Advanced Logic
Lecture 2: Three-Valued Logics

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$f_{\neg}$	
1	0
i	i
0	1

$f_{\wedge}$	1	i	0
1	1	i	0
i	i	i	0
0	0	0	0

$f_{\vee}$	1	i	0
1	1	1	1
i	1	i	i
0	1	i	0

$f_{\supset}$	1	i	0
1	1	i	0
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Overview

Motivation for a third truth value, in addition to 0 and 1

Notation and logics

The three-valued logics of Kleene and Łukasiewicz

Priest's three-valued logic of paradox LP and the logic RM_3

Conditionals

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- Truth-value gluts
- Truth-value gaps

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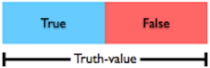

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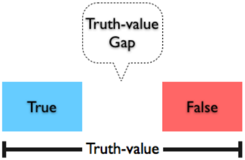

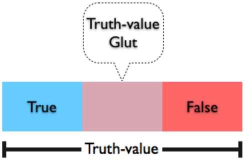

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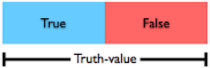

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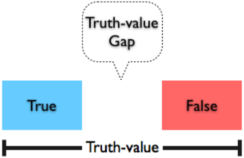
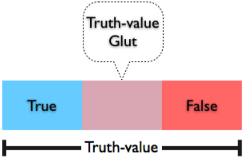
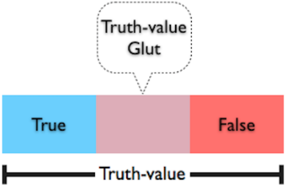




Let us discuss examples of both, from daily life and philosophy.

Examples of truth-value gluts

Here, the new truth value *i* is interpreted as ‘both true and false’.



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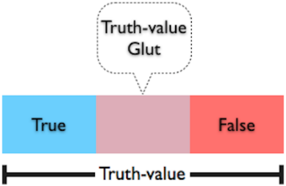

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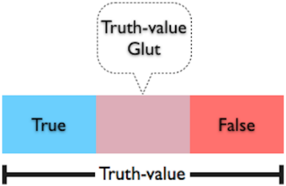

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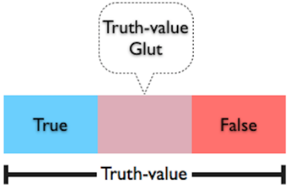

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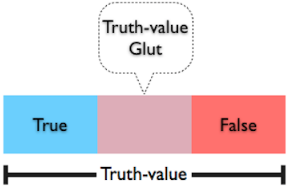

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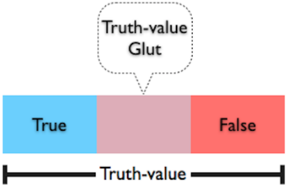

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  By the law of excluded middle ($p \vee \neg p$), one of these must be true. In either case, p is both true and false.

Another example of a truth-value glut: Russell's paradox

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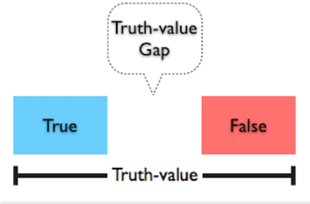

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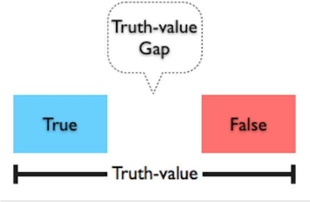

Truth-value gaps occur in the following example situation:

- Denotation failure:
  "The greatest integer is even".
  This sentence is neither true nor false, because 'the greatest integer' does not exist (Frege 1892)

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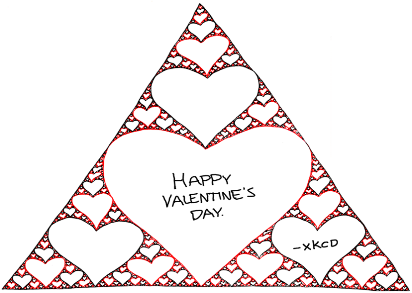

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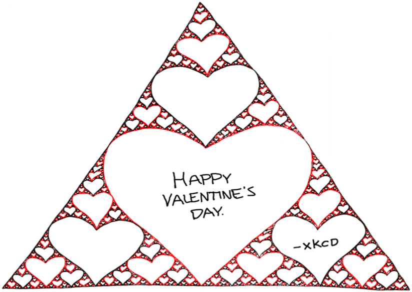

Suppose you have reasons to believe that Chris sent you this card, but not enough evidence to be sure.

Then you may evaluate the sentence "Chris sent me this card" as neither true nor false.

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In Artificial Intelligence applications and robotics, it is often useful to have a truth value i for "uncertain"/"not yet known".

Future contingents continued: Aristotle's sea-battle argument in *On Interpretation*

Let $\varphi :=$ There will be a sea battle tomorrow.

$\Box\varphi$ stands for: "necessarily φ".

Aristotle made the following argument:

1. $\varphi \vee \neg\varphi$
2. $\varphi \rightarrow \Box\varphi$
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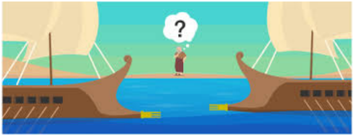

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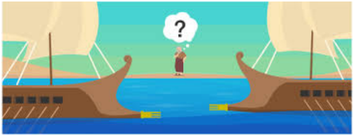

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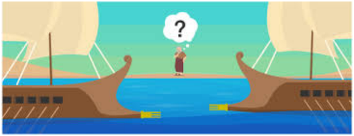

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Nowadays, it is more usual to give up 2 and 3:
When φ is true now, φ is not necessarily so. Same for $\neg\varphi$.

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- Notation in the book 'Introduction to Non-Classical Logics'
- Set theory: reminder
- General structure
- Truth tables and valuations / interpretations
- Validity of inferences

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Notation in the book 'Introduction to Non-Classical Logics' (NCL) by Priest

The language of propositional logic in Priest's book is slightly different from the one we know from LPL (Intro to Logic). Here follow the elements of Priest's notation:

- propositional parameters (atoms): $p, q, r, \ldots$
- arbitrary formulas: $A, B, C, \ldots$
- sets of formulas: $\Sigma, \ldots$
- sentential operators (propositional connectives):
  - negation: $\neg$
  - conjunction: $\wedge$
  - disjunction: $\vee$
  - material conditional: $\supset$
  - material equivalence (bi-implication): $\equiv$;
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Also, Priest uses truth values 1 and 0 instead of T and F.

Set theory: reminder

In order to define logical structures, we need a bit of set theory (see Lecture 5 of Intro to Logic). Here's a reminder:

- set: $\{1, 2, 3\}$
- element: $2 \in \{1, 2, 3\}$, $4 \notin \{1, 2, 3\}$
- abstraction: $\{x : A(x)\}$;
  in other books often represented as $\{x \mid A(x)\}$
- empty set: $\emptyset$
- subset: $\subseteq$
- strict subset: $\subset$ (note: in Introduction to Logic, we used $\subsetneq$)
- intersection: $\cap$
- union: $\cup$
- ordered n-tuple: $\langle 1, \ldots, n \rangle$

The general structure of a logic

Structure

A logic is defined by a triple structure $\langle \mathcal{V}, \mathcal{D}, \{f_c : c \in \mathcal{C}\} \rangle$, where

- $\mathcal{V}$ is a set of truth values,
- $\mathcal{D}$ is a set of designated ('good') values (with $\mathcal{D} \subseteq \mathcal{V}$)
- $\mathcal{C}$ is a set of connectives
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Example (Classical propositional logic)

- $\mathcal{V} = \{0, 1\}$
- $\mathcal{D} = \{1\}$
- $\mathcal{C} = \{\wedge, \vee, \neg, \supset\}$ where

$$\begin{array}{lcl} f_\wedge(x, y) & = & \min(x, y) \\ f_\vee(x, y) & = & \max(x, y) \\ f_\neg(x) & = & 1 - x \\ f_\supset(x, y) & = & \max(1 - x, y) \end{array}$$

Truth tables

Remember that for every n-ary connective $c \in \mathcal{C}$, there is a *truth function* $f_c : \mathcal{V}^n \to \mathcal{V}$.

Example (truth functions for negation and conjunction)

There is a unary truth function $f_\neg$ and there is a binary truth function $f_\wedge$.

For classical two-valued propositional logic, they can be depicted in truth tables as follows:

$f_\neg$	
1	0
0	1

$f_\wedge$	1	0
1	1	0
0	0	0

Valuations / interpretations

Valuations/interpretations

A *valuation* or *interpretation* is a function $v : \mathcal{P} \to \mathcal{V}$, where:

- $\mathcal{P}$ is the set of propositional parameters (atoms) and
- $\mathcal{V}$ is the set of truth values.

The interpretation v is extended to all formulas by an inductive definition as follows, using truth functions f_c for each n-ary connective c:

$$v(c(A_1, \ldots, A_n)) = f_c(v(A_1), \ldots, v(A_n))$$

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Example (Classical propositional logic)

Suppose there is a valuation (interpretation) v such that: $v(p) = 1$ and $v(q) = 0$.

Now $v(p \land q) = f_\land(v(p), v(q)) = f_\land(1, 0) = \min(1, 0) = 0$.

Validity of inferences

Now we can define validity of an inference from a set of premises Σ to a conclusion A as follows:

Validity

Standard definition: $\Sigma \models A$ ("A is a logical consequence of the set of premises Σ") iff
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Example (Validity in classical two-valued propositional logic)

We have $p, p \supset q \models q$ ("modus ponens").
Do we have $p \vee q \models (p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge q)$?
Do we have $p \supset q \models \neg p \supset \neg q$?
Do we have $\models p \vee \neg p$?

Introducing many-valued logics as logical structures

Reminder: Structure

A logic is defined by a triple structure $\langle \mathcal{V}, \mathcal{D}, \{f_c : c \in \mathcal{C}\}\rangle$, where

- $\mathcal{V}$ is a set of truth values,
- $\mathcal{D}$ is a set of designated ('good') values (with $\mathcal{D} \subseteq \mathcal{V}$)
- $\mathcal{C}$ is a set of connectives
- for each n-ary connective $c \in \mathcal{C}$, there is a *truth function* $f_c : \mathcal{V}^n \to \mathcal{V}$

If $\mathcal{V}$ has n members, it is said to be an n-valued logic.
In the rest of this lecture, we focus on different logics with $|\mathcal{V}| = 3$: *three-valued logics*.

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In Lecture 3, we will focus on several four-valued logics.

Introducing many-valued logics as logical structures

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In the rest of this lecture, we focus on different logics with $|\mathcal{V}| = 3$: *three-valued logics*.

In Lecture 3, we will focus on several four-valued logics.

In Lecture 4, we will turn to an infinitely many-valued logic, namely, fuzzy logic.

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Motivation for a third truth value, in addition to 0 and 1

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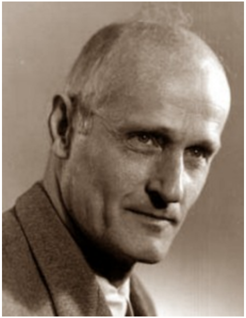
Conditionals

The three-valued logic of Kleene

Stephen Cole Kleene (1909-1994)



Take $\mathcal{V} = \{0, i, 1\}$, $\mathcal{D} = \{1\}$.
i: neither true nor false (mnemonic: "i" for "indeterminate")

$f_\neg$	
1	0
i	i
0	1

$f_\wedge$	1	i	0
1	1	i	0
i	i	i	0
0	0	0	0

$f_\vee$	1	i	0
1	1	1	1
i	1	i	i
0	1	i	0

$f_\supset$	1	i	0
1	1	i	0
i	1	i	i
0	1	1	1

Truth tables in three-valued logics

Truth tables generally have 3^n rows, where n is the number of propositional variables that occur in the formulas one wants to check.

Each row in the truth table corresponds to an interpretation (valuation) v; for two propositional atoms p, q, there are 9 different interpretations, arranged as follows:

p	q	
1	1	
1	i	
1	0	
i	1	
i	i	
i	0	
0	1	
0	i	
0	0	

Checking inferences for Kleene's logic

Let's check some arguments to see whether they are valid in K_3.

Remember that for Kleene's logic, $\mathcal{D} = \{1\}$. So by definition:

$\Sigma \models A$ iff

every interpretation v is such that if $v(B) \in \{1\}$ for all $B \in \Sigma$, then $v(A) \in \{1\}$, iff

every interpretation v is such that if $v(B) = 1$ for all $B \in \Sigma$, then $v(A) = 1$.

Do we have contraposition in Kleene's logic?

We want to check whether: $\dfrac{p \supset q}{\neg q \supset \neg p}$

Do we have contraposition in Kleene's logic?

We want to check whether: $\dfrac{p \supset q}{\neg q \supset \neg p}$

Let us make a truth table:

p	q	$p \supset q$	$\neg q$	$\supset$	$\neg p$
1	1	**1**	0	**1**	0
1	i	**i**	i	**i**	0
1	0	**0**	1	**0**	0
i	1	**1**	0	**1**	i
i	i	**i**	i	**i**	i
i	0	**i**	1	**i**	i
0	1	**1**	0	**1**	1
0	i	**1**	i	**1**	1
0	0	**1**	1	**1**	1

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We want to check whether: $\dfrac{p \supset q}{\neg q \supset \neg p}$

Let us make a truth table:

p	q	$p \supset q$	$\neg q$	$\supset$	$\neg p$
1	1	**1**	0	**1**	0
1	i	**i**	i	**i**	0
1	0	**0**	1	**0**	0
i	1	**1**	0	**1**	i
i	i	**i**	i	**i**	i
i	0	**i**	1	**i**	i
0	1	**1**	0	**1**	1
0	i	**1**	i	**1**	1
0	0	**1**	1	**1**	1

Conclusion: $p \supset q \models_{K_3} \neg q \supset \neg p$ because for each v with $v(p \supset q) = 1$ (in lines 1, 4, 7, 8, 9), we also have $v(\neg q \supset \neg p) = 1$.

Checking other arguments in Kleene's logic

Do we have the law of excluded middle?

$$\vdash p \vee \neg p$$

Let us check using a truth table:

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p	p	$\vee$	$\neg$	p
1		1	0	
i		i	i	
0		1	1	

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Do we have the law of excluded middle?

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Let us check using a truth table:

p	p	$\vee$	$\neg$	p
1		1	0	
i		i	i	
0		1	1	

Conclusion: we have $\not\models_{K_3} p \vee \neg p$. Take as counter-example an interpretation v with $v(p) = i$.
For this v, we have $v(p \vee \neg p) = i$, and $i \notin \mathcal{D}$;
(remember that $\mathcal{D} = \{1\}$).

Checking other arguments in Kleene's logic, continued

Do we have:

$$\models p \supset p$$

Let us check using a truth table:

Checking other arguments in Kleene's logic, continued

Do we have:

$$\vdash p \supset p$$

Let us check using a truth table:

p	$p \supset p$
1	1
i	i
0	1

Checking other arguments in Kleene's logic, continued

Do we have:

$$\models p \supset p$$

Let us check using a truth table:

p	$p \supset p$
1	1
i	i
0	1

Conclusion: we have $\not\models_{K_3} p \supset p$. Take as counter-example an interpretation v with $v(p) = i$.
For this v, we have $v(p \supset p) = i$, and $i \notin \mathcal{D}$.
(remember that $\mathcal{D} = \{1\}$)

Checking other arguments in Kleene's logic, continued

Do we have:

$$\vdash p \supset (q \supset p)$$

Let us check using a truth table:

Checking other arguments in Kleene's logic, continued

Do we have:

$$\models p \supset (q \supset p)$$

Let us check using a truth table:

p	q	p	$\supset$	$(q$	$\supset$	$p)$
1	1		1		1	
1	i		1		1	
1	0		1		1	
i	1		i		i	
i	i		i		i	
i	0		1		1	
0	1		1		0	
0	i		1		i	
0	0		1		1	

Checking other arguments in Kleene's logic, continued

Do we have:

$$\models p \supset (q \supset p)$$

Let us check using a truth table:

p	q	p	$\supset$	$(q$	$\supset$	$p)$
1	1		1		1	
1	i		1		1	
1	0		1		1	
i	1		i		i	
i	i		i		i	
i	0		1		1	
0	1		1		0	
0	i		1		i	
0	0		1		1	

Conclusion: we have $\not\models_{K_3} p \supset (q \supset p)$. Take as counter-example an interpretation v with $v(p) = i$ and $v(q) = i$ (or $v(q) = 1$). For such v, we have $v(p \supset (q \supset p)) = i$, and $i \notin \mathcal{D} = \{1\}$.

Important question about Kleene's logic K_3

Is there *any* formula P such that for all interpretations v, $v(P) = 1$?

That is, are there any logical truths in K_3?

There are no logical truths in Kleene's logic

In Exercise 7.14.3 (p. 140 in Priest's book) you will be asked to prove by induction on formula P that K_3 has no logical truths (in this context that's the same as no tautologies) at all.

That means that there is no formula P such that for all interpretations v, $v(P) = 1$.

There are no logical truths in Kleene's logic

In Exercise 7.14.3 (p. 140 in Priest's book) you will be asked to prove by induction on formula P that K_3 has no logical truths (in this context that's the same as no tautologies) at all.

That means that there is no formula P such that for all interpretations v, $v(P) = 1$.

As a consolation, classical logical truths will never get the truth value 0 in K_3.

The relation between $\supset$, $\neg$ and $\vee$ in Kleene's logic

We can also check whether $p \supset q$ and $\neg p \vee q$ have the same truth tables in K_3, just like in classical logic:

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p	q	p	$\supset$	q	$\neg p$	$\vee$	q
1	1		1		0	1	
1	i		i		0	i	
1	0		0		0	0	
i	1		1		i	1	
i	i		i		i	i	
i	0		i		i	i	
0	1		1		1	1	
0	i		1		1	1	
0	0		1		1	1	

The relation between $\supset$, $\neg$ and $\vee$ in Kleene's logic

We can also check whether $p \supset q$ and $\neg p \vee q$ have the same truth tables in K_3, just like in classical logic:

p	q	p	$\supset$	q	$\neg p$	$\vee$	q
1	1		1		0	1	
1	i		i		0	i	
1	0		0		0	0	
i	1		1		i	1	
i	i		i		i	i	
i	0		i		i	i	
0	1		1		1	1	
0	i		1		1	1	
0	0		1		1	1	

Note that the columns under the main connectives of $p \supset q$ and $\neg p \vee q$ do have the same values.

The three-valued logic of Łukasiewicz

Jan Łukasiewicz (1878-1956)

Same truth tables as for Kleene's three-valued logic K_3, except the table for $\supset$:

$f_\neg$	
1	0
i	i
0	1

$f_\wedge$	1	i	0
1	1	i	0
i	i	i	0
0	0	0	0

$f_\vee$	1	i	0
1	1	1	1
i	1	i	i
0	1	i	0

$f_\supset$	1	i	0
1	1	i	0
i	1	1	i
0	1	1	1

Some properties of Łukasiewicz' logic

In Łukasiewicz' logic, we also have $\mathcal{V} = \{0, i, 1\}$, $\mathcal{D} = \{1\}$, just like in Kleene's logic. Now, just like Łukasiewicz wanted: $\models_{Ł_3} p \supset p$.

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$p :=$ "The temperature will be $+8°$ C in Groningen on Feb 22".

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$p :=$ "The temperature will be $+8^\circ$ C in Groningen on Feb 22".
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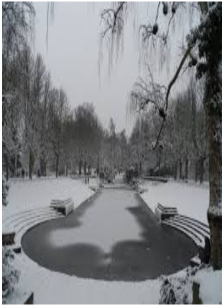
Today (Feb 11), we have $v(p) = v(q) = i$: both are uncertain.



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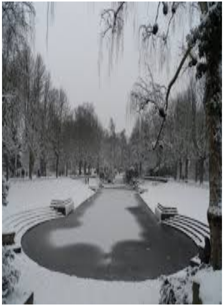

Truth table for $f_\supset$

$f_\supset$	1	i	0
1	1	i	0
i	1	1	i
0	1	1	1

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In Łukasiewicz' logic, we also have $\mathcal{V} = \{0, i, 1\}$, $\mathcal{D} = \{1\}$, just like in Kleene's logic. Now, just like Łukasiewicz wanted: $\models_{Ł_3} p \supset p$.

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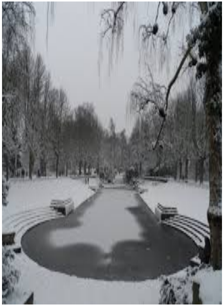

Truth table for $f_\supset$

$f_\supset$	1	i	0
1	1	i	0
i	1	1	i
0	1	1	1

According to the truth table, $v(p \supset q) = 1$, which is undesired.

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- Priest's three-valued logic of paradox *LP*
- Dunn's three-valued logic R Mingle RM_3

Conditionals

Priest's three-valued logic of paradox *LP*

Graham Priest (1948–)

$\mathcal{V} = \{0, i, 1\}$, $\mathcal{D} = \{i, 1\}$

i: both true and false

1: true and true only

0: false and false only

Same truth tables as for Kleene's three-valued logic K_3:

$f_\neg$	
1	0
i	i
0	1

$f_\wedge$	1	i	0
1	1	i	0
i	i	i	0
0	0	0	0

$f_\vee$	1	i	0
1	1	1	1
i	1	i	i
0	1	i	0

$f_\supset$	1	i	0
1	1	i	0
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Checking arguments for Priest's logic

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i		i	i	
0		1	1	

Checking arguments for Priest's logic

Do we have the law of excluded middle?

$$\models p \vee \neg p$$

Let us check using a truth table:

p	p	$\vee$	$\neg$	p
1		1	0	
i		i	i	
0		1	1	

Conclusion: we have $\models_{LP} p \vee \neg p$. For all v, we have $v(p \vee \neg p) \in \{i, 1\}$; (remember that $\mathcal{D} = \{i, 1\}$).
So Priest's logic LP, in contrast to Kleene's logic K_3, does have some tautologies (logical truths).

Checking other arguments in Priest's logic, continued

Do we have "ex falso sequitur quodlibet":

$p \wedge \neg p \models_{LP} q$

Let us check using a truth table:

Checking other arguments in Priest's logic, continued

Do we have "ex falso sequitur quodlibet":

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Let us check using a truth table:

p	q	p	$\wedge$	$\neg p$
1	1		0	0
1	i		0	0
1	0		0	0
i	1		i	i
i	i		i	i
i	0		i	i
0	1		0	1
0	i		0	1
0	0		0	1

Checking other arguments in Priest's logic, continued

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$p \wedge \neg p \models_{LP} q$

Let us check using a truth table:

p	q	p	$\wedge$	$\neg p$
1	1		0	0
1	i		0	0
1	0		0	0
i	1		i	i
i	i		i	i
i	0		i	i
0	1		0	1
0	i		0	1
0	0		0	1

Conclusion: we have $p \wedge \neg p \not\models_{LP} q$. Take as counter-example an interpretation v with $v(p) = i$ and $v(q) = 0$.

For such v, we have $v(p \wedge \neg p) = i \in \mathcal{D} = \{i, 1\}$, but $v(q) \notin \mathcal{D}$.

Some more properties of Priest's logic LP

Do we have $p, p \supset q \models_{LP} q$ (modus ponens)?
And how about $\neg q, p \supset q \models_{LP} \neg p$ (modus tollens)? Let's check:

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1	i	i	0	i
1	0	0	0	1
i	1	1	i	0
i	i	i	i	i
i	0	i	i	1
0	1	1	1	0
0	i	1	1	i
0	0	1	1	1

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i	1	1	i	0
i	i	i	i	i
i	0	i	i	1
0	1	1	1	0
0	i	1	1	i
0	0	1	1	1

Conclusion: $p, p \supset q \not\models_{LP} q$.
Take as counter-example a v with $v(p) = i$, $v(q) = 0$.
Then $v(p) = i$ and $v(p \supset q) = i \in \mathcal{D} = \{i, 1\}$, but $v(q) = 0 \notin \mathcal{D}$.

Some more properties of Priest's logic LP

Do we have $p, p \supset q \models_{LP} q$ (modus ponens)?
And how about $\neg q, p \supset q \models_{LP} \neg p$ (modus tollens)? Let's check:

p	q	$p \supset q$	$\neg p$	$\neg q$
1	1	1	0	0
1	i	i	0	i
1	0	0	0	1
i	1	1	i	0
i	i	i	i	i
i	0	i	i	1
0	1	1	1	0
0	i	1	1	i
0	0	1	1	1

Conclusion: $p, p \supset q \not\models_{LP} q$.
Take as counter-example a v with $v(p) = i$, $v(q) = 0$.
Then $v(p) = i$ and $v(p \supset q) = i \in \mathcal{D} = \{i, 1\}$, but $v(q) = 0 \notin \mathcal{D}$.

Also, $\neg q, p \supset q \not\models_{LP} \neg p$. To show this, take $v(p) = 1, v(q) = i$.
Then $v(\neg q) = i$ and $v(p \supset q) = i \in \mathcal{D}$, but $v(\neg p) = 0 \notin \mathcal{D}$.

Dunn's three-valued logic R Mingle RM_3

John Michael Dunn
(1941–2021)

$\mathcal{V} = \{0, i, 1\}$, $\mathcal{D} = \{i, 1\}$

i: both true and false

Same truth tables as for Kleene's three-valued logic K_3, except the table for $\supset$:

$f_\neg$	
1	0
i	i
0	1

$f_\wedge$	1	i	0
1	1	i	0
i	i	i	0
0	0	0	0

$f_\vee$	1	i	0
1	1	1	1
i	1	i	i
0	1	i	0

$f_\supset$	1	i	0
1	1	0	0
i	1	i	0
0	1	1	1

Some properties of Dunn's logic RM_3

Do we have $p, p \supset q \models_{RM_3} q$ (modus ponens)?

And how about $\neg q, p \supset q \models_{RM_3} \neg p$ (modus tollens)? Let's check:

Some properties of Dunn's logic RM_3

Do we have $p, p \supset q \models_{RM_3} q$ (modus ponens)?

And how about $\neg q, p \supset q \models_{RM_3} \neg p$ (modus tollens)? Let's check:

p	q	$p \supset q$	$\neg p$	$\neg q$
1	1	1	0	0
1	i	0	0	i
1	0	0	0	1
i	1	1	i	0
i	i	i	i	i
i	0	0	i	1
0	1	1	1	0
0	i	1	1	i
0	0	1	1	1

Some properties of Dunn's logic RM_3

Do we have $p, p \supset q \models_{RM_3} q$ (modus ponens)?

And how about $\neg q, p \supset q \models_{RM_3} \neg p$ (modus tollens)? Let's check:

p	q	$p \supset q$	$\neg p$	$\neg q$
1	1	1	0	0
1	i	0	0	i
1	0	0	0	1
i	1	1	i	0
i	i	i	i	i
i	0	0	i	1
0	1	1	1	0
0	i	1	1	i
0	0	1	1	1

Conclusion: $p, p \supset q \models_{RM_3} q$: modus ponens is valid.

There are 3 interpretations v for which both premises are in $\mathcal{D}$:
lines 1, 4, 5. There, we also have $v(q) \in \mathcal{D} = \{i, 1\}$.

Some properties of Dunn's logic RM_3

Do we have $p, p \supset q \models_{RM_3} q$ (modus ponens)?

And how about $\neg q, p \supset q \models_{RM_3} \neg p$ (modus tollens)? Let's check:

p	q	$p \supset q$	$\neg p$	$\neg q$
1	1	1	0	0
1	i	0	0	i
1	0	0	0	1
i	1	1	i	0
i	i	i	i	i
i	0	0	i	1
0	1	1	1	0
0	i	1	1	i
0	0	1	1	1

Conclusion: $p, p \supset q \models_{RM_3} q$: modus ponens is valid.

There are 3 interpretations v for which both premises are in $\mathcal{D}$: lines 1, 4, 5. There, we also have $v(q) \in \mathcal{D} = \{i, 1\}$.

Also, $\neg q, p \supset q \models_{RM_3} \neg p$: modus tollens is valid.

There are 3 interpretations for which both premises are in $\mathcal{D}$: lines 5, 8, 9. There, we also have $v(\neg p) \in \mathcal{D} = \{i, 1\}$.

Our Four Three-valued Logics and Conditionals

Which inferences are valid for which logics?

		K_3	Ł$_3$	LP	RM_3
(1)	$q \models p \supset q$	✓	✓	✓	×
(2)	$\neg p \models p \supset q$	✓	✓	✓	×
(3)	$(p \wedge q) \supset r \models (p \supset r) \vee (q \supset r)$	✓	✓	✓	✓
(4)	$(p \supset q) \wedge (r \supset s) \models (p \supset s) \vee (r \supset q)$	✓	✓	✓	✓
(5)	$\neg(p \supset q) \models p$	✓	✓	✓	✓
(6)	$p \supset r \models (p \wedge q) \supset r$	✓	✓	✓	✓
(7)	$p \supset q, q \supset r \models p \supset r$	✓	✓	×	✓
(8)	$p \supset q \models \neg q \supset \neg p$	✓	✓	✓	✓
(9)	$\models p \supset (q \vee \neg q)$	×	×	✓	×
(10)	$\models (p \wedge \neg p) \supset q$	×	×	✓	×

See page 126 of the book (and Problem 1, p. 140).

What's next?

Thank you for your attention today.

Next: **Tutorial** on three-valued logics, this Wednesday.

Then: **Lecture 3** this Thursday, February 13.
Topic: four-valued logics and tableaux.
Associated tutorials on this Thursday/Friday.

Due next Tuesday February 18: **Homework assignment 2** about three-valued logics. The assignment will be made available at latest this Wednesday in the Schedule.