Branching dynamics

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1 Introduction

To run your own simulations varying the parameters, or see other sets of parameters already simulated see this Github repository
We establish the ACL model and then make a section for each of the following questions.

Does Branching dynamics split the energy eigenspace between the branches in a particular way? Do the worlds share it equally? Proportional to their respective probabilities?

Having established that they share the eigenstates weirdly. Why? What drives this dynamics? Does it generalize beyond the ACL model in which we observe this dynamics?

Can we get it all from the energy spectrum and the distribution of the global state in it? Energy spectrum + state defined in energy basis → sharing of Hilbert space by worlds → explain the phenomena we observed

2 A measurement model

2.1 The Adapted Caldeira-Leggett dynamics

To answer these questions, we consdier the adapted Caldeira-Leggett model, a system $\mathcal{S}$ + environment $\mathcal{E}$ where the system is a truncated SHO. Its Hamiltonian takes the form,

$$H_{tot} = H_s + H_e + E_I H_I \tag{1}$$

We choose no self interaction for the truncated SHO, so $H_s = \mathbb{1} \otimes \mathbb{1}$, and we choose a random hermitian matrix for the environment self interaction $H_e = \mathbb{1} \otimes H_r$, with H_r a random hermitian matrix. The interaction Hamiltonian is $H_I = H_q \otimes H'_r$, with $(H_q)_{ij} = q_i \delta_{ij}$ and H'_r another random hermitian matrix. This makes it so that the environment acts on the system depending on its state.
Taking the eigenstates of H_q to be $\{|q_i\rangle\}\forall$, there is a basis like $\{|q_i\rangle|k_i\rangle\}$, such that H_{tot}, H_e and H_I take the a block diagonal form. $i \in [0, d_1 - 1], j \in [0, d_2 - 1]$, d_1 and d_2 the dimensions of the system and environment respectively, giving a total dimension $d = d_1 * d_2$.

2.2 The state

We choose our initial state to be a separble state with $|e\rangle$ an environment state,

$$|\psi(0)\rangle = (\sqrt{w}|q_1\rangle + \sqrt{1-w}|q_2\rangle)|e\rangle \tag{2}$$

this corresponds to a superposition of energy eigenstates not yet entangled with an environment. w is the probability of being in $|q_1\rangle$. We also define here the eigenstates of H_{tot} to be $\{|q_i\rangle|k_i\rangle\}$. Note the fact that $|k_i\rangle$ depends on q_i. This is because each block of H_{tot} is different, meaning that it diagonalizes in different basis. More precisely, we can isolate the d_1 Hamiltonians that act on our environment depending on q_i, but since they do not commute, they do not share a common eigenbasis. Note that we can use a basis defined without q_i dependance in the k terms in the H_I dominated limit or the $E_I = 0$ limit (that last one is less interesting).

(a) Heat map of H_e (b) Heat map of H_I

Figure 1

Figure 2: Heat map of the Hamiltonians H_e and H_I in the eigenbasis of H_{tot}, for $d_1 = 10, d_2 = 200$. We note the d_1 degeneracy of the spectrum of H_e and the no degeneracy of H_I. By our constrution, there is no degeneracy for H_{tot}

2.3 Time evolution and decoherence

With this in mind, our initial state is only defined in $2d_2$ out of d energy eigenstates. Thus to understand the action of the $e^{-itH_{tot}}$ on our state, we only need to consider 3 different $d_2 * d_2$ matrices, $H_r, E_I q_1 H'_r, E_I q_2 H'_r$ (or 2 if we consider H_{tot}). We can write the time evolution,

$$
\begin{aligned}
|\psi(t)\rangle &= e^{-itH_{tot}}|\psi(0)\rangle && (3)\\
&= e^{-itH_{tot}}|q_1\rangle\sqrt{w}|e\rangle + e^{-itH_{tot}}|q_2\rangle\sqrt{1-w}|e\rangle && (4)\\
&= |q_1\rangle\sqrt{w}e^{-it(H_e^{(1)})}|e\rangle + |q_2\rangle\sqrt{1-w}e^{-it(H_e^{(2)})}|e\rangle && (5)\\
&= |q_1\rangle\sqrt{w}|\mathcal{E}_1(t)\rangle + |q_2\rangle\sqrt{1-w}|\mathcal{E}_2(t)\rangle && (6)\\
& && (7)
\end{aligned}
$$

We defined, $H_e^{(1)} = H_r + E_I q_1 H'_r$ and $H_e^{(2)} = H_r + E_I q_2 H'_r$, two matrices acting on $\mathcal{E}$.

2.3.1 Existence of pointer states

Block diagonal Hamiltonians always allow for certain pointer states (See "A Search for Classical Subsystems in Quantum Worlds" Appendix A)

We find that $|q_i\rangle|\mathcal{E}(t)\rangle = |p_i(t)\rangle$ are global pointer states since the time evolution leaves the state as a separable state. In the total energy eigenbasis, a pointer state is defined in a d/d_1 subspace at most (and would need to be fine tuned to be defined in a smaller state). Technically, pointer states would be defined on the system, but we consider here an extension to the full Hilbert space, since our system has no self interaction.

For all bipartide decomposition you can always find a unitary that makes the factorization produce a block diagonal Hamiltonian. Thus the way we make our Hamiltonian restricts that a little bit by making the blocks not independant from each other. The Hamiltonian is then such that the system density becomes diagonal in the pointer basis. The quantum measurement limit, ie the case where the interaction Hamiltonian is dominant corresponds to the case where the limit where the blocks of the block diagonal Hamiltonian are proportional to each other. The

other limits to look at are the decoupling limit, and the intermediate limit.

In either the intermediate or the measurement limit the density matrix of the system becomes diagonal in the pointer basis. We see later that the pointer states being defined in limited subsets of the energy eigenbasis of the total Hamiltonian it means that the worlds split the energy eigenspace with each other. (Note, When we are not in this limit a useful idea is that of a decoherence free subspace.)

In the factorization of Hilbert space that makes the Hamiltonian block diagonal, the system state does not evolve in time and the environment state does but is confined to the subspace defined by the pointer state. This means that studying the pointer states, we will see the restriction to only parts of the total energy eigenspaces.

From the subsystems paper: For any Hamiltonian $H = \sum_k E_k|E_k\rangle\langle E_k|$, defining a factorization $|E_k\rangle = |i\rangle_s|j\rangle_e$, we get,

$$H = \sum_k E_k|E_k\rangle\langle E_k| = \sum_i |i\rangle_s\langle i| \otimes \sum_j |j\rangle_s\langle j|E_{ij} \tag{8}$$

$$= \sum_i |i\rangle_s\langle i| \otimes H_e^{(i)} = \sum_i |i\rangle_s\langle i| \otimes H_e + \sum_i |i\rangle_s\langle i| \otimes \tilde{H}_e^{(i)} \tag{9}$$

With $\tilde{H}_e^{(i)} = \sum_j E_{ij}|j\rangle_s\langle j| - H_e$. With this we recover a Hamiltonian in the form of the ACL model. If H_e and $\tilde{H}_e^{(i)}$ are such that decoherence happens, we have branching and the branches are defined on a reduced energy eigenspace. So we end up with having to define conditions on theses Hamiltonians. After that, the condition on theses 2 Hamiltonians can be seen as a condition on the total Hamiltonian spectrum for a certain factorization.

2.3.2 Decoherence

The initial state is separable with,

$$\rho(0) = |\psi(0)\rangle\langle\psi(0)|$$
$$\rho_s(0) = Tr_e[\rho(0)]$$
$$\rho_e(0) = Tr_s[\rho(0)]$$

We have in the $\{|q_i\rangle\}$ basis,

$$\rho_s(0) = (|q_1\rangle\sqrt{w} + |q_2\rangle\sqrt{1-w})(\langle q_1|\sqrt{w} + \langle q_2|\sqrt{1-w}) = \begin{pmatrix} 0 & ... & ... & 0 \\ ... & w & ... & \sqrt{w(1-w)} & ... \\ ... & 0 & ... & 0 & ... \\ ... & \sqrt{w(1-w)} & ... & (1-w) & ... \\ 0 & ... & ... & 0 \end{pmatrix} \tag{10}$$

In a basis $\{|q_1\rangle\sqrt{w} + |q_2\rangle\sqrt{1-w}, ...\}$ it is diagonal with only one non-zero eigenvalue $\lambda_0 = 1$. After some time we get,

$$\rho(t) = (|q_1\rangle\sqrt{w}|\mathcal{E}_1(t)\rangle + |q_2\rangle\sqrt{1-w}|\mathcal{E}_2(t)>) * (\cdots)^\dagger \tag{11}$$

$$\implies \rho_s(t) = Tr_e[\rho(t)] = \begin{pmatrix} 0 & ... & ... & 0 \\ ... & w & ... & \sqrt{w-w^2}\sum_k\langle k|\mathcal{E}_2(t)\rangle\langle\mathcal{E}_1(t)|k\rangle & ... \\ ... & 0 & ... & 0 & ... \\ ... & \sqrt{w-w^2}\sum_k\langle k|\mathcal{E}_1(t)\rangle\langle\mathcal{E}_2(t)|k\rangle & ... & 1-w & ... \\ 0 & ... & ... & 0 \end{pmatrix} \tag{12}$$

$$\tag{13}$$

We used here $\{|k\rangle\}$ a basis for the environment that is not one of the $\{|k_i\rangle\}$. This means that $|\mathcal{E}_1\rangle = \sum_k c_k^{\mathcal{E}_1}(t)|k\rangle$ with the $c_k^{\mathcal{E}_1}(t)$'s not just phases and actually varying in amplitude. We see this using the expression of the $|k_1\rangle$

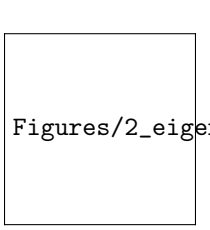


Figure 3: plot of $1 - 2c_1\sqrt{1-c_1^2}$

basis states in the $|k\rangle$ basis,

$$|\mathcal{E}_1\rangle = \sum_{k_1} e^{-itk_1} c_{k_1} |k_1\rangle \tag{14}$$

$$= \sum_{k_1} e^{-itk_1} c_{k_1} \sum_k a_k^{(k_1)} |k\rangle \tag{15}$$

$$= \sum_k (\sum_{k_1} e^{-itk_1} c_{k_1} a_k^{(k_1)}) |k\rangle \tag{16}$$

The term $(\sum_{k_1} e^{-itk_1} c_{k_1} a_k^{(k_1)}) = c_k^{\mathcal{E}_1}(t)$ has varying norm.

Or density matrix is already in the pointer basis $|p_i\rangle = |q_i\rangle$ so decoherence would be diagonalization in this basis, ie $|\langle\mathcal{E}_1|\mathcal{E}_2\rangle|^2 = |r(t)|^2 \to 0$. We get for the diagonal term,

$$\langle\mathcal{E}_1|\mathcal{E}_2\rangle(t) = \sum_k (c_k^{\mathcal{E}_1})^\dagger(t) c_k^{\mathcal{E}_2}(t) \tag{17}$$

$$|r(t)|^2 = |\langle\mathcal{E}_1|\mathcal{E}_2\rangle(t)|^2 \tag{18}$$

2.3.3 A quick result on decoherence time

Using $|\mathcal{E}_i(t)\rangle = e^{-itH_e^{(i)}}|e\rangle$ and $H_e^{(i)} = H_r - E_I q_i H_r'$, we have

$$r(t) = \langle\mathcal{E}_1(t)|\mathcal{E}_2(t)\rangle = \langle e|e^{-it(H_e^{(2)} - H_e^{(1)})}|e\rangle \tag{19}$$

$$= \langle e|e^{-itE_I H_r'(q_2 - q_1)}|e\rangle \tag{20}$$

Which means that the decoherence time can be expressed as $T_{dec} = \frac{T_{dec,0}}{E_I(q_2 - q_1)}$.

2.3.4 When do we have decoherence?

We have decoherence when the matrix $E_I H_r'(q_2 - q_1)$ generates a unitary such that the environment state can become orthogonal to its initial state. This depends on the state of the environemnt, as imagining the environment as starting in an eigenstate of H_r', then $|r(t)|^2 = 1 \forall t$. This means that we cannot get a condition on H_r' that necessarily implies decoherence. We need to take into account both the state and Hamiltonian. What about if the state is not an energy eigenstate of H_r'? For $|e\rangle = c_1|1\rangle + c_2|2\rangle$ with $|i\rangle$ an energy eigenstate of H_r' and c_i real and the eigenenergies are E_1 and E_2 (inclding inside the parameter dependance), we get (using normalization $c_1^2 + c_2^2 = 1$ and $c_2 = \sqrt{1-c_2^2}$),

$$|r(t)|^2 = 1 - 2c_1\sqrt{1-c_1^2}\cos t(E_1 - E_2) \tag{21}$$

This is minimized for $T = \frac{\pi(2k+1)}{E_1 - E_2}$ but only achieves a zero if $c_1 = c_2 = \frac{1}{\sqrt{2}}$ Fig. (**??**).

We again see that not any state allows for full decoherence. (Maybe full decoherence is too strong to look at?) We do the same thing with 3 eigenstates (defining $E_i - E_j = e_{ij}$),

$$|r(t)|^2 = 1 + 2c_1c_2cos(te_{12}) + 2c_1\sqrt{1-c_1^2-c_2^2}cos(te_{13}) + 2c_2\sqrt{1-c_1^2-c_2^2}cos(te_{23}) \tag{22}$$

It would be simpler to find the types of states that allow for full decoherence if there was a time for which all 3 cosines go to -1, so let us look for one. For a T such that $Te_{12} = (2k_{12}+1)$, $Te_{13} = (2k_{13}+1)$, then using the fact

that $Te_{23} = T(e_{13} - e_{12})$ we have,

$$Te_{23} = 2k_{13} + 1 - 2k_{12} - 1 = 2(k_{13} - k_{12}) \tag{23}$$

which is even.

===Continuing in this directions what is the goal. tell something about decoherence from the spectrum. ok. This might be a valide direction. The idea would be to get some measure of decoherence, maybe how low $|r(t)|^2$ can go to. this is dependant on the c_k's and the e_{ij}'s which is exactly what I want, and also makes a tonn of sense. it is state dependant, and from it maybe I can defined a state that has the nice property of doing a double decoherence. Then fro this I may start figuring out if it is hard or not to "branch" n times. or if a typical Hamiltonian does it.

2.3.5 Simplifying the spectrum

We start by simplifying the problem, from our finite spectrum of reals $\{E_i\}$, we note that adding a constant to it does not change the physics and multiplying it all by a constant simply inversely scales the time. So from that we can set the first $E_0 = 0$. To make all terms rationals we redefine it $\{E_i\} \to \{(\Pi_{k \text{ s.t } E_k \text{ irrational}} E_k)E_i\}$. Now we can write $\{E_i = \frac{a_i}{b_i}\}$ and similarly we make it a spectrum of integers $\{E_i\} \to \{(\Pi_k b_k)\frac{a_i}{b_i}\} = \{(\Pi_{k\neq i} b_k)a_i\}$.

2.3.6 Decoherence by variance

What do we know about $c_k^{\mathcal{E}_i}$? It is a complex number with norm varying between 0 and 1 with a normalization condition $\sum_k |c_k^{\mathcal{E}_i}|^2 = 1$. Suppose after some time t, they are Gaussian distributed $\mathbb{E}(c_k^{\mathcal{E}_i}) = 0$. Then $\mathbb{E}(\sum_k c_k^{\mathcal{E}_i}) = 0$, and we know that $|Var(\sum_k c_k^{\mathcal{E}_i})| \leq 1$ .We can also suppose that after some time, the $c_k^{\mathcal{E}_1}$ and $c_k^{\mathcal{E}_2}$ are independant. Then,
FIND WAY TO BOUND VARIANCE EXPONENTIALLY IN D_2..
Note, we neglected the independance of the $c_k^{\mathcal{E}_i}$ (leading to probabilities).

2.3.7 Decoherence by $\langle |r(t)|^2 \rangle$

$$|\langle \mathcal{E}_1 | \mathcal{E}_2 \rangle|^2 = |\langle e | e^{-itH_r' E_I (q_2 - q_1)} | e \rangle|^2 \tag{24}$$

$$= \frac{1}{d_2^2} \sum_{ij} |c_k|^2 |c_j|^2 e^{-it(E_k - E_j)} \tag{25}$$

$$= \frac{1}{d_2^2} [d_2 + 2 \sum_{k \geq j} |c_k|^2 |c_j|^2 cos(t(E_k - E_j))] \tag{26}$$

$$\tag{27}$$

We used the normalization $\sum_k |c_k|^2 = 1$ and used E_k as the k'th eigenvalue of $H_r' E_I (q_2 - q_1)$ As in Zurek's seminal paper, we look at $\langle |r(t)|^2 \rangle$,

$$\langle |r(t)|^2 \rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T |r(t)|^2 \tag{28}$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{1}{d_2^2} [d_2 + 2 \sum_{k \geq j} |c_k|^2 |c_j|^2 cos(t(E_k - E_j)) \tag{29}$$

$$= \frac{1}{d_2} + \lim_{T \to \infty} \frac{2}{d_2^2} \sum_{k \geq j} |c_k|^2 |c_j|^2 \frac{sin(T(E_k - E_j))}{T(E_k - E_j)}] \tag{30}$$

$$\propto \frac{1}{d_2} \tag{31}$$

Which supposing our environment is made up of qubits, we have that $d_2 = 2^N$, making $|r(t)|^2$ exponentially decreasing in number of qubits forming our environment.

Figure 4: For $d_2 = 5$, plot of $\sum_{k=1}^{d_2}(d_2 - k)cos(ktf)$. It would seems reasonable to define the decoherence time the first local minima around 0.

2.3.8 Decoherence by unit sphere

Other way of explaining away decoherence, we give each term the same amplitude and just look at phases. We have a sum of a bunch of random phases. Or simply, we can see it as the overlap of 2 random quantum states, ie 2 random vectors on the unit sphere of the Hilbert space (of dimension $2d_2-1$). Doing that, we set $|\mathcal{E}_1\rangle = (1, 0, ..., 0)$, meaning we now only need to care about the zero value of a random vector on the unit sphere ($|\mathcal{E}_2\rangle = (r_1, ...r_2)$). It turns out that the distribution of r_i for points randomly selected on the sphere (the r_i's are points in $2n-1$ real space) is such that it is distributed according to approximately $(1+r_1)^{d_2}(1-r_1)^{d_2}$, which means that the mean of r_1 is approximately $B(d_2 + 1/2, d_2 + 1/2) \approx \frac{\Gamma(d_2-1/2)\Gamma(d_2+1/2)}{\Gamma(2*d_2)}$.

2.3.9 Decoherence other

Bosonis bath: A la Zurek "Coherent States via Decoherence".

2.3.10 Decoherence time

We study a simple case, the limit where the interaction strength dominates. This implies that the blocks in the Hamiltonian are diagonalizable together (because of our specific construction). We get from this, that $|\mathcal{E}_i\rangle = \frac{1}{\sqrt{d_2}}\sum_k e^{-itE_k^{(i)}}|k\rangle$. (Where we also assumed that we started in a uniformly distributed state). Of note, is that this implies that our 2 worlds's porbability vecotrs (in total energy eigenspace) will have constant non-zero overlap. Still this makes the state overlap (ie orthogonality in Hilbert space) easier to study,

$$r(t) = \langle\mathcal{E}_1|\mathcal{E}_2\rangle = \frac{1}{d_2}\sum_k e^{-it(E_k - E'_k)} \tag{32}$$

$$\tag{33}$$

We study the case where the energy spectrums of $H_e^{(i)}$ is uniform, $E_k = q_1 k$ and $E'_k = q_2 k$, also defining $q_1 - q_2 = f$. We apply the next simplification. We now look at $|r(t)|^2$,

$$|r(t)|^2 = \frac{1}{d_2^2}|\sum_k e^{-itkf}|^2 \tag{34}$$

$$= \frac{1}{d_2^2}\sum_{kj} e^{-itk(1-f)+itjf} \tag{35}$$

$$= \frac{1}{d_2^2}(d_2 + \sum_{k\neq j} e^{-it(k-j)f}) \tag{36}$$

$$= \frac{1}{d_2^2}(d_2 + 2\sum_{k=1}^{d_2}(d_2 - k)cos(ktf)) \tag{37}$$

$$\tag{38}$$

Where what we end up with is a sum that has terms of higher importance than others (the low k's have a higher weight.). Fully time symmetric as expected of unitary evolution, as well as periodic with period $T_p = \frac{2\pi}{f}$. For $d_2 = 5$, we get that $\sum_{k=1}^{d_2}(d_2 - k)cos(ktf)$ has the form of Fig. (**??**)

Decoherence time dependence on q_2 and q_1: By time symmetry of the solution, we choose to consider q_2 superior to q_1 and redefine $f \to -f$ just to keep things positive. Suppose we have a T_{dec}, such that $|r(T_{dec})|^2 = \epsilon$, where ϵ is choosen to satisfy our decoherence condition, or T_{dec} defined as the time of the first minima around 0 in Fig. (**??**). Then varying the difference $q_2 - q_1$ rescales time in the above equation, making the decoherence time depend on the energy difference f as,

$$T_{dec}(f) = \frac{T_{dec}(1)}{f} \tag{39}$$

This is to be interpret as $T_{dec}(q_1, q_2) = \frac{T_dec(q_2'=q_1'+1)}{q_2-q_1}$. We see this as meaning that increasing q_2 and leaving q_1 unchanged re-scales the decoherence time by their difference. So the decoherence time depends on the energy scale difference of the 2 interactions and not the energy ratio or something more complex. Fundamentally this is a trivial result, but we got a spectrum $\rightarrow$ decoherence time type of result. We now want to go to more complex cases, not interaction dominated, with non trivial spectrum (not determined by only 1 variable, the difference in energy scale.). Note again that this decoherence time from the spectrum approach neglects the whole idea that the pointer states are actually defined on totally different eigenstates.

About defining the decoherence time: We suggest using an ϵ bound on the overlap of the environment states after which we consider decoherence good. Or, as suggested by the form of the the function represented in Fig. (**??**), we look for the first minimum. We note the following useful relations,

$$\sum_{k=1}^{n} cos(kx) = \frac{1}{2} + \frac{sin(n+\frac{1}{2})x}{2sin(\frac{x}{2})} \tag{40}$$

$$\sum_{k=1}^{n} kcos(kx) = \frac{d}{dx}(\sum_{k=1}^{n} sin(kx)) = \frac{d}{dx}(\frac{sin(\frac{nx}{2})sin(\frac{(n+1)x}{2})}{sin(\frac{x}{2})}) \tag{41}$$

This means that we can write the following,

$$|r(t)|^2 = \frac{1}{d_2^2}(d_2 + 2\sum_{k=1}^{d_2}(d_2-k)cos(ktf)) \tag{42}$$

$$= \frac{1}{d_2^2}(d_2 + 2d_2\sum_{k=1}^{d_2} cos(kt') - 2\sum_{k=1}^{d_2} kcos(kt')) \tag{43}$$

$$= \frac{1}{d_2^2}(d_2 + 2d_2(\frac{1}{2} + \frac{sin(d_2+\frac{1}{2})t'}{2sin(\frac{t'}{2})}) - 2(\frac{d}{dt'}(\frac{sin(\frac{d_2t'}{2})sin(\frac{(d_2+1)t'}{2})}{sin(\frac{t'}{2})}))) \tag{44}$$

$$\tag{45}$$

where we used $t' = tf$. Definning the decoherence time as the first minima, we find it analytically, (see mathematica)

About the relative importance of the terms of this function: Note that,

$$\sum_{k=1}^{n} = \frac{n(n+1)}{2} \tag{46}$$

from which we can look at,

$$\Delta = \frac{d_2(d_2-1)}{2} - \frac{a(a-1)}{2} \tag{47}$$

Δ represents some measure of the error coming from neglecting the $d_2 - a$ terms. For large d_2 and small a (we want to neglect as much as we can), we get,

$$\Delta \propto \frac{d_2^2}{2} - \frac{a^2}{2} \tag{48}$$

This leads to,

$$a \propto d_2\sqrt{1 - \frac{\Delta}{d_2^2}} \tag{49}$$

Where we find that taking $\Delta = 0$ leads to $a = d_2$, meaning that to get no error, we need to consider the full sum, as expected. Inverting the formula,

$$\Delta' = \frac{2\Delta}{d_2^2} = \frac{d_2^2}{2}\sqrt{1 - \frac{a^2}{d_2^2}} \tag{50}$$

Defining This has Δ' grows faster closer to $a = d_2$ than $a = 0$, which is again an argument that the error is less sensitive to neglecting the small a's. We can use this formula to get a bound on the error for a certain precision. Note, we care about having to put less of the terms in the sum to 0, because they imply havig a lower T_0 for the equation.

2.4 Schmidt states in the ACL

2.4.1 System Schmidt states and Global Schimdt states

Given a system and an environment $\mathcal{S}+\mathcal{E}$ and a pure state defined over both $|\psi(t)\rangle$. It can always be decomposed in its Schmidt basis,

$$|\psi(t)\rangle = \sum_i \sqrt{p_i(t)}|s_i(t)\rangle|s_i^{(e)}(t)\rangle \tag{51}$$

The important points are that the sum is over one index, that the sum is over dimension of the smaller of both systems ($\mathcal{S}$ or $\mathcal{E}$) at most, and that the basis itself depends on the state, which implies that it varies in time.

While $\rho_s(0)$ Eq. (**??**) can be diagonalized and have only 1 eigenvalue $\lambda = 0$, after some time t, we get $\rho_s(t)$ Eq. (**??**) which now has 2 eigenvalues. This is due to the fact that the environment evolves differently for the 2 states in superposition, we get entanglement between the evolving environment and the state of the system. The eigenvectors of this density matrix are the Schmitd states of the system, and take the form,

$$|s_1(t)\rangle = u_1(t)|q_1\rangle + u_2(t)|q_2\rangle \tag{52}$$
$$|s_2(t)\rangle = v_1(t)|q_1\rangle + v_2(t)|q_2\rangle \tag{53}$$
$$\implies \tag{54}$$
$$|q_1\rangle = (\frac{v_2}{\alpha})|s_1\rangle - (\frac{u_2}{\alpha})|s_2\rangle \tag{55}$$
$$|q_2\rangle = -(\frac{v_1}{\alpha})|s_1\rangle + (\frac{u_1}{\alpha})|s_2\rangle \tag{56}$$

With $\alpha = (u_1v_2 - u_2v_1)$ By unitary evolution our global state stay pure and the Schmidt decomposition tells us that it can always be seen as the superposition of 2 separable states (in this particular bipartide decomposition, and because of the fact that the system has at most 2 non-zero Schmidt values). Going from the time evolution of the global state Eq. (**??**), we can express our $|q_i\rangle$'s from $|s_i\rangle$'s and get the following,

$$|\psi(t)\rangle = \sqrt{w}|q_1\rangle|\mathcal{E}_1(t)\rangle + \sqrt{1-w}|q_2\rangle|\mathcal{E}_2(t)\rangle \tag{57}$$
$$= \frac{1}{\alpha}[|s_1(t)\rangle(\sqrt{w}v_2(t)|\mathcal{E}_1(t)\rangle - \sqrt{1-w}v_1(t)|\mathcal{E}_2(t)\rangle) + |s_2(t)\rangle(-\sqrt{w}u_2(t)t|\mathcal{E}_1(t)\rangle + \sqrt{1-w}u_1(t)|\mathcal{E}_2(t)\rangle)] \tag{58}$$
$$= |s_1(t)\rangle|s_1^{(e)}(t)\rangle + |s_2(t)\rangle|s_2^{(e)}(t)\rangle \tag{59}$$

We define the following objects as the Schmidt states of our system + environment,

$$|S_1(t)\rangle = |s_1(t)\rangle|s_1^{(e)}(t)\rangle$$
$$|S_2(t)\rangle = |s_2(t)\rangle|s_2^{(e)}(t)\rangle$$

Getting $|s_i^{(e)}\rangle$: To find $|s_i^{(e)}\rangle$ we can find A_i, u_i, v_i and $|\mathcal{E}_i\rangle$ and use them to express it. Diagonalize ρ_s and ρ_e, find their eigenvector and associate them to each other by matching their eigenvalue. Or we can do the following (which is implemeted numerically.)

To find $|s_i^{(e)}(t)\rangle$, we consider $P_i|\psi(t)\rangle$, with $P_i(t) = |s_i(t)\rangle\langle s_i(t)|$ using the orthogonality $\langle s_1|s_2\rangle = 0$. And finding the first nonzero term in $|s_i(t)\rangle$ we isolate the vector $|s_i^{(e)}(t)\rangle$.

$|s_i^{(e)}\rangle$ are also simply the eigenstates of ρ_e. Thus they are orthogonal as well.

2.4.2 Decoherence induces convergence of global Schmidt to global pointer states

The basic idea is the following:

- **Schmidt states** are the eigenstates of ρ_s
- **Pointer states** are states of the system such that entanglement between them and the environment does not grow with time.
- Decoherence causes ρ_s to become diagonal in the pointer basis

- $\implies$ Schmidt states converge to pointer states

What about the global forms of the Schmidt and Pointer states? They are both extended to their global definition in the following way,

$$|\psi(t)\rangle = |s_1(t)\rangle|s_1^{(e)}(t)\rangle + |s_2(t)\rangle|s_2^{(e)}(t)\rangle = |S_1(t)\rangle + |S_2(t)\rangle \tag{60}$$

$$|\psi(t)\rangle = |p_1\rangle|p_1^{(e)}(t)\rangle + |p_2\rangle|p_2^{(e)}(t)\rangle = |P_1(t)\rangle + |P_2(t)\rangle \tag{61}$$

At time $t' > T_{dec}$, decoherence diagonalizes the system density matrix in the pointer basis, meaning $|s_i(t')\rangle = |p_i\rangle = cst$. This also implies equality between our global Schmidts and pointers at the time,

$$|\psi(t')\rangle = |s_1(t')\rangle|s_1^{(e)}(t')\rangle + |s_2(t')\rangle|s_2^{(e)}(t')\rangle = |p_1\rangle|s_1^{(e)}(t')\rangle + |p_2\rangle|s_2^{(e)}(t')\rangle \tag{62}$$

$$= |p_1\rangle|p_1^{(e)}(t')\rangle + |p_2\rangle|p_2^{(e)}(t')\rangle \tag{63}$$

$$\implies |s_i^{(e)}(t')\rangle = |p_i^{(e)}(t')\rangle \tag{64}$$

$$\implies |S_i(t')\rangle = |P_i(t')\rangle \tag{65}$$

Note that our global pointer states varies, while our system one is constant.

This is quite obvious, but it means that worlds (if they are our global pointer states) split the total energy eigenspace when the pointer states do. The question becomes, when do the global pointer states do so?

The definition of being a global pointer state is to stay separable + have the system part be a system pointer state. Staying separable means having an evolution of the form,

$$|P(t)\rangle = |p\rangle|p^{(e)}(t)\rangle \tag{66}$$

A sufficient conditions for this, is that we can write our energy eigenstates $|p\rangle|k\rangle$ and that the $|k\rangle$'s here for an environment basis. This allows us to put all the evolution into $|p^{(e)}(t)\rangle$. It is also possible it using degeneracies of the total Hamiltonian, ie we need $|p\rangle = \sum_i b_i|b_i\rangle$ where $|b_i\rangle$ are a basis of the system such that $|b_i\rangle|e\rangle$ all form total basis states with the same energy. This allows us to absorbe the evolution of $|p\rangle$ into a single phase which we can pass on to $|p^{(e)}(t)\rangle$.

To see it more mathematically: we explain how $|S_i\rangle \approx |p_i\rangle$ for post-decoherence times. Note that the pointer states are defined by their weights in the total energy eigenbasis, the phase associated to each weight is not important to their property of staying separable.

A potential confusion is the following. Our global pointer states are states that are separable according to our bipartide system+environment separation, but this also seems to match out global Schmidt states, which are also separable by definition. The thing to keep in mind is that Pointer states are stable in their separability, while if we froze our Schmidt state at a certain time and evolved it in time, we would have that the separability doesn't last. (TODO add graphs showing that.)

By the form of pointer states being $|q_j\rangle|$any environment state$\rangle$, we can show that the Schmidt states converge to pointer states simply by showing $|s_i\rangle \to a_i|q_j\rangle$. This mean that we dont actually care about the form of $|s_i^{(e)}\rangle$ of Eq. (**??**) and all we are asking is that either $|u_1(t)|$ or $|u_2(t)|$ tends to 0. And vice versa for v. This is equivalent to showing that the off

We start at $|u_1(0)| = \frac{1}{\sqrt{w}}$ and $|u_2(0)| = \frac{1}{\sqrt{1-w}}$.

We construct $\rho_s(t)$ from its Schmidt form, and express it in the $\{|q_i\rangle\}$ basis

$$= \begin{pmatrix} 0 & ... & ... & 0 \\ ... & |u_1|^2 + |v_1|^2 + u_1v_1^* + v_1u_1^* & ... & u_1u_2^* + v_1v_2^* + u_1v_2^* + v_1u_2^* & ... \\ ... & 0 & ... & 0 & ... \\ ... & u_2u_1^* + v_2v_1^* + u_2v_1^* + v_2u_1^* & ... & |u_2|^2 + |v_2|^2 + u_2v_2^* + v_2u_2^* & ... \\ 0 & ... & ... & 0 \end{pmatrix} \tag{67}$$

Matching eq. (**??**) and Eq. (**??**), we find the following equalities,

$$\begin{aligned}
|u_1|^2 + |v_1|^2 + u_1 v_1^* + v_1 u_1^* &= w \\
|u_2|^2 + |v_2|^2 + u_2 v_2^* + v_2 u_2^* &= 1 - w \\
u_1 u_2^* + v_1 v_2^* + u_1 v_2^* + v_1 u_2^* &= \sqrt{w - w^2}\langle \mathcal{E}_1(t)|\mathcal{E}_2(t)\rangle \\
u_2 u_1^* + v_2 v_1^* + u_2 v_1^* + v_2 u_1^* &= \sqrt{w - w^2}\langle \mathcal{E}_2(t)|\mathcal{E}_1(t)\rangle
\end{aligned}$$

At a time T_{dec}, we have $\langle \mathcal{E}_1(T_{dec})|\mathcal{E}_2(T_{dec})\rangle \approx 0$. Then we have solution to this system of equation $u_2 = v_1 = 0$ and $u_1 = \sqrt{w}$ $v_2 = \sqrt{1-w}$ makes the equations work. It becomes the only solution if we add the 2 conditions $|u_1| + |u_2| = |v_1| + |v_2| = 1$ and impose no phases to u_1 and v_2 (8 eqs for 8 unknowns.). (TODO prove that)

We thus conclude that the global Schmidt states converge to global pointer states due to decoherence. For $t \geq T_{dec}$,

$$\begin{aligned}
|S_1(t)\rangle &= |s_1(t)\rangle |s_1^{(e)}(t)\rangle \\
&= |q_1\rangle |\mathcal{E}_1(t)\rangle \\
|S_2(t)\rangle &= |s_2(t)\rangle |s_2^{(e)}(t)\rangle \\
&= |q_2\rangle |\mathcal{E}_2(t)\rangle
\end{aligned}$$

This can be trivially summarized to: Schmidt states converge to pointer states because pointer states are the stable eigenstates of the density matrix post decoherence, meaning the Schmidt states (eigenstates of the density matrix) will naturally converge to pointer states by decoherence. The fact that our global Schmidt and pointer states follow the same rule comes from the way that the environment part of theses global states is found.

2.5 Seeing the convergence of Schmidt states into pointer states

Each possible pointer states is non-zero on d_2 total energy eigenstates. Another way to see this is that the pointer states are defined in the spectrum of $E_I q_i H_r'$, which for $i = 1, 2, ..., 10$ is shown in Fig. (**??**). For $i = 1$, it corresponds to a thin distribution in the energy spectrum, like the spectrum represented by a blue line. For $i = 2$ it corresponds to one of the wide spectra, like the orange one. So as the Schmdit states converge to the pointer states we expect to see them go from being defined in the spectrum of $E_I q_1 H_r'$ and $E_I q_1 H_r'$ together, to being defined in one each. This will be visible by the fact that the spectra have different width.

So we expect to see this very well in the total energy eigenbasis in the interaction dominated limit, but as shown in Fig. (**??**), we also see it in the intermediate regime when we look in the interaction basis.

Note that this is only true in the interaction dominated picture and approximately true in the intermediate regime. Still, in the intermediate regime, we have pointer states defined in the basis of $H_i^{(e)}$, and theses Hamiltonians by our definition are not degenerate with each other (all eigenstates are different), making the pointer states live in different energy subspace.

TODO perturbation theory to justify that it is apprioximately true.

2.6 Discussion

2.6.1 When do we have splitting of the energy eigenspaces between branches

Taking the pointer states that our Schmidt states converge to as the classical worlds post measurement, we have that theses branches split the energy eigenspaces with each other if the pointer states split the energy eigenspace. And from the definition of our global pointer state, we get that when we have a separable energy eigenbasis, such that the system eigenstates are the system part of the total energy basis (equivelent to asking for no self interaction). (We also requier no "degeneracy" in the environment basis).

One thing with the global Schnmidt states is that they allow us to carry the whole state, ie dont leave anything behind

(a) Distribution before decoherence

(b) Distribution after decoherence

Figure 5

Figure 6: Freeze frame on the distribution of a Schmidt state in the energy eigenbasis of H_I before and after decoherence. We see the distribution converge to that of a pointer state.

Figure 7: The set of wigner semi-circles who's sum gives us the spectrum of H_I. The wider ones are for higher q_i's.

2.6.2 Ramification of splitting the total energy eigenstate

The branches of the wavefunction loose access to half of the energy eigenbasis. Implication about loss of access to some dimension of the Hilbert space? This is the very special case where the total energy basis states can be written a product states of the system energy basis states and some environment state, AND there being no self interaction of the system. But this begs the question. Everytime we are able to define a measurement with this model, then the dimension of our accessible Hilbert space in the totla energy eigenbasis is halfed? (Simple example of no self interaction superposition of state, 2 photons in a cavity.)
Note that post decoherence the branches have a different effective Hamiltonian, ie one made of the sum of 2 block diagonal terms (one of the block diagonal si different for diff branches). So the total Hamiltonian wrt which you loose access to part of the Hilbert space changes after each measurement. This I think, makes the continuous loss of Hilbert space narrative no longer true.

$H_{tot,1} \rightarrow$ branching of state defined over $2 * d_2$ energy eigenstates of H_{tot} leads to 2 worlds $|p_1\rangle$ and $|p_2\rangle$ defined over d_1 eigenstates of H_{tot} and evolving under

2.7 Was energy splitting the only way?

Worlds need to be **orthogonal** and **classical**. Orthogonality is clear, classical here is simply having the system in a pointer state (ie a state selected as stably non-quantum). Do theses two conditions suffice to explain the splitting? Or is splitting one possible way only to respect theses conditions.

We can get orthogonality with overlap. Imposing pointer states as system state, then we can also make that work with overlap, but what we find is that the environemnt state converges to being defined in a limited energy

subspace, BY DECOHERENCE WHY ONLY LIVING IN A SUBSPACE!!!!??? WAS THAT NOT EXPLAINED?

TODO If you show equivalence between decoherence and "schmidts into pointer states" then you have that the pointer environmen states converge to liing in a subspace only.

$$|w_i(t)\rangle = \sum_{k \in O_i} c_k e^{e^{-itE_k}} |k\rangle \tag{68}$$

O_i defines the set of total hamiltonian eigenstates in which world i is defined. If we take that we have 2 worlds and that they share an energy eigenstate (j), then the overlap is constant and worth $\langle w_1|w_2\rangle(t) = |c_j|^2$. If we have that the overlap is in n eigenstates, it is still constant. Basically, the only way for our worlds to not have an overlap is that they do not share an energy eigenbasis.

-Discuss the fact (and why) do worlds become orthogonal to each other by splitting the hilbert space. Make the latex discussion of it clear. (can I get a conclusion somehow that tells us that if they did not they would never stay orthogonal, or that it requires very strong fine tunning.) If I find that, i think we can conclude that our conclusion generalizes further than the model.

So the "worlds" in this model are orthogonal in this energy eigenspace in this specific splitting manner because the specific pointer state of the problem are. Can you have pointer states that are also orthogonal (condition we want of worlds) but that are not orthogonal in this specific manner? Yes, but the overlap terms must annihilate with each other. Which puts conditions on the complex phases and amplitudes of their definition.

In our specific case, to be a state that stays separable (pointer state) is equivalent to a state that has a "classical system". This is no longer true for a system with self interaction. In a system with self interaction, the eigenstates of the system will not be the classical ones.... Or maybe How does this relate to the question of living in different eigenspaces?

2.8 Characterizing the probability distribution of the global state

The state, and Schmidts are define as non-zero on only part of the energy eigenspace because of our model. The Hamiltonian has random matrices acting on only the environment subspace, and no self interaction for the system. By starting and staying defined over 2 out of 10 energy eigenstates of the system, the global state is always defined over 1/5th the space (for $d_1 = 10$). Is that important at all?

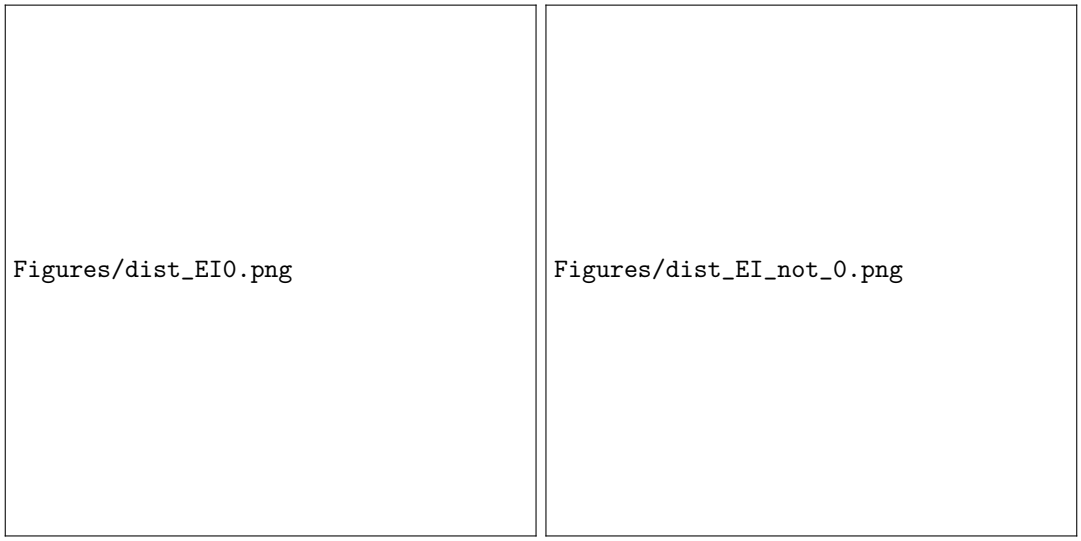


Figure 8

Impact of E_I: Varying E_I changes the total energy eigenbasis. Taking $E_I = 0$ we have as expected a delta, as the environment starts and stays as a total energy eigenstate. As we vary E_I, it takes a bell shape and its standard deviation rises. See Fig. (**??**), which shows us how an H_e eigenstate is defined in the eigenbasis of $H_e + H'_e$.

Impact of w: It is a factor in front of the 2 branches, so changes the relative importance of each branches when we vary it.

2.9 The preffered basis problem

Einselection solves it, but can there be multiple prefere basis together? Basically I need to learn better what einselction is. This is another formlulation of the question: "Q1: can a change of basis leave us seeing the our 2 branches as more than 2 branches? Less?"

In the notebook, I have expressed it as a basis transformation problem without realizing I was just asking about the prefered basis problem.

2.10 Locality from the spectrum

The question is, does looking at things in the einselected basis somehow also gives us locality? Subsystems paper has a little paragraph on that.

My first thought was that we could look at the ACL model as a SHO and look at the position basis there. But it turns out that different systems with different physics and local basis have the same spectrum. For one, the spectrum of the HO describes a quantum HO, or phonons in a lattice.

Since the interaction Hamiltonian is the one who's spectrum matches theses 2 systems, making its eigenstates the states that end up selected, we look at the physical interpretation of the energy eigenstates of a truncated HO and a phonon on a lattice.

Maybe even a simpler third system could be considered. It is a simple lattice on which an electon can hope, and the lattice is also in an enlectric field that looks like a linear slope. There, each energy eigenstate is a position eigenstate.

HO: energy eigenstates are excitation levels of a small mass on a spring. An energy eigenstate has a position spread out in the well, in a way that increases with energy. So position is best localized for lowest energy state. Generally the position is delocalized, but not maximally so I think. Why? Because basically the position function for an energy eigenstate is some sort of sine wave with higher frequency as energy increases. Fundamentally, if I change my energy eigenstate, where I can expect to find my mass changes.

Phonon: A phonon is a massless scalar field with fixed momentum. I think it will be different from the HO because in the HO energy depends on position from center, while here there is no potential dependant on position to worry about. Here the energy eigenstates should all correspond to totally delocalized states. Ok, phonon subtelty is that you can incerase energy by adding a phonon or incerasing the momentum of a phonon.

2.10.1 Are the pointer states local in the HO?

(about limit. the weak/intermediate/strong limit is a limit defined based on the ratio of energy scales for the interaction and self interaction Hamiltonian. Since we have not introduced a self interaction we are always in the strong interaction limit.)

2.10.2 Are the pointer states local for the phonon?

3 Does Branching dynamics split the energy eigenspace between the branches in a particular way?

This question is motivated by the behavior of $Neff_{schmidt,i}/Neff_{total} = \delta_i$, with $Neff_{state} = \frac{1}{\sum_i p_i^2}$, p_i the probability of finding the state we are considering in total energy eigenstate i. See Fig. (**??**). The idea of $Neff$ being to measure the spread of the Schmidts over the energy eigenspace. Neff is maximized by a uniform distribution and minimized by $p_i = 1$ for $i = a$ for some integer a. δ_i is going to depend on how the distribution of Schmidt i evolves in the energy eigenspace, which is what we want to study.

Figure 9: In this specific run, the curves join and stay together. With different parameters, we see the 2 curves stabilize with some distance, positive or negative

What we saw looking at the graphs for multiple runs of the simulation with varied parameters, is that sometimes both δ_i's would match up, sometimes after decoherence $\delta_1 > \delta_2$ or $\delta_2 > \delta_1$.

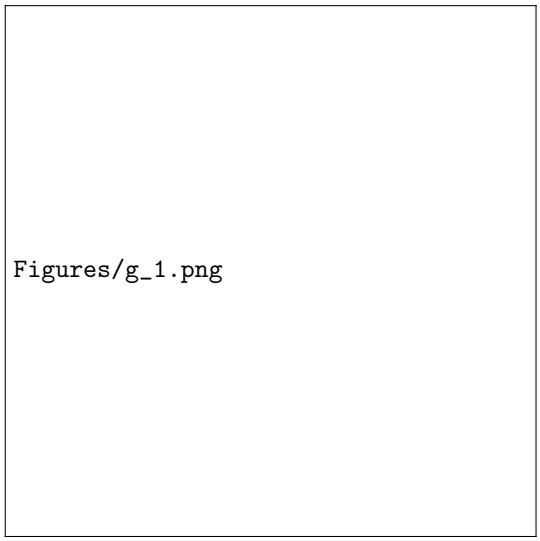
It would be interesting to characterize $Neff$ to understand its behavior. Are there necessary conditions for the convergence of the 2 curves? Why does Neff have different behaviors for different E_I, w or the initial environment state?

Figure 10: Graphs of δ for varying parameters E_I and w. Top left $w = 0.1, E_I = 0.03$, $\rightarrow$ rising E_I, $\downarrow$ rising w.

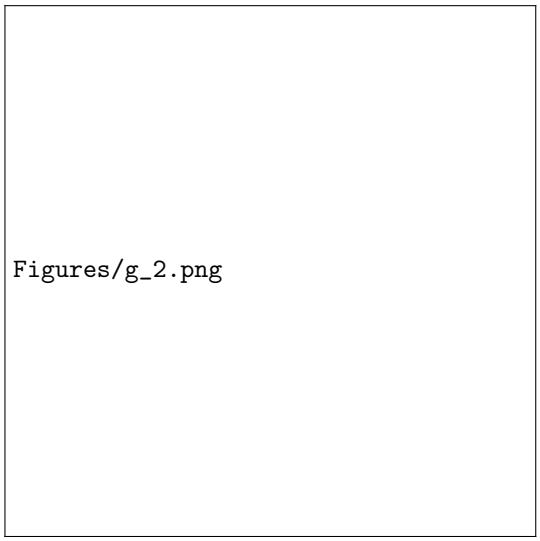
We see in Fig. (**??**) that δ_1 always starts at 1 since it is the global state at $t = 0$. It then varies a little from its base value to some equilibrium value slightly above or under 1. The fact that δ_1 barely varies comes from the fact that the shape of the distribution of Schmidt 1 also barely varies. δ_2 varies (from initial to equilibrium) by a factor from 10 to 100.

3.1 Other hints that eigenspace is shared in an interesting way

Looking at the difference between how things are a little after interaction starts and after all has settled, Fig. (**??**) we see that the Schmidt states seem to have shared the energy eigenspace.



(a) Zoomed view of the probability distribution of Schmidt 1 and 2 in the total energy eigenbasis at frame 11/100



(b) Zoomed view of the probability distribution of Schmidt 1 and 2 in the total energy eigenbasis at frame 89/100

Figure 11: There is smaller probability overlap at frame 89 than 11

3.2 Are probabilities not just shared following a conservation law?

Since we have unitary evolution, wont the probability of being in an energy eigenstate for both Schmidt states just sum up to what it is for the global state? **They wont during decoherence because of interference, but they will after decoherence.**

"""This makes it weird that my total interference doesn't actually go to zero... this is equivalent to having my Schmidt states not exactly converging to pointer states...."""

We look at the state in its Schmidt decomposition, defining $|S_i'\rangle$ the normalized form of $|S_i\rangle$,

$$|\psi(t)\rangle = \sqrt{p_1(t)}|S_1\rangle > +\sqrt{p_2(t)}|S_2\rangle \tag{69}$$

Taking a total energy eigenstate $|q_i, k_i\rangle$, The probability that the state is in this energy eigenstate takes the form,

$$P(|\psi(t)\rangle \text{ in } |q_i, k_i\rangle) = |\langle q_i, k_i|\psi(t)\rangle|^2 \tag{70}$$

$$= p_1(t)|\langle q_i, k_i|S_1'(t)\rangle|^2 + p_2(t)|\langle q_i, k_i|S_2'(t)\rangle|^2 \tag{71}$$

$$+ \sqrt{p_1(t)p_2(t)}(\langle q_i, k_i|S_1'(t)\rangle\langle S_2'(t)|q_i, k_i\rangle^* + \langle q_i, k_i|S_2'(t)\rangle\langle S_1'(t)|q_i, k_i\rangle^*) \tag{72}$$

which we can see as,

$$P(|\psi(t)\rangle \text{ in } |q_i, k_i\rangle) = p_1(t)P(|Schmidt_1\rangle \text{ in } |q_i, k_i\rangle) + p_2(t)P(|Schmidt_2\rangle \text{ in } |q_i, k_i\rangle) + \sqrt{p_0(t)p_1(t)}(Interferences)$$

Where the leftover are interference terms. Interestingly, $\sqrt{s_0 s_1}$ is maximal for $s_0 = s_1$ at maximal entanglement (post decoherence). But this is countered by the fact that post decoherence, since $|S_j(t)\rangle = |q_j\rangle|(E)_j(t)\rangle$, we get that either both, or 1 of $\langle q_i, k_i|S_1'(t)\rangle\langle$ or $\langle q_i, k_i|S_2'(t)\rangle\langle$ are zero (depending on $|q_i\rangle$). This makes the interference terms go to zero by decoherence.

We can visualize this interference in Fig. (**??**) and Fig. (**??**).

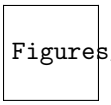


Figure 12: In this plot we see the importance of interference effects for the value of probability that a Schmidt has to be in an energy eigenstate. (Chosen for its high interference.)

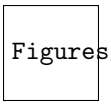


Figure 13: Graph of the total interference between Schmidt states

3.3 Other metrics to describe the sharing of eigenspace

We want to confirm that there is indeed interesting dynamics to find in the sharing of energy eigenspace + we want the right tool to characterize it.

Mean and standard deviation To characterize the shape of the distributions, we look at the variation of the mean and standard deviation of both distributions in time. The question is "How do the shapes of the distributions evolve in time?"

Figure 14: Graphs of the means and standard deviations of the probability distributions of Schmidt 1 and 2 for parameters $E_I = [0.05, 0.06, 0.07, 0.08]$ and $w = [0.2,0.25,0.3,0.35]$.$\rightarrow$ rising E_I, $\downarrow$ rising w.

What this confirms is that the shape of the distribution of Schmidt 1 varies little in time, while Schmidt 2 varies much at the beginning.

Overlap of the probability distributions The overlap is the projected value of the square root of the probability vectors of Schmidt 1 and 2 with themselves and with the global state. The question investigated is "Are Schmidt 1 and 2 defined over the same eigenbasis?". In more detail, they are of course defined in the same Hilbert space, but do they both have high probability in the same eigenbasis? The green curve, shows that the Schmidt

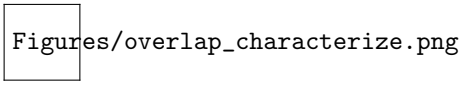


Figure 15: Graphs of the overlap of the probability distributions of the Schmidt states and the global state. Parameters $w = [0.1, 0.2, 0.3, 0.4], EI = [0.03, 0.05, 0.07, 0.09]$. $\rightarrow$ rising E_I, $\downarrow$ rising w.

states start by becoming similar likely as Schmidt 2 catch's up quickly to the shape it is supposed to have, but then give way to each other, do not overlap much in the energy eigenbasis. The distributions evolve to similar shapes, but do not occupy the same eigenstates.

Occupation of the Hilbert space For the occupation measure of the Hilbert space, we count the number of total energy eigenstates in which Schmidts have probability above a threshold ϵ. The question investigated is, "Does Schmidt 1 gives some of its space to Schmidt 2?". No. All we see, is that no the number of eigenbasis states "occupied" by Schmidt 1 does not really change. And for Schmidt 2 we see the expected behavior, knowing that it goes from a uniform distribution to a more concentrated one. **??**

Figure 16: Graphs of the occupation of Hilbert space by Schmidt 1 and 2 ($\epsilon = 0.001$).Parameters $w = [0.1, 0.2, 0.3, 0.4], EI = [0.03, 0.05, 0.07, 0.09]$. $\rightarrow$ rising E_I, $\downarrow$ rising w.

Gifs Gifs of how the distributions of both Schmidts changes in time. Cannot put them in a pdf but we can discuss them here.

The distribution of Schmidt 1 starts as the distribution of the global state. As the second Schmidt value goes from 0 to not 0, Schmidt 2 gains in importance, it takes over some of the energy eigenstates on which Schmidt 1 is defined and takes a similar shape. So we have 2 distributions of the same shape but not defined over the same space.

3.3.1 General characterization of these metrics

Interference and w We generally see these metrics vary in unison with the amount of interference between Schmidt 1 and 2. We also note an inversion in behavior at equilibrium for a change of w to 1-w. As an example of this law, is the plots of occupation with w=0.03 and 0.07 (which also have underneath the plots of interference, and we clearly see the same shape.) **??**

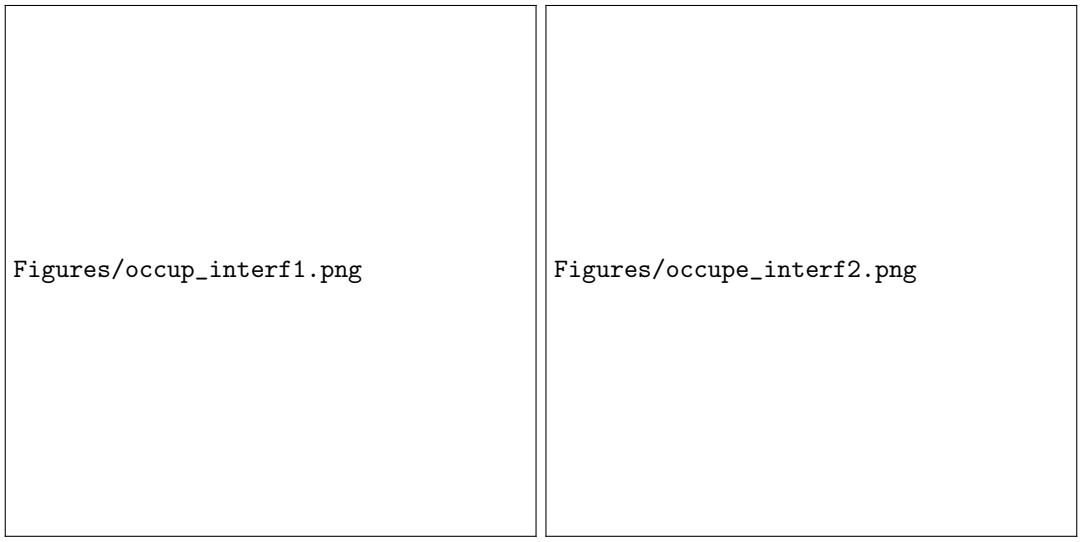


Figure 17: We see that the occupation number varies with total interference. We also see that we have an inversion of equilibrium behavior for w=0.3 and w=0.7

Interference makes sense of the common behavior.
Inverting w the probability, inverts teh equilibrium behavior.

General solution The weight in front of both states are equal if,

$$q_1 = 2(\frac{\sqrt{w}}{\sqrt{1-w}} - 1) + q_2 \frac{\sqrt{w}}{\sqrt{1-w}} \tag{73}$$

Then,

$$|\psi(t)\rangle = |q_1\rangle(A) \sum_j c_j(t)|e_j\rangle + |q_2\rangle(A) \sum_j c'_j(t)|e_j\rangle \tag{74}$$

We can make that happen while keeping E_I and w invariant by changing either the energy between the eigenstates of the SHO, or the states we choose for the superposition (which is what we do in practice). The idea being that we eliminated the asymmetry between the 2 Schmidt states, which makes δ_1 and δ_2 behave the same at equilibrium (as well as other metrics).

We verify this by making a plot of δ where the curves converge and another where they are inversed while keeping $w = 0.41$ and $E_I = 0.06$, typically values where we get distinct equilibrium behaviors for Schmidt 1 and 2.
See notebook "The same"

3.4 Conclusion:

The "energy" of the superimposed states, and their probability (in our case $q_1, q_2, w, (w-1)$) has an impact on the distributions of Schmidts in the energy eigenbasis.

(a) For weight difference of 0.01 (b) For a weight difference of 10

Figure 18: We see a that by making Schmidt 1 and 2 as "important" we can get them to agree on Neff. And we see the opposite happen when we maximize their weight difference.

Figure 19: The distribution of Schmidts has the possibility under certain conditions of going from uniformly distributed to ending and staying in this distribution.

3.5 Pseudo pointer in the interaction spectrum

We find an interesting evolution of the distribution in this basis. See Fig. (**??**). The motivation for looking in this basis was to look for pointer like behavior of converging towards an energy eigenstate of H_I. We see that it has a behavior like that, only not so complet.

3.5.1 Characterizing the behavior of the distributions in the interaction energy basis.

env does not matter

characterize behavior to see if it is weight dependant. interestingly you can get both to do so by macthing the weihghts. can you get neither to do so?

We look back at the equation,

$$|\psi(t)\rangle = |q_1\rangle(\frac{2+q_1}{\sqrt{w}})\sum_j c_j(t)|e_j\rangle + |q_2\rangle(\frac{2+q_2}{\sqrt{1-w}})\sum_j c_j(t)'|e_j\rangle \tag{75}$$

We have that the eigenstates of H_e are $\{|e_i\rangle\}$, the eigenstates of H_I are $\{|q_i\rangle|A_j\rangle\}$ We can re-write $|e_j\rangle = \sum_k a_{kj}|A_k\rangle$, meaning,

$$|\psi(t)\rangle = |q_1\rangle(\frac{2+q_1}{\sqrt{w}})\sum_{jk} c_j(t)a_{kj}|A_k\rangle + |q_2\rangle(\frac{2+q_2}{\sqrt{1-w}})\sum_{jk} c_j(t)'a_{kj}|A_k\rangle \tag{76}$$

Which leads to ,

$$P(|s_2(t)\rangle \text{ is in } |A_l\rangle) = |\frac{2+q_2}{\sqrt{1-w}}\sum_{jk} c'_j(t)a_{kj}\delta_{lk}|^2 = |\frac{2+q_2}{\sqrt{1-w}}|^2|\sum_{jl} c'_j(t)a_{lj}|^2 \tag{77}$$

Somehow, we have that $|\sum_{jl} c'_j(t)a_{lj}|^2$ is suppressed for a certain set of l's. The l's that correspind to outside the bounds we can see in Fig. (**??**). To study this, we need to understand the time evolution of the $c_j(t)$'s.

This is not explained by our numerical or model (at least trivially), since we do see that in principle the a_{lj}'s are not zero.

What could explain the supression? Not interferences. For multiple reasons. interferences go to 0 on the out

We would like to see how it evolves in the basis of H_I

To do that theoretically, we would need to Haussdorf approximate out unitary of non-commuting op.

4 Why do they share the energy eigenstates in this way?

What explains this way of sharing the eigenspace? Interferences between Schmidt states?

Do the probabilities in individual eigenstates vary less once equilibrium is reached? Do near zeros stay at near zero? Could it be that they spread in the eigenbasis and since they dont have many high probabilities, they dont overlap meaningfully simply by chance?

5 Can we get it all from the energy spectrum and the distribution of the global state in it?

Energy spectrum + state defined in energy basis -¿ sharing of Hilbert space by worlds -¿ explain the phenomenas we observed

Can we get the decoherence time from the spectrum?

5.1 Getting decoherence time from the spectrum

Note on getting things from the spectrum.... the decoherence time should be essentially defined by how fast we can consider the phases in front of the decomposition of $|\mathcal{E}_1(t)\rangle$ and $|\mathcal{E}_2(t)\rangle$ in $|k\rangle$, independent/randomised. $|\mathcal{E}_1(t)\rangle$ and $|\mathcal{E}_2(t)\rangle$ evolve according to $H_e^{(1)}$ and $H_e^{(2)}$ which have different spectrum. it might be possible to estimate from the spectrum distribution the time after which the relative phase of both vectors (compared in a common basis) look randomly selected, at which point we would have decoherence. Note that I need to prove that this is sufficient for decoherence. There is also a complication, which is that there is no common eigenbasis for $H_e^{(1)}$ and $H_e^{(2)}$. But it should be easy to look at in the H_I dominated limit.

5.1.1 H_I dominated limit

In this limit $H_e^{(1)} = E_I q_1 H_r$ and $H_e^{(2)} = E_I q_2 H_r$, meaning that they have a common eigenbasis. Taking this common eigenbasis to be $|k\rangle$, it means that we have,

$$|\mathcal{E}_1(t)\rangle = \frac{1}{\sqrt{d_2}} \sum_k e^{-itE_k} |k\rangle \tag{78}$$

$$|\mathcal{E}_2(t)\rangle = \frac{1}{\sqrt{d_2}} \sum_k e^{-itE'_k} |k\rangle \tag{79}$$

Since both environment states start in the same state, they have the same amplitudes in any basis. We simplify the problem by taking that amplitudes to be uniform in the basis we consider. The differentiation between both states then only comes from the phases.

The spectrum of $H_e^{(1)}$ and $H_e^{(2)}$ are simply different by a factor of $\frac{q_2}{q_1} = f$.

6 Other

6.1 Why the ACL model?

The point of the Caldeira-Leggett model is to study the intermediate regime of measurement $H_s \approx H_I$. So i do not think I can do a perturbation theory on the interaction energy

6.1.1 Condition on there being an energy splitting

Bsically, if we have a Hamiltonian that can be split as the tensor product of a diagonal matrix and a complex Hamiltonian, and that in the same bassi have the state of the universe be a superposition product some random state, we will get this splitting happening. TODO specify better the conditions for the splitting happening, and look into if that is a common condition? if we go from a hamiltonian that has that, do the world afterwards and the effective hamltonians they live in also have that?

6.1.2 More on pointer states

Pointer states are states selected by the predictability sieve, which selects states of the system which deteriorate the least into mixtures over time. (or stay separable if we consider the global state.) By this condition, they are the state with the highest degree of reversibility in the system, whiich is a property of classical systems. (no growing entropy associated to reversibility). Fundamentally, we justify looking at global pointer states because they are defiend by properties we ask of classical worlds. Can we get an useful defintion of worlds by developing our thinking on pointer states?

We ask of a world to be separable into many classical subsystems. This can be translated into the following condition. Choosing a scale d (ie number of dimension of Hilbert space above which we consider a system classical), and a subdivision of the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes ... \otimes \mathcal{H}_n$, with each $\mathcal{H}_i$ has dimension d we can alternatively consider each Hilbert space $\mathcal{H}_i$ as a subsystem, in which the state is a pointer state einselected by the rest of the Hilbert space. Furthermore, the evolution of the global state is separated (meaning you can see it like an operator phase in front of both term in the separable state). This must be true for all sub-system, meaning that there is a global pointer state basis in which the global state can be written like this,

$$|w\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes ... \otimes |\psi_n\rangle \tag{80}$$

$$|w(t)\rangle = e^{-itH_1}|\psi_1\rangle \otimes e^{-itH_2}|\psi_2\rangle \otimes ... \otimes e^{-itH_n}|\psi_n\rangle \tag{81}$$

If we go back to the table analogy and the subsystems as pieces of the table big enough to be considered classical, then we note that there are different ways to cut a table up in chunks of set size. That is we also need to consider $\mathcal{H} = \mathcal{H}_{1'} \otimes ... \otimes \mathcal{H}_{n'} \neq \mathcal{H}_1 \otimes ... \otimes \mathcal{H}_n$. And the conclusion is that we also need our world to have a global pointer state basis in which the world can be written,

$$|w\rangle = |\psi'_1\rangle \otimes |\psi'_2\rangle \otimes ... \otimes |\psi'_n\rangle \tag{82}$$

$$|w(t)\rangle = e^{-itH'_1}|\psi'_1\rangle \otimes e^{-itH'_2}|\psi'_2\rangle \otimes ... \otimes e^{-itH'_n}|\psi'_n\rangle \tag{83}$$

This imposes a condition on $|\psi\rangle$ and H_{tot}. What is that condition? is it that the world is separable in any possible subdivision of the Hilbert space? I dont know....

What has been discussed here is true for local subdivisions (pieces of table I can hold in my hands), but what if we consider $\mathcal{H} = \mathcal{H}_{1''} \otimes ... \otimes \mathcal{H}_{n''}$ a non-local decomposition of the Hilbert space.

the simple idea being that a table can be cut down in many "classical" pieces in many ways. The basic question is: What happens when I consider global pointer states defined by being product state of pointer states of many subsystems. And do I get anything else from asking for a variable decomposition (setting maybe the size of the subsytems to be classical).

NOTE, THIS MIGHT VERY MUCH BE RELATED TO THE QUANTUM ZENO PARADOX.

6.1.3 Pointer states are worlds

Our research question mentions worlds, yet we can only say things about pointer states and Schmidt states. Are pointert states worlds? or are worlds made of LC of poionter states?

Pointer states as definition of worlds https://journals.aps.org/prd/abstract/10.1103/PhysRevD.40.1071

Our global Schmidt states have the great property of being orthogonal, evolving to states with determined measurement results (pointer states) in the time it takes the state to decohere. And the global pointer states, to which the global schmidt states converge have the nice property of being separable (and constantly so). This makes it so that post decoherence we do not recover quantumness (in the form of uncertainty about the state of the system). Finally our global definition of theses terms leaves no quantum state behind

In "A search for classical subsytems in quantum worlds" they are looking for classical subsystems (ie pointer states). I expand this definition and call my worlds whatever environment state is connected to a classical subsystem. (In truth we would "make classical" every subsytems of the environment of sufficient size and make that into a definition of a classical world.)

Figure 20: Graphs of the similarity between s1, parameters $E_I = 0.06$ and $w = 0.3$.

6.2 Why do we get no overlape?

World overlape in the total energy eigenbasis. We defien our 2 worlds as $|w_1\rangle(t) = \sum_k c_k e^{-ite_k}|k\rangle$ and $|w_1\rangle = \sum_k c'_k e^{-ite_k}|k\rangle$. Then their overlap is,

$$\langle w_1|w_2\rangle(t) = \sum_k c^*_k c'_k \tag{84}$$

Note that the time dependance of the worlds is such that there is time independance of their overlap.

With only 1 eigenstate where the worlds are non-zero together, they have nonzero overlap $\langle w_1|w_2\rangle(t) = cst = c^*_i c'_i$. With more than one such eigenstate there exists a possible balance. ie, $\langle w_1|w_2\rangle(t) = \sum_{i\in nz} c^*_i c'_i = 0$ is possible.

6.3 What does it imply?

If the worlds are truly defined in different eigenstate of the total energy eigenbasis, what are the implications? Do we loose most of the space after each measurement?

What are the conditions for such measurement events happening? We need a Hamiltonian that can be decomposed in the form of the ACL model,.... what exactly is needed to reproduce theses effects... + condition that in this basis the state can be written in the manner we specify at the beginning. (I can work in the general case.)

Can there be an integrability condition for the Hamiltonian having this level of separability?

Anyways, once you have this Hamiltonian, you can imagine

6.3.1 Confusion regarding the loss of dimension

The state of the worlds are themselves defined in less dimensions, ie using less numbers in the total energy eigenbasis. Is that recoverable? Do they keep on loosing more? My confusion comes from the fact that $|0\rangle = \frac{|0\rangle}{2} + |1\rangle + \frac{|0\rangle}{2}|1\rangle = |+\rangle + |-\rangle$. and if somehow $|\pm\rangle$ happened to be the worlds post another measurement, then theses worlds have recovered 1 dimension... But I think, the point is mostly that my world is defined in 1 dim (here) but also only time evolves according to a Hamiltonian that is a scalar, which is incapable of creating worlds defined in more dimensions.

Measure of fluctuation in time of the distributions Similarity in time, a sort of measure of fluctuation. Maybe useful to characterize the behaviors of s1 and s2 distribution themselves.

Potential selection mechanism for schmidt states that fluctuate less, ie are stable in time?

6.4 ...

We have a certain sharing of the Hilbert space under measurement This sharing implies certain difference in behavior for quantities such as Neff. And if possible, it would be amazing to recover this sharing from the shape of the spectrum and the distribution of the global state! mad dog evrietian style.

Something that troubles me. q and w have a large impact on the "shape "of the distribution. The way I used to see things. Is that the shape of the distribution was the determining factor to how the world would be, ie it would define it classically, ie macroscopically. And to me the 2 worlds couldn't me so different. But, this macro looking factor (distribution shape), might just be irrelevant.

Could there be something to say about the fact that we are looking at overlap in the probability space? This is because of course the 2 worlds will be orthogonal in the Hilbert space. So our fundamental object of study is really their distribution in the total energy eigenbasis.

Pointer states: found using predicatibility sieve. They are the states that devellope the least entanglement with the environment in a robust manner.

So I guess we dont have pointer states in our case, since we dont have self interaction to vary the system state.

This dynamics of Schmdit basis states towards the pointer states might be worth studying. In the case of no self interaction for the system, the pointer states are eigenstates of the interaction Hamiltonian. Might be interesting to check.

Note the Schmidt basis does not have to be orthogonal. we have always orthogonal schmidt basis in our ACL case due to orthogonal system state + no self interaction.

(Andreas Albrecht study of Schmidt states dynamics under measurement: https://journals.aps.org/prd/abstract/10.1103/Phy https://journals.aps.org/prd/abstract/10.1103/PhysRevD.48.3768)

6.4.1 A discussion of Schmidt might not be enough

How pointers might not really be branches https://arxiv.org/pdf/gr-qc/9610028.pdf Is a paper, looking at the possibility of s=using the schmidt decomposition to seelct the physics set of consistent histories. The find is that it would not be enought. The schmidt basis is induced by ...

The pointer basis is induced by decoherence, or more precisely is einselected by the environment-system interaction

issue, isn't einselection only defined for S+M+E? while we are working here with S+E?

looking at the schmidt states has the nice property of always looking at 2 orthogonal states.

Are pointer states worlds? https://arxiv.org/abs/gr-qc/9610028 says that no

6.5 Decoherence and Schmidt state

We can, in general terms take our unitary evolution as,

$$U(t) = \sum_j |a_j\rangle\langle a_j| \otimes U_j^{\mathcal{E}}(t) \tag{85}$$

Starting the system in $|b\rangle$, then $U_j^{\mathcal{E}}|b>= |b_j(t)\rangle$. This means that the reduced density matrix of the system $\mathcal{S}$, evovlves as,

$$\rho_{ij} = \langle b_i(t)|b_i(t)\rangle \rho_{ij}(0) \tag{86}$$

This results in the evolution of the system density matrix to a diagonal form. This diagonal form, explains the apparent classicality of the unique measurement result. The 2 results, after some decoherence time, belong to 2 different orthogonal states.

Decoherence has it that $\langle b_j(t)|b_i(t)\rangle \to \delta_{ij}$ for $t \to \infty$.

A Spectrum of H_{tot}

A.1 Characterizing the spectrum of the Hamiltonian

The Hamiltonian who's spectrum we are trying to characterize is composed of 3 parts, $\mathbb{1}, \mathbb{1} \otimes H_e, E_I H_q \otimes H'_e$. We are interested in the shape that the distribution of eigenvalues will take. $\mathbb{1}$ will simply shift that distribution by $+1$, since it commutes with the other 2.

We will use the fact that the random matrices that appear in these Hamiltonians, are Gaussian Unitary Ensemble (GUE) matrices, and are known to have semi-circle spectrum distribution. Wigner semi-circle law:

$$f(E) = \frac{2}{\pi R^2}\sqrt{E^2 - R^2} \tag{87}$$

distribution between $[-R, R]$ bounds.

Considering $\mathbb{1} \otimes H_e$, H_e has spectrum $\sigma_2 = \{\lambda_1, \lambda_2, ...\lambda_{d_1}\}$ and the identity is here defined over Hilbert space of dimension d_1. This makes the final spectrum $\sigma_{1,2} = \lambda_1, \lambda_1, ..., \lambda_1, \lambda_2, ..., \lambda_2, \lambda_3, ...$, where each element of the spectrum is repeated d_1 times. The distribution of the spectrum of the first term is then an elevated Wigner semi-circle.

$$f_{env}(E) = \frac{2}{\pi R^2}\sqrt{R^2 - E^2} + d_1 \tag{88}$$

Considering $H_q \otimes H'_e$, with the spectrum of the random matrix $\sigma_2 = \{\lambda'_1, \lambda'_2, ...\lambda'_{d_1}\}$. We get a new spectrum, $\sigma_{1,2} = 0 * \sigma_2 \bigcup \sigma_2 \bigcup 2\sigma_2 \bigcup ... \bigcup (d_1 - 1)\sigma_2$ ie $\sigma_{1,2} = \{0, ..., 0, \lambda'_1, \lambda'_2, ..., 2\lambda'_1, 2\lambda'_2, ..., (d_1 - 1)\lambda'_1, ...\}$. This produces a sum of Wigner semi-circles with $R_i = i * R_1$ for $i = 0, 1.., d - 1$.

$$f_{int}(E) = E_I \sum_{j=1}^{d_1 - 1} \frac{2}{\pi j^2 R^2} \sqrt{j^2 R^2 - E^2} + \delta_{E,0} d_2 \tag{89}$$

The $\delta_{E,0} d_2$ comes from our SHO having ground state energy $E_0 = 0$. The E_I factor is added to the distribution here. Since these distributions are centered on 0, the factor E_I acts as a spread factor. (ie the max and min are split $2E_I * R_{effective}$)

We verify that we do indeed get this in Fig. (**??**).

Figures/Wigner_sum.png

Figure 21: The set of wigner semi-circles who's sum gives us the spectrum of H_I. The wider ones are for higher q_i's.

With our simulation we can also get them numerically as seen in Fig. (**??**), we choose $E_I = 0.06$ and $w = 0.3$.

About Fig. (**??**), it looks like $\sigma_{total} = \sigma_{env} + E_I \sigma_{int} + 1$ (+1 from identity matrix). But that would only be the case if both Hamiltonians where diagonalizable in a common basis, which they are not, being different independant random matrices. Is it still possible to consider the final spectrum as such? I do not see another way to characterize it.

The spectra look like they sum, even though the Hamiltonians do not commute

If we take the summing of the spectra as a correct approximation, we have characterized the shape of the total spectrum depending on E_I.

$$f_{total}(E) = \frac{2}{\pi R^2} \sqrt{R^2 - E^2} + E_I \sum_{j=1}^{d_1 - 1} \frac{2}{\pi j^2 R^2} \sqrt{j^2 R^2 - E^2} + \delta_{E,0} d_2 + d_1 \theta(R^2 - E^2) \tag{90}$$

$\theta(x)$ is the Heaviside function. For the $H_{rand,d=200}$ $R \approx 10$.

Potential change to the model. The identity spectrum does not add anything interesting we could remove it. It might also be beneficial to make the GS of the SHO different from 0.

A.1.1 We use bounds on the values of the sum of 2 matrices.

For 2 hermitian matrices A, B such that $A + B = C$. with ordered eigenvalues, we have,

$$c_{i+j-1} \leq a_i + b_j \quad \text{and} \quad c_{n-i-j} \geq a_{n-i} + b_{n-j} \tag{91}$$

This means for us that

$$f_{total}(E) \leq \frac{2}{\pi R^2} \sqrt{R^2 - (E - e)^2} + d_1 + E_I \sum_{j=1}^{d_1 - 1} \frac{2}{\pi j^2 R^2} \sqrt{j^2 R^2 - e^2} + \delta_{E,0} d_2 \tag{92}$$

Figures/Spectra.png

Figure 22: First is the distribution of the state in the total energy eigenbasis. Second is the spectrum of that total energy eigenbasis. Third is the distribution of ythe spectrum of $\mathbb{1} \otimes H_e$ in its own energy eigenbasis. We do see a Wigner semi-circle lifted up by 10. Finally is the distribution of the spectrum of $H_q \otimes H'_e$, which also matches a sum of 10 Wigner semi-circles with growing R. We also see the $200 * \delta_{E,0}$ predicted at the center. Note that this last distribution has to be scaled down by the factor $E_I = 0.06$ (in this case)

A.2 Overlap $|r(t)|^2$ for H_I dominated but no conditions on energy spectrums

We again take a case where both blocks of H_I in question are diagonalizable together but have different energy spectrum, we dont specify the spectrum shape this time. Note again that this not in the context of worlds that are not defined the same same eigenstate, in this one they are and we might show here that it is very difficult to achieve decoherence then... ie, is it a delicate balance to have orthogonality and sharing of eigenspaces?

$$|w_1\rangle = \frac{1}{\sqrt{d_2}} \sum_k e^{-itE_k} |k\rangle \tag{93}$$

$$|w_2\rangle = \frac{1}{\sqrt{d_2}} \sum_k e^{-itE'_k} |k\rangle \tag{94}$$

and,

$$|r(t)|^2 = |\langle w_2(t)|w_1(t)\rangle|^2 = \frac{1}{d_2^2} \sum_{kj} e^{-it((E_k - E_j) - (E'_k - E'_j))} \tag{95}$$

$$= \frac{1}{d_2^2} (\sum_{kj} e^{-itA_{kj}}) \tag{96}$$

$$= \frac{1}{d_2^2} (d_2 + 2 \sum_{k=1}^{d_2} \sum_{j=0}^{k-1} cos(tA_{kj})) \tag{97}$$

From this can we study how finely tunned the frequencies have to be to cause decoherence? Or is there always a decoherence like behavior here?