The wind-driven circulation

Abstract

Wind stress along the sea surface is an important part of ocean dynamics. Subtropical gyres and equatorial currents are deeply dependant of the wind stress. Several different approaches have been used to understand the role of the wind for the general circulation. Based on the course, the aim of this paper is to present the historical approaches of Stommel and Sverdrup for this problem, and to present the modern tools used to understand the wind-driven circulation, especially in the case of the subtropical gyres.

1 Sverdrup's approach (1947)

1.1 Equations of the problem

Harald Sverdrup was a Norwegian oceanographer who is known for his work on the influence of the wind on the water masses transport. His name was given to the transport unit, 1 Sv corresponding to 10^6 m^3/s. In 1947, he published an article on the wind-driven circulation, with an application to the equatorial currents [1]. It is one of the first to consider a baroclinic ocean for this problem, with variations of velocities with depth. Sverdrup assumed an ocean where the acceleration terms are negligible, but not the Coriolis force. It corresponds to the hypothesis of a Rossby number $Ro \ll 1$. He also assumed a term of viscosity depending only on z. The influence of the wind on the circulation is inside this term. The considered problem is at mesoscale. The horizontal Reynolds number being very large ($Re \sim 10^{10}$), the horizontal terms of viscosity can be neglected. The momentum equations are then :

$$\begin{cases} 0 = -\partial_x p + f\rho v + \partial_z(A\partial_z u) & \text{(1a)} \\ 0 = -\partial_y p - f\rho u + \partial_z(A\partial_z v) & \text{(1b)} \end{cases}$$

where x is the horizontal axis pointing eastward, y pointing northward, u and v are the velocity components respectively on the x-axis and the y-axis, p is the pressure field, $f = 2\Omega \sin\phi$ is the Coriolis parameter, ρ is the ocean density, A is the vertical viscosity. Sverdrup considers a baroclinic ocean by assuming that everything can depend on z, but that the pressure gradient and velocities go to zero after a given depth, called d. Integrating the equations from $z = -d$ to $z = 0$ (surface) :

$$\begin{cases} \displaystyle\int_{-d}^{0} \partial_x p \,\mathrm{d}z = f \int_{-d}^{0} \rho v \,\mathrm{d}z + \int_{-d}^{0} \partial_z(A\partial_z u)\,\mathrm{d}z & \text{(2a)} \\ \displaystyle\int_{-d}^{0} \partial_y p \,\mathrm{d}z = -f \int_{-d}^{0} \rho u \,\mathrm{d}z + \int_{-d}^{0} \partial_z(A\partial_z v)\,\mathrm{d}z & \text{(2b)} \end{cases}$$

Because d doesn't depend on x or y, the integral and the derivative can be inverted. We can rewrite the left hand side term as $\partial_x P$ for equation (2a), and $\partial_y P$ for equation (2b), with $P = \int_{-d}^{0} p\,\mathrm{d}z$. The first term of the right hand side is the transport (meridional for (2a), zonal for (2b)). The second term is easy to integrate : at the surface, it corresponds to the wind stress τ, and at d-depth, the vertical gradient of velocity is null. To close the problem, we also need the continuity equation.

Integrating it on the vertical, and assuming that the vertical velocity is null at $z = 0$ and $z = -d$ (rigid lid), it leads to the set of equations :

$$\begin{cases} \partial_x P = f M_y + \tau_x & \text{(3a)} \\ \partial_y P = -f M_x + \tau_y & \text{(3b)} \\ \partial_x M_x + \partial_y M_y = 0 & \text{(3c)} \end{cases}$$

with $M_x = \int_{-d}^{0} \rho u$, $M_y = \int_{-d}^{0} \rho v$. With the boundary condition $u = 0$ for $x = 0$, these equations are the basis of the article of Sverdrup. After having presented them, he applies these equations to the equatorial dynamics, and shows their relevance in the case of the equatorial currents. This part of the article won't be detailed here, but instead, we propose to apply these equations to the Stommel's problem (see section 3).

2 Stommel's approach (1948)

Henry Stommel was an American oceanographer, younger than Sverdrup, who notably contributed to explain the circulation of planetary currents such as the Kuroshio or the Gulf Stream. His article of 1948 is the first to explain their acceleration on the western boundaries of the ocean. It differs from the Sverdrup's approach by adding a bottom friction, by considering a free surface, but also by considering that the ocean is one homogeneous layer. The main result of this article is that the boundary currents accelerate because of a non uniform Coriolis parameter.

Stommel starts from the momentum equations integrated over the vertical component z. He considers a rectangular basin with horizontal dimensions of $\lambda \times b$ and a height at rest H. He needed an analytical expression for the wind forcing. He only considered trade winds blowing westward and midlatitude winds blowing eastward. A possible analytical expression for these winds is to set the wind stress profile as $-F \cos\left(\frac{\pi y}{b}\right)$, where F is a constant, represented in figure 1. Momentum equations and the continuity equation are then integrated over the vertical :

$$\begin{cases} 0 = f(H+\eta)v - F\cos\left(\frac{\pi y}{b}\right) - Ru - g(H+\eta)\partial_x \eta & \text{(4a)} \\ 0 = -f(H+\eta)u - Rv - g(H+\eta)\partial_y \eta & \text{(4b)} \\ 0 = \partial_x[(H+\eta)u] + \partial_y[(H+\eta)v] & \text{(4c)} \end{cases}$$

where R is the bottom friction parameter, g the acceleration of gravity and η the free surface. This term replaces the pressure gradient. These equations differ from the equations of Sverdrup by the bottom drag term, not present in his model because of the baroclinic assumption he made. Also, Stommel didn't assume a rigid lid. The integration is also different because u and v don't depend on z.

Taking the cross differenciation of (4a) and (4b) using (4c) leads to :

$$v(H+\eta)\frac{\mathrm{d}f}{\mathrm{d}y} + \frac{F\pi}{b}\sin\left(\frac{\pi y}{b}\right) + R(\partial_x v - \partial_y u) = 0 \tag{5}$$

Because $\eta \ll H$, we can simplify this equation. Dividing by R, it leads to :

$$\alpha v - \gamma \sin\left(\frac{\pi y}{b}\right) + \partial_x v - \partial_y u = 0 \tag{6}$$

where $\alpha = \frac{H}{R}\frac{\mathrm{d}f}{\mathrm{d}y}$ and $\gamma = -\frac{F\pi}{Rb}$. The streamfunction ψ is introduced as :

$$u = -\partial_y \psi \qquad v = \partial_x \psi \tag{7}$$

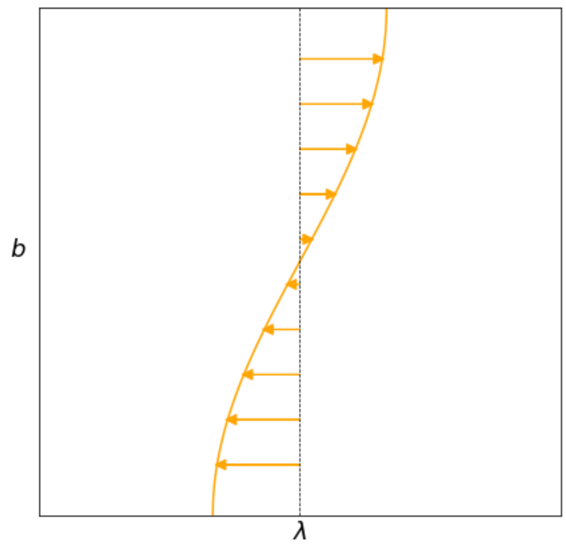

Figure 1: Illustration of the analytical profile of wind stress chosen by Stommel, representing the trade winds at low latitudes, and westerlies at mid-latitudes.

The equation to solve becomes then :

$$\nabla^2\psi + \alpha\partial_x\psi = \gamma \sin\left(\frac{\pi y}{b}\right) \tag{8}$$

This equation is, as system (3) for Sverdrup, the basis of Stommel's work, with boundary conditions being $\psi = 0$ at each boundary. He then solves this problem by making the β-plane approximation, which enables α to be constant. How he solves it will not be investigated in this work because it is essentially mathematical considerations that are not important to understand the approach and the physical arguments involved.

3 Application of the Sverdrup's equations on the Stommel's problem

The objective of this part is to compare the two different approaches on the same problem. Starting from equations (3), with $\tau_x = -\tau_0 \cos\left(\frac{\pi y}{b}\right)$, $\tau_y = 0$.

$$\begin{cases} \partial_x P = f M_y - \tau_0 \cos\left(\frac{\pi y}{b}\right) & \text{(9a)} \\ \partial_y P = -f M_x & \text{(9b)} \\ \partial_x M_x + \partial_y M_y = 0 & \text{(9c)} \end{cases}$$

where $\tau_0 = F\rho_0$ is a constant. Taking the cross differenciation of (9a) and (9b) leads to :

$$0 = -\frac{\tau_0 \pi}{b} \sin\left(\frac{\pi y}{b}\right) + M_y \frac{\mathrm{d}f}{\mathrm{d}y} \tag{10}$$

To be consistent with Stommel's results, we make the β-plane approximation, so $\frac{\mathrm{d}f}{\mathrm{d}y} = \frac{2\Omega\cos\phi_0}{R} = \beta$, where R is the Earth radius and ϕ_0 the mean latitude.

$$M_y = \frac{\tau_0 \pi}{b\beta} \sin\left(\frac{\pi y}{b}\right) \tag{11}$$

The continuity equation (3c) gives the expression of M_x. After integration, we get :

$$M_x = -\frac{\tau_0 \pi^2}{b^2\beta} \cos\left(\frac{\pi y}{b}\right) x \tag{12}$$

with the boundary condition $u = 0$ for $x = 0$. If we don't consider the small variations of density with depth, M_x and M_y can be seen as the barotropic component of the velocity field in Sverdrup's baroclinic theory. We can then introduce a barotropic streamfunction ψ_{bt} as :

$$M_x = -\partial_y \psi_{bt} \qquad M_y = \partial_x \psi_{bt} \tag{13}$$

To know the expression of ψ_{bt}, we can integrate relations (13), using (11) and (12). Both lead to the same expression for ψ_{bt}, which is illustrated in figure 2 :

$$\psi_{bt} = \frac{\tau_0 \pi}{b\beta} \sin\left(\frac{\pi y}{b}\right) x \tag{14}$$

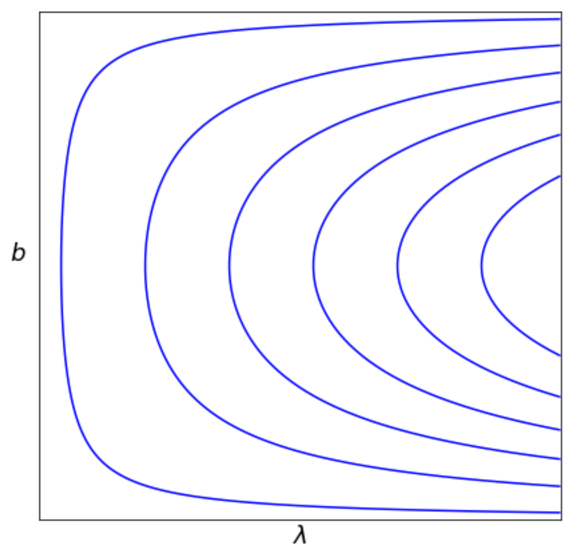

Figure 2: Streamlines representing the Sverdrup solution in the Stommel's problem, with the boundary condition $u = 0$ for $x = 0$.

4 Comparison between the two approaches

Stommel's solution to his problem is given in figure 3, in both f and β planes. It is clear that the β-effect is responsible for the acceleration of the subtropical gyre on Western boudaries of basins.

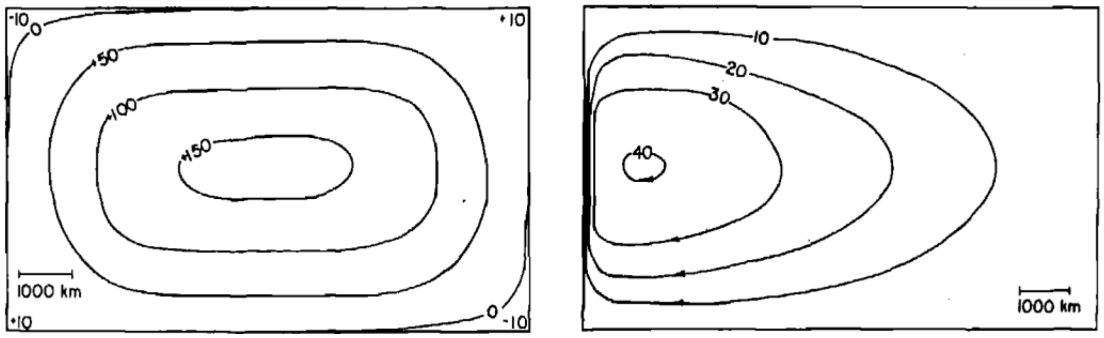

Figure 3: Stommel solution in the f-plane (right) and in the β-plane (left). From Stommel [2].

This solution seems more realistic, with a zonal velocity null at Eastern and Western boundaries. Sverdrup's solution is only valid if we consider a half-opened domain, with a coast only on one side of it. Figure 2 shows that with a Western boundary, the zonal velocity is not null on the Eastern side. The result is symmetric if we consider an Eastern boundary, and a completely closed domain with both boundary conditions can't lead to a physically acceptable result (no movement). This default in Sverdrup's theory comes from the fact that he considered that velocities and pressure gradients are null deeper than $z = -d$. With this assumption, the bottom drag term disappears. In fact, this term is really important, especially in the boundary layer, where viscosity becomes more important as the Reynolds number decreases.

Furthermore, this solution, even in the β-plane, doesn't explain the acceleration of subtropical gyres at the Western boundary. The comparison with the Stommel's theory shows that both bottom drag and a varying Coriolis parameter explain this acceleration. This is why the article of Stommel was so important to understand this phenomenon, as other theories presented this kind of major defaults until then.

But Stommel's solution is still imperfect. Indeed, Sverdrup's circulation shows streamlines that are not closed. If we only consider the barotropic solution, water masses accumulate on the North Eastern corner. So his solution shows an important behavior of the subtropical gyres : their baroclinicity. For example, in the case of the Gulf Stream, water masses dive South of the Labrador Sea. It creates a countercurrent flowing deeper than the surface current. With the Stommel's solution, all this part of the subtropical gyres remains unsolved, as he considered a barotropic ocean. But the Sverdrup's one shows this water masses downwelling as it is observed.

5 Modern research on the wind driven circulation.

Both articles were published in the middle of the 20$^{\text{th}}$ century. Since then, computer capacities have been much improved, enabling more and more complex simulations. On top of that, satellite observations have been a lot developed too. Both aspects created new research methods during the last decades. Especially, models of the ocean to understand its dynamics, corroborated and improved thanks to satellite observations, enable a control of each parameter of the ocean. The pre-printed article of C. De Marez and M. Le Corre [3] has been chosen to show the difference in the modern approach compared to Stommel and Sverdrup's.

This article shows why the Earth can't be flat with ocean physics arguments on a realistic simulation performed thanks to a CROCO model (see www.croco-ocean.org for more details). The main important consequence of a flat Earth is that the Coriolis parameter is uniform. Two different simulations have been performed, one in the f-plane (FLAT), one with a Coriolis parameter $f = 2\Omega \sin \phi$ varying as the Earth were spherical (BETA). The results of the simulation are compared to satellite observations to see which simulation fits the best. The results are presented in figure 4. The

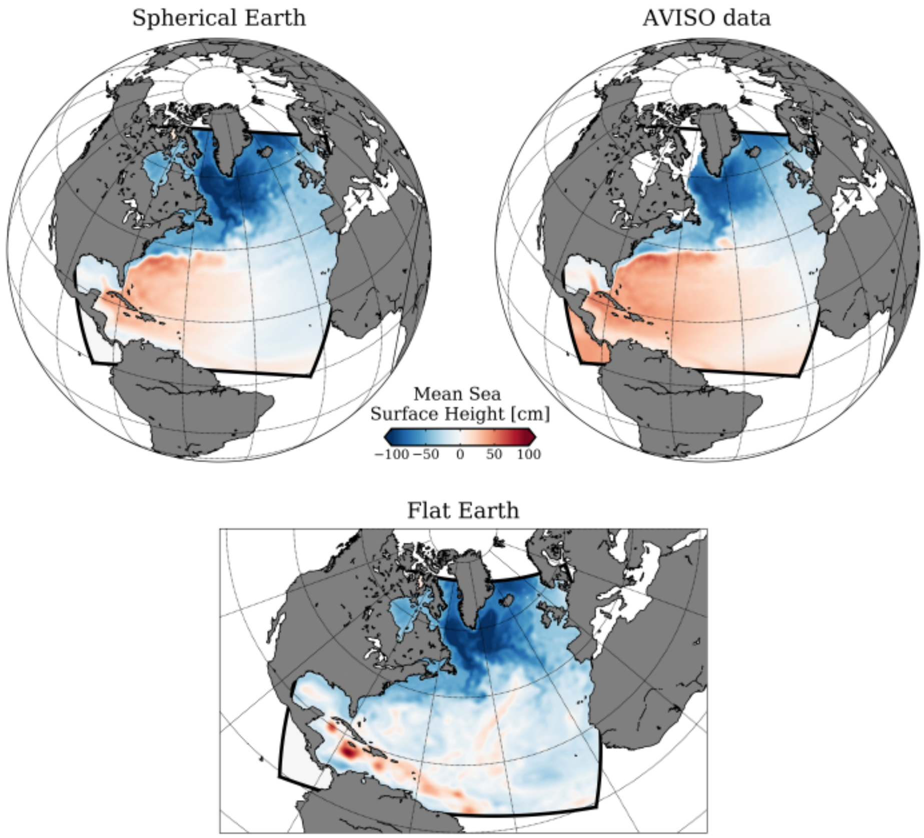

Figure 4: Sea Surface Height (SSH) for a varying Coriolis parameter $f = 2\Omega \sin\phi$ (upper left), for a constant Coriolis parameter (bottom), and satellite data (upper right).

Sea Surface Height (SSH) can be related to the currents at first order *via* the geostrophic balance. The stronger the SSH gradient, the more intense the currents. The FLAT simulation shows the same result as figure 3 : without a varying Coriolis parameter, characteristic of a non flat earth, the subtropical gyres are not accelerated on the western boundary of the basin. The result in the BETA simulation is quite closer to the observed SSH, which confirms the importance of considering a non flat earth for the subtropical gyres dynamics. This result is known in oceanography since Stommel's article, but the approach of this paper is very different from his. Even if it wasn't needed, it gives more credit to Stommel's results, and confirms how brilliant this 1948 short article was.

6 Conclusion

The principle of falsifiability is fundamental in science. It states that it is impossible to establish something as true, but only to establish it as wrong. Then, physicists need theories, on which knowledge is constructed, to get forecasting results. If the results given by these theories are representative of the reality, physicists can keep searching new results based on them. Some results will themselves show the limits of a theory, which is improved in consequence to become stronger. This is what happened with the Sverdrup's theory, very useful for equatorial dynamics, but not relevant enough for subtropical gyres. Stommel tried then to improve his theory by changing assumptions. Thanks to this, he explained the surface dynamics, but couldn't explain the baroclinic behavior of the gyres, contrary to Sverdrup. What is new in the article of C. De Marez and M. Le

Corre is not the result, but the method used to demonstrate it.

Subtropical gyres are accelerated on their Western boundary due to a varying Coriolis parameter, so to a non flat Earth. It is a proof among thousands, but one more, that gives credit to the spherical Earth theory, and one more that falsifies the concept of a flat Earth, despite the belief of the Flat Earth Society.

Bibliography

[1] H. U. Sverdrup, *Wind-driven currents in a baroclinic ocean ; with application to the equatorial currents on the Eastern Pacific.*

[2] H. Stommel, *The westward intensification of wind-driven ocean currents.*

[3] C. De Marez, M. Le Corre, *Can the earth be flat ? A physical oceanographer's perspective*