

The number of such boxes is greater or equal to

$$\left(\lfloor \frac{\varepsilon_n^{-1}-1}{2} \rfloor + 1\right)^{d-1} \sim (2\varepsilon_n)^{-(d-1)} = 2n.$$

Therefore, one can choose n of these boxes, say $V_1, \ldots, V_n$. The main feature of these boxes is that

$$\forall i \neq j \, \forall y \in V_i, y' \in V_j \; : \; \max_{1\leq m\leq d-1} |y^{(m)} - (y')^{(m)}| \geq \varepsilon_n. \tag{9}$$

Preparation: support of the d-th component. Let c be the bound of the density (as in (8)). It follows immediately from Lemma 14 (with $\delta = 1/4$) that for any $\beta \in [0, \frac{1}{4c}]$ we have

$$\mathrm{vol}_1 \left\{ x \in [0,1] : \mathbb{P}[R \in [x, x+\beta]] > \frac{\beta}{4} \right\} > \frac{1}{4c}.$$

This implies that for $\beta < 1/(8c)$ we have

$$\begin{aligned} \frac{1}{8c} &< \mathrm{vol}_1 \left\{ x \in [0, 1-\beta] : \mathbb{P}[R \in [x, x+\beta]] > \frac{\beta}{4} \right\} \\ &= \mathrm{vol}_1 \left\{ z \in [0, 1-\beta] : \mathbb{P}[R + z \in [1-\beta, 1]] > \frac{\beta}{4} \right\}. \end{aligned}$$

Let us denote

$$Z(\beta) := \left\{ z \in [0, 1-\beta] : \mathbb{P}[R + z \in [1-\beta, 1]] > \frac{\beta}{4} \right\}.$$

We shall use this set as the support of the d-th component in the scenario we will construct. It will be used for $\beta = \beta_n \to 0$ so that the assumption $\beta < 1/(8c)$ is satisfied for n large enough.

Definition of the scenario and evaluation of its probability. Define the 'tubes'

$$W_i := V_i \times Z(\beta_n), \qquad i = 1, \ldots, n,$$

where β_n is chosen later according to the involved norm.

Consider the following scenario:

$$E := E_n := \{N = n\} \cap \bigcup_{\pi \text{ permutation of } \{1,\ldots,n\}} E_\pi,$$

where

$$E_\pi := \{\xi_i \in W_{\pi(i)}, \xi_i^{(d)} + R_i \in [1-\beta_n, 1], \forall i = 1, \ldots, n\}.$$

By using the projection $\sigma(x^{(1)}, \ldots, x^{(d)}) := (x^{(1)}, \ldots, x^{(d-1)})$, we can estimate the probability of E as follows:

$$\begin{aligned} \mathbb{P}[E] &\geq \frac{\lambda^n}{n!} e^{-\lambda} \cdot n! \cdot \left(\mathbb{P}[\sigma(\xi_1) \in V_1] \cdot \mathbb{P}[\xi_1^{(d)} \in Z(\beta_n), \xi_1^{(d)} + R_1 \in [1-\beta_n, 1]] \right)^n \\ &\geq \lambda^n e^{-\lambda} \left(\varepsilon_n^{d-1} \cdot \mathbb{P}[\xi_1^{(d)} \in Z(\beta_n)] \inf_{z \in Z(\beta_n)} \mathbb{P}[z + R_1 \in [1-\beta_n, 1]] \right)^n \\ &\geq \lambda^n e^{-\lambda} \left(\varepsilon_n^{d-1} \cdot \frac{1}{8c} \frac{\beta_n}{4} \right)^n \\ &= \left(\frac{\lambda}{2^{d+5}c} \right)^n e^{-\lambda} n^{-n} \beta_n^n, \end{aligned} \tag{10}$$

where we used that the ξ_i are uniformly distributed in $[0, 1]^d$ in the second and the third step. Later, we will choose β_n (polynomially decaying in n) according to the involved norm.

Scenario E implies $K \geq n$. We now proceed to showing that the scenario $E = E_n$ implies $K \geq n$.

For this, it is sufficient to show that under E, for any i the following auxiliary point x_i belongs to the ball $B(\xi_i, R_i)$ but it is not covered by any other ball. Thus, none of the balls $B(\xi_i, R_i)$ can be left out when representing the picture S.

Define $x_i := \xi_i + (0, \ldots, 0, R_i)$. Clearly,

$$\|\xi_i - x_i\|_\infty = \|\xi_i - x_i\|_2 = \|\xi_i - x_i\|_1 = R_i,$$

thus $x_i \in B(\xi_i, R_i)$. It remains to show that $x_i \notin B(\xi_j, R_j)$ for any $j \neq i$, i.e.

$$\|x_i - \xi_j\|_q > R_j \quad \forall j \neq i, \tag{11}$$

for respective $q \in \{1; 2\}$. This will be achieved separately for the different norms and with different choices of the sequence (β_n).

Proof of (11) for ℓ_1-norm. Here we choose $\beta_n := \varepsilon_n/2$.

Assume for the sake of contradiction that $\|x_i - \xi_j\|_1 \leq R_j$ for some $i \neq j$.

Since $\xi_j^{(d)} + R_j \in [1 - \beta_n, 1]$ on the event E, for any $z \in [1 - \beta_n, 1]$ we have

$$\beta_n \geq |(\xi_j^{(d)} + R_j) - z| = |R_j - (z - \xi_j^{(d)})| \geq R_j - |z - \xi_j^{(d)}|$$

and so $|z - \xi_j^{(d)}| \geq R_j - \beta_n$. Letting $z := x_i^{(d)} = \xi_i^{(d)} + R_i \in [1 - \beta_n, 1]$, we have

$$|x_i^{(d)} - \xi_j^{(d)}| \geq R_j - \beta_n. \tag{12}$$

It follows that

$$\begin{aligned} R_j &\geq \|x_i - \xi_j\|_1 = \sum_{m=1}^{d-1} |x_i^{(m)} - \xi_j^{(m)}| + |x_i^{(d)} - \xi_j^{(d)}| \\ &\geq \sum_{m=1}^{d-1} |\xi_i^{(m)} - \xi_j^{(m)}| + R_j - \beta_n \\ &\geq \max_{1 \leq m \leq d-1} |\xi_i^{(m)} - \xi_j^{(m)}| + R_j - \beta_n. \end{aligned}$$

Hence,

$$\max_{1 \leq m \leq d-1} |\xi_i^{(m)} - \xi_j^{(m)}| \leq \beta_n = \varepsilon_n/2,$$

in contradiction to (9). Therefore, we must have $\|x_i - \xi_j\|_1 > R_j$.

Proof of (11) for ℓ_2-norm. Here we choose $\beta_n := \varepsilon_n^2/3$.

Assume for the sake of contradiction that $\|x_i - \xi_j\|_2 \leq R_j$ for some $j \neq i$.

Using (12) again, we have

$$\begin{aligned} R_j^2 &\geq \|x_i - \xi_j\|_2^2 = \sum_{m=1}^{d-1} |x_i^{(m)} - \xi_j^{(m)}|^2 + |x_i^{(d)} - \xi_j^{(d)}|^2 \\ &\geq \sum_{m=1}^{d-1} |\xi_i^{(m)} - \xi_j^{(m)}|^2 + (R_j - \beta_n)^2 \\ &\geq \max_{1 \leq m \leq d-1} |\xi_i^{(m)} - \xi_j^{(m)}|^2 + R_j^2 - 2R_j\beta_n + \beta_n^2. \end{aligned}$$

Hence,

$$\max_{1 \leq m \leq d-1} |\xi_i^{(m)} - \xi_j^{(m)}|^2 \leq 2R_j\beta_n \leq 2\varepsilon_n^2/3 < \varepsilon_n^2,$$

in contradiction to (9). Therefore, we must have $\|x_i - \xi_j\|_2 > R_j$.

Rate of $\mathbb{P}[E]$. Finally, it is simple to see that the rate in (10) with the choices $\beta_n = \varepsilon_n/2$ (ℓ_1-norm) and $\beta_n = \varepsilon_n^2/3$ (ℓ_2-norm) leads to the asserted rates in the statement of the proposition. □

3. Upper bounds for the large deviation results

3.1. Generic radius

The following result based on an assumption on the radius concentration function is valid for arbitrary norms in $\mathbb{R}^d$.

Proposition 15. *If* R_1 *is such that* $\sup_{x>0} \mathbb{P}\big[R_1 \in [x, x+r]\big] \le cr^\alpha$ *for some* $c > 0, \alpha \in (0, 1]$ *and all* $r > 0$*, then*

$$\mathbb{P}[K \ge n] \le \exp(-(1 + \alpha/d)n \log n(1 + o(1))), \qquad \text{as } n \to \infty.$$

Remark 16. If, for example, R_1 has a bounded density, then the assumption of proposition holds with $\alpha = 1$.

Note that a bound $\le cr^\alpha$ in the assumption of the proposition cannot hold for $\alpha > 1$.

Proof. *Step 1: Initial definitions.*

In order to avoid cumbersome notations, in this proof we assume that $n^{1/d}$ is an integer.

Let us divide the unit cube into n equal boxes of side length $n^{-1/d}$. Denote by $J_i \in \{1, \dots, n\}$ the number of the box that contains ξ_i. Further, denote by $N(k, j) := \#\{i \le k : J_i = j\}$ the number of balls among the first k that have their centres in the j-th box. If $N(k, j) > 0$ we define

$$R^*(k, j) := \max\{R_i : i \le k, J_i = j\}$$

which is the maximal radius of the balls having centres in the box j among the first k balls.

Step 2: Building a collection.

We shall now gather certain balls (identified by their numbers) into a *collection*. We proceed by looking in the k-th step at the k-th ball, possibly adding it to the collection and possibly deleting another ball from the collection. During the whole time, the collection will maintain the following important properties:

(1) In each step k, every ball (among the first k balls) that is not included in the collection is covered by some other ball (from the first k balls).
(2) In each step k, for every box j, if $N(k, j) > 0$ then a ball corresponding to the maximal radius with centre in that box, $R^*(k, j)$, *is* included in the collection.

Let us now describe the inspection procedure that will lead to a collection. In this procedure the diameter (with respect to the norm we consider) of the unit cube will be involved. We denote it $\mathfrak{D}$.

At step 0, we start with an empty collection, which certainly satisfies (1) and (2).

When moving from k to $k+1$, we first identify the box of the next ball, J_{k+1}. If $N(k, J_{k+1}) = 0$ (i.e. there was no ball with the centre located in the box J_{k+1} so far), we include the ball $k+1$ into the collection. Further, note that this step certainly does preserve properties (1) and (2). Also note that for each box j the case $j = J_{k+1}$ and $N(k, J_{k+1}) = 0$ happens at most once.

If $N(k, J_{k+1}) > 0$, we compare R_{k+1} with $R^*(k, J_{k+1})$, i.e. the radius of the current ball $k+1$ with the maximal radius in box J_{k+1}:

- If $R_{k+1} > R^*(k, J_{k+1}) + \mathfrak{D}n^{-1/d}$, then ball $k+1$ is large enough to cover the ball that corresponds to the maximal radius $R^*(k, J_{k+1})$, because the distance of their centres is at most $\mathfrak{D}n^{-1/d}$ (as the centres are in the same box). So, we may delete from the collection the balls that correspond to the current maximal radius $R^*(k, J_{k+1})$ (it *is* in the collection by property (2)) and add the $(k+1)$-th ball. At the same time, R_{k+1} becomes the maximal radius, i.e. $R^*(k+1, J_{k+1}) = R_{k+1}$. Certainly, this preserves (1), as any ball that we deleted from the collection is covered by the ball that was added to the collection. It also preserves property (2), since the newly added ball is the one that corresponds to the maximal radius now.

- If $R_{k+1} < R^*(k, J_{k+1}) - \mathfrak{D}n^{-1/d}$, then the $(k+1)$-th ball is not added to the collection. Note that it is covered by the ball that corresponds to the maximal radius, because the distance between their centres is at most $\mathfrak{D}n^{-1/d}$ (the centres being in the same box). This shows that property (1) is preserved, and certainly (2) is preserved, because the maximal radius is unchanged, $R^*(k+1, J_{k+1}) = R^*(k, J_{k+1})$.
- Finally, if $R_{k+1} \in [R^*(k, J_{k+1}) - \mathfrak{D}n^{-1/d}, R^*(k, J_{k+1}) + \mathfrak{D}n^{-1/d}]$, we do add the $(k+1)$-th ball into the collection. It may have the new maximal radius or not, in both cases (since we do not exclude any ball from the collection) properties (1) and (2) are preserved.

We now count how often the last of the three cases above occurs. Let us define the corresponding event

$$A_{k+1} := \left\{ N(k, J_{k+1}) > 0,\, R_{k+1} \in \left[R^*(k, J_{k+1}) - \frac{\mathfrak{D}}{n^{1/d}},\, R^*(k, J_{k+1}) + \frac{\mathfrak{D}}{n^{1/d}} \right] \right\}$$

and denote by

$$S_m := \sum_{k=1}^{m} \mathbb{1}_{A_k}$$

the number of occurrences of the last of the three cases up to m steps of the algorithm. Let us further denote by $K_c(m)$ the size of the collection after m steps, and set

$$K'(m) := m - \sum_{k=1}^{m} \mathbb{1}_{\{\exists i \neq k : B(\xi_k, R_k) \subseteq B(\xi_i, R_i)\}}$$

for the number of balls (with index $\leq m$) that are not covered by *some* other ball. The major observation is that, because of property (1) and the fact that in each step of the algorithm the number of balls increased by at most one (either because we had $N(k, J_{k+1}) = 0$ or because A_{k+1} occurred), we have, respectively,

$$K'(m) \leq K_c(m) \leq n + S_m. \tag{13}$$

Another important observation is that for K from the statement of the proposition and the Poisson random variable N we have almost surely (as on a set of measure zero two or more balls might coincide)

$$K \leq K'(N). \tag{14}$$

Step 3: Evaluation of the probability of one A-event given the past.
Let the σ-fields $\mathcal{F}_k$ be defined by

$$\mathcal{F}_k := \sigma((\xi_i)_{i \leq k+1}; (R_i)_{i \leq k}), \qquad k = 0, 1, 2, \ldots.$$

These σ-fields represent the information about the radii up to step k and the centres up to step $k+1$.

We show that there is a number $\kappa > 0$ (only depending on the law of the radii) such that for each k

$$p_{k+1} := \mathbb{P}[A_{k+1} | \mathcal{F}_k] \leq \kappa\, n^{-\alpha/d}, \tag{15}$$

To see (15), note that

$$\begin{aligned} p_{k+1} &:= \mathbb{P}[A_{k+1} | \mathcal{F}_k] \\ &= \mathbb{P}[R_{k+1} \in [R^*(k, J_{k+1}) - \mathfrak{D}n^{-1/d}, R^*(k, J_{k+1}) + \mathfrak{D}n^{-1/d}] \big| \mathcal{F}_k] \mathbb{1}_{\{N(k, J_{k+1}) > 0\}}. \end{aligned}$$

Note that $N(k, J_{k+1})$ and $R^*(k, J_{k+1})$ are measurable w.r.t. $\mathcal{F}_k$ and R_{k+1} is independent of $\mathcal{F}_k$. Therefore, we can estimate

$$p_{k+1} \leq \sup_{x>0} \mathbb{P}[R_{k+1} \in [x - \mathfrak{D}n^{-1/d}, x + \mathfrak{D}n^{-1/d}]] \leq (2\mathfrak{D})^\alpha\, c\, n^{-\alpha/d} =: \kappa\, n^{-\alpha/d},$$

where we used the bound for the concentration function of R_{k+1} from the proposition's assumption.

Step 4: Majorization of S_m.
Set $\Delta = \Delta(n) := \kappa n^{-\alpha/d}$.
We shall prove by induction that for any $\gamma > 0$ and any $m \in \mathbb{N}$ we have

$$\mathbb{E}\,\exp(\gamma S_m) \le (\Delta(e^\gamma - 1) + 1)^m. \tag{16}$$

Clearly the estimate holds for $m = 0$. Assume the estimate holds for $m = k$. Then

$$\begin{aligned}\mathbb{E}\,\exp(\gamma S_{k+1}) &= \mathbb{E}\,[\mathbb{E}\,[\exp(\gamma(S_k + \mathbb{1}_{A_{k+1}}))|\mathcal{F}_k]] \\ &= \mathbb{E}\,[\exp(\gamma S_k)\mathbb{E}\,[\exp(\gamma \mathbb{1}_{A_{k+1}})|\mathcal{F}_k]] \\ &= \mathbb{E}\,[\exp(\gamma S_k)\mathbb{E}\,[e^\gamma \mathbb{1}_{A_{k+1}} + \mathbb{1}_{A_{k+1}^c}\big|\mathcal{F}_k]] \\ &= \mathbb{E}\,[\exp(\gamma S_k)(e^\gamma p_{k+1} + (1 - p_{k+1}))] \\ &= \mathbb{E}\,[\exp(\gamma S_k)((e^\gamma - 1)p_{k+1} + 1)] \\ &\le \mathbb{E}\,[\exp(\gamma S_k)((e^\gamma - 1)\Delta + 1)]\end{aligned}$$

where we used (15) in the last step. This shows the claim in (16).

Step 5: Final computations.
Fix $B \in \mathbb{N}$, $B \ge 2$. Then due to (14), (13), and (16) we have for any $\gamma > 0$

$$\begin{aligned}\mathbb{P}[K \ge Bn] &\le \mathbb{P}[K'(N) \ge Bn] \\ &= \sum_{m=0}^{\infty} \mathbb{P}[N = m] \cdot \mathbb{P}[K'(m) \ge Bn] \\ &\le \sum_{m=0}^{\infty} \mathbb{P}[N = m] \cdot \mathbb{P}[S_m \ge (B-1)n] \\ &\le \sum_{m=0}^{\infty} \mathbb{P}[N = m] \cdot \mathbb{E}\,[\exp(\gamma S_m)]\, e^{-\gamma(B-1)n} \\ &\le \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} e^{-\lambda} \cdot (\Delta(e^\gamma - 1) + 1)^m\; e^{-\gamma(B-1)n} \\ &= e^{-\lambda-\gamma(B-1)n} \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \cdot (\Delta(e^\gamma - 1) + 1)^m \\ &= \exp\left(-\gamma(B-1)n + \lambda\Delta(e^\gamma - 1)\right).\end{aligned}$$

Choosing $\gamma := \log((B-1)n/(\lambda\Delta))$ (and n large enough to ensure that $\gamma > 0$), we obtain the estimate

$$\log \mathbb{P}[K \ge Bn] \le -(B-1)n \log((B-1)n/(\lambda\Delta)) + (B-1)n - \lambda\Delta.$$

Note that $(B-1)n/\Delta \sim (B-1)n^{1+\alpha/d}/\kappa$, as $n \to \infty$. This shows that for any $\varepsilon > 0$ and n large enough,

$$\log \mathbb{P}[K \ge Bn] \le -(B-1)(1 + \alpha/d) \cdot (n \log n)(1 - \varepsilon).$$

Now, let $\ell \in \mathbb{N}$. Then there exists a unique $n = n_\ell \in \mathbb{N}$ such that $Bn \le \ell < B(n+1)$. Then for large enough ℓ (and thus n) we obtain

$$\begin{aligned}\frac{\log \mathbb{P}[K \ge \ell]}{\ell \log \ell} &\le \frac{\log \mathbb{P}[K \ge Bn]}{B(n+1) \log B(n+1)} \\ &= \frac{\log \mathbb{P}[K \ge Bn]}{n \log n} \cdot \frac{n \log n}{B(n+1) \log B(n+1)} \\ &\le -(B-1)(1 + \alpha/d)(1 - \varepsilon) \cdot \frac{n \log n}{B(n+1) \log B(n+1)}.\end{aligned}$$

Taking the $\limsup_{\ell\to\infty}$ and using that $n = n_\ell \to \infty$ with $\ell \to \infty$ we obtain

$$\limsup_{\ell\to\infty} \frac{\log \mathbb{P}[K \geq \ell]}{\ell \log \ell} \leq -\frac{(B-1)(1+\alpha/d)(1-\varepsilon)}{B}.$$

Letting $B \to \infty$ and $\varepsilon \to 0$ show the claim. □

Remark 17. We did not use the fact that the (ξ_i) are uniformly distributed, i.e. any distribution on $[0, 1]^d$ works.

Remark 18. The same argument works if the (ξ_i) take values in a totally bounded metric space: Let us denote the covering numbers of that space by $N(\varepsilon)$, $\varepsilon > 0$. Let

$$Q_R(r) := \sup_{x>0} \mathbb{P}\big[R_1 \in [x, x+r]\big]$$

be the concentration function of R. Then the above proof can be modified to show that for any $\varepsilon > 0$

$$K(m) \leq N(\varepsilon) + S_m$$

where S_m is a random variable satisfying

$$\mathbb{E}\, e^{\gamma S_m} \leq (Q_R(\varepsilon)(e^\gamma - 1) + 1)^m, \qquad \forall m \in \mathbb{N}\, \forall \gamma > 0$$

and

$$K(m) := \min\{r \geq 1 | \exists i_1, \ldots, i_r \in \{1, \ldots, m\} : \bigcup_{s=1}^{m} B(\xi_{i_s}, R_{i_s}) = \bigcup_{s=1}^{r} B(\xi_{i_s}, R_{i_s})\},$$

3.2. *Constant radius for ℓ_1-, ℓ_2-norms*

3.2.1. *Constant radius: ℓ_1-norm*

The following result completely matches the lower bound from Proposition 10.

Proposition 19. *Assume that the radius is a.s. constant $R_1 \equiv c < 1$ and we work with ℓ_1-balls. If $d \geq 2$, then*

$$\mathbb{P}[K \geq n] \leq \exp\left(-\left(1 + \frac{1}{d-1}\right) n \log n (1 + o(1))\right), \qquad \text{as } n \to \infty. \tag{17}$$

Proof. The first three steps of the proof are deterministic ones, while probability estimates appear in the fourth step.

Step 1. Combining the balls in groups. For a while, let the norm be arbitrary and let

$$S = \bigcup_{i=1}^{K} B(\theta_i, r_i) \cap [0, 1]^d$$

be an irreducible representation of the picture S. Then for every $i \leq K$ there exists a point $\nu_i \in B(\theta_i, r_i) \cap [0, 1]^d$ such that $\nu_i \notin B(\theta_j, r_j)$ for $j \neq i$. Let $\Delta_i := \nu_i - \theta_i$.

Let $r := \min_{1\leq i\leq K} r_i$ and denote $J_0 := \{i : \|\Delta_i\| \leq r/2\}$. Then for any distinct $i, j \in J_0$ we have

$$r \leq r_i < \|\nu_j - \theta_i\| \leq \|\nu_j - \nu_i\| + \|\nu_i - \theta_i\| = \|\nu_j - \nu_i\| + \|\Delta_i\| \leq \|\nu_j - \nu_i\| + r/2.$$

It follows that $\|\nu_j - \nu_i\| > r/2$ and we conclude that $\# J_0 \leq c_1 r^{-d}$ with a constant $c_1 := 2^d/\mathrm{vol}_d B(0, 1)$ depending only on the dimension and the norm.

Let now $i \notin J_0$. Then

$$\max_{1\leq m\leq d} |\Delta_i^{(m)}| = \|\Delta_i\|_\infty \geq c_2 \|\Delta_i\| > c_2 r/2$$

with c_2 depending only on the norm. Therefore, i belongs to one of the $2d$ sets

$$J_m^+ := \{i : \Delta_i^{(m)} > c_2\, r/2\}, \qquad J_m^- := \{i : \Delta_i^{(m)} < -c_2\, r/2\}.$$

From now on we fix one of these sets, say, J_d^+. For $i \in J_d^+$ we have

$$\nu_i^{(d)} - \theta_i^{(d)} = \Delta_i^{(d)} \geq c_2\, r/2.$$

Step 2. Evaluation of coordinate differences. We specify to the case of equal radii $r_1 = \cdots = r_K = r$ and ℓ_1-norm; the subsequent constants c_j are allowed to depend on r.

At this step we give a bound for the difference $|\theta_i^{(d)} - \theta_j^{(d)}|$ for $i, j \in J_d^+$ and show that it can be either quite large or small.

Let $\sigma : \mathbb{R}^d \mapsto \mathbb{R}^d$ be the projection defined by

$$\sigma x := (x^{(1)}, \ldots, x^{(d-1)}, 0).$$

Lemma 20. *Let $i, j \in J_d^+$, $i \neq j$, and set $c_3 := c_2\, r/2$. Then*

$$|\theta_i^{(d)} - \theta_j^{(d)}| \notin [\, \|\sigma\theta_i - \sigma\theta_j\|_1, c_3\,]. \tag{18}$$

Proof of the Lemma. Without loss of generality we may and will assume that $\theta_i^{(d)} \geq \theta_j^{(d)}$. Introduce an auxiliary point $\psi := (\theta_i^{(1)}, \ldots, \theta_i^{(d-1)}, \theta_j^{(d)})$. We have

$$\begin{aligned} \|\nu_j - \psi\|_1 &\leq \|\nu_j - \theta_j\|_1 + \|\theta_j - \psi\|_1 \\ &= \|\nu_j - \theta_j\|_1 + \|\sigma\theta_j - \sigma\theta_i\|_1 \\ &\leq r + \|\sigma\theta_j - \sigma\theta_i\|_1. \end{aligned} \tag{19}$$

Assume, for the sake of contradiction, that

$$\theta_i^{(d)} \in [\,\theta_j^{(d)} + \|\sigma\theta_i - \sigma\theta_j\|_1, \theta_j^{(d)} + c_3\,]. \tag{20}$$

Since $j \in J_d^+$, (20) yields

$$\nu_j^{(d)} \geq \theta_j^{(d)} + c_2 r/2 = \theta_j^{(d)} + c_3 \geq \theta_i^{(d)}.$$

On the other hand,

$$\|\nu_j - \psi\|_1 = \|\sigma\nu_j - \sigma\psi\|_1 + \nu_j^{(d)} - \psi^{(d)} = \|\sigma\nu_j - \sigma\theta_i\|_1 + \nu_j^{(d)} - \theta_j^{(d)}. \tag{21}$$

Therefore, by using (21), (20), (19) we obtain

$$\begin{aligned} \|\nu_j - \theta_i\|_1 &= \|\sigma\nu_j - \sigma\theta_i\|_1 + \nu_j^{(d)} - \theta_i^{(d)} \\ &= \|\nu_j - \psi\|_1 - (\nu_j^{(d)} - \theta_j^{(d)}) + \nu_j^{(d)} - \theta_i^{(d)} \\ &= \|\nu_j - \psi\|_1 - (\theta_i^{(d)} - \theta_j^{(d)}) \\ &\leq \|\nu_j - \psi\|_1 - \|\sigma\theta_i - \sigma\theta_j\|_1 \\ &\leq r + \|\sigma\theta_j - \sigma\theta_i\|_1 - \|\sigma\theta_i - \sigma\theta_j\|_1 = r, \end{aligned}$$

which contradicts the assumption $\nu_j \notin B(\theta_i, r)$ from the definition of ν_j. It follows that (20) does not hold, hence, the assertion of Lemma 20 is true. $\square$

Step 3. Counting boxes' hits.

Let us fix a large $A > 0$ and cover $[0, 1]^d$ with the following collection of cubic boxes:

$$V_{\bar{k},k_d} := \prod_{m=1}^{d} \left[\frac{Ak_m}{n^{1/(d-1)}}, \frac{A(k_m + 1)}{n^{1/(d-1)}} \right],$$

with $k_m \in \{0, \ldots, \lfloor A^{-1} n^{1/(d-1)} \rfloor\}$ for $1 \leq m \leq d$ and multi-index $\bar{k} := (k_1, \ldots, k_{d-1})$.

Let us fix $\bar{k}$, and evaluate the number of the corresponding boxes hit by the ball centres:

$$N(\bar{k}, d, +) := \#\{k : \theta_i \in V_{\bar{k},k} \text{for some } i \in J_d^+\}.$$

Indeed, if $\theta_i \in V_{\bar{k},\kappa_i}$ and $\theta_j \in V_{\bar{k},\kappa_j}$ for some $i, j \in J_d^+$, then

$$\|\sigma\theta_i - \sigma\theta_j\|_1 \le (d-1)An^{-1/(d-1)}$$

and Lemma 20 yields

$$|\theta_i^{(d)} - \theta_j^{(d)}| \notin [(d-1)An^{-1/(d-1)}, c_3].$$

By splitting [0, 1] into $\lceil c_3^{-1} \rceil$ pieces of length less or equal to c_3 we see that if $\theta_i^{(d)}, \theta_j^{(d)}$ belong to the same piece, then $|\theta_i^{(d)} - \theta_j^{(d)}| \le (d-1)An^{-1/(d-1)}$. Hence, $|\kappa_i - \kappa_j| \le d$. It follows immediately that

$$N(\bar{k}, d, +) \le d\lceil c_3^{-1} \rceil =: c_4.$$

By the symmetry of coordinates, the total number of hit boxes admits the estimate

$$\sum_{\bar{k}} \sum_{m=1}^{d} \big(N(\bar{k}, m, +) + N(\bar{k}, m, -)\big) \le (2dc_4) \cdot (n/A^{d-1}) =: \frac{c_5 n}{A^{d-1}}.$$

Let $\mathcal{U}$ denote the ensemble of all possible unions of $\lfloor \frac{c_5 n}{A^{d-1}} \rfloor$ boxes.

Notice that $\#\mathcal{U}$, the number of choices of $\lfloor \frac{c_5 n}{A^{d-1}} \rfloor$ boxes from the total number of $\lceil \frac{n^{d/(d-1)}}{A^d} \rceil$ boxes, admits the bound

$$\begin{aligned} \#\mathcal{U} &\le \exp\left(\frac{c_5 n}{A^{d-1}} \log \left\lceil \frac{n^{d/(d-1)}}{A^d} \right\rceil\right) \\ &= \exp\left(\frac{c_5 dn}{A^{d-1}(d-1)} \log n(1+o(1))\right). \end{aligned} \tag{22}$$

For every $U \in \mathcal{U}$ we have the bound

$$\mathrm{vol}_d(U) \le \frac{c_5 n}{A^{d-1}} \cdot \left(\frac{A}{n^{1/(d-1)}}\right)^d = \frac{Ac_5}{n^{1/(d-1)}}. \tag{23}$$

Step 4. Probabilistic estimates.
Recall that

$$K = \#J_0 + \#\left(\bigcup_{m=1}^{d} (J_d^+ \cup J_d^-)\right) =: K^{(0)} + K^{(\pm)}.$$

Notice that $K^{(\pm)}$ centres simultaneously belong to some random $U \in \mathcal{U}$, which can be written as

$$N_U := \#\{i : \xi_i \in U\} \ge K^{(\pm)}.$$

Recall that $K^{(0)} \le c_1 r^{-d} =: c_6$. Therefore, we obtain

$$\begin{aligned} \mathbb{P}[K \ge n] &\le \mathbb{P}[K^{(\pm)} \ge n - c_6] \\ &\le \sum_{U\in\mathcal{U}} \mathbb{P}[N_U \ge n - c_6] \\ &\le \#\mathcal{U} \cdot \max_{U\in\mathcal{U}} \mathbb{P}[N_U \ge n - c_6]. \end{aligned}$$

Since for every deterministic U the random variable N_U is a Poissonian one with expectation $\lambda \mathrm{vol}_d(U)$, by using (22), (23) we have

$$\mathbb{P}[K \ge n] \le \exp\left(\frac{c_5 dn}{A^{d-1}(d-1)} \log n(1+o(1))\right) \left(\frac{\lambda \mathrm{vol}_d(U) e}{n - c_6}\right)^{n-c_6}$$

$$\leq \exp\left(\frac{c_5 dn}{A^{d-1}(d-1)}\log n(1+o(1))\right)\left(\frac{(\lambda A c_5\, e)\, n^{-1/(d-1)}}{n-c_6}\right)^{n-c_6}$$
$$\leq \exp\left(\left(\frac{c_5 d}{A^{d-1}(d-1)} - \left(1+\frac{1}{d-1}\right)\right) n\log n(1+o(1))\right).$$

Since A can be chosen arbitrarily large, we get (17). □

3.2.2. Constant radius: ℓ_2-norm

The following result corresponds to the lower bound from Proposition 9 but does not exactly match it.

Proposition 21. *Assume that the radius is a.s. constant $R_1 \equiv r < 1$ and we work with ℓ_2-balls. If $d \geq 2$, then*

$$\mathbb{P}[K \geq n] \leq \exp\left(-\left(1+\frac{1}{d-1}\right) n\log n(1+o(1))\right), \qquad \text{as } n \to \infty.$$

Proof. Steps 1 and 4 of the proof are exactly the same as in the proof of Proposition 19. Step 3 is almost identical, up to an appropriate modification of the constant c_4. We do have to modify Step 2, where the particular form of the norm is used. The following is an appropriate modification of Lemma 20.

Lemma 22. *Let $i, j \in J_d^+$, $i \neq j$, set $c_3 := c_2\, r/2$, $c_3' := (2r+\sqrt{d-1})/c_3$. Then*

$$|\theta_i^{(d)} - \theta_j^{(d)}| \notin [\, c_3' \|\sigma\theta_i - \sigma\theta_j\|_2, c_3\,]. \tag{24}$$

Proof of the Lemma. Without loss of generality we may and do assume that $\theta_i^{(d)} \geq \theta_j^{(d)}$. Introduce the same auxiliary point $\psi := (\theta_i^{(1)}, \ldots, \theta_i^{(d-1)}, \theta_j^{(d)})$ as before. We have

$$\begin{aligned} \|\nu_j - \psi\|_2 &\leq \|\nu_j - \theta_j\|_2 + \|\theta_j - \psi\|_2 \\ &= \|\nu_j - \theta_j\|_2 + \|\sigma\theta_j - \sigma\theta_i\|_2 \\ &\leq r + \|\sigma\theta_j - \sigma\theta_i\|_2. \end{aligned} \tag{25}$$

Assume temporarily that

$$\theta_i^{(d)} \in [\,\theta_j^{(d)} + c_3'\|\sigma\theta_i - \sigma\theta_j\|_2, \theta_j^{(d)} + c_3\,]. \tag{26}$$

Since $j \in J_d^+$, (26) yields

$$\nu_j^{(d)} \geq \theta_j^{(d)} + c_2 r/2 = \theta_j^{(d)} + c_3 \geq \theta_i^{(d)}. \tag{27}$$

On the other hand,

$$\|\nu_j - \psi\|_2^2 = \|\sigma\nu_j - \sigma\psi\|_2^2 + (\nu_j^{(d)} - \psi^{(d)})^2 = \|\sigma\nu_j - \sigma\theta_i\|_2^2 + (\nu_j^{(d)} - \theta_j^{(d)})^2. \tag{28}$$

Therefore, by using (28), (25), (26), (27) and the definition of c_3', we obtain

$$\begin{aligned} \|\nu_j - \theta_i\|_2^2 &= \|\sigma\nu_j - \sigma\theta_i\|_2^2 + (\nu_j^{(d)} - \theta_i^{(d)})^2 \\ &= \|\nu_j - \psi\|_2^2 - (\nu_j^{(d)} - \theta_j^{(d)})^2 + (\nu_j^{(d)} - \theta_i^{(d)})^2 \\ &= \|\nu_j - \psi\|_2^2 - (\theta_i^{(d)} - \theta_j^{(d)})(2\nu_j^{(d)} - \theta_i^{(d)} - \theta_j^{(d)}) \\ &\leq \left(r + \|\sigma\theta_j - \sigma\theta_i\|_2\right)^2 - (\theta_i^{(d)} - \theta_j^{(d)})(\nu_j^{(d)} - \theta_j^{(d)}) \\ &\leq r^2 + (2r+\sqrt{d-1})\|\sigma\theta_j - \sigma\theta_i\|_2 - c_3'\|\sigma\theta_i - \sigma\theta_j\|_2 \cdot c_3 \\ &= r^2, \end{aligned}$$

which contradicts the assumption $\nu_j \notin B(\theta_i, r)$ from the definition of ν_j. It follows that (26) does not hold, hence, the assertion of Lemma 22 is true. □

The rest of the proof of Proposition 21 goes exactly as in the ℓ_1-norm case. □

4. Dimension $d = 1$

Proof of Theorem 1. The lower bound follows from Proposition 12 which is valid for any dimension. The upper bound is based on the following elementary lemma.

Lemma 23. *Let*

$$S = \bigcup_{i=1}^{K} [x_i - r_i, x_i + r_i] \cap [0, 1] \tag{29}$$

be an irreducible representation of a one-dimensional picture S. Then

$$\sum_{i=1}^{K} \min\{r_i, 1\} \le 2. \tag{30}$$

Proof of Lemma 23. We notice first that any point $x \in [0, 1]$ is covered by at most two intervals $[x_i - r_i, x_i + r_i]$. The corresponding indices are those where $\min_{i:x\in[x_i-r_i,x_i+r_i]}(x_i - r_i)$ and $\max_{i:x\in[x_i-r_i,x_i+r_i]}(x_i + r_i)$ are attained. Would there be another interval covering x, it would be covered by those two we have chosen which contradicts to irreducibility of (29).

Using that for any $x \in [0, 1]$, $r > 0$

$$\mathrm{vol}_1 ([x - r, x + r] \cap [0, 1]) \ge \min\{r, 1\},$$

we have by integration

$$\begin{aligned}
\sum_{i=1}^{K} \min\{r_i, 1\} &\le \sum_{i=1}^{K} \mathrm{vol}_1 ([x_i - r_i, x_i + r_i] \cap [0, 1]) \\
&= \sum_{i=1}^{K} \int_0^1 \mathbf{1}_{\{x\in[x_i-r_i,x_i+r_i]\}} dx \\
&= \int_0^1 \left(\sum_{i=1}^{K} \mathbf{1}_{\{x\in[x_i-r_i,x_i+r_i]\}} \right) dx \le \int_0^1 2\, dx = 2,
\end{aligned}$$

and (30) follows. □

We derive now the upper bound in (1). Fix a large $M > 0$ and observe the identity

$$K = \#\{i : r_i > 2M/n\} + \#\{i : r_i \le 2M/n\} =: K_1 + K_2.$$

The bound (30) ensures that $K_1 \le n/M$ for $n \ge 2M$, while K_2 is bounded by the total number of balls of radius less or equal to $2M/n$ in the *initial* representation of the picture S. The latter is a Poissonian random variable with expectation $a := \lambda \int_0^{2M/n} p(z)dz \le c\,(M/n)^{\alpha}$. Therefore,

$$\begin{aligned}
\mathbb{P}[K \ge n] &\le \mathbb{P}[K_2 \ge (1 - 1/M)n] \\
&\le e^{-a} \frac{a^{(1-1/M)n}}{\lfloor (1 - 1/M)n \rfloor !} (1 + o(1)) \\
&\sim \frac{a^{(1-1/M)n} e^{(1-1/M)n}}{\sqrt{2\pi(1 - 1/M)n}[(1 - 1/M)n]^{(1-1/M)n}} \\
&\le \frac{[c\,(M/n)^{\alpha}]^{(1-1/M)n} e^{(1-1/M)n}}{[(1 - 1/M)n]^{(1-1/M)n}} \\
&= \exp(-(1 + \alpha)(1 - 1/M)n \log n\,(1 + o(1))).
\end{aligned}$$

Letting $M \to \infty$ proves the required upper bound. □

Remark 24. If $\mathbb{P}[R_i \ge r] = 1$ for some $r > 0$, then it follows from (30) that $\mathbb{P}[K \le \frac{2}{\min\{r,1\}}] = 1$. This means that if the radii are separated from zero, the large deviations for K are impossible.

5. The coding problem

5.1. *Proof of* Theorem 4

Notice that m in the theorem is the maximal possible number of disjoint intervals that compose our random set S.

Proof. Let us start with the case $c < 1/2$.

Upper bound. Let $\varepsilon \in (0,1)$ and let $G_\varepsilon := \{\varepsilon, 2\varepsilon, \ldots, \lfloor \varepsilon^{-1} \rfloor \varepsilon\}$ be the corresponding grid. We produce a dictionary

$$\mathcal{C} := \left\{ \bigcup_{j=1}^{k} [a_j, b_j],\ 1 \le k \le m, a_j \le b_j, a_j, b_j \in G_\varepsilon \right\} \cup \{\emptyset\}.$$

If our random set S is not empty, it is a union of k disjoint intervals with $1 \le k \le m$ and we always have a set $C \in \mathcal{C}$ such that $d_H(S, C) \le \varepsilon$.

On the other hand, we have

$$\#\mathcal{C} \le \sum_{k=1}^{m} \lfloor \varepsilon^{-1} \rfloor^{2k} \le m\, \varepsilon^{-2m}.$$

For given large r, we choose ε from equation $m\,\varepsilon^{-2m} = \exp(r)$, i.e. $\varepsilon := m^{1/2m} e^{-r/2m}$. We conclude with the required bound

$$D^{(q)}(r) \le m^{1/2m} e^{-r/2m} := c_2\, e^{-r/2m}.$$

Lower bound. By the definition of m there exists a sufficiently small δ such that $\delta < c$ and $m(2c+4\delta) < 1$. Therefore, we may place m disjoint intervals $[z_k - (c+2\delta), z_k + (c+2\delta)]$, $1 \le k \le m$, into $[0, 1]$. In the following, we will consider the case when S is a union of pairwise overlapping $2m$ intervals of length $2c$ with centres $\xi_{2k-1} \in [z_k - \delta, z_k]$ and $\xi_{2k} \in [z_k, z_k + \delta]$, for $1 \le k \le m$. We have

$$S = \bigcup_{k=1}^{m} [a_{2k-1}, a_{2k}],$$

where

$$\begin{aligned} a_{2k-1} &= \xi_{2k-1} - c \in [z_k - c - \delta, z_k - c], \\ a_{2k} &= \xi_{2k} + c \in [z_k + c, z_k + c + \delta]. \end{aligned}$$

Moreover, the random points a_k, $1 \le k \le 2m$, are independently and uniformly distributed on the corresponding intervals. We will denote $a := (a_k)_{1 \le k \le 2m}$ the corresponding $2m$-dimensional random vector uniformly distributed on a cube of side length δ.

Let us now fix a non-random closed set $C \subset [0, 1]$ and consider two cases.

(a) Assume first that all sets $C \cap [z_k - (c+2\delta), z_k + (c+2\delta)]$ are non-empty. Then we may define a $2m$-dimensional deterministic vector $b = b(C) := (b_k)_{1 \le k \le 2m}$ by

$$\begin{aligned} b_{2k-1} &:= \min\{x \big| x \in C \cap [z_k - (c+2\delta), z_k + (c+2\delta)]\}, \\ b_{2k} &:= \max\{x \big| x \in C \cap [z_k - (c+2\delta), z_k + (c+2\delta)]\}. \end{aligned}$$

The main observation is as follows: we have

$$d_H(S, C) \ge \min\{\|a - b\|_\infty; \delta\} \ge \delta \cdot \|a - b\|_\infty,$$

where the minimum with δ appears because of possible points in C outside each interval $[z_k - (c - 2\delta), z_k + (c + 2\delta)]$.

(b) If for some $k \le m$ the set $C \cap [z_k - (c+2\delta), z_k + (c+2\delta)]$ is empty, we simply have

$$d_H(S, C) \ge \delta$$

and choosing $b = b(C)$ arbitrarily in the unit cube we still have

$$d_H(S, C) \geq \delta \geq \delta \cdot \|a - b\|_\infty.$$

We conclude that inequality $d_H(S, C) \geq \delta \cdot \|a - b\|_\infty$ holds in all cases. It follows that for any dictionary $\mathcal{C}$ we have

$$\mathbb{E} \min_{C \in \mathcal{C}} d_H(S, C) \geq \delta \, \mathbb{E} \min_{C \in \mathcal{C}} \|a - b(C)\|_\infty.$$

By using the well known bound for the quantization error of finite-dimensional vectors uniformly distributed on cubes (see e.g. Lemma 22 in [2] or [11]), it follows immediately that

$$D^{(q)}(r) \geq \delta \, D^{(q)}(a, \|.\|_\infty; r) \geq c_1 \, e^{-r/2m},$$

where c_1 depends on m and on δ.

The case $c \geq 1/2$. For this case, the result is the same as in (2) with $2m$ replaced by 1. The reason is that in this case S consists of a unique interval with only *one* random end. □

5.2. Proof of the upper bound in Theorem 7

We shall use the following general result on the Hausdorff distance of balls.

Lemma 25. *For any norm $\|.\|$ on $\mathbb{R}^d$, the respective balls $B(\cdot, \cdot)$ and the respective Hausdorff distance, we have for all centres $x, x' \in \mathbb{R}^d$ and radii $r, r' > 0$*

$$d_H(B(x, r), B(x', r')) = \|x - x'\| + |r - r'|. \tag{31}$$

Proof of Lemma 25. The upper bound follows from the triangle inequality in Hausdorff distance,

$$\begin{aligned} d_H(B(x, r), B(x', r')) &\leq d_H(B(x, r), B(x', r)) + d_H(B(x', r), B(x', r')) \\ &\leq \|x - x'\| + |r - r'|. \end{aligned}$$

The lower bound follows from the triangle inequality in $\mathbb{R}^d$. Let $v \in B(x, r)$ be an arbitrary point and choose $v' \in B(x', r')$ such that x, x', v' lay on the same line, $\|x' - v'\| = r'$, and x' is situated between x and v'. Then

$$\begin{aligned} r + \|v - v'\| &\geq \|x - v\| + \|v - v'\| \geq \|x - v'\| \\ &= \|x - x'\| + \|x' - v'\| = \|x - x'\| + r', \end{aligned}$$

which implies $\|v - v'\| \geq \|x - x'\| + r' - r$. Therefore,

$$d_H(B(x, r), B(x', r')) \geq \inf_{v \in B(x,r)} \|v - v'\| \geq \|x - x'\| + r' - r.$$

By the full symmetry, we obtain

$$d_H(B(x, r), B(x', r')) \geq \|x - x'\| + |r' - r|,$$

as required. □

The upper bound follows from the next general lemma, which relates an upper bound for the asymptotics of $\mathbb{P}[K \geq n]$ for $n \to \infty$ to the upper bound for the quantization error.

Lemma 26. *Assume that for some $a > 0$*

$$\mathbb{P}[K \geq n] \leq \exp(-a \cdot n \log n(1 + o(1))), \qquad \text{as } n \to \infty.$$

Then

$$D^{(q)}(r) \leq \exp\left(-\sqrt{\frac{2a}{d+1} \, r \log r} \cdot (1 + o(1))\right), \qquad \text{as } r \to \infty. \tag{32}$$

Moreover, if the radius is constant, then

$$D^{(q)}(r) \le \exp\left(-\sqrt{\frac{2a}{d}\, r\log r}\cdot(1+o(1))\right), \qquad \text{as } r\to\infty. \tag{33}$$

Proof of Lemma 26. The proof relies on the following coding strategy. Recall that K is the number of balls needed in order to produce the random picture S *without any error*. We shall encode the positions and radii of these K balls *approximately* and thus retrieve the picture S approximately. In particular, if $K=k$, then our random picture admits a representation

$$S=\bigcup_{i=1}^{k} B(\widetilde{\xi}_i,\widetilde{R}_i)$$

and one has to encode a vector in $[0,1]^{dk}\times\mathbb{R}_+^k$ (the centres $\widetilde{\xi}_i$, $i=1,\dots,k$ and the radii $\widetilde{R}_i$, $i=1,\dots,k$). It is clear that one does not have to encode radii that are larger than the diameter $\mathfrak{D}$ of the unit cube $[0,1]^d$, as in that case the picture is trivial, i.e. $S=[0,1]^d$.

Note that Lemma 22 in [2] (also see [11]) gives an explicit bound on the quantization error in $[0,1]^\ell$ with respect to ℓ_∞-norm, namely, for any random element $X\in[0,1]^\ell$ and any $\rho\ge 0$ there exists a dictionary $\mathcal{C}(\ell,\rho)$ such that $\#\mathcal{C}(\ell,\rho)\le e^\rho$ and

$$\mathbb{E}\min_{x\in\mathcal{C}(\ell,\rho)}\|X-x\|_\infty \le e^{-\rho/\ell}. \tag{34}$$

Fix k with $1\le k\le r$. Using (34) with $\ell=(d+1)k$, $\rho:=r-k$, we can choose sets $\mathcal{C}_k=\{(p_i^j,r_i^j)_{i=1,\dots,k}, j=1,\dots,\#\mathcal{C}_k\}$ with $\#\mathcal{C}_k\le e^{r-k}$ such that

$$\mathbb{E}\left[\min_j\|(\widetilde{\xi}_i,\widetilde{R}_i/\mathfrak{D})_{i=1}^k-(p_i^j,r_i^j/\mathfrak{D})_{i=1}^k\|_\infty\Big|K=k\right]\le e^{-(r-k)/(k(d+1))}. \tag{35}$$

Furthermore, we build a sub-dictionary of pictures,

$$\mathcal{C}_k^\circ=\left\{\bigcup_{i=1}^k B(p_i^j,r_i^j),\, j=1,\dots,\#\mathcal{C}_k\right\}.$$

Set $\mathcal{C}_0^\circ:=\{[0,1]^d\}$ and define the full dictionary by $\mathcal{C}^\circ:=\bigcup_{k=0}^r\mathcal{C}_k^\circ$. Then $\#\mathcal{C}^\circ\le 1+\sum_{k=1}^r e^{r-k}\le e^r$ for large enough r.

Furthermore, (31) in Lemma 25 yields an upper estimate of the same type for the unions of balls, namely,

$$d_H\left(\bigcup_{i=1}^k B(x_i,r_i),\bigcup_{i=1}^k B(x_i',r_i')\right)\le\max_{1\le i\le k}[\,\|x_i-x_i'\|+|r_i-r_i'|\,]$$

Applying this estimate to any element of $\mathcal{C}_k^\circ$ and to S, we obtain

$$\begin{aligned}
d_H\left(\bigcup_{i=1}^k B(p_i^j,r_i^j),S\right) &= d_H\left(\bigcup_{i=1}^k B(p_i^j,r_i^j),\bigcup_{i=1}^k B(\widetilde{\xi}_i,\widetilde{R}_i)\right)\\
&\le \max_{1\le i\le k}[\,\|\widetilde{\xi}_i-p_i^j\|+|\widetilde{R}_i-r_i^j|\,]\\
&\le \max_{1\le i\le k}\left[d\mathfrak{D}\,\|\widetilde{\xi}_i-p_i^j\|_\infty+\mathfrak{D}\left|\frac{\widetilde{R}_i}{\mathfrak{D}}-\frac{r_i^j}{\mathfrak{D}}\right|\right]\\
&\le (d+1)\mathfrak{D}\,\|(\widetilde{\xi}_i,\widetilde{R}_i/\mathfrak{D})_{i=1}^k-(p_i^j,r_i^j/\mathfrak{D})_{i=1}^k\|_\infty.
\end{aligned}$$

By (35) we obtain

$$\begin{aligned}
&\mathbb{E}\left[\min_{C\in\mathcal{C}_k^\circ} d_H(C,S)\Big|K=k\right]\\
&\le (d+1)\mathfrak{D}\,\mathbb{E}\left[\min_j \|(\xi_i,R_i/\mathfrak{D})_{i=1}^k-(p_i^j,r^j/\mathfrak{D})_{i=1}^k\|_\infty\Big|K=k\right]\\
&\le (d+1)\mathfrak{D}\cdot e^{-(r-k)/(k(d+1))}.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
D^{(q)}(r) &\le \mathbb{E}\min_{C\in\mathcal{C}^\circ} d_H(C,S)\\
&= \sum_{k=0}^{r}\mathbb{P}[K=k]\cdot\mathbb{E}\,[\min_{C\in\mathcal{C}^\circ} d_H(C,S)|K=k]\\
&\quad+\sum_{k=r+1}^{\infty}\mathbb{P}[K=k]\cdot\mathbb{E}\,[\min_{C\in\mathcal{C}^\circ} d_H(C,S)|K=k]\\
&\le \sum_{k=0}^{r}\mathbb{P}[K=k]\cdot\mathbb{E}\,[\min_{C\in\mathcal{C}_k^\circ} d_H(C,S)|K=k]+\sum_{k=r+1}^{\infty}\mathbb{P}[K=k]\cdot\mathfrak{D}\\
&\le (d+1)\mathfrak{D}\sum_{k=0}^{r}\mathbb{P}[K=k]e^{-(r-k)/(k(d+1))}\\
&\quad+\mathfrak{D}e^{1/(d+1)}\sum_{k=r+1}^{\infty}\mathbb{P}[K=k]\cdot e^{-r/(k(d+1))}\\
&= ((d+1)+1)e^{1/(d+1)}\mathfrak{D}\sum_{k=0}^{\infty}\mathbb{P}[K=k]e^{-r/(k(d+1))}\\
&= (d+2)e^{1/(d+1)}\mathfrak{D}\,\mathbb{E}\,e^{-r/(K(d+1))}.
\end{aligned}$$

By a standard Tauberian theorem (cf. [3], Theorem 4.12.9), the assumption $\mathbb{P}[K\ge k]\le\exp(-ak\log k(1+o(1)))$, as $k\to\infty$, gives the conclusion (32). Moreover, if the radius is constant, the same reasoning applies with $\ell=dk$ instead of $\ell=(d+1)k$ and we arrive at (33). □

5.3. *Proof of the lower bound in Theorem 7*

In the following, c is used to denote a constant not depending on r and n that may change at each occurrence.

Consider the following collection of boxes:

$$\prod_{m=1}^{d}\left[\frac{k_m}{(2n)^{1/d}}+\frac{1/4}{(2n)^{1/d}},\frac{k_m}{(2n)^{1/d}}+\frac{3/4}{(2n)^{1/d}}\right],\qquad k_m\in\{0,\dots,\lfloor(2n)^{1/d}\rfloor-1\}.$$

The number of boxes being of order $2n$, we may choose among them n distinct boxes, say $V_1,\dots,V_n$. In the sequel we will also need larger boxes $W_1,\dots,W_n$ from the collection

$$\prod_{m=1}^{d}\left[\frac{k_m}{(2n)^{1/d}},\frac{k_m+1}{(2n)^{1/d}}\right],\qquad k_m\in\{0,\dots,\lfloor(2n)^{1/d}\rfloor-1\}.$$

such that $W_i\supseteq V_i$ for each i.

Define the following event:

$$E:=\bigcup_{\pi\text{ permutation of }\{1,\dots,n\}} E_\pi,$$

where

$$E_\pi := \left\{N = n, \xi_i \in V_{\pi(i)}, i = 1, \ldots, n, R_i \in [c_1\theta n^{-1/d}, c_2\theta n^{-1/d}]\right\},$$

$c_2 := 2^{-3-1/d}$, $c_1 := c_2/2 > 0$, and $\theta > 0$ is chosen so that $\|x\| \geq \theta\|x\|_\infty$ for all $x \in \mathbb{R}^d$.

In view of the lower bound in the assumption on the density p, the probability of this event admits the following bound:

$$\begin{aligned}
\mathbb{P}[E] &= \frac{\lambda^n}{n!}e^{-\lambda} \cdot n! \cdot \left(\frac{1/2}{(2n)^{1/d}}\right)^{dn} \cdot \mathbb{P}[R_1 \in [c_1\theta n^{-1/d}, c_2\theta n^{-1/d}]]^n \\
&= (\lambda/2^{d+1})^n e^{-\lambda} n^{-n} \cdot \left(\int_{c_1\theta n^{-1/d}}^{c_2\theta n^{-1/d}} p(z)\mathrm{d}z\right)^n \\
&\geq (\lambda/2^{d+1})^n e^{-\lambda} n^{-n} \cdot \left(c\, n^{-\alpha/d}\right)^n \\
&= \exp(-(1+\alpha/d)n\log n \cdot (1+o(1))).
\end{aligned} \tag{36}$$

Consider

$$\begin{aligned}
D^{(q)}(r) &= \inf_{\#\mathcal{C}\leq e^r} \mathbb{E}\,[\min_{C\in\mathcal{C}} d_H(C,S)] \\
&\geq \inf_{\#\mathcal{C}\leq e^r} \mathbb{E}\,[\min_{C\in\mathcal{C}} d_H(C,S)\mathbb{1}_E] \\
&= \inf_{\#\mathcal{C}\leq e^r} \mathbb{E}\,[\min_{C\in\mathcal{C}} d_H(C,S)|E] \cdot \mathbb{P}[E].
\end{aligned}$$

Further, denoting by $\mathcal{K}$ the set of all measurable subsets of $[0,1]^d$ note that for any dictionary $\mathcal{C}$ with $\#\mathcal{C} \leq e^r$ and any $\delta > 0$

$$\begin{aligned}
\mathbb{E}\,[\min_{C\in\mathcal{C}} d_H(C,S)|E] &\geq \delta \cdot \mathbb{P}[\forall C \in \mathcal{C} : d_H(C,S) \geq \delta|E] \\
&= \delta \cdot (1 - \mathbb{P}[\exists C \in \mathcal{C} : d_H(C,S) < \delta|E]) \\
&\geq \delta \cdot (1 - \#\mathcal{C} \cdot \sup_{C\in\mathcal{K}} \mathbb{P}[d_H(C,S) < \delta|E]). \\
&\geq \delta \cdot (1 - e^r \cdot \sup_{C\in\mathcal{K}} \mathbb{P}[d_H(C,S) < \delta|E]) \\
&\geq \delta \cdot (1 - e^r \cdot \sup_{C\in\mathcal{K}} \sup_{\pi} \mathbb{P}[d_H(C,S) < \delta|E_\pi]).
\end{aligned}$$

Combining this bound with the last estimate yields

$$D^{(q)}(r) \geq \mathbb{P}[E] \cdot \delta \cdot (1 - e^r \sup_{C\in\mathcal{K}} \sup_{\pi} \mathbb{P}[d_H(C,S) < \delta|E_\pi]). \tag{37}$$

Now we estimate $\mathbb{P}[d_H(C,S) < \delta|E_\pi]$ for a fixed set C, fixed permutation π and

$$\delta < \frac{\theta}{8(2n)^{1/d}}. \tag{38}$$

We first show that under E_π for each $i \leq n$ one has

$$B(\xi_i, R_i + \delta) \subset W_{\pi(i)}. \tag{39}$$

Indeed, if $x \in B(\xi_i, R_i + \delta)$, then

$$\|x - \xi_i\|_\infty \leq \theta^{-1}\|x - \xi_i\| \leq \theta^{-1}(R_i + \delta) \leq c_2 n^{-1/d} + \delta/\theta \leq 2^{-2}(2n)^{1/d}.$$

Since $\xi_i \in V_{\pi(i)}$, we obtain $x \in W_{\pi(i)}$ and (39) follows.

We see from (39) that all balls in the representation

$$S = \bigcup_{i=1}^n B(\xi_i, R_i)$$

are not only disjoint but δ-separated. Therefore, if $d_H(C, S) < \delta$, then $C \cap W_{\pi(i)} \neq \emptyset$ for all i and so

$$d_H(C \cap W_{\pi(i)}, S \cap W_{\pi(i)}) = d_H(C \cap W_{\pi(i)}, B(\xi_i, R_i)) < \delta.$$

Since C is deterministic, when π is fixed there exists a deterministic ball $B(x_i, r_i)$ such that $d_H(C \cap W_{\pi(i)}, B(x_i, r_i)) < \delta$.

Indeed, let U (here $U = C \cap W_{\pi(i)}$ for short) be a deterministic set such that $\mathbb{P}[d_H(U, B(x(\omega), r(\omega))) < \delta] > 0$. Take a countable set of balls $(B(x_k, r_k))_{k\in\mathbb{N}}$ which is d_H-dense in the set of all balls. We clearly have

$$\begin{aligned}\sum_{k\in\mathbb{N}} \mathbb{P}[d_H(U, B(x_k, r_k)) < \delta] &\geq \mathbb{P}(\inf_k d_H(U, B(x_k, r_k)) < \delta)\\ &\geq \mathbb{P}[d_H(U, B(x(\omega), r(\omega))) < \delta] > 0.\end{aligned}$$

Obviously, there exists some $k \in \mathbb{N}$ such that $\mathbb{P}[d_H(U, B(x_k, r_k)) < \delta] > 0$.

But both sets, U and $B(x_k, r_k)$, are deterministic. Therefore, we simply have $d_H(U, B(x_k, r_k)) < \delta$, as required.

Hence, $d_H(C, S) < \delta$ yields, by the triangle inequality,

$$\begin{aligned}&d_H(B(x_i, r_i), B(\xi_i, R_i))\\ &\quad \leq d_H(B(x_i, r_i), C \cap W_{\pi(i)}) + d_H(C \cap W_{\pi(i)}, B(\xi_i, R_i)) < 2\delta.\end{aligned}$$

The equality (31) yields now $\|\xi_i - x_i\| \leq 2\delta$ and $\|R_i - r_i\| \leq 2\delta$. Recall that x_i, r_i are deterministic and depend only on C and π.

Even after conditioning on E_π, the ensembles of centres $(\xi_i)_{1\leq i\leq n}$ and radii $(R_i)_{1\leq i\leq n}$ remain independent; while ξ_i is uniformly distributed on $V_\pi(i)$ and R_i is distributed on $[c_1\theta n^{-1/d}, c_2\theta n^{-1/d}]$ with a density proportional to p.

These observations show that

$$\begin{aligned}&\mathbb{P}\left(d_H(S, C) < \delta \middle| E_\pi\right)\\ &\leq \mathbb{P}\left(\|\xi_i - x_i\| \leq 2\delta, |R_i - r_i| \leq 2\delta, 1 \leq i \leq n \middle| E_\pi\right)\\ &= \prod_{i=1}^{n} \mathbb{P}\left(\|\xi_i - x_i\| \leq 2\delta \middle| E_\pi\right) \cdot \prod_{i=1}^{n} \mathbb{P}\left(|R_i - r_i| \leq 2\delta \middle| E_\pi\right).\end{aligned}$$

We clearly have

$$\mathbb{P}\left(\|\xi_i - x_i\| \leq 2\delta \middle| E_\pi\right) \leq \frac{\mathrm{vol}_d(B(0,1))(2\delta)^d}{\mathrm{vol}_d(V_1)} =: c\,\delta^d n.$$

Using the upper bound in the assumption on the density p we obtain

$$\mathbb{P}\left[|R_i - r_i| < 2\delta \middle| E_\pi\right] \leq c \int_{r_i-2\delta}^{r_i+2\delta} z^{\alpha-1} \mathrm{d}z (c\, n^{-\alpha/d})^{-1} \leq c\,\delta n^{1/d}.$$

Hence,

$$\mathbb{P}\left(d_H(S, C) < \delta \middle| E_\pi\right) \leq (c\,\delta^d n)^n \cdot (c\,\delta n^{1/d})^n =: c^n\, \delta^{(d+1)n}\, n^{(1+1/d)n}. \tag{40}$$

Putting estimates (36), (37), (40) together yields

$$D^{(q)}(r) \geq \exp(-(1+\alpha/d)n \log n(1+o(1))) \cdot \delta \cdot \left(1 - e^r c^n\, \delta^{(d+1)n}\, n^{(1+1/d)n}\right).$$

Now we choose δ such that

$$e^r c^n \delta^{(d+1)n} n^{(1+1/d)n} = 1/2$$

and obtain $\delta = c\, e^{-r/((d+1)n)} n^{-1/d}$, which gives

$$\begin{aligned}D^{(q)}(r) &\geq \exp\left(-(1+\alpha/d)n \log n(1+o(1)) - r/((d+1)n)\right)\\ &=: \exp\left(-An \log n(1+o(1)) - r/(Bn)\right).\end{aligned}$$

Now we optimize in n by letting $n \sim \sqrt{\frac{2r}{AB\log r}}$ and obtain

$$D^{(q)}(r) \geq \exp\left(-\sqrt{\frac{2Ar\log r}{B}}\,(1+o(1))\right)$$
$$= \exp\left(-\sqrt{\frac{2(1+\alpha/d)}{d+1}\,r\log r}\,(1+o(1))\right),$$

as required in the assertion of the theorem. It remains to notice that the choice of δ agrees with required property (38) for large n and r.

5.4. Proof of Theorem 6, ℓ_1-balls part

The upper bound follows from the claim (33) of Lemma 26 where we may let $a := \frac{d}{d-1}$ by Proposition 19.

For getting the lower bound we use the construction from the proof of Proposition 10 and the proof scheme of the lower bound in Theorem 7. We repeat everything for completeness.

Consider the following collection of boxes:

$$\left\{\prod_{m=1}^{d-1}\left[\frac{k_m}{(2n)^{1/(d-1)}}+\frac{1/4}{(2n)^{1/(d-1)}},\frac{k_m}{(2n)^{1/(d-1)}}+\frac{3/4}{(2n)^{1/(d-1)}}\right]\right\}\times\left[0,\frac{c_2}{n^{1/(d-1)}}\right],$$

and the larger tubes

$$\left\{\prod_{m=1}^{d-1}\left[\frac{k_m}{(2n)^{1/(d-1)}},\frac{k_m+1}{(2n)^{1/(d-1)}}\right]\right\}\times[0,1],$$

with $k_m \in \{0,\ldots,\lfloor(2n)^{1/(d-1)}\rfloor-1\}$. Here, $c_2 := 2^{-(4+1/(d-1))}$.

The number of boxes being of order $2n$, we may choose among them n distinct boxes, say $V_1,\ldots,V_n$ and use the corresponding tubes $U_1,\ldots,U_n$ such that $V_i \subset U_i$, $i=1,\ldots,n$.

As before, we consider the event E,

$$E := \bigcup_{\pi \text{ permutation of } \{1,\ldots,n\}} E_\pi,$$

where

$$E_\pi := \left\{N=n,\ \xi_i\in V_{\pi(i)},\ i=1,\ldots,n\right\},$$

and recall from (3) the bound

$$\mathbb{P}[E] \geq \exp\left(-\frac{d}{d-1}\,n\log n(1+o(1))\right). \tag{41}$$

We will use inequality (37) with this E and these E_π. Note that its derivation does not depend on the concrete event E, but it holds for any event.

As in the previous proof, we have to estimate $\mathbb{P}[d_H(C,S)<\delta|E_\pi]$ for a fixed set C, fixed permutation π and small δ, however using very different geometric arguments.

Recall that notation c is used for a constant not depending on r or n that may change at each occurrence. Instead of (40), we will prove

$$\mathbb{P}\left(d_H(S,C)<\delta\,\middle|\,E_\pi\right) \leq \left(c\,\delta^d n^{d/(d-1)}\right)^n. \tag{42}$$

Putting estimates (37), (41), (42) together yields

$$D^{(q)}(r) \geq \exp\left(-\frac{d}{d-1}\,n\log n(1+o(1))\right)\cdot\delta\cdot\left(1-e^r\left(c\,\delta^d n^{d/(d-1)}\right)^n\right).$$

Now we choose δ such that

$$e^r \left(c\,\delta^d n^{d/(d-1)}\right)^n = 1/2$$

and obtain $\delta = c\,2^{-1/dn}\,e^{-r/(dn)} n^{-1/(d-1)}$, which gives

$$\begin{aligned} D^{(q)}(r) &\geq \exp\left(-\frac{d}{d-1}\,n\log n(1+o(1)) - r/(dn)\right) \\ &=: \exp\left(-An\log n(1+o(1)) - r/(Bn)\right) \end{aligned}$$

with $A := \frac{d}{d-1}$ and $B = d$. We optimize in n as before, by letting $n \sim \sqrt{\frac{2r}{AB\log r}}$ and obtain

$$\begin{aligned} D^{(q)}(r) &\geq \exp\left(-\sqrt{\frac{2Ar\log r}{B}}\,(1+o(1))\right) \\ &= \exp\left(-\sqrt{\frac{2}{d-1}\,r\log r\,(1+o(1))}\right), \end{aligned}$$

as required in the assertion of the theorem.

It remains to prove (42). To this aim, we fix a deterministic set C and a permutation π. Assume that

$$d_H(S, C) < \delta \tag{43}$$

with a small δ such that

$$\delta < \frac{\theta}{2^6 (2n)^{1/(d-1)}}. \tag{44}$$

For every $i \leq n$ we have the following. Let

$$y_i := \operatorname{argmax}\{y^{(d)} | y \in C \cap U_{\pi(i)}\}$$

be a local top point of C and let $x_i := \xi_i + (0, \ldots, 0, R)$ be the top point of the ball $B(\xi_i, R)$. We will show that y_i and x_i are close.

First, we prove that

$$y_i^{(d)} \geq x_i^{(d)} - \delta. \tag{45}$$

Indeed, by (43) there exists $y \in C$ such that $\|y - x_i\|_1 \leq \delta$. Using $\xi_i \in V_{\pi(i)}$ and the inequality $\delta \leq \frac{1/4}{(2n)^{1/(d-1)}}$, we see that $y \in U_{\pi(i)}$. Hence,

$$y_i^{(d)} \geq y^{(d)} \geq x_i^{(d)} - \|y - x_i\|_1 \geq x_i^{(d)} - \delta.$$

Second, for any $b \in B(\xi_i, R)$ it is true that

$$\begin{aligned} R - \|x_i - b\|_1 &\geq \|b - \xi_i\|_1 - \|x_i - b\|_1 = |b^{(d)} - \xi_i^{(d)}| - (x_i^{(d)} - b^{(d)}) \\ &\geq (b^{(d)} - \xi_i^{(d)}) - (x_i^{(d)} - b^{(d)}) = 2(b^{(d)} - x_i^{(d)}) + R, \end{aligned}$$

hence,

$$b^{(d)} \leq x_i^{(d)} - \|x_i - b\|_1/2. \tag{46}$$

Third, by (43) there exists a $b_i \in S$ such that $\|b_i - y_i\|_1 \leq \delta$. In particular,

$$y_i^{(d)} \leq b_i^{(d)} + \delta. \tag{47}$$

Moreover, it is true that $b_i \in B(\xi_i, R)$. Indeed, assume that $b_i \in B(\xi_j, R)$ for some $j \neq i$. Then

$$b_i^{(d)} \leq x_j^{(d)} - \|x_j - b_i\|_1/2 \leq x_i^{(d)} + \frac{c_2}{n^{1/(d-1)}} - (\|x_j - y_i\|_1 - \delta)/2$$

$$\leq x_i^{(d)} + \frac{c_2}{n^{1/(d-1)}} - \frac{1}{8(2n)^{1/(d-1)}} + \delta/2$$
$$= x_i^{(d)} - \frac{1}{2^4(2n)^{1/(d-1)}} + \delta/2 < x_i^{(d)} - 2\delta.$$

Here we used inequality (46) with $b = b_i$ and with j instead of i, the definition of c_2, and the bound (44) for δ. The result contradicts

$$b_i^{(d)} \geq y_i^{(d)} - \delta \geq x_i^{(d)} - 2\delta$$

and we see that $b_i \in B(\xi_j, R)$, for $j \neq i$, is impossible.

Fourth, by applying (46) to $b = b_i$ and combining with (47), it follows that

$$y_i^{(d)} \leq x_i^{(d)} - \|x_i - b_i\|_1/2 + \delta.$$

By comparing this inequality with (45) we obtain $\|x_i - b_i\|_1 \leq 4\delta$, and so by the definition of b_i we get $\|x_i - y_i\|_1 \leq 5\delta$. The latter is equivalent to $\|\xi_i - z_i\|_1 \leq 5\delta$, where a deterministic point z_i is defined by $z_i := y_i - (0, \ldots, 0, R)$.

These observations show that

$$\mathbb{P}\left(d_H(S, C) < \delta \middle| E_\pi\right) \leq \mathbb{P}\left(\|\xi_i - z_i\|_1 \leq 5\delta, 1 \leq i \leq n \middle| E_\pi\right)$$
$$= \prod_{i=1}^{n} \mathbb{P}\left(\|\xi_i - z_i\|_1 \leq 5\delta \middle| E_\pi\right),$$

and using that ξ_i is uniformly distributed in $V_{\pi(i)}$ on E_π we get

$$\mathbb{P}\left(\|\xi_i - z_i\|_1 \leq 5\delta \middle| E_\pi\right) \leq \frac{\mathrm{vol}_d(B(0,1))(5\delta)^d}{\mathrm{vol}_d(V_1)} =: c\,\delta^d n^{d/(d-1)}.$$

Hence,

$$\mathbb{P}\left(d_H(S, C) < \delta \middle| E_\pi\right) \leq \left(c\,\delta^d\, n^{d/(d-1)}\right)^n,$$

as required in (42). □

5.5. *Proof of Theorem 6, ℓ_2-balls part*

This proof closely follows the previous one with two minor changes. For getting the upper bound, we refer to Proposition 21 instead of Proposition 19. For getting the lower bound we use the construction from the proof of Proposition 9 instead of Proposition 10.

Furthermore, the geometric properties of the ℓ_2-norm come into play. We must use inequality

$$b^{(d)} \leq x_i^{(d)} - \|x_i - b\|_2^2/(2R_1)$$

instead of (46). All other arguments go through exactly as before. □

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