

# Introduction to Bayesian inference

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Joachim Vandekerckhove

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- Special issue of Psychonomic Bulletin & Review (volume 25, 2018)

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- Not to be confused with “Bayes-in-the-head”, a set of psychological theories about how lay humans perform inference in daily life



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- In one meaning—the Bayesian meaning—probability is a **degree of belief**
- In another meaning—the classical, or frequentist meaning—probability is a statement of **expected frequency over many repetitions**

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- We are not usually interested in the frequency with which a well-defined process will achieve a certain outcome

## The Sum and Product Rules of probability

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These notations can be combined:  $P(A, B|\neg C, \neg D)$  is the probability that  $A$  and  $B$  are both true assuming that  $C$  and  $D$  are both false.



# The Product Rule of probability

With this notation in mind, we introduce the **Product Rule of probability**:

$$P(A, B) = P(B)P(A|B)$$

In words: the probability that  $A$  and  $B$  are both true is equal to the probability of  $B$  multiplied by the conditional probability of  $A$  *assuming  $B$  is true*.

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Of course,

$$\begin{aligned} P(A, B) &= P(B)P(A|B) \\ &= P(B, A) = P(A)P(B|A). \end{aligned}$$

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  - Possible outcomes of a coin flip:  $\{\text{heads}, \text{tails}\}$
  - Possible outcomes of a roll of a six-sided die:  $\{1, 2, 3, 4, 5, 6\}$
  - The truth of some hypothesis  $H$ , which must be either true or false:  $\{H, \neg H\}$

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- The probability of a single one of these events alone, say  $A$ , can be found by adding up the probabilities of all of the joint events that contain  $A$  as follows:

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- In words, the probability it rains today is the sum of two joint probabilities: (1) the probability it rains today and tomorrow, and (2) the probability it rains today but not tomorrow.

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In general, if  $\{B_1, B_2, \dots, B_K\}$  is a disjunctive set, the **Sum Rule of probability** states

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That is, the probability of event  $A$  alone is the sum of all the joint probabilities between  $A$  and the elements of a disjunctive set.

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Together, these two rules allow us to calculate probabilities in an incredible variety of circumstances. One combination of the two rules in particular is useful for scientific inference is **hypothesis testing**.

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  - *Conditional* on the truth of an hypothesis, likelihood functions specify the probability of a given outcome and are usually only interpretable in relation to other hypotheses' likelihoods
  - Of interest is the probability that  $H$  is true, given the data  $X$ , or  $P(H|X)$ .

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This is one common formulation of **Bayes' Rule**, and analogous versions can be written for each of the other competing hypotheses; for example, Bayes' Rule for  $\neg H$  is

$$P(\neg H|X) = \frac{P(\neg H)P(X|\neg H)}{P(X)}.$$

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- If the hypothesis is well-described it should attach a particular probability to the observed data
- What remains is the denominator  $P(X)$

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- The name varies by how one uses Bayes' Rule
- When one uses it to explain Bayes' Rule, the prior predictive probability of the data  $P(X)$  is the probability of observing a given outcome in the experiment, taking into account all the possible hypotheses we are considering

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Which gives a weighted-average probability of observing the outcome.

## A more complete formulation of Bayes' Rule

$$P(H|X) = \frac{P(H)P(X|H)}{P(H)P(X|H) + P(\neg H)P(X|\neg H)}.$$

Bayes' Rule is obtained as a necessary consequence of the Product Rule and the Sum Rule of probability.

**Why is Bayes better?**

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# The Lady Tasting Tea



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So, can she truly tell the difference or not? How do we decide?

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  - In other words, set out to show that  $p = P(X|H_0)$  is small

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    - The result is said to be “significant” with  $p = .016$

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  - Fisher realized this absurdity, and made a second attempt

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  - No longer “significant” at .05!

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  - $p = 5.7 \times 10^{75} \times 2^{-256} \approx .049$ , and we again reject  $H_0$  for every possible outcome

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  - We compute  $p$  as the probability of observing data that is **at least as extreme** as the real data, assuming that  $H_0$  is true
  - Somewhat absurdly, we now use as evidence an imaginary data pattern (RRRRRR) **that we did not observe and that no hypothesis we hold predicts** (more on this in a moment)

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- For the lady tasting tea:
  - For the outcome RRRRRW, there are 5 others as extreme and 1, with no errors, more extreme, giving 7 cases in all and a total probability of  $7 \times 2^{-6} = .109$ , not significant at 5%

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- In this example, Fisher takes other possibilities with 6 pairs of cups
- But why fix 6? Did they decide that in advance, or did Dr. Bristol have a meeting to go to after tea? Had the cups been prepared less efficiently, might she have done fewer?

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- This is absurd! What does it matter what might have happened, but didn't?

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But  $p(x|H_0)$  is not what we are after—we are interested in  $p(H_0|x)$ .

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- To answer this consider another lady...