RADIAL BEHAVIOR OF MAHLER FUNCTIONS

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Abstract. Many papers have been recently devoted to the study of the radial behavior as $z \to 1^-$ of transcendental r-Mahler functions holomorphic in the open unit disk. In particular, Bell and Coons showed in 2017 that, in a generic sense, r-Mahler functions behave like $(1+o(1))C(z)/(1-z)^\rho$ for some $\rho \in \mathbb{C}$ and $C(z)$ is a real analytic function of $z \in (0,1)$ such that $C(z) = C(z^r)$. They did not provide a formula for $C(z)$ which was made explicit only in a few examples of r-Mahler functions of order 1 and 2, and for specific values of r. In this paper, we first provide an explicit expression of $C(z)$ as an exponential of a Fourier series in the variable $\log\log(1/z)/\log(r)$ for every r-Mahler function of order 1. Then, extending to a large setting a method introduced by Brent-Coons-Zudilin in 2016 to compute $C(z)$ associated to the Dilcher-Stolarsky function (a 4-Mahler function of order 2 in $\mathbb{Q}[[z]]$), we provide an explicit expression of $C(z)$ for every r-Mahler function of order 2 under mild assumptions on the coefficients in $\mathbb{R}(z)$ of the underlying r-Mahler equations. This applies in particular to the generating function of the Baum-Sweet sequence. We do the same for r-Mahler functions solutions of inhomogeneous Mahler equations of order 1.

1. Introduction

1.1. **Context.** We fix an integer $r \geq 2$. A r-Mahler function of order d is a formal series $f \in \mathbb{C}[[z]]$ which is solution of a r-Mahler equation of order d, that is a functional equation of the form

$$\sum_{j=0}^{d} p_j(z) y(z^{r^j}) = 0 \tag{1.1}$$

where the p_j's are in $\mathbb{C}[z]$ and $p_0 p_d \neq 0$. (From now on, we will drop "r-" in front of Mahler when we refer to functions or equations because there will be no ambiguity.) If there exists a formal solution $f \in \mathbb{C}[[z]]$ of (1.1) then, unless $f(z)$ is a rational function, $f(z)$ is a transcendental function holomorphic at $z = 0$, meromorphic in the unit open disk $D(0,1)$ and the unit circle is a natural boundary for f (this is a theorem of Randé [17, Theorem 4.3]; see also [6]). The generating functions of automatic sequences with values in $\mathbb{C}$ (generated by a finite-state automaton) are known to be r-Mahler functions for some $r \geq 2$ (which can be any integer such that the r-kernel of the sequence is finite), but the converse is not true, see [3].

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When f is meromorphic in $D(0,1)$, many papers have been devoted to understand the asymptotic behavior of $f(z)$ as $z \to 1$ radially, and more generally as z tends radially to z_0 of modulus 1 (usually, z_0 is a root of unity). For instance, Bell and Coons proved the following result. Let $a_j := p_j(1)$ and let $P(X)$ be the characteristic polynomial $a_0X^d + \cdots + a_{d-1}X + a_d$ of (1.1). Assume that $a_0a_d \neq 0$ and that $P(X)$ has only one non-zero root of greatest modulus and let λ be this root. Then [5, Theorem 1] provides the following asymptotic result for a solution f of (1.1):

$$f(z) = \frac{C(z)}{(1-z)^{\log_r(\lambda)}}(1 + o(1)), \quad z \to 1^- . \tag{1.2}$$

They proved in particular that the function $C(z)$ is real analytic in $(0,1)$, such that $0 < c_1 \leq C(z) \leq c_2 < \infty$ for some constants c_1, c_2 that depend on f, and $C(z) = C(z^r)$. But they did not give an explicit expression for $C(z)$. An asymptotic expansion like in (1.2) has many applications, for instance in transcendance theory or to compute effective solutions to Mahler equations; see [1, 4, 8].

In this paper, we shall be interested in making more precise Bell and Coons' result by giving explicitly the function $C(z)$ in the form of the exponential of a Fourier series in the variable $\log\log(1/z)/\log(r)$. This is already a difficult task for Mahler functions of order 1 and our first main result deals with this case in full generality. A general treatment of Mahler functions of order ≥ 2 seems currently out of reach, though results have been obtained in special cases. We shall also determine $C(z)$ associated to Mahler functions solutions of an equation of order 2 with relatively mild assumptions on the coefficients $p_j(z)$ of this equation, though we emphasize here that they are not necessary as the example analysed in detail in [7] shows. Our final result will deal with the intermediate case, i.e. Mahler functions solutions of an inhomogeneous Mahler equation of order 1, $y(z) = p(z)y(z^r) + q(z)$ with $p(z), q(z) \in \mathbb{C}(z)$: again, we shall determine $C(z)$ with relatively mild assumptions on $p(z)$ and $q(z)$.

The problem of making explicit $C(z)$ for Mahler equations of order ≥ 3 is open in full generality; the methods of the present paper can in principle be adapted to this case as well but with stronger and stronger restrictions on the coefficients $p_j(z)$ when the order of the equation increases. Our method is an elaboration of that used by Bell, Coons and Zudilin in [7] for the generating series of an analogue of Stern sequences, introduced in [11]. Roughly speaking, their main idea is to deduce the asymptotic behavior of a Mahler function $y(z)$ from that of $y(z)/y(z^r)$: the latter satisfies a non-linear Mahler equation of order 1 when y satisfies a linear Mahler equation of order 2, which greatly simplifies the analysis. We refer to Sections 1.2, 1.3 and 1.4 of this introduction for the statements of our results for Mahler equations of order 1, order 2 and order 1 in the inhomogeneous case respectively.

Notations. In this paper, r is an integer greater than or equal to 2. We define $\log(z) = \ln|z| + i\arg(z)$ with the principal determination of the argument $-\pi < \arg(z) < \pi$ and $\log_r(z) := \log(z)/\log(r)$. Of importance in the sequel is the polylogarithm function defined by $\mathrm{Li}_s(\alpha) := \sum_{n=1}^{\infty} \alpha^n/n^s$ for any $s \in \mathbb{C}$ and any $\alpha \in \mathbb{C}$ such that $|\alpha| < 1$. The analytic continuation of this function with respect to s and/or α was studied by Jonquière [14]. The (meromorphic continuations of the) Gamma function $\Gamma(s) := \int_0^{+\infty} t^{s-1}e^{-t}\mathrm{d}t$ and of the Riemann zeta function $\zeta(s) := \sum_{n=1}^{\infty} 1/n^s$ will also be used. We shall need the following facts. If $s \in \mathbb{Z}_{\leq 0}$, $\mathrm{Li}_s(\alpha)$ reduces to a rational function of α with only one pole, at $\alpha = 1$. If $0 < |\alpha| < 1$ is fixed, $\mathrm{Li}_s(\alpha)$ is an entire function of s. If s is fixed and $\Re(s) > 0$, then $\mathrm{Li}_s(\alpha)$ is an analytic function of $\alpha \in \mathbb{C} \setminus [1, +\infty)$, and in fact it is a holomorphic function of (s, α) in this region. We set $\ell(\alpha) := \frac{\partial \mathrm{Li}_s(\alpha)}{\partial s}\big|_{s=1}$, which is analytic in $\alpha \in \mathbb{C} \setminus [1, +\infty)$. In particular,

$$\mathrm{Li}_{1+z}(\alpha) = -\log(1-\alpha) + \ell(\alpha)z + \mathcal{O}(z^2), \quad z \to 0.$$

For any $r \in \mathbb{R}$, we set $D(0, r) := \{z \in \mathbb{C} : |z| < r\}$ and $\mathcal{H}_r := \{z \in \mathbb{C} : \Re(z) > -r\}$. We denote $v_0(f)$ the order of vanishing of a formal series $f(z) \in \mathbb{C}[[z]]$: it is the largest integer $v \in \mathbb{N}$ such that f belongs to $z^v\mathbb{C}[[z]]$. Finally, in a big O estimate of the form $\mathcal{O}_\varepsilon(f(x))$, the presence of ε means that the implicit constant depends on ε.

We now review our main results in the following three subsections.

1.2. **Mahler equations of order 1.** The setting is as follows. We consider a Mahler equation of order 1:

$$y(z^r) = P(z)y(z) \tag{1.3}$$

where $P \in \mathbb{C}(z)$ is such that $P(0) = 1$. We write

$$P(z) := \frac{\prod_{i=1}^{n}(1-\alpha_i z)}{\prod_{i=1}^{m}(1-\beta_i z)}$$

where $\alpha_i, \beta_i \in \mathbb{C} \setminus \{0\}$. Since $P(0) = 1$, we know that

$$f(z) := \prod_{k=0}^{+\infty} P\left(z^{r^k}\right)^{-1} \tag{1.4}$$

is a solution of (1.3), holomorphic at $z = 0$ and meromorphic in the open unit disk. Conversely, any function of this form is solution of an equation like (1.3). We want to understand the asymptotic behavior of the Mahler function $f(z)$ when $z \to 1^-$ radially. We shall consider this problem only when none of the α's and β's are in $]1, +\infty[$. Otherwise $f(z)$ has a sequence of real zeros or poles accumulating at $z = 1$ and we cannot say much.

In order to do that, we first look at the case $P(z) = 1 - \alpha z$, $\alpha \neq 0$, and the Mahler function $f_\alpha(z) := \prod_{k=0}^{+\infty}\left(1 - \alpha z^{r^k}\right)^{-1}$, where the integer $r \geq 2$ is fixed from now on. We use the change of variables $z = e^{-s}$ with $s > 0$.

Theorem 1.1. *For all $\alpha \in \mathbb{C} \setminus [1, +\infty)$, $\alpha \neq 0$, and any sufficiently small $s > 0$, we have*

$$f_\alpha(e^{-s}) = \exp\big(\varphi_1(\alpha, s) + \varphi_2(\alpha, s) + \varphi_3(\alpha, s)\big), \tag{1.5}$$

where

$$\varphi_1(\alpha,s) := \frac{1}{\ln(r)} \sum_{k\in\mathbb{Z}\setminus\{0\}} \Gamma\Big(\frac{2\pi k i}{\ln(r)}\Big) \mathrm{Li}_{1+\frac{2\pi k i}{\ln(r)}}(\alpha) s^{-\frac{2\pi k i}{\ln(r)}}$$

$$\varphi_2(\alpha,s) := \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{1}{1-r^n} \mathrm{Li}_{1-n}(\alpha) s^n$$

$$\varphi_3(\alpha,s) := \big(\gamma + \ln(s)\big) \log_r(1-\alpha) - \frac{\log(1-\alpha)}{2} + \frac{\ell(\alpha)}{\ln(r)}.$$

The function $\exp(\varphi_1(\alpha,s))$ *is invariant under the change of variables* $s \mapsto rs$.

The "sufficiently small" condition on s depends on α and will be made more precise in the proof. Our method does not technically work for $\alpha = 1$, where a kind of "phase transition" occurs: a single pole becomes a double pole and the analysis of the situation must be performed differently. It turns out that this analysis was done long ago by de Bruijn [10] who obtained the following result: for every s such that $0 < s < 2\pi r$, we have

$$\log\big(f_1(e^{-s})\big) = \frac{\ln(s)^2}{2\ln(r)} - \frac{1}{2}\ln(s) + \frac{1}{\ln(r)} \sum_{k\in\mathbb{Z}\setminus\{0\}} \Gamma\Big(\frac{2\pi k i}{\ln(r)}\Big) \zeta\Big(1 + \frac{2\pi k i}{\ln(r)}\Big) s^{-\frac{2\pi k i}{\ln(r)}} + \sum_{n=1}^{\infty} \frac{B_n}{n! n} \frac{s^n}{r^n - 1}, \tag{1.6}$$

where $(B_n)_{n\geq 1}$ is the sequence of the Bernoulli numbers. In other words,

$$f_1(z) = \frac{e^{(\ln\ln(1/z))^2/(2\ln(r))}}{\sqrt{\ln(1/z)}} C(z)(1+o(1)), \quad z \to 1^- \tag{1.7}$$

where $C(e^{-s})$ is the exponential of the Fourier series in (1.6). Observe that the expansion (1.7) is not of the form covered by [5, Theorem 1] because the characteristic polynomial of $y(z) = (1-z)y(z^r)$ is X, which is not admissible because the assumption $a_0 a_d \neq 0$ is not fulfilled. It is possible that (1.6) could be deduced from (1.5) by letting $\alpha \to 1$ in a suitable way, but this seems a non-trivial task. The Fourier series involving the Γ and ζ functions in (1.6) is invariant under $s \mapsto rs$, and is an analogue of $\varphi_1(\alpha,s)$ in Theorem 1.1.

From Theorem 1.1, we deduce the following consequence.

Corollary 1.2. *Let* $\alpha_1, \dots, \alpha_n$, $\beta_1, \dots, \beta_m$ *in* $\mathbb{C} \setminus [1, +\infty)$. *For any sufficiently small* $s > 0$ *(with respect to the* α_j*'s and* β_j*'s), we have*

$$f(e^{-s}) = \exp\Big(\sum_{j=1}^{n} \big(\varphi_1(\alpha_j,s) + \varphi_2(\alpha_j,s) + \varphi_3(\alpha_j,s)\big) - \sum_{j=1}^{m} \big(\varphi_1(\beta_j,s) + \varphi_2(\beta_j,s) + \varphi_3(\beta_j,s)\big)\Big),$$

where $f(z)$ *is defined in* (1.4).

Moreover, in this formula, it is possible to enable some of (or all) the α_j's to be equal to 1, respectively some of (or all) the β_j's to be equal to 1, provided the corresponding term $\varphi_1(\alpha_j,s)+\varphi_2(\alpha_j,s)+\varphi_3(\alpha_j,s)$, respectively the corresponding term $\varphi_1(\beta_j,s)+\varphi_2(\beta_j,s)+\varphi_3(\beta_j,s)$, are replaced by the right-hand side of (1.6).

Remark 1.3. • In particular, the function $C(z)$ ($z\in(0,1)$) in the Bell-Coons asymptotic expansion (1.2) (see also [9] in this "order 1" case), is given in our situation by

$$C(e^{-s}) := \exp\Big(\sum_{j=1}^{n}\Big(\varphi_1(\alpha_j,s)+\gamma\log_r(1-\alpha_j)-\frac{\log(1-\alpha_j)}{2}+\frac{\ell(\alpha_j)}{\ln(r)}\Big) - \sum_{j=1}^{m}\Big(\varphi_1(\beta_j,s)+\gamma\log_r(1-\beta_j)-\frac{\log(1-\beta_j)}{2}+\frac{\ell(\beta_j)}{\ln(r)}\Big)\Big),$$

using instead, when some of the α_j's or β_j's are equal to 1, the series over $k\in\mathbb{Z}\backslash\{0\}$ in (1.6). It is clearly invariant under $s\mapsto rs$, which corresponds to the invariance $C(z^r)=C(z)$, and for all $z\in(0,1)$, we have $0<|C(z)|<+\infty$. Moreover, for $z>0$, $z\to 1^-$, we have

$$f(z)=\frac{C(z)(1+o(1))}{(1-z)^{\sum_{j=1}^{m}\log_r(1-\beta_j)-\sum_{j=1}^{n}\log_r(1-\alpha_j)}}=\frac{C(z)}{(1-z)^{\log_r(1/P(1))}}(1+o(1))$$

when none of the α_j's or β_j's is equal to 1.

1.3. **Mahler equations of order 2.** The authors of [7] studied a 4-Mahler function

$$F(z) := 1+z+z^2+z^5+z^6+z^8+z^9+z^{10}+\cdots,$$

which is a solution of $y(z)=(1+z+z^2)y(z^4)-z^4y(z^{16})$, a Mahler equation of order 2. This function had been introduced in [11] where it is proved that its coefficients are in $\{0,1\}$. In [7, Proposition 1], the asymptotic behavior of $F(z)$ is given as $z\to 1^-$ radially. The second aim of this article is to extend the approach of [7] to a general context. The function $F(z)$ satisfies all but Assumption (H1) below, but see the remarks after Theorem 1.4.

Given an integer $r\geq 2$, we consider Mahler equations of order 2 of the form

$$y(z)=a(z)y(z^r)+b(z)y(z^{r^2}) \tag{1.8}$$

where $a(z), b(z)\in\mathbb{R}(z)$ are such that

(H1) $a(z), b(z)\in\mathbb{R}^+[[z]]$;
(H2) $a(0)+b(0)=1$;
(H3) $a(z)$ and $b(z)$ are defined at $z=1$;
(H4) $a(z)$ and $b(z)$ have no pole in $D(0,1)$;
(H5) $a(z)$ and $b(z)$ are not both constant;
(H6) For all $z\in[0,1]$, $|rz^{r-1}b(z)|<a(z^r)^2$.

For instance, Assumptions (H1)–(H6) hold ([1]) when $a(z)=1+rz/2$ and $b(z)=z$. If they hold for a particular instance of (1.8), then (H6) shows that there are only finitely many

[1]Throughout the paper, when we write "(Pn)–(Pm) hold" where P∈{A,H} and $n<m$ are integers, we mean that for $k=n$ to m, all the assumptions (Pk) hold simultaneously.

integers $s \geq r$ such that they also hold for the equation $y(z) = a(z)y(z^s) + b(z)y(z^{s^2})$. Without loss of generality, we shall assume that $b(1) > 0$, otherwise necessarily $b(z) = 0$ identically and (1.8) reduces to a Mahler equation of order 1. Hence, (H6) for $z = 1$ implies that

$$0 < rb(1) < a(1)^2 < \mu_1^2$$

where $\mu_1 := (a(1) + \sqrt{a(1)^2 + 4b(1)})/2$ is the largest root of the characteristic polynomial $X^2 - a(1)X - b(1)$ (these roots are real numbers because $a(1)^2 + 4b(1) > 0$). Let us define $\alpha > 1$ such that $r^\alpha b(1) = a(1)^2$. We still have $r^\alpha b(1) < \mu_1^2$.

We shall prove in Proposition 3.1 in §3.1 that the Mahler equation (1.8) has a solution $f(z)$ holomorphic on the open unit disk, which is unique up to a multiplicative constant equal to $f(0)$. Without loss of generality, we assume that $f(0) = 1$ so that, as we shall prove, all the Taylor coefficients of $f(z)$ are in $\mathbb{R}^+$ and $f(z) > 0$ for every $z \in [0, 1)$. As in [7] for the function $F(z)$, to inspect the asymptotic behavior of $f(z)$ when $z \to 1^-$, we introduce $\mu(z) := f(z)/f(z^r)$ which is well defined and positive for $z \in [0, 1)$ and we set

$$\mathcal{M}(s) := \int_0^{+\infty} \ln(\mu(e^{-t}))t^{s-1}\mathrm{d}t,$$

which is defined for $s \in \mathcal{H}_0$, as we shall see in Section 3.2. Note that $\mu(z)$ is a solution of the non-linear Mahler equation $y(z) = a(z) + b(z)/y(z^r)$, which together with the value $\mu(0) = 1$ completely determines $\mu(z)$ without making reference to the function $f(z)$. We shall prove the following result.

Theorem 1.4. *Under Assumptions (H1)–(H6), $\mathcal{M}(s)$ can be meromorphically continued to $\mathcal{H}_{\min(\alpha,2)}$ (with simple poles at $s = 0$ and $s = -1$) and there exist explicit constants c_0 and c_1 such that, for any $\varepsilon > 0$, we have*

$$f(e^{-s}) = \exp\Big(\log_r(\mu_1)\ln(1/s) + c_0 + \frac{1}{\ln(r)}\sum_{k\in\mathbb{Z}\setminus\{0\}} \mathcal{M}\Big(\frac{2\pi ki}{\ln(r)}\Big)s^{-\frac{2\pi ki}{\ln(r)}} + c_1 s + \mathcal{O}_\varepsilon(s^{\min(\alpha,2)-\varepsilon})\Big), \quad (1.9)$$

when $s > 0$, $s \to 0^+$.

Remark 1.5. • The proof shows that $c_0 = \kappa_0/\ln(r) + \ln(\mu_1)/2$ where κ_0 is the constant term in the Laurent expansion of $\mathcal{M}(s)$ at $s = 0$, and that $c_1 = \mu_1'/((r-1)\mu_1)$ where μ_1' is the left-derivative of $\mu(z)$ at $z = 1$ (that will be proved to exist and to be finite).

• The series over $k \in \mathbb{Z} \setminus \{0\}$ in (1.9) converges absolutely for all $s > 0$ and is invariant under the change of variables $s \mapsto rs$. As a consequence of (1.9), the function $C(z)$ in (1.2) is given by

$$C(e^{-s}) = \exp\Big(c_0 + \frac{1}{\ln(r)}\sum_{k\in\mathbb{Z}\setminus\{0\}} \mathcal{M}\Big(\frac{2\pi ki}{\ln(r)}\Big)s^{-\frac{2\pi ki}{\ln(r)}}\Big) \quad (1.10)$$

and, for $z > 0$, $z \to 1^-$, we have

$$f(z) = \frac{C(z)}{(1-z)^{\log_r(\mu_1)}}(1 + o(1)).$$

• The core of the method in [7] is to prove that $\mu(z) > 0$ for all $z \in [0,1)$, and that

$$\mu(e^{-s}) = \mu_1 - s\mu_1' + \mathcal{O}(s^{\min(\alpha,2)}), \quad \mu'(e^{-s}) = \mu_1' + \mathcal{O}(s^{\min(\alpha-1,1))}),$$
$$\mu''(e^{-s}) = \mathcal{O}(s^{\min(\alpha-2,0)}) \qquad (1.11)$$

as $s \to 0^+$, for some constants $\alpha > 1$, $\mu_1 > 1$, $\mu_1' > 0$. We shall prove that such estimates hold under assumptions (H1)–(H6). An asymptotic expansion of the form (1.9) in Theorem 1.4 can then be obtained by methods of Analysis of Singularities, using standard properties of the Mellin transform. Our method to prove (1.9) follows the steps of [7], but is often different on the verification of the technical details.

• Our assumptions (H1)–(H6) on the rational functions $a(z)$ and $b(z)$ are sufficient to ensure that (1.11) holds, but they are not necessary. For instance, (H2)–(H6) hold for the specific function $F(z)$ examined in [7], and not (H1). Nonetheless, these authors directly checked the validity of (1.11) in this case and proved an expansion of the form (1.9) for $F(z)$.

• (H1), (H2), (H4) and (H5) ensure that (1.8) has a non-trivial solution in $\mathbb{R}[[z]]$ with radius of convergence 1 and with Taylor coefficients all of the same sign or 0 (see Proposition 3.1 in §3.1), which is a crucial property for us in order to define the function $\mu(z)$ for all $z \in [0,1)$.

• We could also assume (H3)–(H6), the existence of a non-trivial solution with Taylor coefficients all of the same sign or 0, $b(1) \neq 0$ and not necessarily (H1) or (H2). Our proof of Theorem 1.4 would still work (see the footnote in Lemma 3.6). Such assumptions are satisfied by the function $F(z)$.

• We shall also prove in §4 that (H1)–(H5), but not (H6), hold for the generating function of the celebrated Baum-Sweet sequence. We will then directly check that (1.11) holds so that Theorem 1.4 can thus also be applied to this generating function. The specific method used in this case could likely be used in other interesting situations.

1.4. **Inhomogeneous Mahler equations of order 1.** Between Mahler equations of order 1 and those of order 2, lies the class of inhomogeneous Mahler equations of order 1:

$$y(z) = p(z)y(z^r) + q(z) \qquad (1.12)$$

where $p(z), q(z) \in \mathbb{C}(z)$. A solution of such an equation is also solution of a Mahler equation of order 2. Before [5], a few results had been obtained in the literature concerning the asymptotic behavior as $z \to 1^-$ of holomorphic solution in $D(0,1)$ of such inhomogeneous equations. For instance, Hardy [13] examined the series $\sum_{n=0}^{\infty} z^{r^n}$, solution of the equation $f(z) = f(z^r) + z$; we recall his result in §5.1. So far, there is no uniform treatment of solutions of (1.12) and, in spirit of the results in §1.3, we now present a general result when $p(z), q(z)$ are in $\mathbb{R}(z)$ and satisfy the following assumptions:

(A1) $p(z), q(z) \in \mathbb{R}^+[[z]]$;
(A2) $q(0) = 0$;
(A3) $p(z)$ and $q(z)$ are defined at $z = 1$:
(A4) $p(z)$ and $q(z)$ have no pole in $D(0,1)$;
(A5) $p(1) > r$.

These assumptions are satisfied by $p(z) = rz + 1$ or $p(z)$ a constant $> r$, and $q(z) = z$ for instance.

Remark 1.6. • Though this is not necessary strictly speaking, we shall also assume that $q(z)$ is not constant because otherwise it must be identically 0 by (A2). The problem then reduces to the study of a Mahler equation of order 1.

• It is also possible to relax (A2) to "(A2*) $p(0) = 0$ or $q(0) = 0$", and Theorem 1.7 below still holds. The only new case with respect to (A2) is "$p(0) = 0$ and $q(0) \neq 0$", the solution $f(z)$ to consider being given by (1.14) below. We will not give the details in this case; it requires only minor changes to the proof when the assumption $q(0) = 0$ is used.

• Assumption (A1) can be dropped if we know in advance that we work with a solution $f(z) \in \mathbb{R}[[z]]$ of (1.12) with Taylor coefficients all of the same sign.

We shall prove that, under (A2)–(A4), (1.12) has solutions in $\mathbb{C}[[z]]$ which turn out to be holomorphic in the open unit disk $D(0,1)$. If $p(0) = 1$, then the solutions are given by

$$f(z) := f_0 \prod_{n=0}^{\infty} p(z^{r^n}) + \sum_{n=0}^{\infty} p(z)p(z^r)\cdots p(z^{r^{n-1}})q(z^{r^n}), \tag{1.13}$$

where f_0 is a free parameter, which turns out to be equal to $f(0)$. On the other hand, if $p(0) \neq 1$, then necessarily $f(0) = 0$ (by (1.12) evaluated at $z = 0$) and the unique possible solution in $\mathbb{C}[[z]]$ is then given by

$$f(z) := \sum_{n=0}^{\infty} p(z)p(z^r)\cdots p(z^{r^{n-1}})q(z^{r^n}). \tag{1.14}$$

Hence, in all cases the choice of f_0 determines a unique solution $f(z) \in \mathbb{C}[[z]]$ of (1.12). We shall assume that $f_0 \geq 0$, so that (A1) implies that $f(z) \in \mathbb{R}^+[[z]]$ ([2]). We define $\omega := v_0(f) \geq 0$. If $\omega \geq 1$, then $\omega = v_0(q)$. We have $f(z) = f_\omega z^\omega + \mathcal{O}(z^{\omega+1})$ with $f_\omega > 0$, and we normalize $f(z)$ by defining $\widehat{f}(z) = f(z)/(f_\omega z^\omega) \in 1 + z\mathbb{R}^+[[z]]$.

We define $\widehat{\mu}(z) := \widehat{f}(z)/\widehat{f}(z^r) = z^{(r-1)\omega} f(z)/f(z^r) \in 1 + z\mathbb{R}[[z]]$. For all $z \in [0,1)$, we have $\widehat{\mu}(z) > 0$. We set

$$\widehat{\mathcal{M}}(s) := \int_0^{+\infty} \ln(\widehat{\mu}(e^{-t}))t^{s-1}\mathrm{d}t,$$

which is defined and analytic for $s \in \mathcal{H}_0$ as we shall see in §5. We set $\eta := \log_r(p(1)) > 1$ (by (A5)).

Theorem 1.7. *Under Assumptions (A1)–(A5) and $f_0 \geq 0$, $\widehat{\mathcal{M}}(s)$ can be meromorphically continued to $\mathcal{H}_{\min(\eta,2)}$ (with a simple pole at $s = 0$) and there exist explicit constants $\widehat{c}_0$*

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[2]If one takes $f_0 < 0$, the positivity of the Taylor coefficients of $f(z)$ is no longer guaranteed and this leads to various complications we don't know how to overcome in a simple manner.

and $\widehat{c}_1$ *such that, for any* $\varepsilon > 0$, *we have*

$$f(e^{-s}) = \exp\Big(\eta\ln(1/s) + \widehat{c}_0 + \frac{1}{\ln(r)}\sum_{k\in\mathbb{Z}\setminus\{0\}} \widehat{\mathcal{M}}\Big(\frac{2\pi ki}{\ln(r)}\Big)s^{-\frac{2\pi ki}{\ln(r)}} + \widehat{c}_1 s + \mathcal{O}_\varepsilon(s^{\min(\eta,2)-\varepsilon})\Big), \quad (1.15)$$

when $s > 0$, $s \to 0^+$.

Remark 1.8. • The proof shows that $\widehat{c}_0 = \widehat{\kappa}_0/\ln(r) + \ln(p(1))/2$ where $\widehat{\kappa}_0$ is the constant term in the Laurent expansion of $\widehat{\mathcal{M}}(s)$ at $s = 0$, and that $\widehat{c}_1 = \omega + p'(1)/((r-1)p(1))$.

• The series over $k \in \mathbb{Z}\setminus\{0\}$ in (1.15) converges absolutely for all $s > 0$ and is invariant under the change of variables $s \mapsto rs$. As a consequence of (1.15), the function $C(z)$ in (1.2) is given by

$$C(e^{-s}) = \exp\Big(\widehat{c}_0 + \frac{1}{\ln(r)}\sum_{k\in\mathbb{Z}\setminus\{0\}} \widehat{\mathcal{M}}\Big(\frac{2\pi ki}{\ln(r)}\Big)s^{-\frac{2\pi ki}{\ln(r)}}\Big)$$

and, for $z > 0$, $z \to 1^-$, we have

$$f(z) = \frac{C(z)}{(1-z)^{\log_r(p(1))}}(1 + o(1)).$$

• Our assumptions (A1)–(A5) on the rational functions $a(z)$ and $b(z)$ are sufficient to ensure that (1.11) holds (with $\mu(z)$ replaced by $\widehat{\mu}(z)$), but they are not necessary. On the other hand, a solution of the equation $y(z) = p(z)y(z^r) + q(z)$ is also a solution of

$$y(z) = \frac{q(z) + q(z^r)p(z)}{q(z^r)}y(z^r) - \frac{q(z)p(z^r)}{q(z^r)}y(z^{r^2}).$$

By (A2), $q(z)p(z^r)/q(z^r)$ has a pole at $z = 0$ and Theorem 1.4 cannot be applied. Hence, Theorem 1.7 enables us to determine the asymptotic behavior as $z \to 1^-$ of certain solutions of Mahler equations of order 2 that do not fall under the scope of Theorem 1.4.

The rest of the paper is organized as follows. The aim of §2 is to prove Theorem 1.1. We first present in §2.1 various lemmas used for the proof of Theorem 1.1 given in §2.2. We adopt the same presentation in §3, where lemmas are first proved in §3.1 and then Theorem 1.4 is proved in §3.2. In §4, we show how to adapt the proof of Theorem 1.4 to the case of the generating function of the Baum-Sweet sequence. In §5, we first recall a classical result of Hardy on the series $\sum_{n=0}^{\infty} z^{r^n}$, and then we give the proof of Theorem 1.7: as it is similar to the proof of Theorem 1.4, we adopt a less formal presentation. Finally, in §6.1, we present interesting Mahler functions of order 2 to which our results can not be applied, and in §6.2 we display a strategy to extend our theorems to Mahler equations of higher order.

2. Proof of Theorem 1.1

2.1. **Preparatory results.** We assume for the moment that $\alpha \in (-1,1)$ and $s>0$ are real numbers; the assumption on α can be relaxed in certain lemmas proved below, while s can be subject to certain restriction that depend on α. We define

$$G_{\alpha,n}(s) := -\sum_{k=0}^{n} \log\left(1-\alpha\exp(-sr^k)\right)$$

and since we deal with real numbers, we have

$$\lim_{n\to+\infty} G_{\alpha,n}(s) = \ln(f_\alpha(e^{-s})).$$

For simplicity, we set $G_\alpha(s) = \ln(f_\alpha(e^{-s}))$.

Lemma 2.1. *Fix $\alpha \in (-1,1)$, $s>0$. For any $a>0$, we have*

$$G_\alpha(s) = \frac{1}{2\pi i}\int_{a-\infty i}^{a+\infty i} s^{-z}\frac{\Gamma(z)\mathrm{Li}_{1+z}(\alpha)}{1-r^{-z}}\,\mathrm{d}z\,.$$

Proof. We have

$$G_\alpha(s) = -\sum_{k=0}^{+\infty}\log\left(1-\alpha\exp\left(-sr^k\right)\right) = \sum_{k=0}^{+\infty}\sum_{m=1}^{+\infty} m^{-1}\alpha^m e^{-sr^k m}.$$

Using the Cahen-Mellin formula (see [10])

$$e^{-\omega} = \frac{1}{2\pi i}\int_{a-\infty i}^{a+\infty i}\Gamma(z)\omega^{-z}\,\mathrm{d}z \quad \text{for } a>0, \omega>0, \tag{2.1}$$

and the fact that

$$\int_{a-\infty i}^{a+\infty i}\sum_{k=0}^{+\infty}\sum_{m=1}^{+\infty}\left|\Gamma(z)m^{-1}\alpha^m\left(sr^k m\right)^{-z}\right|\,\mathrm{d}z$$
$$\leq \int_{a-\infty i}^{a+\infty i}\sum_{k=0}^{+\infty}\sum_{m=1}^{+\infty}\left|\Gamma(z)m^{-1}\left(sr^k m\right)^{-z}\right|\,\mathrm{d}z < +\infty,$$

we have

$$G_\alpha(s) = \frac{1}{2\pi i}\int_{a-\infty i}^{a+\infty i}\sum_{k=0}^{+\infty}\sum_{m=1}^{+\infty}\Gamma(z)m^{-1}\alpha^m\left(sr^k m\right)^{-z}\,\mathrm{d}z$$

because the interchange of integral and series is justified. Now, the series on k is a geometric one and the series on m is a polylogarithm, so that

$$G_\alpha(s) = \frac{1}{2\pi i}\int_{a-\infty i}^{a+\infty i} s^{-z}\frac{\Gamma(z)\mathrm{Li}_{1+z}(\alpha)}{1-r^{-z}}\,\mathrm{d}z$$

as expected. □

We now want to shift the vertical line $a+\mathbb{R}i$ to the left in order to obtain another expression for $G_\alpha(s)$. We need a few lemmas, that hold for α a complex number.

Lemma 2.2. *Let $\alpha \in \mathbb{C}$ be such that $0 < |\alpha| < 1$. Then,*

$$|\mathrm{Li}_{1+z}(\alpha)| \le \begin{cases} \frac{1}{1-|\alpha|} & \text{if } \Re(z) \ge -1 \\ \zeta(2)\left(1-\Re(z)\right)^{1-\Re(z)}\left(e\ln(1/|\alpha|)\right)^{\Re(z)-1} & \text{otherwise.} \end{cases}$$

Proof. We recall that

$$\mathrm{Li}_{1+z}(\alpha) = \sum_{n=1}^{+\infty} \frac{\alpha^n}{n^{1+z}}.$$

The first case is clear since $\left|\frac{\alpha^n}{n^{1+z}}\right| = \frac{|\alpha|^n}{n^{1+\Re(z)}} \le |\alpha|^n$. For the second case,

$$|\mathrm{Li}_{1+z}(\alpha)| \le \max_{n\ge 1}\left(\frac{|\alpha|^n}{n^{\Re(z)-1}}\right)\zeta(2)\,.$$

Let $t := 1-\Re(z) > 2$ and $y := |\alpha| < 1$. The maximum of the function $u : x \mapsto y^x x^t$ for $x > 0$ is

$$u\Big(-\frac{t}{\ln(y)}\Big) = \Big(\frac{t}{-e\ln(y)}\Big)^t,$$

which concludes the proof. ∎

Lemma 2.3. *Assume that $\alpha \in \mathbb{C}$ is such that $0 < |\alpha| < 1$. Let s be such that $0 < s < r\ln(1/|\alpha|)$. We have*

$$\lim_{n\to+\infty} \frac{1}{2\pi i}\int_{-n+1/2-\infty i}^{-n+1/2+\infty i} s^{-z}\frac{\Gamma(z)\mathrm{Li}_{1+z}(\alpha)}{1-r^{-z}}\,\mathrm{d}z = 0.$$

Proof. We introduce

$$I_n(\alpha) := \int_{\mathbb{R}} s^{n-1/2-yi}\frac{\Gamma(1/2-n+yi)\mathrm{Li}_{3/2-n+yi}(\alpha)}{1-r^{n-1/2-yi}}\,\mathrm{d}y\,.$$

We have

$$|\Gamma(1/2-n+yi)|^2 = \frac{\pi}{\cosh(\pi y)}\frac{1}{\prod_{k=1}^{n}\left((k-1/2)^2+y^2\right)} \le \frac{\pi}{\cosh(\pi y)}\frac{1}{1/4(n-1)!^2}\,,$$

and, from Lemma 2.2, for $n \ge 2$,

$$\left|\mathrm{Li}_{3/2-n+yi}(\alpha)\right| \le \zeta(2)\,(n+1/2)^{n+1/2}\left(e\ln(1/|\alpha|)\right)^{-n-1/2}.$$

For $n \ge 2$, we obtain

$$|I_n(\alpha)| \le \frac{c}{\ln(1/|\alpha|)}\left(\frac{s}{r\ln(1/|\alpha|)}\right)^{n-1/2}\underbrace{\frac{(n+1/2)^{n+1/2}e^{-n}}{(n-1)!}}_{:=d(n)}\frac{r^{n-1/2}}{r^{n-1/2}-1}$$

with $c := 2\sqrt{\pi}\zeta(2)e^{-1/2}\int_{\mathbb{R}}\frac{1}{\sqrt{\cosh(\pi y)}}\,\mathrm{d}y < +\infty$. From Stirling's formula,

$$d(n) \underset{n\to+\infty}{\sim} \frac{e^{1/2}}{\sqrt{2\pi}}n$$

and, since $0 < s < r\ln(1/|\alpha|)$, we have $\lim\limits_{n\to+\infty} I_n(\alpha) = 0$. $\square$

Lemma 2.4. *Assume that $\alpha \in \mathbb{C}$ is such that $0 < |\alpha| < 1$. Let s be such that $0 < s < r\ln(1/|\alpha|)$ and write*

$$J_n(\alpha) := \frac{1}{2\pi i}\int_{-\infty+ni}^{-1+ni} s^{-z}\frac{\Gamma(z)\mathrm{Li}_{1+z}(\alpha)}{1-r^{-z}}\,\mathrm{d}z\,.$$

We have

$$\lim_{n\to+\infty} J_n(\alpha) = 0 \quad \textit{and} \quad \lim_{n\to-\infty} J_n(\alpha) = 0\,.$$

Proof. We study the following integral:

$$\int_1^{+\infty} s^{x-ni}\frac{\Gamma(ni-x)\mathrm{Li}_{1-x+ni}(\alpha)}{1-r^{x-ni}}\,\mathrm{d}x.$$

Let $x \geq 1$. We have:

- $\left|\frac{s^{x-ni}}{1-r^{x-ni}}\right| \leq 2\frac{s^x}{r^x}$,
- $|\mathrm{Li}_{1-x+ni}(\alpha)| \leq \zeta(2)(1+x)^{1+x}\left(e\ln(1/|\alpha|)\right)^{-1-x}$ for $x > 1$ from Lemma 2.2,
- $\Gamma(ni-x) = (ni-x-1)\Gamma(ni-x-1)$ and

$$\Gamma(ni-x-1) \sim \sqrt{2\pi}(ni-x-1)^{ni-x-3/2}e^{-ni+1+x}$$

as $x \to +\infty$. Hence,

$$|\Gamma(ni-x)| \underset{x\to+\infty}{\sim} \sqrt{2\pi}\frac{e^{x+1}}{\sqrt{(x+1)^2+n^2}^{x+1/2}}e^{-n\arg(ni-x-1)}.$$

It remains to study

$$\left(\frac{s}{r\ln(1/|\alpha|)}\right)^x \underbrace{\left(\frac{(1+x)^2}{(1+x)^2+n^2}\right)^{\frac{x}{2}+\frac{1}{4}}}_{\leq 1}\sqrt{1+x}e^{-n\arg(ni-x-1)}.$$

If $n \geq 0$ then $-n\arg(ni-x-1) \leq -n\frac{\pi}{2}$, which is negative. If $n < 0$ then $-n\arg(ni-x-1) \leq n\frac{\pi}{2}$, which is also negative. We obtain the desired result. $\square$

Lemma 2.5. *Assume $\alpha \in \mathbb{C}$ is such that $0 < |\alpha| < 1$. Let $s > 0$ and $a > 0$. Let*

$$K_y(\alpha) := \frac{1}{2\pi i}\int_{-1+yi}^{a+yi} s^{-z}\frac{\Gamma(z)\mathrm{Li}_{1+z}(\alpha)}{1-r^{-z}}\,\mathrm{d}z.$$

There exists an increasing to $+\infty$ (resp. decreasing to $-\infty$) sequence $(u_n)_{n\in\mathbb{N}}$ (resp. $(v_n)_{n\in\mathbb{N}}$) such that $u_n \in \mathbb{N}$ (resp. $v_n \in \mathbb{Z}_{\leq 0}$) and such that

$$\lim_{n\to+\infty} K_{u_n}(\alpha) = 0 \quad \textit{and} \quad \lim_{n\to+\infty} K_{v_n}(\alpha) = 0.$$

Proof. We want to avoid the poles of $\frac{1}{r^{x+ni}-1}$ for $x \in [-a, 1]$. We have

$$\begin{aligned} |r^{x+in} - 1|^2 &= r^{2x} - 2r^x \cos(n \ln(r)) + 1 \\ &\geq \sin^2(n \ln(r)) \end{aligned}$$

(because the minimum of $X^2 - 2X\cos(n\ln(r)) + 1$ is attained at $X = \cos(n\ln(r))$).

Now $\frac{\ln(r)}{2\pi} \notin \mathbb{Q}$ otherwise we would have $e^\pi \in \overline{\mathbb{Q}}$ (a contradiction), so that by Weyl's equidistribution theorem, there exists an increasing sequence $(u_n)_{n\in\mathbb{N}}$ such that $u_n \in \mathbb{N}$ and such that $|r^{x+u_n i} - 1| \geq 1/2$ for all $n \in \mathbb{N}$. Moreover,

- $|\mathrm{Li}_{1-x+ni}(\alpha)| \leq \frac{1}{1-|\alpha|}$ by Lemma 2.2,
- $|\Gamma(ni - x)| \underset{n\to+\infty}{\sim} \sqrt{2\pi} e^x |ni - x|^{-(x+1/2)} e^{-n\arg(ni-x)}$.

Since if $n \neq 0$, $-n\arg(ni - x) < -|n|c$ where c is a positive constant, the first point is proved. The second point is similar. □

Lemma 2.6. *Let $\alpha \in \mathbb{C} \setminus [1, +\infty)$. Let ω_α be any number in $(0, \pi/2)$ if $|\alpha| \leq 1$, and any number in $(0, \arctan(|\arg(\alpha)|/\ln|\alpha|))$ if $|\alpha| > 1$. Then, for any $y \in \mathbb{R}$, $x > 0$, we have*

$$|\Gamma(x + yi)\mathrm{Li}_{x+yi}(\alpha)| \leq e^{-\omega_\alpha |y|} I_{sign(y)\omega_\alpha}(x)$$

where

$$I_\omega(x) := |\alpha| \int_0^{+\infty} \frac{u^{x-1}}{|e^{ue^{i\omega}} - \alpha|} du < +\infty.$$

Proof. For any $\alpha \in \mathbb{C} \setminus [1, +\infty)$ and any s such that $\Re(s) > 0$, we have

$$\mathrm{Li}_s(\alpha) = \frac{\alpha}{\Gamma(s)} \int_0^{+\infty} \frac{t^{s-1}}{e^t - \alpha} dt.$$

(This provides the analytic continuation of $\mathrm{Li}_s(\alpha)$ to this region.)

If $|\alpha| \leq 1$, then $e^t - \alpha$ vanishes for no t such that $\Re(t) > 0$. Then, by Cauchy's formula, we have

$$\begin{aligned} \int_0^{+\infty} \frac{t^{s-1}}{e^t - \alpha} dt &= \int_0^{e^{i\omega}\infty} \frac{t^{s-1}}{e^t - \alpha} dt \\ &= e^{i\omega s} \int_0^{+\infty} \frac{u^{s-1}}{e^{ue^{i\omega}} - \alpha} du. \end{aligned}$$

for any $\omega \in (-\pi/2, \pi/2) \setminus \{0\}$. Therefore, we have

$$|\Gamma(x + yi)\mathrm{Li}_{x+yi}(\alpha)| \leq e^{-\omega y} I_\omega(x)$$

where

$$I_\omega(x) := |\alpha| \int_0^{+\infty} \frac{u^{x-1}}{|e^{ue^{i\omega}} - \alpha|} du.$$

If $y > 0$, we take $\omega > 0$ while if $y < 0$, we take $\omega < 0$, and eventually we set $\omega_\alpha := |\omega|$.

If $|\alpha| > 1$, then $e^t - \alpha$ vanishes for no t such that $\Re(t) > 0$ and

$$\arg(t) \in \Big(-\arctan\Big(\frac{|\arg(\alpha)|}{\ln|\alpha|}\Big), \arctan\Big(\frac{|\arg(\alpha)|}{\ln|\alpha|}\Big)\Big).$$

We conclude as above. ☐

2.2. **Completion of the proof of Theorem 1.1.** The function

$$g_{\alpha,s}(z) = s^{-z}\frac{\Gamma(z)\mathrm{Li}_{1+z}(\alpha)}{1-r^{-z}}$$

has the following poles in the plane $\mathrm{Re}(z) < a$ ($a > 0$):

- simple poles at $0, -1, -2, \ldots, -n, \ldots$ of Γ whose residue is $\frac{(-1)^n}{n!}$;
- simple poles at $z_k := \frac{2\pi k i}{\ln(r)}$, $k \in \mathbb{Z}$, of $\frac{1}{1-r^{-z}}$.

The residue of $g_{\alpha,s}$ at

- $z = -n$, n a positive integer, is
$$s^n\frac{(-1)^n}{n!}\frac{1}{1-r^n}\mathrm{Li}_{1-n}(\alpha)\,,$$
where $\mathrm{Li}_{1-n}(\alpha)$ is a rational function (of the variable α).
- $z = \frac{2\pi k i}{\ln(r)}$, k a nonzero integer, is
$$\frac{1}{\ln(r)}s^{-\frac{2\pi k i}{\ln(r)}}\Gamma\left(\frac{2\pi k i}{\ln(r)}\right)\mathrm{Li}_{1+\frac{2\pi k i}{\ln(r)}}(\alpha)\,.$$
- $z = 0$ (double pole) is
$$\big(\gamma + \ln(s)\big)\log_r(1-\alpha) - \frac{\log(1-\alpha)}{2} + \frac{\ell(\alpha)}{\ln(r)}.$$

Let us define the rectangle $C_{m,n}$ with vertices $-m+\frac{1}{2}+u_n i$, $-m+\frac{1}{2}+v_n i$, $a+v_n i$ and $a+u_n i$ where $m \geq 2$ is an integer and $(u_n)_{n\geq 0}$, $(v_n)_{n\geq 0}$ are defined as in Lemma 2.5. We have

$$\begin{aligned}\frac{1}{2\pi i}\int_{C_{m,n}} g_{\alpha,s}(z)dz &= \sum_{\ell=1}^{m-1} s^\ell\frac{(-1)^\ell}{\ell!}\frac{1}{1-r^\ell}\mathrm{Li}_{1-\ell}(\alpha)\\ &+\frac{1}{\ln(r)}\sum_{k\in\mathbb{Z}\setminus\{0\},v_n<|\frac{2\pi k}{\ln(r)}|<u_n} s^{-\frac{2\pi k i}{\ln(r)}}\Gamma\left(\frac{2\pi k i}{\ln(r)}\right)\mathrm{Li}_{1+\frac{2\pi k i}{\ln(r)}}(\alpha)\\ &+\big(\gamma+\ln(s)\big)\log_r(1-\alpha) - \frac{\log(1-\alpha)}{2} + \frac{\ell(\alpha)}{\ln(r)}. \qquad (2.2)\end{aligned}$$

We shall now let $m, n \to +\infty$ in (2.2). We first need two lemmas.

Lemma 2.7. *The series*

$$\varphi_1(\alpha, s) := \frac{1}{\ln(r)}\sum_{k\in\mathbb{Z}\setminus\{0\}} s^{-\frac{2\pi k i}{\ln(r)}}\Gamma\left(\frac{2\pi k i}{\ln(r)}\right)\mathrm{Li}_{1+\frac{2\pi k i}{\ln(r)}}(\alpha)$$

converges for all $\alpha \in \mathbb{C}\setminus[1,+\infty)$ and all s in a sector $\{s \in \mathbb{C}^* : |\arg(s)| < \Omega_\alpha\}$, where $\Omega_\alpha := \frac{\pi}{2}$ if $|\alpha| \leq 1$, and $\Omega_\alpha := \arctan(|\arg(\alpha)|/\ln|\alpha|)$ if $|\alpha| > 1$. This defines domains of $\mathbb{C}$ on which φ_1 is a holomorphic function of one of its variable when the other is fixed.

Proof. By Lemma 2.6 applied to $x + yi := 1 + \frac{2\pi k i}{\ln(r)}$, for every $\alpha \in \mathbb{C} \setminus [1, +\infty)$, for all $k \in \mathbb{Z} \setminus \{0\}$, we have

$$\Big|\Gamma\Big(\frac{2\pi k i}{\ln(r)}\Big)\mathrm{Li}_{1+\frac{2\pi k i}{\ln(r)}}(\alpha)\Big| \ll_{\alpha,r} \exp\Big(-\frac{2\pi\omega_\alpha}{\ln(r)}|k|\Big),$$

where ω_α is defined in Lemma 2.6. Assuming that $|\arg(s)| < \omega_\alpha$, we have for all $k \in \mathbb{Z}\setminus\{0\}$

$$\Big|\Gamma\Big(\frac{2\pi k i}{\ln(r)}\Big)\mathrm{Li}_{1+\frac{2\pi k i}{\ln(r)}}(\alpha) s^{-\frac{2\pi k i}{\ln(r)}}\Big|$$
$$\ll_{\alpha,r} \exp\Big(-\frac{2\pi}{\ln(r)}\big(|k|\omega_\alpha - k\arg(s)\big)\Big) \le \exp\Big(-\frac{2\pi|k|}{\ln(r)}\big(\omega_\alpha - |\arg(s)|\big)\Big).$$

The result follows because ω_α can be chosen arbitrarily close to Ω_α. ∎

Lemma 2.8. *The series*

$$\varphi_2(\alpha, s) := \sum_{n=1}^{\infty} s^n \frac{(-1)^n}{n!}\frac{1}{1-r^n}\mathrm{Li}_{1-n}(\alpha)$$

converges for all $\alpha \in \mathbb{C} \setminus \{1\}$ and all $s \in \mathbb{C}$ such that $|s| < r|\alpha - 1|\min(1, 1/|\alpha|)$. This defines domains of $\mathbb{C}$ on which φ_2 is a holomorphic function of one of its variable when the other is fixed.

Proof. Without loss of generality, we can assume below that $n \ge 2$. We have

$$\mathrm{Li}_{1-n}(\alpha) = \frac{1}{(1-\alpha)^n}\sum_{k=0}^{n-2} A(n-1,k)\alpha^{n-1-k},$$

where $A(n-1,k)$ are Eulerian numbers (see for example the introduction of [15]). Since $\sum_{k=0}^{n-2} A(n-1,k) = (n-1)!$,we have

$$|\mathrm{Li}_{1-n}(\alpha)| \le |\alpha|\frac{\max(1,|\alpha|)^{n-2}}{|\alpha-1|^n}(n-1)!$$

and since $\frac{1}{r^n-1} \le \frac{2}{r^n}$ for all $n \ge 2$, we obtain

$$|\varphi_2(\alpha,s)| \le \frac{|s|}{(r-1)|\alpha-1|} + 2|\alpha|\sum_{n\ge 2}\frac{1}{n}\left(\frac{s}{r|\alpha-1|}\right)^n \max(1,|\alpha|)^{n-2}.$$

The result follows. ∎

We also set

$$\varphi_3(\alpha,s) := \big(\gamma + \ln(s)\big)\log_r(1-\alpha) - \frac{\log(1-\alpha)}{2} + \frac{\ell(\alpha)}{\ln(r)}.$$

This is a holomorphic function of $(\alpha, s) \in (\mathbb{C} \setminus [1, +\infty)) \times (\mathbb{C} \setminus (-\infty, 0])$.

On the one hand, by Lemmas 2.7 and 2.8, as $m, n \to +\infty$, the right-side of (2.2) converges to $\varphi_1(\alpha, s) + \varphi_2(\alpha, s) + \varphi_3(\alpha, s)$ for any $\alpha \in \mathbb{C} \setminus [1, +\infty)$ and any $s \in \mathbb{C}$ such that $|\arg(s)| < \Omega_\alpha$ and $0 < |s| < r|\alpha - 1| \min(1, 1/|\alpha|)$.

On the other hand, Lemmas 2.1 to 2.5 ensure that, as $m, n \to +\infty$, the left-hand side of (2.2) converges to $\ln(f_\alpha(e^{-s}))$ for any $0 \neq \alpha \in (-1, 1)$ and any s such that $0 < s < r \log(1/|\alpha|)$.

Therefore, the identity

$$f_\alpha(e^{-s}) = \exp\big(\varphi_1(\alpha, s) + \varphi_2(\alpha, s) + \varphi_3(\alpha, s)\big) \tag{2.3}$$

holds at least for every non-zero $\alpha \in (-1, 1)$ and every $s \in \mathbb{R}$ such that $0 < s < r \ln(1/|\alpha|)$ and $0 < s < r|\alpha - 1| \min(1, 1/|\alpha|)$. Now, the function $f_\alpha(e^{-s})$ is holomorphic in $(\alpha, s) \in \mathbb{C} \times \{\Re(s) > 0\}$. Hence given the analyticity properties of the involved functions, the assumptions on α and s under which (2.3) holds can be relaxed to $0 \neq \alpha \in \mathbb{C} \setminus [1, +\infty)$ and $0 < s < r \max(\ln(1/|\alpha|), |\alpha - 1| \min(1, 1/|\alpha|))$. This completes the proof of Theorem 1.1.

3. Proof of Theorem 1.4

3.1. **Preparatory results.** Let $r \geq 2$ be an integer. We look at equations of the form

$$y(z) = a(z) y(z^r) + b(z) y\big(z^{r^2}\big) \tag{3.1}$$

and we want to find conditions on a and b to apply a construction similar to [7] for giving an asymptotic expansion of a solution f of (3.1) when $z \to 1^-$.

We recall that we assume $a(z), b(z) \in \mathbb{R}(z)$ to be such that

- (H1) $a(z), b(z) \in \mathbb{R}^+[[z]]$;
- (H2) $a(0) + b(0) = 1$;
- (H3) $a(z)$ and $b(z)$ are defined at $z = 1$;
- (H4) $a(z)$ and $b(z)$ have no pole in $D(0, 1)$;
- (H5) $a(z)$ and $b(z)$ are not both constant;
- (H6) For all $z \in [0, 1]$, $|r z^{r-1} b(z)| < a(z^r)^2$.

Without loss of generality, we assume $a(z), b(z) \neq 0$ because if one of them is identically 0 then Eq. (3.1) is reduced to an equation of order 1.

We write

$$a(z) = \sum_{n=0}^{+\infty} \alpha_n z^n \quad \text{and} \quad b(z) = \sum_{n=0}^{+\infty} \beta_n z^n$$

their Taylor series expansions. We first prove the existence of a unique holomorphic solution of Eq. (3.1) in the open unit disk.

Proposition 3.1. *Under Assumptions (H1)–(H6), Eq.* (3.1) *has a unique solution f (up to multiplying f by a constant) which is holomorphic on the open unit disk $D(0, 1)$.*

Let $\sum_{i=0}^{+\infty} f_n z^n$ be the Taylor expansion of $f(z)$ at $z = 0$. If $f_0 > 0$, respectively $f_0 < 0$, then the coefficients f_n are non-negative, respectively non-positive, and the radius of convergence is equal to 1. If $f_0 = 0$, then f is identically equal to 0.

Remark 3.2. • Only Assumptions (H1), (H2), (H4) and (H5) are needed in the proof of that proposition. In fact, Assumption (H5) is only needed to prove that the radius of convergence is equal to 1, see the remark after the proof.

Proof. From (3.1), we deduce a recurrence relation between the coefficients f_n: for any $n \geq 0$, we have

$$f_n = \sum_{k=0}^{n} \alpha_{n-k} \widetilde{f_k} + \sum_{k=0}^{n} \beta_{n-k} \widehat{f_k} \tag{3.2}$$

where

$$\widetilde{f_k} := \begin{cases} f_{k/r} & \text{if } r \text{ divides } k \\ 0 & \text{otherwise} \end{cases}$$

and

$$\widehat{f_k} := \begin{cases} f_{k/r^2} & \text{if } r^2 \text{ divides } k \\ 0 & \text{otherwise}\,. \end{cases}$$

If $n = 0$, then $f_0 = \alpha_0 f_0 + \beta_0 f_0$: this relation is satisfied if $f_0 = 0$, and also if $f_0 \neq 0$ because $\alpha_0 + \beta_0 = a(0) + b(0) = 1$ by assumption. For $n \geq 1$, we see that on the right-hand side of (3.2) we have the numbers f_m for $m \leq n-1$ and not for $m = n$, so that the sequence $(f_n)_{n \geq 0}$ is defined by (3.2) in a unique way from the value f_0. Moreover it is readily check by induction on n that $f_n \geq 0$ for all $n \geq 0$ if $f_0 > 0$, respectively $f_n \leq 0$ for all $n \geq 0$ if $f_0 < 0$, because $\alpha_n \geq 0$ and $\beta_n \geq 0$ for all $n \geq 0$. Moreover, if $f_0 = 0$, then it follows again by induction that $f_n = 0$ for all $n \geq 0$.

We now prove that f is holomorphic on a neighborhood of 0 to begin with when $f_0 \neq 0$. Since a and b are holomorphic functions at 0, there exist $u_1, u_2 \in \mathbb{R}^+$ and $v_1, v_2 \in \mathbb{R}^+$ such that for all $k \in \mathbb{N}$, we have $|\alpha_k| \leq u_1 v_1^k$ and $|\alpha_k| \leq u_2 v_2^k$. Up to taking larger numbers v_1 and v_2, we can assume that $u_1 = u_2 = 1$ and $v_1 = v_2 := v$. Moreover, we assume that $v \geq 27$. Let $d \in \mathbb{N}$ be such that $|f_0| \leq d$. Let us prove by induction on $n \in \mathbb{N}$ that for all n, we have

$$|f_n| \leq d(3v)^n\,.$$

This result holds for $n = 0$. Let us prove that it holds for $n \geq 1$ if we assume that it holds for all $k \leq n-1$. From Equation (3.2), we have

$$\begin{aligned} |f_n| &\leq d \sum_{k=0}^{n} v^{n-k} (3v)^{k/r} + d \sum_{k=0}^{n} v^{n-k} (3v)^{k/r^2} \\ &\leq 2dv^n \sum_{k=0}^{n} \left(\frac{\sqrt[r]{3}}{v^{(r-1)/r}} \right)^k = 2dv^n \frac{1}{\left(v^{(r-1)/r}\right)^n} \frac{\left(v^{(r-1)/r}\right)^{n+1} - \sqrt[r]{3}^{n+1}}{v^{(r-1)/r} - \sqrt[r]{3}} \\ &\leq 2dv^n \frac{v^{(r-1)/r}}{v^{(r-1)/r} - \sqrt[r]{3}}. \end{aligned}$$

Since $v \geq 27 \geq 3^{(r+1)/(r-1)}$, we have $v^{(r-1)/r} \geq 3\sqrt[r]{3}$ so that $2v^{(r-1)/r} \leq 3(v^{(r-1)/r} - \sqrt[r]{3})$ and $2\frac{v^{(r-1)/r}}{v^{(r-1)/r} - \sqrt[r]{3}} \leq 3 \leq 3^n$. Thus, $|f_n| \leq d(3v)^n$, which concludes the induction. Hence, f is holomorphic on $D(0, \varepsilon)$, the open unit disk of radius ε for a certain $\varepsilon \in (0, 1)$.

From Eq. (3.1), we shall deduce that the function f can be continued to a holomorphic function on the open unit disk $D(0,1)$. Indeed, $z \mapsto f(z^r)$ and $z \mapsto f(z^{r^2})$ are holomorphic functions on $D(0, \sqrt[r]{\varepsilon})$ and on $D(0, \sqrt[r^2]{\varepsilon})$ respectively. By Eq. (3.1) and the fact that $a(z)$ and $b(z)$ are holomorphic on $D(0,1)$ (at least), this implies that $f(z)$ can be continued to a holomorphic function on $D(0, \sqrt[r]{\varepsilon})$. We repeat this process by replacing ε by $\sqrt[r]{\varepsilon}$ etc. Since $\sqrt[r^n]{\varepsilon} \to 1$ when $n \to +\infty$, we obtain that f can be analytically continued to a holomorphic function on $D(0,1)$. It follows that the radius of convergence of the series $\sum_{n\geq 0} f_n z^n$ is ≥ 1.

From [17, Theorem 4.3] (see also [6]) either the Taylor expansion of f has a radius of convergence equal to 1, or f is a rational function. Let us now prove that this radius is equal to 1 when $f_0 > 0$. We first observe that there exists an integer $c \geq 1$ such that $f_c > 0$. Indeed, taking the terms corresponding to $k = 0$ in the recurrence relation (3.2), we have $f_c \geq (\alpha_c + \beta_c) f_0$ for any $c \geq 1$. Since a and b are not both constant, there exists at least one value of $c \geq 1$ such that $(\alpha_c + \beta_c) f_0 > 0$, so that $f_c > 0$.

If $\beta_0 = 0$ (hence $\alpha_0 = 1$), we take $n = cr^{m+1}$ in (3.2) for any $m \geq 0$, and only the term for $k = n$ in the two sums: we obtain $f_{cr^{m+1}} \geq f_{cr^m}$ so that $f_{cr^m} \geq f_c > 0$ for all $m \geq 0$ and thus the radius of convergence of $\sum_{n\geq 0} f_n z^n$ must be ≤ 1.

We now assume that $\beta_0 \neq 0$. We take $n = cr^{m+2}$ in (3.2) for any $m \geq 0$, and only the term for $k = n$ in each of the two sums: we have

$$f_{cr^{m+2}} \geq \alpha_0 f_{cr^{m+1}} + \beta_0 f_{cr^m}.$$

By induction on $m \geq 0$, we deduce that $f_{cr^m} \geq g_m$ for all $m \geq 0$ where the sequence $(g_m)_{m\geq 0}$ is defined by the Fibonacci-like recurrence relation $g_{m+2} = \alpha_0 g_{m+1} + \beta_0 g_m$ and $g_0 = f_c > 0$, $g_1 = f_{cr} \geq 0$. There exists γ_1 and γ_2 such that $g_m = \gamma_1 \delta_1^m + \gamma_2 \delta_2^m$ where δ_1, δ_2 are the roots of the characteristic polynomial $X^2 - \alpha_0 X - \beta_0$. Since $\alpha_0 + \beta_0 = 1$, we have $\delta_1 = 1$ and $\delta_2 = -\beta_0$. Moreover, γ_1 and γ_2 are determined by the equations $\gamma_1 + \gamma_2 = g_0 = f_c$ and $\gamma_1 - \beta_0 \gamma_2 = g_1 = f_{cr}$. We deduce that, for all $m \geq 0$,

$$g_m = \frac{\beta_0 f_c + f_{cr}}{1+\beta_0} + \frac{f_c - f_{cr}}{1+\beta_0}(-\beta_0)^m.$$

If $f_c \geq f_{cr}$, then $f_{cr^m} \geq g_m \geq \frac{\beta_0 f_c + f_{cr}}{1+\beta_0} > 0$ for all even integers $m \geq 0$. If $f_c < f_{cr}$, then $f_{cr^m} \geq g_m \geq \frac{\beta_0 f_c + f_c r}{1+\beta_0} > 0$ for all odd integers $m \geq 1$. Hence, in both cases, the radius of convergence of $\sum_{n\geq 0} f_n z^n$ must be ≤ 1. $\square$

Remark 3.3. • In the above proof, the assumption that $a(z)$ and $b(z)$ are not both constant is only used to ensure that the radius of convergence of the Taylor expansion of $f(z)$ at $z = 0$ is ≤ 1, while it had already been proved to be ≥ 1. Let us now keep all our assumptions on $a(z)$ and $b(z)$, except that we assume $a(z)$ and $b(z)$ to be constant functions, denoted by $a \geq 0$ and $b \geq 0$ respectively. Then the solutions $f(z) := \sum_{n=0}^{\infty} f_n z^n$ of the equation $y(z) = ay(z^r) + by(z^{r^2})$ are reduced to the constant functions. Indeed, the function $f(z) = 0$ is a solution, and if we seek a non-zero solution, then we have seen in the proof of Proposition 3.1 that we must have $a + b = 1$ and $f_0 \neq 0$. We now assume that $f_0 > 0$, the case $f_0 < 0$ being dealt in a similar way. We know that f is holomorphic at the origin with radius of convergence of its Taylor expansion ≥ 1, and let us assume that f is not a constant. Since $f_n \geq 0$ for all $n \geq 0$ and $f_m > 0$ for at least one $m \geq 1$, f is increasing on $[0,1)$. Thus for any $z \in (0,1)$, we have $f(z^r) < f(z)$. But $f(z) = af(z^r) + bf(z^{r^2}) \leq (a+b)f(z^r) = f(z^r)$, contradiction.

Without loss of generality, we assume from now on $f_0 > 0$ (up to replacing $f(z)$ by $-f(z)$ in Eq. (3.1) if $f_0 < 0$). We now set

$$\mu(z) := f(z)/f(z^r).$$

This is a well-defined function for $z \in [0,1)$ because $f(z) > 0$ on $[0,1)$. We have $\mu(z) \geq 1$ for all $z \in [0,1)$ because f is increasing on $[0,1)$. Moreover, $\mu(z)$ is independent of the choice of $f_0 > 0$ and it is C^∞ on $[0,1)$. (The radius of convergence of the Taylor expansion of μ at the origin is $|\xi^{1/r}|$ where ξ is a zero of minimal modulus of f in $D(0,1)$ such that $f(\xi^{1/r}) \neq 0$; if there is no such ξ, then μ is analytic in $D(0,1)$.) From (3.1), we have

$$\mu(z) = a(z) + \frac{b(z)}{\mu(z^r)}. \tag{3.3}$$

This non-linear Mahler equation is crucial to understand the analytic properties of $\mu(z)$ for $z \in [0,1)$, which are stated in the next lemmas.

Lemma 3.4. *The limits of $\mu(z)$ and of $\mu'(z)$ when $z \to 1^-$ exist. They are denoted by μ_1 and μ_1' respectively: we have $\mu_1 > 1$ and $\mu_1' > 0$.*

Proof. Let $\ell := \liminf_{z\to 1^-} \mu(z)$ and $L := \limsup_{z\to 1^-} \mu(z)$. Since $\mu(z) \geq 1$, we have $1 \leq \ell \leq L$. From (3.3), if $L = +\infty$ then $\ell = a(1) < +\infty$ and thus $L = a(1) + b(1)/\ell < +\infty$, which is a contradiction. Therefore, $L < +\infty$ and $\ell < +\infty$. From (3.3), we obtain

$$\ell = a(1) + \frac{b(1)}{L} \quad \text{and} \quad L = a(1) + \frac{b(1)}{\ell}$$

so that $a(1)(\ell - L) = 0$. Since $a(1) > 0$ (because $a(z) \in \mathbb{R}^+[[z]] \setminus \{0\}$), we thus have $\ell = L$ and the limit of $\mu(z)$ when $z \to 1^-$ exists. Moreover, ℓ is a solution of $X^2 - a(1)X - b(1) = 0$. This equation has only one nonnegative solution, hence

$$\mu_1 = \frac{a(1) + \sqrt{a(1)^2 + 4b(1)}}{2} > 0.$$

The function $\mu'(z)$ satisfies

$$\mu'(z) = \rho(z) + \sigma(z)\mu'(z^r) \quad \text{where} \quad \rho(z) = a'(z) + \frac{b'(z)}{\mu(z^r)}, \quad \sigma(z) = -rz^{r-1}\frac{b(z)}{\mu(z^r)^2}. \tag{3.4}$$

We have

$$\mu'(z) = \sum_{n=0}^{N} (\sigma\sigma_r \ldots \sigma_{r^{n-1}}\rho_{r^n})(z) + \mu'\left(z^{r^{N+1}}\right) \prod_{n=0}^{N} \sigma_{r^n}(z)$$

where $\rho_k(z) := \rho(z^k)$ and $\sigma_k(z) := \sigma(z^k)$. Since we assume (H6), i.e. $|rz^{r-1}b(z)| < a(z^r)^2$ for all $z \in [0,1]$, we have for all $z \in [0,1]$

$$|rz^{r-1}b(z)| < a(z^r)^2 \leq \mu(z^r)^2,$$

the last inequality coming from the fact $0 \leq a(z) \leq \mu(z)$ by (3.3). Thus, for all $z \in [0,1]$, $|\sigma(z)| < 1$. Since σ is a continuous function on $[0,1]$, there exists $c \in (0,1)$ such that $|\sigma(z)| \leq c$. Therefore, for all $z \in [0,1]$,

$$\lim_{N\to+\infty} \prod_{n=0}^{N} \sigma_{r^n}(z) = 0.$$

Moreover, $\lim_{z\to 0} \mu'(z) = f_1/f_0 < +\infty$. Thus, for all $z \in [0,1)$,

$$\mu'(z) = \sum_{n=0}^{+\infty} m_n(z) \quad \text{where} \quad m_n := \sigma\sigma_r \ldots \sigma_{r^{n-1}}\rho_{r^n}. \tag{3.5}$$

We know that μ' is a continuous function on $[0,1)$. Moreover, there exists $d \in \mathbb{R}^+$ such that $|\rho(z)| \leq d$ for all $z \in [0,1]$. Thus, the series given in (3.5) is normally convergent on $[0,1]$ because $\sum_{n=0}^{+\infty} ||m_n||_\infty \leq d \sum_{n=0}^{+\infty} c^n < +\infty$. Therefore, the limit of $\mu'(z)$ when $z \to 1^-$ exists and

$$\lim_{z\to 1^-} \mu'(z) = \sum_{n=0}^{+\infty} m_n(1) = \left(a'(1) + \frac{b'(1)}{\mu_1}\right) \sum_{n=0}^{+\infty} \left(-\frac{rb(1)}{\mu_1^2}\right)^n = \mu_1 \frac{a'(1)\mu_1 + b'(1)}{\mu_1^2 + rb(1)}.$$

Moreover, since a and b are in $\mathbb{R}^+[[z]]$ and are not both constants, we have either $a(1) > a(0)$ or $b(1) > b(0)$, and either $a'(1) > 0$ or $b'(1) > 0$. It follows that

$$\mu_1 = \frac{a(1) + \sqrt{a(1)^2 + 4b(1)}}{2} > \frac{a(0) + \sqrt{a(0)^2 + 4b(0)}}{2}.$$

Hence, since $a(0) + b(0) = 1$ (and in particular $a(0) \leq 1$), we have that $\sqrt{a(0)^2 + 4b(0)} = |a(0) - 2| = 2 - a(0)$. Consequently, $\mu_1 > \frac{a(0)+2-a(0)}{2} = 1$. Finally, $\mu_1' > 0$ because

$$\mu_1' = \mu_1 \frac{a'(1)\mu_1 + b'(1)}{\mu_1^2 + rb(1)} > 0.$$

This completes the proof. □

Remark 3.5. • The existence and the computation of the value of the limit of $\mu(z)$ as $z \to 1^-$ can be obtained from Theorem 1 of [5], which can be applied here because the roots of $X^2 - a(1)X - b(1)$ have distinct modulus. We then have

$$\frac{f(z)}{f(z^r)} = \frac{C(z)(1-z^r)^{\log_r(\mu_1)}}{C(z^r)(1-z)^{\log_r(\mu_1)}}(1+o(1)) = \frac{(1-z^r)^{\log_r(\mu_1)}}{(1-z)^{\log_r(\mu_1)}}(1+o(1)) \to \mu_1, \quad z \to 1^-$$

because $C(z) = C(z^r)$. Our proof does not use that theorem.

• The existence and the computation of the value of the limit of $\mu'(z)$ as $z \to 1^-$ could be much simplified if we knew *a priori* that $\mu'(z) \geq 0$ in an interval $[1-\varepsilon, 1)$. Indeed, let $\ell' = \liminf_{z\to 1^-} \mu'(z)$ and $L' = \liminf_{z\to 1^-} \mu'(z)$. Taking lim sup in (3.4), we obtain $L' = \rho(1)+\sigma(1)\ell' \leq \rho(1)$ because $\sigma(1) \leq 0$ and $\ell' \geq 0$. Hence L' is finite and taking lim inf in (3.4) we obtain that $\ell' = \rho(1) + \sigma(1)L'$. It follows that $(\sigma(1) + 1)(L' - \ell') = 0$. But $\sigma(1) = -rb(1)/\mu(1)^2 \neq -1$ by (H6), so that $L' = \ell'$.

Lemma 3.6. *Under Assumptions (H1)–(H6), for any $\beta \in (0, 2]$ such that $b(1)r^\beta < \mu_1^2$, we have ([3])*

$$\mu''(e^{-t}) = \mathcal{O}(t^{\beta-2}), \quad t \to 0^+.$$

Proof. Let $\delta := 1 + b(1)r^\beta/\mu_1^2$. By construction, we have $1 < \delta < 2$. Let $d > 0$ be such that $(1 + d)\delta < 2 + d$. Since $z \mapsto 2 + z - \delta(1 + z)$ takes a positive value at $z = 0$, there exists such a d. Differentiating both sides of (3.4) gives

$$\mu''(z) = A(z) + B(z)\mu''(z^r)$$

where $A(z)$ is a continuous function on $[0, 1]$ and $B(z) = -r^2z^{2(r-1)}b(z)/\mu(z^r)^2$. Let $M \in \mathbb{R}^+$ be such that $|A(z)| \leq M$ for all $z \in [0, 1]$. Let $z = e^{-t}$ with $t \in (0, +\infty)$. Thus, from the previous equation,

$$\mu''(e^{-t}) = \mathcal{A}(t) + \mathcal{B}(t)\mu''(e^{-rt}) \tag{3.6}$$

where $\mathcal{A}(t) := A(e^{-t})$ and $\mathcal{B}(t) := B(e^{-t})$.

Since

$$\mathcal{B}(t) = -r^2e^{-2(r-1)t}\frac{b(e^{-t})}{\mu(e^{-rt})^2} = -\frac{r^2b(1)}{\mu_1^2} + \mathcal{O}(t), \quad t \to 0^+,$$

there exists $0 < \varepsilon < 1$ such that

$$\forall t \in (0, \varepsilon), \quad |\mathcal{B}(t)| < \frac{r^2b(1)}{\mu_1^2}(1 + d).$$

Let $t_0 \in (0, \varepsilon)$. The function $|\mu''(e^{-t})|/t^{\beta-2}$ is non-negative and continuous on $[t_0, \varepsilon]$. We denote by m its maximum on $[t_0, \varepsilon]$. We choose

$$C > \max\Big(m, \frac{M}{2 + d - \delta(1 + d)}\Big).$$

We prove by induction on $k \geq 0$ that

$$\forall t \in \Big[\frac{t_0}{r^k}, \varepsilon\Big), \quad \big|\mu''(e^{-t})\big| \leq Ct^{\beta-2}.$$

The case $k = 0$ is an immediate consequence of the choice of C. Assume it is true for $k = k_0 \geq 0$, then if $t \in \big[\frac{t_0}{r^{k_0+1}}, \frac{t_0}{r^{k_0}}\big]$, we have by Eq. (3.6)

$$\begin{aligned} |\mu''(e^{-t})| &\leq M + |\mathcal{B}(t)|C(rt)^{\beta-2} \\ &\leq M + \tfrac{b(1)}{\mu_1^2}r^\beta(1 + d)Ct^{\beta-2} = M + C(\delta - 1)(1 + d)t^{\beta-2}. \end{aligned}$$

However, since $0 \leq M \leq C(2 + d - \delta(1 + d))$ and $0 \leq t^{2-\beta} \leq 1$, we have

$$t^{2-\beta}M \leq C(2 + d - \delta(1 + d)) = C(1 + (1 - \delta)(1 + d))$$

[3] In this Lemma, the real assumption used is $0 < |b(1)|r^\beta < \mu_1^2$, which is equivalent to $b(1)r^\beta < \mu_1^2$ in our situation; this could be useful in other contexts.

so that $M + C(\delta - 1)(1 + d)t^{\beta-2} \le Ct^{\beta-2}$ and

$$\left|\mu''(e^{-t})\right| \le Ct^{\beta-2},$$

which concludes the induction and proves the lemma. $\square$

Corollary 3.7. *Under Assumptions (H1)–(H6), we have*

$$\mu'(e^{-t}) = \mu_1' + \mathcal{O}(t^{\min(\alpha-1,1)}) \quad \textit{and} \quad \mu(e^{-t}) = \mu_1 - t\mu_1' + \mathcal{O}(t^{\min(\alpha,2)}), \quad t \to 0^+$$

where $\alpha > 1$ is defined by $r^\alpha b(1) = a(1)^2$.

Proof. We can apply Lemma 3.6 with $\beta := \min(\alpha, 2) \in (1, 2]$ because $r^\alpha b(1) < \mu_1^2$. Hence,

$$\mu''(e^{-t}) = \mathcal{O}\left(t^{\beta-2}\right), \quad t \to 0^+.$$

With $t = \ln(1/z) = 1 - z + o(1 - z)$ when $z \to 1^-$, we deduce that

$$\mu''(z) = \mathcal{O}\left((1 - z)^{\beta-2}\right), \quad z \to 1^-.$$

Because $\beta > 1$, we can integrate twice over the interval $[z, 1]$ and we obtain

$$\mu'(z) = \mu_1' + \mathcal{O}\left((1 - z)^{\beta-1}\right), \quad \mu(z) = \mu_1 + \mu_1'(z - 1) + \mathcal{O}\left((1 - z)^{\beta}\right), \quad z \to 1^-$$

We now make the change of variables $z = e^{-t}$: we have

$$\mu'(e^{-t}) = \mu_1' + \mathcal{O}(t^{\beta-1}), \quad \mu(e^{-t}) = \mu_1 + \mu_1'(-t + \mathcal{O}(t^2)) + \mathcal{O}\left(t^{\beta}\right), \quad t \to 0^+$$

and the result follows. $\square$

3.2. **Completion of the proof of Theorem 1.4.** In order to obtain the precise asymptotic behavior of $f(z)$ as $z \to 1^-$, we consider the two Mellin transforms:

$$\mathcal{F}(s) := \int_0^{+\infty} \ln(f(e^{-t}))t^{s-1}\mathrm{d}t$$

and

$$\mathcal{M}(s) := \int_0^{+\infty} \ln(\mu(e^{-t}))t^{s-1}\mathrm{d}t.$$

The integrands are well defined because $f(e^{-t}) > 0$ and $\mu(e^{-t}) > 0$ on $(0, +\infty)$. These integrals are convergent for $s \in \mathcal{H}_0$ because:

1) $f(z) = 1 + \mathcal{O}(z)$ and $\mu(z) = 1 + \mathcal{O}(z)$ as $z \to 0^+$, which ensures the convergence of both integrals at $t = +\infty$ because $\ln(f(e^{-t}))$ and $\ln(\mu(e^{-t}))$ are both $\mathcal{O}(e^{-t})$ when $t \to +\infty$.

2) $\mu(1)$ is finite and, since the equation $X^2 - a(1)X - b(1)$ has two distinct roots with $\mu_1 > 1$ the one of largest absolute value, Theorem 1 of [5] implies that $f(z) = \mathcal{O}\big((1 - z)^{-\log_r(\mu_1)}\big)$ when $z \to 1^-$. Hence $\ln(\mu(e^{-t})) = \mathcal{O}(1)$ and $\ln(f(e^{-t})) = \mathcal{O}(\ln(1/t))$ when $t \to 0^+$. This ensures the convergence of both integrals at $t = 0$.

Therefore both $\mathcal{F}(s)$ and $\mathcal{M}(s)$ define analytic functions on the half plane $\mathcal{H}_0$. Our goal is to meromorphically continue $\mathcal{F}$ and $\mathcal{M}$ to a larger domain. By definition of $\mu(z)$, we trivially have

$$(1 - r^{-s})\mathcal{F}(s) = \mathcal{M}(s), \quad \Re(s) > 0.$$

We define

$$\widetilde{\mu}(t) := \ln(\mu(e^{-t})) - \ln(\mu_1)e^{-\lambda t}, \tag{3.7}$$

where

$$\lambda := \frac{\mu_1'}{\mu_1 \ln(\mu_1)} > 0$$

by Lemma 3.4.

Lemma 3.8. *Under Assumptions (H1)–(H6), we have*

$$\widetilde{\mu}(t) = \mathcal{O}(e^{-\min(\lambda,1)t}), \quad t \to +\infty.$$

and

$$\widetilde{\mu}(t) = \mathcal{O}(t^{\min(\alpha,2)}), \quad t \to 0^+.$$

Proof. Since $\ln(\mu(e^{-t})) = \mathcal{O}(e^{-t})$, we deduce from (3.7) that $\widetilde{\mu}(t) = \mathcal{O}(e^{-\min(\lambda,1)t})$ when $t \to +\infty$. By Corollary 3.7, we have

$$\widetilde{\mu}(t) \underset{t\to 0^+}{=} (\lambda \ln(\mu_1) - \mu_1'/\mu_1)t + \mathcal{O}(t^2) + \mathcal{O}(t^{\min(\alpha,2)}) = \mathcal{O}(t^{\min(\alpha,2)})$$

because $\lambda \ln(\mu_1) - \mu_1'/\mu_1 = 0$. □

Now, we have for $s \in \mathcal{H}_0$:

$$\mathcal{M}(s) = \widetilde{\mathcal{M}}(s) + \ln(\mu_1)\lambda^{-s}\Gamma(s) \tag{3.8}$$

where

$$\widetilde{\mathcal{M}}(s) := \int_0^{+\infty} \widetilde{\mu}(t)t^{s-1}\mathrm{d}t.$$

Under Assumptions (H1)–(H6), Lemma 3.8 implies that $\widetilde{\mathcal{M}}(s)$ converges for s in $\mathcal{H}_{\min(\alpha,2)}$ on which it is an analytic function. Consequently, $\mathcal{M}(s)$ and $\mathcal{F}(s)$ can both be meromorphically extended to $\mathcal{H}_{\min(\alpha,2)}$. Since $\widetilde{\mathcal{M}}(s)$ has no singularities in $\mathcal{H}_{\min(\alpha,2)}$, the singularities of $\mathcal{M}(s)$ are those of $\ln(\mu_1)\lambda^{-s}\Gamma(s)$. As $\mathcal{F}(s) = \frac{\mathcal{M}(s)}{1-r^{-s}}$, the singularities of $\mathcal{F}$ in $\mathcal{H}_{\min(\alpha,2)}$ are:

(1) a double pole at $s = 0$ coming from the pole of Γ and the fact that $1 - r^{-s}$ vanishes at $s = 0$,
(2) simple poles at $s = 2\pi ki/\ln(r)$ for $k \in \mathbb{Z} \setminus \{0\}$, where $1 - r^{-s}$ vanishes,
(3) a simple pole at $s = -1$, which is a pole of Γ.

Lemma 3.9. *Let $x \in \mathbb{R}$ and $y \in \mathbb{R}^*$. Under (H1)–(H6), the functions $\mathcal{M}(x + yi)$ and $\widetilde{\mathcal{M}}(x + yi)$ are both $\mathcal{O}_x(|y|^{-2})$ when $y \to \pm\infty$ for any $x > -\min(\alpha, 2)$.*

Proof. The proof is similar to the one given in [7, p. 9]. Under (H1)–(H6), by lemma (3.8), we have that $\widetilde{\mu}(t) = \mathcal{O}(t^{\min(\alpha,2)})$ as $t \to 0^+$ and $\widetilde{\mu}(t) = \mathcal{O}(e^{-\min(\lambda,1)t})$ as $t \to +\infty$. Hence, for $x > -\min(\alpha, 2)$ and $y \in \mathbb{R}^\star$, an integration by parts gives

$$\widetilde{\mathcal{M}}(x+yi) = \int_0^{+\infty} \widetilde{\mu}(t) t^{x+yi-1} \mathrm{d}t$$
$$= \left[\widetilde{\mu}(t) \frac{t^{x+yi}}{x+yi}\right]_0^{+\infty} - \int_0^{+\infty} \widetilde{\mu}'(t) \frac{t^{x+yi}}{x+yi} \mathrm{d}t = -\frac{1}{x+yi} \int_0^{+\infty} \widetilde{\mu}'(t) t^{x+yi} \mathrm{d}t.$$

The second integral converges for $x > -\min(\alpha, 2)$ because $\widetilde{\mu}'(t) = -e^{-t}\mu'(e^{-t})/\mu(e^{-t}) + \lambda \ln(\mu_1) e^{-\lambda t} = \mathcal{O}(t^{\min(\alpha-1,1)})$ as $t \to 0^+$ and $\widetilde{\mu}'(t) = \mathcal{O}(e^{-\min(\lambda,1)t})$ as $t \to +\infty$. We can perform a second integration by parts for $x > -\min(\alpha, 2)$:

$$\widetilde{\mathcal{M}}(x+yi) = \left[-\frac{\widetilde{\mu}'(t) t^{x+yi+1}}{(x+yi)(x+1+yi)}\right]_0^{+\infty} + \int_0^{+\infty} \frac{\widetilde{\mu}''(t) t^{x+yi+1}}{(x+yi)(x+1+yi)} \mathrm{d}t$$
$$= \frac{1}{(x+yi)(x+1+yi)} \int_0^{+\infty} \widetilde{\mu}''(t) t^{x+yi+1} \mathrm{d}t.$$

The second integral converges (absolutely) because $\widetilde{\mu}''(t) = \mathcal{O}(t^{\min(\alpha-2,0)})$ as $t \to 0^+$ and $\widetilde{\mu}''(t) = \mathcal{O}(e^{-\min(\lambda,1)t})$ as $t \to +\infty$. Therefore, for all $x > -\min(\alpha, 2)$ and $y \neq 0$, we have

$$\left|\widetilde{\mathcal{M}}(x+yi)\right| \leq \frac{1}{|y|^2} \int_0^{+\infty} |\widetilde{\mu}''(t)| t^{x+1} \mathrm{d}t = \mathcal{O}_x(1/|y|^2), \quad y \to \pm\infty.$$

Moreover, by Stirling's formula and $\lambda > 0$, $\lambda^{-(x+yi)}\Gamma(x+yi) = \mathcal{O}_x(e^{-\pi y/2})$ when $y \to \pm\infty$. From Eq. (3.8), we obtain the desired bound for $\mathcal{M}(x+yi)$ itself. $\square$

Proof of Theorem 1.4. First, we look at the double pole of $\mathcal{F}$ at $s = 0$. We have

$$\Gamma(s) = \frac{1}{s} - \gamma + \mathcal{O}(s) \quad \text{and} \quad \frac{1}{1 - r^{-s}} = \frac{1}{\ln(r)} \frac{1}{s} + \frac{1}{2} + \mathcal{O}(s).$$

Thus,

$$\mathcal{F}(s) = \frac{\ln(\mu_1)}{\ln(r)} \frac{1}{s^2} + \frac{c_0}{s} + \mathcal{O}(1),$$

where

$$c_0 = \frac{\kappa_0}{\ln(r)} + \frac{\ln(\mu_1)}{2}$$

and κ_0 is the constant term in the Laurent expansion of $\mathcal{M}(s)$ at $s = 0$. From the "Mellin dictionary" in [12, pp 762–765], this contributes to the term $\log_r(\mu_1) \ln(1/s) + c_0$ in the expansion (1.9).

The simple pole at $s = 2\pi k i / \ln(r)$ of $\mathcal{F}$ for $k \in \mathbb{Z} \setminus \{0\}$ has residue

$$\frac{1}{\ln(r)} \mathcal{M}\Big(\frac{2\pi k i}{\ln(r)}\Big).$$

Thus, for the simple pole at $2\pi k i / \ln(r)$, the dictionary provides the term

$$M_k(s) := \frac{1}{\ln(r)} \mathcal{M}\Big(\frac{2\pi k i}{\ln(r)}\Big) s^{-\frac{2\pi k i}{\ln(r)}}.$$

in the expansion (1.9). The series $\sum_{k\in\mathbb{Z}\setminus\{0\}} M_k(s)$ converges because, by Lemma 3.9 with $x = 0$, $\mathcal{M}(yi) = \mathcal{O}\left(|y|^{-2}\right)$ as $y \to \pm\infty$.

Finally, the simple pole of $\mathcal{F}$ at $s = -1$ has residue $c_1 := \lambda \ln(\mu_1)/(r-1) = \mu_1'/((r-1)\mu_1)$, and this contributes to the term $c_1 s$ in the expansion (1.9).

Adding all these contributions proves the expansion (1.9) of $\ln(f(e^{-s}))$ as $s \to 0^+$. The error term $\mathcal{O}_\varepsilon(s^{\min(\alpha,2)-\varepsilon})$ is a consequence of the inverse Mellin transform properties (see [12, p. 764, (47)]): for any $c > -\min(2, \alpha)$ and any $s > 0$, we have

$$\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} s^{-z}\widetilde{\mathcal{M}}(z)\mathrm{d}z = \mathcal{O}_c(s^{-c})$$

because $\widetilde{M}(z)$ is analytic in $\mathcal{H}_{\min(\alpha,2)}$ and its modulus decays like $1/\Im(z)^2$ as $\Im(z) \to \pm\infty$ by Lemma 3.9. □

4. The generating function of the Baum-Sweet sequence

The Baum-Sweet sequence $(b_n)_{n\geq 0} \in \{0,1\}^{\mathbb{N}}$ is a celebrated automatic sequence introduced in [2] and defined recursively by the relations $b_0 = 1$, $b_{2n+1} = b_n$, $b_{4n} = b_n$ and $b_{4n+2} = 0$. The generating function $S(z) := \sum_{n=0}^{\infty} b_n z^n$ is a solution of the 2-Mahler equation of order 2:

$$y(z) = zy(z^2) + y(z^4). \tag{4.1}$$

We have $S(z) = 1 + z + z^3 + z^4 + z^7 + \ldots \in 1 + z\mathbb{R}^+[[z]]$, so that $\mu(z) := S(z)/S(z^2) = 1 + z - z^2 + 2z^4 - 3z^6 + z^8 - 4z^{10} + 6z^{12} + \ldots$ is holomorphic at $z = 0$, ≥ 1 on $[0, 1)$ and in $C^\infty([0, 1))$. ([4]) Eq. 4.1 satisfies Assumptions (H1)–(H5) but not (H6). From the proof of Theorem 1.4, it follows that $\mu_1 := \lim_{z\to 1^-} \mu(z)$ exists and is equal to the largest root of $X^2 - X - 1$, i.e. $\mu_1 = \frac{1}{2}(\sqrt{5} + 1) > 1$. Though (H6) does not hold, we shall now prove by a direct computation that

$$\mu_1' := \lim_{z\to 1^-} \mu'(z) = \frac{1}{1+\delta} > 0, \tag{4.2}$$

where $\delta := 2/\mu_1^2 \approx 0.76$, and that there exists $\alpha \in (1, 2]$ such that

$$\mu''(e^{-t}) = \mathcal{O}(t^{\alpha-2}). \tag{4.3}$$

We fix $z_0 \in (0, 1)$; its value is irrelevant in the sequel. On $[z_0, z_0^{1/2}]$, we have $u_0 := \min \mu'(z) \leq \mu'(z) \leq \max \mu'(z) =: v_0$, where min and max are taken on this interval. We are going to define by induction two particular sequences $(u_k)_{k\geq 0}$ and $(v_k)_{k\geq 0}$ such that for all $k \geq 0$ and all $z \in [z_0^{1/2^k}, z_0^{1/2^{k+1}}]$ we have $u_k \leq \mu'(z) \leq v_k$.

This is already done for $k = 0$ and let us assume u_k and v_k are defined for $k = n$. Let $x_n, y_n \in [z_0^{1/2^n}, z_0^{1/2^{n+1}}]$ be such that $\mu(x_n)$, respectively $\mu(y_n)$, is the minimal, respectively the maximal value taken by μ on this interval. Since $\mu \geq 1$, we have in particular $\mu(x_n) \neq$

[4]From the functional equation $\mu(z) - z = 1/\mu(z^2)$, it is clear that $\mu(z) - z = \sum_{n=0}^{\infty} m_n z^{2n}$. It seems that for all $n \geq 0$ $(-1)^n m_n \geq 0$ (and possibly > 0); this property is not essential for us. Since we prove that $\lim_{z\to 1^-} \mu(z)$ exists and is finite, the sign of m_n cannot eventually always be the same.

0 and $\mu(y_n) \neq 0$. Note that clearly, $\lim_{n\to+\infty} \mu(x_n) = \lim_{n\to+\infty} \mu(y_n) = \mu_1$. For all $z \in [z_0^{1/2^n}, z_0^{1/2^{n+1}}]$, we have

$$0 < t_n := \frac{2z_0^{1/2^{n+1}}}{\mu(y_n)^2} \leq \frac{2z^{1/2}}{\mu(z)^2} \leq \frac{2z_0^{1/2^{n+2}}}{\mu(x_n)^2} =: s_n$$

and $u_n \leq \mu'(z) \leq v_n$. Let

$$\alpha_n := \begin{cases} s_n & \text{if } v_n \geq 0 \\ t_n & \text{if } v_n < 0 \end{cases} \qquad \text{and} \quad \beta_n := \begin{cases} t_n & \text{if } u_n \geq 0 \\ s_n & \text{if } u_n < 0. \end{cases}$$

By construction, we have

$$\beta_n u_n \leq \frac{2z^{1/2}}{\mu(z)^2}\mu'(z) \leq \alpha_n v_n$$

for all $z \in [z_0^{1/2^n}, z_0^{1/2^{n+1}}]$. Now, from the functional equation (4.1) for $S(z)$, we deduce that $\mu'(z^{1/2}) = 1 - \frac{2z^{1/2}}{\mu(z)^2}\mu'(z)$. Hence, for all $z \in [z_0^{1/2^n}, z_0^{1/2^{n+1}}]$,

$$u_{n+1} \leq \mu'(z^{1/2}) \leq v_{n+1} \qquad \text{where} \qquad \begin{cases} u_{n+1} := 1 - \alpha_n v_n \\ v_{n+1} := 1 - \beta_n u_n. \end{cases}$$

In other words, for all $z \in [z_0^{1/2^{n+1}}, z_0^{1/2^{n+2}}]$, we have $u_{n+1} \leq \mu'(z) \leq v_{n+1}$. This completes the recursive definition of the sequences $(u_k)_{k\geq 0}$ and $(v_k)_{k\geq 0}$.

We shall now prove that whatever is the chosen value for $z_0 \in (0,1)$, we have

$$\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} v_n = \frac{1}{1+\delta}. \tag{4.4}$$

Since $u_n \leq \mu'(z) \leq v_n$ for all $z \in [z_0^{1/2^n}, z_0^{1/2^{n+1}}]$ and $z_0^{1/2^n} \to 1$ as $n \to +\infty$, Eq. (4.2) will follow. (Note that the value of μ_1' is of course the one given by the functional equation $\mu'(z) = 1 - 2z\mu'(z^2)/\mu(z^2)^2$ when we let $z \to 1^-$ on both sides.)

Since $u_{n+1} = 1 - \alpha_n v_n$ and $v_n = 1 - \beta_{n-1} u_{n-1}$, we have

$$u_{n+1} = 1 - \alpha_n + \alpha_n \beta_{n-1} u_{n-1}. \tag{4.5}$$

Let us assume n is odd. Iterating (4.5), we have

$$u_{n+1} = u_0 \prod_{k=0}^{(n-1)/2} (\alpha_{n-2k}\beta_{n-2k-1}) + \sum_{j=0}^{(n-1)/2} \Big((1-\alpha_{n-2j}) \prod_{k=0}^{j-1} (\alpha_{n-2k}\beta_{n-2k-1})\Big).$$

Since α_n and $\beta_n \to \delta := 2/\mu_1^2 < 1$ because both t_n and $s_n \to \delta := 2/\mu_1^2$, we have

$$\lim_{n\to+\infty} \prod_{k=0}^{(n-1)/2} (\alpha_{n-2k}\beta_{n-2k-1}) = 0.$$

Moreover for any fixed $j \geq 0$,

$$\lim_{n\to+\infty}(1-\alpha_{n-2j})\prod_{k=0}^{j-1}(\alpha_{n-2k}\beta_{n-2k-1}) = (1-\delta)\delta^{2j}$$

and for all $n, j \geq 0$, we have

$$|1-\alpha_{n-2j}|\prod_{k=0}^{j-1}(\alpha_{n-2k}\beta_{n-2k-1}) < C(0.6)^j$$

(the left-hand side is even equal to 0 if $j > (n-1)/2$) because there exists $k_0 \geq 0$ (that depends on z_0 only) such that for all $k \geq k_0$, we have $0 < \alpha_k\beta_{k-1} < 0.6$. The constant C depends only on $\alpha_1, \beta_0, \ldots, \alpha_{k_0}, \beta_{k_0-1}$ and neither on j nor on n. Since $\sum_j (0.6)^j < +\infty$, we can apply Tannery's theorem (i.e. dominated convergence for series) and deduce that

$$\lim_{n\to+\infty, n\,\mathrm{odd}} u_{n+1} = (1-\delta)\sum_{j=0}^{\infty}\delta^{2j} = \frac{1}{1+\delta}.$$

We proceed similarly when n is even with minor changes, and also for the sequence $(v_n)_{n\geq 0}$ which satisfies $v_{n+1} = 1-\beta_n+\alpha_{n-1}\beta_n v_{n-1}$. We eventually obtain Eq. (4.4) and then Eq. (4.2) as expected.

Let us now prove that there exists $\alpha \in (1,2]$ such that (4.3) holds. Indeed, the equation $2^\beta = \mu_1^2$ has a unique solution $\beta_0 := 2\log_2(\mu_1) \approx 1.39 \in (1,2]$. We can thus apply Lemma 3.6 with β any fixed number in $(1,\beta_0)$. The proof of Lemma 3.6 can then be readily adapted *mutatis mutandis* and this proves (4.3). Since $\beta \in (1,2]$, we are now exactly in the same situation as in the proof of Theorem 1.4 starting from Corollary 3.7, with of course a different definition of α because (H6) does not hold (not even for $z=1$). Therefore, the same analysis enables us to deduce that the conclusions of Theorem 1.4 hold for $S(z)$ with $\alpha := \beta_0$ (because of $\varepsilon > 0$); we have in particular

$$S(z) = \frac{C(z)}{(1-z)^{\log_2(\sqrt{5}+1)-1}}(1+o(1)), \quad z\to 1^-,$$

where the function $C(z)$ is given by (1.10).

Remark 4.1. Another way to study the generating function S of the Baum-Sweet sequence is to use the identity $S(z) = zF(z^3)+G(z^3)$, shown in [16], where F and G are the functions defined in [7].

5. Proof of Theorem 1.7

Using the variation of constants method, we see that the general solution of an inhomogeneous equation of order 1 of the form $y(z) = p(z)y(z^r)+q(z)$ (where $p(z), q(z) \in \mathbb{C}(z)$) is formally given by

$$c\prod_{n=0}^{\infty}p(z^{r^n}) + \sum_{n=0}^{\infty}p(z)p(z^r)\cdots p(z^{r^{n-1}})q(z^{r^n}), \quad c \in \mathbb{C}.$$

This formula defines an analytic solution in the open unit disk $D(0,1)$ when, for instance, $p(z) \in 1 + z\mathbb{C}[[z]]$ and $q(z) \in z\mathbb{C}[[z]]$ both do not have poles in this disk. The product falls under the scope of Theorem 1.1 but when $z \to 1^-$, there does not seem to exist a general method to study the precise behavior of the series.

In the rest of this section, we recall as a starter a result of Hardy, the proof of which is similar to the proof of Theorem 1.1. Then we proceed with the proof of Theorem 1.7.

5.1. **Hardy's expansions.** A classical case is $p(z) = 1$ and $q(z) = z$: the series

$$H_r(z) := \sum_{n=0}^{\infty} z^{r^n}$$

is solution of $y(z) = y(z^r) + z$. It is also solution of the order 2 Mahler equation $z^{r-1}y(z) = (1 + z^{r-1})y(z^r) - y(z^{r^2})$, which cannot be treated by Theorem 1 of [5] because 1 is a double root of the characteristic polynomial $X^2 - 2X + 1$ of the equation. However, in [13, p. 283], Hardy showed that, for any $s > 0$,

$$H_r(e^{-s}) = \sum_{n=1}^{\infty} \frac{(-s)^n}{n!(1-r^n)} - \frac{\ln(s)}{\ln(r)} + \frac{1}{2} - \frac{\gamma}{\ln(r)} - \frac{1}{\ln(r)} \sum_{k\in\mathbb{Z}\setminus\{0\}} \Gamma\Big(\frac{2\pi ki}{\ln(r)}\Big) s^{-\frac{2\pi ki}{\ln(r)}}, \qquad (5.1)$$

which provides the exact behavior of $H_r(z)$ as $z \to 1^-$. Hardy's method might have inspired de Bruijn because to prove (5.1) Hardy first justified that, for any $a > 0$,

$$H_r(e^{-s}) = \frac{1}{2\pi i} \int_{a-\infty i}^{a+\infty i} \frac{\Gamma(z)s^{-z}}{1 - r^{-z}} dz,$$

where (2.1) is used. This is an expression similar to the integral identity in Lemma 2.1.

Eq. (5.1) follows by the residue theorem applied to the poles of the integral on the right-hand side: the poles are at 0 (double), $2\pi ki/\ln(r)$ ($k \in \mathbb{Z} \setminus \{0\}$, simple) and $k \in \mathbb{Z}_{\le -1}$ (simple). The method can be generalized to the series $H_{r,\beta}(z) := \sum_{n=0}^{\infty} \beta^n z^{r^n}$, $|\beta| \ge 1$ because

$$H_{r,\beta}(e^{-s}) = \frac{1}{2\pi i} \int_{a-\infty i}^{a+\infty i} \frac{\Gamma(z)s^{-z}}{1 - \beta r^{-z}} dz.$$

Hardy considered in detail the case $\beta = -1$ in [13, pp. 276–282]. However, his method is very specific and it cannot be applied to the series $\sum_{n=0}^{\infty} p(z)p(z^r)\cdots p(z^{r^{n-1}})q(z^{r^n})$ in general. A solution of an equation $y(z) = p(z)y(z^r) + q(z)$ is also a solution of

$$y(z) = \frac{q(z) + q(z^r)p(z)}{q(z^r)} y(z^r) - \frac{q(z)p(z^r)}{q(z^r)} y(z^{r^2}). \qquad (5.2)$$

But it seems difficult to apply Theorem 1.4 to this equation. This explains our more direct approach to the solutions of the equation $y(z) = p(z)y(z^r) + q(z)$. This approach, reflected by Theorem 1.7, works in particular for $H_{r,\beta}(z)$ for any $\beta > r$.

We conclude with the following remark. The function

$$F(z) := \sum_{n=0}^{\infty} (1 - z^{1/2^n})$$

is defined and holomorphic in $\mathbb{C} \setminus (-\infty, 0]$, and it is a solution of the Mahler equation $y(z) = y(z^2) + 1 - z$. This equation looks similar to the above equation for $H_2(z)$, but in fact it does not have a solution defined at $z = 0$ (simply because $y(0) \neq y(0) + 1$). The asymptotic expansion of $F(z)$ for $z \to 0^+$ is given in [12, p. 765, (49)]: for all $\varepsilon > 0$, we have

$$F(e^{-s}) = \frac{\ln(s)}{\ln(2)} + \frac{\gamma}{\ln(2)} + \frac{1}{2} + \frac{1}{\ln(2)} \sum_{k \in \mathbb{Z} \setminus \{0\}} \Gamma\Big(\frac{2\pi k i}{\ln(2)}\Big) s^{-\frac{2\pi k i}{\ln(2)}} + \mathcal{O}_\varepsilon(s^\varepsilon), \quad s \to +\infty.$$

Observe the similarity with (5.1) for $H_2(e^{-s})$, when $s \to 0^+$ this time.

5.2. **Proof of Theorem 1.7.** We recall that we consider an inhomogeneous Mahler equation of order 1

$$y(z) = p(z)y(z^r) + q(z) \tag{5.3}$$

where $p(z), q(z) \in \mathbb{R}(z)$ satisfy the following assumptions:

- (A1) $p(z), q(z) \in \mathbb{R}^+[[z]]$;
- (A2) $q(0) = 0$;
- (A3) $p(z)$ and $q(z)$ are defined at $z = 1$:
- (A4) $p(z)$ and $q(z)$ have no pole in $D(0, 1)$;
- (A5) $p(1) > r$;

Note that if q is a constant, it is identically equal to 0 by (A2). Hence (5.3) reduces to a Mahler equation of order 1, which is the subject of Theorem 1.1. Hence, there is no real loss of generality in the sequel in assuming that q is not a constant, and we make this assumption from now on.

a) We set $p(z) = \sum_{n=0}^\infty \alpha_n z^n$ and $q(z) = \sum_{n=0}^\infty \beta_n z^n$. Given any $f_0 \in \mathbb{C}$, Equation (5.3) has a unique solution $f(z) = \sum_{n=0}^\infty f_n z^n$ where the sequence $(f_n)_{n\ge 0}$ satisfies the recurrence relation

$$f_n = \sum_{k=0}^{n} \alpha_{n-k} \widetilde{f}_k + \beta_n \tag{5.4}$$

$$\widetilde{f}_k := \{ \begin{array}{rl} f_{k/r} & \text{if } r \text{ divides } k \\ 0 & \text{otherwise.} \end{array}$$

The case $n = 0$ reads $f_0 = \alpha_0 f_0 + \beta_0 = \alpha_0 f_0$: if $\alpha_0 = 1$, f_0 is a free parameter, while if $\alpha_0 \neq 1$, $f_0 = 0$ necessarily. For $n \geq 1$, f_n appears on left-hand side of (5.4) while on the right-hand side there are only values f_m with $m < n$; hence the sequence $(f_n)_{n\ge 0}$ is uniquely determined once the value of f_0 is fixed. By the same method used in §3.1 it can be proved that $f(z)$ is analytic in $D(0, 1)$.

b) We are in fact in a situation where the formal solution of (5.3) is an analytic one, i.e. we have

$$f(z) = f_0 \prod_{n=0}^\infty p(z^{r^n}) + \sum_{n=0}^\infty p(z)p(z^r)\cdots p(z^{r^{n-1}})q(z^{r^n}), \quad z \in D(0, 1). \tag{5.5}$$

The series converges on $D(0,1)$ and defines an analytic function because $q(0) = 0$. The product defines an analytic function of $D(0,1)$ when $p(0) = 1$, while if $p(0) \neq 1$, then necessarily $f_0 = 0$ and it is then understood that the right-hand side of (5.5) reduces to the series. It is then clear from the expression of $f(z)$ in (5.5) that the radius of convergence of $\sum_{n=0}^{\infty} f_n z^n$ is equal to 1 by positivity of the Taylor coefficients of $p(z)$ and $q(z)$.

c) Let us assume that $p(z)$ is not a constant. Then we have $p'(z) = \sum_{n=1}^{\infty} n\alpha_n z^n \in \mathbb{R}^+[[z]]$ and at least one of $n\alpha_n$ is positive (for $n = n_0 \geq 1$ say). Hence,

$$p'(1) = \lim_{x\to 1^-} \sum_{n=1}^{\infty} n\alpha_n x^{n-1} \geq \lim_{x\to 1^-} n_0\alpha_{n_0} x^{n_0-1} = n_0\alpha_{n_0} > 0.$$

If $p(z)$ is a constant, it is $> r$ by (A5) and this forces $f_0 = 0$; we shall see how to use this information in **h)** below.

d) We assume from now on that $f_0 \geq 0$. By non negativity of α_n and β_n for all $n \geq 0$, f_n is also non negative for all n. Moreover, since q is not a constant, there exists $n_1 \geq 1$ such that $\beta_{n_1} > 0$, which implies that $f_{n_1} > 0$, hence that $f(z)$ is not a constant. The function $f(z)$ is in $f_0 + z\mathbb{R}^+[[z]]$, is increasing and > 0 on $(0,1)$, all its derivatives are in $\mathbb{R}^+[[z]]$, increasing and ≥ 0 on $[0,1)$.

e) Recall that $f(z)$ is solution of the second order Mahler equation (5.2). Following [5], the associated characteristic equation is $X^2 - (p(1)+1)X + p(1) = 0$, whose solutions are 1 and $p(1) > r \geq 2$. Hence, by Theorem 1 of [5], we have

$$f(z) = \frac{C(z)}{(1-z)^{\log_r(p(1))}}(1+o(1)), \quad z \to 1^-,$$

where there exist two constants c_1, c_2 such that $0 < c_1 \leq C(z) \leq c_2 < +\infty$ for all $z \in (0,1)$. We deduce from this that, for all $z \in [1/2, 1)$,

$$\frac{c_3}{(1-z)^{\log_r(p(1))}} \leq f(z) \leq \frac{c_4}{(1-z)^{\log_r(p(1))}} \tag{5.6}$$

for some constants $c_3, c_4 > 0$. Moreover, for all $z \in [1/2, 1)$,

$$0 \leq f'(z) \leq \frac{c_5}{(1-z)^{\log_r(p(1))+1}}, \qquad 0 \leq f''(z) \leq \frac{c_6}{(1-z)^{\log_r(p(1))+2}} \tag{5.7}$$

for some constants $c_5, c_6 > 0$. Indeed, by the mean value theorem, for all $z \in [1/\sqrt{2}, 1)$, there exists $\zeta \in (z^2, z)$ such that

$$0 \leq f'(z^2) \leq f'(\zeta) = \frac{f(z) - f(z^2)}{z - z^2} \leq \frac{f(z)}{z - z^2} \leq \sqrt{2}c_4(1-z)^{-\log_r(p(1))-1}$$

and similarly, there exists $\zeta' \in (z^2, z)$ such that

$$0 \leq f''(z^2) \leq f''(\zeta') = \frac{f'(z) - f'(z^2)}{z - z^2} \leq \frac{f'(z)}{z - z^2} \leq 2c_4(1-z)^{-\log_r(p(1))-2}.$$

(We used the fact that f, f' and f'' are increasing on $[0,1)$.)

f) We define $\omega := v_0(f) \geq 0$. If $\omega \geq 1$ (i.e. if $f_0 = 0$), then $\omega = v_0(q)$. We have $f(z) = f_\omega z^\omega + \mathcal{O}(z^{\omega+1})$ with $f_\omega > 0$, and we normalize f by defining $\widehat{f}(z) = f(z)/(f_\omega z^\omega) \in 1 + z\mathbb{R}^+[[z]]$. We also define $\widehat{\mu}(z) := \widehat{f}(z)/\widehat{f}(z^r) = z^{(r-1)\omega} f(z)/f(z^r) \in 1 + z\mathbb{R}[[z]]$; for all $z \in [0,1)$, we have $\widehat{\mu}(z) \geq 1$, and moreover $\widehat{\mu}(z)$ is holomorphic at $z = 0$ and

$$\widehat{\mu}(z) = \widehat{p}(z) + \frac{\widehat{q}(z)}{f(z^r)}. \tag{5.8}$$

where $\widehat{p}(z) = z^{(r-1)\omega} p(z)$ and $\widehat{q}(z) = z^{(r-1)\omega} q(z)$.

Remark 5.1. • Note that the non-linear Mahler equation (deduced from Eq. (5.2))

$$\widehat{\mu}(z) = \Big(z^{(r-1)\omega} \frac{q(z)}{q(z^r)} + z^{(r-1)\omega} p(z) \Big) - z^{(r^2-1)\omega} \frac{q(z)p(z^r)}{q(z^r)} \cdot \frac{1}{\widehat{\mu}(z^r)} \tag{5.9}$$

and the value $\widehat{\mu}(0) = 1$ if $\omega \geq 1$ (respectively the value $\widehat{\mu}^{(\ell)}(0) = p^{(\ell)}(0) + q^{(\ell)}(0)/f_0$ if $\omega = 0$, where $\ell := v_0(q) \geq 1$) uniquely determine the coefficients of the Taylor expansion $\widehat{\mu}(z) := \sum_{n=0}^{\infty} \delta_n z^n$ (hence $\widehat{\mu}(z)$ itself in $D(0,1)$) without any reference to $f(z)$. Indeed, if $\omega \geq 1$ it follows from the fact that

$$z^{(r-1)\omega} \frac{q(z)}{q(z^r)} + z^{(r-1)\omega} p(z) \in \mathbb{R}[[z]] \quad \text{and} \quad z^{(r^2-1)\omega} \frac{q(z)p(z^r)}{q(z^r)} \in \mathbb{R}[[z]].$$

If $\omega = 0$, then $f_0 \neq 0$ by Eq. (5.5), which forces $p(0) = 1$, and $\widehat{\mu} = \mu$. From Eq. (5.8), $\widehat{\mu}(z)$ is of the form $p(z) + z^\ell \frac{\beta_\ell}{f(z^r)} + z^{\ell+1} h(z)$ with $h(z) \in \mathbb{R}[[z]]$, thus $\widehat{\mu}(z)$ and $p(z)$ have the same coefficients in their expansion from the order 0 to the order $\ell - 1$ (included) and the coefficient of order ℓ of $\widehat{\mu}$ is $\widehat{\mu}^{(\ell)}(0)/\ell!$ where $\widehat{\mu}^{(\ell)}(0) = p^{(\ell)}(0) + q^{(\ell)}(0)/f_0$. We know all the coefficients δ_k for $k \leq \ell$ and to determine a recurrence relation for δ_k with $k > \ell$, we look at the coefficient of order $k + \ell(r-1)$ in the following equation:

$$\frac{q(z^r)}{q(z)} \mu(z)\mu(z^r) = \mu(z^r)\Big(1 + \frac{p(z)q(z^r)}{q(z)}\Big) - p(z^r)\,,$$

(which follows from Eq. (5.9)).

g) For simplicity, we set $\eta := \log_r(p(1)) > 1$ (by (A5)). Since $f(z^r) \to +\infty$ when $z \to 1^-$, we deduce from (5.8) that

$$\widehat{\mu}_1 := \lim_{z \to 1^-} \widehat{\mu}(z) = \widehat{p}(1) = p(1) > r.$$

Moreover, we have

$$\widehat{\mu}'(z) = \widehat{p}'(z) + \frac{\widehat{q}'(z)}{f(z^r)} - rz^{r-1}\widehat{q}(z)\frac{f'(z^r)}{f(z^r)^2}. \tag{5.10}$$

The bounds (5.6) and (5.7) for $f(z^r)$ and $f'(z^r)$ for z close to 1 imply that as $z \to 1^-$, we have

$$\left| \frac{f'(z^r)}{f(z^r)^2} \right| \ll (1-z)^{\eta-1} \to 0$$

because $\eta > 1$. Hence from (5.10),

$$\widehat{\mu}'_1 := \lim_{z\to 1^-} \widehat{\mu}'(z) = \widehat{p}'(1) = (r-1)\omega p(1) + p'(1).$$

Moreover,

$$\widehat{\mu}''(z) = \widehat{p}''(z) + \frac{\widehat{q}''(z)}{f(z^r)} - rz^{r-1}\widehat{q}'(z)\frac{f'(z^r)}{f(z^r)^2}$$
$$- (rz^{r-1}\widehat{q}(z))'\frac{f'(z^r)}{f(z^r)^2} - (rz^{r-1})^2\widehat{q}(z)\frac{f''(z^r)}{f(z^r)^2} + 2(rz^{r-1})^2\widehat{q}(z)\frac{f'(z^r)^2}{f(z^r)^3}.$$

Since, as $z \to 1^-$,

$$\left|\frac{f'(z^r)}{f(z^r)^2}\right| \ll (1-z)^{\eta-1}, \quad \left|\frac{f'(z^r)^2}{f(z^r)^3}\right| \ll (1-z)^{\eta-2}, \quad \left|\frac{f''(z^r)}{f(z^r)^2}\right| \ll (1-z)^{\eta-2},$$

and it follows that ([5])

$$\widehat{\mu}''(z) = \widehat{p}''(1) + \mathcal{O}\big((1-z)^{\eta-2}\big), \quad z \to 1^-.$$

Since $\eta - 2 > -1$, we can integrate twice over the interval $[z,1]$, and we obtain

$$\widehat{\mu}'(z) = \widehat{p}'(1) + \widehat{p}''(1)(z-1) + \mathcal{O}\big((1-z)^{\eta-1}\big), \quad z \to 1^-.$$

and

$$\widehat{\mu}(z) = \widehat{p}(1) + \widehat{p}'(1)(z-1) + \frac{1}{2}\widehat{p}''(1)(z-1)^2 + \mathcal{O}\big((1-z)^{\eta}\big), \quad z \to 1^-.$$

With the change of variables $z = e^{-t}$, these estimates become

$$\widehat{\mu}''(e^{-t}) = \widehat{p}''(1) + \mathcal{O}\big(t^{\eta-2}\big), \quad t \to 0^+,$$
$$\widehat{\mu}'(e^{-t}) = \widehat{p}'(1) + \mathcal{O}\big(t^{\min(\eta-1,1)}\big), \quad t \to 0^+.$$

and

$$\widehat{\mu}(e^{-t}) = \widehat{p}(1) - \widehat{p}'(1)t + \mathcal{O}\big(t^{\min(\eta,2)}\big), \quad t \to 0^+.$$

They are analogous to those given in Corollary 3.7.

h) In order to obtain the asymptotic expansion of $f(z)$ as $z \to 1^-$, we consider the two Mellin transforms:

$$\widehat{\mathcal{F}}(s) := \int_0^{+\infty} \ln(\widehat{f}(e^{-t}))t^{s-1}\mathrm{d}t$$

and

$$\widehat{\mathcal{M}}(s) := \int_0^{+\infty} \ln(\widehat{\mu}(e^{-t}))t^{s-1}\mathrm{d}t.$$

The integrands are well defined because $\widehat{f}(e^{-t}) > 0$ and $\widehat{\mu}(e^{-t}) > 0$ on $(0,+\infty)$. These integrals are convergent for $s \in \mathcal{H}_0$ because:

1) $\widehat{f}(z) = 1+\mathcal{O}(z)$ and $\widehat{\mu}(z) = 1+\mathcal{O}(z)$ as $z \to 0^+$, which ensure the convergence of both integrals at $t = +\infty$ because $\ln(\widehat{f}(e^{-t}))$ and $\ln(\mu(e^{-t}))$ are both $\mathcal{O}(e^{-t})$ when $t \to +\infty$.

[5] The term $\widehat{p}''(1)$ is negligible if $\eta \le 2$ but not if $\eta > 2$.

2) $\widehat{\mu}(1)$ is finite and $\widehat{f}(z) = \mathcal{O}\big((1-z)^{-\eta}\big)$ when $z \to 1^-$. Hence $\ln(\widehat{\mu}(e^{-t})) = \mathcal{O}(1)$ and $\ln(\widehat{f}(e^{-t})) = \mathcal{O}(\ln(1/t))$ when $t \to 0^+$. This ensures the convergence of both integrals at $t = 0$.

Therefore both $\widehat{\mathcal{F}}(s)$ and $\widehat{\mathcal{M}}(s)$ define analytic functions on the half plane $\mathcal{H}_0$. Our goal is to meromorphically continue $\widehat{\mathcal{F}}(s)$ and $\widehat{\mathcal{M}}(s)$ to a larger domain. By definition of $\widehat{\mu}(z)$, we have

$$(1 - r^{-s})\widehat{\mathcal{F}}(s) = \widehat{\mathcal{M}}(s), \quad \Re(s) > 0.$$

We define

$$\widehat{\lambda} := \frac{\widehat{\mu}_1'}{\widehat{\mu}_1 \ln(\widehat{\mu}_1)} = \frac{\widehat{p}'(1)}{\widehat{p}(1)\ln(\widehat{p}(1))}.$$

We have $\widehat{p}'(1) = (r-1)\omega p(1) + p'(1) > 0$: indeed, note that if $p(z)$ is not a constant, then $\widehat{p}'(1) \geq r(r-1)\omega + p'(1) \geq p'(1) > 0$ (see **c)** above), while if $p(z)$ is a constant then necessarily $f_0 = 0$, so that $\omega \geq 1$ and thus $\widehat{p}'(1) = (r-1)\omega p(1) > r(r-1) > 0$. Moreover, $\widehat{p}(1) = p(1) > 2$, so that $\widehat{\lambda} > 0$.

We set

$$\widetilde{\widehat{\mu}}(t) := \ln(\widehat{\mu}(e^{-t})) - \ln(p(1))e^{-\widehat{\lambda}t}.$$

We have

$$\widetilde{\widehat{\mu}}(t) = \mathcal{O}(e^{-\min(\widehat{\lambda},1)t}), \quad t \to +\infty \quad \text{and} \quad \widetilde{\widehat{\mu}}(t) = \mathcal{O}(t^{\min(\eta,2)}), \quad t \to 0^+.$$

Now, we have

$$\widehat{\mathcal{M}}(s) = \widetilde{\widehat{\mathcal{M}}}(s) + \ln(p(1))\widehat{\lambda}^{-s}\Gamma(s),$$

where

$$\widetilde{\widehat{\mathcal{M}}}(s) := \int_0^{+\infty} \widetilde{\widehat{\mu}}(t)t^{s-1}\mathrm{d}t.$$

Under Assumptions (A1)–(A5), this last integral converges in the half-plane $\mathcal{H}_{\min(\eta,2)}$, to which $\widehat{\mathcal{M}}(s)$ and $\widehat{\mathcal{F}}(s)$ can now both be meromorphically continued. Since $\widetilde{\widehat{\mathcal{M}}}(s)$ has no singularities in $\mathcal{H}_{\min(\eta,2)}$, the singularities of $\widehat{\mathcal{M}}(s)$ are those of $\ln(p(1))\widehat{\lambda}^{-s}\Gamma(s)$. Since $\widehat{\mathcal{F}}(s) = \frac{\widehat{\mathcal{M}(s)}}{1-r^{-s}}$, the singularities of $\widehat{\mathcal{F}}$ in $\mathcal{H}_{\min(\eta,2)}$ are:

(1) a double pole at $s = 0$ coming from the pole of $\Gamma(s)$ and the fact that $1 - r^{-s}$ vanishes at $s = 0$,
(2) simple poles at $s = 2\pi ki/\ln(r)$ for $k \in \mathbb{Z} \setminus \{0\}$, where $1 - r^{-s}$ vanishes,
(3) a simple pole at $s = -1$, which is a pole of $\Gamma(s)$.

Moreover, an adaptation of the proof of Lemma 3.9 shows that under Assumptions (A1)–(A5), we have for all $x > -\min(\eta, 2)$:

$$\widehat{\mathcal{M}}(x + yi) = \mathcal{O}_x\left(|y|^{-2}\right), \quad y \to \pm\infty.$$

We can now complete the proof of Theorem 1.7 exactly as for Theorem 1.4, *mutatis mutandis.*

6. Beyond our theorems

In this section, we first present examples of interesting Mahler functions of order 2 which are not covered by our theorems. We then discuss the case of Mahler equations of order ≥ 3.

6.1. **Mahler functions of order 2.** Theorem 1.4 does not apply to functions $f(z)$ which are solutions of equations of the form $y(z) = y(z^r) + g(z)$ for a rational function g. Indeed, in this case the function f is solution of the Mahler equation of order 2

$$y(z) = \left(1 + \frac{g(z)}{g(z^r)}\right) y(z^r) - \frac{g(z)}{g(z^r)} y(z^{r^2})$$

and $a(z) := 1 + \frac{g(z)}{g(z^r)}$, $b(z) := -\frac{g(z)}{g(z^r)}$ cannot both have non-negative coefficients (without being both constant). Nonetheless, we have given in Theorem 1.7 sufficient conditions on $g(z)$ to estimate the asymptotic behavior of the solutions of $y(z) = y(z^r) + g(z)$. But neither Theorem 1.4 nor Theorem 1.7 can be applied to

- $f_1(z) = \sum_{n=0}^{\infty} z^{2^n}/(1 + z^{2^n})$ which corresponds to the case $g(z) = \frac{z}{1+z}$, $a(z) = \frac{1+z+2z^2}{z(1+z)}$, $b(z) = -\frac{1+z^2}{z(1+z)}$ and $r = 2$;
- $f_2(z) = \sum_{n=0}^{\infty} z^{2^n}/(1 - z^{2^n})$ which corresponds to the case $g(z) = \frac{z}{1-z}$, $a(z) = \frac{1+2z}{z}$, $b(z) = -\frac{1+z}{z}$ and $r = 2$.

Theorem 1 of [5] does not apply to $f_1(z)$ because 1 is a double root of the characteristic polynomial $X^2 - 2X + 1$. The function $(1 - z)f_2(z)$ is solution of the equation $y(z) = \frac{1}{1+z} y(z^2) + z$: Theorem 1.7 cannot still be applied directly, but it is possible that its proof could be adapted to obtain the precise asymptotic behavior of $f_2(z)$ as $z \to 1^-$, beyond the easy fact that $\lim_{z\to 1^-}(1-z)f_2(z) = 2$ (and thus the associated function $C(z)$ is simply constant equal to 2).

Another recent and very interesting example is given by the inhomogeneous Mahler equation of order 1 of [18], that is $y(z) = 1 + (z - 1)y(z^2)$; it has a power series solution $U_0(z) = \frac{1}{2}(1 + z - z^2 + z^3 + z^4 - z^5 - z^6 + \cdots)$. This equation does not fall under the scope of Theorem 1.7 because (A1), (A2) and (A5) are not satisfied. Theorem 1.4 cannot be applied either: U_0 is also solution of $y(z) = zy(z^2) + (1-z^2)y(z^4)$ which does not satisfy (H1) and (H6). Finally, Theorem 1 of [5] cannot be used as well because the characteristic polynomial is $X^2 - X$ and the condition $a_0 a_2 \neq 0$ is not fulfilled. However, it is possible that *adhoc* arguments like those employed for the generating function of the Baum-Sweet sequence could also allow to determine the precise asymptotic behavior of $U_0(z)$ as $z \to 1^-$.

6.2. **Mahler equations of order ≥ 3.** Even though classical Mahler series are of order less than or equal to 2, in particular those coming from combinatorics and automata theory, it is natural to wonder if the methods of this paper could be extended to the case of Mahler equations of order ≥ 3. This is possible in principle but the technical details lead to conditions (like (H6)) on the coefficients of the equations whose complexity increases with the order and are certainly not best possible. Consider for instance a general r-Mahler equation of order 3:

$$y(z) = a(z)y(z^r) + b(z)y(z^{r^2}) + c(z)y(z^{r^3}) \tag{6.1}$$

where $a, b, c \in \mathbb{C}(z)$.

a) We first have to ensure the existence of a solution $f(z)$ of (6.1) holomorphic in $D(0,1)$. A sufficient condition for this is that a, b, c have no poles in $D(0,1)$. If $a(0)+b(0)+c(0) = 1$, we can also ensure that $v_0(f) = 0$; we assume this for simplicity.

b) It is simpler that $f(z)$ has non-negative Taylor coefficients, and a sufficient condition for this is that $a, b, c \in \mathbb{R}^+[[z]]$.

c) It is also simpler for the analysis to assume that the characteristic equation $X^3 - a(1)X^2 - b(1)X - c(1)$ of (6.1) has roots with pairwise distinct modulus, so that by [5], $f(z) = (1+o(1))C(z)/(1-z)^{\log_r(\mu_1)}$, where $C(z) = C(z^r)$ and μ_1 is the root with largest modulus, which is thus necessarily > 0 because f has real Taylor coefficients. Note that to write down the characteristic equation, it is implicitly assumed that $a(1), b(1)$ and $c(1)$ are defined.

d) Since f is positive and increasing on $[0,1)$, we can define the function $\mu(z) := f(z)/f(z^r) \geq 1$ on $[0,1)$. It satisfies the non-linear Mahler equation

$$\mu(z) = a(z) + \frac{b(z)}{\mu(z^r)} + \frac{c(z)}{\mu(z^r)\mu(z^{r^2})}. \tag{6.2}$$

e) By the Bell-Coons estimate, we have

$$\mu(z) = \frac{C(z)(1-z^r)^{\log_r(\mu_1)}}{C(z^r)(1-z)^{\log_r(\mu_1)}}(1+o(1)) = \frac{(1-z^r)^{\log_r(\mu_1)}}{(1-z)^{\log_r(\mu_1)}}(1+o(1)) \to \mu_1, \quad z \to 1^-.$$

f) The first real difficulty arises when we want to justify that

$$\mu_1' := \lim_{z \to 1^-} \mu'(z)$$

exists and is finite. The only reasonable way to do that is by differentiation of (6.2):

$$\mu'(z) = \rho(z) + \sigma(z)\mu'(z^r) + \tau(z)\mu'(z^{r^2})$$

where

$$\rho(z) := a'(z) + \frac{b'(z)}{\mu(z^r)} + \frac{c'(z)}{\mu(z^r)\mu(z^{r^2})},$$

$$\sigma(z) := -\frac{rz^{r-1}}{\mu(z^r)^2}(b(z) + c(z)/\mu(z^{r^2})), \quad \tau(z) := -\frac{r^2 z^{r^2-1}c(z)}{\mu(z^r)\mu(z^{r^2})^2}.$$

Then one could perform an analysis like in the proof of Theorem 1.4: this requires assumptions on a, b, c like (H6) to succeed.

g) Moreover, we want to prove that $\mu''(z) = \mathcal{O}((1-z)^{\eta-2})$ with $\eta > 1$. This is used to find an asymptotic expansion as $t \to 0^+$ of $\mu'(e^{-t})$ and of $\mu(e^{-t})$ in $\mathcal{O}(t^{\eta-1})$ and in $\mathcal{O}(t^{\alpha})$

respectively, where $\alpha := \min(\eta, 2)$. It is also used to prove the convergence of the series involving the Mellin transform (see Eq. (6.3)) by an analogue of Lemma 3.9.

h) Another difficulty to pursue the analysis is that we need to have $\mu_1'/(\mu_1 \ln(\mu_1)) > 0$. Assuming this holds, we then have everything to prove that

$$f(e^{-s}) = \exp\Big(\log_r(\mu_1)\ln(1/s) + c_0 + \frac{1}{\ln(r)} \sum_{k \in \mathbb{Z}\setminus\{0\}} \mathcal{M}\Big(\frac{2\pi k i}{\ln(r)}\Big) s^{-\frac{2\pi k i}{\ln(r)}} + c_1 s + \mathcal{O}_\varepsilon(s^{\alpha-\varepsilon})\Big), \quad (6.3)$$

where $\alpha \in (1, 2]$, c_0 and c_1 are constants which can be made explicit, and

$$\mathcal{M}(s) = \int_0^{+\infty} \ln(\mu(e^{-t}))t^{s-1}\mathrm{d}t$$

is a priori analytic on $\mathcal{H}_0$ but can be meromorphically continued to $\mathcal{H}_\alpha$.

i) This approach can be generalized to higher order Mahler equations: it is possible to provide sufficient conditions on the coefficients $p_j(z) \in \mathbb{R}(z)$ of the equation

$$\sum_{j=0}^{d} p_j(z) y(z^{r^j}) = 0$$

that ensure the existence of a unique transcendental solution $f(z)$ of the equation, holomorphic in $D(0, 1)$, with $f(0) = 1$ and with asymptotic behavior as $z \to 1^-$ given by (6.3), *mutatis mutandis*.

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