Advanced Logic
Lecture 1: Induction

Rineke Verbrugge

4 February 2025

Introduction to Logic: Learning goals

Welcome!

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What is Advanced Logic about?

1. There are more logics than just first-order logic (FOL) and its sublanguage propositional logic (PL):
   - three-valued logic
   - four-valued logic
   - fuzzy logic
   - several propositional modal and temporal logics
   - quantified modal logics
   - non-monotonic logics, in particular default logic

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   - three-valued logic
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   - non-monotonic logics, in particular default logic

2. Techniques to prove properties about logics:
   - Induction
   - Soundness and completeness proofs

Overview

Practical information

Inductive definitions of sets

Inductive definitions of operators

Proof by induction

Who are we?

There's a whole team of 10 people taking care of you in this course!

Lectures: Rineke Verbrugge
9 lectures and one Q & A, on Tuesdays and Thursdays

Coordination: J.D. Top and K.D. Sijtsma

See Schedule page on Brightspace for timing of lectures and tutorials.
This week: one lecture and two tutorials (Wednesday and Thursday/Friday).

E-mail your TA or J.D. Top for practical questions, e.g., change of tutorial group.

Who are we? continued

Tutorials: 6 tutorial groups, with 12 tutorials each, supervised by:

- Group A: AI yr2 gr1 – Wed 13-15; Fri 9-11 - Jan van Houten
- Group B: AI yr2 gr2 – Wed 13-15; Fri 9-11 - Amber Lubbers & Teun Boersma
- Group C: AI yr2 gr3 – Wed 13-15; Fri 9-11 - Diana Todoran & Aleksandar Todorov
- Group D: AI yr2 gr4 – Wed 13-15; Fri 9-11 - Ula Perovec
- Group E: Math & Guests – Wed 15-17; Thu 15-17 - Yoni Zuidinga & Aleksandar Todorov
- Group F: Philosophy students – Wed 15-17; Thu 15-17 - Klaas-Daniël Sijtsma

Where are we?

All **lectures are on site**, in BB 151.

All 6 **tutorial groups meet on site**. For the rooms, see the schedule.

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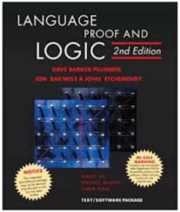
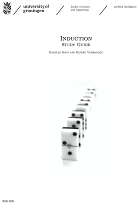
What to do in case of medical conditions

If at some moment due to a medical condition you cannot attend a tutorial of your own group, please e-mail your own TA about it and they will help find a solution.

Literature for the course (see Course Documents)

Lecture 1:





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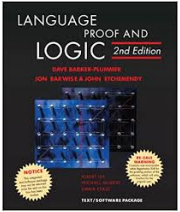
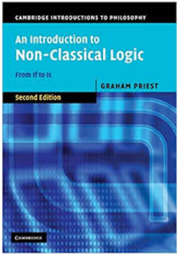
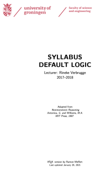


Lectures 2–8:



Lecture 9:



What are we going to do?

Advanced Logic

- Lectures
- Tutorials (supervised by TAs)
- Homework assignments ($\frac{1}{3}$ of final grade; see Course Information for computation)
  - HW 1, 3, 5, 7 are graded by us
  - HW 2, 4, 6 are peer-reviewed by you
- Final exam ($\frac{2}{3}$ of final grade)

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See the **Schedule** on Brightspace for a detailed breakdown of activities (including deadlines for homework!).

The exam will take place on Wednesday April 2, 15:00-17:00, in the Aletta Jacobs Hall.

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Homework is to be made individually;
Homework sets 1, 3, 5 on paper (both handwritten and typed are fine).

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Please submit homework sets 1, 3, 5 in box nr. 2 in the homework cupboard next to room BB 216 by 12:45 PM, the deadline on Tuesdays.

Please submit homework sets 2, 4, 6, 7 as a pdf on Brightspace (HW 7 can also be submitted in the homework cupboard).

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If you cannot submit your homework 1, 3, or 5 in paper form because of medical reasons, please e-mail your TA well before the deadline to make an arrangement.

Make the homework individually

When submitting your homework, please:

- staple you pages together;
- mention your student number, *not* your name, and the letter of your tutorial group (A, B, C, D, E or F) on the first page.

We are in favor of anonymous grading because it diminishes possible biases.

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Do not copy homework answers from other students or from the Internet.

Rules and Regulations from the Board of Examiners

Fraud is an act or omission by the examinee designed to partly or wholly hinder the forming of a correct assessment of his or her knowledge, understanding and skills.

Overview

Practical information

Inductive definitions of sets

Inductive definitions of operators

Proof by induction

Overview

Practical information

Inductive definitions of sets

- Natural numbers
- Terms of the language of arithmetic
- Well-formed formulas (wffs) of propositional logic

Inductive definitions of operators

Proof by induction

Inductive definitions: the natural numbers

Inductive definition of the set of natural numbers

1. 0 is a natural number.
2. If n is a natural number, then $n + 1$ is also a natural number.
3. Nothing is a natural number unless it can be obtained by finitely many repeated applications of 1 and 2.

1 is called the *basic clause*;
2 is called the *inductive clause*;
3 is called the *final clause*.

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Alternative definition of the set of natural numbers

Equivalently, $\mathbb{N}$ is the smallest set S closed under the following conditions:

1. $0 \in S$.
2. If $n \in S$, then $n+1 \in S$.

Terms of the language of arithmetic

Inductive definition of the set of terms of the language of arithmetic

1. The names 0 and 1 are terms
2. If t_1 and t_2 are terms, then so are $(t_1 + t_2)$ and $(t_1 \times t_2)$
3. Nothing is a term unless it is generated by finitely many repeated applications of 1 and 2.

Example (Some terms in the language of arithmetic)

$(1 + 1)$
(0×1)
$(((((0 + 1) + 1) + 1) + 1) + 1)$: the standard way to represent 5

This slide goes with Section 1.7 of 'Language, Proof and Logic'. Note that the book uses $\times$ for the multiplication symbol. In the rest of today's slides, we will use the $\cdot$ symbol. In the 'Induction Study Guide', the multiplication symbol is left out altogether.

Well-formed formulas (wffs) of propositional logic

Inductive definition of the set of propositional wffs

1. Each propositional atom $p_1, p_2, \ldots$ is a wff.
2. If P is a wff, then so is $\neg P$.
3. If P and Q are wffs, then so are $(P \wedge Q)$, $(P \vee Q)$, $(P \rightarrow Q)$, and $(P \leftrightarrow Q)$.
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Example (Some well-formed formulas and some non-wffs)

- $\neg\neg p_1$
- $((p_1 \wedge p_2) \to (p_3 \wedge p_4))$

The above two are both wffs

- $\neg\neg P$

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- $p_1 \vee p_2 \wedge p_3$

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- $p_1 \vee p_2 \wedge p_3$: Disambiguating parentheses are missing

Well-formed formulas (wffs) of propositional logic, cont.

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Steps 2 and 3 of this definition can also be combined into one step:

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- Inductive definitions of operators on natural numbers
- Inductive definitions of operators on formulas

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Inductive definitions of operators on natural numbers: : exponentiation, factorial

Suppose that $\cdot$ has already been inductively defined.

Example (An inductive definition of "2 to the power n")

- $2^0 = 1$
- $2^{n+1} = 2 \cdot 2^n$

Indeed we can now compute $2^4 = 2 \cdot 2^3 = 2 \cdot (2 \cdot 2^2) = 2 \cdot (2 \cdot (2 \cdot 2^1)) = 2 \cdot (2 \cdot (2 \cdot (2 \cdot 2^0))) = 2 \cdot (2 \cdot (2 \cdot (2 \cdot 1))) = 16$.

Inductive definitions of operators on natural numbers: : exponentiation, factorial

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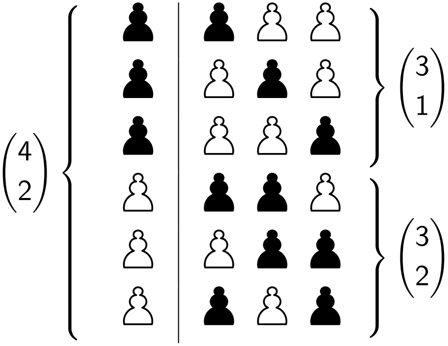
Example (An inductive definition of "n factorial")

- $0! = 1$
- $(n+1)! = (n+1) \cdot n!$

Indeed we can now compute $4! = 4 \cdot 3! = 4 \cdot (3 \cdot 2!) = 4 \cdot (3 \cdot (2 \cdot 1!)) = 4 \cdot (3 \cdot (2 \cdot (1 \cdot 0!))) = 4 \cdot (3 \cdot (2 \cdot (1 \cdot 1))) = 24$.

Informal introduction to binomial coefficients

Informally, $\binom{n}{m}$ represents the number of different ways in which you could take a subset of m objects from a collection of n objects. Say, you have to pick two volunteers out of a group of four pawns (where the black color represents being chosen as volunteer):



There are six $= \binom{4}{2}$ ways to pick two volunteers.

Note: $\displaystyle\binom{4}{2} = \binom{3}{1} + \binom{3}{2}$

Inductive definitions of operators on natural numbers: binomial coeffeicient

Example (An inductive definition of binomial coefficients)

We define $\binom{n}{m}$ by induction on n and m

$$\binom{n}{0} = 1 \qquad \text{for all } n \in \mathbb{N}$$

Inductive definitions of operators on natural numbers: binomial coeffeicient

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\binom{0}{m} &= 0 && \text{for all } m > 0 \\
\binom{n+1}{m+1} &= \binom{n}{m} + \binom{n}{m+1} && \text{for all } n, m \in \mathbb{N}
\end{aligned}
$$

Inductive definitions of operators on formulas

Defining truth valuations on all well-formed formulas

Let v be a function of the set of propositional atoms to $\{0, 1\}$
This *valuation* function is extended to all wffs as follows:

- $v(\neg P) =$

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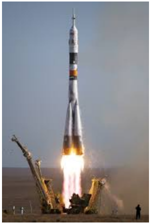
Remark: For ease of writing, we often leave out the extra parentheses when using valuations v and write, for example, $v(P \wedge Q)$.

Recap: the three stages of induction

Induction can be seen as a three-stage rocket:

1. You define a set by induction;
2. You inductively define some operators on the members of that set;
3. By induction, you prove some properties that hold for all members of that set.



So far in the lecture, we have seen a number of examples of stages 1 and 2. Now we move to the most important stage 3: **Proof by induction**.

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Proof by induction
- Proving properties of the natural numbers
- Inductive proofs on formulas

Formal inductive proofs in Peano Arithmetic (LPL 1.7)

Consider the following argument:

$$
\begin{array}{|l}
Q(0) \\
\forall x(Q(x) \rightarrow Q(x+1)) \\
\hline
Q(((0+1)+1)+1)
\end{array}
$$

This argument is valid, and we can make a formal proof:

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1.	$Q(0)$	
2.	$\forall x(Q(x) \to Q(x+1))$	
3.	$Q(0) \to Q(0+1)$	$\forall$ Elim (2)
4.	$Q(0+1)$	$\to$ Elim (1,3)
5.	$Q(0+1) \to Q((0+1)+1)$	$\forall$ Elim (2)
6.	$Q((0+1)+1)$	$\to$ Elim (4,5)
7.	$Q((0+1)+1) \to Q(((0+1)+1)+1)$	$\forall$ Elim (2)
8.	$Q(((0+1)+1)+1)$	$\to$ Elim (6,7)

We could continue the proof with 4 extra steps and show that

$$Q(((((0+1)+1)+1)+1)+1).$$

Formal inductive proofs in Peano arithmetic, contd.

So we have:

$$\begin{array}{|l} Q(0) \\ \forall x(Q(x) \to Q(x+1)) \\ \hline Q(((((0+1)+1)+1)+1)+1) \end{array}$$

In a sense, we can prove that Q holds for every natural number:

The number 0 is represented as 0;
the number 1 is represented as $0+1$;
the number 2 is represented as $(0+1)+1$; and so on.

Now for every natural number n represented this way we can show that n has property Q, by repeating the proof pattern above.

Since we can prove that all natural numbers have property Q, we would also like to conclude that $\forall x Q(x)$, for domain of discourse the set of natural numbers. Unfortunately, it is impossible to prove this conclusion in a formal proof (such as $\mathcal{F}$ in LPL).

Formal inductive proofs in Peano arithmetic, contd.

Question How to solve the problem that even if we can show $Q(n)$ for every natural number n, we cannot prove $\forall x Q(x)$?
Answer We can add this type of argument to the formal proof system in the form of the following axiom scheme of induction:

$$(Q(0) \wedge \forall x(Q(x) \rightarrow Q(x+1))) \rightarrow \forall x Q(x)$$

Induction scheme

Let $Q(x)$ represents a (possibly) complex formula in the FOL language of arithmetic, in which x occurs as a free variable. For such Q, it holds that *if*

- 0 has property Q and
- for all n, if number n has the property Q, then its successor $n+1$ also has property Q,

then the property Q holds for all natural numbers.

$Q(x)$ has the same role as, for instance, P(x) in the proof rule Universal Introduction (see page 560 of LPL).

Proving properties of the natural numbers I

Theorem For all $n \in \mathbb{N}$:

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

Proof.

Basis step $\sum_{i=0}^{0} i = 0 = \frac{0(0+1)}{2}$

Inductive hypothesis Take an arbitrary $n \in \mathbb{N}$ and suppose that $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$.

Inductive step $\sum_{i=0}^{n+1} i = n + 1 + \sum_{i=0}^{n} i$. By the inductive hypothesis this equals $n + 1 + \frac{n(n+1)}{2}$. Since $n + 1 = \frac{2(n+1)}{2}$, it must be that $n + 1 + \frac{n(n+1)}{2} = \frac{2(n+1)}{2} + \frac{n(n+1)}{2} = \frac{(n+1)(n+2)}{2}$.

Conclusion Therefore, $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

$\square$

Proving properties of the natural numbers: general structure

In general, an inductive proof of a statement such as "for all $n \in \mathbb{N}$ it holds that $Q(n)$" has the following structure:

Basis step:	A proof that $Q(0)$.
Inductive hypothesis:	Take an arbitrary $n \in \mathbb{N}$ and suppose that $Q(n)$.
Inductive step:	A proof that $Q(n+1)$ follows from the induction hypothesis.

Proving properties of the natural numbers: general structure

In general, an inductive proof of a statement such as "for all $n \in \mathbb{N}$ it holds that $Q(n)$" has the following structure:

Basis step:	A proof that $Q(0)$.
Inductive hypothesis:	Take an arbitrary $n \in \mathbb{N}$ and suppose that $Q(n)$.
Inductive step:	A proof that $Q(n+1)$ follows from the induction hypothesis.

You end the proof with a
Conclusion: Therefore, for all $n \in \mathbb{N}$, it holds that $Q(n)$.

Proving properties of the natural numbers II

Theorem For all $n \in \mathbb{N}$:

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

Proof.

Basis step $\sum_{i=0}^{0} 2^i = 1 = 2^1 - 1$

Inductive hypothesis Take an arbitrary $n \in \mathbb{N}$ and suppose that $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$.

Inductive step $\sum_{i=0}^{n+1} 2^i = 2^{n+1} + \sum_{i=0}^{n} 2^i$. By the inductive hypothesis this equals $2^{n+1} + 2^{n+1} - 1$. Now $2^{n+1} + 2^{n+1} - 1 = 2(2^{n+1}) - 1 = 2^{n+2} - 1$.

Conclusion Therefore, $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$ for all $n \in \mathbb{N}$.

$\square$

Inductive proofs on formulas; general scheme

The structure of inductive proofs on formulas is as follows:

Basis step: A proof that all propositional letters (atoms) have the property E.

Inductive hypothesis: Take arbitrary formulas P and Q and suppose that they have property E.

Inductive step: A proof that $\neg P$ has property E follows from the inductive hypothesis;
and for each binary operator $\otimes$ in the language, a proof that $(P \otimes Q)$ has property E follows from the inductive hypothesis.

Inductive proofs on formulas; general scheme

The structure of inductive proofs on formulas is as follows:

Basis step:	A proof that all propositional letters (atoms) have the property E.
Inductive hypothesis:	Take arbitrary formulas P and Q and suppose that they have property E.
Inductive step:	A proof that $\neg P$ has property E follows from the inductive hypothesis; and for each binary operator $\otimes$ in the language, a proof that $(P \otimes Q)$ has property E follows from the inductive hypothesis.

You end the proof with a

Conclusion: Therefore, property E holds for all formulas P.

Inductive proof: How is the number of binary connectives in a wff related to the number of parentheses?

Let us first inductively define *bincon*(P), the number of binary connectives and *parn*(P), the number of parentheses in wff P:

Inductive proof: How is the number of binary connectives in a wff related to the number of parentheses?

Let us first inductively define *bincon*(P), the number of binary connectives and *parn*(P), the number of parentheses in wff P:

Defining the number of binary connectives for all wffs

- *bincon*$(p_i) =$

Inductive proof: How is the number of binary connectives in a wff related to the number of parentheses?

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- $bincon((P \to Q)) = bincon(P) + bincon(Q) + 1$
- $bincon((P \leftrightarrow Q)) = bincon(P) + bincon(Q) + 1$

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Defining the number of parentheses for all wffs

- $parn(p_i) = 0$ for all propositional atoms p_i
- $parn(\neg P) = parn(P)$
- $parn((P \otimes Q)) =$

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Defining the number of parentheses for all wffs

- $parn(p_i) = 0$ for all propositional atoms p_i
- $parn(\neg P) = parn(P)$
- $parn((P \otimes Q)) = parn(P) + parn(Q) + 2$ for $\otimes \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$

Inductive proof that $parn(P) = 2 \cdot bincon(P)$ for all wffs

We will use the definitions of $bincon(P)$ and $parn(P)$ in the proof:

Proof.

Basis step For all atomic propositions p_i, we have $parn(p_i) = 0 = 2 \cdot 0 = 2 \cdot bincon(p_i)$.

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Proof.

Basis step For all atomic propositions p_i, we have $parn(p_i) = 0 = 2 \cdot 0 = 2 \cdot bincon(p_i)$.

Inductive hypothesis Suppose that for arbitrary wffs P and Q, we have $parn(P) = 2 \cdot bincon(P)$ and $parn(Q) = 2 \cdot bincon(Q)$.

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Inductive hypothesis Suppose that for arbitrary wffs P and Q, we have $parn(P) = 2 \cdot bincon(P)$ and $parn(Q) = 2 \cdot bincon(Q)$.

Inductive step Then we have:

- $parn(\neg P) = parn(P)$ (by def. of $parn$) $= 2 \cdot bincon(P)$ (by IH) $=$ $2 \cdot bincon(\neg P)$ (by def of $bincon$)

Inductive proof that $parn(P) = 2 \cdot bincon(P)$ for all wffs

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- $parn((P \otimes Q)) = parn(P) + parn(Q) + 2$ (def $parn$) $= 2 \cdot bincon(P) + 2 \cdot bincon(Q) + 2$ (by IH) $= 2 \cdot (bincon(P) + bincon(Q) + 1)$ $= 2 \cdot bincon((P \otimes Q))$ (by def. of $bincon$), for $\otimes \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$

Inductive proof that $parn(P) = 2 \cdot bincon(P)$ for all wffs

We will use the definitions of $bincon(P)$ and $parn(P)$ in the proof:

Proof.

Basis step: For all atomic propositions p_i, we have $parn(p_i) = 0 = 2 \cdot 0 = 2 \cdot bincon(p_i)$.

Inductive hypothesis: Suppose that for arbitrary wffs P and Q, we have $parn(P) = 2 \cdot bincon(P)$ and $parn(Q) = 2 \cdot bincon(Q)$.

Inductive step: Then we have:

- $parn(\neg P) = parn(P)$ (by def. of $parn$) $= 2 \cdot bincon(P)$ (by IH) $= 2 \cdot bincon(\neg P)$ (by def of $bincon$)
- $parn((P \otimes Q)) = parn(P) + parn(Q) + 2$ (def $parn$) $= 2 \cdot bincon(P) + 2 \cdot bincon(Q) + 2$ (by IH) $= 2 \cdot (bincon(P) + bincon(Q) + 1)$ $= 2 \cdot bincon((P \otimes Q))$ (by def. of $bincon$), for $\otimes \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$

Conclusion: Therefore, for all well-formed formulas P, we have $parn(P) = 2 \cdot bincon(P)$.

Inductive proofs on formulas: $\{\neg, \wedge\}$ is functionally complete

Theorem For each well-formed formula P, there is a well-formed formula P' such that:
P' contains no operators except $\neg$ and $\wedge$ and
for all valuations v, $v(P) = v(P')$.
(That is, every formula is logically equivalent to a formula where as operators, only $\neg$ and $\wedge$ can appear.)

Proof.

Proof 1 We start by defining P' inductively:

- $p_i' = p_i$ for p_i a propositional atom
- $(\neg P)' = \neg P'$
- $(P \wedge Q)' = (P' \wedge Q')$
- $(P \vee Q)' = \neg(\neg P' \wedge \neg Q')$
- $(P \rightarrow Q)' = \neg(P' \wedge \neg Q')$
- $(P \leftrightarrow Q)' = \neg(P' \wedge \neg Q') \wedge \neg(Q' \wedge \neg P')$

Inductive proof that $\neg$ and $\wedge$ suffice

You can easily prove by induction that the formulas P' contain no operators except $\neg$ and $\wedge$ (we skip that here). Now let v be any valuation. We need to prove by induction that for all well-formed formulas P, $v(P) = v(P')$.

Basis step For propositional atoms p_i, we have $p_i' = p_i$ so $v(p_i') = v(p_i)$.

Inductive hypothesis Take arbitrary formulas P and Q and suppose that $v(P) = v(P')$ and $v(Q) = v(Q')$.

Inductive proof that $\neg$ and $\wedge$ suffice: Inductive step

1. negation: $v(\neg P) = 1 - v(P) =$ (by induction hypothesis) $1 - v(P') = v(\neg P') =$ (by definition of $(\neg P)'$) $= v((\neg P)')$.

Inductive proof that $\neg$ and $\wedge$ suffice: Inductive step

1. negation: $v(\neg P) = 1 - v(P) =$ (by induction hypothesis) $1 - v(P') = v(\neg P') =$ (by definition of $(\neg P)'$) $= v((\neg P)')$.
2. conjunction
   $v(P \wedge Q) = min\{v(P), v(Q)\} =$ (by IH)
   $min\{v(P'), v(Q')\} = v(P' \wedge Q') =$ (by definition of $(P \wedge Q)'$)
   $v((P \wedge Q)')$.

Inductive proof that $\neg$ and $\wedge$ suffice: Inductive step

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   $v((P \wedge Q)')$.
3. disjunction
   $v(P \vee Q) = max\{v(P), v(Q)\} =$ (by IH)
   $max\{v(P'), v(Q')\} = v(\neg(\neg P' \wedge \neg Q')) =$ (by def. $(P \vee Q)'$)
   $v((P \vee Q)')$.

Inductive proof that $\neg$ and $\wedge$ suffice: Inductive step

1. negation: $v(\neg P) = 1 - v(P) =$ (by induction hypothesis) $1 - v(P') = v(\neg P') =$ (by definition of $(\neg P)'$) $= v((\neg P)')$.
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   $max\{v(P'), v(Q')\} = v(\neg(\neg P' \wedge \neg Q')) =$ (by def. $(P \vee Q)'$)
   $v((P \vee Q)')$.
4. conditional
   $v(P \rightarrow Q) = max\{1 - v(P), v(Q)\} =$ (by IH)
   $max\{1 - v(P'), v(Q')\} = v(\neg(P' \wedge \neg Q')) =$
   (by def. $(P \rightarrow Q)'$) $= v((P \rightarrow Q)')$.

Inductive proof that $\neg$ and $\wedge$ suffice: Inductive step

1. negation: $v(\neg P) = 1 - v(P) =$ (by induction hypothesis) $1 - v(P') = v(\neg P') =$ (by definition of $(\neg P)'$) $= v((\neg P)')$.
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   $v(P \wedge Q) = min\{v(P), v(Q)\} =$ (by IH)
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   (by def. $(P \rightarrow Q)'$) $= v((P \rightarrow Q)')$.
5. biconditional
   $v(P \leftrightarrow Q) = 1 - \mid v(P) - v(Q) \mid =$ (by IH)
   $1 - \mid v(P') - v(Q') \mid = v(\neg(P' \wedge \neg Q') \wedge \neg(Q' \wedge \neg P')) =$
   (by definition of $(P \leftrightarrow Q)'$) $= v((P \leftrightarrow Q)')$.

Inductive proof that $\neg$ and $\wedge$ suffice: Conclusion

Conclusion

Therefore, for each well-formed formula P, there is a formula P' that contains no operators except $\neg$ and $\wedge$ such that for every valuation v, $v(P) = v(P')$;
that is, for every well-formed formula, the formulas P and P' are logically equivalent.

Alternative proof of the same result, without an explicit inductive definition of P'

We can also give a differently formulated proof of the same result that every well-formed formula (wff) P is logically equivalent to a wff P' where no operators occur except $\wedge$ and $\neg$. This time the proof proceeds without an explicit separate inductive definition of P'; instead, that definition is woven into the proof step by step.

Proof.

Basis step In propositional atoms, no connectives occur. Since every formula is equivalent to itself, every propositional atom is logically equivalent to a wff where no operators occur, so certainly no operators except $\wedge$ and $\neg$.

Inductive hypothesis Take arbitrary wffs P and Q and suppose that they are logically equivalent to P' and Q' respectively, where no operators occur in P' and Q' except $\wedge$ and $\neg$.

□

Alternative proof of the same result: inductive step, 1

- negation Consider the formula $\neg P$. We know from the inductive hypothesis that P is equivalent to P' and that $\wedge$ and $\neg$ are the only operators that occur in P'. Clearly $\neg P$ is equivalent to $\neg P'$, and in $\neg P'$ no operators except $\wedge$ and $\neg$ occur.
- conjunction Consider the formula $(P \wedge Q)$. We know from the inductive hypothesis that P is equivalent to P' and that Q is equivalent to Q' where $\wedge$ and $\neg$ are the only operators that occur in P' and Q'. By the principle of substitution of logical equivalents, $(P \wedge Q)$ is equivalent to $(P' \wedge Q')$ and in $(P' \wedge Q')$ no operators except $\wedge$ and $\neg$ occur.
- disjunction Consider the formula $(P \vee Q)$. By the De Morgan laws, this formula is equivalent to $\neg(\neg P \wedge \neg Q)$. By the inductive hypothesis and the principle of substitution of equivalents, this formula is equivalent to $\neg(\neg P' \wedge \neg Q')$. Since $\wedge$ and $\neg$ are the only operators that occur in P' and Q', they are also the only operators that occur in $\neg(\neg P' \wedge \neg Q')$.

Alternative proof of the same result: inductive step, 2

- conditional Consider the formula $(P \to Q)$. With a truth table one can see that this formula is equivalent to $\neg(P \wedge \neg Q)$. By the inductive hypothesis and the principle of substitution of equivalents, this formula is equivalent to $\neg(P' \wedge \neg Q')$. Since $\wedge$ and $\neg$ are the only operators that occur in P' and Q', they are also the only operators that occur in $\neg(P' \wedge \neg Q')$.
- biconditional Consider the formula $(P \leftrightarrow Q)$. With a truth table one can see that this formula is equivalent to $\neg(P \wedge \neg Q) \wedge \neg(\neg P \wedge Q)$. By the inductive hypothesis and the principle of substitution of equivalents, this formula is equivalent to $\neg(P' \wedge \neg Q') \wedge \neg(\neg P' \wedge Q')$. Since $\wedge$ and $\neg$ are the only operators that occur in P' and Q', they are also the only operators that occur in $\neg(P' \wedge \neg Q') \wedge \neg(\neg P' \wedge Q')$.

Conclusion Therefore, every wff is equivalent to a wff where no operators occur except $\wedge$ and $\neg$.

What's next?

Thank you for your attention today.

Next: two **Tutorials** on induction, this Wednesday and Thursday/Friday (that is, two sessions for each tutorial group).

Then: **Lecture 2** next Tuesday, February 11. Topic: three-valued logics.

Due next Tuesday: **Homework assignment 1** about induction, to be made individually and submitted on site by Tuesday, February 11th, 12:45 PM.

Please submit your homework on paper in box nr. 2 in the homework cupboard next to room BB 216.

The assignment will be made available tonight in the 'Schedule'.