Solutions

Induction Study Guide

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General remark

In Exercises 1 to 8, we use a property Q. However, this is not necessary; it is also possible to complete these exercises without making use of an explicitly formulated property Q.

1 Exercise 1

Show by induction on n that for all $n \in \mathbb{N}$

$$\binom{n}{n} = 1$$

Define property $Q(n) : \binom{n}{n} = 1$, where $n \in \mathbb{N}$.

Basic step

For this, we fill in 0 for n: $Q(0) : \binom{0}{0} = 1$.

Definition 1 tells us that $\binom{n}{0} = 1$ for every $n \in \mathbb{N}$, so $\binom{0}{0} = 1$.

Inductive hypothesis

Take an arbitrary $n \in \mathbb{N}$ and suppose $Q(n) : \binom{n}{n} = 1$.

Inductive step

We want to show that $Q(n+1)$.

For this, we fill in $n+1$ for n: $Q(n+1) : \binom{n+1}{n+1} = 1$.

$$\begin{aligned} \binom{n+1}{n+1} &= \binom{n}{n} + \binom{n}{n+1} && (1.1) \\ &= 1 + 0 && (1.2) \\ &= 1 && (1.3) \end{aligned}$$

(1.1) makes use of Definition 1;

(1.2) makes use of the IH (inductive hypothesis) and formula (1) on page 4 of the Induction Study Guide.

Conclusion

We conclude by induction that $\binom{n}{n} = 1$ for every $n \in \mathbb{N}$.

2 Exercise 2

Show by induction on n that for all $n \in \mathbb{N}$

$$\binom{n}{1} = n$$

Define property $Q(n) : \binom{n}{1} = n$, where $n \in \mathbb{N}$.

Basic step

For this, we fill in 0 for n: $Q(0) : \binom{0}{1} = 0$.
Definition 1 tells us that $\binom{0}{m} = 0$ for every $m \in \mathbb{N}$ with $m > 0$, so $\binom{0}{1} = 0$.

Inductive hypothesis

Take an arbitrary $n \in \mathbb{N}$ and suppose $Q(n) : \binom{n}{1} = n$.

Inductive step

We want to show that $Q(n+1)$.
For this, we fill in $n+1$ for n: $Q(n+1) : \binom{n+1}{1} = n+1$.

$$\begin{aligned} \binom{n+1}{1} &= \binom{n}{1} + \binom{n}{1-1} && (2.1) \\ &= n + \binom{n}{0} && (2.2) \\ &= n + 1 && (2.3) \end{aligned}$$

(2.1) and (2.3) make use of Definition 1;
(2.2) makes use of the IH.

Conclusion

We conclude by induction that $\binom{n}{1} = n$ for every $n \in \mathbb{N}$.

3 Exercise 3

Show by induction on n that for all $n \in \mathbb{N}$

$$\binom{n+1}{n} = n+1$$

Define property $Q(n) : \binom{n+1}{n} = n+1$, where $n \in \mathbb{N}$.

Basic step

For this, we fill in 0 for n: $Q(0) : \binom{0+1}{0} = 0+1$.

Definition 1 tells us that $\binom{n}{0} = 1$ for every $n \in \mathbb{N}$, so $\binom{1}{0} = 1$.

Inductive hypothesis

Take an arbitrary $n \in \mathbb{N}$ and suppose $Q(n) : \binom{n+1}{n} = n+1$.

Inductive step

We want to show that $Q(n+1)$.

For this, we fill in $n+1$ for n: $Q(n+1) : \binom{n+1+1}{n+1} = n+1+1$.

$$\begin{aligned}
\binom{n+1+1}{n+1} &= \binom{n+1}{n} + \binom{n+1}{n+1} && (3.1)\\
&= (n+1)+1 && (3.2)\\
&= n+1+1 && (3.3)
\end{aligned}$$

(3.1) makes use of Definition 1;
(3.2) makes use of the IH and Exercise 1;
(3.3) eliminating the brackets.

Conclusion

We conclude by induction that $\binom{n+1}{n} = n+1$ for every $n \in \mathbb{N}$.

4 Exercise 4

Show by induction on n that for all $n, m \in \mathbb{N}$, if $m \leq n$, then

$$\binom{n}{m} = \binom{n}{n-m}$$

Define property $Q(n) : \binom{n}{m} = \binom{n}{n-m}$, where $n, m \in \mathbb{N}$, and $m \leq n$.

Basic step

For this, we fill in 0 for n: $Q(0) : \binom{0}{m} = \binom{0}{0-m}$.
$m \leq n$ and $n = 0$, so $m = 0$. Therefore: $\binom{0}{m} = \binom{0}{0}$, so we have to prove that $\binom{0}{0} = \binom{0}{0-0}$.
This is clearly true.

Inductive hypothesis

Take an arbitrary $n \in \mathbb{N}$ and suppose $Q(n) : \binom{n}{m} = \binom{n}{n-m}$ for all $m \leq n$.

Inductive step

We want to show that $Q(n+1)$.
For this, we fill in $n+1$ for n: $Q(n+1) : \binom{n+1}{m} = \binom{n+1}{n+1-m}$ for all m with $m \leq n+1$.
Because binomial coefficients are not defined for negative numbers, we distinguish two cases:

- $m = 0$

$$\begin{aligned}
\binom{n+1}{m} = \binom{n+1}{0} &= 1 && (4.1)\\
&= \binom{n+1}{n+1} && (4.2)\\
&= \binom{n+1}{n+1-m} && (4.3)
\end{aligned}$$

  (4.1) and (4.2) make use of Definition 1; (4.3) is a rewriting of the formula.

- $m > 0$

$$\begin{aligned}
\binom{n+1}{m} &= \binom{n}{m-1} + \binom{n}{m} && (4.4)\\
&= \binom{n}{n-(m-1)} + \binom{n}{n-m} && (4.5)\\
&= \binom{n}{n-m+1} + \binom{n}{n-m} && (4.6)\\
&= \binom{n+1}{n-m+1} && (4.7)\\
&= \binom{n+1}{n+1-m} && (4.8)
\end{aligned}$$

  (4.4) makes use of Definition 1;
  (4.5) uses the IH twice;

(4.6) is a rewriting of the formula;
(4.7) makes use of Definition 1 once more, but in reverse direction;
(4.8) is a second rewriting of the formula.

Conclusion

We conclude by induction that $\left(\begin{array}{c} n \\ m \end{array} \right) = \left(\begin{array}{c} n \\ n-m \end{array} \right)$ for every $n, m \in \mathbb{N}$, if $m \leq n$.

5 Exercise 5

Show by induction on n that for all $n, m \in \mathbb{N}$

$$\sum_{k=0}^{n} \binom{k}{m} = \binom{n+1}{m+1}$$

Define property $Q(n) : \sum_{k=0}^{n} \binom{k}{m} = \binom{n+1}{m+1}$, where $n, m \in \mathbb{N}$.

Basic step

For this, we fill in 0 for n: $Q(0) : \sum_{k=0}^{0} \binom{k}{m} = \binom{0+1}{m+1} = \binom{1}{m+1}$. After rewriting the first part, we get: $\sum_{k=0}^{0} \binom{k}{m} = \binom{0}{m}$.

If $m = 0$, we have to prove that: $\binom{0}{0} = \binom{1}{0+1}$. Using Exercise 1, we see that the result is 1 in both cases.

If $m > 0$, we have to prove that: $\binom{0}{m} = \binom{1}{m+1}$. According to Definition 1, $\binom{0}{m} = 0$, and according to formula (1) on page 4 in the Induction Study Guide, $\binom{n}{m} = 0$, as long as $m > n$. In this case $m + 1 > 1$, since $m > 0$. Therefore, $\binom{1}{m+1} = 0$.

Inductive hypothesis

Take an arbitrary $n \in \mathbb{N}$ and suppose $Q(n) : \sum_{k=0}^{n} \binom{k}{m} = \binom{n+1}{m+1}$ for all $m \in \mathbb{N}$.

Inductive step

We want to show that $Q(n + 1)$.

For this, we fill in $n + 1$ for n: $Q(n + 1) : \sum_{k=0}^{n+1} \binom{k}{m} = \binom{n+1+1}{m+1}$ for all $m \in \mathbb{N}$.

$$\begin{aligned}
\sum_{k=0}^{n+1} \binom{k}{m} &= \sum_{k=0}^{n} \binom{k}{m} + \binom{n+1}{m} && (5.1)\\
&= \binom{n+1}{m+1} + \binom{n+1}{m} && (5.2)\\
&= \binom{n+1+1}{m+1} && (5.3)
\end{aligned}$$

(5.1) is a rewriting of the formula;
(5.2) makes use of the IH;
(5.3) makes use of Definition 1, in reverse direction.

Conclusion

We conclude by induction that $\sum_{k=0}^{n} \binom{k}{m} = \binom{n+1}{m+1}$ for every $n, m \in \mathbb{N}$.

6 Exercise 6

Show by induction on n that for all $n \in \mathbb{N}$

$$\sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) = 2^n$$

Define property $Q(n) : \sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) = 2^n$, where $n \in \mathbb{N}$.

Basic step

For this, we fill in 0 for n: $Q(0) : \sum_{m=0}^{0} \left(\begin{array}{c} m \\ 0 \end{array} \right) = \left(\begin{array}{c} 0 \\ 0 \end{array} \right) = 1 = 2^0$. Here, we rewrote the left and the right part. That $\left(\begin{array}{c} 0 \\ 0 \end{array} \right) = 1$, is given in Exercise 1.

Inductive hypothesis

Take an arbitrary $n \in \mathbb{N}$ and suppose $Q(n) : \sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) = 2^n$.

Inductive step

We want to show that $Q(n+1)$.
For this, we fill in $n+1$ for n: $Q(n+1) : \sum_{m=0}^{n+1} \left(\begin{array}{c} n+1 \\ m \end{array} \right) = 2^{n+1}$.

$$\begin{aligned}
\sum_{m=0}^{n+1} \left(\begin{array}{c} n+1 \\ m \end{array} \right) &= \left(\begin{array}{c} n+1 \\ 0 \end{array} \right) + \sum_{m=1}^{n+1} \left(\begin{array}{c} n+1 \\ m \end{array} \right) && (6.1)\\
&= 1 + \sum_{m=1}^{n+1} \left(\left(\begin{array}{c} n \\ m-1 \end{array} \right) + \left(\begin{array}{c} n \\ m \end{array} \right) \right) && (6.2)\\
&= 1 + \sum_{m=1}^{n+1} \left(\begin{array}{c} n \\ m-1 \end{array} \right) + \sum_{m=1}^{n+1} \left(\begin{array}{c} n \\ m \end{array} \right) && (6.3)\\
&= 1 + \sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) + \sum_{m=1}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) + \left(\begin{array}{c} n \\ n+1 \end{array} \right) && (6.4)\\
&= \left(\begin{array}{c} n \\ 0 \end{array} \right) + \sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) + \sum_{m=1}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) + 0 && (6.5)\\
&= \sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) + \sum_{m=1}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) + \left(\begin{array}{c} n \\ 0 \end{array} \right) && (6.6)\\
&= \sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) + \sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) && (6.7)\\
&= 2^n + 2^n && (6.8)\\
&= 2 \times 2^n && (6.9)\\
&= 2^{n+1} && (6.10)
\end{aligned}$$

(6.1) is a rewriting of the formula;
(6.2) makes use of Definition 1 twice;
(6.3) and (6.4) are both a rewriting of the formula;
(6.5) uses Definition 1 and formula (1) from page 4 of the Induction Study Guide. Since it is clear that

$n+1>n$, we know $\begin{pmatrix} n \\ n+1 \end{pmatrix} = 0$;
(6.6) is a rearrangement of terms;
(6.7) is another rewriting of the formula;
(6.8) uses the IH twice;
(6.9) and (6.10) are again a rewriting of the formula.

Conclusion

We conclude by induction that $\sum_{m=0}^{n} \begin{pmatrix} n \\ m \end{pmatrix} = 2^n$ for every $n \in \mathbb{N}$.

7 Exercise 7

Show by induction on n that for all $n, m \in \mathbb{N}$ it holds that if $m \leq n$, then

$$\left(\begin{array}{c} n \\ m \end{array} \right) = \frac{n!}{(n-m)!m!}$$

Define property $Q(n) : \left(\begin{array}{c} n \\ m \end{array} \right) = \frac{n!}{(n-m)!m!}$, where $n, m \in \mathbb{N}$, and $m \leq n$.

Basic step

For this, we fill in 0 for n: $Q(0) : \left(\begin{array}{c} 0 \\ m \end{array} \right) = \frac{0!}{(0-m)!m!}$.

Since $m \leq n$ and $n = 0$, we know that $m = 0$. According to Definition 1, $\left(\begin{array}{c} n \\ 0 \end{array} \right) = 1$ for every $n \in \mathbb{N}$, so $\left(\begin{array}{c} 0 \\ 0 \end{array} \right) = 1$. As for the righthand part: $\frac{0!}{(0-m)!m!} = \frac{0!}{0!0!}$. According to Definition 2 (as specified before Exercise 7), we have: $0! = 1$. Filling this in yields: $\frac{0!}{0!0!} = \frac{1}{1*1} = 1$.

Inductive hypothesis

Take an arbitrary $n \in \mathbb{N}$ and suppose $Q(n) : \left(\begin{array}{c} n \\ m \end{array} \right) = \frac{n!}{(n-m)!m!}$, for all m with $m \leq n$.

Inductive step

We want to show that $Q(n+1)$.
For this, we fill in $n+1$ for n: $Q(n+1) : \left(\begin{array}{c} n+1 \\ m \end{array} \right) = \frac{(n+1)!}{(n+1-m)!m!}$ for m with $m \leq n+1$. Because binomial coefficients are not defined for negative numbers, we distinguish two cases:

- $m = 0$

$$\left(\begin{array}{c} n+1 \\ m \end{array} \right) = \left(\begin{array}{c} n+1 \\ 0 \end{array} \right) = 1 = \frac{(n+1)!}{(n+1)!} \tag{7.1}$$

$$= \frac{(n+1)!}{(n+1-0)!0!} \tag{7.2}$$

$$= \frac{(n+1)!}{(n+1-m)!m!} \tag{7.3}$$

(7.1) makes use of Definition 1 and basic mathematics;
(7.2) and (7.3) are rewritings of the formula.

- $m > 0$

$$\begin{aligned}
\begin{pmatrix} n+1 \\ m \end{pmatrix} &= \begin{pmatrix} n \\ m-1 \end{pmatrix} + \begin{pmatrix} n \\ m \end{pmatrix} && (7.4)\\
&= \frac{n!}{(n-(m-1))!(m-1)!} + \frac{n!}{(n-m)!m!} && (7.5)\\
&= \frac{n!(n-m)!m! + n!(n-(m-1))!(m-1)!}{(n-(m-1))!(m-1)!(n-m)!m!} && (7.6)\\
&= \frac{n!(n-m)!m! + n!(n+1-m)!(m-1)!}{(n+1-m)!(m-1)!(n-m)!m!} && (7.7)\\
&= \frac{n!(n-m)!(m-1)!m + n!(n-m)!(n+1-m)(m-1)!}{(n+1-m)!m!(m-1)!(n-m)!} && (7.8)\\
&= \frac{n!(m-1)!(n-m)!(m+(n+1-m))}{(n+1-m)!m!(m-1)!(n-m)!} && (7.9)\\
&= \frac{n!(m+n+1-m)}{(n+1-m)!m!)} && (7.10)\\
&= \frac{(n+1)n!}{(n+1-m)!m!} && (7.11)\\
&= \frac{(n+1)!}{(n+1-m)!m!} && (7.12)
\end{aligned}$$

(7.4) makes use of Definition 1;
(7.5) uses the IH twice;
(7.6) after equalizing the denominators, adds up the fractions;
(7.7) is a rewriting of the formula;
(7.8) uses Definition 2 twice, and the denominator is rewritten;
(7.9) the brackets around $(m-1)!$ and $(n-m)!$ are eliminated;
(7.10) the terms discussed in (7.6) are eliminated;
(7.11) is another rewriting of the formula. The terms m and $-m$ are removed;
(7.12) makes use of Definition 2.

Conclusion

We conclude by induction that for every $n, m \in \mathbb{N}$, if $m \leq n$, it holds that $\begin{pmatrix} n \\ m \end{pmatrix} = \frac{n!}{(n-m)!m!}$.

8 Exercise 8

Show that for all $n, m \in \mathbb{N}$ if $m \leq n$, then

$$\frac{\begin{pmatrix} n \\ m+1 \end{pmatrix}}{\begin{pmatrix} n \\ m \end{pmatrix}} = \frac{n-m}{m+1}$$

Thus, we should use the result of Exercise 7. If $m+1 \leq n$, then

$$\frac{\begin{pmatrix} n \\ m+1 \end{pmatrix}}{\begin{pmatrix} n \\ m \end{pmatrix}} = \frac{\frac{n!}{(n-(m+1))!(m+1)!}}{\frac{n!}{(n-m)!m!}} \tag{8.1}$$

$$= \frac{n!}{(n-(m+1))!(m+1)!} \times \frac{(n-m)!m!}{n!} \tag{8.2}$$

$$= \frac{n!(n-m)!m!}{(n-(m+1))!(m+1)!n!} \tag{8.3}$$

$$= \frac{n!(n-m)(n-m-1)!m!}{(n-m-1)!(m+1)m!n!} \tag{8.4}$$

$$= \frac{n-m}{m+1} \tag{8.5}$$

(8.1) uses Exercise 7's result twice;
(8.2) to divide by a fraction is to multiply by the inverse;
(8.3) multiply fractions;
(8.4) uses Definition 2 twice;
(8.5) is a rewriting of the formula.

If $m+1 > n$, that implies $m+1 = n+1$ so $m = n$, because we assumed $m \leq n$. In that case,

$$\frac{\begin{pmatrix} n \\ m+1 \end{pmatrix}}{\begin{pmatrix} n \\ m \end{pmatrix}} = \frac{\begin{pmatrix} n \\ n+1 \end{pmatrix}}{\begin{pmatrix} n \\ n \end{pmatrix}} = \frac{0}{1} = \frac{n-m}{m+1} \tag{8.6}$$

where the result of Exercise 1 and formula (1) on page 4 of the Induction Study Guide are used.

9 Exercise 9

Consider the language of propositional logic without negation, i.e. the neg-free-wffs.

i Each propositional letter p is a neg-free-wff.

ii If A and B are neg-free-wffs, then so are $(A \wedge B)$, $(A \vee B)$, $(A \to B)$ and $(A \leftrightarrow B)$.

iii Nothing is a neg-free-wff unless it is generated by repeated applications of i and ii.

Also consider the valuation v that assigns truth value 1 to all propositional letters. Show by induction that $v(A) = 1$ for all neg-free-wffs.

Basic step

By definition, propositional letters have truth value 1 (given in the exercise description).

Inductive hypothesis

Suppose that A and B are two arbitrary neg-free-wffs for which it holds that $v(A) = 1$ and $v(B) = 1$.

Inductive step

We have to show that $v(A \wedge B) = 1$, $v(A \vee B) = 1$, $v(A \to B) = 1$, and $v(A \leftrightarrow B) = 1$.

For $A \wedge B$: $v(A \wedge B) = 1$ if and only if (iff) $v(A) = 1$ and $v(B) = 1$. These two requirements are satisfied by the IH. Now, we can infer that $v(A \wedge B) = 1$.

For $A \vee B$: $v(A \vee B) = 1$ iff $v(A) = 1$ or $v(B) = 1$. Since $v(A) = 1$ and $v(B) = 1$ (see IH), we can infer that $v(A \vee B) = 1$.

For $A \to B$: $v(A \to B) = 1$ iff $v(A) = 0$ or $v(B) = 1$. We know that $v(B) = 1$ (see IH), so we can infer that $v(A \to B) = 1$.

For $A \leftrightarrow B$: $v(A \leftrightarrow B) = 1$ iff both $v(A) = 1$ and $v(B) = 1$, or both $v(A) = 0$ and $v(B) = 0$. We know that $v(A) = 1$ and $v(B) = 1$ (see IH), so we can infer that $v(A \leftrightarrow B) = 1$.

Conclusion

We conclude by induction that, given that all propositional letters are assigned truth value 1 by valuation v, we have $v(A) = 1$ for all neg-free-wffs.

10 Exercise 10

Show by induction that for every wff P:

$$\#a(P) \leq l(P)$$

$\#a(P)$ is the number of propositional letters in the formula P, and $l(P)$ is the length of the formula P.

Basic step

We have to prove that $\#a(p) \leq l(p)$, for propositional letters p (atomic wffs).
By definition, $a(p) = \{p\}$. This set thus consists of 1 element, which means $\#a(p) = 1$. In addition, by definition, $l(p) = 1$. Since both are 1, we have $\#a(p) \leq l(p)$.

Inductive hypothesis

Suppose that P and Q are two arbitrary wffs for which it holds that $\#a(P) \leq l(P)$ and $\#a(Q) \leq l(Q)$.

Inductive step

We have to show that the property $\#a(P) \leq l(P)$ holds for every wff.

For $\neg P$: By the definitions, $a(\neg P) = a(P)$ and $l(\neg P) = 1+l(P)$. By the IH, we know that $\#a(P) \leq l(P)$. Therefore, $\#a(P) \leq 1 + l(P)$. Hence, we can infer that $\#a(\neg P) \leq l(\neg P)$.

For $P \wedge Q$: By the definitions, $a(P \wedge Q) = a(P) \cup a(Q)$. We also know that $\#(a(P) \cup a(Q)) \leq \#a(P) + \#a(Q)$, since the number of elements in $a(P)$ plus the number of elements in $a(Q)$ can only be greater than (or equal to) the number of elements in the union of both sets. By definition, $l(P \wedge Q) = l(P) + 1 + l(Q)$. By the IH, we know that $\#a(P) \leq l(P)$ and $\#a(Q) \leq l(Q)$. Hence, we know that $\#a(P) + \#a(Q) \leq l(P) + l(Q)$, and so $\#(a(P) \cup a(Q)) \leq l(P) + l(Q)$. Since $l(P) + l(Q) \leq l(P) + l(Q) + 1$, we can infer that $\#a(P \wedge Q) \leq l(P \wedge Q)$.

For $P \vee Q$, $P \rightarrow Q$ and $P \leftrightarrow Q$, the inference is analogous to that for $P \wedge Q$.

Conclusion

We conclude by induction that, given the definitions of $l(P)$, $a(P)$, and $\#(S)$, it holds for every wff P that $\#a(P) \leq l(P)$.

11 Exercise 11

Show by induction that for every wff P:

$$h(P) < l(P)$$

$h(P)$ is the height of the tree.

Basic step

We have to prove that $h(p) < l(p)$ for propositional letters p (atomic wffs). By the definitions, $h(p) = 0$ and $l(p) = 1$. It is obvious that $0 < 1$, so $h(p) < l(p)$.

Inductive hypothesis

Assume that P and Q are two arbitrary wffs for which it holds that $h(P) < l(P)$ and $h(Q) < l(Q)$.

Inductive step

We have to show that the property $h(P) < l(P)$ holds for every formula generated by $\neg P$, $P \wedge Q$, $P \vee Q$, $P \to Q$, and $P \leftrightarrow Q$.

For $\neg P$: By definition, $h(\neg P) = 1 + h(P)$ and $l(\neg P) = 1 + l(P)$. By the IH, we know that $h(P) < l(P)$, so $1 + h(P) < 1 + l(P)$ is also true. From this, we can infer that $h(\neg P) < l(\neg P)$.

For $P \wedge Q$: By definition, $h(P \wedge Q) = 1 + max(h(P), h(Q))$ and $l(P \wedge Q) = l(P) + 1 + l(Q)$. In the case that $h(P) \leq h(Q)$, we have $h(P \wedge Q) = 1 + h(Q)$. By the IH, we know that $h(Q) < l(Q)$. From this, we can infer that $1 + h(Q) < l(P) + 1 + l(Q)$ and so $h(P \wedge Q) < l(P \wedge Q)$. In the other case that $h(Q) < h(P)$, we have $h(P \wedge Q) = 1 + h(P)$. By the IH, we know that $h(P) < l(P)$. From this, we can infer that $1 + h(P) < l(P) + 1 + l(Q)$ and so $h(P \wedge Q) < l(P \wedge Q)$.

For $P \vee Q$, $P \to Q$ and $P \leftrightarrow Q$, the inference is analogous to the one for $P \wedge Q$.

Conclusion

We conclude by induction that, given the definitions of $l(P)$ and $h(P)$, it holds for every wff P that $h(P) < l(P)$.

12 Exercises 12,13 & 14, 16

You can solve these yourself. Good luck!

13 Exercise 15

$$\begin{aligned}
\textit{connective-depth}(P) &= 0 \text{ if } P \text{ is an atomic wff} \\
\textit{connective-depth}(\neg P) &= 1 + \textit{connective-depth}(P) \\
\textit{connective-depth}((P \wedge Q)) &= 1 + \max(\textit{connective-depth}(P), \textit{connective-depth}(Q)) \\
\textit{connective-depth}((P \vee Q)) &= 1 + \max(\textit{connective-depth}(P), \textit{connective-depth}(Q)) \\
\textit{connective-depth}((P \to Q)) &= 1 + \max(\textit{connective-depth}(P), \textit{connective-depth}(Q)) \\
\textit{connective-depth}((P \leftrightarrow Q)) &= 1 + \max(\textit{connective-depth}(P), \textit{connective-depth}(Q))
\end{aligned}$$

Notice that for each wff P, it holds that $\textit{connective-depth}(P) = h(P)$ (where h is as defined in Exercise 11).