Recall

Let's remember what we mean by an Autoregressive Process of Order p, *AR(p)*

- The Z_t's are white noise $Z_t \sim iid(0, \sigma^2)$
- Include the current innovation and some history of the process

$$X_t = Z_t + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p}$$

Bringing in the Back-Shift Operator

You should be eager at this point to express this in terms of the backwards shift. We build the individual terms

$$\begin{gathered} X_{t-1} = B\,X_t \\ X_{t-2} = B^2 X_t \\ \vdots \\ X_{t-p} = B^p X_t \end{gathered}$$

Now form the expression

$$X_t = Z_t + \phi_1 B X_t + \cdots + \phi_p B^p X_t = Z_t + \left(\phi_1 B + \cdots + \phi_p B^p\right) X_t$$

That's a nice notational convenience- we've expressed our history with an operator. Push a little further. With a tiny bit of algebra, it is clear that we can express the innovation at time t, Z_t as

$$Z_t = \left(1 - \phi_1 B - \cdots - \phi_p B^p\right) X_t = \Phi(B) X_t$$

But using the algebra of shift operators, we can then write

$$X_t = \frac{1}{\left(1 - \phi_1 B - \cdots - \phi_p B^p\right)} Z_t = \frac{1}{1 - \left(\phi_1 B + \cdots + \phi_p B^p\right)} Z_t$$

Will we ever escape the geometric series?

$$\frac{1}{1-a} = 1 + a + a^2 + \cdots \qquad \text{when } |a| < 1$$

Good notation, like the backward shift operator, doesn't just let us write things more compactly! It suggests results and allows us to proceed much faster, and with greater clarity, than we were previously able. The Leibnitz notation in Calculus is a great example, and so is the above.

Push your result just a little further and see that an autoregressive process of order p, *AR(p)*, may be thought of as an (infinite order) moving average process

$$X_t = (1 + \theta_1 B + \theta_2 B^2 + \cdots)\, Z_t$$

We can show this by substitution for an *AR(1)* process pretty easily.

$$X_t = Z_t + \phi\, BX_t = Z_t + \phi X_{t-1}$$

$$X_t = Z_t + \phi\, (Z_{t-1} + \phi X_{t-2}) = Z_t + \phi Z_{t-1} + \phi^2\, X_{t-2}$$

$$X_t = Z_t + \phi Z_{t-1} + \phi^2\, (Z_{t-2} + \phi X_{t-3}) = Z_t + \phi Z_{t-1} + \phi^2 Z_{t-2} + \phi^3 X_{t-3}$$

Etc.

And, following the geometric series approach:

$$X_t = Z_t + \phi\, BX_t$$

$$(1 - \phi\, B)X_t = Z_t$$

$$X_t = \frac{1}{(1 - \phi\, B)} Z_t = (1 + \phi B + \phi^2 B^2 + \cdots)Z_t$$

$$X_t = Z_t + \phi Z_{t-1} + \phi^2 Z_{t-2} + \cdots$$

OK, we will in general have to find the correct infinite order *MA()* coefficients, but still…proof of concept…examples will be forthcoming.

In the meantime, the beauty of treating the *AR(p)* process like this is we quickly inherit several results from our work with *MA(q)* processes.

- Suppose you'd like to find the average of an *AR(p)* process. Just recall $E[Z_t] = 0$ and take

$$E[X_t] = E[\,(1 + \theta_1 B + \theta_2 B^2 + \cdots)\,Z_t]$$

$$E[X_t] = \; E[\,Z_t] + \theta_1 E[Z_{t-1}] + \cdots + \theta_k E[Z_{t-k}] + \cdots \; = 0$$

- How about the variance? The Z_t are independent, so using $V[aX] = a^2 V[X]$

$$V[X_t] = V[\,(1 + \theta_1 B + \theta_2 B^2 + \cdots)\,Z_t]$$

$$V[X_t] = \; V[\,Z_t] + \theta_1^2 V[Z_{t-1}] + \cdots + \theta_k^2 V[Z_{t-k}] + \cdots$$

$$V[X_t] = \sigma_Z^2 (1 + \theta_1^2 + \cdots + \theta_k^2 + \cdots) = \sigma_Z^2 \sum_{i=0}^{\infty} \theta_i^2$$

We obviously took $\theta_0 = 1$. Evidently we have a *necessary* condition for stationarity, that is, we need the infinite series to converge.

- Finally, how about autocorrelation and autocovariance? We saw, for a $MA(q)$ process,

$$\gamma(k) = \sigma_Z^2 \cdot \sum_{i=0}^{q-k} \theta_i\,\theta_{i+k} \quad \text{(where appropriate)}$$

So, we have, for an *AR(p)* process

$$\gamma(k) = \sigma_Z^2 \cdot \sum_{i=0}^{\infty} \theta_i\,\theta_{i+k} \quad \text{(where appropriate)}$$

Sneaking in a result from real analysis (though maybe this is a little beyond Calculus II), the series converges when the more basic series is absolutely convergent:

$$\sum_{i=0}^{\infty} |\theta_i|$$

Summarizing the conversation so far, for an *AR(p)* process, find the corresponding *MA()* process and then express

$$E[X_t] = 0 \qquad V[X(t)] = \sigma_Z^2 \sum_{i=0}^{\infty} \theta_i^2 \qquad \gamma(k) = \sigma_Z^2 \cdot \sum_{i=0}^{\infty} \theta_i \, \theta_{i+k}$$

Time for some examples. Start with

$$X_t = Z_t + 0.4X_{t-1} = Z_t + 0.4\, B\, X_t$$

Take advantage of our operator notation to solve for X_t

$$Z_t = (1 - .4B)X_t$$

$$X_t = \left\{\frac{1}{1 - .4\, B}\right\} Z_t$$

Remembering, yet again, our geometric series

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$$

We write the autoregressive process as an infinite order *MA(q)* as

$$X_t = \left\{\sum_{k=0}^{\infty} (0.4B)^k\right\} Z_t = \{1 + .4\, B + (.4)^2 B^2 + (.4)^3 B^3 + \cdots\}\, Z_t$$

We are trading the ϕ coefficient: $\phi_1 = 0.4$ for an infinite set of β coefficients

$$\theta_0 = 1,\;\; \theta_1 = .4,\;\; \theta_2 = 0.16,\;\; \theta_3 = .064, \dots, \theta_k = .4^k, \dots$$

Can we work out the autocovariance and autocorrelation functions?

Our sum depends upon index *i*, with *k* constant with respect to the sum, so

$$\gamma(k) = \sigma_Z^2 \cdot \sum_{i=0}^{\infty} \theta_i \, \theta_{i+k} \quad = 1 \cdot \sum_{i=0}^{\infty} .4^i \, .4^{i+k} = .4^k \sum_{i=0}^{\infty} (.4^2)^i$$

$$\gamma(k) = .4^k \frac{1}{1-.16}$$

And now scale for the autocorrelations

$$\rho(k) = \frac{\sum_{i=0}^{\infty} \theta_i \, \theta_{i+k}}{\sum_{i=0}^{\infty} \theta_i \, \theta_i}$$

We already worked out the numerator. For the denominator

$$\sum_{i=0}^{\infty} \theta_i \, \theta_i = \sum_{i=0}^{\infty} .4^i \cdot .4^i = \sum_{i=0}^{\infty} .16^i = \frac{1}{1-.16}$$

This leads to a surprisingly simple result

$$\rho(k) = \frac{.4^k \frac{1}{1-.16}}{\frac{1}{1-.16}} = 0.4^k$$

Let this sink in a bit. There is nothing special about $\phi_1 = 0.4$, so we have really shown that, for a first order autoregressive process

$$X_t = Z_t + \phi_1 X_{t-1}$$

We have the autocorrelation function

$$\rho(k) = \phi_1^k$$

In tabular format

k	1	2	3	...	k	...
$\rho(k)$	$.4$	$.16$	.064		0.4^k	

How do these compare with our estimates from the previous lecture?

Recall that we wrote the code:

```
set.seed(2016);   N=1000;  phi = .4;
Z = rnorm(N,0,1)
X=NULL;

X[1] = Z[1];

for (t in 2:N) {
        X[t] = Z[t] + phi*X[t-1] ;
}

X.ts = ts(X)
X.acf = acf(X.ts)
```

A quick call to *(r.coef = X.acf$acf)* tells me

$$r_1 = 0.401713170$$

$$r_2 = 0.177573718$$

$$r_3 = 0.119641043$$

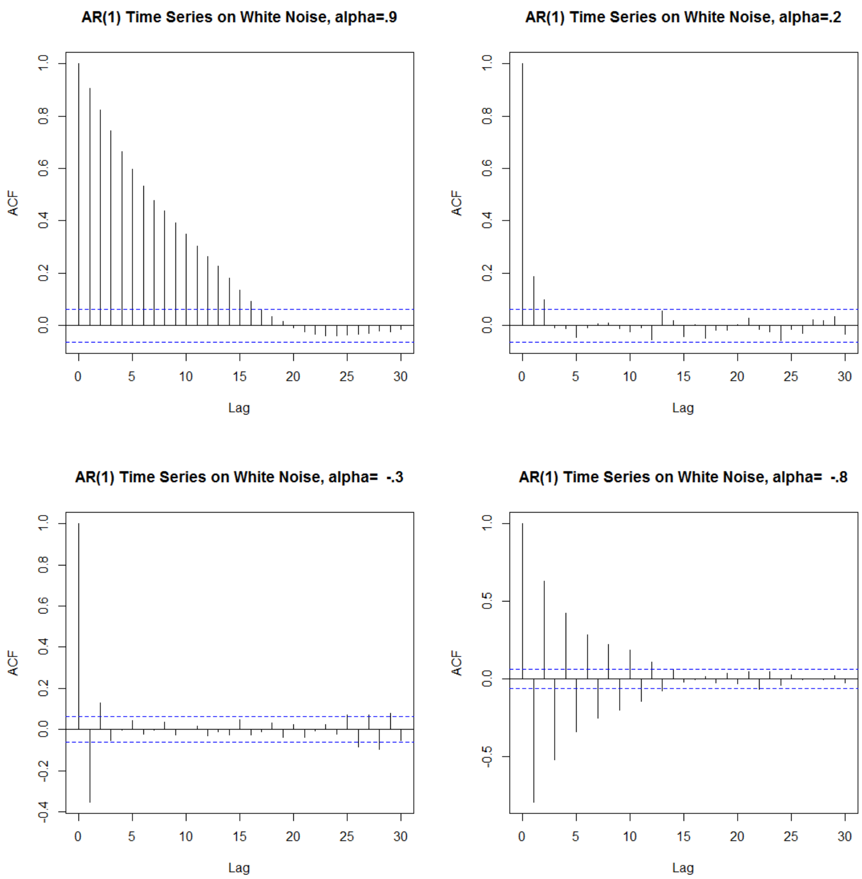
Run the code several times without setting the seed to see whether you believe our results are supported. We also made predictions for the mean and variance.

	predicted	from simulation
mean	$\mu = 0$	$\bar{x} = 0.0113660$
variance	$V[X_t] = \sigma_Z^2 \sum_{i=0}^{\infty} \theta_i^2 = \frac{1}{1-.16} = 1.190476$	$s^2 = 1.091754$

This is pretty nice agreement!

Now that we know how to proceed, let's look at how the parameter in a *AR(p=1)* process affects the autocorrelation function. Given

$$\rho(k) = \phi_1^k$$



When $\phi_1 \approx 1$ then we have something close to the random walk we studied previously. Recall that the random walk $\phi = 1$ would give us a nonstationary process (growing variance). As $\phi_1 \downarrow 0$ the correlations decay more quickly. Note that $\phi_1 = 0$ brings us back to white noise. For negative values $\phi_1 < 0$ we have alternating positive and negative correlations as the terms "flip back and forth".