Advanced Bayesian modeling

Joachim Vandekerckhove

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$$\sum_{r=1}^{2} \left(\sum_{c=1}^{3} x_{rc} \right) = \sum_{c=1}^{3} \left(\sum_{r=1}^{2} x_{rc} \right)$$

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Sum of products vs. product of sums:

$$\sum_{r=1}^{2} \left(\prod_{c=1}^{3} x_{rc} \right) \neq \prod_{c=1}^{2} \left(\sum_{r=1}^{3} x_{rc} \right)$$

Order of operations

In general, order of operations matters:

$$f \circ g(x) \neq g \circ f(x)$$

Estimating parameters is an operation

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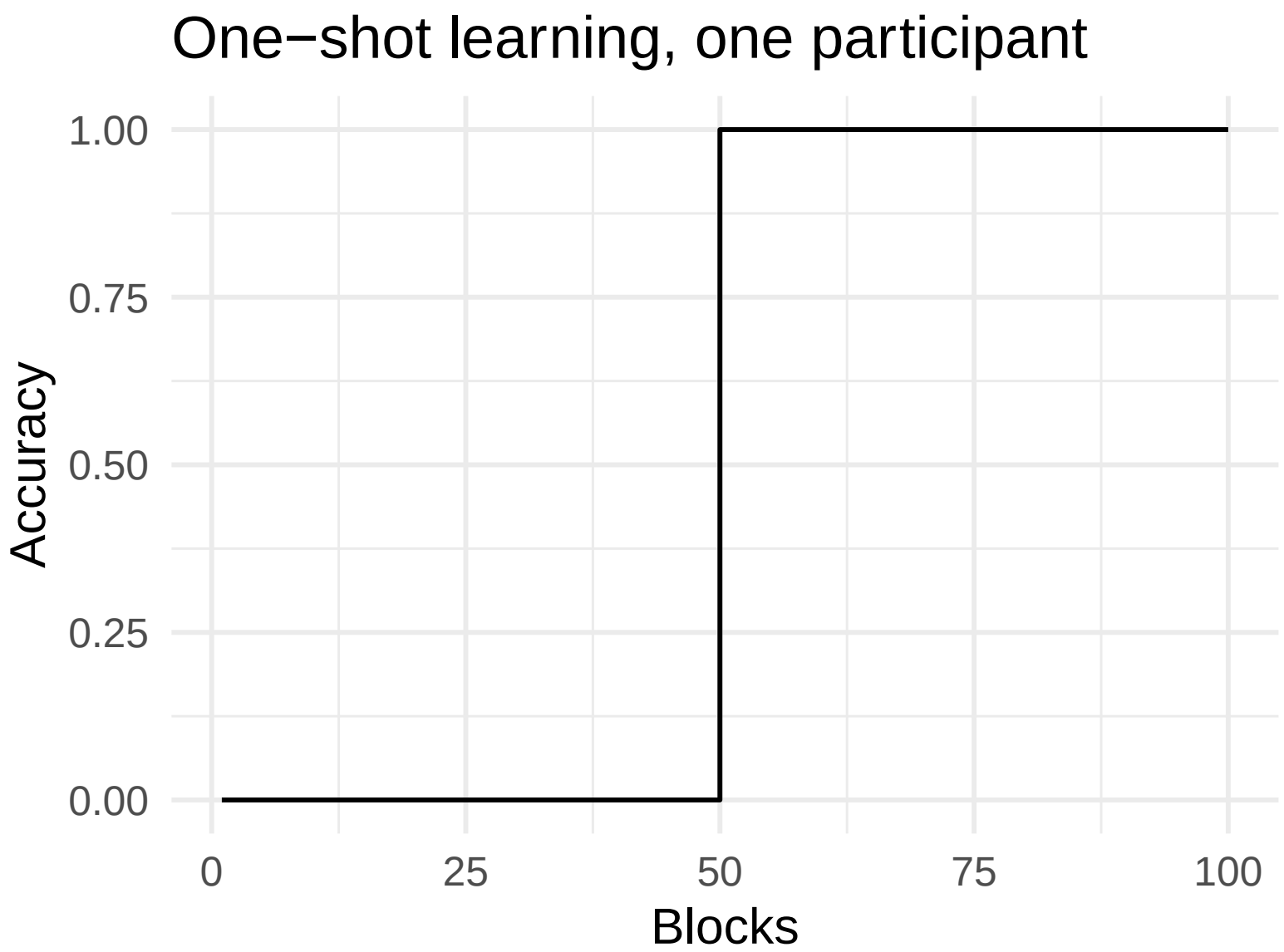
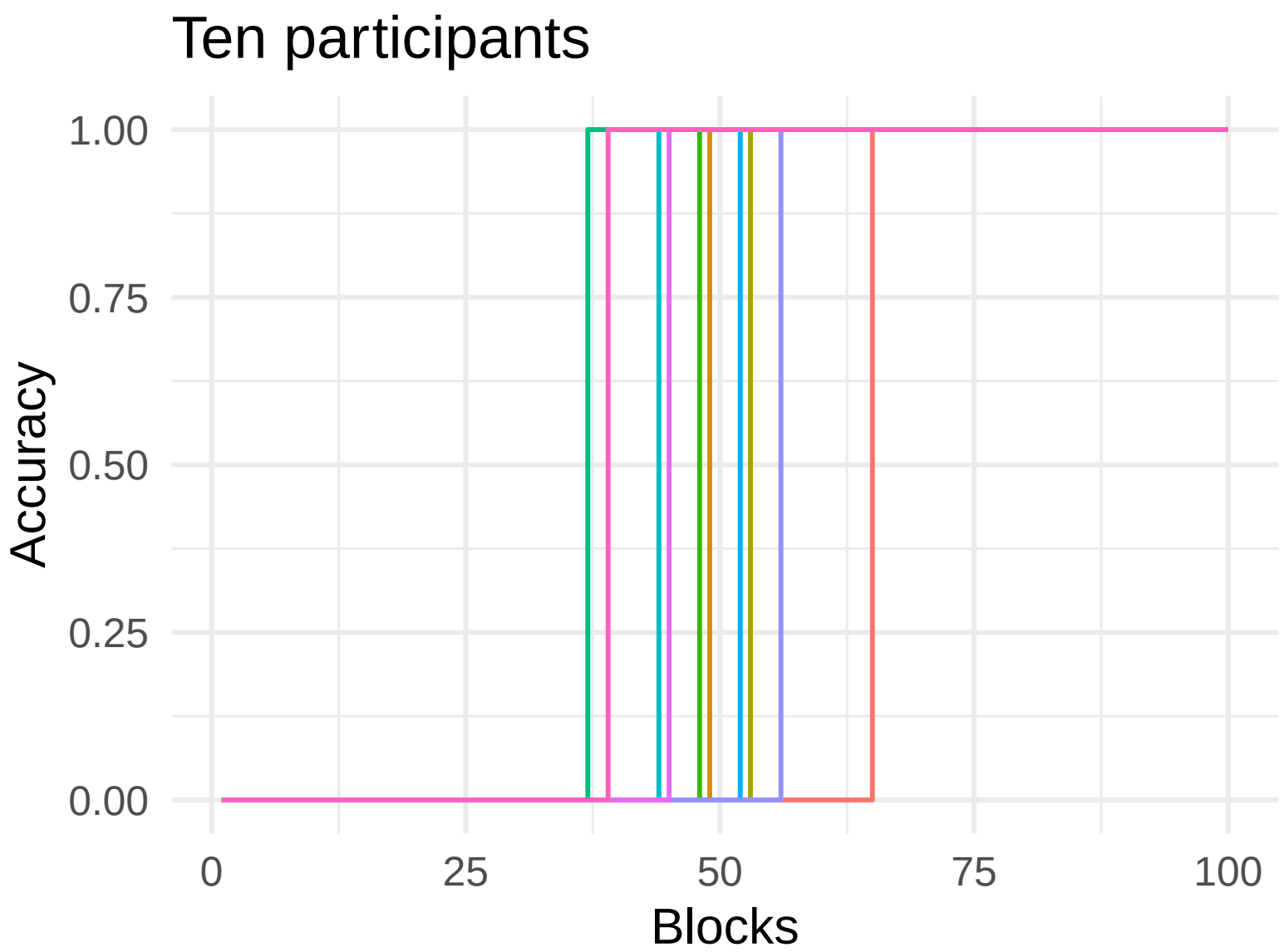
Do we want the average model parameters of the data or the model parameters of the average data?

$$\overline{f(x)} \neq f\left(\overline{x}\right)$$

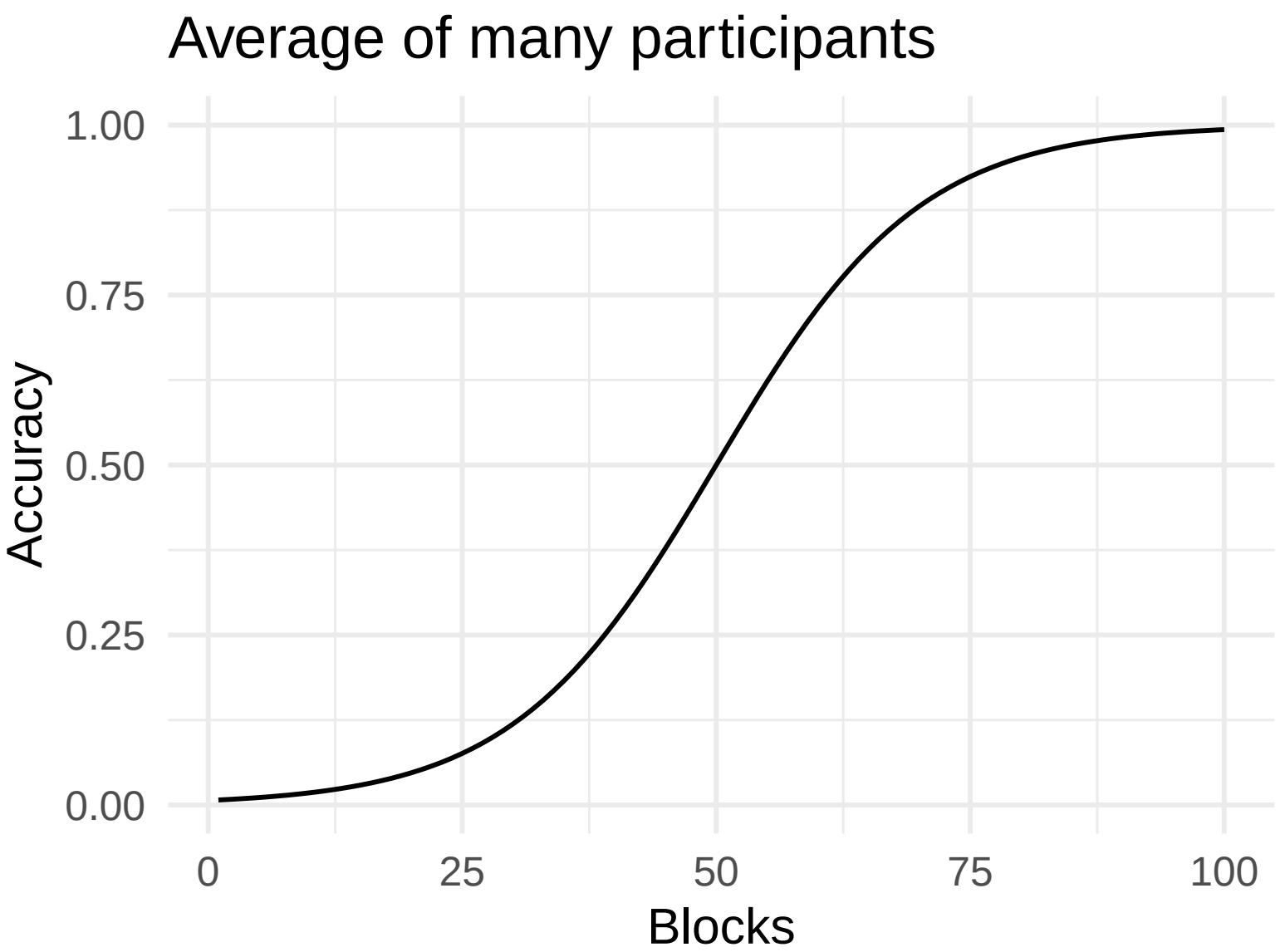
Insight learning



Insight learning

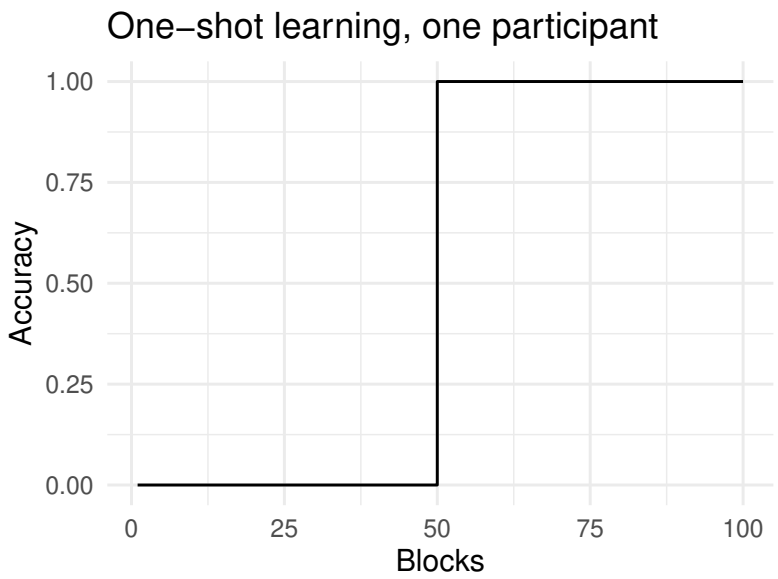


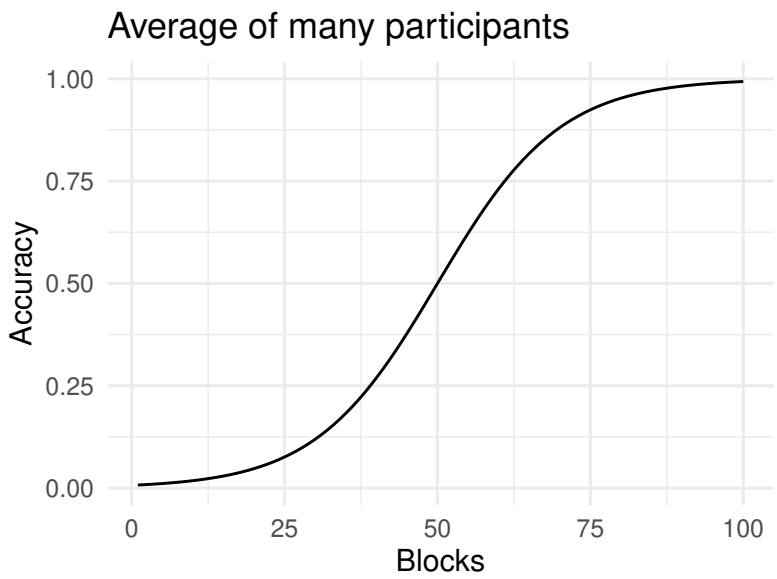
Insight learning



Insight learning

The "average" learning curve looks nothing like the person-specific learning curve!





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Instead, what we want here is to acknowledge that each person has their own trajectory, and then say something about the (average) properties of the trajectories.

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We want to make an abstraction of the data, which are something complicated that is generated by a process with parameters θ_p (for person p), and instead focus on parameters.

Basic hierarchy

$$\mathcal{M}_h : \begin{cases} x_p \sim \text{one-shot}\,(\theta_p) \\ \theta_p \sim N(\mu, \tau) \end{cases}$$

The hierarchical model contains one set of assumptions about the data given the model (the likelihood level), and another set of assumptions about structure among parameters.

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Here, the hierarchical parameters μ and τ tell us something about the population of participants, each with their own change point θ_p.

Population assumptions

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If $n_t > 2$, a measurement error may have occurred.

Insight learning

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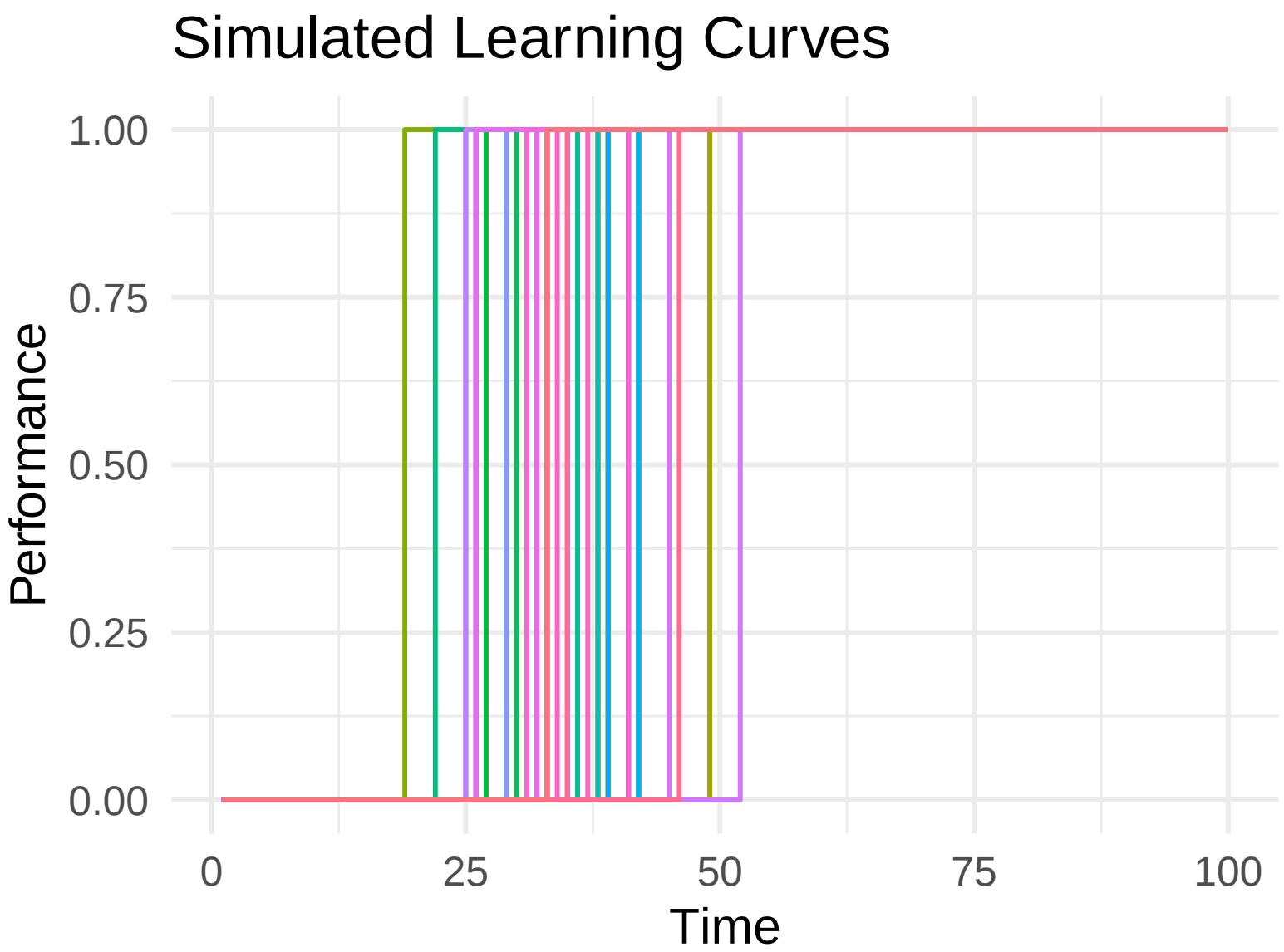
Back to the one-shot learning, we could try to characterize the population of participants in terms of their change point distribution $\theta_p \sim N(\mu, \tau)$.

- Let's generate learning curves for a population of $n = 50$ participants, each with a random change point.
- Let's use a step function to represent the learning curves.

Insight learning



Hierarchical model of insight learning

- We define a hierarchical model in JAGS to estimate each participant's change point.

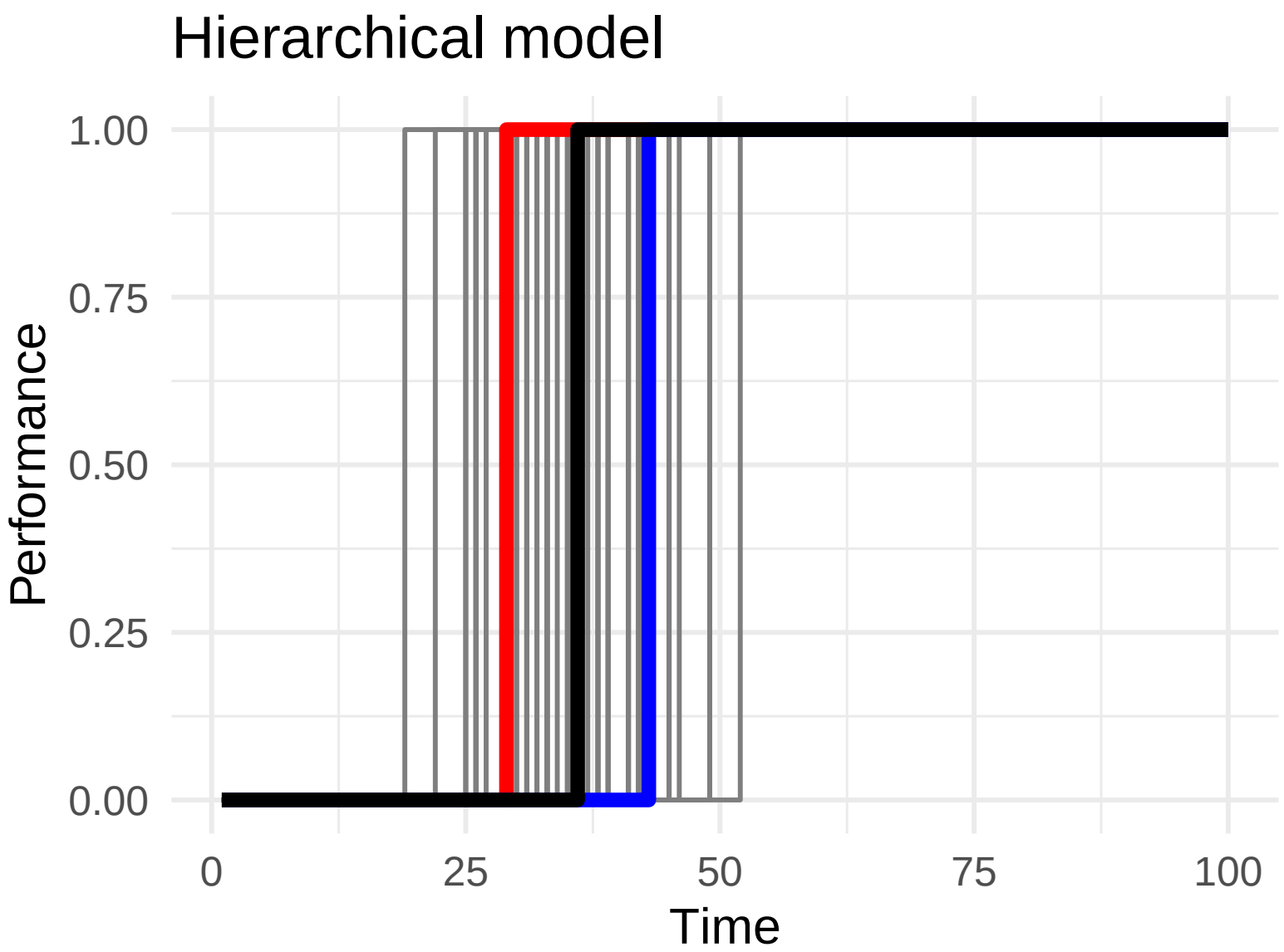
Hierarchical model of insight learning

- We define a hierarchical model in JAGS to estimate each participant's change point.
- The change point for each participant is modeled as a uniform distribution between 1 and 100.

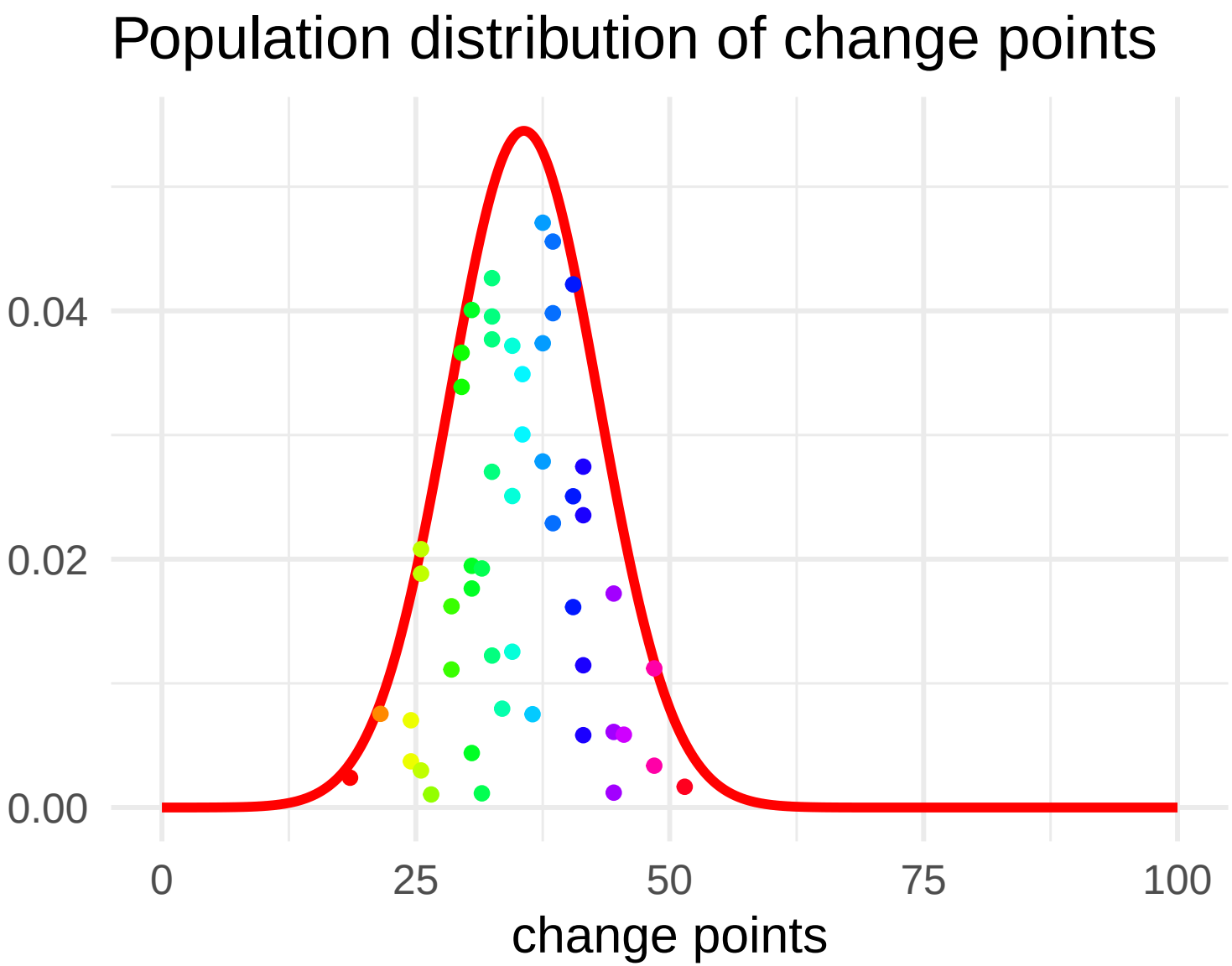
Hierarchical model of insight learning

```
##
## model {
##   for (j in 1:P) {
##     for (i in 1:N) {
##       y[i, j] ~ dbern(p[i, j])
##       p[i, j] <- ifelse(time[i] < theta[j], 0, 1)
##     }
##     theta[j] ~ dnorm(mu, tau)T(0,100)
##   }
##   mu  ~ dnorm(50, 0.05)T(0,100)
##   tau ~ dnorm(20, 0.10)T(0,)
##   sigma <- pow(tau, -0.5)
## }
```

Hierarchical model of insight learning



Population results



Parameter estimates

```
summary(samples_hrcl)[1]$
  statistics[c("mu","sigma"),
             c("Mean", "SD", "Time-series SE")] %>%
  print()
```

```
##              Mean        SD Time-series SE
## mu     35.642928 1.0245142    0.011413396
## sigma   7.320065 0.7517616    0.008193004
```

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Compare to:

- Simulated $\mu = 35$

Parameter estimates

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summary(samples_hrcl)[1]$
  statistics[c("mu","sigma"),
             c("Mean", "SD", "Time-series SE")] %>%
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```

```
##              Mean        SD Time-series SE
## mu     35.642928 1.0245142    0.011413396
## sigma   7.320065 0.7517616    0.008193004
```

Compare to:

- Simulated $\mu = 35$
- Simulated $\sigma = 8$

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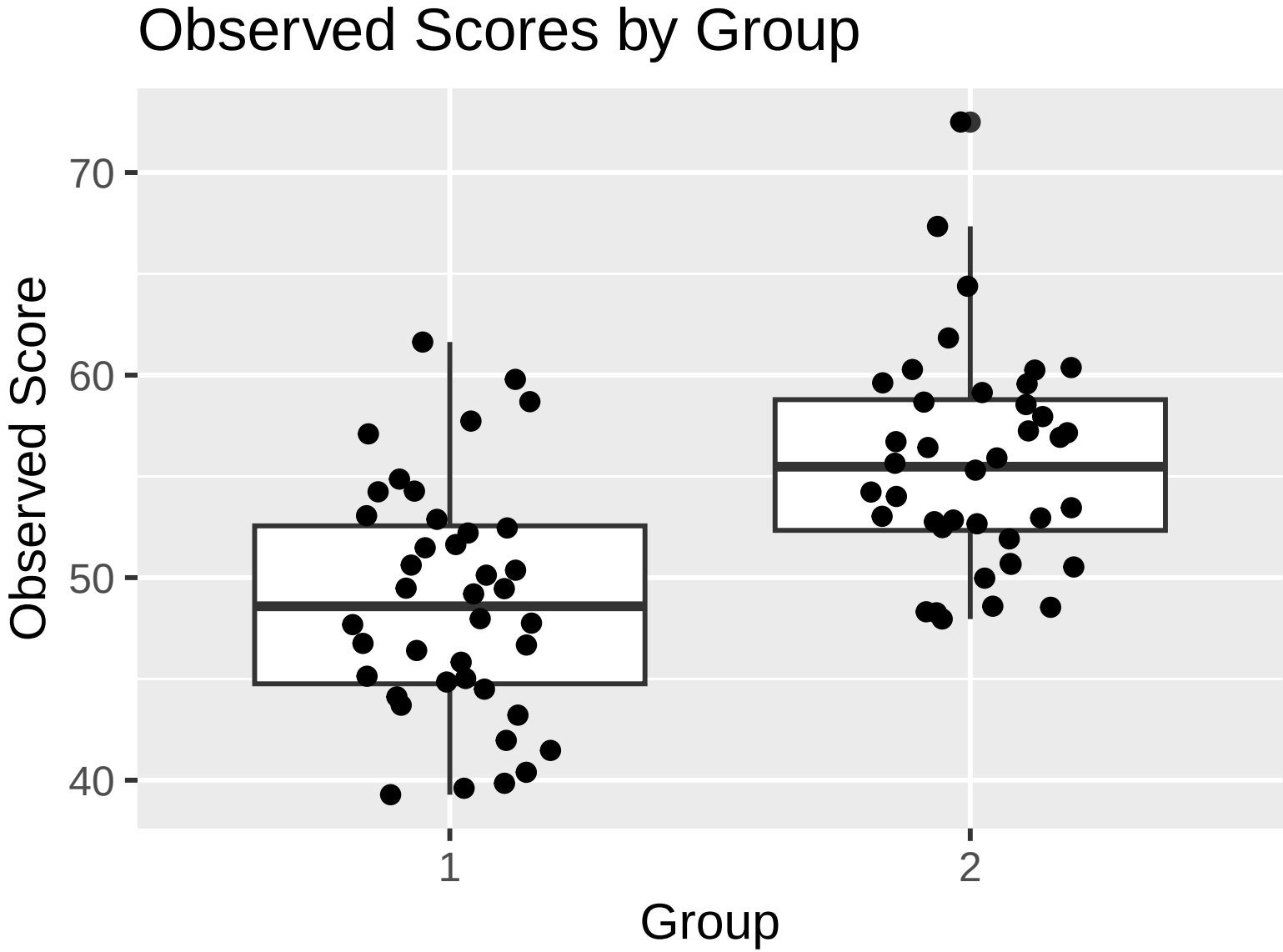
- Accumulate evidence for the absence of an effect
- Draw valid conclusions with almost no data
- Say things about means without knowing much about the basic units

Means without knowing much about the basic units

```r
N <- 80
group <- rep(1:2, each = N/2) # Make two groups
# True score for each participant depends on group
true_score <-
  rnorm(N,
        mean = rep(c(50, 55), each = N/2),
        sd   = 2)
# Observations are noisy
observed_score <- true_score +
  rnorm(N, mean = 0, sd = 5)

data <- data.frame(participant = 1:N,
                   group = factor(group),
                   true_score, observed_score)
```


Observed Scores by Group
Observed Score
70
60
50
40
1
2
Group

```
model_string <- "
model {
  for (i in 1:N) {
    # Likelihood
    observed_score[i] ~ dnorm(true_score[i], tau)
    # Hierarchical model
    true_score[i]     ~ dnorm(mu[group[i]], tau_true)
  }
  diff_mu ~ dnorm(0, 0.001)
  mu[1]   ~ dunif(0, 100)
  mu[2] <- mu[1] + diff_mu
  tau       ~ dgamma(0.1, 0.1)
  tau_true ~ dgamma(0.1, 0.1)
}
"
```

```r
data_list <- list(
  N = N,
  observed_score = data$observed_score,
  group = as.numeric(data$group)
)

# Initial values
inits <- function() {
  list(
    tau        = rgamma(1, 0.1, 0.1),
    tau_true   = rgamma(1, 0.1, 0.1),
    true_score = runif(N, 0, 100)
  )
}
```

```
jags_model <- jags.model(textConnection(model_string),
                         data     = data_list,
                         inits    =     inits,
                         n.chains =         3 ,
                         n.adapt  =      1000 )
```

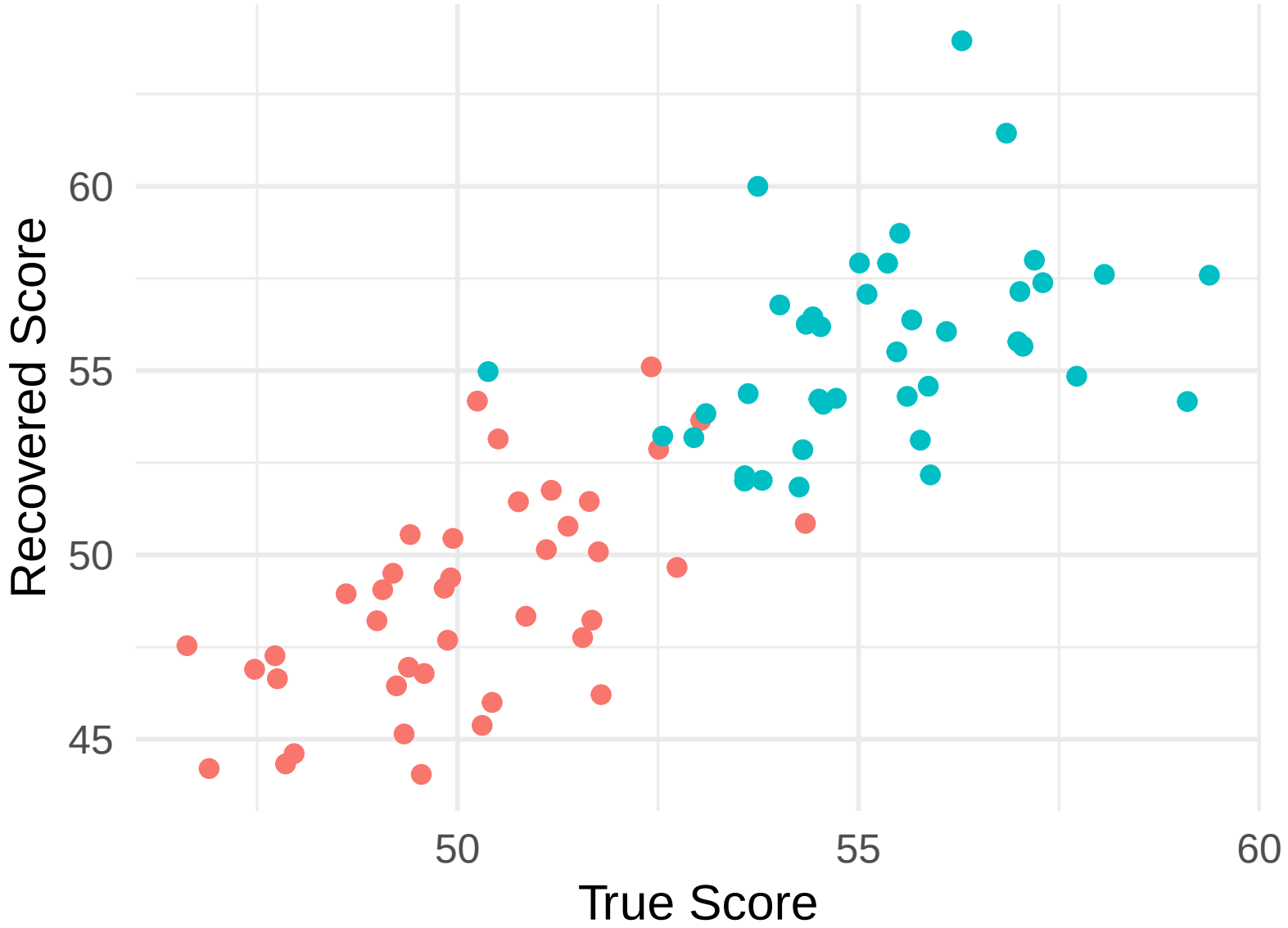
```
## Compiling model graph
##    Resolving undeclared variables
##    Allocating nodes
## Graph information:
##    Observed stochastic nodes: 80
##    Unobserved stochastic nodes: 84
##    Total graph size: 250
##
## Initializing model
```

```r
update(jags_model, n.iter = 1000)
samples <- coda.samples(jags_model,
                        variable.names = c("diff_mu",
                                           "true_score"),
                        n.iter = 5000)
```

```r
# Summarize the results
summary_diff_mu <- summary(samples)$
  statistics["diff_mu", ]
print(summary_diff_mu)
```

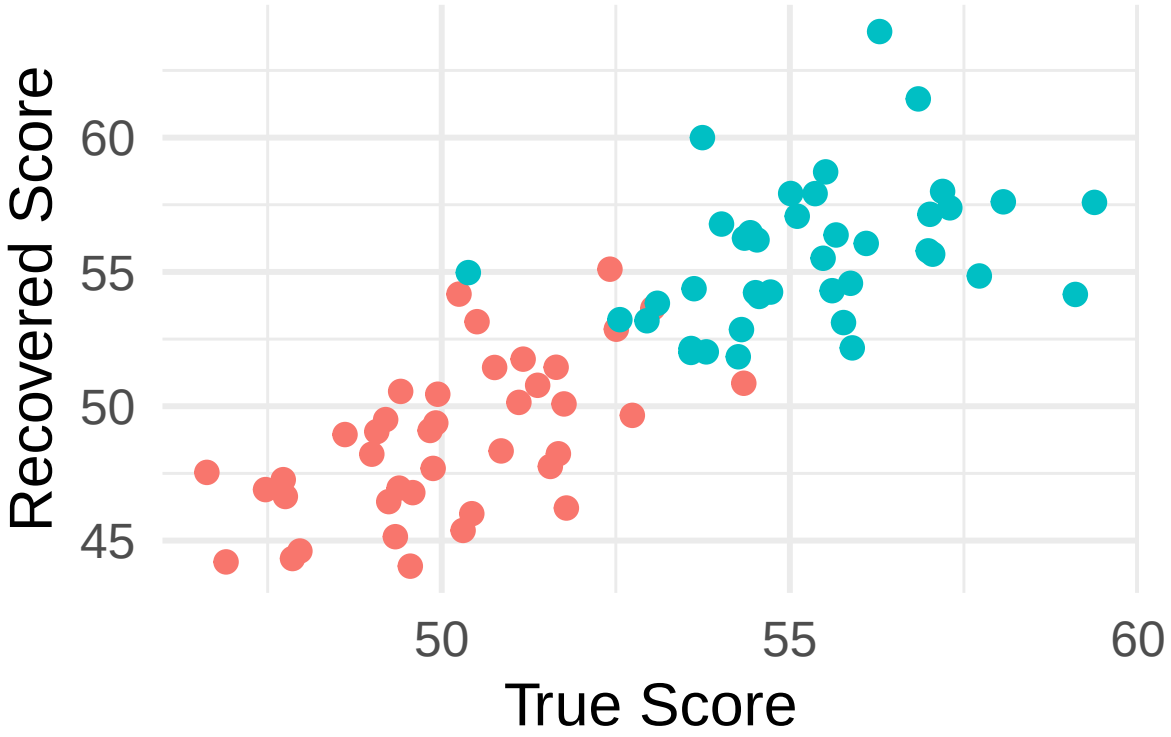
```
##           Mean             SD       Naive SE Time-series SE
##    6.875807265    1.214229771    0.009914145    0.045074384
```

```r
# Extract true scores from the posterior samples
true_scores_posterior <- as.data.frame(as.matrix(samples)) %>%
  select(starts_with("true_score")) %>%
  apply(2, mean)

# Add the recovered true scores to the data frame
data$recovered_true_score <- true_scores_posterior

# Scatter plot of true scores vs recovered true scores
p2 <- ggplot(data, aes(x = true_score, y = recovered_true_score,
                       color = group)) +
  geom_point() +
  labs(x = "True Score", y = "Recovered Score") +
  theme_minimal() +
  theme(legend.position = "none")
```


Recovered Score
60
55
50
45
50
55
60
True Score

Hierarchical recovery beats individual recovery

The difference between groups is well recovered as 6.8758073 ± 1.2142298, even though the true scores within each group are poorly recovered.



Perform inference with no actual data

Suppose these were scores from a test (and suppose the two groups are an "on-track" group and an "advanced" group). One student from the advanced group missed class. What do we know about student $N + 1$?

```
model_string <- "
model {
  for (i in 1:N) {
    # Likelihood
    observed_score[i] ~ dnorm(true_score[i], tau)
    # Hierarchical model
    true_score[i]     ~ dnorm(mu[group[i]], tau_true)
  }
  diff_mu ~ dnorm(0, 0.001)
  mu[1]   ~ dunif(0, 100)
  mu[2] <- mu[1] + diff_mu
  tau      ~ dgamma(0.1, 0.1)
  tau_true ~ dgamma(0.1, 0.1)

  true_score[N+1] ~ dnorm(mu[2], tau_true)
  observed_score[N+1] ~ dnorm(true_score[N+1], tau)
}
```

```r
data_list <- list(
  N = N,
  observed_score = c(data$observed_score, NA),
  group = as.numeric(data$group)
)

# Initial values
inits <- function() {
  list(
    tau         = rgamma(1, 0.1, 0.1),
    tau_true    = rgamma(1, 0.1, 0.1),
    true_score = runif(N+1, 0, 100)
  )
}
```

```
jags_model <- jags.model(textConnection(model_string),
                         data     = data_list,
                         inits    =     inits,
                         n.chains =         3 ,
                         n.adapt  =      1000 )
```

```
## Compiling model graph
##    Resolving undeclared variables
##    Allocating nodes
## Graph information:
##    Observed stochastic nodes: 80
##    Unobserved stochastic nodes: 86
##    Total graph size: 252
##
## Initializing model
```

```r
update(jags_model, n.iter = 1000)
samples <- coda.samples(jags_model,
                        variable.names = c("diff_mu",
                                           "true_score",
                                           "observed_score"),
                        n.iter = 5000)
```

```r
# Summarize the results
summary_new <- summary(samples)$
  statistics[c("true_score[81]","observed_score[81]"),
             c("Mean", "SD")]
print(summary_new)
```

```
##                         Mean       SD
## true_score[81]      55.62454 4.451111
## observed_score[81] 55.66355 5.828375
```

True score and observed score