Model based detection of homogeneous portions in trajectories

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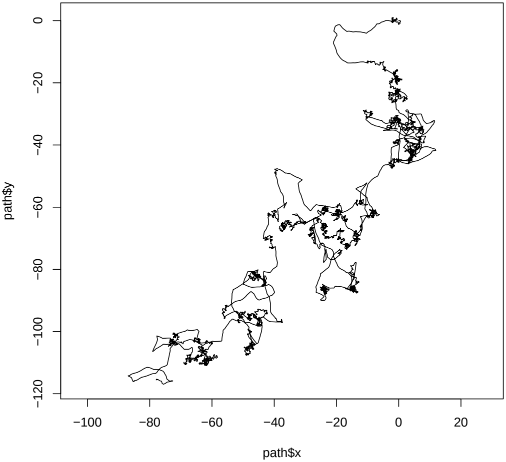
Movement Ecology Workshop 2015 - Port Elizabeth

Detection of homogenous region in trajectories

Why ?

- Different behaviour
- Link with different activities
- Link with different environmental condition



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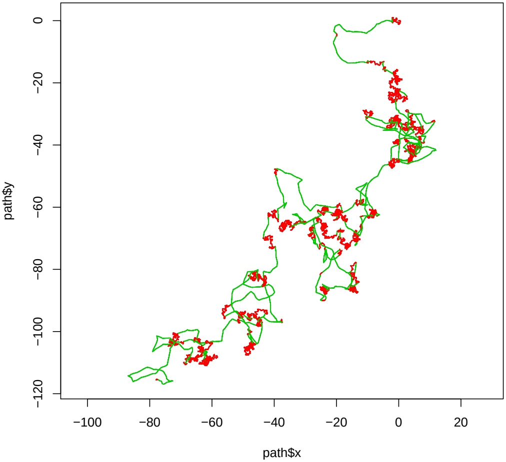
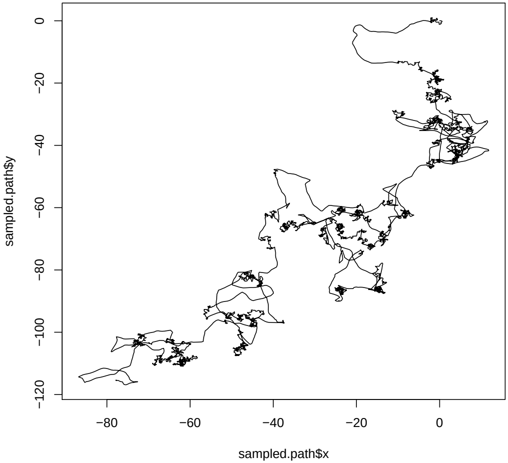


Effect of sampling step



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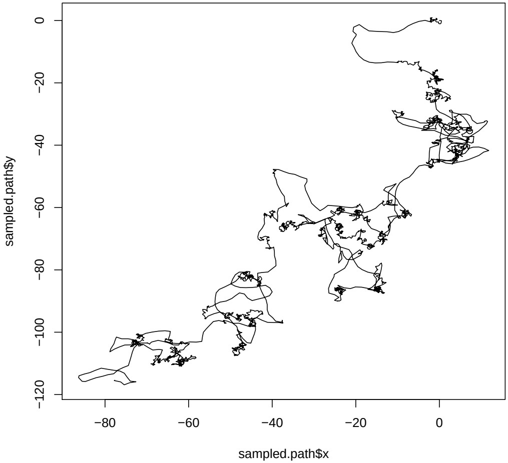

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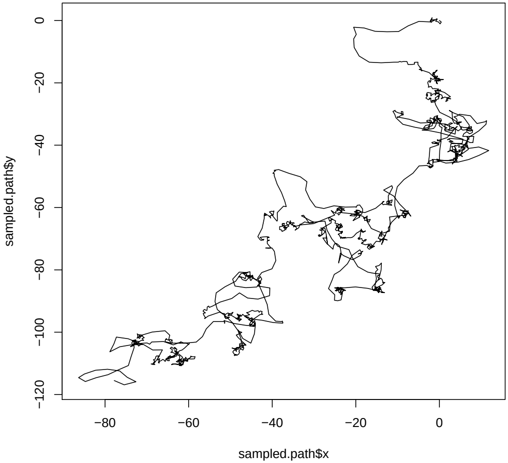

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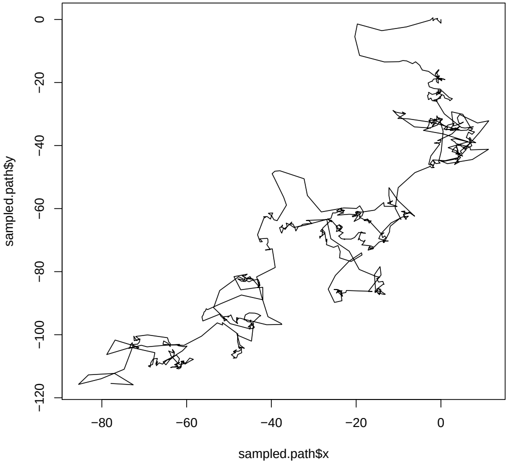

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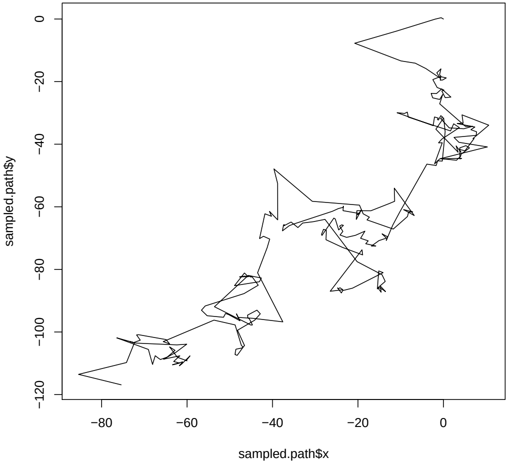

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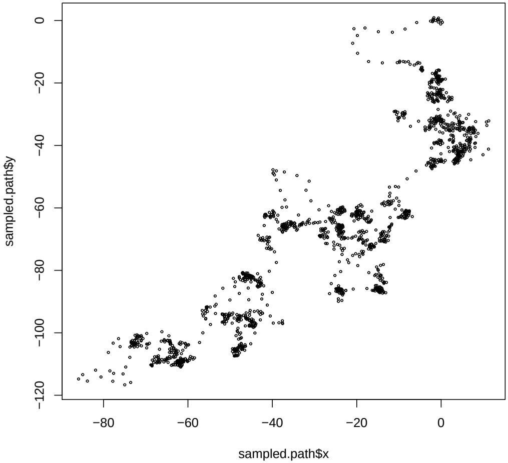


Summarising trajectories

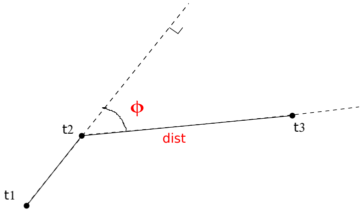
$(t_1, \ldots, t_N)$ denotes the time acquisition and $((x_1, y_1), \ldots, (x_N, y_N))$ the position at those times.

Trajetories as Turning angle and Speed

$$\mathbf{\Phi} = (\phi_2, \ldots, \phi_N)$$

$$\mathbf{S} = (S_2, \ldots, \S_N),$$

with $S_i = dist_i/(t_i - t_{i-1})$



Summarising trajectories

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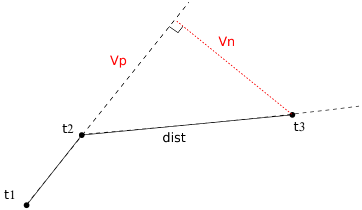
Trajetories as Persistent and Normal Velocity

$$\mathbf{V}^P = (V_2^P, \ldots, V_N^P)$$

$$\mathbf{V}^N = (V_2^N, \ldots, V_N^N)$$

with

$$V_i^P = S_i \, cos(\phi_i)$$
$$V_i^N = S_i \, sin(\phi_i)$$



From Trajectories data to signal - Loosing information on spatial location

How trajectories data migh be considered?

- A sequence of (time, position)
- Turning angle and speed sequences
- Persistent and Normal Velocity sequences

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Model approach:

- Most methods don't consider the two phenomena : movement and sampling process.
- Results will be closely dependent of the sampling step.

Model and parameters

The model is a tentative to represent the main characteristics of potential data.

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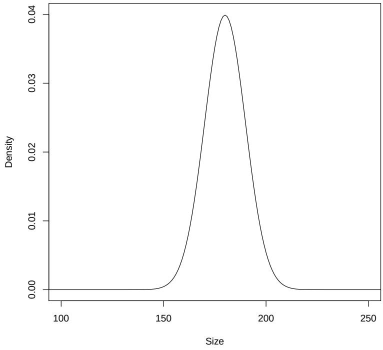
If we observe the size of people Y and assume $Y \sim \mathcal{N}(\mu, \sigma)$



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If people are sampled at random (with no relationship)

$$Y_i \overset{i.i.d}{\sim} \mathcal{N}(\mu, \sigma)$$

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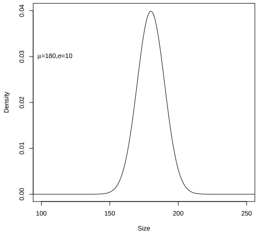

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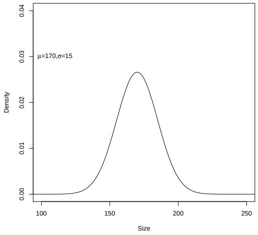

Estimation

Simulation context: with a given model M, and given parameters $\boldsymbol{\theta}$, you can produce fake data.
From $(M, \boldsymbol{\theta})$ to $\mathbf{Y}$.

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What is a good value for $\boldsymbol{\theta}$? with a given model M, and some data $\mathbf{Y}$, you want to give a score to each possible value of $\boldsymbol{\theta}$.
The likelihood as a measure of the quality of $\boldsymbol{\theta}$: with a given model M, and some data $\mathbf{Y}$, for each value of $\boldsymbol{\theta}$ you can compute the probability

$$\mathbb{P}(\mathbf{Y}; \boldsymbol{\theta})$$

Statistics and Hidden variables

A model $(M, \boldsymbol{\theta})$ produce $\mathbf{Y}$ and $\mathbf{Z}$.

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Questions are

- Parameters: Is it still possible to estimate $\boldsymbol{\theta}$?
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- Information on $\mathbf{Z}$: is it possible to "reconstruct" the unobserved data $\mathbf{Z}$?

Bayes formula is the key :

$$\mathbb{P}(\mathbf{Y}, \mathbf{Z}) = \mathbb{P}(\mathbf{Y}|\mathbf{Z})\mathbb{P}(\mathbf{Z}) = \mathbb{P}(\mathbf{Z}|\mathbf{Y})\mathbb{P}(\mathbf{Y})$$

Convention

Notation:

- $\mathbf{Y} = (Y_1, \ldots, Y_n)$ = observed data (typically Speed)
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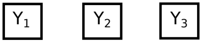
Graphical Representation (DAG):

Change point



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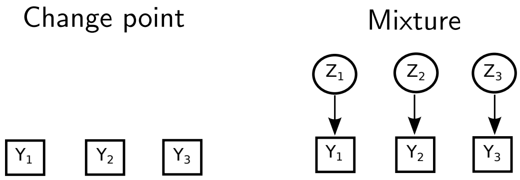

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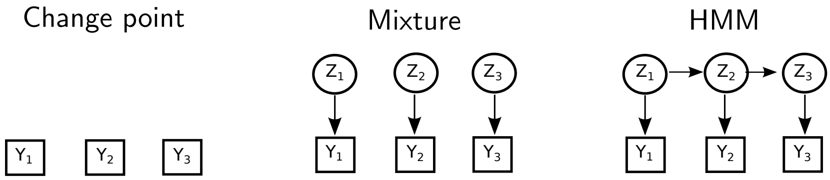
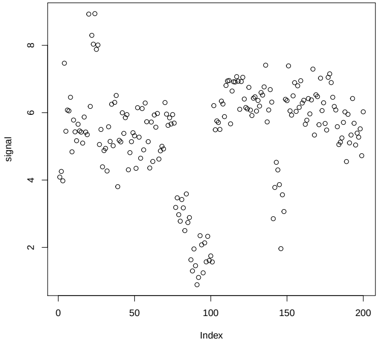


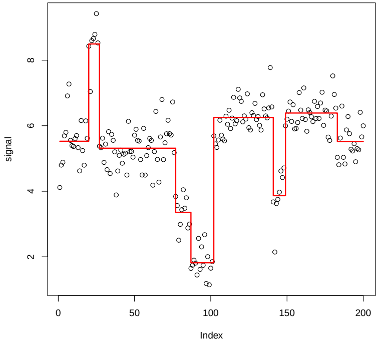
Plan

Change point detection context



Goal : Identifying homogenous region and abrupt changes in the signal.

Change point detection context



These regions may be interpreted afterwards.

Underlying model

Modelling :

- Data $Y_1, \ldots, Y_n$ are drawn from a given pdf, driven by unknown parameter θ

$$Y_i \overset{i.i.d}{\sim} f_\theta(.)$$

θ values change at $K-1$ unknow instants, the change point : $t_1, \ldots, t_{K-1}$:

$$Y_t \sim f(\theta_k) \text{ if } t \text{ in region } I_k = [t_{k-1}+1, t_k]$$

Underlying model

Change point detection in the trend (and/or in variance) :

- Data $Y_1, \ldots, Y_n$ are drawn from a given pdf, driven by unknown parameter θ

$$Y_i \overset{i.i.d}{\sim} f_\theta(.)$$

θ values change at $K-1$ unknow instants, the change point : $t_1, \ldots, t_{K-1}$:

$$Y_t \overset{ind}{\sim} \mathcal{N}(\mu_k, \sigma^2) \text{ if } t \text{ in portion } I_k, \text{ for } k = 1, \ldots, K.$$

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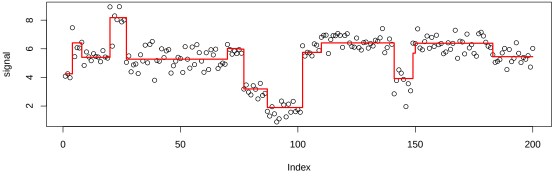


Remark : $K-1$ change points $\Leftrightarrow$ K regions.

Hybrid algorithm

2 levels of statistical units

- The inference of the breakpoints T is made at the position level t;
- The inference of the groups (status) (Θ, τ_{kp}) is made at the segment level k.

Hybrid algorithm

Alternate parameters estimation with K and P known

1. When $\mathbf{T}$ is fixed, the Expectation-Maximisation (EM) algorithm estimates $\boldsymbol{\theta}$;

$$\hat{\boldsymbol{\theta}}^{(h+1)} = \arg\max_{\boldsymbol{\theta}} \left\{ \log \mathcal{L}_{KP} \left(\boldsymbol{\theta}, T^{(h)} \right) \right\}.$$

$$\log \mathcal{L}_{KP}(\hat{\boldsymbol{\theta}}^{(h+1)}; \hat{\mathbf{T}}^{(h)}) \geq \log \mathcal{L}_{KP}(\hat{\boldsymbol{\theta}}^{(h)}; \hat{\mathbf{T}}^{(h)})$$

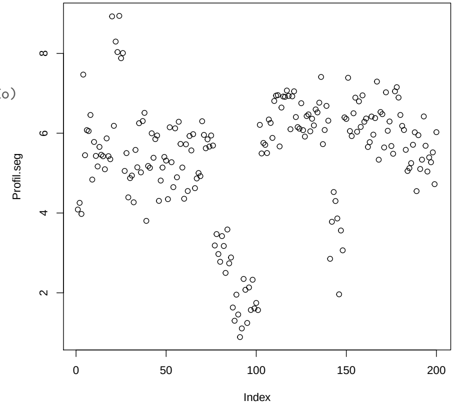
2. When $\boldsymbol{\theta}$ is fixed, dynamic programming estimates $\mathbf{T}$;

$$\hat{\mathbf{T}}^{(h+1)} = \underset{\mathbf{T}}{\mathrm{argmax}} \left\{ \log \mathcal{L}_{KP} \left(\hat{\boldsymbol{\theta}}^{(h+1)}, \mathbf{T} \right) \right\}.$$

$$\log \mathcal{L}_{KP}(\hat{\boldsymbol{\theta}}^{(h+1)}; \hat{\mathbf{T}}^{(h+1)}) \geq \log \mathcal{L}_{KP}(\hat{\boldsymbol{\theta}}^{(h+1)}; \hat{\mathbf{T}}^{(h)})$$

How to perform this segmentation/clustering approach ?

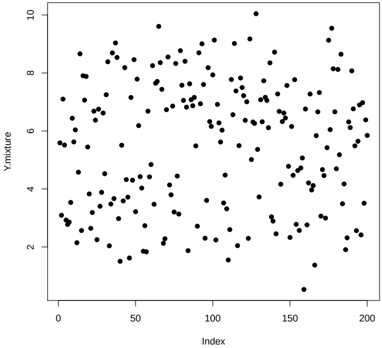
```
## format data into CGHdata
calling(CGHo)<- TRUE ## no classification

Error in calling(CGHo) <- TRUE: object
'CGHo' not found

CGHo@nblevels=2

Error in CGHo@nblevels = 2:  object 'CGHo'
not found

segSignal   <- uniseg(.Object=signalCGH,CGHo=CGHo)

Error in uniseg(.Object = signalCGH, CGHo
= CGHo):  error in evaluating the argument
'.Object' in selecting a method for
function 'uniseg':  Error:  object
'signalCGH' not found

segSignalProf <- getsegprofiles(segSignal)

Error in getsegprofiles(segSignal):  error
in evaluating the argument '.Object' in
selecting a method for function
'getsegprofiles':  Error:  object
'segSignal' not found
```



Do it yourself

Plan

Problem presentation



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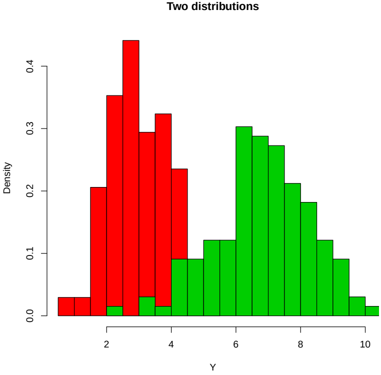
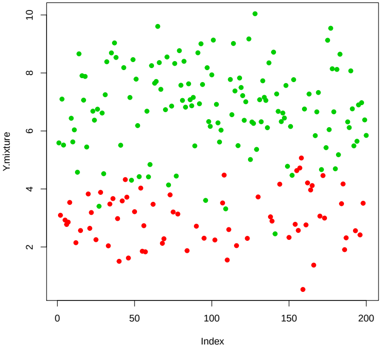




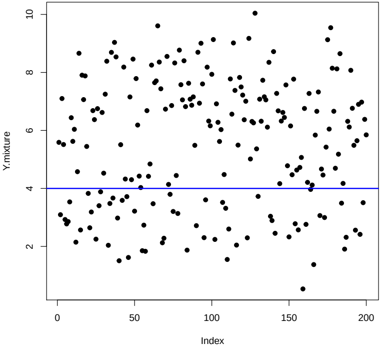
Problem presentation

Basic idea :

"Expert" threshold s

$$State_i = 1 \quad \text{if} Y_i < s$$

$$State_i = 2 \quad \text{if} Y_i \geq s$$



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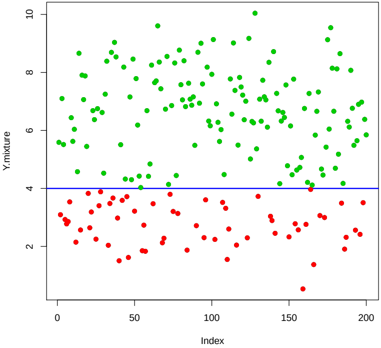

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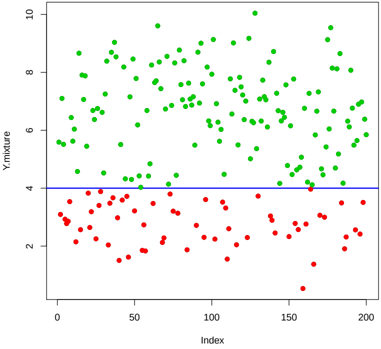
Improvement:

Estimating the threshold s and reconstruction of the hidden state (colour)

Compute the probability to belong to State 1 or 2.



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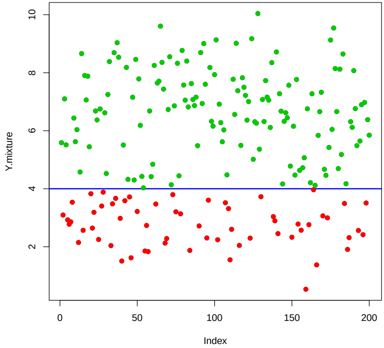
$\Longrightarrow$ Mixture Model



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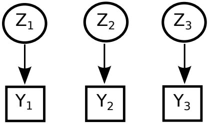

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- Modelling Z: $\pi_k = \mathbb{P}(Z_i = k), \quad k = 1, \ldots, K, \quad \sum_k \pi_k = 1$
  $Z_i \overset{i.i.d}{\sim} \mathcal{M}(1, \boldsymbol{\pi})$
- Modelling Y: The $Y_i's$ are assumed to be independent conditionnaly to $\mathbf{Z}$: $(Y_i|Z_i = k) \overset{i.i.d}{\sim} f_{\gamma_k}()$.

```
K <- 2; N <- 100; mu <- c(3, 7); sigma <- c(1,1.5)
Z <- sample(1:2, size = N, replace=T, prob=c(0.3, 0.7))
plot(Z, col=Z+1, pch=15, cex=0.8)
Y.mixture <- rnorm(N, mean=mu[Z], sd=sigma[Z])
plot(Y.mixture, col=Z+1, pch=19)
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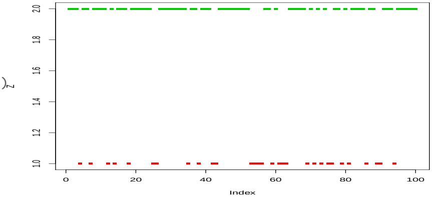

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```
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Z <- sample(1:2, size = N, replace=T, prob=c(0.3, 0.7))
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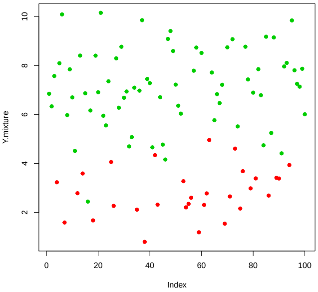

Model Properties

- Couples $\{(Y_i, Z_i)\}$ are i.i.d.
- Label switching:
  the model is invariant for any permutation of the labels $\{1, \dots, K\} \Rightarrow$ the mixture model has $K!$ equivalent definitions.
- Distribution of a Y_i:

$$P(Y_i) = \sum_{k=1}^{K} P(Y_i, Z_i = k) = P(Z_i = k) P(Y_i | Z_i = k)$$

- Distribution of $\mathbf{Y}$:

$$\begin{aligned} P(\mathbf{Y}; \boldsymbol{\theta}, \boldsymbol{\pi}) = \prod_{i=1}^{n} \sum_{k=1}^{K} P(Y_i, Z_i = k) &= \prod_{i=1}^{n} \sum_{k=1}^{K} P(Z_i = k) P(Y_i | Z_i = k) \\ &= \prod_{i=1}^{n} \sum_{k=1}^{K} \pi_k f_{\gamma_k}(Y_i) \end{aligned}$$

Statistical inference of incomplete data models

Maximum likelihood estimate: We are looking for

$$(\widehat{\boldsymbol{\theta}}, \widehat{\boldsymbol{\pi}}) = \arg\max_{\boldsymbol{\theta}, \boldsymbol{\pi}} \log P(\mathbf{Y}; \boldsymbol{\theta}, \boldsymbol{\pi})$$

- Likelihood of the observed data (or observed likelihood):

$$\log P(\mathbf{Y}; \boldsymbol{\theta}, \boldsymbol{\pi}) = \sum_{i=1}^{n} \log \left[\sum_{k=1}^{K} \pi_k f_{\gamma_k}(Y_i) \right]$$

- No analytical estimators.
- Brute force algorithm is not the way

And what if **Z** were observed ?

The complete likelihood is

$$\begin{aligned}\log P(\mathbf{Y}, \mathbf{Z}; \boldsymbol{\theta}, \boldsymbol{\pi}) &= \log P(\mathbf{Z}; \boldsymbol{\pi}) + \log P(\mathbf{Y}|\mathbf{Z}; \boldsymbol{\theta}) \\ &= \sum_i \sum_k Z_{ik} \log \pi_k + \sum_i \sum_k Z_{ik} \log f_{\gamma_k}(Y_i) \\ &= \sum_i \sum_k Z_{ik} [\log \pi_k + \log f_{\gamma_k}(Y_i)].\end{aligned}$$

Now, the sum contains nK (200 if $n = 100$ and $K = 2$) terms. It is much easier.

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Now, the sum contains nK (200 if $n = 100$ and $K = 2$) terms. It is much easier.

Unfortunately **Z** are unknown.

Idea: Replace Z_i, by our best guess, that is :

$$\tau_{ik} := \mathbb{E}(Z_i = k|Y_i) = P(Z_i = k|Y_i)$$

Idea of EM algorithm

Quantity to be maximized :

$$\sum_i \sum_k Z_{ik} [\log \pi_k + \log f_{\gamma_k}(Y_i)].$$

Idea of EM algorithm

Quantity to be maximized :

$$\sum_i \sum_k \tau_{ik} [\log \pi_k + \log f_{\gamma_k}(Y_i)].$$

where τ_{ik} is a good approximation of Z_{ik}.

Idea of EM algorithm

Quantity to be maximized :

$$Q(\boldsymbol{\theta}, \boldsymbol{\theta}_0) = \sum_i \sum_k \tau_{ik}^{(0)} [\log \pi_k + \log f_{\gamma_k}(Y_i)]$$

where τ_{ik} is a good approximation of Z_{ik}.

If $\boldsymbol{\theta}_0$ is known, a good approximation is
$\tau_{ik}^{(0)} = \mathbb{E}_{\boldsymbol{\theta}_0} \{Z_{ik}|\mathbf{Y}\} = \mathbb{P}_{\boldsymbol{\theta}_0} \{Z_i = k|\mathbf{Y}\}$,

Idea of EM algorithm

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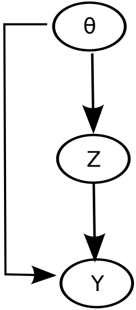
Idea : iterative algorithm, for a value of $\boldsymbol{\theta}^{(l)}$, compute $\tau_{ik}^{(l)}$, and then update $\boldsymbol{\theta}^{(l)}$ to $\boldsymbol{\theta}^{(l+1)}$.

More generally - EM algorithm



Bayes Formula

$$\begin{aligned} P(\mathbf{Y},\mathbf{Z};\boldsymbol{\theta}) &= P(\mathbf{Y}|\mathbf{Z};\boldsymbol{\theta})P(\mathbf{Z};\boldsymbol{\theta}), \\ &= P(\mathbf{Z}|\mathbf{Y};\boldsymbol{\theta})P(\mathbf{Y};\boldsymbol{\theta}). \end{aligned}$$

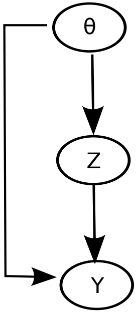
Therefore,

$$\begin{aligned} \log P(\mathbf{Y};\boldsymbol{\theta}) &= \log\left\{P(\mathbf{Y},\mathbf{Z};\boldsymbol{\theta})/P(\mathbf{Z}|\mathbf{Y};\boldsymbol{\theta})\right\} \\ &= \log P(\mathbf{Y},\mathbf{Z};\boldsymbol{\theta}) - \log P(\mathbf{Z}|\mathbf{Y};\boldsymbol{\theta}) \end{aligned}$$

For a given $\boldsymbol{\theta}_0$, we may compute $P_{\boldsymbol{\theta}_0} = P(\mathbf{Z}|\boldsymbol{\theta}_0,\mathbf{Y})$ and

$$\begin{aligned} \log P(\mathbf{Y};\boldsymbol{\theta}) &= \mathbb{E}_{\boldsymbol{\theta}_0}(\log P(\mathbf{Y},\mathbf{Z};\boldsymbol{\theta})) - \mathbb{E}_{\boldsymbol{\theta}_0}(\log P(\mathbf{Z}|\mathbf{Y};\boldsymbol{\theta})) \\ &= Q(\boldsymbol{\theta},\boldsymbol{\theta}_0) - H(\boldsymbol{\theta},\boldsymbol{\theta}_0) \end{aligned}$$

More generally - EM algorithm



Since

$$\log P(\mathbf{Y};\boldsymbol{\theta}) = Q(\boldsymbol{\theta},\boldsymbol{\theta}_0) - H(\boldsymbol{\theta},\boldsymbol{\theta}_0),$$

and $H(\boldsymbol{\theta},\boldsymbol{\theta}_0)$ achieves its maximum in $\boldsymbol{\theta}_0$,

$$\log P(\mathbf{Y};\boldsymbol{\theta}) - \log P(\mathbf{Y};\boldsymbol{\theta}_0) = (Q(\boldsymbol{\theta},\boldsymbol{\theta}_0) - Q(\boldsymbol{\theta},\boldsymbol{\theta}_0)) + (H(\boldsymbol{\theta}_0,\boldsymbol{\theta}_0) - H(\boldsymbol{\theta},\boldsymbol{\theta}_0)).$$

Expectation - Maximization algorithm

1. Phase E :
   Calculate $Q(\boldsymbol{\theta},\boldsymbol{\theta}^k)$ for every $\boldsymbol{\theta}$.
2. Phase M :
   Define $\boldsymbol{\theta}^{k+1} = argmax\, Q(\boldsymbol{\theta},\boldsymbol{\theta}^k)$

EM algorithm for independent mixture model

Recall that $\tau_{ik}^{(\ell)} := P_{\boldsymbol{\theta}^{(\ell)}}(Z_i = k | Y_i)$

$$Q(\boldsymbol{\theta}; \boldsymbol{\theta}^{(\ell)}) = \sum_i \sum_k \tau_{ik}^{(\ell)} \log \pi_k + \sum_i \sum_k \tau_{ik}^{(\ell)} \log f_{\gamma_k^{(\ell)}}(Y_i)$$

$\rightarrow$ Need to estimate $\tau_{ik}^{(\ell)}$

EM algorithm for independent mixture model

- Initialisation of $\boldsymbol{\theta}^{(0)} = (\pi_1, ..., \pi_K, \gamma_1, ..., \gamma_K)^{(0)}$.

EM algorithm for independent mixture model

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- Alternate

  E-step Calculation of

$$\tau_{ik}^{(\ell)} = P(Z_i = k | y_i, \boldsymbol{\theta}^{(\ell-1)}) = \frac{\mathbb{P}(Z_i = k, y_i; \boldsymbol{\theta}^{(\ell-1)})}{\mathbb{P}(y_i, \boldsymbol{\theta}^{(\ell-1)})}$$

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&= \frac{\mathbb{P}(y_i | Z_i = k; \boldsymbol{\theta}^{(\ell-1)})\mathbb{P}(Z_i = k; \boldsymbol{\theta}^{(\ell-1)})}{\mathbb{P}(y_i; \boldsymbol{\theta}^{(\ell-1)})} \\
&= \frac{\pi_k^{(\ell-1)} f_{\theta_k^{(\ell-1)}}(y_i)}{\sum_{k'} \pi_{k'}^{(\ell-1)} f_{\theta_{k'}^{(\ell-1)}}(y_i)}
\end{aligned}$$

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M-step Maximization of

$$(\boldsymbol{\pi}, \gamma) \longmapsto \sum_i \sum_k \tau_{ik}^{(\ell)} [\log \pi_k + \log f(x_i; \gamma_k)]$$

In the example

- $Z \in \{1,2\}$: $P(Z=1)=\pi_1$ and $P(Z=2)=1-\pi_1$
- For $k=1$ or 2, $(X|Z=k) \sim \mathcal{N}(\mu_k, \sigma_k^2)$
- The parameter vector is $\boldsymbol{\theta} = (\pi, \mu_1, \mu_2, \sigma_2^2, \sigma_2^2)$

$$
\begin{aligned}
\widehat{\pi}_1^{(\ell+1)} &= \frac{1}{n}\sum_{i=1}^{n} \tau_{i1}^{(\ell)}, \\
\widehat{\mu}_k^{(\ell+1)} &= \frac{1}{\sum_{i=1}^{n} \tau_{ik}^{(\ell)}} \sum_{i=1}^{n} \tau_{ik}^{(\ell)} y_i \\
\widehat{\sigma}^2_{k^{(\ell+1)}} &= \frac{1}{\sum_{i=1}^{n} \tau_{ik}^{(\ell)}} \sum_{i=1}^{n} \tau_{ik}^{(\ell)} (y_i - \widehat{\mu}_k^{(\ell)})^2
\end{aligned}
$$

$\rightarrow$ They are a weighted version of the usual maximum likelihood estimates.

Back on earth - Practically speaking

```
#library('mclust')
library('mixtools')
Y.clustering <- normalmixEM (Y.mixture, lambda = NULL, mu = NULL, sigma = NULL, k = 2,
                             mean.constr = NULL, sd.constr = NULL,
                             epsilon = 1e-07, maxit = 1000, maxrestarts=20)
```

```
number of iterations= 94
```

```
summary(Y.clustering)
```

```
summary of normalmixEM object:
         comp 1
lambda 0.644837
mu     7.366618
sigma  1.374730
         comp 2
lambda 0.355163
mu     3.072069
sigma  1.142811
loglik at estimate:  -220.6712
```

```
Y.clustering$posterior
```

```
             comp.1
  [1,] 0.9969687292
  [2,] 0.9851514346
```

Do it yourself

Reconstruction of hidden state **Z**

Since $\pi_{ik} = \mathbb{P}(Z_i = k)$, and $\hat{\pi}_{ik}$ is an estimation of π_{ik},

$$\widehat{Z}_i = \operatorname{argmax} k = 1, K = \widehat{\pi}_{ik}$$

Plan

Markov chain model

Modelling the dependence in state sequence: If an animal is feeding at time i, he has more chance to be feeding at time $i+1$ than if he was travelling at time i.

$$P(Z_{i+1} = 1 | Z_i = 1) \neq P(Z_{i+1} = 1 | Z_i = 2)$$

Markov Chain definition $\mathbf{Z}$ is a Markov chain if

$$P(Z_{i+1} | Z_{1:i}) = P(Z_{i+1} | Z_i)$$

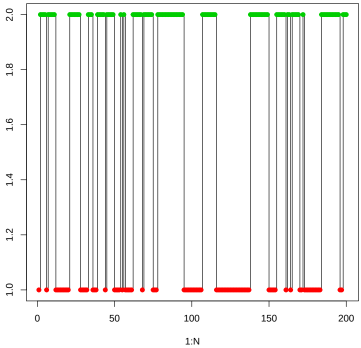
$\mathbf{Z}$ is completely defined by the distribution $\nu_1 = P(Z_1)$ and the transition matrix

$$\Pi = \begin{bmatrix} \pi_{11} & 1 - \pi_{11} \\ 1 - \pi_{22} & 1 - \pi_{22} \end{bmatrix}$$

Markov chain simulation

```r
### Hidden State simulation
set.seed(6)
N <- 200
pi11 <- 0.8
pi22 <- 0.9
## initial distribution
mu1 <- c(0.5, 0.5)
##transition matrix
PI <- matrix(c(pi11, 1-pi11, 1-pi22, pi22), ncol=2, byrow = T)

##initialisation of Z
Z <- rep(NA, N)
Z[1] <- sample(1:2, size=1, prob = mu1)
for( i in 1:(N-1))
{
  Z[i+1] <- sample(1:2, size=1, prob = PI[Z[i],])
}
plot(1:N, Z, "s")
points(1:N, Z, col=Z+1, pch=19)
```



Hidden Markov Chain model

Model For a given number of states K,

- Hidden States $\mathbf{Z}$ model: $\mathbf{Z}$ is assumed to follow a Markov Chain model with unknown initial distribution $\boldsymbol{\nu}$ and transition matrix $\boldsymbol{\Pi}$.
- Observations $\mathbf{Y}$ model: The $Y_i's$ are assumed to be independent conditionnaly to $\mathbf{Z}$: $(Y_i|Z_i = k) \overset{i.i.d}{\sim} f_{\gamma_k}()$.

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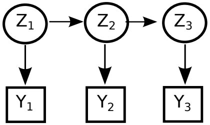

Hidden Markov Chain simulation

```
### observation simulation
mu <- c(3, 7)
sigma <- c(1,1.5)
Y.hmm <- rnorm(N, mean=mu[Z], sd=sigma[Z])
plot(Y.hmm, pch=19)
plot(Y.hmm, pch=19, col=Z+1)
```

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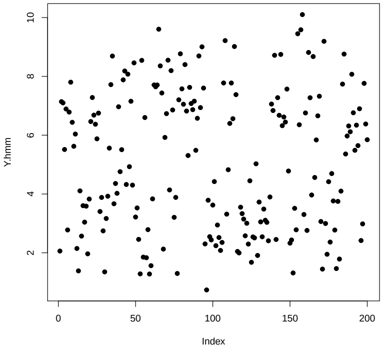

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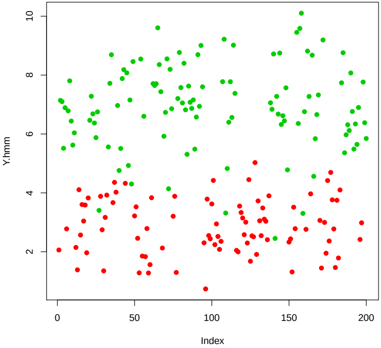

Sojourn time properties

T_i, the sojourn time in State i follows a geometric distribution

$$\mathbb{P}(T_i = l) = (\Pi_{ii})^{l-1}(1 - \Pi_{ii})$$

```r
switchTime <- which(diff(Z)!=0)
sojournTime <- diff(switchTime)
sojournState <- rep(c(3-Z[1], Z[1]),
                    length.out = length(sojournTime))
br <- unique(quantile(sojournTime,
                      p<- seq(1/N, 1, length.out = 6)))

abc <- seq(1, max(sojournTime)+10)
lapply(1:2, function(i){
  var <- sojournTime[sojournState==i]
  br <- sort(unique(var))
  hist(var, col=i+1, freq=F,
       xlim=range(sojournTime),
       ylim=c(0,0.4),
       main=paste0("Sojourn Time, State ", i),
       breaks=br)
  lines(abc, dgeom(abc-1, prob = 1-PI[i,i]),
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})
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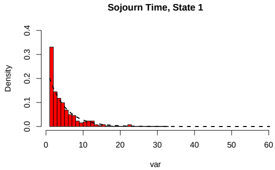

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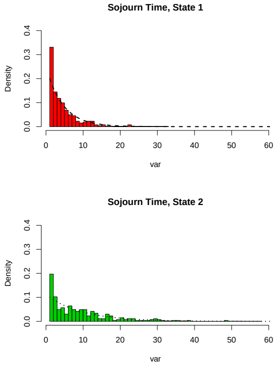

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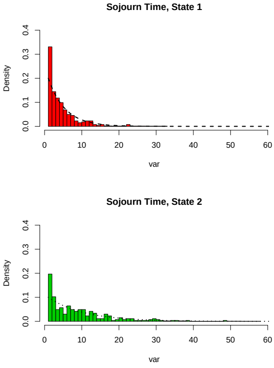

see Semi Hidden Markov Model for removing this assumption

Statistical inference of incomplete data models

Maximum likelihood estimate: We are looking for

$$(\widehat{\gamma}, \widehat{\boldsymbol{\Pi}}, \widehat{\boldsymbol{\nu}}) = \arg\max_{\boldsymbol{\theta}, \boldsymbol{\Pi}, \boldsymbol{\nu}} \log P(\mathbf{Y}; \boldsymbol{\theta}, \boldsymbol{\Pi}, \boldsymbol{\nu})$$

$$\begin{aligned}
\log P(\mathbf{X}, \mathbf{Z}; \boldsymbol{\theta}) \quad = \quad & \sum_k Z_{1k} \log \nu_k \\
& + \sum_{i>1} \sum_{k,\ell} Z_{i-1,k} Z_{i,\ell} \log \pi_{k\ell} \\
& + \sum_i \sum_k Z_{ik} \log f(X_i; \gamma_k)
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\mathbb{E}\left(\log P(\mathbf{X}, \mathbf{Z}) | Y_{1:N}\right) \quad = \quad & \sum_k \mathbb{E}\left(Z_{1k} | Y_{1:N}\right) \log \nu_k \\
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& + \sum_i \sum_k \mathbb{P}\left(Z_i = k | Y_{1:N}\right) \log f\left(X_i; \gamma_k\right)
\end{aligned}$$

EM Algorithm (Baum Welch)

- Initialisation of $\boldsymbol{\theta}^{(0)} = (\Pi, \gamma_1, ..., \gamma_K)^{(0)}$.
- While the convergence is not reached

E-step Calculation of

$$\begin{aligned}\tau_{ik}^{(\ell)} &= P(Z_i = k | Y_{1:n}, \boldsymbol{\theta}^{(\ell-1)}) \\ \eta_{ikh}^{(\ell)} &= \mathbb{E}[Z_{i-1,k} Z_{ih} | Y_{1:n}, \boldsymbol{\theta}^{(\ell-1)}]\end{aligned}$$

M-step Maximization in $\boldsymbol{\theta} = (\boldsymbol{\pi}, \boldsymbol{\gamma})$ of

$$\sum_k \tau_{1k}^{(\ell)} \log \nu_k + \sum_{i>1} \sum_{k,h} \eta_{ikh}^{(\ell)} \log \pi_{kh} + \sum_i \sum_k \tau_{ik}^{(\ell)} \log f(x_i; \gamma_k)$$

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E-step Calculation of Smart algorithm Forward-Backward algorithm

M-step Maximization in $\boldsymbol{\theta} = (\boldsymbol{\pi}, \boldsymbol{\gamma})$ of

$$\sum_k \tau_{1k}^{(\ell)} \log \nu_k + \sum_{i>1} \sum_{k,h} \eta_{ikh}^{(\ell)} \log \pi_{kh} + \sum_i \sum_k \tau_{ik}^{(\ell)} \log f(x_i; \gamma_k)$$

Reconstruction of hidden state $\mathbf{Z}$

Most credible value for Z_i: We are interested in

$$\underset{k}{\operatorname{argmax}}\, \mathbb{P}(Z_i = k | Y_{1:n}) = \underset{k}{\operatorname{argmax}}\, \tau_{ik}.$$

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But force brut algorithm is not possible

$\longrightarrow$ a smart algorithm : Viterbi algorithm

Viterbi algorithm

Key quantity: The probability of the best hidden path from time 1 to i who finished in k

Viterbi algorithm

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$$\delta_i(k) = \max_{k_1,\ldots k_{i-1}} \mathbb{P}(Y_{1:i}, Z_1 = k_1, \ldots, Z_{i-1} = k_{i-1}, Z_i = k)$$

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- *Initialisation*

$$\begin{aligned} \delta_1(k) &= \mathbb{P}(Z_1 = k, Y_1) = \mathbb{P}(Z_1 = k)\mathbb{P}(Y_1 | Z_1 = k) = \nu(k) f_{\gamma_k}(y_1) \\ \psi_1(k) &= 0 \end{aligned}$$

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- *Recurrence, for $i = 1, \dots n-1$*

$$\begin{array}{rcl} \delta_{i+1}(k) & = & \max_{k_1,\dots,k_i} \mathbb{P}(Y_{1:i}, Z_1 = k_1, \dots, Z_{i-1} = k_{i-1}, Z_i = k_i, Z_{i+1} = k) \\ & = & \max_{k_i} \{\delta_i(k_i)\mathbb{P}(Y_i | Z_i = k_i)\mathbb{P}(Z_{i+1} | Z_i = k_i)\} \\ \psi_{i+1}(k) & = & \operatorname{argmax}_j \{\delta_i j \Pi_{jk}\} \end{array}$$

Estimation of HMM with R

```
library('depmixS4')

df <- data.frame(Y=Y)
K=2
m1 <- depmix(Y~1,data=df, nstates=2, family=gaussian())
fit.model <- fit(m1)
summary(fit.model)
Z.hat <- viterbi(fit.model)[,1]
table(Z, Z.hat)
```

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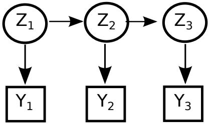
- **Z** is a sequence of hidden states and the observations **Y** are ruled by this sequence.



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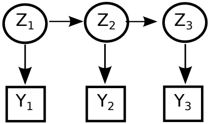

Z could be thougth as the actual locations and **Y** the observed locations.

- Estimation tools are EM algorithm or Bayesian framework (with monte carlo based estimation technics)

Plan

And more ...

- It is possible to include dependency in $\mathbf{Y}$.
- Markovian property could be removed (SHMM)
- Presented methods may be used on several signals (ex *Speed* and *angle*)
- Work in progress for continuous time Markov model and dependent observations.

But

Limitations

Trajectories are in continuous space and continuous time.

- Mostly, discrete time : effects of the sampling step and assumption of regularity.
- Segmentation methods consider a signal in time,
- Spatial information is lost.
- Those methods are useful to identify different regimes of movement. This difference may be due to behaviour or the environment or interaction.

Many thanks to Emilie Lebarbier, Marie-Laure Martin-Magniette, Stéphane Robin for some contents of the slides.

Many thanks to Andrea and Linda for the invitation and organisation.