Model based detection of homogeneous portions in trajectories

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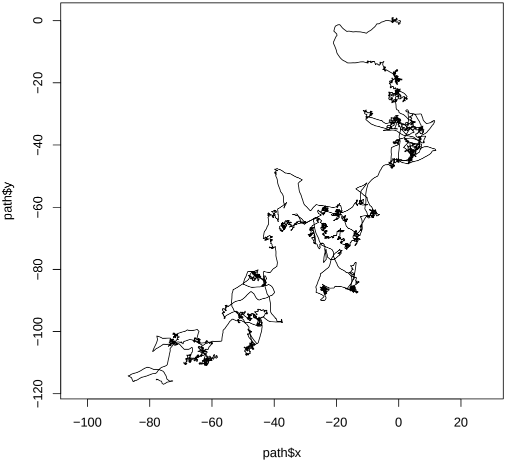
Detection of homogenous regions in trajectories

Why ?

- Different behaviours.
- Link with different activities.
- Link with different environmental conditions.



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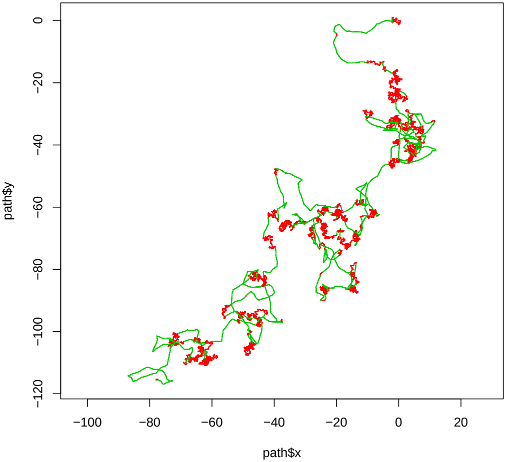
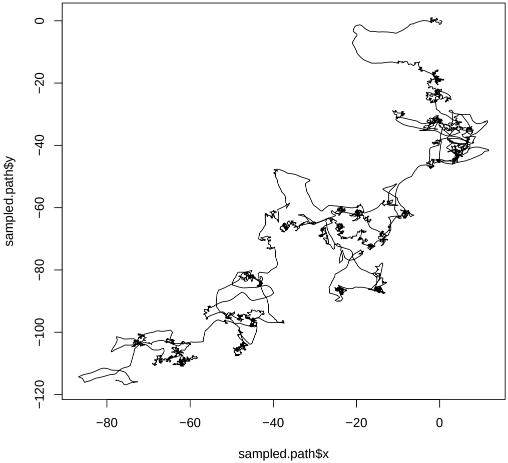


Effect of sampling step



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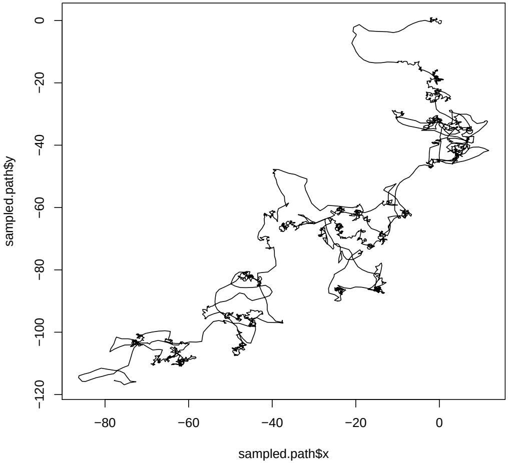

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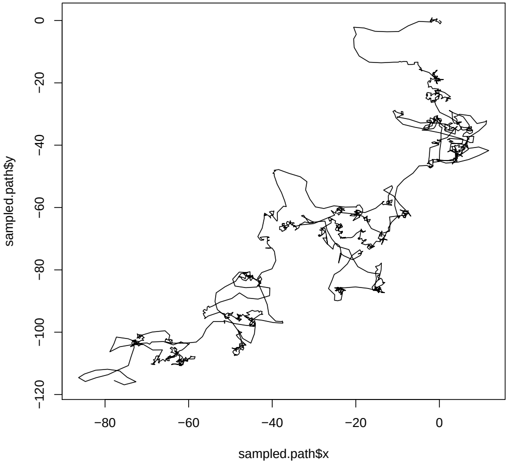

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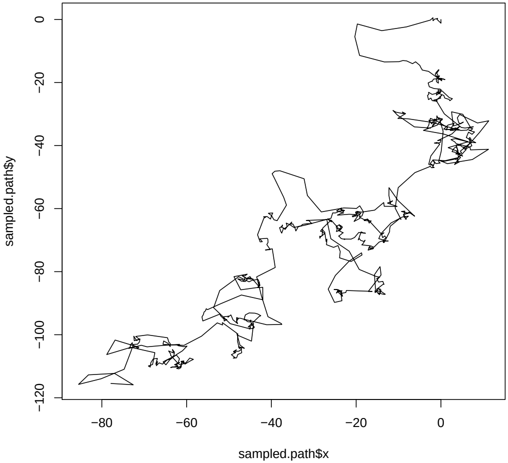

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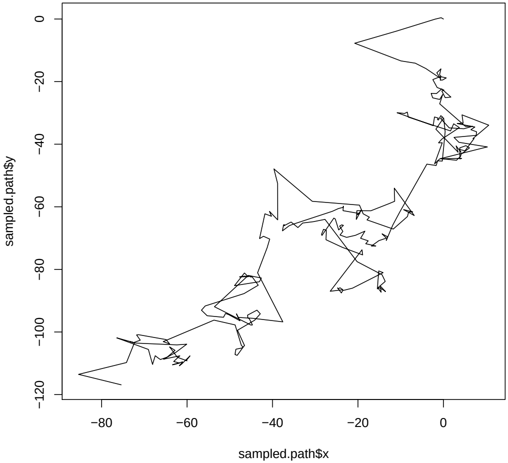

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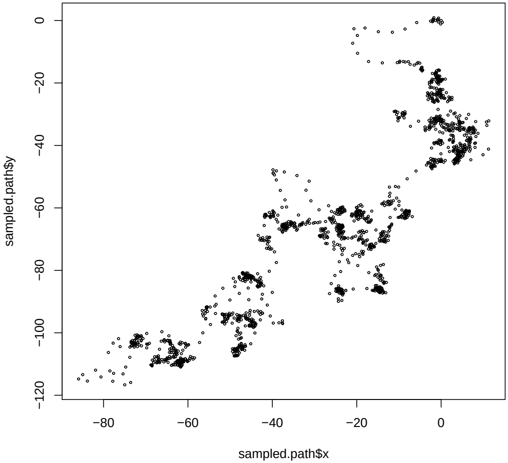


Summarising trajectories

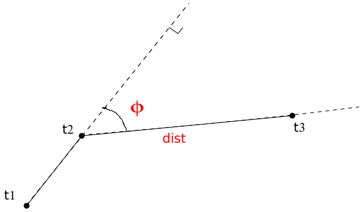
$(t_1, \ldots, t_N)$ denotes the time acquisition and $((x_1, y_1), \ldots, (x_N, y_N))$ the position at those times.

Trajetories as Turning angle and Speed

$$\mathbf{\Phi} = (\phi_2, \ldots, \phi_N)$$

$$\mathbf{S} = (S_2, \ldots, \S_N),$$

with $S_i = dist_i/(t_i - t_{i-1})$



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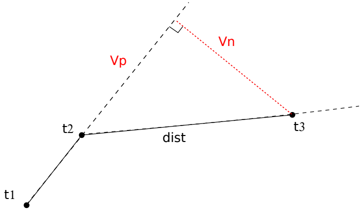
Trajetories as Persistent and Normal Velocity

$$\mathbf{V}^P = (V_2^P, \ldots, V_N^P)$$

$$\mathbf{V}^N = (V_2^N, \ldots, V_N^N)$$

with

$$V_i^P = S_i\, cos(\phi_i)$$
$$V_i^N = S_i\, sin(\phi_i)$$



From Trajectories data to signal - Loosing information on spatial location

How trajectories data migh be considered?

- A sequence of (time, position)
- Turning angle and speed sequences
- Persistent and Normal Velocity sequences

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Model approach:

- Most methods don't consider the two phenomena : movement and sampling process.
- Results will be closely dependent of the sampling step.

Model and parameters

The model is a tentative to represent the main characteristics of potential data.

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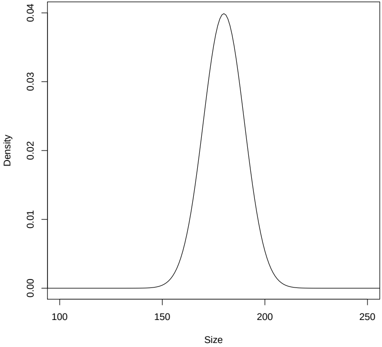
If we observe the size of people Y and assume $Y \sim \mathcal{N}(\mu, \sigma)$



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If people are sampled at random (with no relationship)

$$Y_i \overset{i.i.d}{\sim} \mathcal{N}(\mu, \sigma)$$

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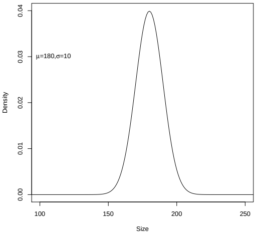
The parameters rule the behaviour of the model.



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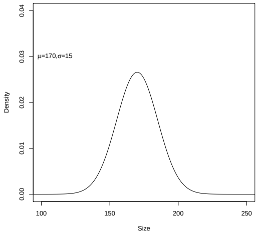

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Simulation context: with a given model M, and given parameters $\boldsymbol{\theta}$, you can produce fake data.
From $(M, \boldsymbol{\theta})$ to $\mathbf{Y}$.

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The likelihood as a measure of the quality of $\boldsymbol{\theta}$: with a given model M, and some data $\mathbf{Y}$, for each value of $\boldsymbol{\theta}$ you can compute the probability

$$\mathbb{P}(\mathbf{Y}; \boldsymbol{\theta})$$

Statistics and Hidden variables

A model $(M, \boldsymbol{\theta})$ produce $\mathbf{Y}$ and $\mathbf{Z}$.

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Questions are

- Parameters: Is it still possible to estimate θ ?
- Information on $\mathbf{Z}$: is it possible to "reconstruct" the unobserved data $\mathbf{Z}$?

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- Parameters: Is it still possible to estimate $\boldsymbol{\theta}$?
- Information on $\mathbf{Z}$: is it possible to "reconstruct" the unobserved data $\mathbf{Z}$?

Bayes formula is the key :

$$\mathbb{P}(\mathbf{Y}, \mathbf{Z}) = \mathbb{P}(\mathbf{Y}|\mathbf{Z})\mathbb{P}(\mathbf{Z}) = \mathbb{P}(\mathbf{Z}|\mathbf{Y})\mathbb{P}(\mathbf{Y})$$

Convention

Notations:

- $\mathbf{Y} = (Y_1, \dots, Y_n)$ = observed data (typically Speed)
- $\mathbf{Z} = (Z_1, \dots, Z_n)$ unobserved data (typically State, for mixture and Hidden Markov model)
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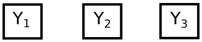
Graphical Representation (DAG):

Change point



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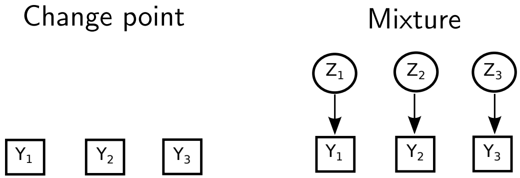

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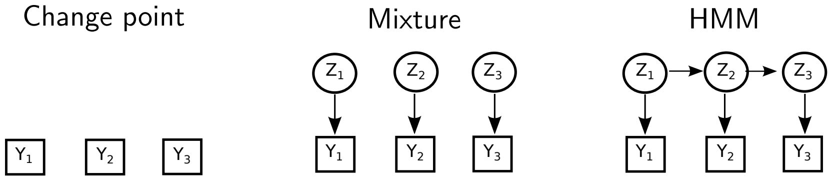
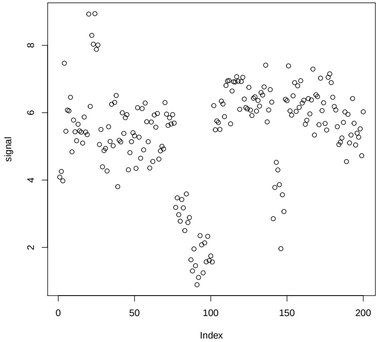


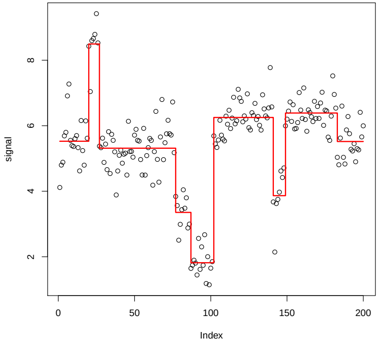
Plan

Change point detection context



Goal: Identifying homogenous regions and abrupt changes in the signal.

Change point detection context



These regions may be interpreted afterwards.

Underlying model

Modelling :

- Data $Y_1, \ldots, Y_n$ are drawn from a given pdf, driven by unknown parameter θ

$$Y_i \overset{i.i.d}{\sim} f_\theta(.)$$

θ values change at $K-1$ unknow instants, the change point : $t_1, \ldots, t_{K-1}$:

$$Y_t \sim f(\theta_k) \ \text{ if } t \text{ in region } I_k = [t_{k-1}+1, t_k]$$

Underlying model

Change point detection in the trend (and/or in variance) :

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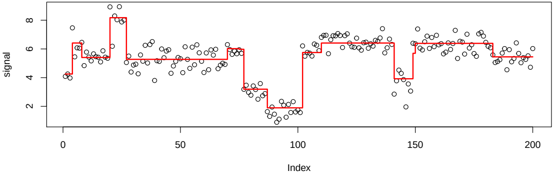

Remark : $K-1$ change points $\Leftrightarrow$ K regions.

Estimation procedure

- Unknown parameters : $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_K)$, σ, and $\mathbf{T} = (T_1, \ldots, T_K)$, but also K itself.

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- Estimation Procedure
  - For given K and $\mathbf{T}$, $\boldsymbol{\theta}$ is estimated using maximum likelihood.

Estimation procedure

- Unknown parameters : $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_K)$, σ, and $\mathbf{T} = (T_1, \ldots, T_K)$, but also K itself.
- Estimation Procedure
  - For given K, compute the maximum likelihood for any possible position for $\mathbf{T}$, But,

$$\binom{n-1}{K-1}$$

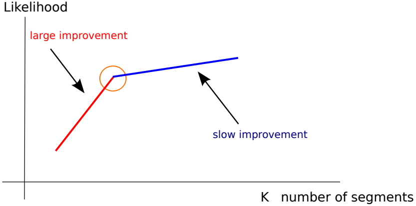
possible choices for the $K-1$ positions, that is 10^{30} for $K = 10$, $n = 200$, ($\approx 10^{11}$ years on a 2014 computer).

$\Rightarrow$ Practically impossible even for small K and n

Dynamic programming

Estimation procedure

- Unknown parameters : $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_K)$, σ, and $\mathbf{T} = (T_1, \ldots, T_K)$, but also K itself.
- Estimation Procedure
  - Estimating K. Likelihood increases with the number of segment K, use a penalized likelihood criterion and



Estimation procedure

Likelihood

$$
\begin{aligned}
2log(P_K(\mathbf{Y}, \mathbf{T}, \boldsymbol{\theta})) &= 2\sum_{k=1}^{K} \log f(\{Y_t\}_{t\in I_k}; \theta_k) = 2\sum_{k=1}^{K}\sum_{t\in I_k} \log f(Y_t; \theta_k) \\
&= -n\log\sigma^2 - \frac{1}{\sigma^2}\sum_{k=1}^{K}\sum_{t\in I_k}(Y_t - \mu_k)^2 + \text{cst.}
\end{aligned}
$$

Estimations

$$
(\hat{\mathbf{T}}, \hat{\boldsymbol{\theta}}) = \underset{(\mathbf{T},\boldsymbol{\theta})}{\operatorname{argmax}}\, log(P_K(\mathbf{Y}, \mathbf{T}, \boldsymbol{\theta}))
$$

If the change points are known

$$
\widehat{\mu}_k = \frac{1}{n_k}\sum_{t\in I_k} Y_t \qquad\qquad \widehat{\sigma}^2 = \frac{1}{n}\sum_{k=1}^{K}\sum_{t\in I_k}(Y_t - \widehat{\mu}_k)^2
$$

Finding the K-1 change points

Considering all possible segmentations, the best segmentation minimizes

$$J_k(1, n) = \sum_{k=1}^{K} \sum_{t \in I_k} (Y_t - \widehat{\mu}_k)^2.$$

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Dynamic programming, with complexity ($\mathcal{O}(n^2)$).

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Dynamic programming, with complexity ($\mathcal{O}(n^2)$).

Finding the K-1 change points

Sub-paths of the optimal path are themselves optimal, Bellmann optimality

Initialisation: Compute for $0 \leq i < j \leq n$, cost of portion I_{ij} :

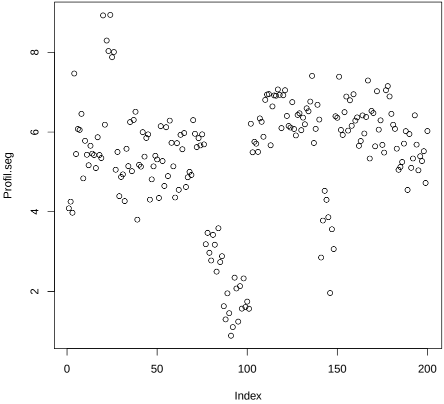
$$J_1(i,j) = \sum_{t=i+1}^{j} (Y_t - \widehat{\mu})^2$$

Etape k: Compute for $2 \leq k \leq K$, $J_k(i,j)$ the cost of the best segmentation in k segments between i and j.

$$J_k(i,j) = \min_{i<h<j} \left[J_{k-1}(i,h) + J_1(h+1,j) \right].$$

How to perform this segmentation approach ?

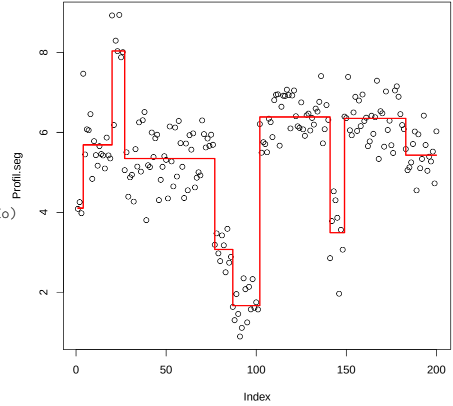
```
load("../Data/dataSegmentation.Rd")
summary(Profil.seg)


   Min. 1st Qu.
 0.8904  4.8720
 Median    Mean
 5.6990  5.3660
3rd Qu.    Max.
 6.3070  8.9400


plot(Profil.seg)
```



How to perform this segmentation approach ?

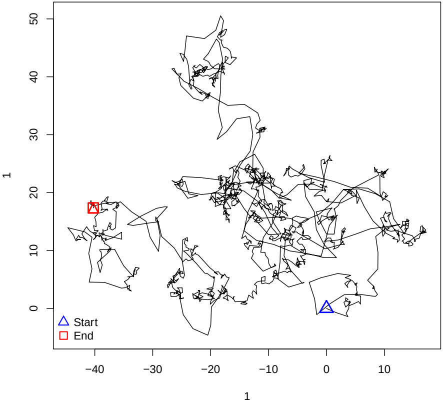
```r
library('cghseg')
## format data into CGHdata
signalCGH    <- new("CGHdata",Y=Profil.seg)
CGHo         <- new("CGHoptions")
calling(CGHo)<- FALSE ## no classification

segSignal    <- uniseg(.Object=signalCGH,CGHo=CGHo)
segSignalProf <- getsegprofiles(segSignal)
plot(Profil.seg)
lines(1:length(segSignalProf),
      segSignalProf, type="s", col=2, lwd=2)
```

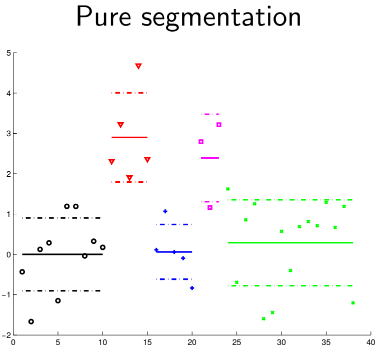


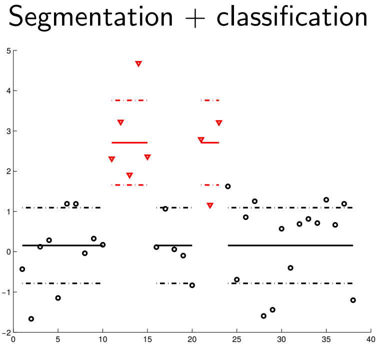
Do it yourself

```
load(file="../Data/trajEx.Rd")
plot(traj.ex, addpoints = F)
legend("bottomleft",pch=c(2, 0), col=c(4,2),
       legend=c("Start", "End"), bty = "n",
       pt.lwd = c(1.5,1.5), pt.cex = c(1.5,1.5))
```



When Segmentation is not sufficient - clustering segmentation model





Segmentation-Clustering

- The distribution of the signal given the group of the segment is

$$
\begin{aligned}
t \in I_k, k \in p \qquad \Rightarrow \qquad & Y_t \sim \mathcal{N}(m_p, \sigma^2) \\
Y^k | Z_{kp} = 1 \sim \mathcal{N}(m_p, \sigma^2).
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$$Y^k | Z_{kp} = 1 \sim \mathcal{N}(m_p, \sigma^2).$$

- It is a model of segmentation / clustering.
- Model parameters are $\boldsymbol{\theta} = (\boldsymbol{\pi}, \boldsymbol{\gamma})$ and the breakpoint positions $\mathbf{T} = (t_1, ..., t_{K-1})$.

Hybrid algorithm

2 levels of statistical units

- The inference of the breakpoints T is made at the position level t;
- The inference of the groups (status) (γ, τ_{kp}) is made at the segment level k.

Hybrid algorithm

Alternate parameters estimation with K and P known

1. When $\mathbf{T}$ is fixed, the Expectation-Maximisation (EM) algorithm estimates $\boldsymbol{\theta}$;

$$\hat{\boldsymbol{\theta}}^{(h+1)} = \arg\max_{\boldsymbol{\theta}} \left\{ \log \mathcal{L}_{KP} \left(\boldsymbol{\theta}, T^{(h)} \right) \right\}.$$

$$\log \mathcal{L}_{KP}(\hat{\boldsymbol{\theta}}^{(h+1)}; \hat{\mathbf{T}}^{(h)}) \geq \log \mathcal{L}_{KP}(\hat{\boldsymbol{\theta}}^{(h)}; \hat{\mathbf{T}}^{(h)})$$

2. When $\boldsymbol{\theta}$ is fixed, dynamic programming estimates $\mathbf{T}$;

$$\hat{\mathbf{T}}^{(h+1)} = \operatorname*{argmax}_{\mathbf{T}} \left\{ \log \mathcal{L}_{KP} \left(\hat{\boldsymbol{\theta}}^{(h+1)}, \mathbf{T} \right) \right\}.$$

$$\log \mathcal{L}_{KP}(\hat{\boldsymbol{\theta}}^{(h+1)}; \hat{\mathbf{T}}^{(h+1)}) \geq \log \mathcal{L}_{KP}(\hat{\boldsymbol{\theta}}^{(h+1)}; \hat{\mathbf{T}}^{(h)})$$

How to perform this segmentation/clustering approach ?

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## format data into CGHdata
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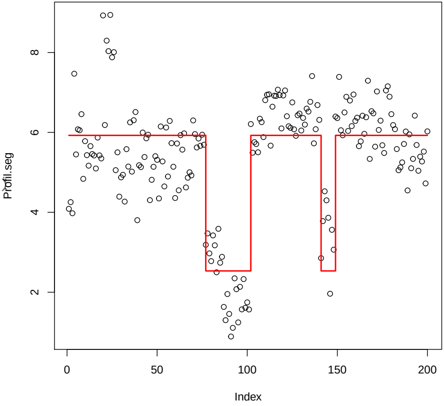


Do it yourself

Plan

Markov chain model

Modelling the dependence in state sequence: If an animal is feeding at time i, he has more chance to be feeding at time $i+1$ than if he was travelling at time i.

$$P(Z_{i+1} = 1 | Z_i = 1) \neq P(Z_{i+1} = 1 | Z_i = 2)$$

Markov Chain definition **Z** is a Markov chain if

$$P(Z_{i+1} | Z_{1:i}) = P(Z_{i+1} | Z_i)$$

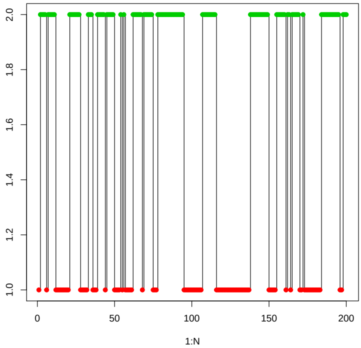
Z is completely defined by the distribution $\nu_1 = P(Z_1)$ and the transition matrix

$$\Pi = \begin{bmatrix} \pi_{11} & 1-\pi_{11} \\ 1-\pi_{22} & 1-\pi_{22} \end{bmatrix}$$

Markov chain simulation

```
### Hidden State simulation
set.seed(6)
N <- 200
pi11 <- 0.8
pi22 <- 0.9
## initial distribution
mu1 <- c(0.5, 0.5)
##transition matrix
PI <- matrix(c(pi11, 1-pi11, 1-pi22, pi22), ncol=2, byrow = T)

##initialisation of Z
Z <- rep(NA, N)
Z[1] <- sample(1:2, size=1, prob = mu1)
for( i in 1:(N-1))
{
  Z[i+1] <- sample(1:2, size=1, prob = PI[Z[i],])
}
plot(1:N, Z, "s")
points(1:N, Z, col=Z+1, pch=19)
```



Hidden Markov Chain model

Model For a given number of states K,

- Hidden States $\mathbf{Z}$ model: $\mathbf{Z}$ is assumed to follow a Markov Chain model with unknown initial distribution $\boldsymbol{\nu}$ and transition matrix $\boldsymbol{\Pi}$.
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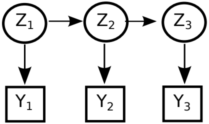

Hidden Markov Chain simulation

```
### observation simulation
mu <- c(3, 7)
sigma <- c(2,3)
Y.hmm <- rnorm(N, mean=mu[Z], sd=sigma[Z])
plot(Y.hmm, pch=19)
plot(Y.hmm, pch=19, col=Z+1)
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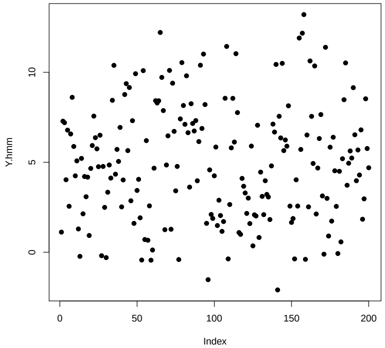

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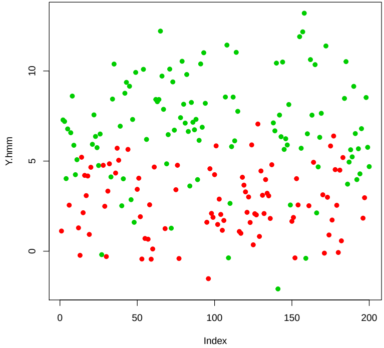

Sojourn time properties

T_i, the sojourn time in State i follows a geometric distribution

$$\mathbb{P}(T_i = l) = (\Pi_{ii})^{l-1}(1 - \Pi_{ii})$$

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switchTime <- which(diff(Z)!=0)
sojournTime <- diff(switchTime)
sojournState <- rep(c(3-Z[1], Z[1]),
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br <- unique(quantile(sojournTime,
                      p<- seq(1/N, 1, length.out = 6)))

abc <- seq(1, max(sojournTime)+10)
lapply(1:2, function(i){
  var <- sojournTime[sojournState==i]
  br <- sort(unique(var))
  hist(var, col=i+1, freq=F,
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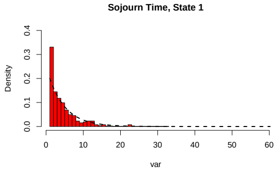

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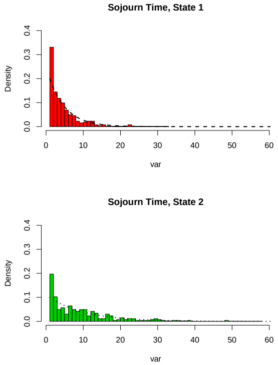

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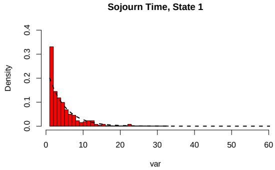

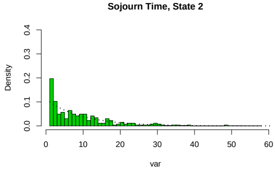

see Semi Hidden Markov Model for removing this assumption

Statistical inference of incomplete data models

Maximum likelihood estimate: We are looking for

$$(\widehat{\gamma}, \widehat{\mathbf{\Pi}}, \widehat{\boldsymbol{\nu}}) = \arg \max_{\boldsymbol{\gamma}, \mathbf{\Pi}, \boldsymbol{\nu}} \log P(\mathbf{Y}; \gamma, \mathbf{\Pi}, \boldsymbol{\nu})$$

$$\begin{aligned} \log P(\mathbf{Y}, \mathbf{Z}; \boldsymbol{\theta}) \quad = \quad & \sum_k Z_{1k} \log \nu_k \\ & + \sum_{i>1} \sum_{k,\ell} Z_{i-1,k} Z_{i,\ell} \log \pi_{k\ell} \\ & + \sum_i \sum_k Z_{ik} \log f(y_i; \gamma_k) \end{aligned}$$

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$$\begin{aligned}\mathbb{E}\left(\log P(\mathbf{Y}, \mathbf{Z}) | Y_{1:N}\right) \quad = \quad & \sum_k \mathbb{E}\left(Z_{1k} | Y_{1:N}\right) \log \nu_k \\ & + \sum_{i>1} \sum_{k,\ell} \mathbb{E}\left(Z_{i-1,k} Z_{i,\ell} | Y_{1:N}\right) \log \pi_{k\ell} \\ & + \sum_i \sum_k \mathbb{E}\left(Z_{ik} | Y_{1:N}\right) \log f\left(X_i; \gamma_k\right)\end{aligned}$$

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\mathbb{E}\left(\log P(\mathbf{Y}, \mathbf{Z}) | Y_{1:N}\right) \quad = \quad & \sum_k \mathbb{P}\left(Z_1 = k | Y_{1:N}\right) \log \nu_k \\
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& + \sum_i \sum_k \mathbb{P}\left(Z_i = k | Y_{1:N}\right) \log f(X_i; \gamma_k)
\end{aligned}$$

EM Algorithm (Baum Welch)

- Initialisation of $\boldsymbol{\theta}^{(0)} = (\Pi, \gamma_1, ..., \gamma_K)^{(0)}$.
- While the convergence is not reached

E-step Calculation of

$$\begin{aligned}\tau_{ik}^{(\ell)} &= P(Z_i = k | \mathbf{Y}, \boldsymbol{\theta}^{(\ell-1)}) \\ \eta_{ikh}^{(\ell)} &= \mathbb{E}[Z_{i-1,k} Z_{ih} | \mathbf{Y}, \boldsymbol{\theta}^{(\ell-1)}]\end{aligned}$$

M-step Maximization in $\boldsymbol{\theta} = (\boldsymbol{\pi}, \boldsymbol{\gamma})$ of

$$\sum_k \tau_{1k}^{(\ell)} \log \nu_k + \sum_{i>1} \sum_{k,h} \eta_{ikh}^{(\ell)} \log \pi_{kh} + \sum_i \sum_k \tau_{ik}^{(\ell)} \log f(y_i; \gamma_k)$$

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  E-step Smart algorithm Forward-Backward algorithm

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Reconstruction of hidden state $\mathbf{Z}$

Most credible value for Z_i: We are interested in

$$\underset{k}{\operatorname{argmax}} \, \mathbb{P}(Z_i = k|\mathbf{Y}) = \underset{k}{\operatorname{argmax}} \, \tau_{ik}.$$

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But force brut algorithm is not possible

$\longrightarrow$ a smart algorithm : Viterbi algorithm

Viterbi algorithm

Key quantity: The probability of the best hidden path from time 1 to i who finished in k

Viterbi algorithm

Key quantity:

$$\delta_i(k) = \max_{k_1,\ldots k_{i-1}} \mathbb{P}(Y_{1:i}, Z_1 = k_1, \ldots, Z_{i-1} = k_{i-1}, Z_i = k)$$

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- *Initialisation*

$$\begin{aligned} \delta_1(k) &= \mathbb{P}(Z_1 = k, Y_1) = \mathbb{P}(Z_1 = k)\mathbb{P}(Y_1 | Z_1 = k) = \nu(k) f_{\gamma_k}(y_1) \\ \psi_1(k) &= 0 \end{aligned}$$

Viterbi algorithm

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- *Recurrence, for $i = 1, \dots n-1$*

$$\begin{array}{rcl}\delta_{i+1}(k) & = & \max_{k_1,\dots,k_i} \mathbb{P}(Y_{1:i}, Z_1 = k_1, \dots, Z_{i-1} = k_{i-1}, Z_i = k_i, Z_{i+1} = k) \\ & = & \max_{k_i} \{\delta_i(k_i)\mathbb{P}(Y_i | Z_i = k_i)\mathbb{P}(Z_{i+1} | Z_i = k_i)\} \\ \psi_{i+1}(k) & = & \operatorname{argmax}_j \{\delta_i j \Pi_{jk}\}\end{array}$$

Estimation of HMM with R

```
library('depmixS4')

df <- data.frame(Y=Y)
K=2
m1 <- depmix(Y~1,data=df, nstates=2, family=gaussian())
fit.model <- fit(m1)
summary(fit.model)
Z.hat <- viterbi(fit.model)[,1]
table(Z, Z.hat)
```

State Space model

This is the trendy name for HMM, with potentially continuous value for **Z**.

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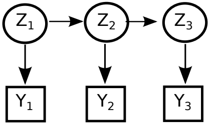
- **Z** is a sequence of hidden states and the observations **Y** are ruled by this sequence.



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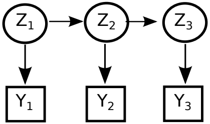

Z could be thougth as the actual locations and **Y** the observed locations.

- Estimation tools are EM algorithm or Bayesian framework (with monte carlo based estimation technics)

Plan

And more ...

- It is possible to include dependency in $\mathbf{Y}$.
- Markovian property could be removed (SHMM)
- Presented methods may be used on several signals (ex *Speed* and *angle*)
- Work in progress for continuous time Markov model and dependent observations.

But

Limitations

Trajectories are in continuous space and continuous time.

- Mostly, discrete time : effects of the sampling step and assumption of regularity.
- Segmentation methods consider a signal in time,
- Spatial information is lost.
- Those methods are useful to identify different regimes of movement. This difference may be due to behaviour or the environment or interaction.

Many thanks to Emilie Lebarbier, Marie-Laure Martin-Magniette, Stéphane Robin for some contents of the slides.

Many thanks to Andrea and Linda for the invitation and organisation.