GASP Codes for Secure Distributed Matrix Multiplication

Introduction to Blockchain course

Team project

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What is the GASP and SDMM?

SDMM

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \times \begin{bmatrix} 6 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 12 & 8 & 6 \\ 18 & 12 & 9 \\ 24 & 16 & 12 \end{bmatrix}$$

GASP

$$\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \tfrac{1}{2}(3x^2 - 1) \\
P_3(x) &= \tfrac{1}{2}(5x^3 - 3x) \\
P_4(x) &= \tfrac{1}{8}(35x^4 - 30x^2 + 3) \\
P_5(x) &= \tfrac{1}{8}(63x^5 - 70x^3 + 15x) \\
P_6(x) &= \tfrac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5) \\
P_7(x) &= \tfrac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x) \\
P_8(x) &= \tfrac{1}{128}(6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35) \\
P_9(x) &= \tfrac{1}{128}(12155x^9 - 25740x^7 + 18018x^5 - 4620x^3 + 315x) \\
P_{10}(x) &= \tfrac{1}{256}(46189x^{10} - 109395x^8 + 90090x^6 - 30030x^4 + 3465x^2 - 63) \\
P_{11}(x) &= \tfrac{1}{256}(88179x^{11} - 230945x^9 + 218790x^7 - 90090x^5 + 15015x^3 - 693x)
\end{aligned}$$

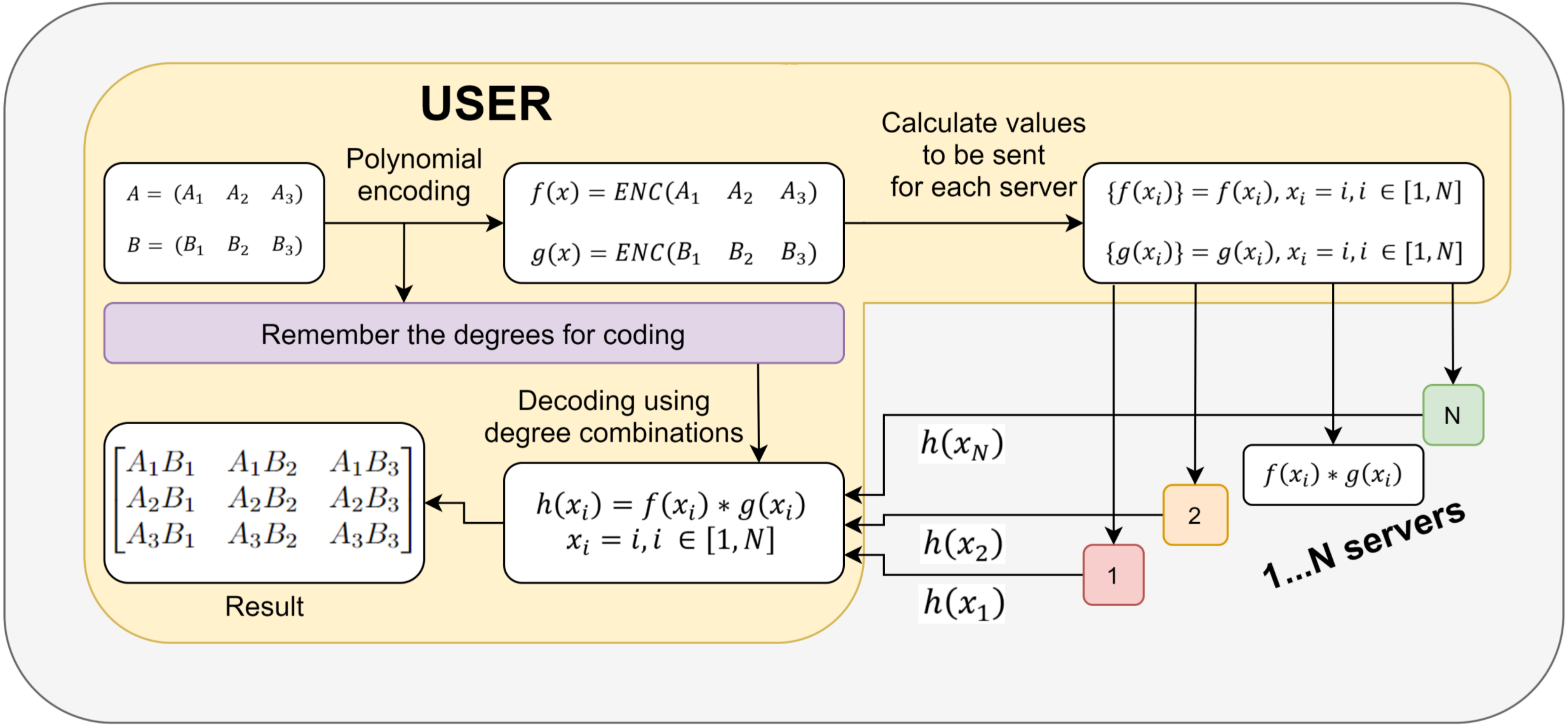
Formulation of the problem

$$A = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & B_2 & B_3 \end{bmatrix}$$ **- User has**

$$AB = \begin{bmatrix} A_1B_1 & A_1B_2 & A_1B_3 \\ A_2B_1 & A_2B_2 & A_2B_3 \\ A_3B_1 & A_3B_2 & A_3B_3 \end{bmatrix}$$ **- We need to calculate**

How to calculate AB?

GASP Block-diagram



Algorithm

Consider K = L = 3, T = 2

$$f(x) = A_1 x^{\alpha_1} + A_2 x^{\alpha_2} + A_3 x^{\alpha_3} + R_1 x^{\alpha_4} + R_2 x^{\alpha_5}$$
$$g(x) = B_1 x^{\beta_1} + B_2 x^{\beta_2} + B_3 x^{\beta_3} + S_1 x^{\beta_4} + S_2 x^{\beta_5}$$

$$h(x_i) = f(x_i) * g(x_i), x_\mathrm{i} \epsilon [1, N]$$

N points: $\begin{cases} h(x_1) = A_1 B_1 {x_1}^0 + \ldots + A_3 B_3 {x_1}^8 + C_9 {x_1}^9 + \ldots + C_{22} {x_1}^{22} \\ \circ \quad \circ \quad \circ \\ h(x_N) = A_1 B_1 {x_N}^0 + \ldots + A_3 B_3 {x_N}^8 + C_9 {x_N}^9 + \ldots + C_{22} {x_N}^{22} \end{cases}$, where

$h(x_N) - we\ know\ in\ N\ points, N\ -\ an\ amount\ of\ servers$

$h(x_N)\ -\ some\ matrix\ coding\ the\ part\ of\ \ result$

Degree table: sum of degrees combination

Left top square - **decodability**

	β_1	β_2	β_3	β_4	β_5
α_1	$\alpha_1+\beta_1$	$\alpha_1+\beta_2$	$\alpha_1+\beta_3$	$\alpha_1+\beta_4$	$\alpha_1+\beta_5$
α_2	$\alpha_2+\beta_1$	$\alpha_2+\beta_2$	$\alpha_2+\beta_3$	$\alpha_2+\beta_4$	$\alpha_2+\beta_5$
α_3	$\alpha_3+\beta_1$	$\alpha_3+\beta_2$	$\alpha_3+\beta_3$	$\alpha_3+\beta_4$	$\alpha_3+\beta_5$
α_4	$\alpha_4+\beta_1$	$\alpha_4+\beta_2$	$\alpha_4+\beta_3$	$\alpha_4+\beta_4$	$\alpha_4+\beta_5$
α_5	$\alpha_5+\beta_1$	$\alpha_5+\beta_2$	$\alpha_5+\beta_3$	$\alpha_5+\beta_4$	$\alpha_5+\beta_5$

Right top, right bottom, left bottom **- sequrity**

	$\beta_1=0$	$\beta_2=3$	$\beta_3=6$	$\beta_4=9$	$\beta_5=10$
$\alpha_1=0$	0	3	6	9	10
$\alpha_2=1$	1	4	7	10	11
$\alpha_3=2$	2	5	8	11	12
$\alpha_4=9$	9	12	15	18	19
$\alpha_5=12$	12	15	18	21	22

The degree table used for decoding

Decoding

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{22} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{22} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{18} & x_{18}^2 & \cdots & x_{18}^{22} \end{pmatrix} \begin{pmatrix} A_1 B_1 \\ \vdots \\ R_5 S_5 \end{pmatrix} = \begin{pmatrix} h(x_0) \\ \vdots \\ h(x_{18}) \end{pmatrix}$$

$$V * a = h \Rightarrow$$

$$V^{-1} * V * a = V^{-1} * \mathrm{h} \Rightarrow$$

$$a = V^{-1} * \mathrm{h}$$

Vandermond matrix has powers from the degree table for 18 x values (19 servers) – 18 by 18 matrix

Performance of GASP

Download rate – main performance metric

Let N – amount of servers

$$R = \frac{SIZE\_OF\_DESIRED\_RESULT_{bits}}{TOTAL\_DOWNLOADET\_DATA_{bits}} = \mathrm{R(K, L, T), where}$$

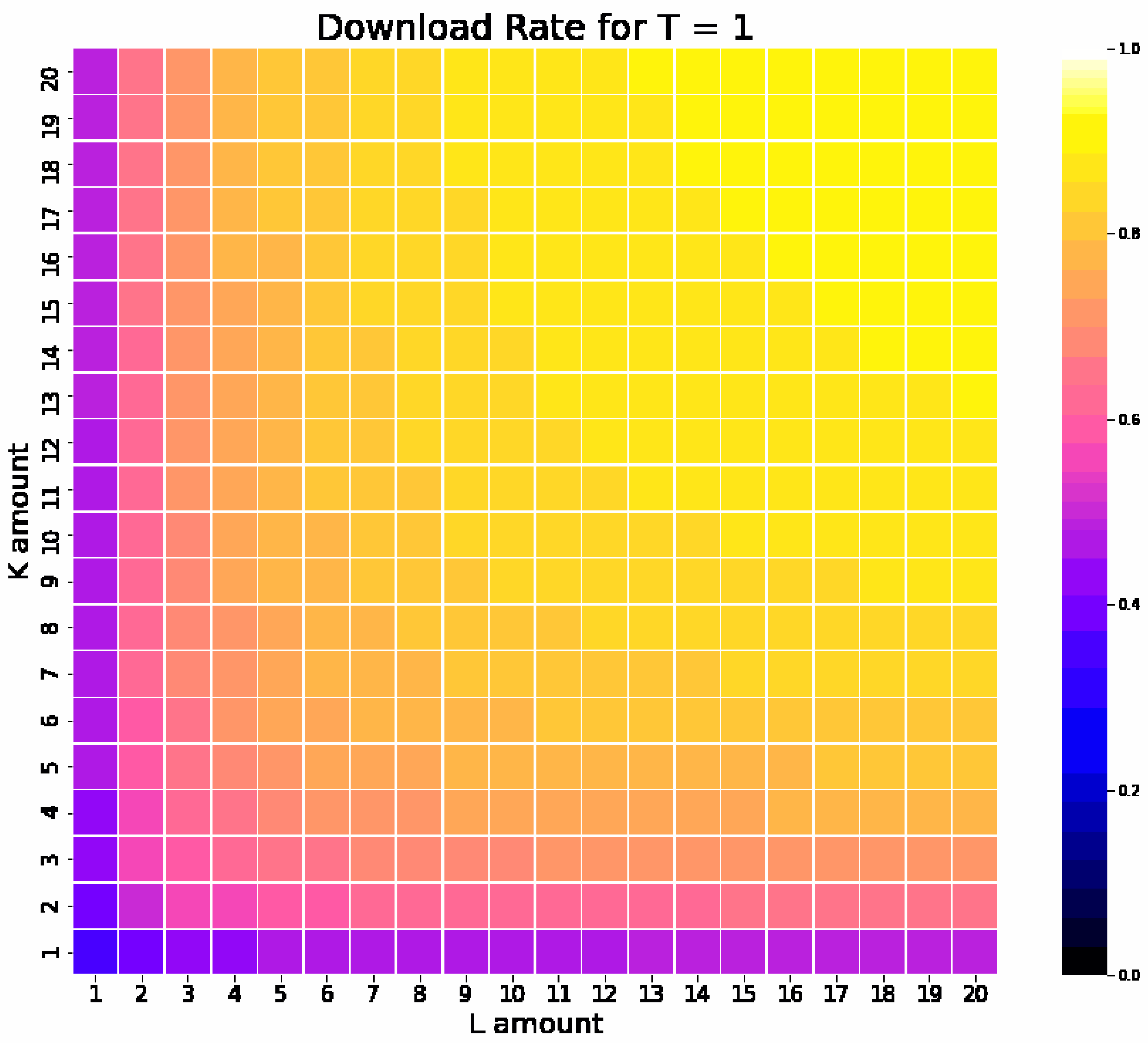
K $-$ A submatrixies
L $-$ B submatrixies
T $-$ T of N servers which may collude

Polynomial for Big and Small T

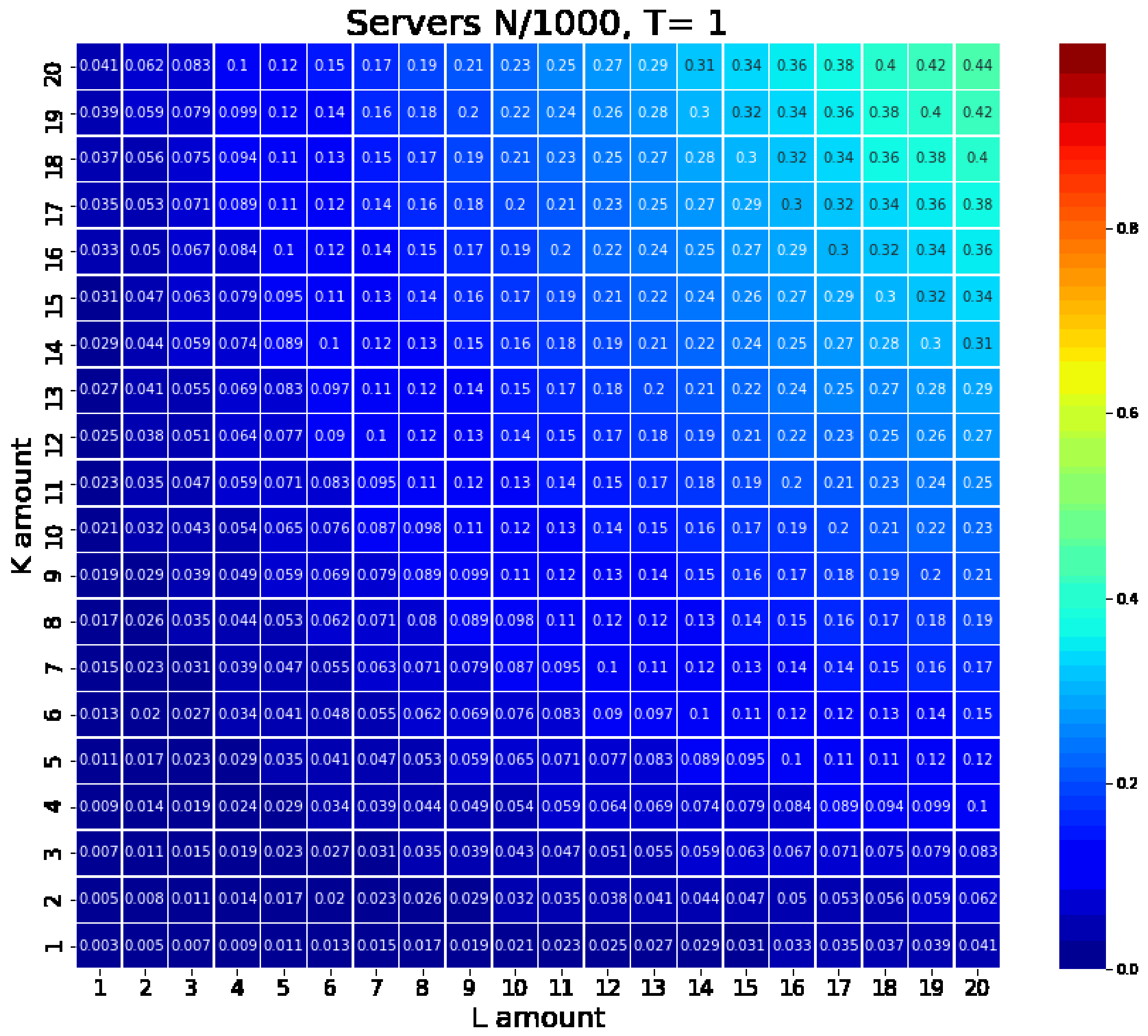
$$GASP = \begin{cases} GASP_{small} \; if \; T < \min\{K, L\} \\ GASP_{big} if \; T \geq \min\{K, L\} \end{cases} - GASP_{small} \;\; and GASP_{big} \;\; Criteria$$

$$N, R \begin{cases} If \; GASP_{small} \Rightarrow \begin{cases} N_{small} = F_{small}(K, L, T) \\ R_{small} = F_{small}(K, L, T) \end{cases} \\ If \; GASP_{big} \quad \Rightarrow \begin{cases} N_{big} = F_{big}(K, L, T) \\ R_{big} = F_{big}(K, L, T) \end{cases} \end{cases}$$

Download rate GASP (big + small)



An amount of servers N(K, L, T)



What is already implemented?

- Generating matrix A and B with K and L sizes:

$$A = A[K,1] \quad B = B[1,L]$$

- Generating submatrices A_k and $B_l, k \in [1,K], l \in [1,L]$:

$$A = \begin{bmatrix} A_1 \\ \vdots \\ A_K \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & \cdots & B_L \end{bmatrix}$$

- Generating submatrices S_t and R_t for T - security, $t \in [1,T]$,

size $A_k =$ size R_t, size $B_k =$ size S_t :

$$R = \begin{bmatrix} R_1 \\ \dots \\ R_T \end{bmatrix}$$

$$S = \begin{bmatrix} S_1 & \dots & S_T \end{bmatrix}$$

What is already implemented?

- GASP algorithms, which consists of $GASP_{big+small}$ for getting of degree sets and the optimal quantity of servers N:

$$\{\alpha\} = [\alpha_1 \quad \dots \quad \alpha_{T+K}]$$
$$\{\beta\} = [\beta_1 \quad \dots \quad \beta_{T+L}]$$
$$N = \mathrm{GASP}(K, L, T)$$

- Generating:
  - degree table
  - polynomials f(x) and g(x) for all servers:

$$f(x_i), g(x_i), x_i = i, i \in [1, N]$$

- Distributed multiplication of f(x) by g(x) and getting h(x):

$$h(x_i) = f(x_i) * g(x_i), x_i = i, i \in [1, N]$$

- Getting result of multiplication with using inverse Vandermond matrix to restore result coefficients:

$$\begin{bmatrix} A_1B_1 & A_1B_2 & A_1B_3 \\ A_2B_1 & A_2B_2 & A_2B_3 \\ A_3B_1 & A_3B_2 & A_3B_3 \end{bmatrix}$$

The main difficulties

- problems with the justification of mathematical aspects
- the need to study additional functionality of used libraries
- problems associated with selection the Vandermond matrix (regular Numpy function np.vander() experienced overflow)

Future plans

- To compare different ways of calculations:
  - Strassen algorithm
  - MPI matrix multiplication
  - Regular python functions
  - Etc…

- Try experimenting on real servers

Thank you for your attention!

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