The Feynman-Kac Theorem and Black-Scholes Formula Derivation

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Abstract

This document provides a comprehensive derivation of the Black-Scholes formula using the Feynman-Kac theorem. We begin with the fundamental partial differential equation (PDE) governing option pricing and demonstrate how stochastic calculus transforms this into a probabilistic expectation. The derivation culminates in the famous Black-Scholes formula for European call options.

Contents

1 The Black-Scholes Partial Differential Equation

The Black-Scholes equation for the price $f(t, S)$ of a derivative security is given by:

The Black-Scholes PDE

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} - rf = 0 \tag{1}$$

with boundary condition: $f(T, S) = \Phi(S)$ (payoff function)

where:

- S is the stock price
- t is time
- r is the risk-free interest rate
- σ is the volatility
- T is the maturity time
- $\Phi(S)$ is the payoff function at maturity

2 The Feynman-Kac Theorem

The Feynman-Kac theorem provides a stochastic representation for solutions to certain parabolic PDEs.

Theorem 2.1 (Feynman-Kac Theorem)**.** *Consider the PDE:*

$$\frac{\partial v}{\partial t} + \frac{1}{2}a^2(x)\frac{\partial^2 v}{\partial x^2} + b(x)\frac{\partial v}{\partial x} - rv = 0 \tag{2}$$

with terminal condition $v(T, x) = g(x)$.

If X_t satisfies the SDE:

$$dX_s = b(X_s)ds + a(X_s)dW_s \tag{3}$$

Then the solution is:

$$v(t, x) = \mathbb{E}\left[e^{-r(T-t)}g(X_T) \mid X_t = x\right] \tag{4}$$

2.1 Intuitive Understanding

Physical Interpretation

Think of $v(t, x)$ as representing a particle starting at position x at time t:

1. The particle evolves according to the stochastic process $dX_s = b\,ds + a\,dW_s$
2. At time T, apply function g to get $g(X_T)$
3. Discount back to present value: $e^{-r(T-t)}g(X_T)$
4. Take expectation: this gives us $v(t, x)$

What does "discount back" mean?

The factor $e^{-r(T-t)}$ is the continuous compounding discount factor. If you're promised $g(X_T)$ dollars at time T, its present value at time t is $e^{-r(T-t)}g(X_T)$.

3 Deriving the Feynman-Kac Formula

3.1 Setting up the Martingale Property

From the definition of $v(t,x)$, we have:

$$v(t,x) = \mathbb{E}\left[e^{-r\cdot dt} \cdot v(t+dt, X_{t+dt}) \mid X_t = x\right] \tag{5}$$

Using the approximation $e^{-r\cdot dt} \approx 1 - r \cdot dt$ for small dt:

$$v(t,x) = \mathbb{E}[(1 - r\cdot dt)(v(t,x) + dv)] \tag{6}$$

3.2 Applying Itô's Formula

For the process X_t satisfying $dX = b\,dt + a\,dW$, Itô's formula gives:

$$\begin{aligned} dv &= \frac{\partial v}{\partial t}dt + \frac{\partial v}{\partial x}dX + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}(dX)^2 && (7) \\ &= \left[\frac{\partial v}{\partial t} + b\frac{\partial v}{\partial x} + \frac{1}{2}a^2\frac{\partial^2 v}{\partial x^2}\right]dt + a\frac{\partial v}{\partial x}dW && (8) \end{aligned}$$

Taking expectations (noting $\mathbb{E}[dW] = 0$):

$$\mathbb{E}[dv] = \left[\frac{\partial v}{\partial t} + b\frac{\partial v}{\partial x} + \frac{1}{2}a^2\frac{\partial^2 v}{\partial x^2}\right]dt \tag{9}$$

3.3 Deriving the PDE

From the martingale property and ignoring higher-order terms:

$$0 = \mathbb{E}[dv] - r\cdot dt \cdot v(t,x) \tag{10}$$

Substituting and dividing by dt:

The Feynman-Kac PDE

$$\frac{\partial v}{\partial t} + \frac{1}{2}a^2\frac{\partial^2 v}{\partial x^2} + b\frac{\partial v}{\partial x} - rv = 0 \tag{11}$$

4 Application to Black-Scholes

4.1 Matching the Pattern

For the Black-Scholes equation:

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} - rf = 0 \tag{12}$$

We identify:

- $x = S$ (stock price)
- $a(S) = \sigma S$ (diffusion coefficient)
- $b(S) = rS$ (drift coefficient)
- Discount rate $= r$
- Terminal condition: $g(S) = \Phi(S)$

4.2 The Risk-Neutral Process

The corresponding stochastic process is:

Risk-Neutral Stock Price Process

$$dS_s = rS_s\,ds + \sigma S_s\,dW_s \tag{13}$$

This represents the stock price evolution under the risk-neutral measure, where the drift is the risk-free rate r (not the actual expected return μ).

4.3 The Solution Formula

By Feynman-Kac, the option price is:

Black-Scholes Solution

$$f(t, S) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}\Phi(S_T) \mid S_t = S\right] \tag{14}$$

where $\mathbb{Q}$ is the risk-neutral measure.

5 Solving for Geometric Brownian Motion

5.1 Using Itô's Formula on $\ln(S)$

Let $Y_s = \ln(S_s)$. Applying Itô's formula with $f(S) = \ln(S)$:

$$\frac{\partial f}{\partial S} = \frac{1}{S} \tag{15}$$

$$\frac{\partial^2 f}{\partial S^2} = -\frac{1}{S^2} \tag{16}$$

$$\begin{aligned}
d(\ln S_s) &= \frac{1}{S_s}dS_s + \frac{1}{2}\left(-\frac{1}{S_s^2}\right)(dS_s)^2 && (17)\\
&= \frac{1}{S_s}(rS_s\,ds + \sigma S_s\,dW_s) - \frac{1}{2S_s^2}\sigma^2 S_s^2\,ds && (18)\\
&= r\,ds + \sigma\,dW_s - \frac{1}{2}\sigma^2\,ds && (19)\\
&= \left(r - \frac{1}{2}\sigma^2\right)ds + \sigma\,dW_s && (20)
\end{aligned}$$

5.2 The Explicit Solution

Integrating from t to T:

$$\ln(S_T) - \ln(S_t) = \left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma(W_T - W_t) \tag{21}$$

Since $W_T - W_t \sim N(0, T-t)$, we can write $W_T - W_t = \sqrt{T-t} \cdot Z$ where $Z \sim N(0,1)$:

Stock Price at Maturity

$$S_T = S_t \exp\left[\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\sqrt{T-t} \cdot Z\right] \tag{22}$$

where $Z \sim N(0,1)$.

6 The European Call Option

6.1 Setting up the Expectation

For a European call option with strike K, the payoff is $\Phi(S_T) = \max(S_T - K, 0)$.

The option value is:

$$C(t, S_t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[\max(S_T - K, 0)] \tag{23}$$

6.2 Splitting the Expectation

Using the indicator function representation:

$$\max(S_T - K, 0) = (S_T - K) \cdot \mathbf{1}_{S_T > K} \tag{24}$$

$$\mathbb{E}[\max(S_T - K, 0)] = \mathbb{E}[S_T \cdot \mathbf{1}_{S_T > K}] - K \cdot \mathbb{E}[\mathbf{1}_{S_T > K}] \tag{25}$$

We need to evaluate:

1. $\mathbb{E}[\mathbf{1}_{S_T > K}] = \mathbb{P}(S_T > K)$ — probability of exercise
2. $\mathbb{E}[S_T \cdot \mathbf{1}_{S_T > K}]$ — expected stock value if exercised

6.3 Finding the Critical Values

The condition $S_T > K$ is equivalent to:

$$S_t \exp\left[\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\sqrt{T-t} \cdot Z\right] > K \tag{26}$$

$$\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\sqrt{T-t} \cdot Z > \ln\left(\frac{K}{S_t}\right) \tag{27}$$

$$Z > \frac{\ln(K/S_t) - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \tag{28}$$

Critical Values

Define:

$$d_2 = \frac{\ln(S_t/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \tag{29}$$

$$d_1 = d_2 + \sigma\sqrt{T-t} = \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \tag{30}$$

Then $S_T > K$ when $Z > -d_2$, or equivalently, $Z < d_2$.

6.4 Evaluating the Integrals

First integral:

$$\mathbb{E}[\mathbf{1}_{S_T>K}] = \mathbb{P}(Z > -d_2) = \mathbb{P}(Z < d_2) = N(d_2) \tag{31}$$

Second integral: Using a measure change technique (completing the square), we get:

$$\mathbb{E}[S_T \cdot \mathbf{1}_{S_T>K}] = S_t e^{r(T-t)} N(d_1) \tag{32}$$

6.5 The Final Formula

Combining everything:

$$\begin{aligned} C(t, S_t) &= e^{-r(T-t)}[S_t e^{r(T-t)} N(d_1) - K N(d_2)] && (33) \\ &= S_t N(d_1) - K e^{-r(T-t)} N(d_2) && (34) \end{aligned}$$

Black-Scholes Formula for European Call

$$\boxed{C(t, S) = S N(d_1) - K e^{-r(T-t)} N(d_2)} \tag{35}$$

where:

$$d_1 = \frac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \tag{36}$$

$$d_2 = \frac{\ln(S/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t} \tag{37}$$

and $N(\cdot)$ is the cumulative standard normal distribution function.

7 Economic Interpretation

Understanding the Terms

- $SN(d_1)$: Expected stock value if the option is exercised, weighted by the hedge ratio
- $Ke^{-r(T-t)}N(d_2)$: Present value of the strike price, weighted by the risk-neutral probability of exercise
- $N(d_2)$: Risk-neutral probability that $S_T > K$ (option finishes in-the-money)
- $N(d_1)$: The option's delta ($\partial C/\partial S$), representing the hedge ratio

8 Conclusion

The Feynman-Kac theorem provides an elegant bridge between the world of partial differential equations and stochastic processes. By transforming the Black-Scholes PDE into a probabilistic expectation, we can derive the famous Black-Scholes formula through careful application of stochastic calculus and measure theory.

This approach not only yields the formula but also provides deep economic insights into option pricing, revealing the option price as an expected discounted payoff under the risk-neutral measure.