Formulas for the model built on catch declarations

Notations

We model catch declarations D_j (available at coarse resolution through logbooks data) as a sum of Y_i punctual observations (which are unknown i.e. latent variables) each one realised at one fishing position x_i (known through VMS data).

We note:

- $\mathcal{P}_j$: the vector of all fishing positions related to the j^{th} declaration.
- $j \in \{1, ..., l\}$ with l the number of declarations.
- $i \in \{1, ..., m_j\}$ with m_j the number of fishing positions belonging to the j^{th} declaration.

$$D_j = \sum_{i \in \mathcal{P}_j} Y_i$$

Reparameterization of the lognormal distribution

The lognormal distribution is usually written as $Z \sim \mathcal{LN}(\rho; \sigma^2)$ with $Z = e^{\rho + \sigma N}$ and $N \sim \mathcal{N}(0, 1)$. In this case, $E(Z) = e^{\rho + \frac{\sigma^2}{2}}$ and $Var(Z) = (e^{\sigma^2} - 1)e^{2\rho + \sigma^2}$.

We choose to slightly reparameterize the lognormal distribution. Let's define $\rho = ln(\mu) - \frac{sigma^2}{2}$, then:

- $Z = \mu e^{\sigma N - \frac{sigma^2}{2}}$
- $E(Z) = \mu$
- $Var(Z) = \mu^2(e^{\sigma^2} - 1) \Leftrightarrow \sigma^2 = ln(\frac{Var(Z)}{E(Z)^2} + 1)$

D_j probability distribution and moments

We have to express the probability distribution of D_j and its moments as a function of Y_i and its related moments. Let's assume $Y_i = C_i.Z_i$ is a zero-inflated lognormal distribution with C_i a binary random variable and Z_i a lognormal random variable.

$$C_i \sim \mathcal{B}(1 - p_i)$$

with $p_i = exp(-e^{\xi}.S(x_i))$ the probability to obtain a zero value.

$$Z_i \sim \mathcal{LN}(\frac{S(x_i)}{1 - p_i}, \sigma^2)$$

Here, Y_i, C_i and Z_i are observations of a latent field $S(x_i)$ at a sampled point x_i.

Probability of obtaining a zero declaration

$$
\begin{aligned}
P(D_j = 0|S, X) = \prod_{i \in \mathcal{P}_j} P(Y_i = 0|S, X), \\
= \exp\left\{ - \sum_{i \in \mathcal{P}_j} e^{\xi}.S(x_i) \right\} = \pi_j.
\end{aligned}
$$

Expectancy of a positive declaration

Following calculations are supposed to be conditionnal on S and X.

$$
E(D_j|D_j > 0) = \sum_{i \in \mathcal{P}_j} E(C_i Z_i | \exists i \in \mathcal{P}_j, C_i = 1)
$$

$$
\begin{aligned}
E(D_j|D_j > 0) &= E(D_j 1_{\{D_j > 0\}})/P\left(D_j > 0\right), \\
&= E(D_j 1_{\{D_j > 0\}})/\left(1 - \pi_j\right).
\end{aligned}
$$

As $E(D_j 1_{\{D_j > 0\}}) = E(D_j)$,

$$
\begin{aligned}
E(D_j|D_j > 0) &= \left(1 - \pi_j\right)^{-1} E(D_j), \\
&= \left(1 - \pi_j\right)^{-1} \sum_{i \in \mathcal{P}_j} E(C_i Z_i), \\
&= \left(1 - \pi_j\right)^{-1} \sum_{i \in \mathcal{P}_j} (1 - p_i) \frac{S(x_i)}{1 - p_i}, \\
&= \left(1 - \pi_j\right)^{-1} \sum_{i \in \mathcal{P}_j} S(x_i).
\end{aligned}
$$

Variance of a positive declaration

$$
Var(D_j|D_j > 0) = E(D_j^2|D_j > 0) - E(D_j|D_j > 0)^2.
$$

$$
E(D_j^2|D_j > 0) = (1 - \pi_j)^{-1} E(D_j^2 1_{\{D_j > 0\}}) = (1 - \pi_j)^{-1} E(D_j^2)
$$

$$
E(D_j|D_j > 0)^2 = ((1 - \pi_j)^{-1} E(D_j 1_{\{D_j > 0\}}))^2 = (1 - \pi_j)^{-2} E(D_j)^2
$$

Then,

$$
Var(D_j|D_j > 0) = (1 - \pi_j)^{-1} E(D_j^2) - (1 - \pi_j)^{-2} E(D_j)^2 = (1 - \pi_j)^{-1} Var(D_j) - \frac{\pi_j}{(1 - \pi_j)^2} E(D_j)^2.
$$

As the $(Y_i)_{i \in \mathcal{P}_j}$ are independent, $Var(D_j) = \sum_{i \in \mathcal{P}_j} Var(Y_i) = \sum_{i \in \mathcal{P}_j} Var(C_i.Z_i)$.

$$
\begin{aligned}
Var(C_i Z_i) &= E(C_i^2 Z_i^2) - E(C_i Z_i)^2, \\
&= E(C_i^2)E(Z_i^2) - E(C_i)^2 E(Z_i)^2, \\
&= (1-p_i)E(Z_i^2) - (1-p_i)^2 E(Z_i)^2, \\
&= (1-p_i)(Var(Z_i) + E(Z_i)^2) - (1-p_i)^2 E(Z_i)^2, \\
&= \frac{S(x_i)^2}{1-p_i}(e^{\sigma^2} - 1) + \frac{S(x_i)^2}{1-p_i} - S(x_i)^2, \\
&= \frac{S(x_i)^2}{1-p_i}(e^{\sigma^2} - (1-p_i))
\end{aligned}
$$

Sum up of the main formulas

$$
P(D_j = 0|S, X) = \exp\left\{ -\sum_{i \in \mathcal{P}_j} e^{\xi}.S(x_i) \right\} = \pi_j
$$

$$
E(D_j|D_j > 0) = \frac{\sum_{i \in \mathcal{P}_j} S(x_i)}{1 - \pi_j}
$$

$$
Var(D_j|D_j > 0) = \frac{\sum_{i \in \mathcal{P}_j} Var(Y_i)}{1 - \pi_j} - \frac{\pi_j}{(1-\pi_j)^2} E(D_j)^2
$$

$$
Var(Y_i) = \frac{S(x_i)^2}{1-p_j}(e^{\sigma^2} - (1-p_i))
$$

Assuming $D_j|D_j > 0$ also follow a lognroaml distribution we can write:

$$
D_j|D_j > 0 \sim \mathcal{LN}(\mu_j = E(D_j|D_j > 0), \sigma_j^2 = ln(\frac{Var(D_j|D_j > 0)}{E(D_j|D_j > 0)^2} + 1))
$$

0.0.1 Simulations