Introduction to Artificial Intelligence

COMP307

Reasoning Under Uncertainty 1: Probability Basics

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Outline

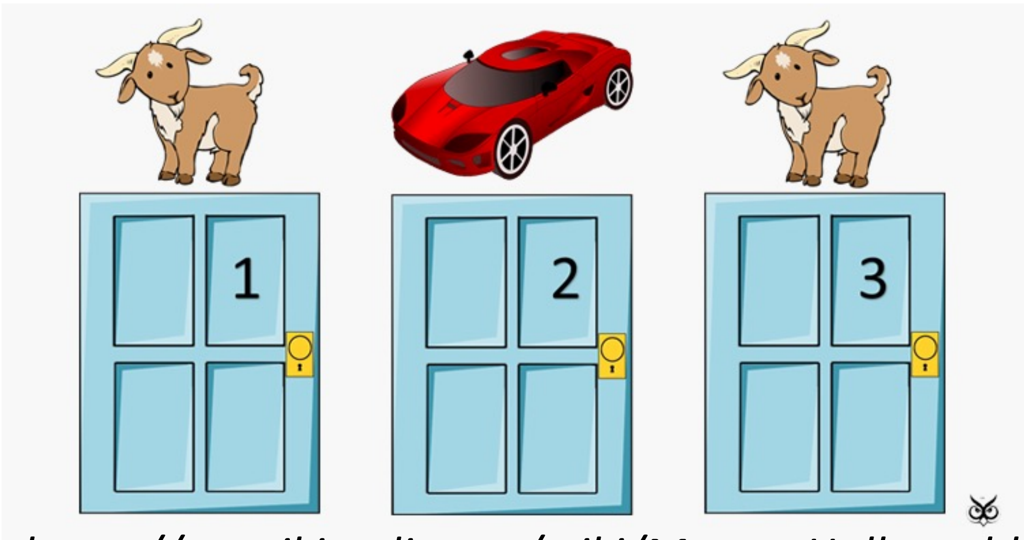
- Introduction
- Product Rule
- Sum Rule
- Normalisation Rule
- Independence
- Summary

Why Reasoning Under Uncertainty

- Make rational decisions
- Many real-world applications

Monty Hall Problem

- Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice? *(Whitaker, 1990, as quoted by vos Savant 1990)*
  - *The host must always open a door that was not picked by the contestant*
  - *The host must always open a door to reveal a goat and never the car.*
  - *The host must always offer the chance to switch between the originally chosen door and the remaining closed door.*



https://en.wikipedia.org/wiki/Monty_Hall_problem

Uncertainty

- Many algorithms are designed as if knowledge is perfect, but it rarely is.
- There are almost always things that are unknown, or not precisely known.
- Reasons of uncertainty
  - **True uncertainty**. e.g., flipping a coin.
  - **Theoretical ignorance**. e.g., medical diagnosis.
  - **Laziness**. The space of relevant factors is very large.
  - **Practical ignorance**. e.g. incomplete information collected
  - …

- Fundamental role of uncertainty in AI
- Probability theory can be applied to many problems

Belief about Propositions

- Rather than reasoning about the truth or falsity of a proposition, reason about the **belief** that a proposition or event is true or false
- For each primitive proposition or event, attach a **degree of belief** to the sentence
- Use **probability theory** as a formal means of manipulating degrees of belief

- Examples:
  – How likely do I believe it will rain tomorrow? (e.g. 50%, 80%, …)
  – How likely do I believe a stock price will rise?
  – …

Probability

- Given a **proposition** A, the probability that A is true is $P(A)$
  - $0 \leq P(A) \leq 1$
  - If A must be true, then $P(A) = 1$; if A must be false, then $P(A) = 0$
  - A is either true or false (binary)
  - $P(A)$ is the degree of belief that A is true

- A common form of **proposition**: "**random variable** = **value**"
- **Domain:** set of values that a random variable can take
- Example
  - $P(weather = rainy) = 0.7$: the probability that the weather will be rainy is believed to be 70%.
    - Proposition: $weather = rainy$: the weather is rainy
    - Random variable: $weather$
    - Domain: $\{rainy, sunny, cloudy, \ldots\}$

  - What is the domain of the outcome of a die?

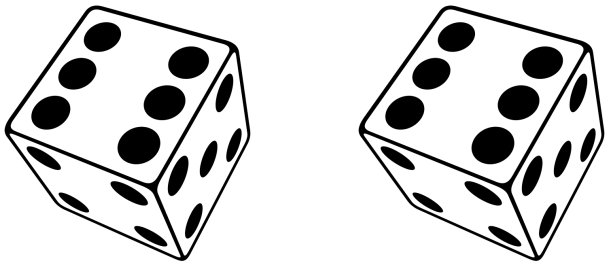
Probability

- Important notations
  - AND: $A \wedge B$. The probability that both A and B are true: $P(A \wedge B)$
  - OR: $A \vee B$. The probability that either A or B is true: $P(A \vee B)$
  - NOT: $\neg A$. The probability that A is false ($\neg A$ is true): $P(\neg A)$

- Axioms of probability theory
  - $0 \leq P(A) \leq 1$
  - $P(true) = 1$, $P(false) = 0$
  - $P(\neg A) = 1 - P(A)$
  - $\sum_{x \in \Omega} P(X = x) = 1$, where Ω is the domain of the random variable X

Question

- If we roll two fair dice, what is the probability that the total number of the two dice is 11?



Dice1	Dice2
1	1
1	2
1	3
1	4
1	5
1	6
2	1
…	…
6	6

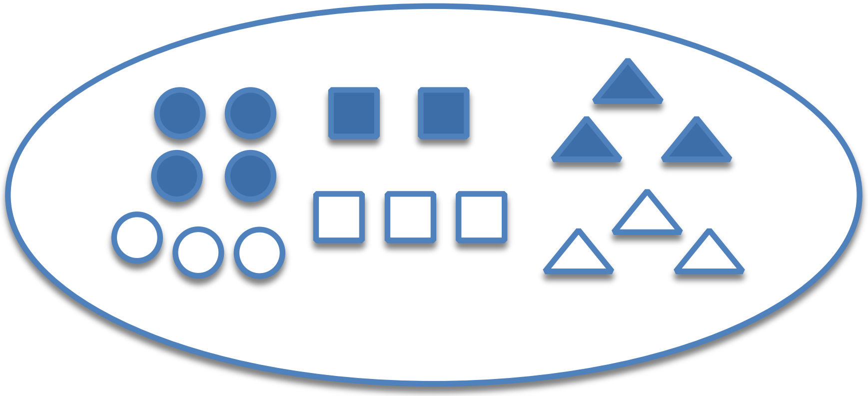
Unconditional/Conditional/Joint Probability

- Unconditional/Prior probability: degrees of belief in propositions in the absence of any other information.
  - E.g. $P(Total = 11)$
- Conditional/Posterior probability: degrees of belief in propositions given some more information (evidence).

- $\boldsymbol{P(A \mid B)}$: the conditional probability that A is true given that B is true
  - E.g. $P(Total = 11 \mid Dice_1 = 6)$, the conditional probability that the total number is 11 given that the first dice gives the number 6

- Joint probability $P(A, B) ::= P(A \wedge B)$: the probability that A is true and B is true

Example

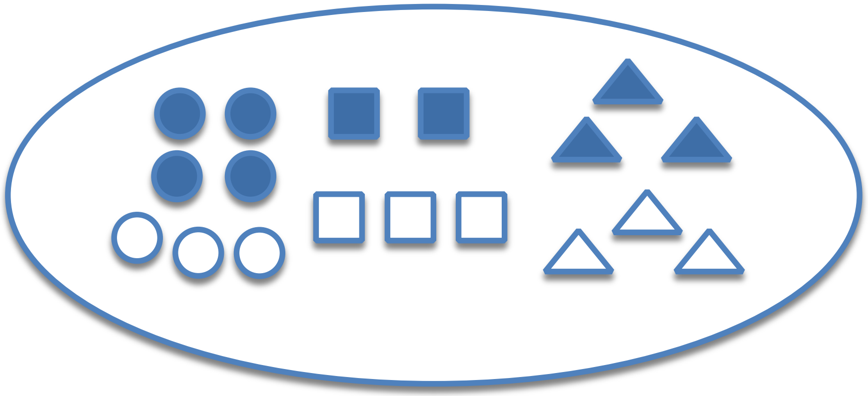


Shape

Colour

	Circle	Square	Triangle
Blue			
White			

Example



- P(Blue, Circle) = 4 / 18
- P(White, Square) = 3 / 18
- P(Circle) = 7 / 18
- P(Circle | Blue) = 4 / 9
- P(Blue | Triangle) = 3 / 6

Shape

		Circle	Square	Triangle	
Colour	Blue	**4**	**2**	**3**	**9**
	White	**3**	**3**	**3**	**9**
		7	**5**	**6**	**18**

Product Rule

- The product rule:
  - $P(A, B) = P(B) * P(A \mid B) = P(A) * P(B \mid A)$

- Check the propositions
  - Simultaneously: $P(A, B)$
  - One by one: $P(B) * P(A \mid B)$ or $P(A) * P(B \mid A)$

Shape

Colour		Circle	Square	Triangle	
	Blue	**4**	**2**	**3**	**9**
	White	**3**	**3**	**3**	**9**
		7	**5**	**6**	**18**

- P(Blue, Circle) = 4 / 18

- P(Blue) = 9 / 18
- P(Circle | Blue) = 4 / 9

- P(Circle) = 7 / 18
- P(Blue | Circle) = 4 / 7

Sum and Normalisation Rule

- The sum rule: the probability of an event is the sum of all the joint probabilities with another event
  - $P(X = x) = \sum_{y \in \Omega} P(X = x, Y = y)$

- The normalisation rule: all the possibilities (given any evidence) sum up to 100%
  - $\sum_x P(X = x) = 1$
  - $\sum_x P(X = x \mid Y = y) = 1$

Question

- There is a biased coin that produces head with probability 0.6 and tail with probability 0.4. If we flip the coin twice, what is the probability that both flips produce head?
  - $P(flip = head) = 0.6$
  - $P(flip = tail) = 0.4$
  - $P(flip_1 = head, flip_2 = head) = ?$

- The probability that stock A rises tomorrow is 0.6. The probability that stock B rises tomorrow is 0.7. What is the probability that both stock A and B rise tomorrow?
  - $P(A = rise) = 0.6$
  - $P(B = rise) = 0.7$
  - $P(A = rise, B = rise) = ?$

Independence

- The product rule: P(A, B) = P(B) * P(A | B) = P(A) * P(B | A)
- If A and B are independent ($A \perp B$) to each other, then
  - P(A | B) = P(A)
  - P(B | A) = P(B)
  - P(A, B) = P(A) * P(B)

- Flip coins twice, flip1 and flip2 are independent
- Stock A rising and stock B rising, they are usually dependent

Quiz

- Predict the weather in the future
- Random variable:
  - X_t: weather for tomorrow
  - X_{t+1}: weather for the day after tomorrow
- Domain: $\{\boldsymbol{S}unny, \boldsymbol{R}ainy\}$

- $P(X_t = S) = 0.2,\ P(X_t = R) = 0.8$
- $P(X_{t+1} = S\ |X_t = S) = 0.6,\ P(X_{t+1} = R\ |X_t = S) = 0.4$
- $P(X_{t+1} = S\ |X_t = R) = 0.3,\ P(X_{t+1} = R\ |X_t = R) = 0.7$

- $P(X_{t+1} = S) = ?$

Summary

- Uncertainty is everywhere
- Product rule
- Sum rule
- Normalisation rule
- Independence