Yule-Walker Equations

PRACTICAL TIME SERIES ANALYSIS

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Objectives

- Introduce Yule – Walker equations
- Obtain ACF of AR processes using Yule – Walker equations

Procedure

- We assume stationarity in advance (a priori assumption)
- Take product of the AR model with X_{n-k}
- Take expectation of both sides
- Use the definition of covariance, and divide by $\gamma(0) = \sigma_X^2$
- Get difference equation for $\rho(k)$, ACF of the process
- This set of equations is called Yule-Walker equations
- Solve the difference equation

Example

We have an AR(2) process

$$X_t = \frac{1}{3}X_{t-1} + \frac{1}{2}X_{t-2} + Z_t \ldots (*)$$

Polynomial

$$\phi(B) = 1 - \frac{1}{3}B - \frac{1}{2}B^2$$

has real roots $\frac{-2 \pm \sqrt{76}}{6}$ both of which has magnitude greater than 1, so roots are outside of the unit circle in $\mathbb{R}^2$. Thus, this AR(2) process is a stationary process.

Example cont.

Note that if $E(X_t) = \mu$, then

$$E(X_t) = \frac{1}{3}E(X_{t-1}) + \frac{1}{2}E(X_{t-2}) + E(Z_t)$$
$$\mu = \frac{1}{3}\mu + \frac{1}{2}\mu$$
$$\mu = 0$$

Multiply both side of $(*)$ with X_{t-k}, and take expectation

$$E(X_{t-k}X_t) = \frac{1}{3}E(X_{t-k}X_{t-1}) + \frac{1}{2}E(X_{t-k}X_{t-2}) + E(X_{t-k}Z_t)$$

Example cont.

Since $\mu = 0$, and assume $E(X_{t-k}Z_t) = 0$,

$$\gamma(-k) = \frac{1}{3}\gamma(-k+1) + \frac{1}{2}\gamma(-k+2)$$

Since $\gamma(k) = \gamma(-k)$ for any k,

$$\gamma(k) = \frac{1}{3}\gamma(k-1) + \frac{1}{2}\gamma(k-2)$$

Divide by $\gamma(0) = \sigma_X^2$

$$\rho(k) = \frac{1}{3}\rho(k-1) + \frac{1}{2}\rho(k-2)$$

This set of equations is called Yule-Walker equations.

Solve the difference equation

We look for a solution in the format of $\rho(k) = \lambda^k$.

$$\lambda^2 - \frac{1}{3}\lambda - \frac{1}{2} = 0$$

Roots are $\lambda_1 = \frac{2+\sqrt{76}}{12}$ and $\lambda_2 = \frac{2-\sqrt{76}}{12}$, thus

$$\rho(k) = c_1\left(\frac{2+\sqrt{76}}{12}\right)^k + c_2\left(\frac{2-\sqrt{76}}{12}\right)^k$$

Finding c_1, c_2

Use constraints to obtain coefficients

$$\rho(0) = 1 \ \Rightarrow c_1 + c_2 = 1$$

And for $k = p - 1 = 2 - 1 = 1$,

$$\rho(k) = \rho(-k)$$

Thus,

$$\rho(1) = \frac{1}{3}\rho(0) + \frac{1}{2}\rho(-1) \ \Rightarrow \rho(1) = \frac{2}{3} \ \Rightarrow \ c_1\left(\frac{2+\sqrt{76}}{12}\right) \ + c_2\left(\frac{2-\sqrt{76}}{12}\right) \ = \frac{2}{3}$$

Solve the system for c_1, c_2

$$\begin{cases} c_1 + c_2 = 1 \\ c_1\left(\dfrac{2+\sqrt{76}}{12}\right) + c_2\left(\dfrac{2-\sqrt{76}}{12}\right) = \dfrac{2}{3} \end{cases}$$

Then,

$$c_1 = \frac{4+\sqrt{6}}{8} \text{ and } c_2 = \frac{4-\sqrt{6}}{8}$$

ACF of the AR(2) model

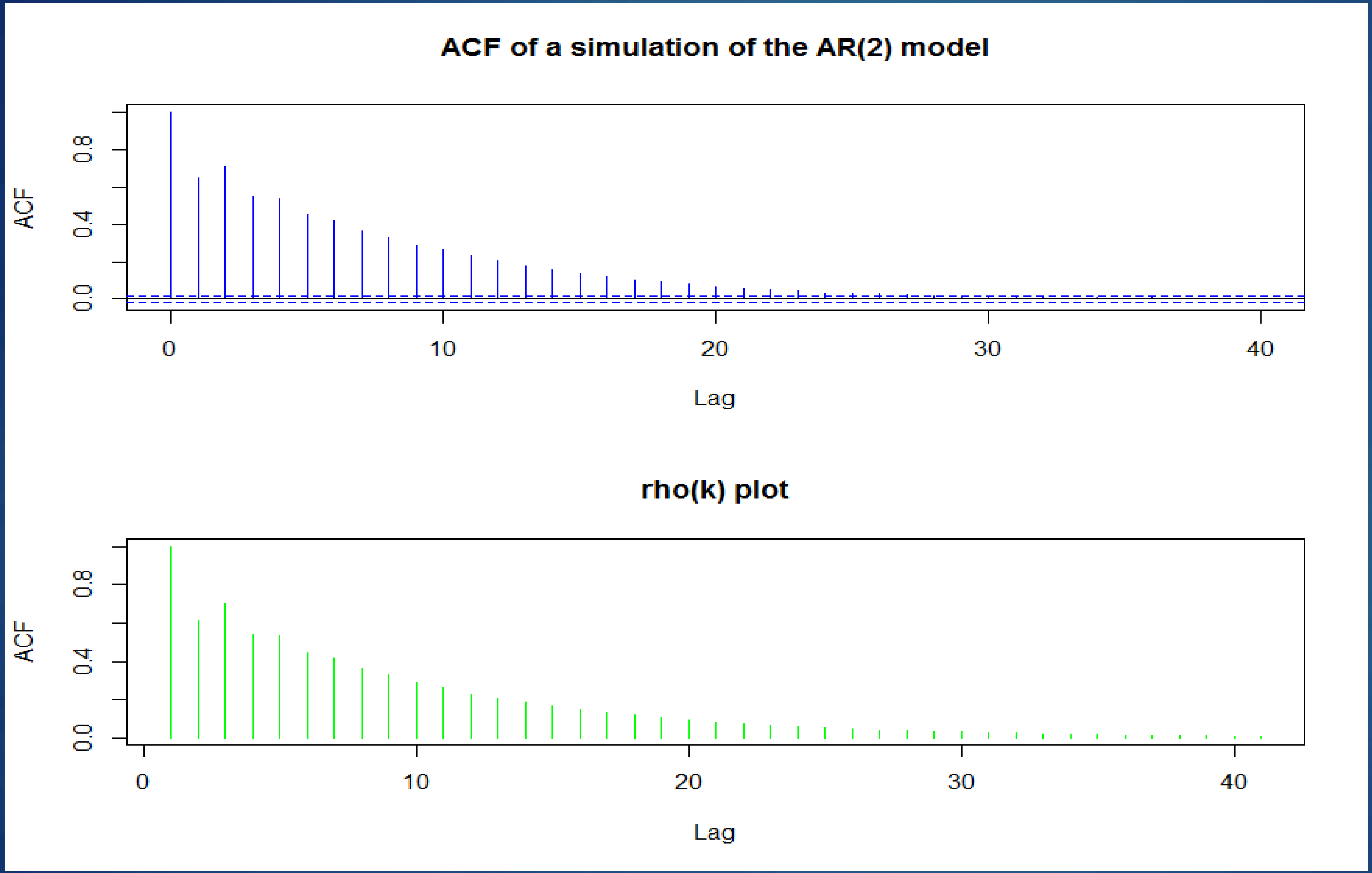
$For\,any\,k \geq 0,$

$$\rho(k) = \frac{4+\sqrt{6}}{8}\left(\frac{2+\sqrt{76}}{12}\right)^k + \frac{4-\sqrt{6}}{8}\left(\frac{2-\sqrt{76}}{12}\right)^k$$

And

$$\rho(k) = \rho(-k)$$

Simulation


ACF of a simulation of the AR(2) model
ACF
0.0
0.4
0.8
0
10
20
30
40
Lag
rho(k) plot
ACF
0.0
0.4
0.8
0
10
20
30
40
Lag

What We've Learned

- Yule- Walker equations is set of difference equations governing ACF of the underlying AR process
- How to find the ACF of an AR process using Yule-Walker equations