- Write only your student number (<u>not your name</u>) on every page you hand in, to enable anonymous grading.
- Please staple pages together.
- Please write your tutorial group letter on the first page.
- **Do not** copy homework answers from someone or something else. This homework is to be done on your own.

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**Advanced Logic 2025 - Homework exercise 5: Normal modal logics**

### Introduction

In this exercise, we introduce two new operators, $\Box_2$ and $\Diamond_2$, as well as a new model restriction, $\sigma_2$. For a detailed description, and how to write it in LaTeX, there is an appendix at the end of the assignment, but for now, I will just give you the tableau rules. The $\Box_2$ and $\Diamond_2$ operators take two steps instead of one, so they use two relations.

The tableau rules for our new operators are as follows. The rule for $\Diamond_2$ involves two new worlds, while the rule for $\Box_2$ involves a set of existing worlds.

Note that the rule for $\Box_2 A, i$ should be applied to *all* $j$ and $k$ such that $irj$ and $jrk$ appear on the branch.

Note that the rule for $\Diamond_2 A, i$ introduces a *new* $j$ and a *new* $k$ such that $irj$ and $jrk$ are introduced on the branch.

$$\begin{array}{c} \Box_2 A, i \\ irj \\ jrk \\ \downarrow \\ A, k \end{array} \qquad \begin{array}{c} \Diamond_2 A, i \\ \downarrow \\ irj \\ jrk \\ A, k \end{array} \qquad \begin{array}{c} \neg\Box_2 A, i \\ \downarrow \\ \Diamond_2 \neg A, i \end{array} \qquad \begin{array}{c} \neg\Diamond_2 A, i \\ \downarrow \\ \Box_2 \neg A, i \end{array}$$

Rule for $\sigma_2$ (two-step symmetric):

$$\begin{array}{c} irj \\ jrk \\ \downarrow \\ kri \end{array}$$

Do not confuse two-step symmetry with regular symmetry, or with transitivity!

**Exercise 1**

By constructing a suitable tableau, determine whether the following inferences are valid in the logic mentioned. Make sure you:

- Develop every branch until it either closes, is open and complete, or is open and infinite. If a branch is open and infinite, develop at least two full repetitions.
- Clearly mark which branches close, which branches are open and complete, and which branches are open and infinite.
- If the inference is invalid, provide a counter-model, and clearly state which branch you read your counter-model off of.
- Draw a conclusion.

1. $\vdash_{K_{\eta\sigma_2}} \boxed{2}\Box p \supset \Diamond_2 \neg p$
   Here we use $\eta$ (extendable), and $\sigma_2$ (two-step symmetry).
2. $\vdash_{K^t_{\sigma_2}} (\langle F\rangle\langle F\rangle p \supset \langle P\rangle p) \wedge (p \vee \neg p)$
   Here we use $\sigma_2$ (two-step symmetry)

## Appendix

Reading this appendix is not needed for doing the assignment, but may be interesting for some students.

In this exercise, we introduce two-step relations in Kripke models. To this end, we invented some new operators, and some new model restrictions.

Our new language is as follows:

1. Each propositional parameter (atom) is a formula.
2. If $A$ is a formula, then so are $\neg A$, $\Box A$, $\Diamond A$, $\boxed{2} A$, and $\Diamond_2 A$.
3. If $A$ and $B$ are formulas, then so are $(A \wedge B)$, $(A \vee B)$, $(A \supset B)$, and $(A \equiv B)$.
4. Nothing is a wff unless it is generated by finitely many repeated applications of 1, 2, and 3.

Note that we have two new operators: $\Diamond_2$, a diamond with a little '2' in it, and $\boxed{2}$, a box with a little '2' in it. These are new modal operators.

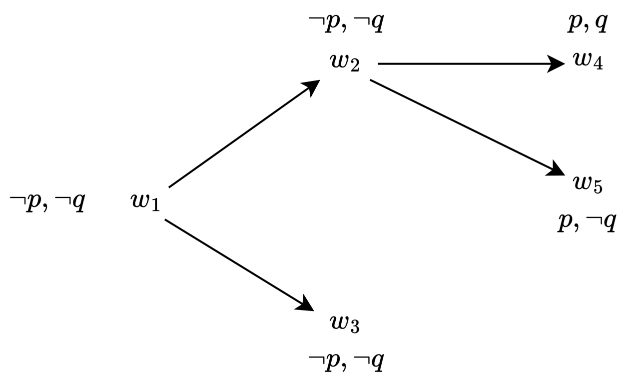
As an example, $(\neg\boxed{2}\neg p \equiv \Diamond_2 p)$ is a formula, and $((\boxed{2} p \wedge \boxed{2}(p \supset q)) \supset \boxed{2} q)$ is a formula.

The valuation of our new operators is defined as follows:

$v_w(\Diamond_2 A) = 1$ iff there is a world $w' \in W$ such that $wRw'$ and
there is a world $w''$ such that $w'Rw''$ and $v_{w''}(A) = 1$

$v_w(\boxed{2} A) = 1$ iff for every world $w' \in W$ such that $wRw'$ and
for every world $w''$ such that $w'Rw''$ it holds that $v_{w''}(A) = 1$

Here is an example Kripke model:

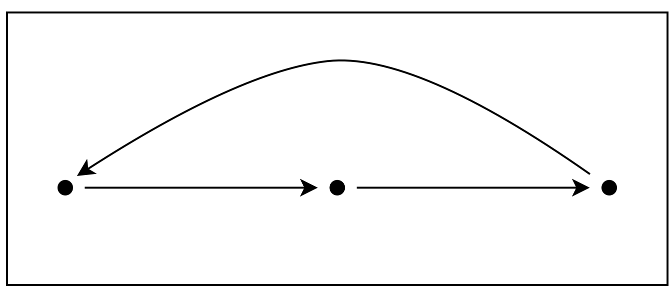



In this example, we have that:

$$\begin{array}{lcll}
v_{w_1}(\boxed{\scriptscriptstyle 2}\,p) & = & 1 & \text{Because } p \text{ is true in } w_4 \text{ and } w_5. \\
v_{w_1}(\Diamond\!\!\!\!{\scriptscriptstyle 2}\;\, q) & = & 1 & \text{Because } q \text{ is true in } w_4. \\
v_{w_1}(\boxed{\scriptscriptstyle 2}\,\neg p) & = & 0 & \text{Because } \neg p \text{ is false in } w_4\text{, and in } w_5. \\
v_{w_1}(\boxed{\scriptscriptstyle 2}\,\Box p) & = & 1 & \text{This is vacuously true, because we cannot take three steps from } w_1. \\
v_{w_1}(\Diamond\!\!\!\!{\scriptscriptstyle 2}\;\,\Diamond p) & = & 0 & \text{No world where } p \text{ is true can be reached in three steps from } w_1.
\end{array}$$

Now we can define a *new* model restriction called 'two-step symmetry':

**Restriction** $\sigma_2$
$R$ is *two-step symmetric* if for all $w_1, w_2, w_3 \in W$:
if $w_1 R w_2$ and $w_2 R w_3$, then $w_3 R w_1$.



This is a lot like transitivity, except the new arrow is pointing backwards.

In LaTeX, you need the package 'scalerel' for our new operators, so make sure you have `\usepackage{scalerel}` at the top of your .tex file. After the package declarations, I added

```
\newcommand\dd{\Diamond\!\!\!\raisebox{0.05cm}{\scaleobj{0.5}{2}}\ }
```

and

```
\newcommand\bb{\Box\!\!\!\raisebox{0.05cm}{\scaleobj{0.5}{2}}\ }
```

Then, you can use the commands `\bb` and `\dd` in math mode to get a box and diamond with a little '2' in it.