Identifying patterns in trajectories

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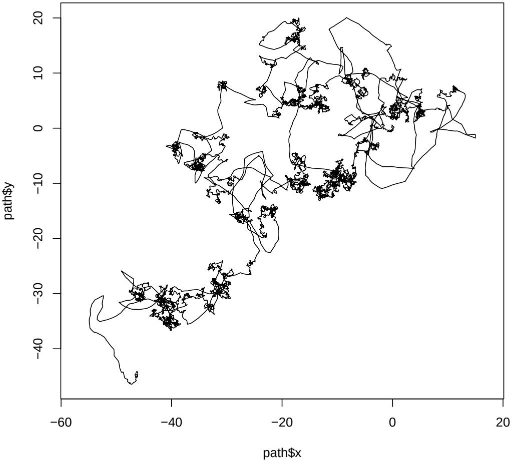
Movement Ecology Workshop 2015 - Port Elizabeth

Detection of homogenous region in trajectories

Why ?

- Different behaviour
- Link with different activities
- Link with different environmental condition



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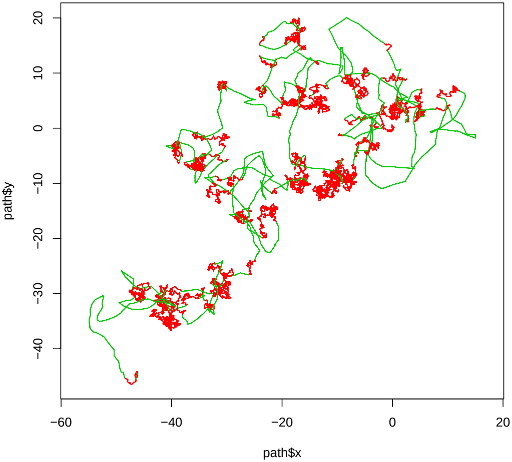

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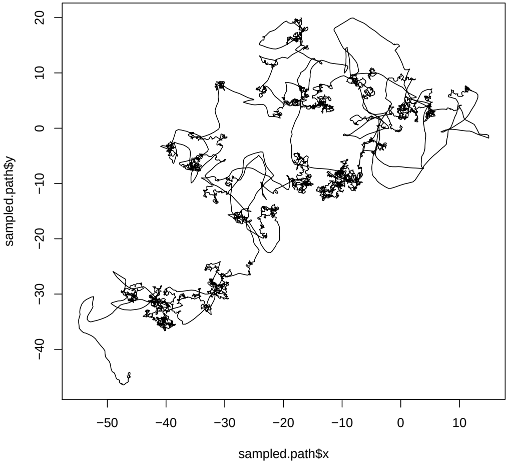

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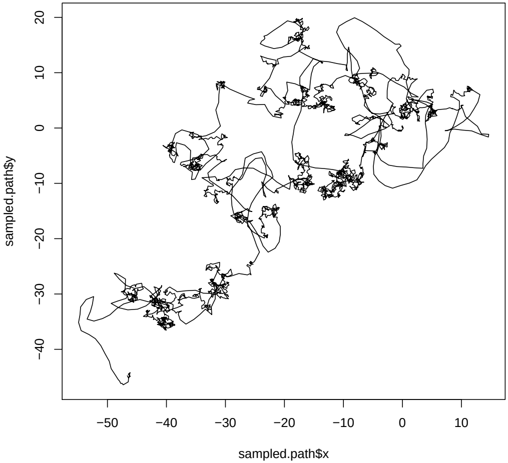

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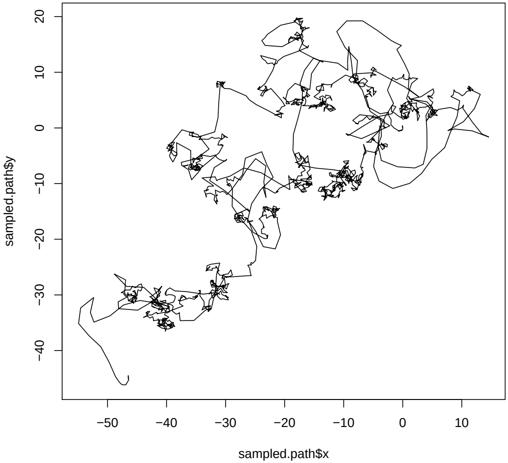

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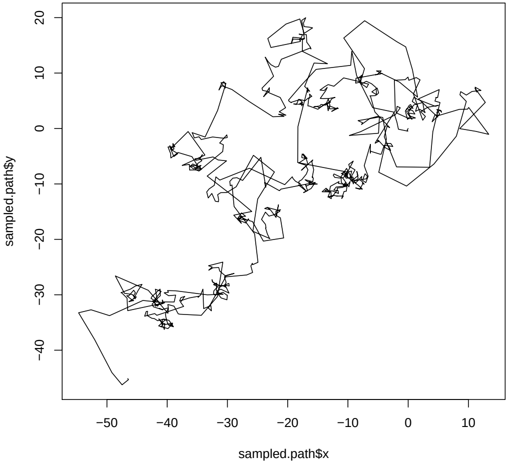

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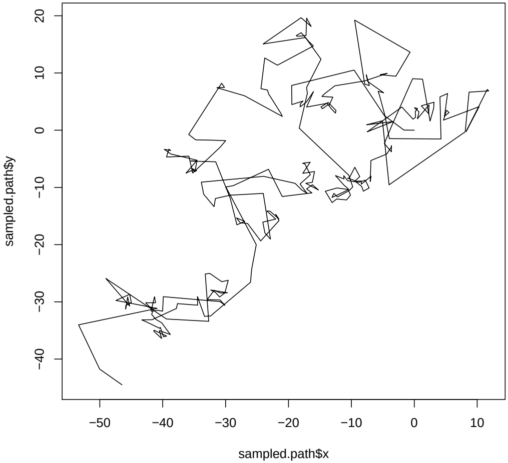

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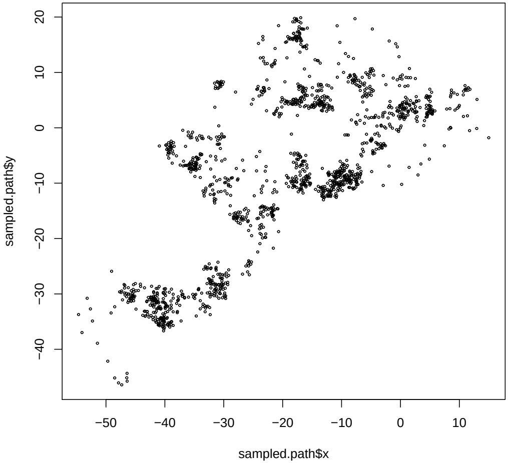


Summarising trajectories

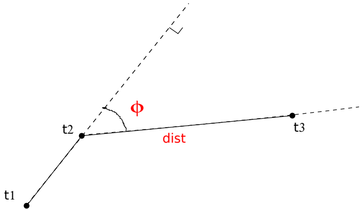
$(t_1, \ldots, t_N)$ denotes the time acquisition and $((x_1, y_1), \ldots, (x_N, y_N))$ the position at those times.

Trajetories as Turning angle and Speed

$$\mathbf{\Phi} = (\phi_2, \ldots, \phi_N)$$

$$\mathbf{S} = (S_2, \ldots, \S_N),$$

with $S_i = dist_i/(t_i - t_{i-1})$



Summarising trajectories

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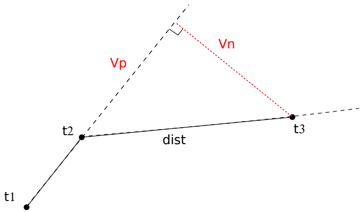
Trajetories as Persistent and Normal Velocity

$$\mathbf{V}^P = (V_2^P, \ldots, V_N^P)$$
$$\mathbf{V}^N = (V_2^N, \ldots, V_N^N)$$

with

$$V_i^P = S_i \, cos(\phi_i)$$
$$V_i^N = S_i \, sin(\phi_i)$$



Trajectories data

How trajectories data migh be considered?

- A sequence of (time, position)
- Turning angle and speed sequences
- Persistent and Normal Velocity sequences

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Model approach:

- Most methods don't consider the two phenomena : movement and sampling process.
- Results will be closely dependent of the sampling step.

Convention

Notation:

- $\mathbf{Y} = (Y_1, \ldots, Y_n) =$ observed data (typically Speed)
- $\mathbf{Z}$ unobserved data (typically State, for mixture and Hidden Markov model)
- $\boldsymbol{\theta}=$ the unknown parameters of $\mathbf{Y}$ and $\mathbf{Z}$.

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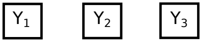
Graphical Representation (DAG):

Change point



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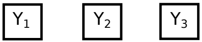
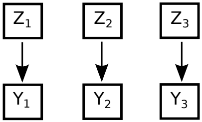


Mixture



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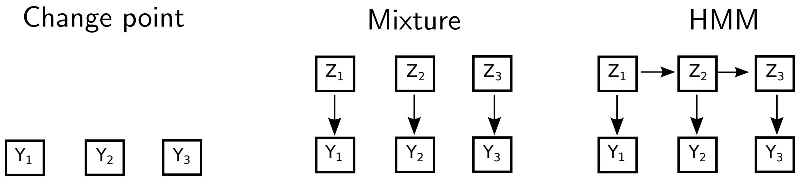
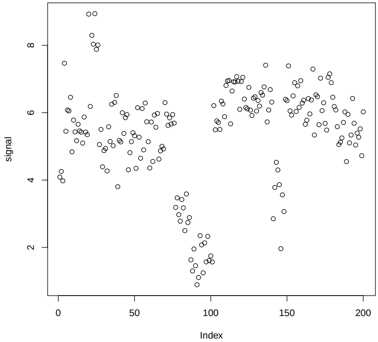


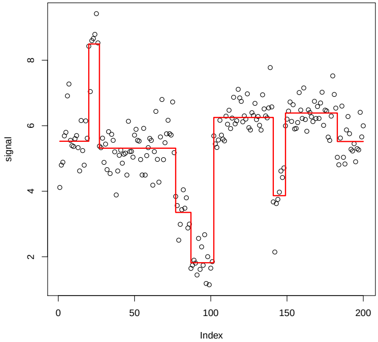
Plan

Change point detection context



Goal : Identifying homogenous region and abrupt changes in the signal.

Change point detection context



These regions may be interpreted afterwards.

Change point detection context

Two possible statistical point of view :

- Bayesian approach : the output is the posterior probability for each point to be a change point.
- *Frequentist* approach : the output is the best segmentation (according to a given critrion)

Underlying model

General framework :

- Data $Y_1, \ldots, Y_n$ are drawn from a given pdf, driven by unknown parameter θ
$$Y_t \sim f_\theta(.)$$
- θ values change at $K-1$ unknow instants, the change point : $t_1, \ldots, t_{K-1}$:
$$Y_t \sim f(\theta_k) \text{ if } t \text{ in region } I_k = [t_{k-1}+1, t_k]$$

Remark : $K-1$ change points $\Leftrightarrow$ K regions.

Underlying model

Change point detection in the trend :

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$$Y_t = \mu_k + E_t \quad \{E_t\}\text{i.i.d.} \sim \mathcal{N}(0, \sigma^2) \text{ if } t \text{ in portion } I_k,$$
for $k = 1, \ldots, K$.

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Estimation procedure

- Unknown parameters : $\mu_1, \ldots, \mu_K$, σ, and $T_1, \ldots, T_K$, but also K itself.

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  - for a given K, $\mu_1, \ldots, \mu_K$, $\sigma_1, \ldots, \sigma_K$, and $T_1, \ldots, T_K$ are estimated using maximum likelihood (and dynanic programming)
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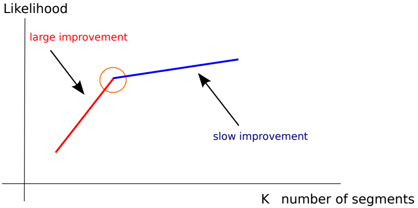

Estimation procedure

Likelihood

$$
\begin{aligned}
2log(P_K(T,\theta)) &= 2\sum_{k=1}^{K} \log f(\{Y_t\}_{t\in I_k};\theta_k) = 2\sum_{k=1}^{K}\sum_{t\in I_k} \log f(Y_t;\theta_k) \\
&= -n\log\sigma^2 - \frac{1}{\sigma^2}\sum_{k=1}^{K}\sum_{t\in I_k}(Y_t-\mu_k)^2 + \text{cst.}
\end{aligned}
$$

Estimations

$$
(\hat{T},\hat{\theta}) = \underset{(T,\theta)}{\operatorname{argmax}}\, log(P_K(T,\theta))
$$

If the change points are known

$$
\widehat{\mu}_k = \frac{1}{n_k}\sum_{t\in I_k} Y_t \qquad \widehat{\sigma}^2 = \frac{1}{n}\sum_{k=1}^{K}\sum_{t\in I_k}(Y_t-\widehat{\mu}_k)^2
$$

Finding the K-1 change points

Considering all possible segmentations, the best segmentation minimizes

$$J_k(1, n) = \sum_{k=1}^{K} \sum_{t \in I_k} (Y_t - \widehat{\mu}_k)^2.$$

Finding the K-1 change points

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But,

$$\binom{n-1}{K-1}$$

possible choices for the $K-1$ positions :

$\Rightarrow$ Practically impossible even for small K and n

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$K = 10,\ n = 200,\ \binom{K-1}{n-1} \approx 2.10^{16}$

Dynamic programming, with complexity $(\mathcal{O}(n^2))$.

Finding the K-1 change points

Dynamic programming, with complexity ($\mathcal{O}(n^2)$).

Sub-paths of the optimal path are themselves optimal,

Bellmann optimality

Finding the K-1 change points

Dynamic programming, with complexity ($\mathcal{O}(n^2)$).

Initialisation: Compute for $0 \leq i < j \leq n$, cost of portion I_{ij} :

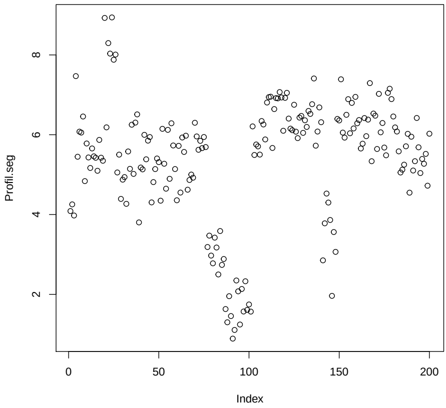
$$J_1(i,j) = \sum_{t=i+1}^{j} (Y_t - \widehat{\mu})^2$$

Etape k: Compute for $2 \leq k \leq K$, $J_k(i,j)$ the cost of the best segmentation in k segments between i and j.

$$J_k(i,j) = \min_{i<h<j} \left[J_{k-1}(i,h) + J_1(h+1,j) \right].$$

How to perform this segmentation approach ?

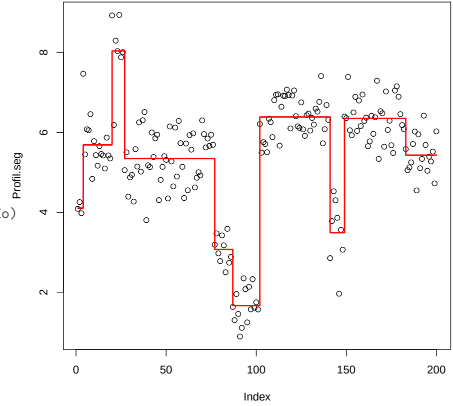
```
load("../Data/dataSegmentation.Rd")
summary(Profil.seg)


   Min. 1st Qu.
 0.8904  4.8720
 Median    Mean
 5.6990  5.3660
3rd Qu.    Max.
 6.3070  8.9400


plot(Profil.seg)
```



How to perform this segmentation approach ?

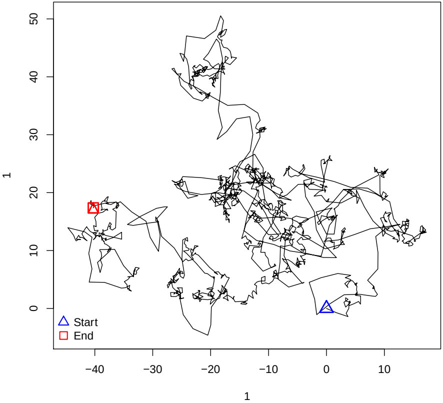
```r
library('cghseg')
## format data into CGHdata
signalCGH    <- new("CGHdata",Y=Profil.seg)
CGHo         <- new("CGHoptions")
calling(CGHo)<- FALSE ## no classification

segSignal   <- uniseg(.Object=signalCGH,CGHo=CGHo)
segSignalProf <- getsegprofiles(segSignal)
plot(Profil.seg)
lines(1:length(segSignalProf),
      segSignalProf, type="s", col=2, lwd=2)
```



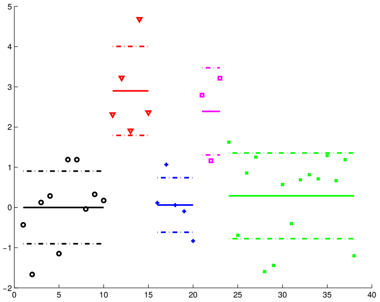
Do it yourself

```
load(file="../Data/trajEx.Rd")
plot(traj.ex, addpoints = F)
legend("bottomleft",pch=c(2, 0), col=c(4,2),
       legend=c("Start", "End"), bty = "n",
       pt.lwd = c(1.5,1.5), pt.cex = c(1.5,1.5))
```

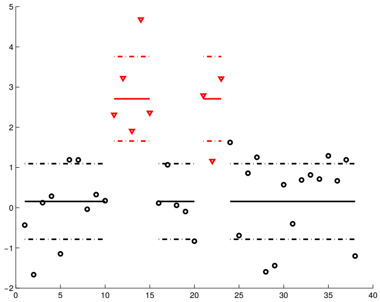


When Segmentation is not sufficient

Pure segmentation



Segmentation + classification



Segmentation-Clustering

- Assuming a secondary underlying structure of the segments into P groups with weights $\pi_1, ..., \pi_P (\sum_p \pi_p = 1)$.
- Let's define hidden variables Z_{kp}, indicators of the group to which segment k belongs.
- π_p denotes the proportion of group p.
- The distribution of the signal given the group of the segment is

$$t \in I_k, k \in p \qquad \Rightarrow \qquad Y_t \sim \mathcal{N}(m_p, \sigma^2)$$
$$Y^k | Z_{kp} = 1 \sim \mathcal{N}(m_p, \sigma^2).$$

- The parameters of this model are

$$\begin{aligned} \text{the breakpoint positions:} \quad T &= (t_1, ..., t_{K-1}), \\ \text{the mixture characteristics:} \quad \Theta &= (\pi_1, \ldots, \pi_P; \mu_1, \ldots, \mu_P, \sigma). \end{aligned}$$

Hybrid algorithm

2 levels of statistical units

- The inference of the breakpoints T is made at the position level t;
- The inference of the groups (status) (Θ, τ_{kp}) is made at the segment level k.

Hybrid algorithm

Alternate parameters estimation with K and P known

1. When T is fixed, the Expectation-Maximisation (EM) algorithm estimates Θ;

$$\hat{\Theta}^{(h+1)} = \arg\max_{\Theta} \left\{ \log \mathcal{L}_{KP} \left(\Theta, T^{(h)} \right) \right\}.$$

$$\log \mathcal{L}_{KP}(\hat{\Theta}^{(h+1)}; \hat{T}^{(h)}) \geq \log \mathcal{L}_{KP}(\hat{\Theta}^{(h)}; \hat{T}^{(h)})$$

2. When Θ is fixed, dynamic programming estimates T;

$$\hat{T}^{(h+1)} = \arg\max_{T} \left\{ \log \mathcal{L}_{KP} \left(\hat{\Theta}^{(h+1)}, T \right) \right\}.$$

$$\log \mathcal{L}_{KP}(\hat{\Theta}^{(h+1)}; \hat{T}^{(h+1)}) \geq \log \mathcal{L}_{KP}(\hat{\Theta}^{(h+1)}; \hat{T}^{(h)})$$

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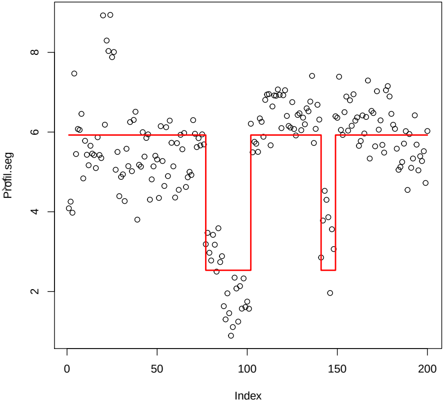

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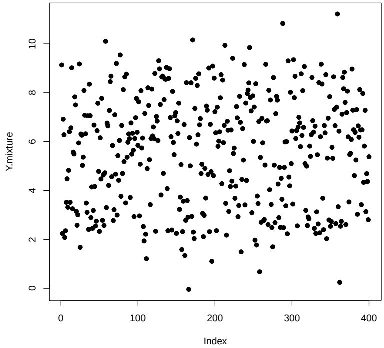
Problem presentation



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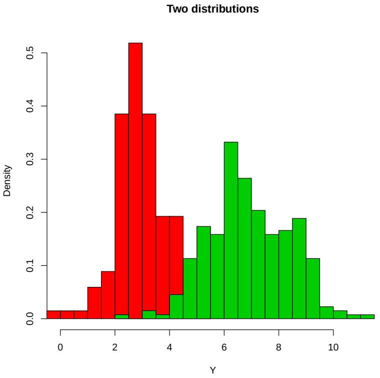

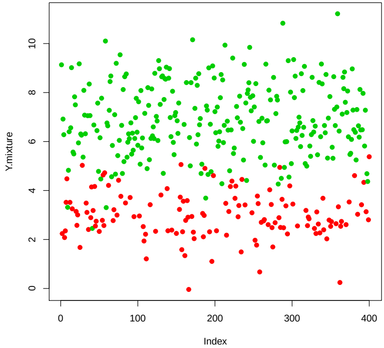

Problem presentation

Basic idea :

"Expert" threshold s

$$State_i = 1 \quad \text{if} Y_i < s$$

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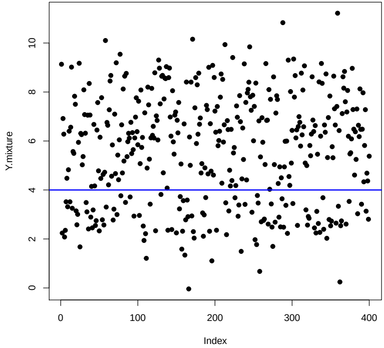

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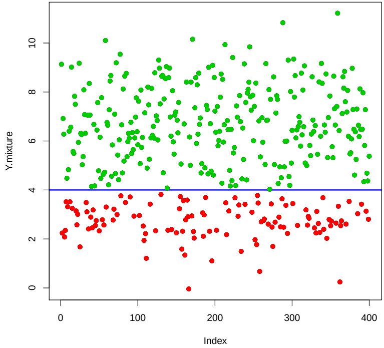

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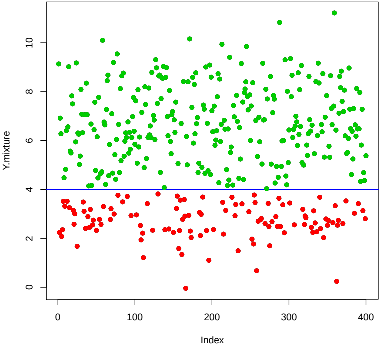
Estimating the threshold s and reconstruction of the hidden state (colour)

Compute the probability to belong to State 1 or 2.



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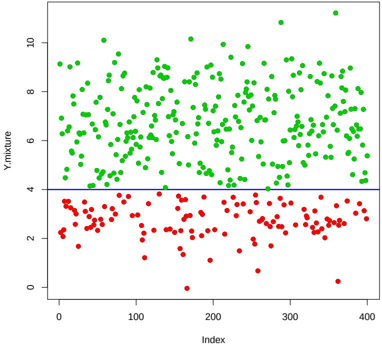
$\Longrightarrow$ Mixture Model



Proposed model

Model For a given number of states K,

- Modelling Z: $\pi_k = \mathbb{P}(Z_i = k), \quad k = 1, \ldots, K, \quad \sum_k \pi_k = 1$
  $Z_i \overset{i.i.d}{\sim} \mathcal{M}(1, \boldsymbol{\pi}), \quad P(Z_{ik} = 1) = \pi_k$
- Modelling Y: The $Y_i' s$ are assumed to be independent conditionnaly to $\mathbf{Z}$: $(Y_i | Z_i = k) \overset{i.i.d}{\sim} f_{\theta_k}()$.

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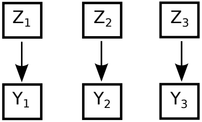

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```
K <- 2; N <- 400; mu <- c(3, 7); sigma <- c(1,1.5)
Z <- sample(1:2, size = N, replace=T, prob=c(0.3, 0.7))
plot(Z, col=Z+1, pch=15)
Y.mixture <- rnorm(N, mean=mu[Z], sd=sigma[Z])
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```

Proposed model

Model For a given number of states K,

- Modelling Z: $\pi_k = \mathbb{P}(Z_i = k), \quad k = 1, \dots, K, \quad \sum_k \pi_k = 1$
  $Z_i \overset{i.i.d}{\sim} \mathcal{M}(1, \boldsymbol{\pi})$
- Modelling Y: The $Y_i's$ are assumed to be independent conditionnaly to $\mathbf{Z}$: $(Y_i | Z_i = k) \overset{i.i.d}{\sim} f_{\theta_k}()$.

```
K <- 2; N <- 400; mu <- c(3, 7); sigma <- c(1,1.5)
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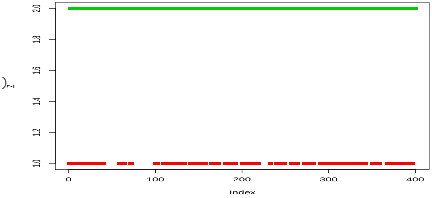

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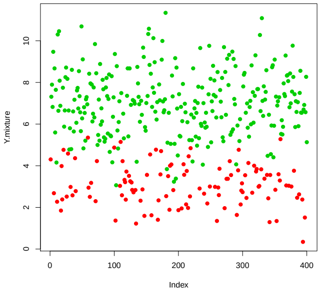

Model Properties

- Couples $\{(Y_i, Z_i)\}$ are i.i.d.
- Label switching:
  the model is invariant for any permutation of the labels $\{1, \ldots, K\} \Rightarrow$ the mixture model has $K!$ equivalent definitions.
- Distribution of a Y_i:

$$P(Y_i) = \sum_{k=1}^{K} P(Y_i, Z_i = k) = P(Z_i = k)P(Y_i | Z_i = k)$$

- Distribution of $\mathbf{Y}$:

$$\begin{aligned} P(\mathbf{Y}; \boldsymbol{\theta}, \boldsymbol{\pi}) = \prod_{i=1}^{n} \sum_{k=1}^{K} P(Y_i, Z_i = k) &= \prod_{i=1}^{n} \sum_{k=1}^{K} P(Z_i = k)P(Y_i | Z_i = k) \\ &= \prod_{i=1}^{n} \sum_{k=1}^{K} \pi_k f_{\theta_k}(Y_i) \end{aligned}$$

Statistical inference of incomplete data models

Maximum likelihood estimate: We are looking for

$$(\widehat{\boldsymbol{\theta}}, \widehat{\boldsymbol{\pi}}) = \arg\max_{\boldsymbol{\theta}, \boldsymbol{\pi}} \log P(\mathbf{Y}; \boldsymbol{\theta}, \boldsymbol{\pi})$$

- Likelihood of the observed data (or observed likelihood):

$$\log P(\mathbf{Y}; \boldsymbol{\theta}, \boldsymbol{\pi}) = \sum_{i=1}^{n} \log \left[\sum_{k=1}^{K} \pi_k f_{\theta_k}(Y_i) \right]$$

- No analytical estimators.
- It is not always possible since this sum typically involves K^n terms : $2^{100} \approx 10^{30}$, the computation will take 10^{10} years on a 2014 computer.
- Brute force algorithm is not the way

And what if **Z** were observed ?

The complete likelihood is

$$\begin{aligned}\log P(\mathbf{Y}, \mathbf{Z}; \boldsymbol{\theta}, \boldsymbol{\pi}) &= \log P(\mathbf{Z}; \boldsymbol{\pi}) + \log P(\mathbf{Y}|\mathbf{Z}; \boldsymbol{\theta}) \\ &= \sum_i \sum_k Z_{ik} \log \pi_k + \sum_i \sum_k Z_{ik} \log f_{\theta_k}(Y_i) \\ &= \sum_i \sum_k Z_{ik} [\log \pi_k + \log f_{\theta_k}(Y_i)].\end{aligned}$$

Now, the sum contains nK (200 if $n = 100$ and $K = 2$) terms. It is much easier.

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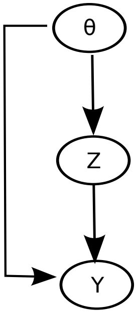
Unfortunately **Z** are unknown.

Idea: Replace Z_i, by our best guess, that is :

$$\tau_{ik} := \mathbb{E}(Z_i = k|Y_i) = P(Z_i = k|Y_i)$$

More generally - EM algorithm



Bayes Formula

$$
\begin{aligned}
P(\mathbf{Y}, \mathbf{Z}; \boldsymbol{\theta}) &= P(\mathbf{Y} | \mathbf{Z}; \boldsymbol{\theta}) P(\mathbf{Z}; \boldsymbol{\theta}), \\
&= P(\mathbf{Z} | \mathbf{Y}; \boldsymbol{\theta}) P(\mathbf{Y}; \boldsymbol{\theta}).
\end{aligned}
$$

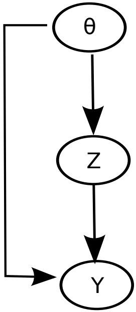
Therefore,

$$
\begin{aligned}
\log P(\mathbf{Y}; \boldsymbol{\theta}) &= \log \left\{ P(\mathbf{Y}, \mathbf{Z}; \boldsymbol{\theta}) / P(\mathbf{Z} | \mathbf{Y}; \boldsymbol{\theta}) \right\} \\
&= \log P(\mathbf{Y}, \mathbf{Z}; \boldsymbol{\theta}) - \log P(\mathbf{Z} | \mathbf{Y}; \boldsymbol{\theta})
\end{aligned}
$$

For a given $\boldsymbol{\theta}_0$, we may compute $P_{\boldsymbol{\theta}_0} = P(\mathbf{Z} | \boldsymbol{\theta}_0, \mathbf{Y})$ and

$$
\begin{aligned}
\log P(\mathbf{Y}; \boldsymbol{\theta}) &= \mathbb{E}_{\boldsymbol{\theta}_0}(\log P(\mathbf{Y}, \mathbf{Z}; \boldsymbol{\theta})) - \mathbb{E}_{\boldsymbol{\theta}_0}(\log P(\mathbf{Z} | \mathbf{Y}; \boldsymbol{\theta})) \\
&= Q(\boldsymbol{\theta}, \boldsymbol{\theta}_0) - H(\boldsymbol{\theta}, \boldsymbol{\theta}_0)
\end{aligned}
$$

More generally - EM algorithm



Since

$$\log P(\mathbf{Y};\boldsymbol{\theta}) = Q(\boldsymbol{\theta},\boldsymbol{\theta}_0) - H(\boldsymbol{\theta},\boldsymbol{\theta}_0),$$

and $H(\boldsymbol{\theta},\boldsymbol{\theta}_0)$ achieves its maximum in $\boldsymbol{\theta}_0$,

$$\log P(\mathbf{Y};\boldsymbol{\theta}) - \log P(\mathbf{Y};\boldsymbol{\theta}_0) = (Q(\boldsymbol{\theta},\boldsymbol{\theta}_0) - Q(\boldsymbol{\theta},\boldsymbol{\theta}_0)) + (H(\boldsymbol{\theta}_0,\boldsymbol{\theta}_0) - H(\boldsymbol{\theta},\boldsymbol{\theta}_0)).$$

Expectation - Maximization algorithm

1. Phase E :
   Calculate $Q(\boldsymbol{\theta},\boldsymbol{\theta}^k)$ for every $\boldsymbol{\theta}$.
2. Phase M :
   Define $\boldsymbol{\theta}^{k+1} = argmax\, Q(\boldsymbol{\theta},\boldsymbol{\theta}^k)$

EM algorithm for independent mixture model

$$
\begin{aligned}
Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(\ell)}) &= \mathbb{E}_{\boldsymbol{\theta}^{(\ell)}}(\log P(\mathbf{Y}, \mathbf{Z}; \boldsymbol{\theta})) \\
&= \mathbb{E}_{\boldsymbol{\theta}^{(\ell)}} \left\{ \sum_i \sum_k Z_{ik} [\log \pi_k + \log f_{\theta_k^{(\ell)}}(Y_i)] \right\} \\
&= \sum_i \sum_k \mathbb{E}_{\boldsymbol{\theta}^{(\ell)}}(Z_i = k | Y_i) \log \left[\pi_k f_{\theta_k^{(\ell)}}(Y_i) \right]
\end{aligned}
$$

Recall that $\tau_{ik}^{(\ell)} := P_{\boldsymbol{\theta}^{(\ell)}}(Z_i = k | Y_i)$

$$
Q(\boldsymbol{\theta}; \boldsymbol{\theta}^{(\ell)}) = \sum_i \sum_k \tau_{ik}^{(\ell)} \log \pi_k + \sum_i \sum_k \tau_{ik}^{(\ell)} \log f_{\theta_k^{(\ell)}}(Y_i)
$$

$\rightarrow$ Need to estimate $\tau_{ik}^{(\ell)}$

EM algorithm for independent mixture model

- Initialisation of $\boldsymbol{\theta}^{(0)} = (\pi_1, ..., \pi_K, \theta_1, ..., \theta_K)^{(0)}$.

EM algorithm for independent mixture model

- Initialisation of $\boldsymbol{\theta}^{(0)} = (\pi_1, ..., \pi_K, \theta_1, ..., \theta_K)^{(0)}$.
- Alternate

E-step Calculation of

$$\tau_{ik}^{(\ell)} = P(Z_i = k | y_i, \boldsymbol{\theta}^{(\ell-1)}) = \frac{\pi_k^{(\ell-1)} f_{\theta_k^{(\ell-1)}}(y_i)}{\sum_{k'} \pi_{k'}^{(\ell-1)} f_{\theta_{k'}^{(\ell-1)}}(x_i)}$$

M-step Maximization of

$$(\boldsymbol{\pi}, \gamma) \longmapsto \sum_i \sum_k \tau_{ik}^{(\ell)} [\log \pi_k + \log f(x_i; \gamma_k)]$$

In our situation

- $Z \in \{1,2\}$: $P(Z=1) = \pi_1$ and $P(Z=2) = 1 - \pi_1$
- For $k = 1$ or 2, $(X|Z=k) \sim \mathcal{N}(\mu_k, \sigma_k^2)$
- the parameter vector is $(\pi, \mu_1, \mu_2, \sigma_2^2, \sigma_2^2)$
  Assume that n observations $y_1, y_2, ..., y_n$ are available
- The parameter estimators at step $(\ell+1)$ of the EM algorithm are given by:

$$
\begin{aligned}
\widehat{\pi}_1^{(\ell+1)} &= \frac{1}{n}\sum_{i=1}^{n} \tau_{i1}^{(\ell)}, \\
\widehat{\mu}_k^{(\ell+1)} &= \frac{1}{\sum_{i=1}^{n} \tau_{ik}^{(\ell)}} \sum_{i=1}^{n} \tau_{ik}^{(\ell)} y_i \\
\widehat{\sigma}^2_{k^{(\ell+1)}} &= \frac{1}{\sum_{i=1}^{n} \tau_{ik}^{(\ell)}} \sum_{i=1}^{n} \tau_{ik}^{(\ell)} (y_i - \widehat{\mu}_k^{(\ell)})^2
\end{aligned}
$$

$\rightarrow$ They are a weighted version of the usual maximum likelihood estimates.

Back on earth - Practically speaking

```
#library('mclust')
library('mixtools')
Y.clustering <- normalmixEM (Y.mixture, lambda = NULL, mu = NULL, sigma = NULL, k = 2,
                             mean.constr = NULL, sd.constr = NULL,
                             epsilon = 1e-07, maxit = 1000, maxrestarts=20)


number of iterations= 62


summary(Y.clustering)


summary of normalmixEM object:
         comp 1
lambda 0.242871
mu     2.908037
sigma  0.896202
         comp 2
lambda 0.757129
mu     6.893074
sigma  1.617269
loglik at estimate:  -870.9465


Y.clustering$posterior


             comp.1
  [1,] 3.825020e-01
  [2,] 3.397032e-06
```

Plan

Markov chain model

Modelling dependence between time step : If an animal is feeding at time i, he has more chance to be feeding at time $i+1$ than if he was travelling at time i.

$$P(Z_{i+1} = 1 | Z_i = 1) \neq P(Z_{i+1} = 1 | Z_i = 2)$$

Markov Chain definition **Z** is a Markov chain if

$$P(Z_{i+1} | Z_{1:i}) = P(Z_{i+1} | Z_i)$$

Z is completely defined by the distribution $\mu_1 = P(Z_1)$ and the transition matrix

$$\Pi = \begin{bmatrix} \pi_{11} & 1-\pi_{11} \\ 1-\pi_{22} & 1-\pi_{22} \end{bmatrix}$$

Markov chain simulation

```r
#
N <- 200
pi11 <- 0.6
pi22 <- 0.7
## initial distribution
mu1 <- c(0.5, 0.5)
##transition matrix
PI <- matrix(c(pi11, 1-pi11, 1-pi22, pi22), ncol=2, byrow = T)

##initialisation of Z
Z <- rep(NA, N)
Z[1] <- sample(1:2, size=1, prob = mu1)
for( i in 1:(N-1))
{
  Z[i+1] <- sample(1:2, size=1, prob = PI[Z[i],])
}
plot(1:N, Z, "s")
points(1:N, Z, col=Z+1, pch=19)
```

Hidden Markov Chain model

Hidden State Modelling: $\mathbf{Z}$ is assumed to follow a Markov Chain model with unknown initial distribution $\boldsymbol{\mu}$ and transition matrix $\mathbf{\Pi}$.

Plan

Plan

Circular Distribution

If $Z \sim WC(\mu, \gamma)$,

$$f_{WC}(\theta; \mu, \gamma) = \sum_{n=-\infty}^{\infty} \frac{\gamma}{\pi(\gamma^2 + (\theta - \mu + 2\pi n)^2)}$$