Lecture notes advanced logic

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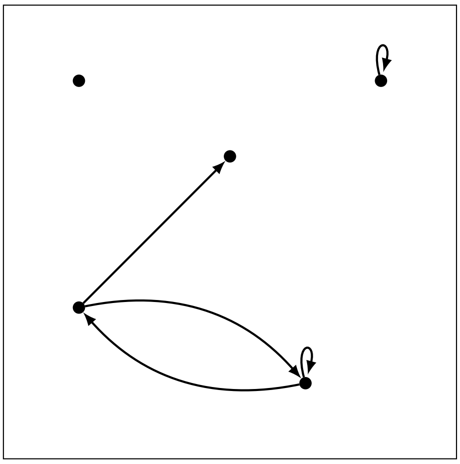
6 Variations of modal logic

6.1 Normal modal logics

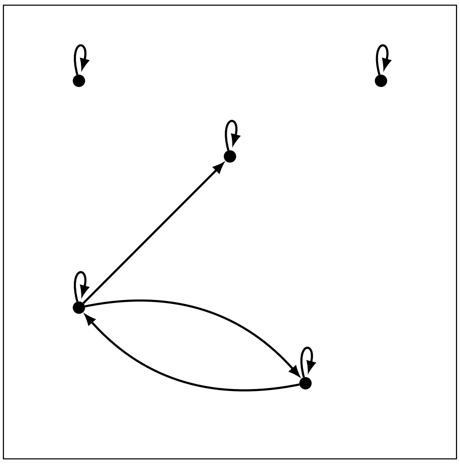
6.1.1 Modal logic K

No restrictions on R



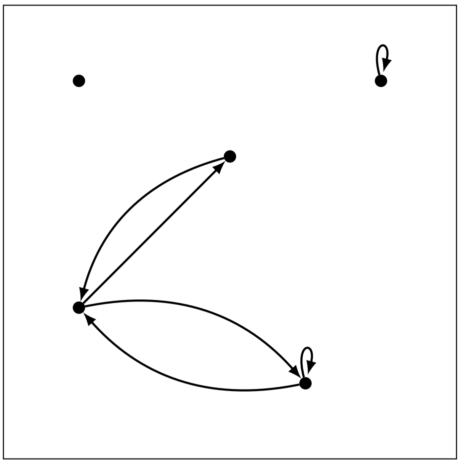
6.1.2 Restriction ρ

R is *reflexive* iff for all $w \in W$, wRw.



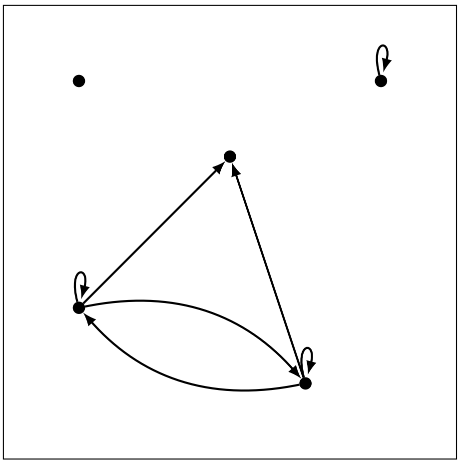
6.1.3 Restriction σ

R is *symmetric* iff for all $w_1, w_2 \in W$, (if $w_1 R w_2$, then $w_2 R w_1$).



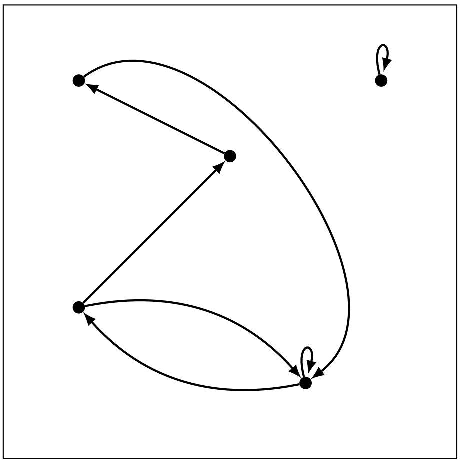
6.1.4 Restriction τ

R is *transitive* iff for all $w_1, w_2, w_3 \in W$,
(if $w_1 R w_2$ and $w_2 R w_3$, then $w_1 R w_3$).



6.1.5 Restriction η

R is *extendable* iff for all $w_1 \in W$ there is a $w_2 \in W$ such that $w_1 R w_2$.



Note that for each accessibility relation R:
If R is reflexive, then R is extendable.

If R is symmetric, transitive and extendable, then R is reflexive.

6.1.6 Normal modal logics: definition

K is a normal modal logic and all *normal modal logics* other than K are obtained by defining validity with respect to truth preservation in a special class of possible worlds models. We have just introduced some examples given by restrictions on the accessibility relation R.

6.1.7 Standard names of modal systems

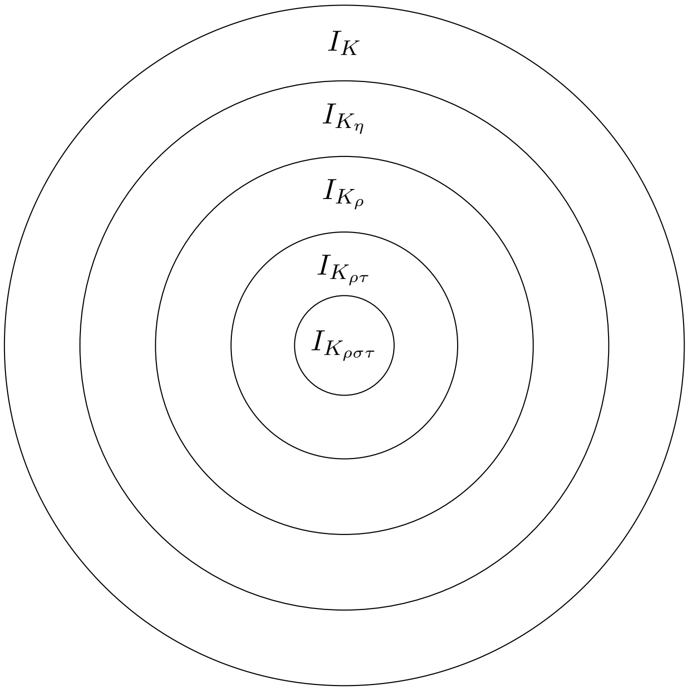
C.I. Lewis (1883–1964)

K	K_ρ	K_η	$K_{\rho\sigma}$	$K_{\rho\tau}$	$K_{\rho\sigma\tau}$
K	T	D	B	S4	S5

6.1.8 Validity

Now we have many notions of validity:
$\models_K$, $\models_{K_\rho}$, $\models_{K_\eta}$, $\models_{K_{\rho\sigma}}$, $\models_{K_{\rho\tau}}$, $\models_{K_{\rho\sigma\tau}}$, $\ldots$

For example:
$\Sigma \models_{K_{\rho\tau}} A$ iff
for all models $\langle W, R, v\rangle$ in which R is reflexive and transitive and all $w \in W$:
(if $v_w(B) = 1$ for all $B \in \Sigma$, then $v_w(A) = 1$).



In the picture above, e.g. I_{K_ρ} stands for the class of possible worlds models that satisfy the restriction ρ. Note that roughly speaking, the more restrictions, the *smaller* the set of corresponding models.

On the other hand, roughly speaking, the more restrictions, the *larger* the set of validities.

For example,

$$\not\models_K \Box p \supset p \text{ but}$$

$$\models_{K_\rho} \Box p \supset p$$

$$\not\models_{K_\rho} \Box p \supset \Box\Box p \text{ but}$$

$$\models_{K_{\rho\tau}} \Box p \supset \Box\Box p$$

More and more inferences are valid as you move towards the centre of the circle, because there are fewer and fewer (counter) models.

Special case: We will see in Section 3.5 of Priest that for restriction υ (universal relation), $I_{K_\upsilon} \subsetneqq I_{K_{\rho\sigma\tau}}$, while K_υ and $K_{\rho\sigma\tau}$ have the same validities.

6.2 Tableaux

6.2.1 reminder

$$
\begin{array}{c} \neg\neg A, i \\ \downarrow \\ A, i \end{array}
\qquad
\begin{array}{c} (A \wedge B), i \\ \downarrow \\ A, i \\ \downarrow \\ B, i \end{array}
\qquad
\begin{array}{c} \neg(A \vee B), i \\ \downarrow \\ \neg A, i \\ \downarrow \\ \neg B, i \end{array}
\qquad
\begin{array}{c} \neg(A \supset B), i \\ \downarrow \\ A, i \\ \downarrow \\ \neg B, i \end{array}
$$

$$
\begin{array}{cc} \multicolumn{2}{c}{\neg(A \wedge B), i} \\ \swarrow & \searrow \\ \neg A, i & \neg B, i \end{array}
\qquad
\begin{array}{cc} \multicolumn{2}{c}{(A \vee B), i} \\ \swarrow & \searrow \\ A, i & B, i \end{array}
\qquad
\begin{array}{cc} \multicolumn{2}{c}{(A \supset B), i} \\ \swarrow & \searrow \\ \neg A, i & B, i \end{array}
\qquad
\begin{array}{cc} \multicolumn{2}{c}{(A \equiv B), i} \\ \swarrow & \searrow \\ A, i & \neg A, i \\ \downarrow & \downarrow \\ B, i & \neg B, i \end{array}
\qquad
\begin{array}{cc} \multicolumn{2}{c}{\neg(A \equiv B), i} \\ \swarrow & \searrow \\ A, i & \neg A, i \\ \downarrow & \downarrow \\ \neg B, i & B, i \end{array}
$$

$$
\begin{array}{c} \neg\Box A, i \\ \downarrow \\ \Diamond\neg A, i \end{array}
\qquad
\begin{array}{c} \neg\Diamond A, i \\ \downarrow \\ \Box\neg A, i \end{array}
\qquad
\begin{array}{c} \Box A, i \\ irj \\ \downarrow \\ A, j \end{array}
\qquad
\begin{array}{c} \Diamond A, i \\ \downarrow \\ irj \\ A, j \end{array}
$$

The rule for $\Box A, i$ should be applied to *all* j for which irj is on the branch.

The rule for $\Diamond A, i$ introduces irj for a *new* j on the branch.

Reminder shortcuts

The rules for $\neg\Diamond A$ and $\neg\Box A$ may also be applied with a shortcut:

$$
\begin{array}{c} \neg\Box A, i \\ \downarrow \\ irj \\ \neg A, j \end{array}
\qquad
\begin{array}{c} \neg\Diamond A, i \\ irj \\ \downarrow \\ \neg A, j \end{array}
$$

Note that the shortcut rule for $\neg\Box A, i$ introduces a *new* j such that irj is introduced on the branch.

Note that the shortcut rule for $\neg\Diamond A, i$ should be applied to *all* j such that irj appears on the branch.

6.2.2 New extra rules corresponding to specific restrictions

Rule for ρ (reflexive)

$$\begin{array}{c} \bullet \\ \downarrow \\ iri \end{array}$$

Here, i is any integer on the tableau.

Rule for σ (symmetric):

$$\begin{array}{c} irj \\ \downarrow \\ jri \end{array}$$

Rule for τ (transitive):

$$\begin{array}{c} irj \\ jrk \\ \downarrow \\ irk \end{array}$$

Now we can show:

- $\vdash_{K_\rho} \Box p \supset p$ (see book)
- $\vdash_{K_\sigma} p \supset \Box\Diamond p$ (see book)
- $\vdash_{K_\tau} \Box p \supset \Box\Box p$ (see book)
- $\vdash_{K_{\sigma\tau}} \Diamond p \supset \Box\Diamond p$ (see book)
- $\nvdash_{K_\tau} \Box\Box p \supset \Box p$
- $\nvdash_{K_{\rho\sigma}} \Box p \supset \Box\Box p$ (see book)

6.3 Infinite tableaux

Rule for η (extendable):

$$\begin{array}{c} \bullet \\ \Big\downarrow \\ irj \end{array}$$

This is applied to any integer i on the branch, provided there was nothing of the form irj on the branch yet.
When applying the rule, i is not new but j is new.

Now we can show:

- $\vdash_{K_\eta} \Box p \supset \Diamond p$
- $\nvdash_{K_\eta} \Box p$.

For the second item, consider the following tableau:

$$\begin{array}{c} \neg\Box p, 0 \\ \Diamond\neg p, 0 \\ 0r1 \\ \neg p, 1 \\ 1r2 \\ 2r3 \\ 3r4 \\ \vdots \end{array}$$

Embrace the infinite and define the extendable countermodel to $\Box p$ as follows:

- $W = \{w_n \mid n \in \mathbb{N}\}$
- $R = \{\langle w_n, w_{n+1}\rangle \mid n \in \mathbb{N}\}$
- $v_{w_n}(p) = 0$ for all $n \in \mathbb{N}$

Note that the above valuation is a correct instance of the official formulation "$v_{w_1}(p) = 0$ and $v_{w_i}(p)$ can be anything one wishes for $i \neq 1$."

Also, when showing that $\nvdash_T \neg(\Diamond p \wedge \Box\Diamond p)$, the tableau method will produce an infinite open branch.

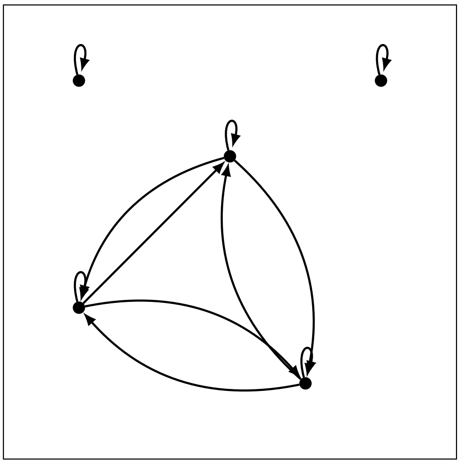
In both these cases, it is also possible to find finite counter-models.

6.4 S5

The system $K_{\rho\sigma\tau}$ is called S5.

- For all $w \in W wRw$ (reflexive);
- For all $w_1, w_2 \in W$, if $w_1 R w_2$, then $w_2 R w_1$ (symmetric);
- For all $w_1, w_2, w_3 \in W$, i if $w_1 R w_2$ and $w_2 R w_3$, then $w_1 R w_3$ (transitive).

Such relations are called *equivalence relations.*



The picture above is divided into three equivalence classes according to R.

In general, for truth of a sentence in a world, only the worlds in the same equivalence class are important. Within an equivalence class, worlds that cannot be reached by R are irrelevant; within the equivalence class, the accessibility relation is *universal.* This helps to explain why K_υ (see below) and $K_{\rho\sigma\tau} = S5$ have the same validities.

6.4.1 General restriction υ (universal)

υ ("upsilon"): $w_1 R w_2$ for all $w_1, w_2 \in W$.

If R is universal, then R is really irrelevant! This is reflected in special tableau rules corresponding to υ:

$$\begin{array}{cc} \Diamond A, i & \Box A, i \\ \downarrow & \downarrow \\ A, j & A, j \end{array}$$

j is new   for all j appearing on the branch

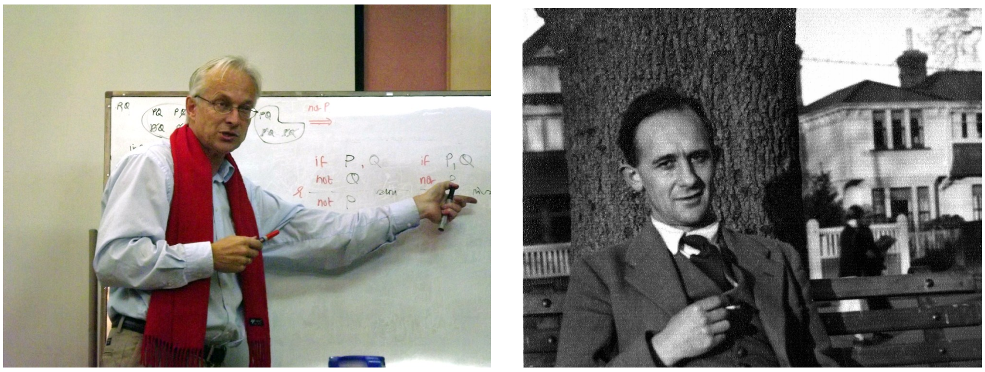


Figure 1: Johan van Benthem (1949 -) and Arthur Prior (1914 - 1969)

6.5 Tense logic

6.5.1 Language: new operators

$[P]A$	at all earlier times A
$[F]A$	at all future times A
$\langle P\rangle A$	at some earlier time A
$\langle F\rangle A$	at some future time A

6.5.2 Semantics

$v_w([P]A) = 1$	iff	for all w' such that $w'Rw$, $v_{w'}(A) = 1$
$v_w([F]A) = 1$	iff	for all w' such that wRw', $v_{w'}(A) = 1$
$v_w(\langle P\rangle A) = 1$	iff	for some w' such that $w'Rw$, $v_{w'}(A) = 1$
$v_w(\langle F\rangle A) = 1$	iff	for some w' such that wRw', $v_{w'}(A) = 1$

6.5.3 Tableau rules for the general system K^t

$$
\begin{array}{cccc}
\begin{array}{c}[F]A, i \\ irj \\ \downarrow \\ A, j\end{array} &
\begin{array}{c}\langle F\rangle A, i \\ \downarrow \\ irj \\ A, j\end{array} &
\begin{array}{c}\neg[F]A, i \\ \downarrow \\ \langle F\rangle\neg A, i\end{array} &
\begin{array}{c}\neg\langle F\rangle A, i \\ \downarrow \\ [F]\neg A, i\end{array}
\end{array}
$$

$$
\begin{array}{cccc}
\begin{array}{c}[P]A, i \\ jri \\ \downarrow \\ A, j\end{array} &
\begin{array}{c}\langle P\rangle A, i \\ \downarrow \\ jri \\ A, j\end{array} &
\begin{array}{c}\neg[P]A, i \\ \downarrow \\ \langle P\rangle\neg A, i\end{array} &
\begin{array}{c}\neg\langle P\rangle A, i \\ \downarrow \\ [P]\neg A, i\end{array}
\end{array}
$$

Note that:

- the rule for $[F]A, i$ should be applied to *all* j such that irj appears on the branch;
- the rule for $[P]A, i$ should be applied to *all* j such that jri appears on the branch;
- the rule for $\langle F\rangle A, i$ introduces a *new* j such that irj is introduced on the branch;
- the rule for $\langle P\rangle A, i$ introduces a *new* j such that jri is introduced on the branch.

Shortcuts rules for tense logic
The rules for $\neg\langle F\rangle A$, $\neg[F]A$, $\neg\langle P\rangle A$, and $\neg[P]A$ may also be applied with a shortcut:

$$
\begin{array}{cccc}
\begin{array}{c}\neg\langle F\rangle A, i \\ irj \\ \downarrow \\ \neg A, j\end{array} &
\begin{array}{c}\neg[F]A, i \\ \downarrow \\ irj \\ \neg A, j\end{array} &
\begin{array}{c}\neg\langle P\rangle A, i \\ jri \\ \downarrow \\ \neg A, j\end{array} &
\begin{array}{c}\neg[P]A, i \\ \downarrow \\ jri \\ \neg A, j\end{array}
\end{array}
$$

Note that the shortcut rules for $\neg[F]A, i$ and $\neg[P]A, i$ both introduce a *new* j such that irj is introduced on the branch.

Note that the shortcut rules for $\neg\langle F\rangle A, i$ and $\neg\langle P\rangle A, i$ should both be applied to *all* j such that irj appears on the branch.

We can show:

- $p \vdash_{K^t} [P]\langle F\rangle p$
- $p \vdash_{K^t} [F]\langle P\rangle p$

6.5.4 Additional rule for dense models: restriction δ

R is *dense* iff for all $w, z \in W$,
(if wRz, then there is a $y \in W$ such that wRy and yRz).

Tableau rule for δ:

$$\begin{array}{c} irj \\ \downarrow \\ irk \\ krj \end{array}$$

To be applied for all irj occurring on the branch;
for each such irj, the rule introduces a *new* k.

6.5.5 Extra rules for non-branching models: restrictions φ, β

Constraint φ:
R is *forward convergent* iff
for all $x, w, y \in W$, if xRy and xRz, then (zRy or $y = z$ or yRz).

Constraint β:
R is *backward convergent* iff
for all $x, w, y \in W$, if yRx and zRx, then (zRy or $y = z$ or yRz).

We will need some auxiliary tableau rules for $=$:

$$\begin{array}{c} \alpha(i) \\ i = j \\ \downarrow \\ \alpha(j) \end{array} \qquad \begin{array}{c} \alpha(i) \\ j = i \\ \downarrow \\ \alpha(j) \end{array}$$

Rule for φ:

$$\begin{array}{ccc} & irj & \\ & irk & \\ \swarrow & \downarrow & \searrow \\ jrk & j = k & krj \end{array}$$

Rule for β:

$$\begin{array}{ccc} & jri & \\ & kri & \\ \swarrow & \downarrow & \searrow \\ jrk & j = k & krj \end{array}$$