Homework : atmospheric dynamics

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Résumé

In this homework, we study the Föhn effect, which takes place when wind blows from one side of a mountain. The wind follows the mountain surface and when it is lifted, water inside the fluid parcels can condensate, which warms the air up. If a part of this condensed water precipitates, the air gets an excess of heat, and the wind will be warmer on the leeward side of the mountain.

1 Notations

We use the indices 1 and 2 for the windward and leeward parts of the mountain respectively. We use the index "s" for the summit, "950" for the data at a 950 hPa altitude, and "LCL" for the Lifted Condensation Level. For the temperature gradients, we write the dry adiabatic gradient as Γ_d, the moist one Γ_m, and the dew one Γ_{dew}.

2 Assumptions

1. The atmosphere is considered as an ideal gas.
2. In the ideal gas law, we don't consider the variations of the molar mass due to the water dissolved in the fluid parcel. So $R = R_d = 287$ J K^{-1} kg^{-1}.
3. The temperature profile is assumed to be piecewise linear.
4. We assume that the atmosphere is on the hydrostatic balance.

3 Common part to the two different case

The objective of this part is to know the temperature at the summit. To know this temperature, we need to estimate the LCL. Because the altitude of the valley is given as a pressure, we also need to estimate it in meters, so we have to compute the pressure field. We can't make the assumption of an isotherm atmosphere, which makes the problem more complex.

Pressure profile First, we have to estimate the value of the altitude where the pressure of 950 hPa is reached. Because the dew temperature is lower than the real temperature, there is no condensation at this level. so the temperature profile is then :

$$T(z) = T_{950,1} - \Gamma_d(z - z_{950})$$

Assuming a hydrostatic balance and that the atmosphere is a perfect gas, we have :

$$\frac{\mathrm{d}P_1}{\mathrm{d}z} = -\rho g \quad \& \quad P_1 = \rho R T_1$$

$$\frac{\mathrm{d}P_1}{\mathrm{d}z} + \frac{g}{RT_1(z)} P_1(z) = 0 \tag{1}$$

We integrate this equation with the boundary condition $P_1(0) = P_0 = 1$ bar. We obtain :

$$P_1(z) = P_0 \left(\frac{1 - \frac{\Gamma_d}{T_{950,1}} z_{950}}{1 - \frac{\Gamma_d}{T_{950,1}} (z_{950} - z)} \right)^{K_d} \tag{2}$$

where $K_d = \frac{g}{R\Gamma_d}$ To know z_{950}, we just apply the relation (2) at the altitude z_{950}. We obtain $\boxed{z_{950} \approx 430 \text{ m}}$.

To simplify the rest of the exercise, we change the origin of the vertical axis to z_{950}, which means that $z_s = 3570$ m now.

Since the temperature profile is not the same below and above the LCL, the pressure profile that we computed is only valid for $z < z_{LCL,1}$. Above the LCL, K_d becomes $K_m = \frac{g}{R_d \Gamma_m} > K_d$. By the same kind of computation, and by continuity of the pressure field in the atmosphere, we get :

$$P_1(z) = \begin{cases} P_{950} \left(\frac{1}{1+z/H_d} \right)^{K_d} & \text{if } z \leq z_{LCL,1} \\ P_{LCL,1} \left(\frac{1 - z_{LCL,1}/H_m}{1-(z-z_{LCL,1})/H_m} \right)^{K_m} & \text{if } z \geq z_{LCL,1} \end{cases}$$

where $H_d = \frac{T_{950,1}}{\Gamma_d}$ and $H_m = \frac{T_{LCL,1}}{\Gamma_m}$.

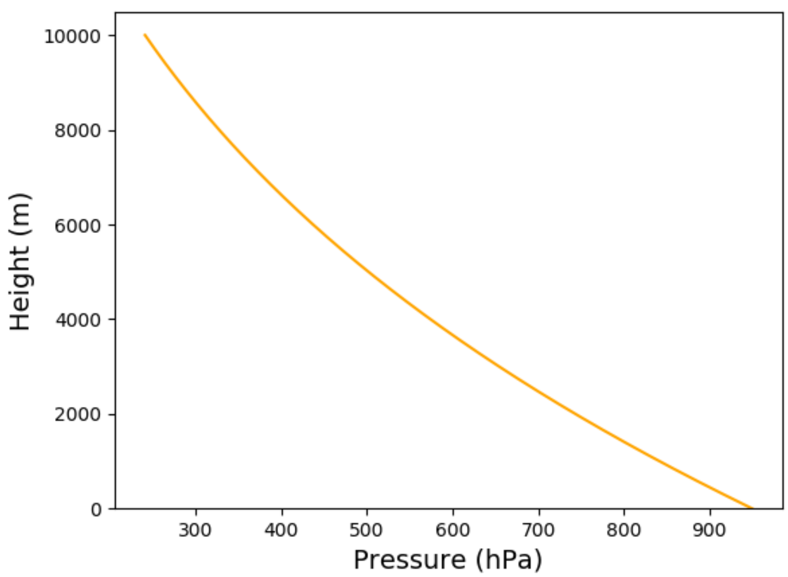

FIGURE 1 – Pressure field for a piecewise linear temperature profile, with $z_{LCL,1} = 500$ m. With this scale, we can't see the angular point for $z = z_{LCL,1}$.

Temperature at the LCL Now that we know precisely the altitude of the valley, we can start the computation. To get $z_{LCL,1}$, we know that :

$$T_{LCL} = T_{950,1} - \Gamma_d z_{LCL,1} = T_{dew}, 1 - \Gamma_{dew} z_{LCL,1}$$

$$\Rightarrow z_{LCL,1} = \frac{T_{950,1} - T_{dew,1}}{\Gamma_d - \Gamma_{dew}} \approx 500 \text{ m}$$

After the LCL, the temperature follows the moist adiabatic temperature gradient till the summit.

$$T_s = T_{LCL} - \Gamma_m(z_s - z_{LCL,1}) = T_{950,1} - \Gamma_d z_{LCL,1} - \Gamma_m(z_s - z_{LCL,1})$$

$$\boxed{T_s \approx -12.5°\mathrm{C}}$$

4 All the water condensed has precipitated before the summit

Temperature in the leeward side valley When it reaches the summit, all the condensed water is gone by precipitation. So the parcels just follow the dry adiabatic temperature gradient.

$$T_{950,2} = T_s + \Gamma_d z_s = T_{950,1} - \Gamma_d z_{LCL,1} - \Gamma_m(z_s - z_{LCL,1}) + \Gamma_d z_s$$

$$\boxed{T_{950,2} \approx 23°\mathrm{C}}$$

5 Half of the water at the LCL

Mass of water in a parcel at the LCL In a parcel at the LCL, we have :

$$\frac{m_v}{m_d} = \frac{\rho_v}{\rho_d} = \frac{M_v}{M_d}\frac{e^*}{p(z_{LCL,1}) - e^*} \tag{3}$$

where M_i is the molar mass of the i component. We already know the pressure profile. To determine the vapor pressure at $z_{LCL,1}$, we use the Clausius-Clapeyron law.

$$e^*(T_{LCL}) = e_0 \exp\left[\frac{L_v}{R_v}\left(\frac{1}{T_0} - \frac{1}{T_{LCL}}\right)\right] \approx 1150 \text{ Pa}$$

With this relation, we get a mixing ratio at the lifted condensation level :

$$\boxed{\frac{m_v}{m_d}(z_{LCL,1}) = 15.2 \text{ g/kg}}$$

Mass of vapor in a parcel at the summit When the parcel is lifted, clouds appear because the vapor pressure decreases faster than the total pressure. We can compute the mixing vapor pressure at the summit :

$$e^*(T_s) = e_0 \exp\left[\frac{L_v}{R_v}\left(\frac{1}{T_0} - \frac{1}{T_s}\right)\right] \approx 236 \text{ Pa}$$

which gives for the mixing ratio at the summit :

$$\boxed{\frac{m_v}{m_d}(z_s) = 4.36 \text{ g/kg}}$$

Mass of condensed water in a parcel at the summit By conservation of the mass, we know that, before precipitation occurs :

$$m_w(z_s) = m_v(z_s) + m_c(z_s) = m_v(z_{LCL,1})$$

Because $m_d(z_{LCL,1}) = m_d(z_s)$, we can divide this relation by m_d, and get for $m_c(z_s)$:

$$\frac{m_c}{m_d}(z_s) = \frac{m_v}{m_d}(z_{LCL,1}) - \frac{m_v}{m_d}(z_s) = 10.8 \text{ g/kg}$$

If we consider that half of the water condensed has precipitated, we have :

$$m_w(z_s) = m_v(z_s) + 0.5\ m_c(z_s)$$

$$\Rightarrow \boxed{\frac{m_w}{m_d}(z_s) = \frac{m_v}{m_d}(z_s) + 0.5\ \frac{m_c}{m_d}(z_s) = 9.76 \text{ g/kg}}$$

Lifted condensation level on the leeward side of the mountain Now that we know the amount of water at the summit after precipitation, we have to know where is the LCL to compute the temperature in the valley. From the ideal gas law applied to water vapor in the parcel, the partial pressure is given by [1] :

$$e(z) = \underbrace{\frac{w}{\epsilon + w}}_{\text{constant}} P_2(z) \approx 9 \cdot 10^{-3} \times P_2(z)$$

where $\epsilon = \frac{M_v}{M_d}$ and $w = \frac{m_w}{m_d}(z_s)$. On the other hand, the Clausius-Clapeyron law gives, for $z \geq z_{LCL,1}$:

$$e^*(z) = \begin{cases} e_0 \exp\left[\frac{L_v}{R_v}\left(\frac{1}{T_0} - \frac{1}{T_{LCL,2} - \Gamma_m(z - z_{LCL,2})}\right)\right] & \text{if } z \leq z_{LCL,2} \\ e_0 \exp\left[\frac{L_v}{R_v}\left(\frac{1}{T_0} - \frac{1}{T_s - \Gamma_m(z - z_s)}\right)\right] & \text{if } z \geq z_{LCL,2} \end{cases}$$

To know $z_{LCL,2}$, we have to solve the equation :

$$e(z_{LCL,2}) = e^*(z_{LCL,2})$$

Computations can be complicated for this part, so two alternative methods are possible. First, we can linearise the pressure profile as we can see on figure 1. The second one, which we will use here, is to let a computer do it. We use a python function to find the zeros of the function $e(z) - e^*(z)$, using for e^* the profile for $z \geq z_{LCL,2}$ because we can't know $T_{LCL,2}$ before knowing $z_{LCL,2}$. For $e(z)$, we have to compute the pressure field from the summit, which gives :

$$P_2(z) = P_s \left(1 - \frac{z - z_s}{H'_m}\right)^{K_m} \qquad \text{if } z \geq z_{LCL,2}$$

Graphically, the situation is illustrated on figure 2 :
The intersection between the two curves gives the LCL. We get :

$$\boxed{z_{LCL,2} \approx 1470 \text{ m}}$$

Once we get the LCL on leeward side of the mountain, we can plot the real partial pressure profile, following the vapor pressure above $z_{LCL,2}$, and following the total pressure profile below the LCL with a K_d growth rate instead of K_m. But $e(z)$ has to be continuous at the LCL, so this is why we only need to solve the equation as we did.

Temperature in the second valley The variations of pressure between the two sides of the mountain are not strong enough to get a different result for the altitude where the pressure of 950 hPa is reached. Then, the temperature in the second valley is :

$$\begin{aligned} T_{950,2} &= T_{LCL,2} - \Gamma_d(z_{950} - z_{LCL,2}) \\ &= T_s - \Gamma_m(z_{LCL,2} - z_s) + \Gamma_d\ z_{LCL,2} \end{aligned}$$

$$\boxed{T_{950,2} \approx 17°\text{C}}$$

Conclusion The Föhn effect appears when some of the water that condensates as air increases its altitude precipitates. The more precipitations there are on the windward side of the mountain, the higher is the LCL on the leeward side of the mountain, and the warmer the wind. Both cases are summarised on the figure 3.

1. This partial pressure is the real one only for $z = z_{LCL,2}$. For $z \geq z_{LCL,2}$, this quantity would be the partial pressure if there were no condensation. For $z \leq z_{LCL,2}$, The condensation occurs when this quantity is equal to the vapor pressure, i.e. at the LCL.

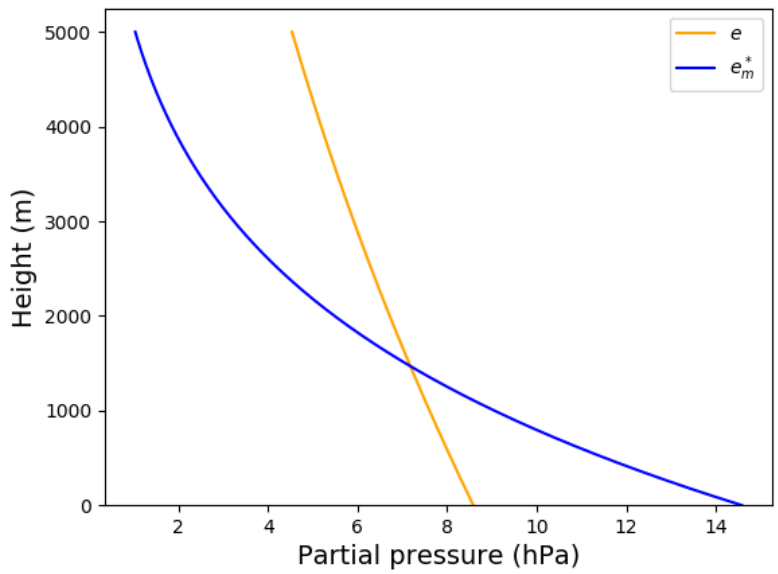

FIGURE 2 – In blue, the vapor pressure for altitudes where condensation takes place. In orange, the partial pressure of water if no condensation occurs. The intersection of these curves gives the LCL. Above the LCL, the real partial pressure follows the blue curve. This is only an artefact to find the LCL.

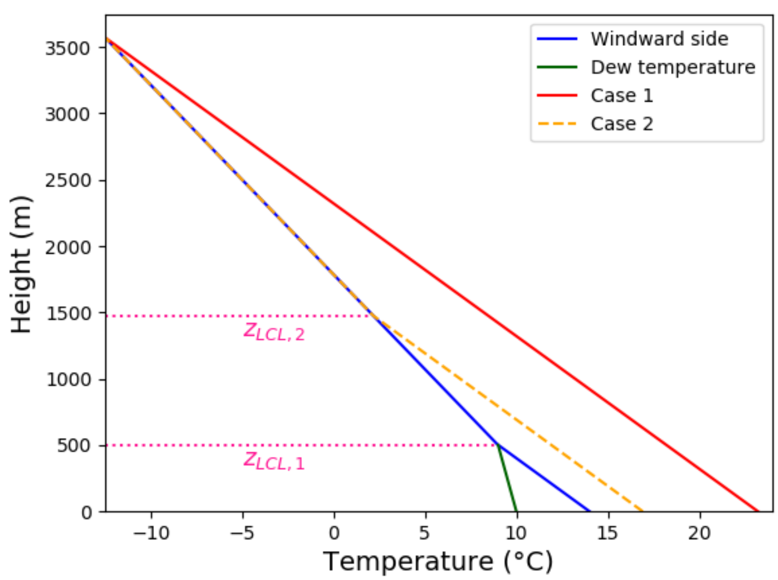

FIGURE 3 – Temperature profiles for case 1 (100 % of the condensed water has precipitated) in red, and case 2 (50 % of the condensed water has precipitated) in orange