Lecture notes advanced logic

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7 Soundness and completeness

We discuss soundness and completeness for classical propositional logic first, as a stepping-stone for soundness and completeness for basic modal logic and normal modal logics.

7.1 Propositional logic

The following goes with Section 1.11 of the book.

7.1.1 Reminder of $\models$ and $\vdash$

Let Σ be a finite set of wffs and A a wff. For classical propositional logic:

- $\Sigma \models A$ iff for every valuation v the following holds:
  If for all $B \in \Sigma$, $v(B) = 1$, then $v(A) = 1$.
- $\Sigma \vdash A$ iff there is a closed tree whose initial list comprises the members of Σ and the negation of A.

7.1.2 Soundness for propositional logic

Definition 1 (Faithful (1.11.1 of the book))**.**
Let v be any propositional interpretation (valuation).
Let b be any branch of a tableau (not necessarily a complete branch).
Define v to be faithful *to b iff for every formula D that occurs on b, $v(D) = 1$.*

Lemma 1 (Soundness lemma (1.11.2 of the book))**.**
If v is faithful to a branch b of a tableau,
and a tableau rule is applied to b,
then v is faithful to at least one of the branches generated, b'.

Theorem 1 (Soundness theorem (1.11.3 of the book)**.**
For finite Σ: if $\Sigma \vdash A$, then $\Sigma \models A$.

The proof is by contraposition, using the Soundness lemma.

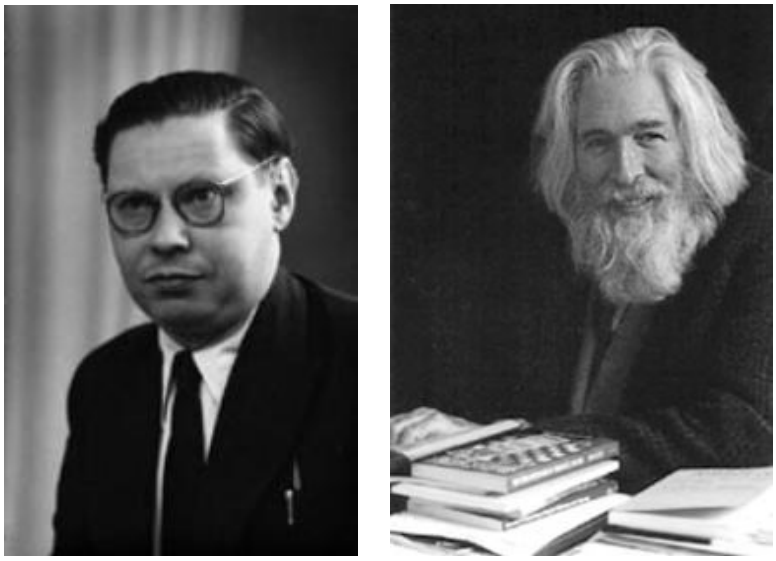

Figure 1: Evert Willem Beth (1908-1964) and Raymond Smullyan (1919-2017) proved soundness and completeness of semantic tableaux for propositional logic.

7.1.3 Completeness for propositional logic

Definition 2 (Induced interpretation (1.11.4 of the book)).
Let b be an open branch of a tableau. An interpretation induced *by b is any interpretation (valuation) v such that for every propositional parameter p:*

if p is at a node on b, then $v(p) = 1$;
and if $\neg p$ is at a node on b, then $v(p) = 0$.
(And otherwise, if neither p nor $\neg p$ appears on the branch, $v(p)$ can be anything (from 0, 1) that one likes.)

Lemma 2 (Completeness lemma (1.11.5 of the book)).
Let b be an open complete branch of a tableau.
Let v be an interpretation induced by b.
Then for all (also complex) wffs D the following hold:

if D is on b, then $v(D) = 1$;
if $\neg D$ is on b, then $v(D) = 0$.

Proof by induction on the complexity of D.

Theorem 2 (Completeness theorem (1.11.6 of the book)).
For finite Σ: if $\Sigma \models A$ then $\Sigma \vdash A$.

Proof by contraposition, using the Completeness lemma.

Figure 2: Kurt Gödel (1906-1978) proved soundness and completeness of natural deduction with respect to first-order logic, in 1929.

7.2 Basic modal logic

The following goes with Section 2.9 of the book.

7.2.1 Reminder of $\models$ and $\vdash$

Let Σ be a finite set of wffs and A a wff. For the basic modal logic K:

- $\Sigma \models A$ iff for all interpretations $\langle W, R, v \rangle$ and all $w \in W$:
  If $v_w(B) = 1$ for all $B \in \Sigma$, then $v_w(A) = 1$.
- $\Sigma \vdash A$ iff there is a closed tree whose initial list comprises:
  1. $B, 0$ for all wffs $B \in \Sigma$ and
  2. $\neg A, 0$.

7.2.2 Soundness of basic modal logic

Definition 3 (Faithful (Subsection 2.9.2 of the book)).
Let $I = \langle W, R, v \rangle$ be any modal interpretation (possible worlds model), and let b be any branch of a tableau.
Then I is faithful *to b iff there is a map f from the natural numbers to W such that:*

- *For every node D, i on b, D is true at world $f(i)$ in I.*
- *If irj is on b, then $f(i)Rf(j)$ in I.*

We say that f shows *that I is faithful to b.*

Lemma 3 (Soundness lemma (Subsection 2.9.3 of the book))**.**
Let b be any branch of a tableau, and let $I = \langle W, R, v\rangle$ be any interpretation (possible worlds model).
If I is faithful to branch b, and a tableau rule is applied to b, then that rule produces at least one extension b' such that I is faithful to b'.

Theorem 3 (Soundness theorem (Subsection 2.9.4 of the book))**.**
For finite Σ: if $\Sigma \vdash A$, then $\Sigma \models A$.

The proof is by contraposition, using the Soundness lemma.

7.2.3 Completeness of basic modal logic

Definition 4 (Induced interpretation (Subsection 2.9.5 of the book))**.**
Let b be an open branch of a tableau.
We say that an interpretation $I = \langle W, R, v \rangle$ is induced *by b iff:*

- *$W = \{w_i : i$ occurs on $b\}$;*
- *$w_i R w_j$ iff irj occurs on b;*
- *If p, i occurs on b, then $v_{w_i}(p) = 1$;*
  if $\neg p, i$ occurs on b, then $v_{w_i}(p) = 0$;
  (and otherwise $v_{w_i}(p)$ can be anything (from 0, 1) that one likes).

Lemma 4 (Completeness lemma (Subsection 2.9.6 of the book))**.**
Let b be an open complete branch of a tableau.
Let $I = \langle W, R, v \rangle$ be an interpretation induced by b.
Then for all (also complex) formulas D and for all i, the following hold:

If D, i is on b, then $v_{w_i}(D) = 1$.
If $\neg D, i$ is on b, then $v_{w_i}(D) = 0$.

Proof by induction on the complexity of D.

Theorem 4 (Completeness theorem (Subsection 2.9.7 of the book))**.**
For finite Σ: if $\Sigma \models A$, then $\Sigma \vdash A$.

The proof is by contraposition, using the Completeness lemma.

7.3 Variations of modal logic

The following goes with Section 3.7 of the book.

Theorem 5 (Soundness for some normal modal logics (Subsection 3.7.1 of the book))**.**
For finite Σ*:*
If $\Sigma \vdash_{K_\rho} A$*, then* $\Sigma \models_{K_\rho} A$*.*
If $\Sigma \vdash_{K_\sigma} A$*, then* $\Sigma \models_{K_\sigma} A$*.*
If $\Sigma \vdash_{K_\tau} A$*, then* $\Sigma \models_{K_\tau} A$*.*
If $\Sigma \vdash_{K_\eta} A$*, then* $\Sigma \models_{K_\eta} A$*.*

Proof: Check the soundness lemma, that if I is faithful to branch b of a tableau for the system in question, and a tableau rule of the system is applied to b, then this produces at least one extension b' such that I is faithful to b'.

Notice that soundness can also be shown for any system combining any subset of the above four restrictions.

Theorem 6 (Completeness for some normal modal logics (Subsection 3.7.3 of the book))**.**

For finite Σ*:*
If $\Sigma \models_{K_\rho} A$*, then* $\Sigma \vdash_{K_\rho} A$*.*
If $\Sigma \models_{K_\sigma} A$*, then* $\Sigma \vdash_{K_\sigma} A$*.*
If $\Sigma \models_{K_\tau} A$*, then* $\Sigma \vdash_{K_\tau} A$*.*
If $\Sigma \models_{K_\eta} A$*, then* $\Sigma \vdash_{K_\eta} A$*.*

Proof: Prove the completeness lemma almost as for K, and for each logic, also check that the induced model has the right properties.

Notice that completeness can also be shown for any system combining any subset of the above four restrictions.

Theorem 7 (The special case of S5 (Subsection 3.7.5 of the book))**.**
$\Sigma \models_{K_{\rho\sigma\tau}} A$ *if and only if* $\Sigma \models_{K_\upsilon} A$*.*