Sequence Algebra

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1 Motivation

The goal of this project is to find a systematic way of predicting the next entry in a sequence of numbers, i.e. finding a rule of how a sequence at hand is constructed. As an example, if we have the sequence

$$5 \quad 10 \quad 20 \quad 35 \quad 55 \quad \ldots ,$$

the rule can easily be found. We can write it as

$$a_{n+1} = a_n + 5 \cdot (n+1) \, , \tag{1}$$

where a_n is the number in the sequence at the index n (starting at zero). With this knowledge, the next number is determined by $55 + 5 \cdot (4+1) = 80$.

The Fibonacci sequence as well as the consecutive construction of multiple numbers by recurrence relations (e.g. to obtain solutions for Pell's equation) are other examples, where the former is typically

$$1 \quad 1 \quad 2 \quad 3 \quad 5 \quad 8 \quad \ldots ,$$

with the famous rule

$$a_{n+1} = a_n + a_{n-1} \, .$$

2 Simple Sequences

2.1 Definition

Let a simple sequence be any sequence of numbers that can be constructed by the rule

$$a^{(m)}_{n+1} = t \cdot a^{(m)}_n + v \cdot n + w \, .$$

The number $a^{(m)}_n$ represents the number at the index n in the sequence m. Define the first index in a sequence to be 0. So a valid simple sequence is fully defined by its rule and a starting number $a^{(m)}_0$.

2.2 Finding the Rule

Since any rule for a simple sequence is only formed by addition and multiplication, some basic strategies to extract relevant information are to either subtract neighbouring numbers from each other or to divide them.

The value of t can be extracted by doing the following:

1. Subtract neighbouring numbers from the starting sequence, which we will call $a^{(0)}$. This will result into a new sequence $a^{(1)}$ that can be defined by

$$a_n^{(1)} = a_{n+1}^{(0)} - a_n^{(0)} = (t-1) \cdot a_n^{(0)} + v \cdot n + w \ .$$

2. Subtract neighbouring numbers from the new sequence $a^{(1)}$ to obtain

$$\begin{aligned} a_n^{(2)} &= a_{n+1}^{(1)} - a_n^{(1)} \\ &= \left[(t-1) \cdot a_{n+1}^{(0)} + v \cdot (n+1) + w\right] - \left[(t-1) \cdot a_n^{(0)} + v \cdot n + w\right] \\ &= (t-1) \cdot \left(a_{n+1}^{(0)} - a_n^{(0)}\right) + v \\ &= (t-1) \cdot a_n^{(1)} + v \ . \end{aligned}$$

From this, we can find the relation

$$a_{n+1}^{(1)} = a_n^{(2)} + a_n^{(1)} = t \cdot a_n^{(1)} + v$$

3. Divide neighbouring numbers of sequence $a^{(2)}$ to find

$$a_n^{(3)} = \frac{a_{n+1}^{(2)}}{a_n^{(2)}} = \frac{(t-1) \cdot a_{n+1}^{(1)} + v}{(t-1) \cdot a_n^{(1)} + v} \ .$$

By using the relation found in step 2, the sequence $a^{(3)}$ will take on the form

$$a_n^{(3)} = \frac{(t-1) \cdot t \cdot a_n^{(1)} + (t-1) \cdot v + v}{(t-1) \cdot a_n^{(1)} + v} = t \ .$$

So, by recursively subtracting neighbouring numbers in a simple sequence two times and dividing them afterwards will yield a constant sequence, where all entries are of the value t. Let $a_0^{(3)}$ represent the known value of t. When we now insert this value into the result from step 2, one can also extract the value of v

$$a_{n+1}^{(1)} = t \cdot a_n^{(1)} + v \quad \Rightarrow \quad v = a_{n+1}^{(1)} - a_0^{(3)} \cdot a_n^{(1)} \ ,$$

and by inserting all results into the original sequence, we obtain

$$a_{n+1}^{(0)} = t \cdot a_n^{(0)} + v \cdot n + w \quad \Rightarrow \quad w = a_{n+1}^{(0)} - a_0^{(3)} \cdot a_n^{(0)} - n \cdot \left(a_{n+1}^{(1)} - a_0^{(3)} \cdot a_n^{(1)}\right) \ ,$$

where a choice for the index n must be made. Since it should not matter where we evaluate these parameters, one can simply choose $n = 0$, which will yield

$$w = a_1^{(0)} - a_0^{(3)} \cdot a_0^{(0)} \qquad \text{and} \qquad v = a_1^{(1)} - a_0^{(3)} \cdot a_0^{(1)} \ .$$

With this algorithm, all unknown parameters t, v and w can be extracted from the starting sequence. However, certain exceptions must still be taken into account for a general solution. For this, see section 2.2.1.

2.2.1 Edge Cases

This approach fails if $t = 1$ and $v = 0$, since then the sequence $a^{(2)}$ will be zero for every index n and we can not divide its numbers in the third step. In this special case we will notice that the sequence $a^{(1)}$ from step 1 will immediately be constant where its entries are w for each index n. Therefore, by observing a constant sequence $a^{(1)}$, we know that a correct assignment of parameters is $t = 1$, $v = 0$ and $w = a_0^{(1)}$.

However, a similar problem occurs when $t = 0$. Then, the general rule and the new sequence after the first step becomes

$$a_{n+1}^{(0)} = v \cdot n + w \quad \Rightarrow \quad a_n^{(1)} = a_{n+1}^{(0)} - a_n^{(0)} = v \cdot (n+1) + w - \big(v \cdot n + w\big) = v \ .$$

Another consequence is that $a_1^{(0)}$ does no longer depend on $a_0^{(0)}$ in this case. Therefore, a sequence like

$$100 \quad 1 \quad 2 \quad 3 \quad 4 \quad \dots \ ,$$

is a valid simple sequence with the starting number $a_0^{(0)} = 100$ and the rule $a_{n+1}^{(0)} = n+1$. This means that the sequence $a^{(1)}$ will only be constant when truncating the index $n = 0$, since otherwise there would be $a_0^{(1)} = a_1^{(0)} - a_0^{(0)}$, with no control over what $a_0^{(0)}$ is. So we can choose the parameters based on the following condition

$$(t, v, w) = \begin{cases} (1, 0, a_0^{(1)}) & \text{, if } a^{(1)} \text{ is constant for every index} \\ (0, a_1^{(1)}, a_1^{(0)}) & \text{, if } a^{(1)} \text{ is constant for all indices } n \geq 1 \\ (a_0^{(3)}, a_1^{(1)} - a_0^{(3)} \cdot a_0^{(1)}, a_1^{(0)} - a_0^{(3)} \cdot a_0^{(0)}) & \text{, otherwise} \end{cases}$$

Note that the first and second case have a certain overlap, namely when $a^{(1)}$ is constant for every index. This shows that the choice of parameters is not unique. As an example, the sequence

$$1 \quad 3 \quad 5 \quad 7 \quad 9 \quad \dots \ ,$$

can be correctly described by one of the two rules

$$a_{n+1}^{(0)} = a_n^{(0)} + 2 \qquad \text{or} \qquad a_{n+1}^{(0)} = 2 \cdot n + 3 \ .$$

2.3 Conditions for Finding Solutions

When we start with a given sequence of N numbers $a_n^{(0)}$ for $0 \leq n \leq N-1$, we can observe that the sequence after step 1 reduces the amount of numbers in the sequence, i.e. $a_n^{(1)}$ will only be known for $0 \leq n \leq N-2$. Likewise, the second step will reduce it by one additional number and the third step will do this as well. This means that at the end of the algorithm proposed in section 2.2 we will have numbers in the sequence $a^{(3)}$ with indices from 0 to $N-4$, which corresponds to $N-3$ numbers in total.

When only $N = 4$ numbers are known initially, we end up with one single number in $a^{(3)}$, which creates a rule that correctly generates the simple sequence. In order to check if the given numbers actually form a simple sequence if $N > 4$, we can use the fact that $a^{(3)}$ has to be the same value for every index. If this is not the case, the numbers do not form a simple sequence. So in order to be able to identify a sequence as a simple sequence, we require at least $N = 5$ known numbers in $a^{(0)}$.

For $N < 4$ there is no unique solution, even if we restrict ourselves to simple sequences only. To see this, consider the sequence

$$1 \quad 2 \quad 4 \quad \dots \ ,$$

which can be constructed by one of the rules

$$a_{n+1}^{(0)} = 2 \cdot n + 2 \qquad \text{or} \qquad a_{n+1}^{(0)} = 2 \cdot a_n^{(0)} \ .$$

These will both form different sequences for $n \geq 3$, which is why we have to require $N \geq 4$ known numbers in $a^{(0)}$ to find a unique way to continue the sequence.

3 Extended Simple Sequences

3.1 Definition

Let an extended simple sequence be any sequence of numbers that can be constructed by the rule

$$a_{n+1}^{(m)} = t_0 \cdot a_n^{(m)} + t_1 \cdot a_{n-1}^{(m)} + v \cdot n + w \ .$$

The notation will be adopted from section 2.1. A valid extended simple sequence is therefore fully defined by its rule and two starting numbers $a_0^{(m)}$ and $a_1^{(m)}$.

3.2 Finding the Rule

To start with a simpler scenario, consider the case where $v = w = 0$. With this, we can define a set of two equations for the parameters t_0 and t_1, which can be written in a matrix form

$$\begin{pmatrix} a_{n+1}^{(0)} \\ a_n^{(0)} \end{pmatrix} = \begin{pmatrix} a_n^{(0)} & a_{n-1}^{(0)} \\ a_{n-1}^{(0)} & a_{n-2}^{(0)} \end{pmatrix} \begin{pmatrix} t_0 \\ t_1 \end{pmatrix} := M \begin{pmatrix} t_0 \\ t_1 \end{pmatrix} \ .$$

For the sake of clearer notation, let $a := a^{(0)}$ in this section. The inverse of this 2×2 matrix M is given by

$$M^{-1} = \frac{1}{a_n \cdot a_{n-2} - a_{n-1}^2} \begin{pmatrix} a_{n-2} & -a_{n-1} \\ -a_{n-1} & a_n \end{pmatrix} \ .$$

With this we can immediately see the solution for the parameters

$$\begin{aligned} t_0 &= \frac{a_{n-2} \cdot a_{n+1} - a_{n-1} \cdot a_n}{a_n \cdot a_{n-2} - a_{n-1}^2} \\ t_1 &= \frac{a_n^2 - a_{n-1} \cdot a_{n+1}}{a_n \cdot a_{n-2} - a_{n-1}^2} \ . \end{aligned}$$

The general case with $v \neq 0 \neq w$ can then always be reconstructed by recursively subtracting neighbouring numbers three times, starting with the starting sequence

$$\begin{aligned} a_n^{(1)} = a_{n+1}^{(0)} - a_n^{(0)} &= (t_0 - 1) \cdot a_n^{(0)} + t_1 \cdot a_{n-1}^{(0)} + v \cdot n + w \\ a_n^{(2)} = a_{n+1}^{(1)} - a_n^{(1)} &= (t_0 - 1) \cdot \left(a_{n+1}^{(0)} - a_n^{(0)}\right) + t_1 \cdot \left(a_n^{(0)} - a_{n-1}^{(0)}\right) + v \\ &= (t_0 - 1) \cdot a_n^{(1)} + t_1 \cdot a_{n-1}^{(1)} + v \\ a_n^{(3)} = a_{n+1}^{(2)} - a_n^{(2)} &= (t_0 - 1) \cdot \left(a_{n+1}^{(1)} - a_n^{(1)}\right) + t_1 \cdot \left(a_n^{(1)} - a_{n-1}^{(1)}\right) + v \\ &= (t_0 - 1) \cdot a_n^{(2)} + t_1 \cdot a_{n-1}^{(2)} \ . \end{aligned}$$

From the equality in the third sequence $a^{(3)}$, we recover the same type of equation as in the beginning of this section.

$$a_{n+1}^{(2)} = t_0 \cdot a_n^{(2)} + t_1 \cdot a_{n-1}^{(2)} \ .$$

Therefore, we can now simply state the general solution of t_0 and t_1 in terms of numbers from $a^{(2)}$

$$\begin{aligned} t_0 &= \frac{a_{n-2}^{(2)} \cdot a_{n+1}^{(2)} - a_{n-1}^{(2)} \cdot a_n^{(2)}}{a_n^{(2)} \cdot a_{n-2}^{(2)} - \left(a_{n-1}^{(2)}\right)^2} \\ t_1 &= \frac{\left(a_n^{(2)}\right)^2 - a_{n-1}^{(2)} \cdot a_{n+1}^{(2)}}{a_n^{(2)} \cdot a_{n-2}^{(2)} - \left(a_{n-1}^{(2)}\right)^2} \ . \end{aligned}$$

Since it should not matter where we evaluate these parameters, one can simply choose $n = 2$. From the equality in the second sequence and the solutions from above, we obtain

$$\begin{aligned} v &= a_{n+1}^{(1)} - t_0 \cdot a_n^{(1)} - t_1 \cdot a_{n-1}^{(1)} \\ &= a_{n+1}^{(1)} - \frac{\left[a_0^{(2)} \cdot a_3^{(2)} - a_1^{(2)} \cdot a_2^{(2)}\right] a_n^{(1)} + \left[\left(a_2^{(2)}\right)^2 - a_1^{(2)} \cdot a_3^{(2)}\right] a_{n-1}^{(1)}}{a_2^{(2)} \cdot a_0^{(2)} - \left(a_1^{(2)}\right)^2} \ . \end{aligned}$$

Here we can take $n = 1$ as an arbitrary choice for the index. And finally, for the value of w one can take the original sequence

$$\begin{aligned} a_{n+1} = t_0 \cdot a_n + t_1 \cdot a_{n-1} + v \cdot n + w \quad &\Rightarrow \quad w = a_{n+1} - t_0 \cdot a_n - t_1 \cdot a_{n-1} - v \cdot n \\ \Rightarrow w &= a_{n+1} - t_0 \cdot a_n - t_1 \cdot a_{n-1} - \left(a_2^{(1)} - t_0 \cdot a_1^{(1)} - t_1 \cdot a_0^{(1)}\right) \cdot n \\ \Rightarrow w &= \left(a_{n+1} - a_2^{(1)}\right) - t_0\left(a_n - a_1^{(1)}\right) - t_1\left(a_{n-1} - a_0^{(1)}\right) \end{aligned}$$

where we can once again choose $n = 1$ for the remaining indices. This then results into the solutions

$$v = a_2^{(1)} - \frac{\left[a_0^{(2)} \cdot a_3^{(2)} - a_1^{(2)} \cdot a_2^{(2)}\right] a_1^{(1)} + \left[\left(a_2^{(2)}\right)^2 - a_1^{(2)} \cdot a_3^{(2)}\right] a_0^{(1)}}{a_2^{(2)} \cdot a_0^{(2)} - \left(a_1^{(2)}\right)^2}$$

$$w = \left(a_2^{(0)} - a_2^{(1)}\right) - \frac{\left[a_0^{(2)} \cdot a_3^{(2)} - a_1^{(2)} \cdot a_2^{(2)}\right]\left(a_1^{(0)} - a_1^{(1)}\right) + \left[\left(a_2^{(2)}\right)^2 - a_1^{(2)} \cdot a_3^{(2)}\right]\left(a_0^{(0)} - a_0^{(1)}\right)}{a_2^{(2)} \cdot a_0^{(2)} - \left(a_1^{(2)}\right)^2}$$

3.2.1 Edge Cases

We observe that when $a_{n-1}^2 = a_n \cdot a_{n-2}$, the above solution for t_0 and t_1 will not be valid. In that case, since we could have chosen any starting index, we have

$$a_{n-1}^2 = a_n \cdot a_{n-2} \quad \Rightarrow \quad \frac{a_n}{a_{n-1}} = \frac{a_{n-1}}{a_{n-2}} \ ,$$

for every index n, so the quotient of two adjacent numbers will be constant throughout the sequence. This means, however, that we can construct this sequence as a simple sequence with the rule

$$a_{n+1}^{(0)} = t \cdot a_n^{(0)} = \frac{a_1^{(0)}}{a_0^{(0)}} \cdot a_n^{(0)} \ .$$