Justification of the criterion for defining isoplanar zones

Let k isoplanar zones (regions where the Point Spread Function (PSF) of the adjacency effect is considered to be constant) have already been specified. Set the boundary of the *k*+1st isoplanar zone based on the relative error δ of the radiance of the received radiation reflected from the ground surface. That is, within the *k*-th zone we set the condition:

$$\frac{\frac{r_{surf}(\varphi_{N,ij},\lambda_{N,ij})E_{sum}(\varphi_{N,ij},\lambda_{N,ij})}{\pi}exp(-\tau(\mu_k))+I_{surf,sc}(\varphi_{N,ij},\lambda_{N,ij},\mu_k)-\frac{r_{surf}(\varphi_{N,ij},\lambda_{N,ij})E_{sum}(\varphi_{N,ij},\lambda_{N,ij})}{\pi}exp(-\tau(\mu_{k+1}))-I_{surf,sc}(\varphi_{N,ij},\lambda_{N,ij},\mu_{k+1})}{\frac{r_{surf}(\varphi_{N,ij},\lambda_{N,ij})E_{sum}(\varphi_{N,ij},\lambda_{N,ij})}{\pi}exp(-\tau(\mu_{k+1}))+I_{surf,sc}(\varphi_{N,ij},\lambda_{N,ij},\mu_{k+1})} \le \delta \quad (1)$$

$$I_{surf,sc}(\varphi_{N,ij},\lambda_{N,ij},\mu_k) = \iint_S \; r_{surf}(r_w,\varphi_w)E_{sum}(r_w,\varphi_w)h\left(r_w,\varphi_w,\mu_k\right)dS$$

where r_{surf} is the ground surface reflectance; $\varphi_{N,ij},\lambda_{N,ij}$ – coordinates of the observed pixel; E_{sum} is the total irradiance of a pixel, $I_{surf,sc}$ is the scattered part of the radiance reflected from the ground surface with the cosine of the angle between the direction of sight and the direction to the nadir equal to μ_k; r_w,φ_w are the surface polar coordinates on the spherical ground surface shown in Figure 1; S is the entire Earth's surface; dS is the surface differential; *h* is the PSF of the adjacency effect; τ - is the optical thickness of the path from the observed surface point to the receiver.

If a homogeneous surface is considered, then $r_{surf}\left(\varphi_{N,ij},\lambda_{N,ij}\right) = const = r$ and (1) is simplified to an expression of the form:

$$\frac{H(\mu_k)-H(\mu_{k+1})}{H(\mu_{k+1})} \le \delta, \qquad (2)$$

$$H(\mu_k) \equiv \frac{1}{\pi}exp\left(-\tau(\mu_k)\right) + \iint_S \; h(r_w,\varphi_w,\mu_k)dS$$

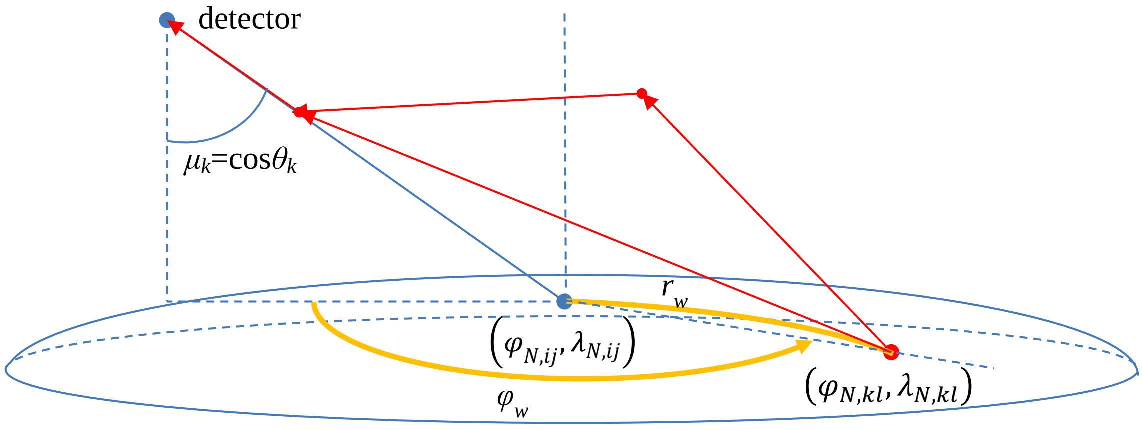

Figure 1 - Polar coordinate system on a spherical surface

Based on calculations performed in our paper [1], we can use an approximation for $H(\mu_k)$:

$$\mathrm{H}(\mu_k) \approx H(1) - C(1-\mu_k)^N, \tag{3}$$

where C, N are approximation constants.

If the boundaries of the regions where the PSF of the adjacency effect can be considered constant, are given by the recursive formula:

$$\begin{cases} H(\mu_k) = H(1) - C(1-\mu_k)^N \\ \mu_{k+1} = 1 - \left[\frac{1}{C}\left(H(1) - \frac{H(\mu_k)}{1+\delta}\right)\right]^{1/N} \end{cases}, \tag{4}$$

then the error in setting the radiance reflected from the ground surface will not exceed δ. The value δ=0.05 was used for the calculations performed.

Justification of the criterion for estimating of the adjacency effect radius.

Set the adjacency effect radius R_k so that the approximate value of the surface luminosity $\tilde{Q}$ would differ from the exact value of Q by less than the given value δ_1:

$$1 \geq \min_{i,j} \frac{Q_{i,j}}{\tilde{Q}_{i,j}} \geq \delta_1. \tag{5}$$

where $\tilde{Q}$ is the approximate value of Q when using R_k.

To simplify the analysis, consider a tighter requirement:

$$\frac{\min_{i,j} Q_{i,j}}{\max_{i,j} \tilde{Q}_{i,j}} \geq \delta_1. \tag{6}$$

If condition (6) is satisfied, then condition (5) is also satisfied. The system of linear equations for determination of the surface luminosity $Q = r_{surf} E_{sum}$ has the form (7) and (8) in the exact and approximate versions, respectively.

$$I_r(\varphi_{N,ij}, \lambda_{N,ij}, \mu_{sun}, \mu_{d,ij}, \varphi_{ij}) - I_{atm}(\mu_{sun}, \mu_{d,ij}, \varphi_{ij}) = \sum_{k=1}^{N_i} \sum_{l=1}^{N_j} Q(\varphi_{N,kl}, \lambda_{N,kl}) A_{ijkl} + \bar{Q} A_{out}(\varphi_{N,ij}, \lambda_{N,ij}) \tag{7}$$

$$I_r(\varphi_{N,ij}, \lambda_{N,ij}, \mu_{sun}, \mu_{d,ij}, \varphi_{ij}) - I_{atm}(\mu_{sun}, \mu_{d,ij}, \varphi_{ij}) = \sum_{k=1}^{Nx_{ij}} \sum_{l=1}^{Ny_{ij}} \tilde{Q}(\varphi_{N,kl}, \lambda_{N,kl}) A_{ijkl} \tag{8}$$

$$A_{out}(\varphi_{N,ij}, \lambda_{N,ij}) = \iint_S h\left(r_w, \varphi_w, \mu_{d,ij}\right) dS - \sum_{k=1}^{N_i} \sum_{l=1}^{N_j} h\left(r_{w,ijkl}, \varphi_{w,ijkl}, \mu_{d,ij}\right)$$

$$A_{ijkl} = \begin{cases} \frac{1}{\pi} exp\left(-\tau\left(\mu_{d,ij}\right)\right) + h\left(0,0,\mu_{d,ij}\right)S_{kl}, & if\ i = k\ and\ j = l \\ h\left(r_{w,ijkl}, \varphi_{w,ijkl}, \mu_{d,ij}\right)S_{kl}, if\ i \neq k\ or\ j \neq l \end{cases}$$

Where I_r is the radiance measured by a satellite receiver; I_{atm} is the radiance of the flux non-interacting with the ground surface; $\bar{Q}$ is the average luminosity of the considered surface fragment in the approximation of the uniform surface, N_i is the total number of lines, N_j is the total number of columns, Nx_{ij} is the number of lines within adjacency effect radius, Ny_{ij} is the number of columns within adjacency effect radius; μ_{sun} is the cosine of the solar zenith angle; μ_d is the cosine of the zenith angle of the receiving system; φ is the relative azimuth angle between the directions to the satellite and to the sun at the observed surface point; S_{kl} is the area of the "pixel" in the *k*-th line and *l*-th column of the considered fragment.

Then the following is true:

$$1 = \frac{\sum_{k=1}^{N_i}\sum_{l=1}^{N_j} Q(\varphi_{N,kl},\lambda_{N,kl})A_{ijkl} + \bar{Q}A_{out}(\varphi_{N,ij},\lambda_{N,ij})}{\sum_{k=1}^{Nx_{ij}}\sum_{l=1}^{Ny_{ij}} \tilde{Q}(\varphi_{N,kl},\lambda_{N,kl})A_{ijkl}} \geq \frac{\min_{i,j} Q_{i,j}\left(\sum_{k=1}^{N_i}\sum_{l=1}^{N_j} A_{ijkl} + A_{out}(\varphi_{N,ij},\lambda_{N,ij})\right)}{\max_{i,j}\tilde{Q}_{i,j}\sum_{k=1}^{Nx_{ij}}\sum_{l=1}^{Ny_{ij}} A_{ijkl}} =$$

$$\frac{\min_{i,j} Q_{i,j}}{\max_{i,j}\tilde{Q}_{i,j}} \frac{\frac{1}{\pi}exp\left(-\tau(\mu_{d,ij})\right) + \iint_S h(r_w,\varphi_w,\mu_k)dS}{\frac{1}{\pi}exp\left(-\tau(\mu_{d,ij})\right) + \iint_{S(R_k)} h(r_w,\varphi_w,\mu_k)dS} \geq \delta_1 \frac{\frac{1}{\pi}exp\left(-\tau(\mu_{d,ij})\right) + \iint_S h(r_w,\varphi_w,\mu_k)dS}{\frac{1}{\pi}exp\left(-\tau(\mu_{d,ij})\right) + \iint_{S(R_k)} h(r_w,\varphi_w,\mu_k)dS} \qquad (9)$$

where $S(R_k)$ is the region on a spherical earth's surface bounded by a radius R_k.

Then after simple transformations, we obtain that in order to fulfill condition (5), it suffices to choose the radius R_k such that the condition is satisfied:

$$k_1(R_k) \equiv \frac{\iint_{S(R_k)} h(r_w,\varphi_w,\mu_k)dS}{\iint_S h(r_w,\varphi_w,\mu_k)dS} \geq \delta_1 + (\delta_1 - 1)\frac{\frac{1}{\pi}exp\left(-\tau(\mu_{d,ij})\right)}{\iint_S h(r_w,\varphi_w,\mu_k)dS}, \qquad (10)$$

The value δ_1=0.95 was used for the calculations performed. According to our estimates made in paper [2] the R_k values for nadir situations (θ_k=0°) can reach 10 km for MODIS band No. 3 and 3 km for MODIS band No. 2. For the angle θ_k=60° R_k can reach 30 km for MODIS band No. 3 and 10 km for MODIS band No. 2. These values also prove the need to take into account the adjacency effect.

Justification of the criterion for estimating the radius of additional irradiance formation.

To set the additional irradiance formation radius R_S we use the following condition:

$$1 \geq \min_{i,j}\frac{r_{surf,ij}}{\tilde{r}_{surf,ij}} \geq \delta_2. \qquad (11)$$

where $\tilde{r}_{surf}$ is the approximate value of r_{surf} when using R_S; δ_2 is the error in determining r_{surf}.

To simplify the analysis, consider a tighter requirement:

$$\frac{\min_{i,j} r_{surf,ij}}{\max_{i,j} \tilde{r}_{surf,ij}} \geq \delta_2. \tag{12}$$

If condition (12) is satisfied, then condition (11) is also satisfied.

The system of equation for determination of the surface reflectance in the exact and approximate versions has the form (13) and (14), respectively.

$$Q_{ij} = r_{surf,ij} E_0(\mu_{sun}) \left(1 + \sum_{k=1}^{N_i} \sum_{l=1}^{N_j} r_{surf,kl} C_{ijkl} + C_{out,ij} \overline{r_{surf}} + \frac{\left(\overline{r_{surf}} \gamma_1\right)^2}{1 - \overline{r_{surf}} \gamma_1}\right) \tag{13}$$

$$Q_{ij} = \tilde{r}_{surf,ij} E_0(\mu_{sun}) \left(1 + \sum_{k=1}^{Mx_{ij}} \sum_{l=1}^{My_{ij}} \tilde{r}_{surf,kl} C_{ijkl}\right) \tag{14}$$

$$C_{ijkl} \equiv h_1(r_{w,ijkl}) S_{kl}$$

where E_0 is the irradiance of the ground surface with multiple reflection neglected; $\overline{r_{surf}}$ is the reflectance averaged over the observed surface fragment in the approximation of the uniform surface; Mx_{ij} is the number of lines within the additional irradiance formation radius R_S; My_{ij} is the number of lines within the additional irradiance formation radius R_S.

Then the following expression is true:

$$1 = \frac{r_{surf,ij}\left(1 + \sum_{k=1}^{N_i} \sum_{l=1}^{N_j} r_{surf,kl} C_{ijkl} + C_{out,ij} \overline{r_{surf}} + \frac{\left(\overline{r_{surf}} \gamma_1\right)^2}{1 - \overline{r_{surf}} \gamma_1}\right)}{\tilde{r}_{surf,ij}\left(1 + \sum_{k=1}^{Mx_{ij}} \sum_{l=1}^{My_{ij}} \tilde{r}_{surf,kl} C_{ijkl}\right)} \geq$$

$$\frac{\min_{i,j} r_{surf,ij}}{\max_{i,j} \tilde{r}_{surf,ij}} \frac{1 + \gamma_1 \min_{i,j} r_{surf,ij} + \frac{\left(\gamma_1 \min_{i,j} r_{surf,ij}\right)^2}{1 - \gamma_1 \min_{i,j} r_{surf,ij}}}{1 + \tilde{\gamma}_1(R_S) \max_{i,j} \tilde{r}_{surf,ij}} \geq \delta_2 \frac{\frac{1}{1 - \gamma_1 \min_{i,j} r_{surf,ij}}}{1 + \tilde{\gamma}_1(R_S) \max_{i,j} \tilde{r}_{surf,ij}} \tag{15}$$

$$\gamma_1 \equiv 2\pi \int_0^{\pi R_e} h_1(r_w) dr_w$$

$$\tilde{\gamma}_1(R_S) \equiv 2\pi \int_0^{R_S} h_1(r_w) dr_w$$

Expressing $\tilde{\gamma}_1(R_S)$ from (15), we get:

$$\tilde{\gamma}_1(R_S) \geq \frac{1}{\max_{i,j} \tilde{r}_{surf,ij}} \left(\frac{\delta_2}{1 - \gamma_1 \min_{i,j} r_{surf,ij}} - 1\right) \geq \delta_2 \min_{i,j} r_{surf,ij} \left(\frac{\delta_2}{1 - \gamma_1 \min_{i,j} r_{surf,ij}} - 1\right) \tag{16}$$

The highest requirement for $\tilde{\gamma}_1(R_S)$ is obtained at $\min_{i,j} r_{surf,ij}$=1, therefore, for any ground surface reflectance, to fulfill condition (11), it is sufficient to fulfill the following condition on R_S:

$$k_2(R_S) \equiv \frac{\tilde{\gamma}_1(R_S)}{\gamma_1} \geq \frac{\delta_2}{\gamma_1}\left(\frac{\delta_2}{1-\gamma_1} - 1\right) \tag{17}$$

The value δ_2=0.95 was used for the calculations performed. The estimates of the R_S value made in [2] show that the R_S value for MODIS band No. 2 can reach 2 km, and for MODIS band No. 3 – 15 km.

Construction of an approximation for I_{atm}.

To construct an approximation for I_{atm}, let's introduce the notation:

$$\begin{cases} I_x = I_{atm}\mu_d\sqrt{1-\mu_d^2}sin\varphi \\ I_y = I_{atm}\mu_d\sqrt{1-\mu_d^2}cos\varphi \\ I_z = I_{atm}\mu_d^2 \end{cases} \tag{18}$$

Numerous calculations performed by us show that the relationship between the values I_x, I_y and I_z is well described by the approximation formula:

$$I_x^2 = \begin{cases} C_{11}I_z^2 + C_{12}I_z + C_{13} + C_{21}I_y^2 + C_{22}I_yI_z + C_{23}I_y,\ 0 \leq \varphi < 90° \\ C_{11}I_z^2 + C_{12}I_z + C_{13},\ \varphi = 90° \\ C_{11}I_z^2 + C_{12}I_z + C_{13} + C_{31}I_y^2 + C_{32}I_yI_z + C_{33}I_y,\ 90 < \varphi \leq 180° \end{cases} \tag{19}$$

where C_{11}, C_{12}, C_{13}, C_{21}, C_{22}, C_{23}, C_{31}, C_{32}, C_{33} are approximation constants.

Then, substituting notation (18) into formula (19), we obtain a quadratic equation for I_{atm}. The positive root of this equation, which is proposed to be used in the approximation, is determined by the formula [3]:

$$I_{atm}(\mu_{sun}, \mu_d, \varphi) = -\frac{B(\mu_d,\varphi)+\sqrt{B(\mu_d,\varphi)^2-4A(\mu_d,\varphi)C_{13}}}{2A(\mu_d,\varphi)\mu_d}, \tag{20}$$

where

$$A(\mu_d, \varphi) =$$

$$\begin{cases} C_{11}\mu_d^2 + C_{21}\left(\sqrt{1-\mu_d^2}cos\varphi\right)^2 + C_{22}\mu_d\sqrt{1-\mu_d^2}cos\varphi - \left(\sqrt{1-\mu_d^2}sin\varphi\right)^2, \varphi \leq 90° \\ C_{11}\mu_d^2 + C_{31}\left(\sqrt{1-\mu_d^2}cos\varphi\right)^2 + C_{32}\mu_d\sqrt{1-\mu_d^2}cos\varphi - \left(\sqrt{1-\mu_d^2}sin\varphi\right)^2, \varphi > 90° \end{cases} \tag{21}$$

$$B(\mu_d, \varphi) = \begin{cases} C_{12}\mu_d + C_{23}\sqrt{1-\mu_d^2}cos\varphi,\ \varphi \leq 90° \\ C_{12}\mu_d + C_{33}\sqrt{1-\mu_d^2}cos\varphi,\ \varphi > 90° \end{cases} \quad (22).$$

Algorithm for calculating the area of the territory closest to the pixel center.

To use the proposed algorithm, it is necessary to determine the areas of the territories S_{ij} closest to the pixel centers. The difficulty lies in the fact that the data series in the image can overlap each other (butterfly effect). In addition, the distance between pixel centers is different for different parts of the image. The calculation is proposed to be performed by the Monte Carlo method. At the first stage, 8 pixels closest to the given one are found. Next, a parallelogram is built containing all 9 pixels (Figure 2). Then N points with random coordinates are thrown uniformly along the affine axes and the number of points n closest to the central pixel is determined. Then the area S_{ij} can be determined by the formula:

$$S_{ij} = \frac{n}{N} S_{par} \tag{23}$$

where S_{par} is the area of the parallelogram.

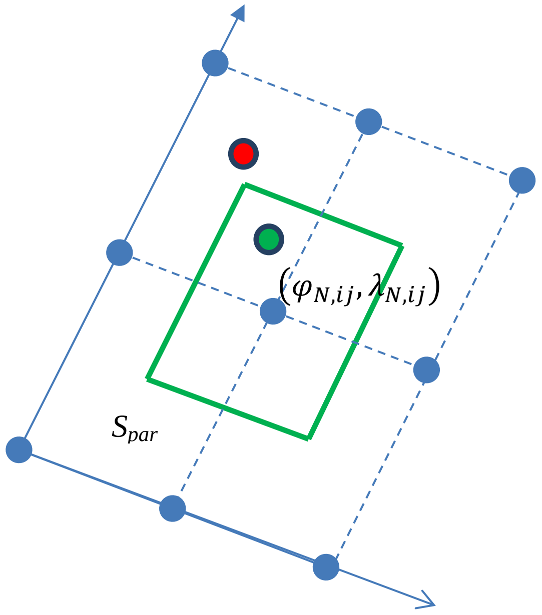

Figure 2 - Scheme for determining the areas S_{ij} by the Monte Carlo method. Red point shows a point outside the area closest to the pixel (i,j), and green point – inside the area.

References

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2. Tarasenkov M. V., Belov V. V., Engel M. V. Algorithm for reconstruction of the Earth surface reflectance from Modis satellite measurements in a turbid atmosphere // Proc. SPIE 10833, 24th International Symposium on Atmospheric and Ocean Optics: Atmospheric Physics, 1083316. 2018. 10 p.

3. Belov V.V., Tarasenkov M.V., Piskunov K.P. Parametric model of solar haze in the visible and UV ranges of the spectrum. // Optika atmosfery i okeana 2010. V. 23. No. 04. P. 294-297. [in russian]