TADI - Scale Space

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Contents

1 Introduction

In this practical we will implement 4 various diffusion numerical schemes seen during the lecture.

2 Exercise One - explicit scheme FTCS

Write a code implementing the 2-D forward and centered numerical scheme of the heat equation (see lecture, Eq. (19) slide 89, for a 1-D scheme). Space step is set to 1, time step is chosen by the user. Experiment the CFL condition.

Heat equation :

$$\frac{L(x,t)}{\partial t} = c\frac{\partial^2 L(x,t)}{\partial x^2}$$

With $c = \frac{1}{2}$

Heat equation FTCS 1D :

$$L_j^{n+1} = L_j^n + c\frac{\Delta t}{\Delta x^2}(L_{j+1}^n - 2L_j^n + L_{j-1}^n)$$

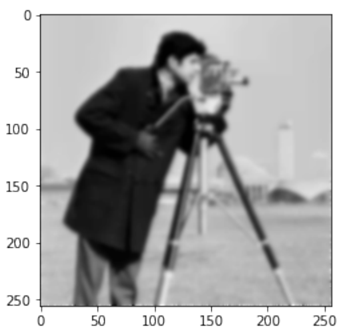
Heat equation FTCS 2D :

$$L_j^{n+1} = L_j^n + c\frac{\Delta t}{\Delta x^2}(\frac{\partial^2 L}{\partial x^2} + \frac{\partial^2 L}{\partial y^2}) = L_j^n + c\frac{}{\Delta x^2}(\frac{L_{j+1,i}^n - 2L_{j,i}^n + L_{j-1,i}^n}{\Delta x^2} + \frac{L_{j,i+1}^n - 2L_{j,i}^n + L_{j,i-1}^n}{\Delta y^2})$$

With Δx and Δy the space step and Δt the time step CLF condition : The scheme is stable (1D) if : $c\frac{\Delta t}{\Delta x^2} < \frac{1}{2}$ The scheme is stable (2D) if : $max(c\frac{\Delta t}{\Delta x^2}, c\frac{\Delta t}{\Delta y^2}) < \frac{1}{2}$

2.1 Observation

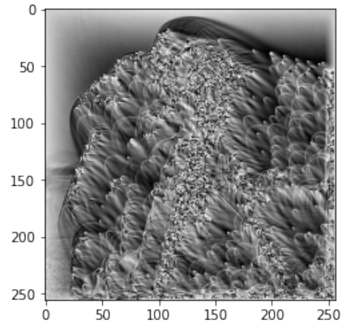
1. The more the smoothing advances, the more the details disappear from the images.
2. Everything is smoothed edges and regions.
3. We can experiement the CLF condition $c\Delta t < \frac{1}{2}$ because we have $\Delta x^2 = 1$ and $\Delta y^2 = 1$:
   - $c\Delta t = 0.15$



(a)

Figure 1: $c\Delta t = 0.15$

- $c\Delta t = 0.5$

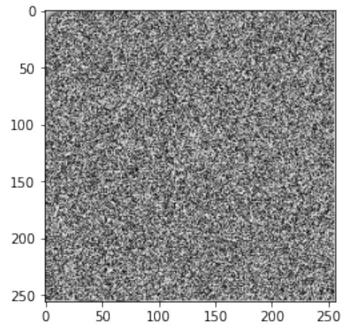


(a)

Figure 2: $c\Delta t = 0.5$

When applying a $c\Delta t = 0.5$ the loss of stability of the model is clear, confirming the CFL condition.

- $c\Delta t = 0.6$



(a)

Figure 3: $c\Delta t = 0.6$

When applying a $c\Delta t = 0.6$ the loss of stability of the model is clear, confirming the CFL condition.

3 Exercise Two - Perona-Malik

Write a code implementing the Perona-Malik scheme, such that given in lecture slides 111. You can use a linear interpolation and/or the simplification given in slide 112, and compare the results.

Isotropic diffusion

$$L_{i,j}^{k+1} = L_{i,j}^{k} + \Delta t[cn\nabla_n L + cs\nabla_s L + ce\nabla_e L + cw\nabla_w L]_{i,j}^{k}$$

With :

$[\nabla_n L]_{i,j}^k = L_{i-1,j}^k - L_{i,j}^k$

$[\nabla_s L]_{i,j}^k = L_{i+1,j}^k - L_{i,j}^k$

$[\nabla_e L]_{i,j}^k = L_{i,j+1}^k - L_{i,j}^k$

$[\nabla_w L]_{i,j}^k = L_{i,j-1}^k - L_{i,j}^k$

$[c_n]_{i,j}^k = g_{i,j}^k$

$[c_s]_{i,j}^k = g_{i+1,j}^k$

$[c_e]_{i,j}^k = g_{i,j+1}^k$

$[c_w]_{i,j}^k = g_{i,j}^k$

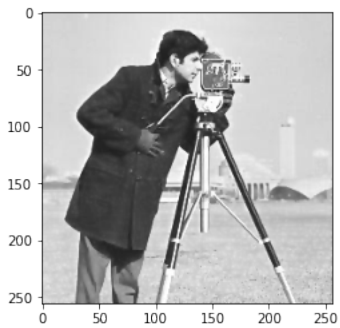
With $g(x,y,t)^k = g(||\nabla L^k(x,y,t)||)$

Tukey conductivity : $g = e^{-(\frac{x}{})^2}$ and Lorentz conductivity : $g = \frac{1}{1+(\frac{x}{})^{1+\alpha}}$

3.1 Observation

1. PM is an isotropic diffusion, the image is preserved over the edges but homogeneous region are smoothed.
2. We have to respect the CFL condition to be stable (with $iter = 10, alpha = 5, kappa = 20$):

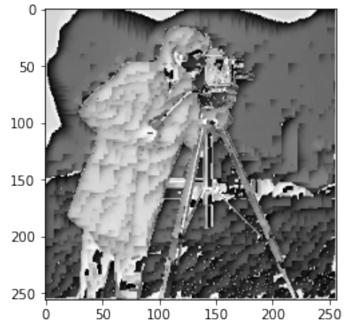
    - $c\Delta t = 0.15$



(a)

Figure 4: $c\Delta t = 0.15$

- $c\Delta t = 0.6$

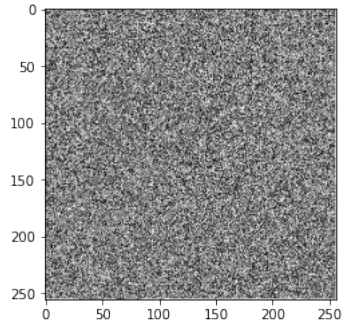


(a)

Figure 5: $c\Delta t = 0.6$

When applying a $c\Delta t = 5$ the loss of stability of the model is clear, confirming the CFL condition.

- $c\Delta t = 5$



(a)

Figure 6: $c\Delta t = 5$

When applying a $c\Delta t = 5$ the loss of stability of the model is clear, confirming the CFL condition.

4 Exercise Three

Write a code implementing the "Edge Enhancing" scheme such that described in slides 130 and 131.

Anisotropic diffusion:

$$L_{i,j}^{k+1} = L_{i,j}^{k} + \Delta t[-\frac{b_{i-1,j}+b_{i,j+1}}{4}L_{i-1,j+1}^{k} + \frac{c_{i,j+1}+c_{i,j}}{2}L_{i,j+1}^{k} + \frac{b_{i+1,j}+b_{i,j+1}}{4}L_{i+1,j+1}^{k} + \frac{a_{i-1,j}+a_{ij}}{2}L_{i-1,j}^{k} -$$

$$\frac{a_{i-1,j}+2a_{i,j}+a_{i+1,j}+c_{i,j-1}+2c_{i,j}+c_{i,j+1}}{2}L_{i,j}^{k} + \frac{a_{i+1,j}+a_{i,j}}{2}L_{i-1,j}^{k} + \frac{b_{i-1,j}+b_{i,j-1}}{4}L_{i-1,j}^{k} +$$

$$\frac{c_{i,j-1}+c_{i,j}}{2}L_{i,j-1}^{k} - \frac{b_{i+1,j}+b_{i,j-1}}{4}L_{i-1,j-1}^{k}]$$

With : $L^{\sigma} = L\frac{\partial G_{\sigma}}{\partial}$

Then : $\nabla^{\sigma} L = (L \star d/dx(g_{\sigma}), L \star d/dy(g_{\sigma}))$

With :

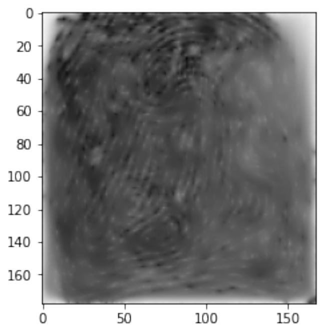
$a = \frac{(\lambda_1 (L_x^{\sigma})^2 + \lambda_2 (L_y^{\sigma})^2)}{||\nabla L^{\sigma}||^2}$

$b = \frac{(\lambda_1 - \lambda_2) L_x^{\sigma} L_y^{\sigma}}{||\nabla L^{\sigma}||^2}$

$c = \frac{(\lambda_2 (L_x^{\sigma})^2 + \lambda_1 (L_y^{\sigma})^2)}{||\nabla L^{\sigma}||^2}$

4.1 Observation

1. Introduction of a blur in the areas of the image, except for the contours which are well preserved.
2. The anisotropic diffusion in the situation of fingerprints for exemple can lead to compensation of the gradients. That can lead to a very blured image even in the edges.



(a) iter=100, dt=0.45, sigma=10

Figure 7: $c\Delta t = 5$

5 Exercise Four - Implicit scheme BTCS

Write the 2-D heat equation using an implicit scheme. Experiment the absence of CFL condition.

Implicit scheme BTCS

1D : $-\alpha L_{j+1}^{n+1} + (1+2\alpha)L_j^{n+1} - \alpha L_{j-1}^{n+1} = L_j^n$

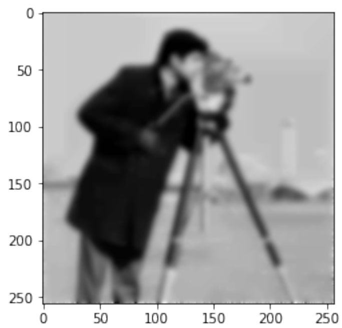
2D : $-\alpha(L_{j+1,i}^{n+1} + L_{j-1,i}^{n+1}) + (1+2\alpha+2\beta)L_{j,i}^{n+1} - \beta(L_{j,i+1}^{n+1} + L_{j,i-1}^{n+1}) = L_{j,i}^n \Leftrightarrow AL^{n+1} = L^n$

5.1 Observation

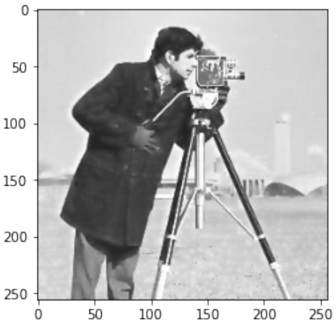
1. The variations of the Alpha parameter and their impact on the result demonstrate the unconditional stability of the model and thus the absence of a CFL condition.
2. The BTCS scheme is always stable but need a matrix inversion, so if the CFL condition is verified is better to use the explicite scheme FCTS else the implicit scheme BTCS.

6 Comparaison of the 4 methods

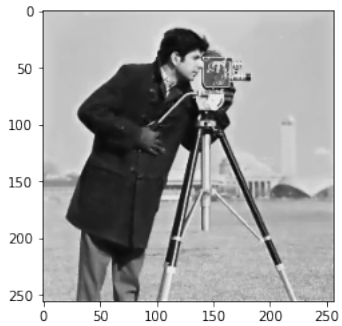
6.1 cameraman



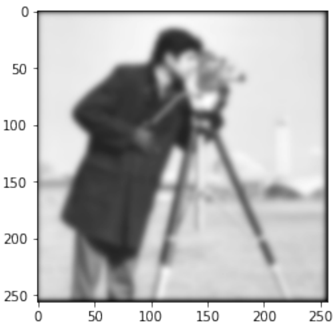
(a) FCTS - iter=20, dt=0.15



(b) P&M - iter=20, dt=0.15, alpha=5, kappa=20, conductivity=lorentz



(c) Edges enhancing - iter=20, dt=0.15, sigma=2



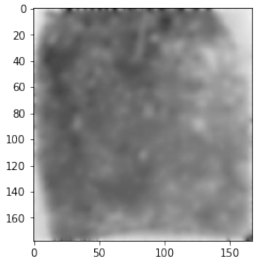
(d) BCTS - iter=20, alpha=0.2, beta=0.2

Figure 8: cameraman

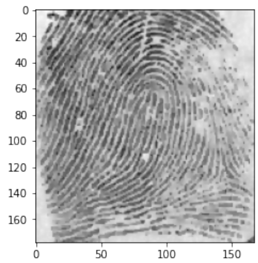
We can verify that we said before :

- FCTS blurs the image without distinction between the contours and the regions
- P&M allows to preserve the contours by blurring the homogeneous regions
- Edges enhancing also allows to preserve the contours by blurring the homogeneous regions but in an even more marked way than P&M.
- BCTS allows just like FCTS to blur the image but does not need parameter dt and is always stable.

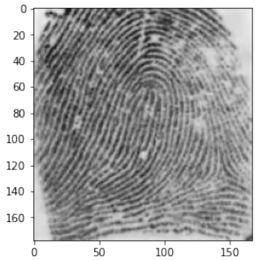
6.2 fingerprint-small



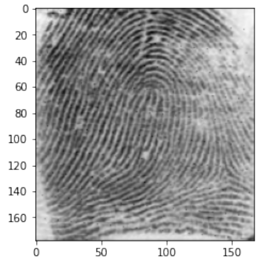
(a) FCTS - iter=20, dt=0.15



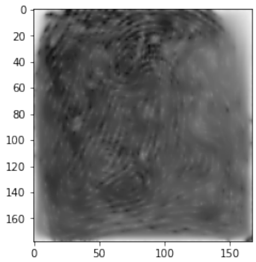
(b) P&M - iter=20, dt=0.15, alpha=5, kappa=20, conductivity=lorentz



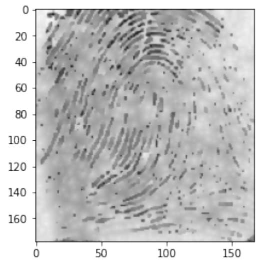
(c) Edges enhancing - iter=20, dt=0.15, sigma=2



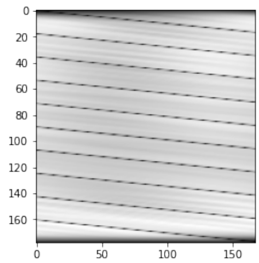
(d) BCTS - iter=20, alpha=0.01, beta=0.01



(e) Edges enhancing - iter=100, dt=0.15, sigma=2



(f) P&M - iter=100, dt=0.15, alpha=5, kappa=20, conductivity=lorentz

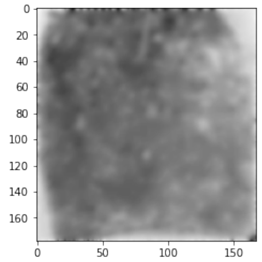


(g) BCTS - iter=20, alpha=0.5, beta=0.5

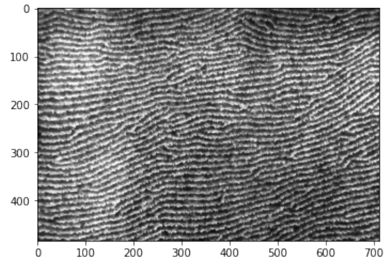
Figure 9: fingerprint

- We can see that FCTS don't preserved the edges at all.
- For P&M we don't need to have too much iterations, we can see very low blur level, indeed there are a lot of edges.
- Edges enhancing enable to preseve approximiativly the edges with low iterations, but when we increase the iteration this scheme can not work well on this type on images because of the direction of gradients that compense each others.
- For BCTS scheme we can see that if we increase too much alpha and beta the algorithm don't work well on this image.

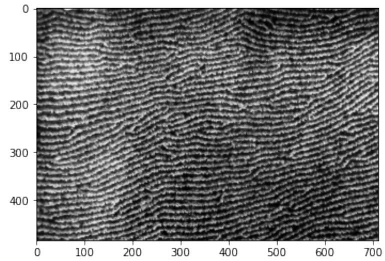
6.3 oyster-s1



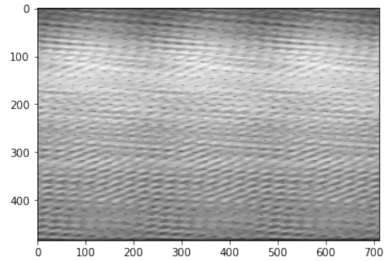
(a) FCTS - iter=20, dt=0.15



(b) P&M - iter=20, dt=0.15, alpha=5, kappa=20, conductivity=lorentz



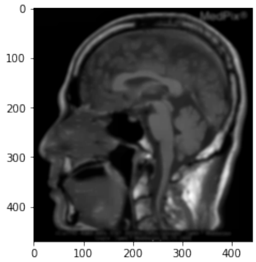
(c) Edges enhancing - iter=20, dt=0.15, sigma=2



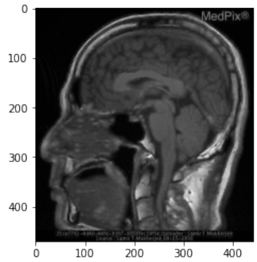
(d) BCTS - iter=20, alpha=0.5, beta=0.5

We have the same kind of result than with fingerprint.

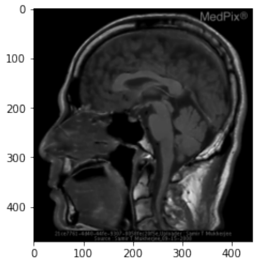
6.4 synpic45657-s1



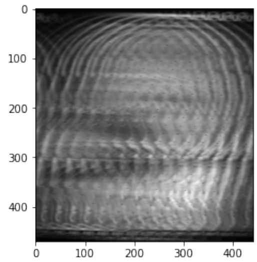
(a) FCTS - iter=20, dt=0.15



(b) P&M - iter=20, dt=0.15, alpha=5, kappa=20, conductivity=lorentz



(c) Edges enhancing - iter=20, dt=0.15, sigma=2



(d) BCTS - iter=20, alpha=0.1, beta=0.1