Advanced Logic 2025 - Homework exercise 6

1. Soundness and completeness

This exercise is about the soundness lemma for the normal modal logic K_σ. Consider the following tableau in K_σ, which contains only one branch which we call b:

$$
\begin{array}{rc}
1: & \neg(\Box p \supset \Box\Box p), 0 \\
2: & \Box p, 0 \\
3: & \neg\Box\Box p, 0 \\
5: & \Diamond\neg\Box p, 0 \\
6: & 0r1 \\
7: & \neg\Box p, 1 \\
8: & \Diamond\neg p, 1 \\
9: & p, 1 \\
10: & 1r2 \\
11: & \neg p, 2 \\
12: & 2r1, 1r0
\end{array}
$$

a) Is b a *complete* branch? Briefly explain why (not).

b) Branch b is *open*. Consider the interpretation $\mathcal{I} = \langle W, R, v \rangle$ with:

- $W = \{w_0, w_1\}$
- $R = \{\langle w_0, w_1 \rangle, \langle w_1, w_0 \rangle\}$
- $v_{w_0}(p) = 1$ and $v_{w_1}(p) = 0$

Moreover, consider partial function f from the natural numbers to W such that $f(1) = w_0$ and $f(0) = f(2) = w_1$.

Question: Does f show that $\mathcal{I}$ is *faithful* to b?
If so, show this step by step, considering each line from b. If not, provide a counterexample and show that it is one.

2. First-order modal logic VK_ρ with variable domain

By constructing a suitable tableau, determine whether the following inference is valid:

$$\forall x \Box Px \land \Diamond Qa \vdash_{VK_\rho} \Diamond(Pb \land Qa)$$

- Develop every branch until it either closes, is open and complete, or is open and infinite. If a branch is open and infinite, develop at least two full repetitions.

- Clearly mark which branches close, which branches are open and complete, and which branches are open and infinite.

- Do not forget to draw a conclusion from the tableau. If the inference is invalid, provide a reflexive counter-model in formal notation, and clearly state which branch you read your counter-model off of.