Learning to Detrend Macroeconomic Data

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Trend-cycle decomposition has been problematic in equilibrium business cycle research. Many models are fundamentally based on the concept of balanced growth, and so have clear predictions concerning the nature of the multivariate trend that should exist in the data if the model is correct. But the multivariate trend that is removed from the data in this literature is not the same one that is predicted by the model. This is understandable, because unexpected changes in trends are difficult to model under a rational expectations assumption. A learning assumption is more appropriate here. We include learning in a standard equilibrium business cycle model with explicit growth. We ask how the economy might react to the important trend-changing events of the postwar era in industrialized economies, such as the productivity slowdown, increased labor force participation by women, and the "new economy" of the 1990s. This tells us what the model says about the trend that should be taken out of the data before the business cycle analysis begins. Thus we use learning to address the trend-cycle decomposition problem that plagues equilibrium business cycle research. We argue that a model-consistent approach, such as the one we suggest here, is necessary if the goal is to obtain an accurate assessment of an equilibrium business cycle model.

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1. INTRODUCTION

1.1. Trend-cycle decomposition via statistical filters

Trend-cycle decomposition has been problematic for equilibrium business cycle research, a wide class of models ranging from the original real business cycle papers to the recent New Keynesian macroeconomics. In nearly all of this work, the economy is viewed as essentially following a balanced growth path, but deviating from that path because of temporary shocks which drive the business cycle. However, when the models are compared to the data, the discipline implied by the notion that the economy is following a balanced growth path is discarded. Instead, atheoretic, statistical filters are employed to detrend the actual data, and render it stationary.[4] This practice has been widely criticized, for instance by Cogley and Nason (1993) and Canova (1998a). The criticisms are not hard to digest:[5]

- The "business cycle facts" are not independent of the statistical filter employed;
- Statistical filters do not remove the same trend from the data that the balanced growth path of the model implicitly requires;
- The data are often detrended one variable at a time, while the model implies a multivariate trend—thus the methodology does not respect the cointegration of the variables that the model requires;
- The filtered trends imply that trend growth rates sometimes change, but the agents in the model are not allowed to react to trend movements. In many models, agents assume a balanced growth path that never changes.

The question is not so much whether these criticisms are correct in principle, for they are patently true. And they cannot be resolved by appeal to alternative atheoretic statistical filters, whatever the properties of those filters might be.[6] The real question is, what can be done to address

[4] For a detailed recent discussion, see King and Rebelo (1999).

[5] Some authors estimate reduced forms of DSGE models, but they also have to face actual data which is nonstationary, and somehow detrend it. The methods employed are not normally related to the balanced growth path implied by the model, and so our methods apply equally well to this situation.

[6] Some recent research has considered, for example, the band-pass filter. See Christiano and Fitzgerald (1999) and Baxter and King (1999).

these criticisms, and, if something can be done, will it make any difference for the evaluation of equilibrium business cycle models currently in wide use?

1.2. Model-consistent detrending

In this paper we take a first step toward a resolution of these problems. Instead of using atheoretic, statistical filters to detrend the data, we take the model seriously and develop a method of *model-consistent detrending*. This means that the trends we remove from the data will be exactly the same ones that are implied by our model. We allow the agents to react to changes in trend growth rates, and we respect the cointegration of the variables that the model implies. We do this in the simplest context available for this issue, but we think our methodology has wide applicability across a range of growth and business cycle models.

More specifically, we study a simple version of an equilibrium business cycle model in which we explicitly allow for growth driven by exogenous productivity improvements and increases in the labor input. We replace the rational expectations assumption with a learning assumption following the methodology of Evans and Honkapohja (2001). We verify that the economy is stable under this learning assumption, meaning that, if there are no changes in the underlying parameters for a period of time, the economy will remain near the balanced growth path as if all agents had rational expectations. We then subject the economy under learning to a few unexpected shocks to the factors driving growth, corresponding to postwar U.S. events such as changing attitudes concerning women in the workforce, the "productivity slowdown," and the "new economy" of the 1990s. These types of shocks occur only once or twice in fifty years, and so it is reasonable to think that agents must learn about them when they occur. Once the shocks occur, the agents adjust to a new balanced growth path and learn the rational expectations equilibrium. Thus in our model the agents are able to track a balanced growth path that is sometimes changing, while simultaneously reacting to ordinary business cycle shocks. When the business cycle shock is turned off, we are able to trace out the multivariate trend implied by the model. We can then remove this same multivariate trend from the data. Thus we provide a model-consistent approach to detrending the macroeconomic data.

1.3. Main findings

We show that our methodology allows us to detrend the data in a relatively smooth fashion, much as currently available statistical filtering techniques do. In this sense, we are able to provide some microfoundations for current practices in the equilibrium business cycle literature. However, we also show how business cycle facts are altered, relative to those commonly reported, when the data are confronted with the trends dictated by our model. We conclude that detrending statistically is not innocuous, and that a model-consistent approach like the one we suggest is necessary to accurately evaluate equilibrium business cycle models. We think that such a conclusion would continue to hold for a wide class of related models, including those in the New Keynesian literature.

1.4. Recent related literature

The literature on detrending and the evaluation of equilibrium business cycle models is large. For critiques of the ability of technology-shock-driven equilibrium business cycle models to reproduce the data, and discussion of related detrending issues, see Cogley and Nason (1993) and Rotemberg and Woodford (1996). The debate between Canova (1998ab) and Burnside (1998) concerned the finding that different statistical filters in general yield a different set of business cycle facts. Canonical discussions of the business cycle facts can be found in Cooley and Prescott (1994), Stock and Watson (1999), and King and Rebelo (1999). King, Plosser, and Rebelo (1988a,b) discuss model-consistent detrending in the same spirit as we do. They investigate a model-consistent, linear trend in their Essay I; we essentially introduce trend breaks and learning into a similar model. Perron (1989) and Hansen (2001) discuss the econometric evidence for characterizing macroeconomic data with log-linear trends coupled with occasional structural change. The macroeconomics learning literature is summarized in Evans and Honkapohja (2001). Pakalen's (2000) thesis studies expectational stability, or learnability, in business cycle models like the one we use. His main focus was on the theoretical stability of irregular equilibria. See Rotemberg (2002) for a recent discussion of the plausibility of assuming shocks to trends are independent of shocks that drive the business cycle. Rotemberg employs a "slow technological diffusion" assumption on the former shocks, an assumption we do not make use of here. For applications of learning about trends to issues in monetary policy, see Lansing (2000, 2002). The effects of a change in trend productivity growth in a rational expectations environment are discussed in Pakko (2002).

2. ENVIRONMENT

2.1. Overview

We study a version of an equilibrium business cycle model with exogenous growth. We stress that our points are methodological, and could be applied to wide variety of models in this general class.

Time is discrete and indexed by $t = 0, 1, 2, \ldots$. The economy consists of many identical households, and the number of households is growing over time. These households make identical decisions, and so we will analyze them as if there was only one decisionmaker. We work in terms of aggregate variables, as opposed to per capita variables. We use capital letters to denote aggregates. Because we have growth explicitly in the model, the aggregate variables output, Y_t, consumption, C_t, investment I_t, and capital, K_t, will be nonstationary. We will transform these variables into their stationary counterparts in order to solve the model. When we do so, we denote the stationary variable by a small case, hatted letter, such as $\hat{c}_t$. With this notation in mind, we write the household problem as maximization of

$$E_t \sum_{t=0}^{\infty} \beta^t \eta^t \left[\ln C_t + \theta \ln \left(1 - \hat{\ell}_t \right) \right] \tag{1}$$

by choice of consumption and leisure at each date subject to constraints which apply at every date t:

$$C_t + I_t \leq Y_t, \tag{2}$$

$$I_t = K_{t+1} - (1 - \delta) K_t, \tag{3}$$

$$Y_t = \hat{s}_t K_t^{\alpha} \left(X_t N_t L_t \right)^{1-\alpha}, \tag{4}$$

$$X_t = \gamma X_{t-1}, \qquad X_0 = 1, \tag{5}$$

$$N_t = \eta N_{t-1}, \qquad N_0 = 1, \tag{6}$$

and

$$\hat{s}_t = \hat{s}_{t-1}^{\rho} \epsilon_t, \qquad \hat{s}_0 = 1, \tag{7}$$

where $\hat{s}_t$ is the technology shock. The household has a time endowment of 1 at each date t, and $\hat{\ell}_t$ is the fraction of this endowment which is supplied to the labor market. The variable X_t is the level of labor-augmenting productivity, or number of efficiency units, in the economy; the growth in this variable will drive real per capita income higher over time. The variable N_t is the size of the labor force, or number of households, where the date 0 size is normalized to unity. The parameter $\beta \in (0,1)$ is the household's discount factor, $\theta > 0$ controls the relative weight in utility placed on leisure, $\delta \in (0,1)$ is the net depreciation rate, $\alpha \in (0,1)$ is the capital share, $\gamma \geq 1$ is the gross rate of growth in productivity, $\eta \geq 1$ is the gross rate of labor force growth, and $\rho \in (0,1)$ controls the degree of serial correlation in the technology shock. The standard expectations operator is denoted E_t. The stochastic term ϵ_t is *iid* and has a mean of unity.

By combining constraints (2) and (3), and using constraint (4), we can write a Lagrangian for the household's problem. Using the first order conditions for this problem, we can write our system in terms of four equations determining C_t, $\hat{\ell}_t$, K_t, and Y_t (along with the definitions of $\hat{s}_t$, X_t, and N_t). In particular, combining (2) and (3) yields

$$K_{t+1} = Y_t + (1-\delta) K_t - C_t, \tag{8}$$

output is produced according to

$$Y_t = \hat{s}_t \left[(K_t)^{\alpha} \left(X_t N_t \hat{\ell}_t \right)^{1-\alpha} \right], \tag{9}$$

and the first order conditions yield

$$C_t = \frac{1-\alpha}{\theta} Y_t \left(\frac{1-\hat{\ell}_t}{\hat{\ell}_t} \right), \tag{10}$$

as well as

$$\frac{1}{C_t} = \beta \eta E_t \left\{ \frac{1}{C_{t+1}} \left[\alpha Y_{t+1} K_{t+1}^{-1} + 1 - \delta \right] \right\}. \tag{11}$$

Our system is given by (8) through (11), along with (5), (6), and (7).

2.2. A linear representation

We now wish to transform equations (8) through (11) along with their definitional counterparts (5), (6), and (7) into a stationary, linearized system so that we may apply the techniques developed by Evans and Honkapohja (2001). We sketch the transformation here, which involves three main steps, and provide the details in Appendix A.

First, we transform equations (8) through (11) into a stationary system by replacing C_t, Y_t, and K_t as appropriate with variables of the form $\hat{c}_t = C_t/(X_t N_t)$, and so on. The hatted variables are in per efficiency unit terms. The resulting system has a nonstochastic steady state which can be calculated directly. We can denote the steady state vector as $\left(\hat{c}_t, \hat{k}_t, \hat{\ell}_t, \hat{y}_t\right) = \left(\bar{c}, \bar{k}, \bar{\ell}, \bar{y}\right), \forall t$. An important feature of the steady state values is that they depend on all parameters of the system, in general, and in particular on the parameters γ and η. Thus, a change in the gross growth rate of productivity, γ, will alter the nonstochastic steady state of the system, as well as important ratios such as the consumption-output ratio or the capital-output ratio.

Next, we linearize about the steady state, using a differences in logarithms approach with variables of the form $\tilde{c}_t = \ln\left(\hat{c}_t/\bar{c}\right)$, and so on. This step requires additional, standard, approximations which are given in detail in Appendix A. The linearized system written in terms of logarithmic deviations from steady state is not satisfactory for our purposes, however. The tilde variables involve steady state values, such as $\bar{c}$, which we have just said depend on the growth rates of productivity and the labor input. If we allow the agents to learn by estimating a VAR using $\left(\tilde{c}_t, \tilde{k}_t, \tilde{\ell}_t, \tilde{y}_t\right)$, then we would in effect be telling them when a change in the steady state had occurred, which is inconsistent with our wish to allow them to learn about such unexpected changes.

Consequently, as a final step we decompose the tilde variables by defining variables of the form $c_t = \ln \hat{c}_t$ and $c = \ln \bar{c}$, and so on. We then collect all terms involving c, k, ℓ, and y into constants for the four equations. Under learning, we will allow the agents to estimate these constant terms, and so they will have to learn the new steady state of the system implied when the growth rates γ or η change unexpectedly. Finishing up, we reduce the four equations down to two, defined in terms of c_t and k_t.

Following these transformations, the system can be written as

$$
\begin{aligned}
c_t &= \mathcal{B}_{10} + \mathcal{B}_{11} E_t c_{t+1} + \mathcal{B}_{12} E_t k_{t+1} + \mathcal{B}_{13} E_t s_{t+1}, && (12)\\
k_t &= \mathcal{D}_{20} + \mathcal{D}_{21} c_{t-1} + \mathcal{D}_{22} k_{t-1} + \mathcal{D}_{23} s_{t-1}, && (13)\\
s_t &= \rho s_{t-1} + \vartheta_t, && (14)
\end{aligned}
$$

with $\vartheta_t = \ln \epsilon_t$, and where the coefficients $\mathcal{B}_{i,j}$, $\mathcal{D}_{i,j}$, $i = 1, 2$; $j = 0, 1, 2, 3$; are agglomerations of the underlying parameters of the model described in detail in Appendix A.[7]

[7] See Pakalen (1999)

3. LEARNING

3.1. The system under recursive learning

We study the system (12)-(14) under a recursive learning assumption, as discussed in Evans and Honkapohja (2001). We imagine that the agents have no specific knowledge of the economy in which they operate, other than the fact that we endow them with a perceived law of motion. The agents we study will be able use this perceived law of motion to learn the rational expectations equilibrium of (12)-(14). In addition, the agents will be able to react to unexpected shocks that hit the economy, such as rare changes in the growth rates of productivity, γ, or the labor input, η, provided that such changes are not too large. We have in mind a situation where shocks to these growth rates are very infrequent, occurring only once or twice in fifty years. This lends plausibility to our assumption that such shocks are largely unexpected and that agents must learn about them when they occur. Our model, then, is one where the economy follows a balanced growth path buffeted by ordinary business cycle shocks, s_t, where the balanced growth path itself changes course infrequently. As we discuss below, we think this model is consistent with econometric evidence developed by Perron (1989) and others, namely, that postwar U.S. data is well-described by a model that has a log-linear trend with occasional trend breaks.

We begin our development of the model under learning by writing the linearized model in equation form as

$$c_t = \mathcal{B}_{10} + \mathcal{B}_{11} E_t^{\star} c_{t+1} + \mathcal{B}_{12} E_t^{\star} k_{t+1} + \mathcal{B}_{13} E_t^{\star} s_{t+1} + \Delta_t \tag{15}$$

$$k_t = \mathcal{D}_{20} + \mathcal{D}_{21} c_{t-1} + \mathcal{D}_{22} k_{t-1} + \mathcal{D}_{23} s_{t-1} \tag{16}$$

$$s_t = \rho s_{t-1} + \vartheta_t \tag{17}$$

In this system, we have added a small shock, Δ_t, to the first equation. The role of this shock is to prevent perfect multicollinearity in the regressions run by the agents using capital and consumption data.[8] The operator $E_t^{\star}$ indicates (possibly nonrational) expectations taken using the information available at date t.

[8] We will keep the standard deviation of this shock three orders of magnitude lower than that of the technology shock. Because it is so small, this shock does not disturb the dynamics we discuss in a quantitatively important way.

We endow the households with a perceived law of motion given by

$$c_t = a_{10} + a_{11}c_{t-1} + a_{12}k_{t-1} + a_{13}s_{t-1}, \qquad (18)$$

$$k_t = a_{20} + a_{21}c_{t-1} + a_{22}k_{t-1} + a_{23}s_{t-1}. \qquad (19)$$

This perceived law of motion is a good one for the agents to use, because it corresponds in form to the equilibrium law of motion for the economy.[9] By repeatedly calculating the coefficients in this vector autoregression as new data becomes available, the agents may be able to correctly infer the equilibrium. The presence of constant terms in the model (15)-(17) and in the perceived law of motion (18)-(19) is effectively saying that the agents must learn the steady state values of variables instead of being given those values. This is important for our results, because it allows the trends we calculate to be smooth.

To obtain the mapping from the perceived law of motion to the actual law of motion, we use the perceived law of motion to obtain expected values and we substitute these into (15)-(17). Consistent with much of the discussion in Evans and Honkapohja (2001), we consider the case where the information available to agents at time t is dated $t-1$ and earlier. The expectations are then given by

$$E_t c_{t+1} = a_{10} + a_{11}E_t c_t + a_{12}E_t k_t + a_{13}E_t s_t \qquad (20)$$

$$E_t k_{t+1} = a_{20} + a_{21}E_t c_t + a_{22}E_t k_t + a_{23}E_t s_t \qquad (21)$$

$$E_t s_{t+1} = \rho E_t s_t \qquad (22)$$

where

$$E_t c_t = a_{10} + a_{11}c_{t-1} + a_{12}k_{t-1} + a_{13}s_{t-1} \qquad (23)$$

$$E_t k_t = a_{20} + a_{21}c_{t-1} + a_{22}k_{t-1} + a_{23}s_{t-1} \qquad (24)$$

$$E_t s_t = \rho s_{t-1} \qquad (25)$$

Substituting appropriately and collecting terms leads to the following actual law of motion for consumption:

$$c_t = T_{10} + T_{11}c_{t-1} + T_{12}k_{t-1} + T_{13}s_{t-1} + \Delta_t \qquad (26)$$

[9] We have written the perceived law of motion so that the agents will estimate the second equation, instead of being given, or "knowing" the coefficients in that equation. We think this is the more natural way to view agents who have limited knowledge about how the economy works. But from the perspective of learning the rational expectations equilibrium, it is only the first equation in the perceived law of motion that matters, because only there do expectations influence actual outcomes through equation (15).

where

$$T_{10} = \mathcal{B}_{10} + \mathcal{B}_{11}\left[a_{10} + a_{11}a_{10} + a_{12}a_{20}\right] + \\ \mathcal{B}_{12}\left[a_{20} + a_{21}a_{10} + a_{22}a_{20}\right], \quad (27)$$

$$T_{11} = \mathcal{B}_{11}\left[a_{11}^2 + a_{12}a_{21}\right] + \mathcal{B}_{12}\left[a_{21}a_{11} + a_{22}a_{21}\right], \quad (28)$$

$$T_{12} = \mathcal{B}_{11}\left[a_{11}a_{12} + a_{12}a_{22}\right] + \mathcal{B}_{12}\left[a_{21}a_{12} + a_{22}^2\right], \quad (29)$$

and

$$T_{13} = \mathcal{B}_{11}\left[a_{11}a_{13} + a_{12}a_{23} + a_{13}\rho\right] + \\ \mathcal{B}_{12}\left[a_{21}a_{13} + a_{22}a_{23} + a_{23}\rho\right] + \mathcal{B}_{13}\left[\rho^2\right]. \quad (30)$$

We write the system under learning as

$$\begin{bmatrix} c_t \\ k_t \\ s_t \end{bmatrix} = \begin{bmatrix} T_{10} \\ \mathcal{D}_{20} \\ 0 \end{bmatrix} + \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ \mathcal{D}_{21} & \mathcal{D}_{22} & \mathcal{D}_{23} \\ 0 & 0 & \rho \end{bmatrix} \begin{bmatrix} c_{t-1} \\ k_{t-1} \\ s_{t-1} \end{bmatrix} \\ + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta_t \\ 0 \\ \vartheta_t \end{bmatrix}. \quad (31)$$

A stationary MSV solution solves

$$T_{1i} = a_{1i}, \quad (32)$$

for $i = 0, 1, 2, 3$, with all eigenvalues of the matrix

$$\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ \mathcal{D}_{21} & \mathcal{D}_{22} & \mathcal{D}_{23} \\ 0 & 0 & \rho \end{bmatrix} \quad (33)$$

inside the unit circle. For the calibrations we study, there is only one such solution.

3.2. Expectational stability

We can calculate expectational stability conditions for this system. Evans and Honkapohja (1995, 2001) have provided general conditions under which expectational stability governs the stability of the system under a wide variety of recursive learning assumptions. Expectational stability is determined by the following matrix differential equation

$$\frac{d}{d\tau}\left(a_{i,j}\right) = T\left(a_{i,j}\right) - \left(a_{i,j}\right), \quad (34)$$

for $i = 1, 2; j = 0, 1, 2, 3$. The fixed points of equation (34) give us the MSV solution. A particular MSV solution $(\bar{a}_{i,j})$ is said to be E-stable if the MSV fixed point of the differential equation (34) is locally asymptotically stable at that point. The conditions for E-stability of the MSV solution are given in Proposition 10.3 of Evans and Honkapohja (2001).

The nontrivial part of the T-map only involves the coefficients in the consumption equation. Using the definitions of $T_{1,j}(a_{i,j}),\ i = 1, 2;\ j = 0, 1, 2, 3$; we can construct the matrix required for evaluating expectational stability as

$$\begin{bmatrix} \begin{matrix}\mathcal{B}_{11}(1+\bar{a}_{11})+\\ \mathcal{B}_{12}\bar{a}_{21}-1\end{matrix} & \mathcal{B}_{11}\bar{a}_{10} & \mathcal{B}_{11}\bar{a}_{20} & 0 \\ 0 & \begin{matrix}2\mathcal{B}_{11}\bar{a}_{11}+\\ \mathcal{B}_{12}\bar{a}_{21}-1\end{matrix} & \mathcal{B}_{11}\bar{a}_{21} & 0 \\ 0 & \mathcal{B}_{11}\bar{a}_{12} & \begin{matrix}\mathcal{B}_{11}(\bar{a}_{11}+\bar{a}_{22})+\\ \mathcal{B}_{12}\bar{a}_{21}-1\end{matrix} & 0 \\ 0 & \mathcal{B}_{11}\bar{a}_{13} & \mathcal{B}_{11}\bar{a}_{23} & \begin{matrix}\mathcal{B}_{11}(\bar{a}_{11}+\rho)+\\ \mathcal{B}_{12}\bar{a}_{21}-1\end{matrix} \end{bmatrix} \tag{35}$$

We verified that the eigenvalues of the above matrix are all negative for the calibration we describe below. This is true for all of all of the values of η and γ used in our analysis. Thus, at our baseline parameter values, the system is indeed expectationally stable. This suggests stability in the real time dynamics under weak conditions. We therefore proceed to real time learning.

3.3. Real time learning

When the agents are learning in real time the parameters $a_{i,j}$ in the recursive updating scheme are time-varying. This means that the T-mapping now becomes

$$\begin{aligned} T_{10}\left(\xi_{t-1}\right) = \mathcal{B}_{10} + \mathcal{B}_{11}\left[a_{10,t-1} + a_{11,t-1}a_{10,t-1} + a_{12,t-1}a_{20,t-1}\right] + \\ \mathcal{B}_{12}\left[a_{20,t-1} + a_{21,t-1}a_{10,t-1} + a_{22,t-1}a_{20,t-1}\right], \end{aligned} \tag{36}$$

$$\begin{aligned} T_{11}\left(\xi_{t-1}\right) = \mathcal{B}_{11}\left[a_{11,t-1}^2 + a_{12,t-1}a_{21,t-1}\right] + \\ \mathcal{B}_{12}\left[a_{21,t-1}a_{11,t-1} + a_{22,t-1}a_{21,t-1}\right], \end{aligned} \tag{37}$$

$$\begin{aligned} T_{12}\left(\xi_{t-1}\right) = \mathcal{B}_{11}\left[a_{11,t-1}a_{12,t-1} + a_{12,t-1}a_{22,t-1}\right] + \\ \mathcal{B}_{12}\left[a_{21,t-1}a_{12,t-1} + a_{22,t-1}^2\right], \end{aligned} \tag{38}$$

and

$$T_{13}\left(\xi_{t-1}\right)=\mathcal{B}_{11}\left[a_{11,t-1}a_{13,t-1}+a_{12,t-1}a_{23,t-1}+a_{13,t-1}\rho\right]+$$
$$\mathcal{B}_{12}\left[a_{21,t-1}a_{13,t-1}+a_{22,t-1}a_{23,t-1}+a_{23,t-1}\rho\right]+\mathcal{B}_{13}\left[\rho^{2}\right]. \quad (39)$$

The actual law of motion is therefore

$$\begin{bmatrix} c_t \\ k_t \\ s_t \end{bmatrix}=\begin{bmatrix} T_{10}\left(\xi_{t-1}\right) \\ \mathcal{D}_{20} \\ 0 \end{bmatrix}+\begin{bmatrix} T_{11}\left(\xi_{t-1}\right) & T_{12}\left(\xi_{t-1}\right) & T_{13}\left(\xi_{t-1}\right) \\ \mathcal{D}_{21} & \mathcal{D}_{22} & \mathcal{D}_{23} \\ 0 & 0 & \rho \end{bmatrix}\begin{bmatrix} c_{t-1} \\ k_{t-1} \\ s_{t-1} \end{bmatrix}$$
$$+\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \Delta_t \\ 0 \\ \vartheta_t \end{bmatrix}. \quad (40)$$

The coefficients ξ_t are updated according to a recursive least squares estimation

$$\xi_t=\xi_{t-1}+t^{-1}R_t^{-1}z_{t-1}z_{t-1}^{'}\left[T\left(\xi_{t-1}\right)-\xi_{t-1}\right], \quad (41)$$

$$R_t=R_{t-1}+t^{-1}\left[z_{t-1}z_{t-1}^{'}-R_{t-1}\right], \quad (42)$$

where

$$z_{t-1}=\begin{bmatrix} 1 \\ c_{t-1} \\ k_{t-1} \\ s_{t-1} \end{bmatrix} \quad (43)$$

and

$$T\left(\xi_{t-1}\right)=\begin{bmatrix} T_{10}\left(\xi_{t-1}\right) \\ T_{11}\left(\xi_{t-1}\right) \\ T_{12}\left(\xi_{t-1}\right) \\ T_{13}\left(\xi_{t-1}\right) \end{bmatrix}. \quad (44)$$

When we study constant gain learning, we replace t^{-1} with a small positive constant g in equations (41) and (42).

In order to simulate this system, we begin with initial, $t-1$, values of capital and consumption. We then obtain k_t from the second equation of (40). Using the third equation of (40), we draw ϑ_t and obtain s_t. Next, we draw a value Δ_t. Then we use equation (42) to obtain time t values for $r_{i,j}$, and equation (41) to obtain time t values for ξ_t. Finally, we use the first equation of (40) to obtain the time t value for c_t. This process is then repeated to generate time series on c_t, k_t, and other variables of interest.

As we have shown, this system is expectationally stable, and so we are counting on the system being stable in the real time recursive learning dynamics as well. This would not be a problem under a recursive least squares scheme in which the agents employ (41) and (42). However, we simulate the constant gain system in which t^{-1} is replaced by a small positive constant g in these equations. Under a constant gain updating scheme, we can no longer be assured that the stability properties of the system will hold. However, if the gain is sufficiently small and the system is in a sufficiently small neighborhood of the rational expectations equilibrium, then we may expect the system to remain in that neighborhood. Moreover, the constant gain will allow the system to respond should an underlying parameter of the model change unexpectedly. This is the property of the system with a constant gain that we think is essential for studying the issues we are interested in for this paper.

In principle, we could now ask how this system would react to any (small enough) change in any parameter of the model. Suppose, for instance, that people became more patient, or that the share of capital in national income increased. Such changes would alter the balanced growth path of the economy (through level effects, for these parameter changes). But the agents in the model would be able to learn the new rational expectations equilibrium implied after the change had taken place.

We now turn to comparing the model with U.S. postwar data.

4. AN EXAMPLE BASED ON POSTWAR U.S. DATA

4.1. Overview

We now wish to illustrate how our model can be used to disentangle the trend and business cycle components of the U.S. data. Since the model is quite simple and does not have some of the important categories of national income that exist in the data, this exercise cannot be completely satisfactory. However, since the model is also a variant of a widely-known benchmark, we can begin to assess how important the detrending issue really is for determining the nature of the business cycle in the data as well as for the performance of the model relative to the U.S. data.

4.2. Calibration

We wish to employ a standard calibration for this model. For this purpose, we turn to Cooley and Prescott (1994). They suggest the following calibration. In preferences, the discount factor $\beta = .987$, and the weight on leisure $\theta = 1.78$. In the technology, the capital share $\alpha = .4$, the serial correlation of the business cycle shock $\rho = .95$, and the standard deviation of the shock is .007. Cooley and Prescott (1994) also calibrate growth rates of labor and technological change, but since we allow changes in these growth rates, the calibration of these features is undertaken separately.

In the learning algorithm we have outlined, the gain sequence would normally be set to $1/t$ to correspond to recursive least squares. However, with such a gain sequence, the agents would not remain sufficiently alert to the possibility that some of the parameters they have assumed to be constant have actually changed, thus altering the balanced growth path. By setting the gain to a small positive constant, we allow the agents to admit that their model may be misspecified, and then to track the balanced growth path should it change from the one they follow during the initial portion of the sample. We set the gain to $g = .00025$. Based on our experience with simulations, this is close to the largest value of the gain that still remains consistent with stability under recursive learning.[10] Quantitatively, the choice of the gain does not seem to have a large impact on our results, so long as it produces a stable system.

Because the model economy does not have all of the major categories of national income that the U.S. national accounts have, direct comparison between the model and the data is not a simple matter. All the data we use are quarterly from 1948q1 to 2002q1. The data are in real terms, 1996 dollars, seasonally adjusted, and chain-weighted. Our model has predictions for aggregates, and so we focus on them. We are quite concerned that the aggregates in the model add up, so that the trends in the labor input and productivity can be viewed as driving the trends in the other variables of interest. We have no government sector in the model, and so we subtract real government purchases from real GDP in the data we use. We also subtract real farm business product from real GDP. This gives us a measure of nonagricultural private sector output. We have a consistent private sector nonagricultural total hours series, based on the establishment survey, for this measure of output. We use this hours series to represent our labor input. Productivity is then quarterly output divided by quarterly aggregate hours. Our model has no international sector, but net exports comprises a nontrivial component of GDP in the data. We add the services portion of net exports to our measure of consumption, and the goods portion of net

[10] Under recursive learning with a constant gain, the system can depart from the basin of attraction for the rational expectations equilibrium, provided a large enough shock occurs, and the agents response to that shock is large enough. We found that for larger values of the gain, such as .001, this type of dynamic occurred relatively often for our baseline calibration.

exports to our measure of investment. In the data where sub-categories of exports and imports are available, capital goods, industrial supplies, and automobiles make up a substantial fraction of goods exports, and so we call this investment for the purposes of our study. Our measure of investment is then gross private domestic investment plus net exports of goods, plus personal consumption expenditures on consumer durables. Our measure of consumption is personal consumption expenditures on services and nondurable goods, plus net exports of services, less farm business product, which is presumably mainly consumption-oriented.

Because of the chain weights, consumption plus investment still may not add up to output. We checked this and found that any discrepancy was neligible after 1980. Before that, the discrepancy can be larger, as much as two percent of output. We therefore allocated any discrepancy to consumption and investment using the consumption-to-output ratio for that year. Thus we end up with time series in which output is indeed equal to consumption plus investment.

4.3. Breaks in the balanced growth path

It is well-known that there was a slowdown in measured productivity growth in the U.S. economy beginning sometime in the early 1970s or late 1960s. The state of the econometric evidence on this question is reviewed in Hansen (2001). A key paper in the literature is Perron (1989), who argued that for postwar U.S. quarterly data, a model with a single trend break in 1973 allowed one to reject a random walk hypothesis for many macroeconomic time series.[11] Another recent attempt to date a structural break during this period is Bai, Lumsdaine, and Stock (1998). Their analysis is multivariate and suggests a trend break sometime between 1966Q2 and 1971Q4.

We have designed our model to allow the economy to adapt to changes of this type. We can alter the growth rate of productivity in the model at a given point in time, and, provided the change is not too large, we can expect the economy to adjust to the new balanced growth path.

How can we go about choosing break dates for our economy? We use the following approach. Our model says that the nature of the balanced growth path—the trend—is dictated by increases in productivity units $X(t)$ and increases in the labor input $N(t)$. For ease of reference, let us call these

[11] Later authors, such as Zivot and Andrews (1992), extended the analysis to the case where the break date was viewed as unknown.

the "actual" productivity and labor input series. When the growth rates of these variables change, the economy must adjust to a new balanced growth path. The model also produces *measured* productivity and a *measured* labor input series. If there were never a trend break, these measured series would have the same trend as the actual series. However, since it takes some time for the economy to adjust to the new balanced growth path, in general there will differences in the trends of the actual and the measured productivity and labor input series. In the data, we have measured increases in productivity and measured increases in the labor input. Thus it seems quite clear that we need the trends in measured productivity and measured labor input from the model to be comparable to the measured productivity and measured labor input trends we have from the data in order to have a satisfactory calibration.

One approach to calibrating the model would be to only allow trend breaks where clear econometric evidence is available. This would probably lead one to posit a trend break in productivity sometime before 1973 (such as the one suggested by Bai, Lumsdaine, and Stock (1998)) and then require the balanced growth path to be log-linear at all other times. We think this may not be the most interesting way to proceed. There could easily be smaller changes in growth rates, economically significant from the standpoint of judging business cycles, but not substantial enough to cause a rejection of a null hypothesis of log-linear growth. One example of this is the greater entry of women into the labor force beginning in the 1960s, which is often cited as one of the major changes in the U.S. economy during the postwar era. For the hours series we employ,[12] a univariate test based on Andrews (1993) cannot reject the null hypothesis of no change in the growth rate of hours across the entire postwar era. A look at the data clarifies the source of this result: The hours series before the 1960s is short and relatively volatile, and any change in the growth rate, if it occurred, is relatively small. Another example of this possibility is the idea of a "new economy" in the 1990s, which is not easy to defend with statistical tests.

Instead of relying on econometric evidence alone, we used a search algorithm, which is described in detail in Appendix B, to choose break dates for the growth factors $X(t)$ and $N(t)$, as well as for growth rates of these factors, based on the principle that the trend in measured productivity from the model should match the trend in measured productivity from the

[12] Total nonagricultural private sector hours from the establishment survey, quarterly, 1948Q1-2002Q1. We thank Jeremy Piger for conducting this test on the hours series.

SEARCH RANGES

Date of first labor input break	Between 1955q1 and 1964q4
Date of "productivity slowdown"	Between 1965q1 and 1974q4
Date of second labor input break	Between 1980q1 and 1990q1
Date of "new economy"	Between 1991q1 and 1997q1
Initial productivity growth, γ_0	$1.00512 \pm .005$
Mid-sample productivity growth, γ_1	$1.00313 \pm .005$
Ending productivity growth, γ_2	$1.00388 \pm .005$
Initial labor input growth, η_0	$1.00319 \pm .005$
Mid-sample labor input growth, η_1	$1.00498 \pm .005$
Ending labor input growth, η_2	$1.00430 \pm .005$

TABLE 1

Search ranges for trend break dates and growth rates. The ranges for possible trend break dates reflect our priors concerning these events. The possible growth rates are quarterly gross rates, and are equal to observed average growth rates in the data, measured from the midpoints of the ranges for possible break dates.

data. We began by specifying some ranges over which we wish to search for trend breaks. We also specified some ranges for possible growth rates between break dates. These ranges are described in Table 1, and reflect our "priors" on when we think reasonable dates for breaks in log linear trends might have occurred. When trend breaks occur in our model, the agents must learn about them, and so we might expect $X(t)$ and $N(t)$ to begin growing at a different rate at a date somewhat before a trend break becomes apparent in the measured series. For this reason, we included years before apparent trend breaks in the data (such as 1965 for the productivity slowdown) as possible trend break dates in our model. We allowed two breaks for productivity, corresponding to the productivity slowdown circa 1970 and the new economy circa 1995. We also allowed two breaks for the labor input, corresponding to changing attitudes toward women in the workforce circa 1960, and to a possible tapping out of that process later in the sample. The growth rate ranges are calculated as the mean quarterly growth rates for hours and productivity in the data, which we allow to possibly be higher or lower by one-half of one percent per quarter. These growth rates are calculated from the data as if the trend break were dated at the midpoint of the ranges in the first portion of the table. Thus, the initial hours growth of 1.00512 is the gross hours growth rate in the data from 1948q1 to 1960q1.

We have two factors driving trend growth in the model, each with two break dates and therefore three distinct periods of different growth rates. This means there is a vector of ten objects we must choose. We begin with a set of candidate solutions. For each candidate solution, we let our model generate a trend. This involves simulating our model with a low noise business cycle shock,[13] but with the trend changes indicated by the candidate vector. We then evaluate each candidate solution according to a fitness criterion. The fitness measure is the sum of mean squared deviations of measured productivity in the data from the implied trend, plus the sum of mean squared deviations of measured hours in the data from the implied trend, which is in correspondence with the above discussion. Hours and productivity thus receive equal weight in this calculation. We then update the set of candidate solutions in the direction of those that tended to generate better fitness scores using standard genetic operators, as discussed further in Appendix B. The process continued until no further fitness improvements could be found.

Table 2 describes our findings. For productivity growth, the break dates are ones which appear often in the literature. Productivity (that is, $X(t)$) grows at a net annual rate of 2.53 percent until 1973, then slows to an annual growth rate of 1.23 percent until 1994, before accelerating to an annual rate of 1.92 percent through the end of the sample. For the labor input (that is, $N(t)$), trend breaks are much less pronounced. The labor input series grows at an annual rate of 1.33 percent initially, before accelerating to 1.88 percent in 1962. Labor input growth actually accelerates again slightly beginning in 1983, but this increase is quite small, just 0.11 percent at an annual rate.

Our first task is to show that the breaks in growth rates we have determined imply reasonable trends for the measured labor input and for measured productivity. Figures 1 and 2 display the trends calculated from the model to the actual data on hours and productivity from the U.S. economy. The trends are generally very smooth and look about like what many economists would have in mind when they say there is a trend in the data.

Our procedure has been to use our theoretical framework to fit trends for measured productivity and measured labor input. But the trends in these growth factors in turn imply trends for output, investment, and consumption. We have not used any fitness criteria for these latter trends.

[13] We used the standard shock s_t, but with the standard deviation reduced by a factor of 1000. We require a small amount of noise in the system so that our VAR systems can still be estimated.

Optimal trend break dates	
Date of first labor input break	1962, Q2
Date of "productivity slowdown"	1973, Q1
Date of second labor input break	1983, Q4
Date of "new economy"	1994, Q4
Initial productivity growth, γ_0	1.00626
Mid-sample productivity growth, γ_1	1.00307
Ending productivity growth, γ_2	1.00476
Initial labor input growth, η_0	1.00332
Mid-sample labor input growth, η_1	1.00466
Ending labor input growth, η_2	1.00493

TABLE 2

Optimal choices for trend breaks and trend growth rates. Growth rates are given in gross, quarterly terms for comparability to Table 1. The annualized growth rates are given in the text.

In addition, the business cycle shock occurs in conjunction with the rare changes in trend we have modelled. We now turn to these issues.

4.4. Implied trends for output, consumption, and investment

While we have fit trends for productivity and hours, we are letting the trends in growth factors dictate the remaining trends in the model. Figures 3, 4, and 5 show how the trends we have calculated compare to the level of output, consumption, and investment, respectively, in the U.S. data.[14] For output, the combination of hours growth and productivity growth with some trend breaks provides a reasonable account of growth, so reasonable in fact that one might think that the trend line was simply drawn through the data by a student of business cycles. It is well known that without the trend breaks, a purely log-linear trend does not provide as reasonable of an account of this data. Figure 3 gives us confidence that a two-factor exogenous growth model is a good one for disentangling trend from cycle in the data.

The division of output between private sector consumption and investment is also dictated by the model. For these variables, the trend lines tend to run through the data in the earlier and middle portions of the sample. In the latter portion of the sample, actual consumption tends to

[14]To compare all of these trends to the data requires a units normalization. We accomplish this normalization by assuming that the model is following a balanced growth path during the initial portion of the sample, before the first trend break occurs.

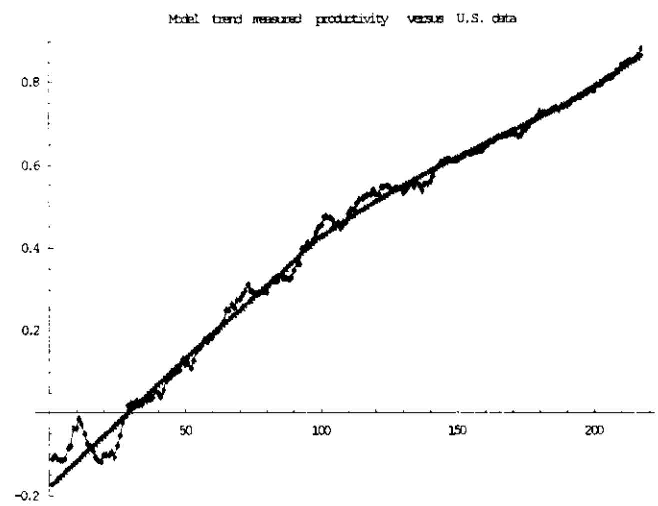

FIG. 1 The calculated trend in measured productivity implied by the model, as compared to the U.S. productivity data. The calculated trend is relatively smooth and not dissimilar to those suggested by statistical filters. The scale on the horizontal axis refers to the 217 quarters of postwar U.S. data we have.

run below trend, while investment tends to run noticeably above trend.[15] It was widely reported that there was an "investment boom" in the 1990s, and the data we have seem to bear this out. Since consumption is the only other component of output here, it must run below trend to accomodate the boom.[16]

We think that these trends are reasonable judgements of what the "actual" trends look like in the data. However, our point is not so much to say that the fit is good, but that we lay bare our assumptions about the growth process that allow us to detrend the data in this manner. Other authors are welcome to provide alternative assumptions on models like this one, or provide alternative growth models, in order to detrend the data in

[15] We considered a few alternative data arrangements to see if this feature of the analysis was robust to changes in the interpretation of "consumption" and "investment". For instance, we considered including consolidated government spending data, allocating using available figures on government consumption versus government investment. We also considered including consumer durable purchases as consumption instead of investment. These types of changes did not alter the qualitative results.

[16] See Cogley (2001) for one approach to using consumption as the basis for determining trend growth changes. Cogley comes to the conclusion that trend growth has been only modestly faster in the 1990s than during the productivity slowdown era.

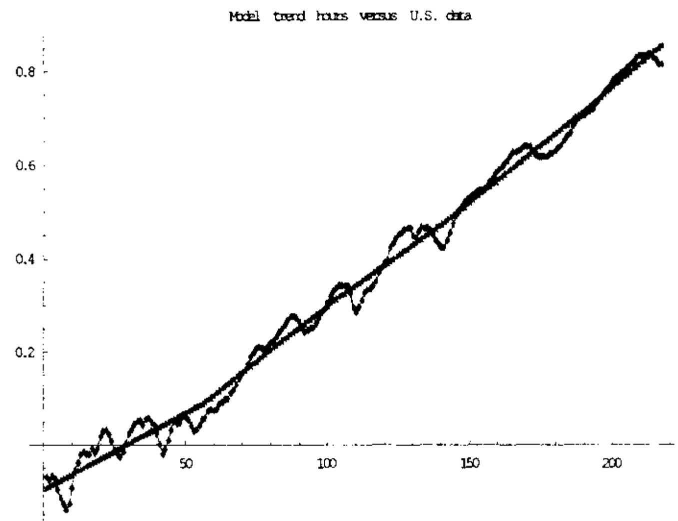

FIG. 2 The trend for hours is also relatively smooth. Our calculations indicate a change in trend in 1962, but otherwise the hours trend has remained approximately linear.

a different manner. Our hope is that constructive work can be done along these lines.

We take the calculated trends as the prediction of our model, so that the deviations from trend are the business cycle components in the data. We now turn to evaluating the properties of these business cycle components.

4.5. Business cycle statistics

The reaction of the economy to changes in the balanced growth path will depend in part on what business cycle shocks occur in tandem with the growth rate changes. In part because of this, we average over a large number of economies in order to calculate business cycle statistics for artificial economies. To generate the artificial data, we simulated the calibrated economy for a large number of periods to verify that the estimated coefficients in the agents' regressions were close to the rational expectations values. We then collected an additional 217 observations, corresponding to the 217 quarters of actual U.S. data we have. During this latter part of the exercise, we allowed the trend breaks as discussed at quarters corresponding to the dates in the postwar data, so that the agents in the economy

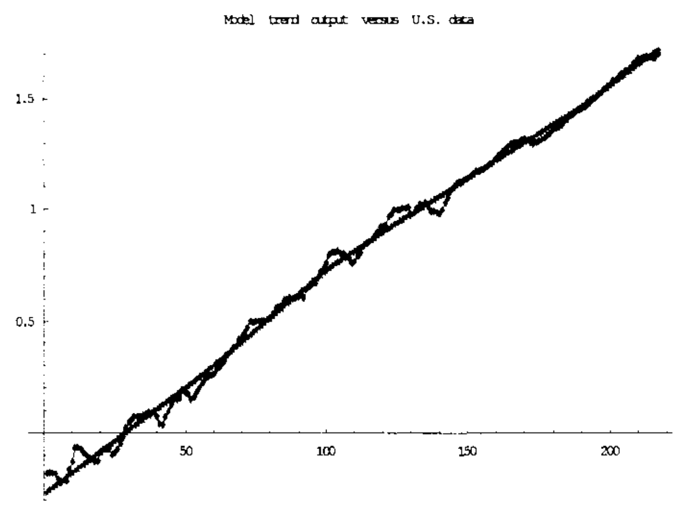

FIG. 3 The calculated output trend compared to the U.S. data.

had to also react to the trend breaks as they were coping with the business cycle shock. The trend that is taken from the artificial data is exactly the same one that is taken from the U.S. data.[17]

In assessing the behavior of equilibrium business cycle models like this one, authors have typically compared volatility and contemporaneous correlation measures from the model to those suggested by the data. We do the same, using our model-consistent trends to calculate percentage deviations of all variables from their trend values. We average our statistics across 500 economies each run for 217 periods with identical trend breaks.

We begin with overall volatility in the actual data and in the artificial data, which is displayed in Table 3. These standard deviations are often more than twice the size of those reported, for instance, by King and Rebelo (1999). The reason for this is simple. The trends we use are essentially log-linear, and so do not attribute a portion of every data movement to the trend component, as many statistical filters do. Thus the portion of the variability in the data that is attributed to business cycle volatility is likely to be larger under our methodology. In this sense, the business cycle shock

[17]An interesting question is whether an econometrician considering the productivity data generated by one of these economies would detect the breaks in trend growth rates that are built into the model. Another interesting question is whether the data generated by the model would be consistent with a random walk hypothesis in the eyes of an econometrician. We hope to investigate these issues in future work.

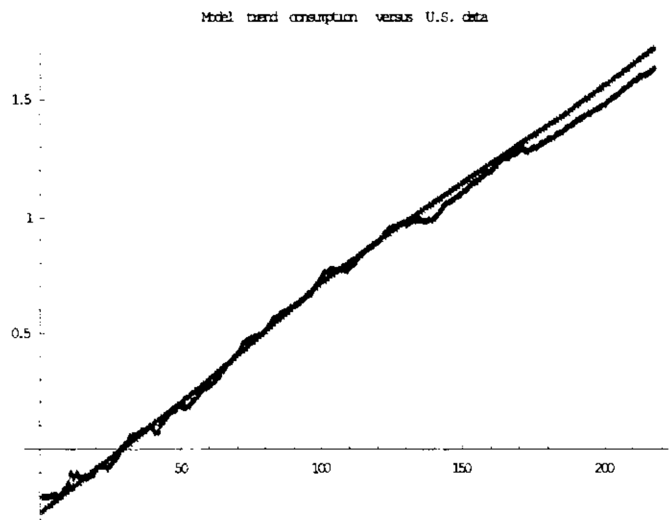

FIG. 4 The calculated consumption trend versus the U.S. data. Consumption tends to fall below trend in the latter portion of the sample.

has to explain more under our approach than under traditional approaches to the detrending question.

A key question for this line of research has been: How much of the variability in the data can be explained by a model of this type? That is, how much variance can we generate by simply assuming a single shock to the production technology along with occasional breaks in trend growth rates? For the model, the average standard deviation for output is 3.26 according to Table 3, while for the data it is 3.34. That suggests that more than 97 percent of the variance of output about the balanced growth path can be explained with a model of this type! That is a high number even compared to other exercises along this line. It suggests that shocks to the technology coupled with the important movements in trend we have observed during the postwar era provide a promising lead on accounting for nearly all of the variability of output around the balanced growth path during the postwar era.[18]

[18] Recent research has argued that the technology portion of the Solow residual is actually less volatile than we have calibrated it, by perhaps a factor of five. If we reduce the standard deviation of the shock to technology, the business cycle volatility of this model will fall proportionately. Again, ours is only an example, which we mainly want to keep comparable to previous research. Interested readers can consult King and Rebelo (1999) for an alternative equilibrium business cycle model that generates similar data

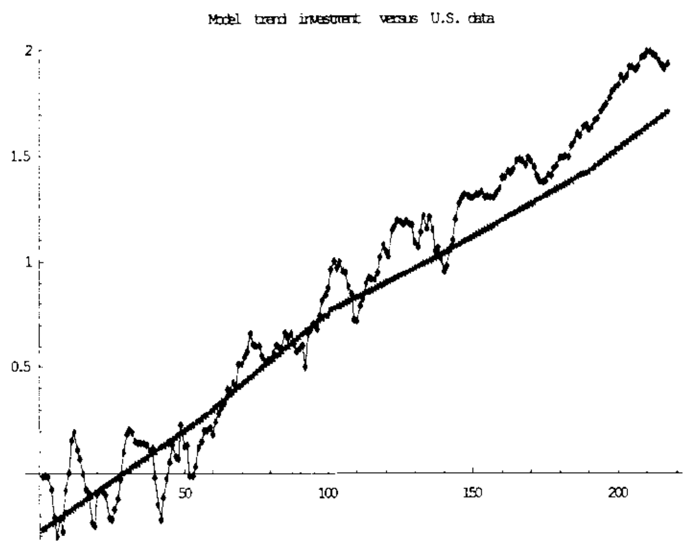

FIG. 5 Investment in the U.S. data, plotted against the calculated trend. Investment boomed in the latter portion of the sample.

Since the trends are log-linear in our model, they tend to be less accomodating to the data than those computed using most statistical filters. We stress that the higher volatility implied by our method applies equally to both the model and the data. This is why the model can still explain a large fraction of the variance in the data, even when that variance has doubled relative to commonly reported statistics.

The volatilities in the data and for the model relative to output volatility are given in Table 4. There are several interesting aspects of the results reported in this table. First, consumption is about as volatile as output in the data, but only two-thirds as variable output in the model. The source of this finding is quite clear from Figure 4, where the U.S. consumption data tends to drift below trend later in the sample. This tends to increase the volatility of the consumption data if it is measured as deviation from trend. The relative volatility of investment is only about half as large in the model as it is in the data; however, in both the model and the data investment is much more volatile than output. Again, the investment boom of the 1990s seems to have contributed quite a lot to the variance of investment in the

with less volatile shocks. That model is still in the balanced growth framework and so our methods would still apply.

Volatility		
	U.S. data	**Model**
Output	3.34	3.26
Consumption	3.42	2.18
Investment	15.40	7.88
Hours	2.65	1.26
Productivity	2.45	2.42

TABLE 3

Overall volatility in the model versus the data. These are standard deviations of detrended series. Volatility is about twice as high in both the data and the model as compared to business cycle statistics commonly reported.

Relative Volatility		
	U.S. data	**Model**
Output	1.00	1.00
Consumption	1.02	0.67
Investment	4.62	2.42
Hours	0.79	0.39
Productivity	0.73	0.74

TABLE 4

Volatility relative to output in the model and in the data. Hours is only about half as volatile in the model as it is in the data. This is the same conclusion reached by Cooley and Prescott (1994).

data.

Hours worked in the data is about three-fourths as volatile as output, somewhat lower than the one-to-one ratio that is often reported in the literature. But the relative volatility of hours in the model is still only about half what it is in the data, that is, .39 in the model versus .79 in the data. Thus one of the key findings of the original equilibrium business cycle literature, that the labor market portion of the model is not satisfactory, holds up in this example.

The contemporaneous correlations with output for both the model and the data are given in Table 5. All variables are procyclical, both in the model and in the data. These statistics tend to be lower than their counterparts reported in the literature, for instance in King and Rebelo (1999), for both the model and the data. The model predicts too much procyclicality across all of the variables, but still, the statistics reported in column two of this table are noticeably lower than those typically reported. One

Contemporaneous Correlations		
	U.S. data	**Model**
Output	1.00	1.00
Consumption	0.59	0.82
Investment	0.67	0.91
Hours	0.69	0.77
Productivity	0.62	0.94

TABLE 5

Contemporaneous correlations in the model and in the data. Hours and productivity move procyclically in the data, and more so in the model, a closer match than commonly reported.

statistic is not lower than typically reported, and that is the correlation of productivity with output in the data, which is .62. Productivity is more strongly procyclical than suggested by Cooley and Prescott (1994) or King and Rebelo (1999). Thus hours and productivity more or less move together both in the model and in the data. Using alternative techniques for detrending, this has not been true, and in fact was judged to be a problem with the model.

5. CONCLUSION

The concept of a balanced growth path has had an enormous influence on macroeconomists. In this paper we have taken this concept, which underlies nearly all macroeconomic models in use today, to the data. Of course, growth rates of important macroeconomic time series are well-known to be inconsistent with purely log-linear growth through the postwar period. For this reason, we have allowed trend breaks where appropriate, and we have used learning via the methodology of Evans and Honkapohja (2001) as a "glue" that holds the resulting various balanced growth paths together. We are thus able to remove the same trend from the data that the balanced growth path of the model requires. We hope our methodology can be used to help resolve some of the puzzles and controversies concerning trends and business cycles in macroeconomics.

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APPENDIX A: LINEAR REPRESENTATION OF THE MODEL

We wish to analyze the system (8)-(11) in which the nonstationary variables, namely capital, consumption, and output, are rendered stationary via

$$\hat{k}_t = \frac{K_t}{X_t N_t}, \qquad \hat{y}_t = \frac{Y_t}{X_t N_t}, \qquad \hat{c}_t = \frac{C_t}{X_t N_t}. \tag{45}$$

If there was no growth in productivity over time, these variables would simply be in per capita terms; with productivity growth they are measured in per total efficiency unit terms. By dividing equations (8) through (11) by $X_t N_t$ appropriately, we can write them in terms of stationary variables as

$$\gamma\eta\hat{k}_{t+1} = \hat{y}_t + (1-\delta)\,\hat{k}_t - \hat{c}_t, \tag{46}$$

$$\hat{y}_t = \hat{s}_t \left(\hat{k}_t\right)^{\alpha} \left(\hat{\ell}_t\right)^{1-\alpha}, \tag{47}$$

$$\hat{c}_t = \frac{(1-\alpha)}{\theta}\hat{y}_t \left(\frac{1-\hat{\ell}_t}{\hat{\ell}_t}\right), \tag{48}$$

and

$$\frac{1}{\hat{c}_t} = \frac{\beta}{\gamma} E_t \left\{ \frac{1}{\hat{c}_{t+1}} \left[\frac{\alpha \hat{y}_{t+1}}{\hat{k}_{t+1}} + 1 - \delta \right] \right\}. \tag{49}$$

A nonstochastic steady state of this transformed system corresponds to a balanced growth path of the original system. The gross rate of growth along the balanced growth path is $\gamma\eta$. We denote the nonstochastic steady state values by $\hat{c}_t = \bar{c}$, $\hat{y}_t = \bar{y}$, $\hat{k}_t = \bar{k}$, $\hat{\ell}_t = \bar{\ell}$, and $\hat{s}_t = \bar{s} = 1$, $\forall t$. These equations can be solved explicitly. Define φ by

$$\varphi = (1+\theta)(\alpha-1)\beta(\delta-1) + \gamma\left[(\alpha-1) + \theta(\alpha\beta\eta-1)\right]. \tag{50}$$

Then

$$\bar{y} = \varphi^{-1}(\alpha-1)\alpha\beta\left(\frac{\beta(\delta-1)+\gamma}{\alpha\beta}\right)^{2+\frac{1}{\alpha-1}}, \tag{51}$$

$$\bar{k} = \varphi^{-1}(\alpha-1)\alpha\beta\left(\frac{\beta(\delta-1)+\gamma}{\alpha\beta}\right)^{\frac{\alpha}{\alpha-1}}, \tag{52}$$

$$\bar{\ell} = \varphi^{-1}(\alpha-1)\left[\beta(\delta-1)+\gamma\right], \tag{53}$$

and

$$\bar{c} = \varphi^{-1}(1-\alpha)\left(\frac{\beta(\delta-1)+\gamma}{\alpha\beta}\right)^{\frac{\alpha}{\alpha-1}}\left[(\alpha-1)\beta(\delta-1) - \gamma + \alpha\beta\gamma\eta\right]. \tag{54}$$

We can deduce that the capital to output ratio along a balanced growth path will be equal to

$$\frac{\bar{k}}{\bar{y}} = \frac{\alpha\beta}{\gamma - \beta(1-\delta)}, \tag{55}$$

that the consumption to output ratio will be

$$\frac{\bar{c}}{\bar{y}} = \frac{\gamma - \beta(1-\delta) - \alpha\beta(\gamma\eta - 1 + \delta)}{\gamma - \beta(1-\delta)}, \tag{56}$$

and that the capital-labor ratio will be

$$\frac{\bar{k}}{\bar{\ell}} = \left(\frac{\gamma - \beta(1-\delta)}{\alpha\beta}\right)^{\frac{1}{\alpha-1}}. \tag{57}$$

Since the growth rates γ and η enter these expressions, growth matters for the calibration of models in this class.[19] Many models that have been studied abstract from growth but calibrate to growth facts such as a constant capital to output ratio.

In order to apply the Evans and Honkapohja (2001) methodology to this problem, we need a linear system. Accordingly, we now proceed with a well-known linearization of this model, expressed in terms of logarithmic deviations from steady state. For this purpose we define

$$\tilde{c}_t = \ln\left(\frac{\hat{c}_t}{\bar{c}}\right), \qquad \tilde{k}_t = \ln\left(\frac{\hat{k}_t}{\bar{k}}\right), \qquad \tilde{\ell}_t = \ln\left(\frac{\hat{\ell}_t}{\bar{\ell}}\right), \tag{58}$$

$$\tilde{y}_t = \ln\left(\frac{\hat{y}_t}{\bar{y}}\right), \text{ and } \tilde{s}_t = \ln\left(\frac{\hat{s}_t}{\bar{s}}\right). \tag{59}$$

By noting that for any of these variables, $\hat{x}_t = e^{\tilde{x}_t}\bar{x}$, using the approximation $e^x \approx 1 + x$, and using the fact that $\bar{y} = (\gamma\eta - 1 + \delta)\bar{k} + \bar{c}$, we can write equation (46) as

$$\gamma\eta\tilde{k}_{t+1} = \frac{\bar{y}}{\bar{k}}\tilde{y}_t + (1-\delta)\tilde{k}_t - \frac{\bar{c}}{\bar{k}}\tilde{c}_t. \tag{60}$$

For equation (47), we can write

$$\tilde{y}_t = \tilde{s}_t + \alpha\tilde{k}_t + (1-\alpha)\tilde{\ell}_t. \tag{61}$$

Using the approximation $\tilde{c}_t\tilde{\ell}_t \approx 0$ and the fact that $\bar{c} = \frac{1-\alpha}{\theta}\bar{y}\frac{1-\bar{\ell}}{\bar{\ell}}$ allows us to write equation (48) as

$$\tilde{c}_t = \tilde{y}_t - \left(\frac{1}{1-\bar{\ell}}\right)\tilde{\ell}_t. \tag{62}$$

[19] See for instance the discussion in Cooley and Prescott (1994).

And finally, for equation (49), we use the fact that $\beta\gamma^{-1}(1-\delta) = 1 - \beta\gamma^{-1}\alpha\bar{y}\bar{k}^{-1}$ as well as approximations of the form $\tilde{x}\tilde{y} \approx 0$ to deduce

$$\tilde{c}_t = E_t\tilde{c}_{t+1} - \beta\gamma^{-1}\alpha\bar{y}\bar{k}^{-1}E_t\tilde{y}_{t+1} + \beta\gamma^{-1}\alpha\bar{y}\bar{k}^{-1}E_t\tilde{k}_{t+1}. \tag{63}$$

An important aspect of our analysis is that we want our agents to learn the new value of the steady state (that is, the vector $(\bar{c}, \bar{y}, \bar{k}, \bar{\ell})$ when a change in growth occurs. With the system in the form of equations (60) through (63), one is in effect assuming that the steady state values are known, and so we cannot leave the system in this form. Instead, we let $c_t = \ln \hat{c}_t$, $k_t = \ln \hat{k}_t$, $y_t = \ln \hat{y}_t$, $\ell_t = \ln \hat{\ell}_t$, and $s_t = \ln \hat{s}_t$, and also $c = \ln \bar{c}$, $k = \ln \bar{k}$, $y = \ln \bar{y}$, $\ell = \ln \bar{\ell}$, and $s = \ln \bar{s} = 0$, and then rewrite equation (60) as

$$k_{t+1} = \kappa_0 + \kappa_1 y_t + \kappa_2 k_t + \kappa_3 c_t, \tag{64}$$

where

$$\kappa_0 = \left(1 - \frac{(1-\delta)}{\gamma\eta}\right)k - \frac{1}{\gamma\eta}\frac{\bar{y}}{\bar{k}}y + \frac{1}{\gamma\eta}\frac{\bar{c}}{\bar{k}}c, \tag{65}$$

$$\kappa_1 = \frac{1}{\gamma\eta}\frac{\bar{y}}{\bar{k}}, \tag{66}$$

$$\kappa_2 = \frac{(1-\delta)}{\gamma\eta}, \tag{67}$$

and

$$\kappa_3 = -\frac{1}{\gamma\eta}\frac{\bar{c}}{\bar{k}}. \tag{68}$$

Equation (61) can be written as

$$y_t = \alpha k_t + (1-\alpha)\ell_t + s_t. \tag{69}$$

For equation (62) we have

$$c_t = \pi_0 + \pi_1 y_t + \pi_2 \ell_t, \tag{70}$$

where

$$\pi_0 = c - y + \frac{\ell}{1-\bar{\ell}}, \tag{71}$$

$$\pi_1 = 1, \tag{72}$$

and

$$\pi_2 = \frac{-1}{1-\bar{\ell}}. \tag{73}$$

Next, equation (63) can be written as

$$c_t = \mu_0 + \mu_1 E_t c_{t+1} + \mu_2 E_t y_{t+1} + \mu_3 E_t k_{t+1}, \tag{74}$$

where

$$\mu_0 = \alpha\beta\gamma^{-1}\frac{\bar{y}}{\bar{k}}(y-k), \tag{75}$$

$$\mu_1 = 1, \tag{76}$$

$$\mu_2 = -\alpha\beta\gamma^{-1}\frac{\bar{y}}{\bar{k}}, \tag{77}$$

and

$$\mu_3 = \alpha\beta\gamma^{-1}\frac{\bar{y}}{\bar{k}}. \tag{78}$$

And finally, the equation for the business cycle shock, (7), can be written as

$$s_t = \rho s_{t-1} + \vartheta_t, \tag{79}$$

where $\vartheta_t = \ln \epsilon_t$.

We now wish to reduce the system to three equations instead of five. Accordingly, we solve equation (70) for ℓ_t, substitute it into equation (69), solve the resulting equation for y_t, and substitute that solution into equations (64) and (74).

This gives the system described in the text,

$$c_t = \mathcal{B}_{10} + \mathcal{B}_{11}E_t c_{t+1} + \mathcal{B}_{12}E_t k_{t+1} + \mathcal{B}_{13}E_t s_{t+1}, \tag{80}$$

$$k_t = \mathcal{D}_{20} + \mathcal{D}_{21}c_{t-1} + \mathcal{D}_{22}k_{t-1} + \mathcal{D}_{23}s_{t-1}, \tag{81}$$

$$s_t = \rho s_{t-1} + \vartheta_t, \tag{82}$$

with $\vartheta_t = \ln \epsilon_t$, and where

$$\mathcal{B}_{10} = \mu_0 + \frac{\mu_2(\alpha-1)\pi_0}{\pi_2 + (1-\alpha)\pi_1}, \tag{83}$$

$$\mathcal{B}_{11} = \mu_1 + \frac{\mu_2(1-\alpha)}{\pi_2 + (1-\alpha)\pi_1}, \tag{84}$$

$$\mathcal{B}_{12} = \mu_3 + \frac{\mu_2\alpha\pi_2}{\pi_2 + (1-\alpha)\pi_1}, \tag{85}$$

$$\mathcal{B}_{13} = \frac{\mu_2\pi_2}{\pi_2 + (1-\alpha)\pi_1}, \tag{86}$$

$$\mathcal{D}_{20} = \kappa_0 + \frac{\kappa_1(\alpha-1)\pi_0}{\pi_2 + (1-\alpha)\pi_1}, \tag{87}$$

$$\mathcal{D}_{21} = \kappa_3 + \frac{\kappa_1(1-\alpha)}{\pi_2 + (1-\alpha)\pi_1}, \tag{88}$$

$$\mathcal{D}_{22} = \kappa_2 + \frac{\kappa_1\alpha\pi_2}{\pi_2 + (1-\alpha)\pi_1}, \tag{89}$$

and

$$\mathcal{D}_{23} = \frac{\kappa_1 \pi_2}{\pi_2 + (1-\alpha)\,\pi_1}. \tag{90}$$

APPENDIX B: SEARCH METHODOLOGY

[Here we describe our genetic algorithm search in more detail.]