Advanced Logic
Lecture 7: Soundness and Completeness

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4 March 2025

Overview

$\vdash$ and $\models$

Propositional logic

Basic modal logic

Normal modal logics; some variations

Provable versus valid: some meta-logic

Remember that also in first-order logic (Introduction to Logic) we had two important notions:

- A is provable (i.e., $\mathcal{F} \vdash A$: there is a formal proof in $\mathcal{F}$ for A);
- A is valid (i.e., $\mathfrak{M} \models A$ for all models $\mathfrak{M}$)

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completeness: valid $\Rightarrow$ provable (rather difficult)

Also: validity (and therefore provability) is <u>undecidable</u> for first-order logic (Turing, Church, 1936).

Some history of completeness proofs

Kurt Gödel (1906-1978) proved soundness and completeness of natural deduction with respect to first-order logic, in 1929.

$\vdash$ and $\models$

In the current lecture, we also look at $\vdash$ and $\models$, but here our proof method represented by $\vdash$ is not natural deduction (Fitch), but **semantic tableaux**.

We discuss soundness and completeness for classical propositional logic first, as a stepping-stone for soundness and completeness for basic modal logic and normal modal logics.

Overview

$\vdash$ and $\models$

Propositional logic

- Soundness for propositional logic: faithful interpretation
- Soundness lemma and soundness theorem
- Completeness for propositional logic: induced interpretation
- Completeness lemma and completeness theorem

Basic modal logic

Normal modal logics; some variations

Propositional logic: Reminders of $\models$ and $\vdash$

The following goes with Section 1.11 of the book.

Reminder of $\models$

Let Σ be a finite set of wffs and A a wff. For classical propositional logic: $\Sigma \models A$ iff for every valuation v the following holds:
If for all $B \in \Sigma$, $v(B) = 1$, then $v(A) = 1$.

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$\Sigma \vdash A$ iff there is a complete tree whose initial list comprises the members of Σ and the negation of A and which is closed.

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Today we will prove the following:

soundness: provable $\Rightarrow$ valid: $\Sigma \vdash A \Rightarrow \Sigma \models A$

completeness: valid $\Rightarrow$ provable: $\Sigma \models A \Rightarrow \Sigma \vdash A$

Soundness for propositional logic: faithful interpretations

Faithful (1.11.1 of the book)

Let v be any propositional interpretation (valuation).
Let b be any branch of a tableau (not necessarily complete).
Define v to be *faithful* to b iff for every formula D that occurs on b, $v(D) = 1$.

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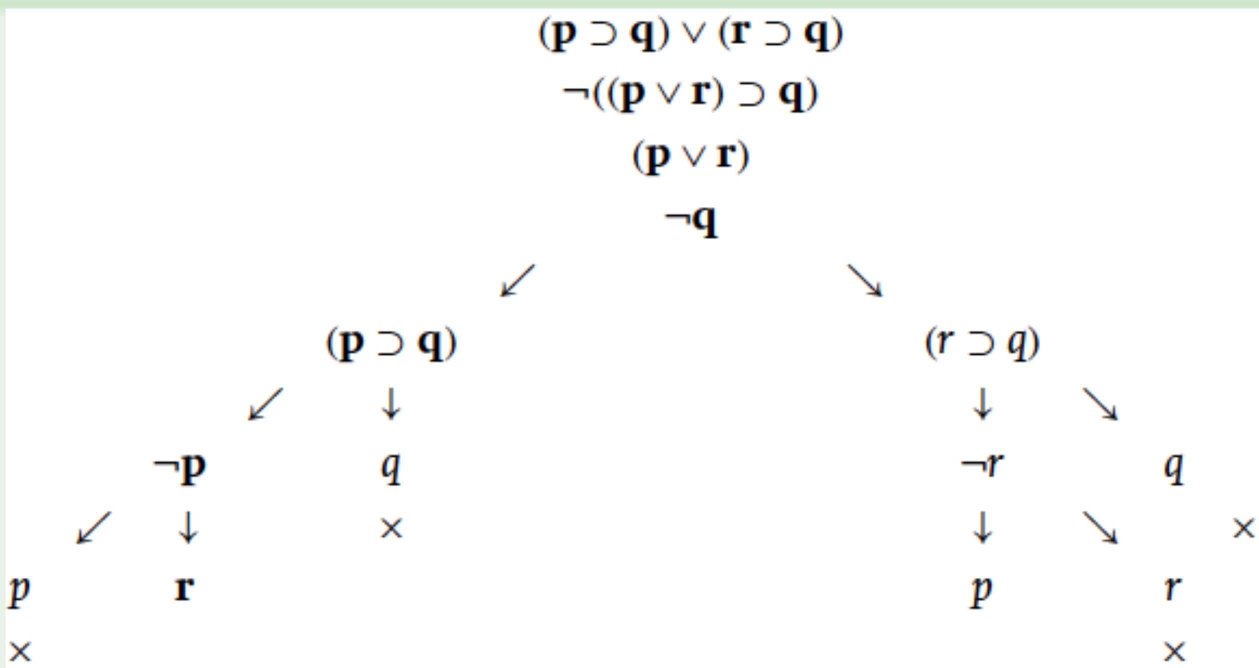

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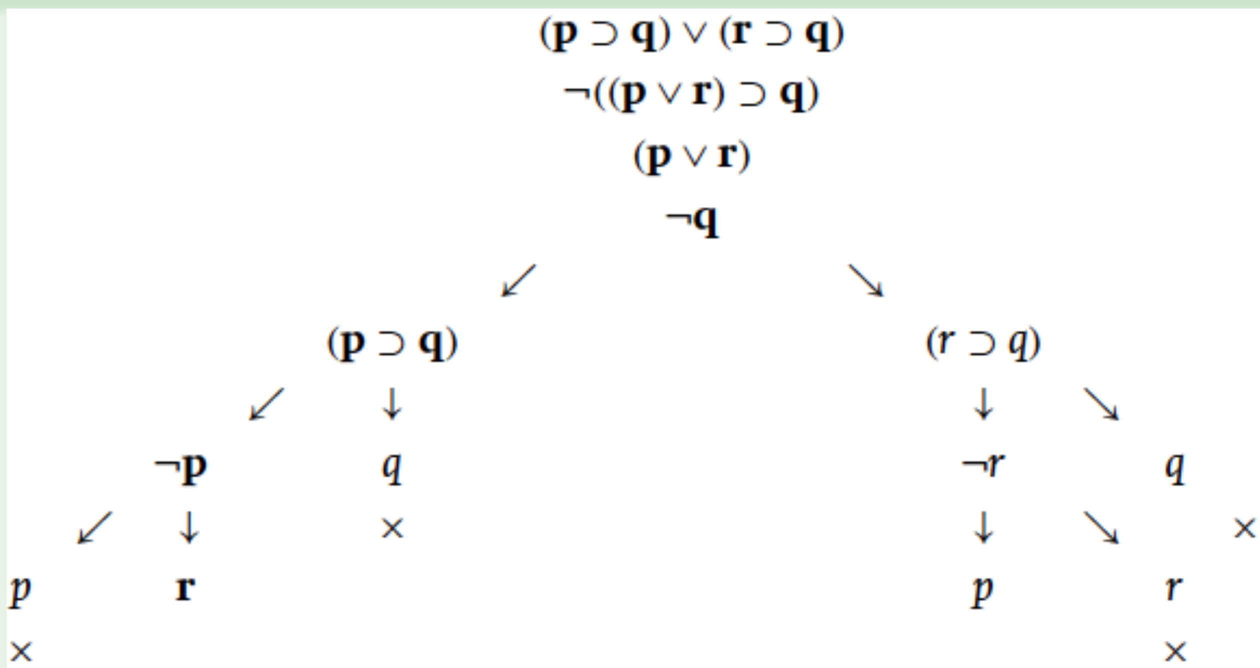

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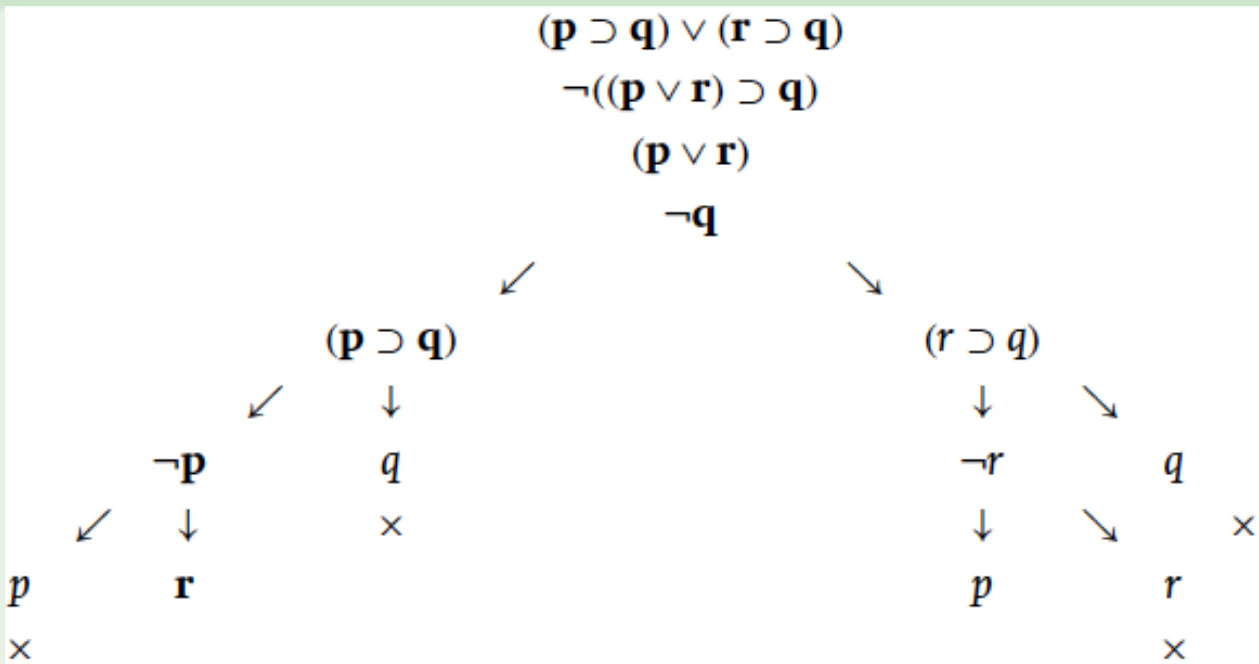

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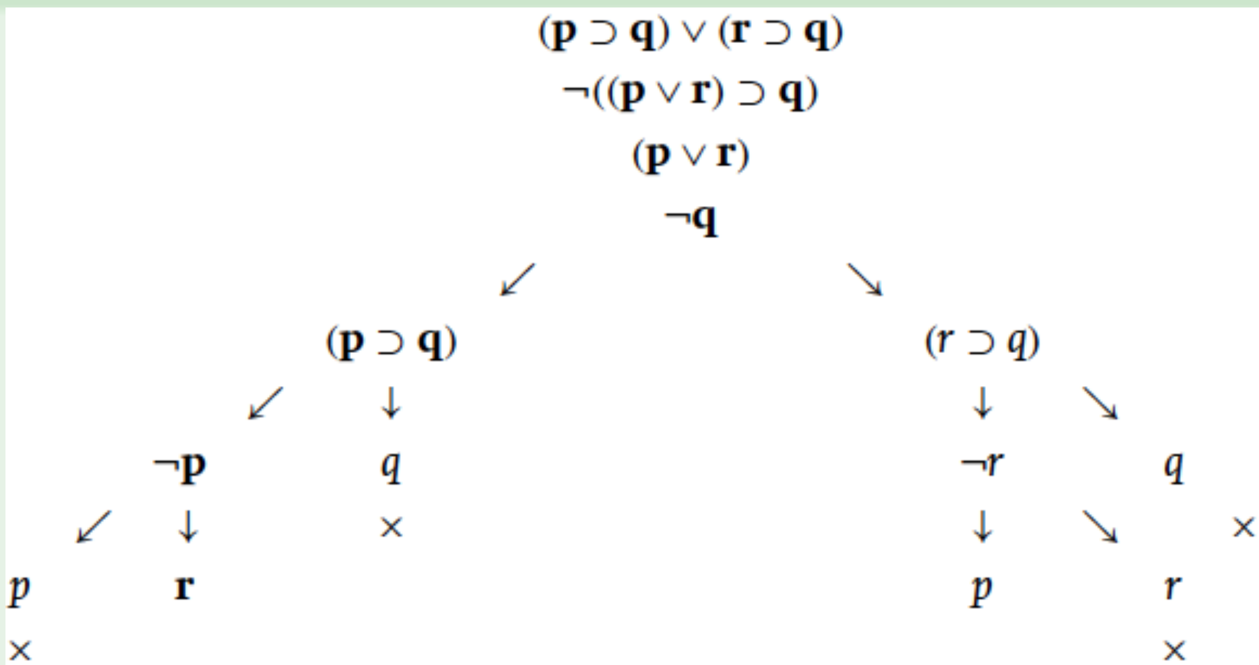


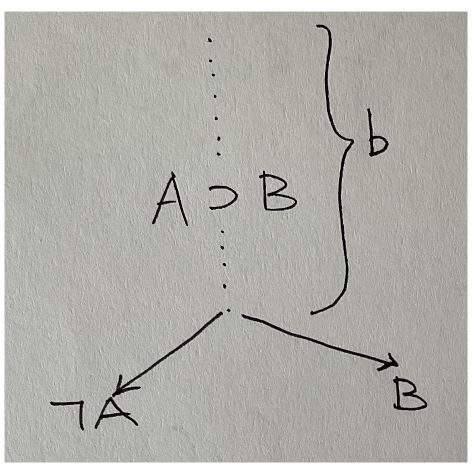
Branch 2 from the left is open and complete.
Now take interpretation v: $v(q) = v(p) = 0$; $v(r) = 1$.
This interpretation v is **faithful** to branch 2.

The soundness lemma

Soundness lemma (1.11.2 of the book)

If v is faithful to a branch b of a tableau,
and a tableau rule is applied to b,
then v is faithful to at least one of the branches generated, b'.

Proof sketch We show this for the two rules for the conditional.



Suppose v is faithful to b, that $A \supset B$ occurs on it, and that we apply a rule to it. Then two branches result:

1. b_L with $\neg A$ and
2. b_R with B.

v is faithful to b, so $v(A \supset B) = 1$, so

- $v(\neg A) = 1$, and v is faithful to b_L; **or**
- $v(B) = 1$, and v is faithful to b_R.

The soundness lemma, continued

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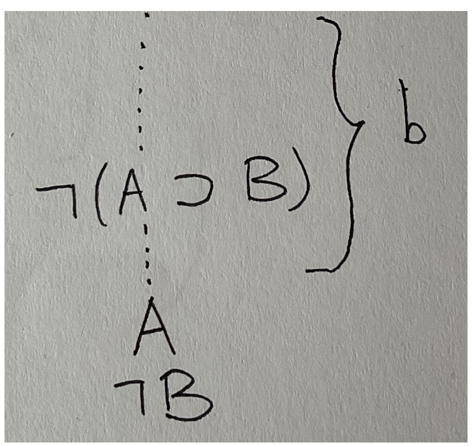
Proof sketch We continue with the other rule for the conditional.



Suppose v is faithful to b and $\neg(A \supset B)$ occurs on it, and that we apply a rule to it. Then one branch b' results, by adding A and $\neg B$ to b.

v is faithful to b, so $v(\neg(A \supset B)) = 1$,
so $v(A) = 1$ and $v(\neg B) = 1$.
So v is faithful to b'.

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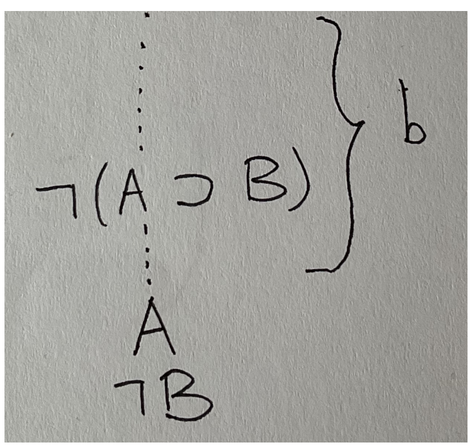

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So v is faithful to b'.

The proof for the other tableau rules is an exercise for the tutorial.

Intermezzo: Proof by contraposition

A conditional is logically equivalent to its contrapositive

Remember that for propositional logic:
$P \supset Q \Leftrightarrow \neg Q \supset \neg P$.

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A conditional is logically equivalent to its contrapositive

Remember that for propositional logic:
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Also in meta-logic, if we want to prove a conditional statement of the form "If P, then Q", it is sometimes easier to prove the equivalent contrapositive statement "If not Q, then not P".

The soundness theorem

Soundness theorem (1.11.3 of the book)

For finite Σ: if $\Sigma \vdash A$, then $\Sigma \models A$.

Proof sketch By contraposition.

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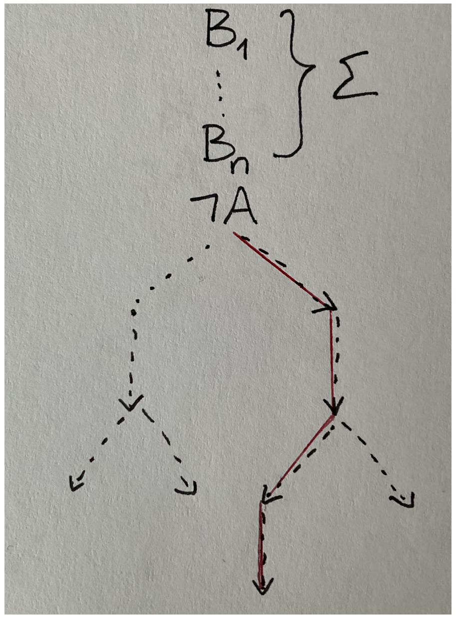

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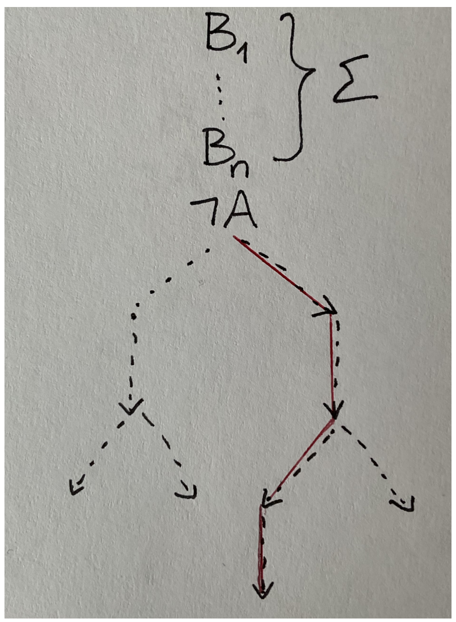

Now consider any *complete* tableau for the inference. v is faithful to the initial list. By the soundness lemma, whenever we apply a rule, we can find at least one extension to which v is faithful. Doing this repeatedly, we find a **complete branch** b such that v is faithful to it.

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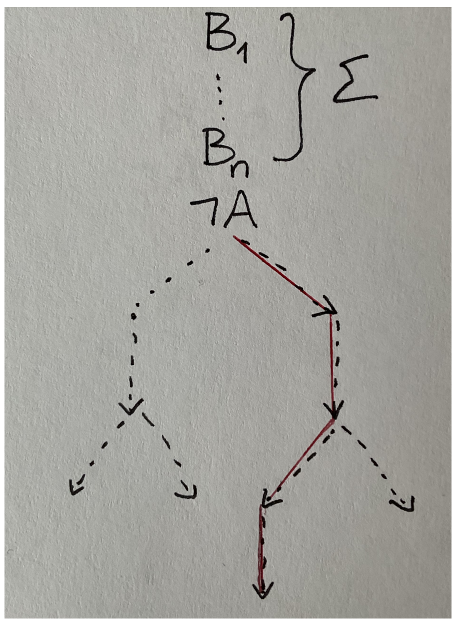

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b cannot be closed. This is because otherwise it would contain some formulas of the form C and $\neg C$. But then, by faithfulness of v to b, we would have $v(C) = v(\neg C) = 1$. Contradiction. So b is complete and open. Therefore, $B_1, \ldots, B_n \nvdash A$ (otherwise there would be at least one complete closed tableau for the inference).

Completeness of propositional logic: Induced interpretation

Definition of induced interpretation (1.11.4 of the book)

Let b be an open branch of a tableau. An interpretation *induced* by b is any interpretation (valuation) v such that for every propositional parameter p:
if p is at a node on b, then $v(p) = 1$;
and if $\neg p$ is at a node on b, then $v(p) = 0$.
(And otherwise, if neither p nor $\neg p$ appears on the branch, $v(p)$ can be anything (from 0, 1) that one likes.)

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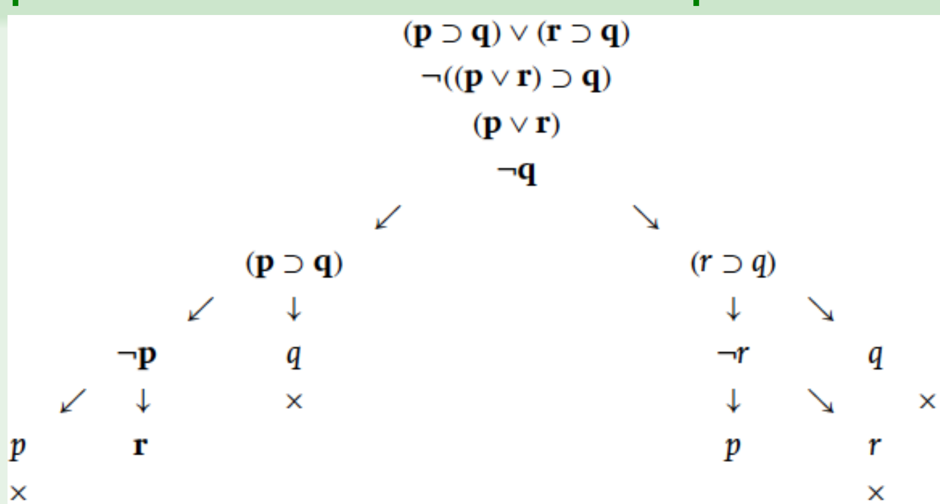

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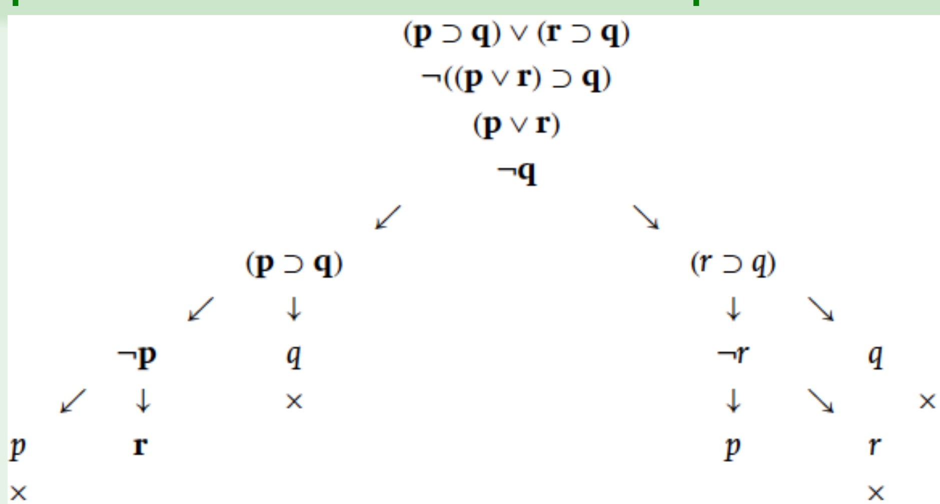

The **induced interpretation** v of branch 4:
$v(q) = v(r) = 0; v(p) = 1.$

Completeness lemma

Completeness lemma (1.11.5 of the book)

Let b be an open complete branch of a tableau.
Let v be an interpretation induced by b.
Then for all (also complex) wffs D the following hold:
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Proof sketch Proof by induction on the complexity of D.

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- **Inductive step** The case for $B \wedge C$: If $B \wedge C$ is on complete branch b, the rule for $B \wedge C$ has been applied, so B and C are on b, so by IH, $v(B) = 1$ and $v(C) = 1$. So $v(B \wedge C) = 1$.

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Completeness theorem

Completeness theorem (1.11.6 of the book)

For finite Σ: if $\Sigma \models A$ then $\Sigma \vdash A$.

Proof sketch: Proof by contraposition.
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Let v be an interpretation induced by b. Then by the completeness lemma, for all wffs D:

- If D is on b, then $v(D) = 1$ and
- if $\neg D$ is on b, then $v(D) = 0$.

So in particular, looking at the initial list containing
$B_1, \ldots, B_n, \neg A$, we have:
$v(B_1) = \ldots = v(B_n) = 1$ and $v(A) = 0$.

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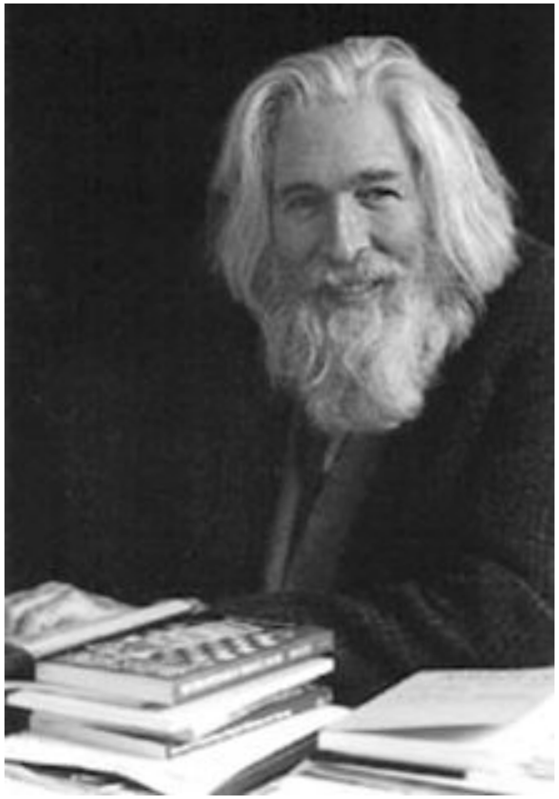
So $B_1, \ldots, B_n \nvDash A$.

Pioneers of tableau systems



Evert Willem Beth (1908-1964) with help of Else Barth (1928-2015) and independently Raymond Smullyan (1919-2017) proved soundness and completeness of semantic tableaux for propositional logic as well as first-order logic.

Overview

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Propositional logic

Basic modal logic

- Soundness of basic modal logic: faithful interpretation
- The soundness lemma and soundness theorem for K
- Completeness of basic modal logic: induced interpretation
- Completeness lemma and completeness theorem for K

Normal modal logics; some variations

Basic modal logic

The following goes with Section 2.9 of the book. In this section, we focus on the basic modal logic K:

Reminder of $\models$ for K

Let Σ be a finite set of wffs and A a wff.
$\Sigma \models A$ iff for all interpretations $\langle W, R, v\rangle$ and all $w \in W$:
If $v_w(B) = 1$ for all $B \in \Sigma$, then $v_w(A) = 1$.

Basic modal logic

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Reminder of $\vdash$ for K

Let Σ be a finite set of wffs and A a wff.
$\Sigma \vdash A$ iff there is a complete, closed tree whose initial list comprises:

1. $B, 0$ for all wffs $B \in \Sigma$ and
2. $\neg A, 0$.

Soundness of basic modal logic: faithful interpretation

Definition faithful interpretation (Subsection 2.9.2 of the book)

Let $I = \langle W, R, v \rangle$ be any modal interpretation (possible worlds model), and let b be any branch of a tableau.
Then I is *faithful* to b iff there is a map f from the natural numbers to W such that:

- For every node D, i on b, sentence D is true at world $f(i)$ in I.
- If irj is on b, then $f(i)Rf(j)$ in I.

We say that f *shows* that I is faithful to b.

Example of a faithful interpretation

Definition faithful interpretation, reminder

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Example (Example of a faithful interpretation)

$$\Diamond\Box\Diamond p,\ 0$$
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$$\Box\neg p,\ 0$$
$$0r1$$
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Now let $I = \langle W, R, v\rangle$ be given by: $W = \{w_1, w_2\}$;
$R = \{\langle w_2, w_1\rangle\}$; $v_{w_1}(p) = 0$, $v_{w_2}(p) = 1$.

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$$\Box\neg p, 0$$
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$$\Box\Diamond p, 1$$
$$\neg p, 1$$

Now let $I = \langle W, R, v\rangle$ be given by: $W = \{w_1, w_2\}$; $R = \{\langle w_2, w_1\rangle\}$; $v_{w_1}(p) = 0$, $v_{w_2}(p) = 1$.
Now let f be such that $f(0) = w_2$ and $f(1) = w_1$. Then f shows that I is faithful to the branch b:

- Only $0r1$ is on b, and indeed $f(0)Rf(1)$ is in I.
- $\neg p, 1$ is on b and indeed $v_{f(1)}(\neg p) = 1$. Similarly for all other D, i on b, up to $v_{f(0)}(\Diamond\Box\Diamond p) = 1$.

The soundness lemma and soundness theorem for K

Soundness lemma (Subsection 2.9.3 of the book)

Let b be any branch of a tableau, and let $I = \langle W, R, v \rangle$ be any interpretation (possible worlds model).
If I is faithful to b, and a tableau rule is applied to b, then that rule produces at least one extension b' such that I is faithful to b'.

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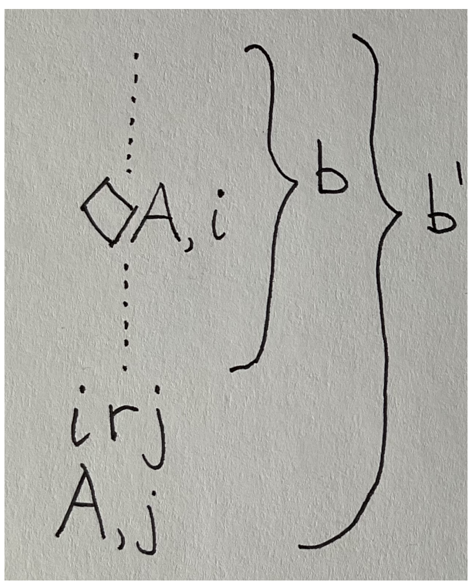

Suppose $\Diamond A, i$ occurs on b, and that we apply the $\Diamond$-rule to get new nodes irj and A, j. f shows that I is faithful to b, so $v_{f(i)}(\Diamond A) = 1$. So there is a $w \in W$ with $f(i)Rw$ and $v_w(A) = 1$. Now let f' be as f, except $f'(j) = w$. This f' also shows that I is faithful to b (where j doesn't occur).

The soundness lemma and soundness theorem for K

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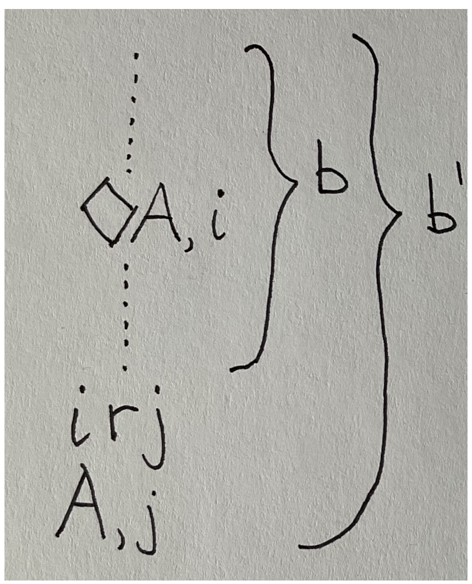

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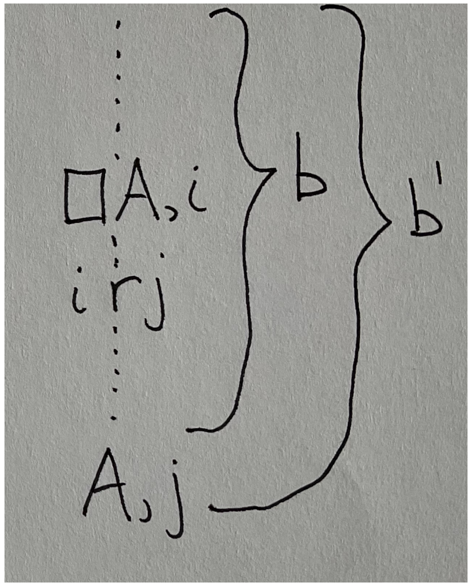
Moreover, f' is faithful to the extended part of b':
$f'(i)Rf'(j)$ because $f(i)Rw$. Also, $v_{f'(j)}(A) = v_w(A) = 1$.

The soundness lemma and soundness theorem for K, contd

Soundness lemma (Subsection 2.9.3 of the book)

Let b be any branch of a tableau, and let $I = \langle W, R, v \rangle$ be any interpretation (possible worlds model).
If I is faithful to b, and a tableau rule is applied to b, then that rule produces at least one extension b' such that I is faithful to b'.

Proof sketch Remember that f shows that I is faithful to b.



Suppose $\Box A, i$ and irj occur on b, and that we apply the $\Box$-rule.
f shows that I is faithful to b, so $\Box A$ is true at $f(i)$.
Also because irj is on b, we have $f(i)Rf(j)$.
By the truth condition for $\Box$, A is true at $f(j)$.

The soundness lemma and soundness theorem for K, contd

Soundness lemma (Subsection 2.9.3 of the book)

Let b be any branch of a tableau, and let $I = \langle W, R, v \rangle$ be any interpretation (possible worlds model).
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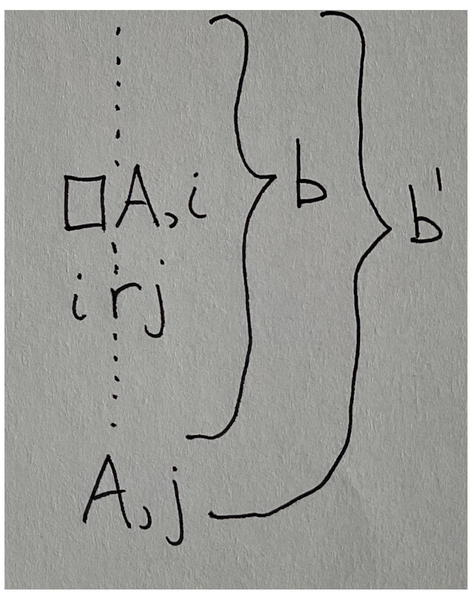
Proof sketch Remember that f shows that I is faithful to b.



Suppose $\Box A, i$ and irj occur on b, and that we apply the $\Box$-rule.
f shows that I is faithful to b, so $\Box A$ is true at $f(i)$.
Also because irj is on b, we have $f(i)Rf(j)$.
By the truth condition for $\Box$, A is true at $f(j)$.

So f itself also shows that I is faithful to b'.

Soundness theorem for K

Soundness theorem (Subsection 2.9.4 of the book)

For finite Σ: if $\Sigma \vdash A$, then $\Sigma \models A$.

Proof sketch By contraposition.

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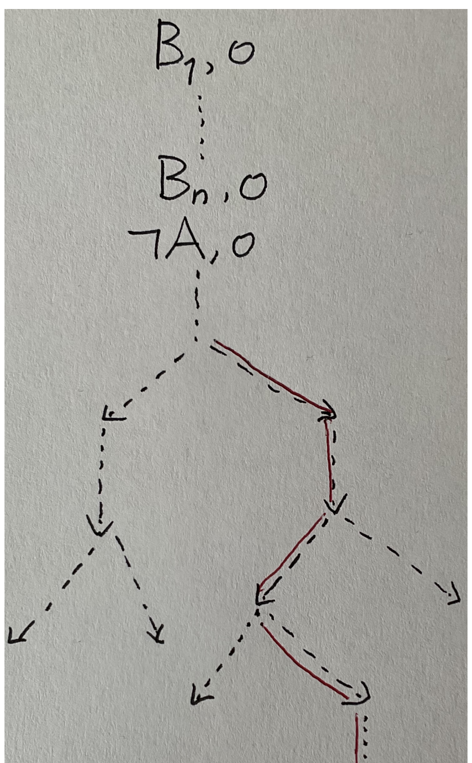

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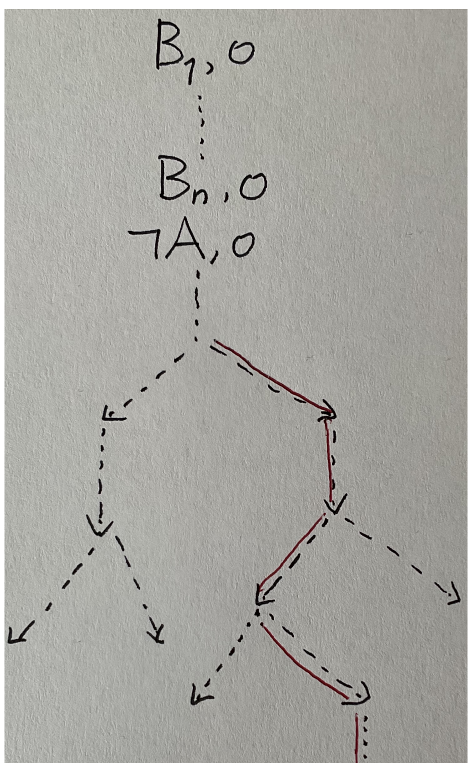

Consider any *complete* tableau for the inference. f shows that I is faithful to the initial list. By the soundness lemma, whenever we apply a rule, we can find at least one extension to which I is faithful. Doing this repeatedly, we find a **complete branch** b to which I is faithful, shown by some f'.

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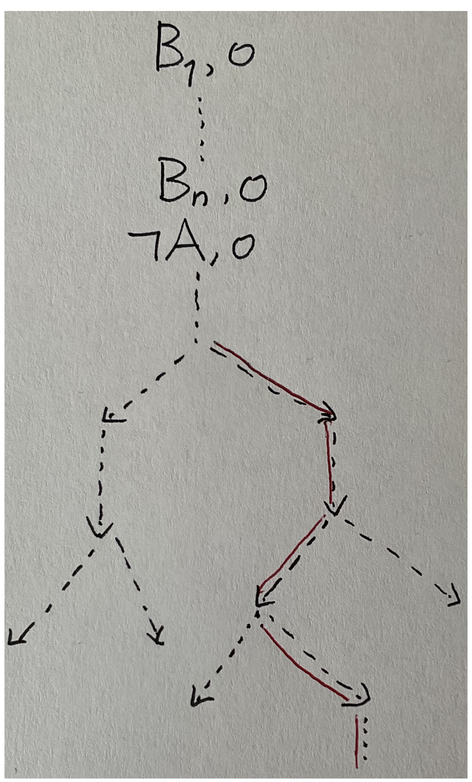

Consider any *complete* tableau for the inference. f shows that I is faithful to the initial list. By the soundness lemma, whenever we apply a rule, we can find at least one extension to which I is faithful. Doing this repeatedly, we find a **complete branch** b to which I is faithful, shown by some f'. b cannot be closed. Because otherwise, it would contain some nodes of the form C, i and $\neg C, i$. But then, by faithfulness of I to b, we would have $v_{f'(i)}(C) = v_{f'(i)}(\neg C) = 1$. Contradiction. So b is complete and open, therefore $B_1, \ldots, B_n \not\vdash A$

Completeness of basic modal logic: induced interpretation

Definition induced interpretation (Subsection 2.9.5)

Let b be an open branch of a tableau.

We say that an interpretation $I = \langle W, R, v \rangle$ is *induced* by b iff:

- $W = \{w_i : i \text{ occurs on } b\}$;
- $w_i R w_j$ iff irj occurs on b;
- If p, i occurs on b, then $v_{w_i}(p) = 1$;
  if $\neg p, i$ occurs on b, then $v_{w_i}(p) = 0$;
  (otherwise $v_{w_i}(p)$ can be anything (from 0, 1) that one likes).

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Example (Example of an induced interpretation)

$$\Diamond\Box\Diamond p, 0$$
$$\neg\Diamond p, 0$$
$$\Box\neg p, 0$$
$$0r1$$
$$\Box\Diamond p, 1$$
$$\neg p, 1$$

Now let $I = \langle W, R, v \rangle$ be given by: $W = \{w_0, w_1\}$;
$R = \{\langle w_0, w_1 \rangle\}$; $v_{w_1}(p) = 0$, $v_{w_0}(p)$ can be anything one likes.
Now I is an induced interpretation of the branch b.

Completeness lemma for K

Completeness lemma (Subsection 2.9.6 of the book)

Let b be an open complete branch of a tableau.
Let $I = \langle W, R, v \rangle$ be an interpretation induced by b.
Then for all (also complex) formulas D and for all i, the following hold:
If D, i is on b, then $v_{w_i}(D) = 1$.
If $\neg D, i$ is on b, then $v_{w_i}(D) = 0$.

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Proof sketch Proof by induction on the complexity of D.

- For atoms p, because $I = \langle W, R, v \rangle$ is induced by b, we have by definition for all i:
  If p, i is on b, then $v_{w_i}(p) = 1$;
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  If $\neg p, i$ is on b, then $v_{w_i}(p) = 0$.
- **Inductive hypothesis** Suppose that for arbitrary wffs B, C, all i:
  if B, i is on b, then $v_{w_i}(B) = 1$;
  if $\neg B$ is on b, then $v_{w_i}(B) = 0$. Same for C.

Completeness lemma for K, contd.

Completeness lemma, reminder

Let b be an open complete branch of a tableau.
Let $I = \langle W, R, v \rangle$ be an interpretation induced by b.
Then for all (also complex) formulas D and for all i, we have:
If D, i is on b, then $v_{w_i}(D) = 1$.
If $\neg D, i$ is on b, then $v_{w_i}(D) = 0$.

Proof sketch, continued Proof by induction on the complexity of D, continued.

- **Inductive step** The case for $\Box B$: Suppose $\Box B, i$ is on open complete branch b. Then because b is complete, for all j such that irj is on b, also B, j is on b. By construction and IH we have $v_{w_j}(B) = 1$ for all j such that $w_i R w_j$, so $v_{w_i}(\Box B) = 1$.

Completeness lemma for K, contd.

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Let b be an open complete branch of a tableau.
Let $I = \langle W, R, v \rangle$ be an interpretation induced by b.
Then for all (also complex) formulas D and for all i, we have:
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Proof sketch, continued Proof by induction on the complexity of D, continued.

- **Inductive step** The case for $\Box B$: Suppose $\Box B, i$ is on open complete branch b. Then because b is complete, for all j such that irj is on b, also B, j is on b. By construction and IH we have $v_{w_j}(B) = 1$ for all j such that $w_i R w_j$, so $v_{w_i}(\Box B) = 1$.
  The case for $\neg\Box B$: If $\neg\Box B, i$ is on open complete branch b, then so is $\Diamond\neg B, i$. The rule for $\Diamond\neg B$ has been applied, so there is a new j with irj and $\neg B, j$ on b. By construction, $w_i R w_j$. Now by IH, $v_{w_j}(B) = 0$. So $v_{w_i}(\Box B) = 0$.

Completeness theorem for K

Completeness theorem (Subsection 2.9.7 of the book)

For finite Σ: if $\Sigma \models A$, then $\Sigma \vdash A$.

Proof sketch: Proof by contraposition.
Suppose $B_1, \ldots, B_n \nvdash A$. Consider a complete open tableau for the inference, and choose an open complete branch in it, call it b.

Completeness theorem for K

Completeness theorem (Subsection 2.9.7 of the book)

For finite Σ: if $\Sigma \models A$, then $\Sigma \vdash A$.

Proof sketch: Proof by contraposition.
Suppose $B_1, \ldots, B_n \not\vdash A$. Consider a complete open tableau for the inference, and choose an open complete branch in it, call it b.

Let $I = \langle W, R, v \rangle$ be an interpretation induced by b. Then by the completeness lemma, for all wffs D:

- If D, i is on b, then $v_{w_i}(D) = 1$ and
- if $\neg D, i$ is on b, then $v_{w_i}(D) = 0$.

So in particular, considering the initial list $B_1, 0; \ldots; B_n, 0; \neg A, 0$, we have: $v_{w_0}(B_1) = \ldots = v_{w_0}(B_n) = 1$ and $v_{w_0}(A) = 0$.

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So $B_1, \ldots, B_n \not\models A$.

Overview

$\vdash$ and $\models$

Propositional logic

Basic modal logic

Normal modal logics; some variations

Soundness for variations of modal logics

Soundness theorems for some normal modal logics (Subsection 3.7.1 of the book)

For finite Σ:

If $\Sigma \vdash_{K_\rho} A$, then $\Sigma \models_{K_\rho} A$.

If $\Sigma \vdash_{K_\sigma} A$, then $\Sigma \models_{K_\sigma} A$.

If $\Sigma \vdash_{K_\tau} A$, then $\Sigma \models_{K_\tau} A$.

If $\Sigma \vdash_{K_\eta} A$, then $\Sigma \models_{K_\eta} A$.

Soundness for variations of modal logics

Soundness theorems for some normal modal logics (Subsection 3.7.1 of the book)

For finite Σ:
If $\Sigma \vdash_{K_\rho} A$, then $\Sigma \models_{K_\rho} A$.
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If $\Sigma \vdash_{K_\tau} A$, then $\Sigma \models_{K_\tau} A$.
If $\Sigma \vdash_{K_\eta} A$, then $\Sigma \models_{K_\eta} A$.

Proof sketch Check the soundness lemma, that if f shows that I is faithful to branch b of a tableau for the system in question, and a tableau rule of the system is applied to b, then this produces at least one extension b' and an f' such that f' shows that I is faithful to b'.

Notice that soundness can also be shown for any system combining any subset of the above four restrictions.

Example: soundness lemma for K_τ

Proof sketch soundness lemma K_τ
Suppose that f shows that some transitive interpretation $I = \langle W, R, v \rangle$ is faithful to a branch b of a tableau for K_τ.

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Suppose that f shows that some transitive interpretation $I = \langle W, R, v \rangle$ is faithful to a branch b of a tableau for K_τ.

Suppose the rule τ is applied to branch b including irj and jrkr:

$$\begin{array}{c} irj \\ jrk \\ \downarrow \\ irk \end{array}$$

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Then because f shows that I is faithful to b, we have $f(i)Rf(j)$ and $f(j)Rf(k)$.

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Suppose the rule τ is applied to branch b including irj and jrkr:

$$\begin{array}{c} irj \\ jrk \\ \downarrow \\ irk \end{array}$$

Then because f shows that I is faithful to b, we have $f(i)Rf(j)$ and $f(j)Rf(k)$.

Hence $f(i)Rf(k)$ since R is transitive. So f also shows that I is faithful to the extension b' including irk.

Completeness for variations of modal logics

Completeness theorems for some normal modal logics (Subsection 3.7.3 of the book)

For finite Σ:

If $\Sigma \models_{K_\rho} A$, then $\Sigma \vdash_{K_\rho} A$.

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If $\Sigma \models_{K_\eta} A$, then $\Sigma \vdash_{K_\eta} A$.

Proof: Prove the completeness lemma almost as for K, and for each logic, also check that the induced model has the right properties.
Notice that completeness can also be shown for any system combining any subset of the above four restrictions.

Example: Completeness lemma for K_τ

Completeness lemma for K_τ

Let b be an open complete branch of a K_τ-tableau.
Let $I = \langle W, R, v \rangle$ be an interpretation induced by b.
Then for all (also complex) formulas D and for all i, the following hold:
If D, i is on b, then $v_{w_i}(D) = 1$.
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Moreover, R is transitive.

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Proof sketch

For the formulas with D, i or $\neg D, i$ on b: similar as for K.

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Moreover, R is transitive.

Proof sketch
For the formulas with D, i or $\neg D, i$ on b: similar as for K.

Transitivity: Suppose for some arbitrary $w_i, w_j, w_k \in W$ we have $w_i R w_j$ and $w_j R w_k$. Then irj and jrk occur on b. But then, because b is complete, irk is on b by the τ-rule. Therefore, $w_i R w_k$.

What's next?

Thank you for your attention today.

Next: **Tutorials** on soundness and completeness, this Wednesday **and** Thursday/Friday.
Also peer review for Homework 4 in the tutorial this week

Then: **Lecture 8** next Tuesday, March 11.
Topic: Quantified modal logics.
(A day before my birthday :-))

Due next Tuesday 11 March: **Homework assignment 5**