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Induction

Study Guide

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1 Introduction

This document provides a guide to studying the material on induction provided in Chapter 16 of *Language, Proof and Logic*. Chapter 16 does not follow the usual introduction to induction. We have found that many students would benefit from a more standard introduction to induction, which starts with induction on the natural numbers. This study guide will provide a guide to reading Chapter 16 in the following order: Formal and informal inductive proofs on natural numbers are discussed first, and then induction is introduced more generally.

After a short refresher on the language of arithmetic in Section 1.7 of *Language, Proof and Logic*, it is a good idea to read Section 16.4 first, and then 16.1–16.3. In 16.3, the focus is again induction on the natural numbers, but now it is viewed from a more general perspective.

2 Formal inductive proofs in Peano Arithmetic

Read Section 1.7 of *Language, Proof and Logic*. Now consider the following argument.

$$\begin{array}{|l}
Q(0) \\
\forall x(Q(x) \to Q(x+1)) \\
\hline
Q(((0+1)+1)+1)
\end{array}$$

This argument is valid, and we can make a formal proof to show this.

$$\begin{array}{|lll}
1. & Q(0) & \\
2. & \forall x(Q(x) \to Q(x+1)) & \\
\hline
3. & Q(0) \to Q(0+1) & \forall \text{ Elim } (2) \\
4. & Q(0+1) & \to \text{ Elim } (1,3) \\
5. & Q(0+1) \to Q((0+1)+1) & \forall \text{ Elim } (2) \\
6. & Q((0+1)+1) & \to \text{ Elim } (4,5) \\
7. & Q((0+1)+1) \to Q(((0+1)+1)+1) & \forall \text{ Elim } (2) \\
8. & Q(((0+1)+1)+1) & \to \text{ Elim } (6,7)
\end{array}$$

This formal proof can be extended with two additional steps to show that

$$Q((((0+1)+1)+1)+1),$$

and we could continue the proof and show that

$$Q(((((0+1)+1)+1)+1)+1).$$

In a sense, we can prove that the property Q holds for every natural number.

The number 0 is represented as 0, the number 1 is represented as $0+1$, the number 2 is represented as $(0+1)+1$, and so on. So all natural numbers can be represented in this way. Therefore, for every natural number n (represented this way) we can show that n has property Q, by repeating the pattern of the proof above.

Since we can prove that all natural numbers have property Q as described above, we would also like to conclude that $\forall x Q(x)$, given that the domain of discourse is the set of natural numbers. Unfortunately, it is impossible to arrive at this conclusion in a formal proof from the system presented in *Language, Proof and Logic*. To solve this problem, we can simply add this type of argument to the formal proof system in the form of the following axiom scheme:

$$(Q(0) \wedge \forall x(Q(x) \rightarrow Q(x+1))) \rightarrow \forall x Q(x)$$

This scheme is called the *induction scheme.* You can apply it to any property Q of natural numbers that can be stated in the language of first-order arithmetic. For such Q, it holds that *if*

- 0 has property Q and
- for all n, if number n has the property Q, then its successor $n+1$ also has property Q,

then the property Q holds for all natural numbers.

Here, $Q(x)$ represents a (possibly) complex formula in which x occurs as a free variable. For instance $x = x$ could be such a formula. $Q(x)$ has the same role as, for instance, $\mathsf{P(x)}$ in the proof rule Universal Introduction (see page 560 of *Language, Proof and Logic*).

Now read Section 16.4 of *Language, Proof and Logic*, in which Peano Arithmetic is introduced. Do exercises 16.19–16.23.

3 Pascal's Triangle

The statements proved in Section 16.4 of *Language, Proof and Logic* are statements that we usually take for granted and for which we would usually not be inclined to give an informal inductive proof.

Interestingly, in the history of mathematics, a lot of very complex mathematics was developed long before inductive proofs were ever used. In 1653, Blaise Pascal wrote a *Traité du Triangle Arithmétique* [4, 3], which is one of the earliest works to use inductive proofs (see [2, 1] for further historical details).

In Figure 1, you can see (part of) Pascal's Triangle as it was originally published. The triangle is built up in the following way. Each cell in the leftmost column and upmost row contains 1. For each other cell the value is determined by taking the sum of the adjacent cells directly above and to the left of the cell. For instance, $\psi = \varphi + \sigma$ $(2 = 1+1)$ and $F = E + C$ $(10 = 6+4)$.

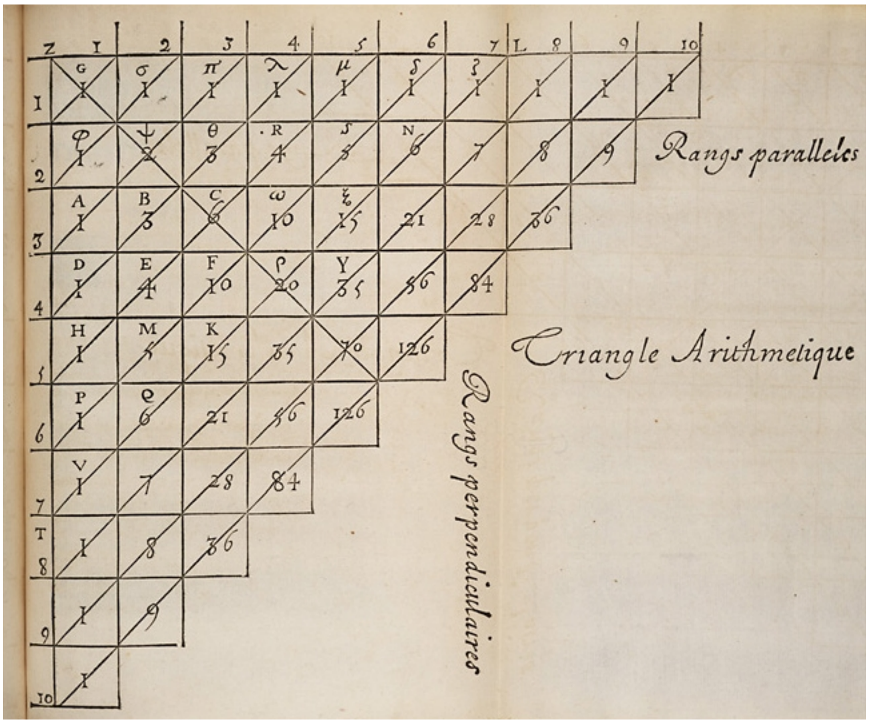


Figure 1: Pascal's Triangle.

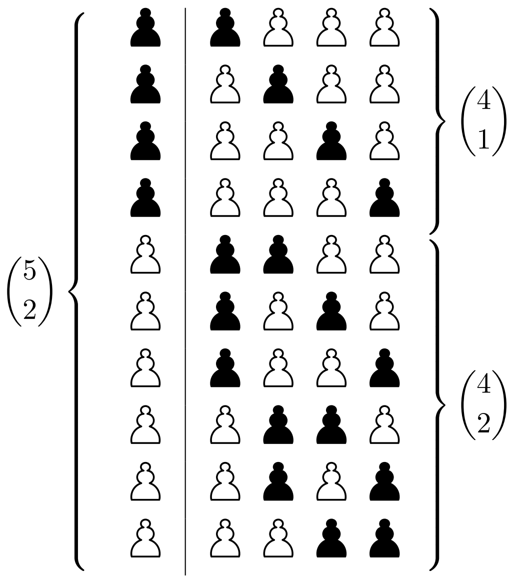


Figure 2: Picking two volunteers from a group of five.

One way to view Pascal's triangle is by viewing the numbers as *binomial coefficients* $\binom{n}{m}$. Informally, the binomial coefficient $\binom{n}{m}$ represents the number of different ways in which you could take a subset of m objects from a collection of n objects. Say, you have to pick two volunteers out of a group of five people. Observe that there are ten ways to pick those two volunteers (see Figure 2). This corresponds to the cell F.

n \ m	0	1	2	3	4	5	6	7	8	9
0	1	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0
2	1	2	1	0	0	0	0	0	0	0
3	1	3	3	1	0	0	0	0	0	0
4	1	4	6	4	1	0	0	0	0	0
5	1	5	10	10	5	1	0	0	0	0
6	1	6	15	20	15	6	1	0	0	0
7	1	7	21	35	35	21	7	1	0	0
8	1	8	28	56	70	56	28	8	1	0
9	1	9	36	84	126	126	84	36	9	1

Table 1: A table with binomial coefficients

When we make a table with the binomial coefficients (Table 1) we see that it is a simple transformation of Pascal's triangle.

The binomial coefficients can be systematically calculated as follows. There is only one way to pick no volunteers from a group of any size. There is no way to pick more than zero volunteers out of a group of zero people. Picking $m + 1$ volunteers from a group of $n + 1$ people can be calculated as follows, as is also illustrated in Figure 2. Take one person apart and suppose that she is one of the volunteers. Then there are still m volunteers to be picked from the n remaining people. If she is not one of the volunteers, then one still has to pick $m+1$ volunteers from the n remaining people.

In this way we arrive at the following definition:

Definition 1 (binomial coefficients)**.**

$$\begin{aligned} \binom{n}{0} &= 1 && \textit{for all } n \in \mathbb{N} \\ \binom{0}{m} &= 0 && \textit{for all } m > 0 \\ \binom{n+1}{m+1} &= \binom{n}{m} + \binom{n}{m+1} && \textit{for all } n, m \in \mathbb{N} \end{aligned}$$

The last clause captures the idea that in Table 1, a cell contains the sum of the cell directly above it and the cell one row higher and one column to the left.

One can show properties of binomial coefficients by inductive proofs. Let us for instance show that for all $n, m \in \mathbb{N}$ it holds that:

$$\binom{n}{m} = 0 \text{ if } m > n \tag{1}$$

Proof. We have to proceed carefully. There are two numbers involved in the statement, n and m, and we have to choose first whether we will show that the statement is true by induction on n or on m. In other words, should we start the proof when $n = 0$ or when $m = 0$? In the definition above we see that when $n = 0$, Definition 1 already states that $\binom{0}{m} = 0$ for all $m > 0$. Therefore we proceed by induction on n. That means we have to prove two things.

A for all $m \in \mathbb{N}$ it holds that $\binom{0}{m} = 0$ if $m > 0$.

B "If the statement is true for some arbitrary number n, then the statement also holds for the next number $n + 1$". Thus, we have to show that for all $n \in \mathbb{N}$, *if* for all $m \in \mathbb{N}$ it holds that $\binom{n}{m} = 0$ if $m > n$, *then* for all $m \in \mathbb{N}$ it holds that $\binom{n+1}{m} = 0$ if $m > n + 1$.

Let us make it clear that this is simply an informal instance of the axiom scheme of induction (page 2). Let us define the property Φ as follows:

$$\Phi(x)\text{: for all } m \in \mathbb{N} \text{ it is the case that } \binom{x}{m} = 0 \text{ if } m > x$$

Now we can reformulate the basis step and the inductive step as follows:

A′ $\Phi(0)$

B′ For all $n \in \mathbb{N}$: if $\Phi(n)$, then $\Phi(n + 1)$.

The principle of induction tells us that once we have proved **A** and **B**, then we can conclude that (1) is true. By **A**, $\Phi(0)$ holds and by **B**, we can then conclude that $\Phi(1)$, and therefore that $\Phi(2)$ and so on, by repeatedly applying **B**. Therefore for all $n \in \mathbb{N}$, it holds that $\Phi(n)$.

We start by proving **A**. Take m arbitrarily. Suppose that $m > 0$. By Definition 1, it now follows that $\binom{0}{m} = 0$.

For the proof of **B**, take n arbitrarily. Suppose that

$$\text{for all } m \in \mathbb{N} \text{ it holds that } \binom{n}{m} = 0 \text{ if } m > n. \tag{*}$$

In order to show that for all $m \in \mathbb{N}$ it holds that $\binom{n+1}{m} = 0$ if $m > n+1$, we take an arbitrary m and suppose that $m > n+1$. By the third clause of Definition 1, $\binom{n+1}{m} = \binom{n}{m-1} + \binom{n}{m}$. From (*) it follows that $\binom{n}{m-1} = 0$ and $\binom{n}{m} = 0$. Therefore $\binom{n+1}{m} = 0$. So the second statement is also true, namely for all $m \in \mathbb{N}$ it holds that $\binom{n+1}{m} = 0$ if $m > n+1$.

Now we can conclude that (1) is true. □

Let us introduce some standard terminology for three elements that you will find in inductive proofs:

- The proof of a statement such as **A** is called the *basis step* of the inductive proof.
- The proof of a statement such as **B** is called the *inductive step* of the inductive proof.
- In the proof of (1), we had supposed that (*) is true. Such a statement is called the *inductive hypothesis.*

As you can see, (*) looks very much like (1). This gives many students the impression that an inductive proof contains a vicious circle. Note, however, that in (1), the n is universally quantified, whereas in (*), the n is not quantified, but it represents some arbitrary given natural number. It may be confusing that in informal proofs there is no strict distinction between variables and individual constants. In a formal proof, one would have used a universally quantified x instead of n in (1), and in (*) one would have used some constant c which occurred nowhere else but in the relevant subproof.

So generally, an inductive proof of a statement such as "for all $n \in \mathbb{N}$ it holds that $\Psi(n)$" has the following structure:

Basis step:	A proof that $\Psi(0)$.
Inductive hypothesis:	Take an arbitrary $n \in \mathbb{N}$ and suppose that $\Psi(n)$.
Inductive step:	A proof that $\Psi(n+1)$ follows from the inductive hypothesis.

Now it is time for you to make inductive proofs. It is possible to *prove* that the top row of the triangle (the diagonal in Table 1) only contains 1:

Exercise 1. *Show by induction on n that for all $n \in \mathbb{N}$*

$$\binom{n}{n} = 1$$

Hint: in the inductive step you will need (1).

It is also possible to prove that the second column of the triangle lists the natural numbers:

Exercise 2. *Show by induction on n that for all $n \in \mathbb{N}$*

$$\binom{n}{1} = n$$

Exercise 3. *Show by induction on n that for all $n \in \mathbb{N}$*

$$\binom{n+1}{n} = n+1$$

This can be further generalized. Note that the whole triangle is symmetric; this can be proved, see the next exercise.

Remark Exercises 4-8 are very challenging if you are not used to working with binomial coefficients. You don't need to worry if you cannot make them independently the first time around.

Exercise 4. *Show by induction on n that for all $n, m \in \mathbb{N}$, if $m \leq n$, then*

$$\binom{n}{m} = \binom{n}{n-m}$$

Exercise 5. *Show by induction on n that for all $n, m \in \mathbb{N}$*

$$\sum_{k=0}^{n} \binom{k}{m} = \binom{n+1}{m+1}$$

There is a connection between Pascal's triangle and powers of 2. Can you prove the following?

Exercise 6. *Show by induction on n that for all $n \in \mathbb{N}$*

$$\sum_{m=0}^{n} \binom{n}{m} = 2^n$$

Another way of thinking about binomial coefficients is in terms of factorials.

Definition 2 (factorials)**.**

$$\begin{array}{lcl} 0! & = & 1 \\ (n+1)! & = & (n+1)n! \quad \textit{for all } n \in \mathbb{N} \end{array}$$

Exercise 7. *Show by induction on n that for all $n, m \in \mathbb{N}$ it holds that if $m \leq n$:*

$$\binom{n}{m} = \frac{n!}{(n-m)!m!}$$

In order to prove the statement in the following exercise, Pascal used induction for the first time in his *Traité*. You can give a proof of this statement using the result in Exercise 7 and then proceeding without further induction.

Exercise 8. *Show that for all $n, m \in \mathbb{N}$ if $m \leq n$, then*

$$\frac{\binom{n}{m+1}}{\binom{n}{m}} = \frac{n-m}{m+1}$$

4 Induction in general

Definitions such as Definitions 1 and 2 are called *inductive* or *recursive* definitions (inductive, because the definition has the same structure as an inductive proof; recursive, because what is being defined also appears on the right-hand side of the inductive clause of the definition). From a more general perspective, not only the natural numbers can be given by way of an inductive definition, which therefore allows inductive proofs. Also formulas of formal languages (such as logical languages and programming languages) and tree structures can be given by an inductive definition. In the *Advanced Logic* course, inductive proofs for logical languages will often be used. The book *Language, Proof and Logic* focuses on these as well, and takes them as a starting point. By now, you should have a good understanding of induction on the natural numbers. Now you can read the introduction of Chapter 16 and Sections 16.1–16.3 (which finishes with induction on the natural numbers). As you are reading, do exercises 16.3–16.9. 16.11–16.13, 16.14–16.18.

5 Induction on formulas

The inductive proofs in this course will mainly be concerned with logical languages. Such inductive proofs also appear in *An Introduction to Non-classical Logic* by Graham Priest. Priest's terminology is slightly different than the terminology we introduced on page 5. Priest uses *base case*, *induction case* and *induction hypothesis* for respectively basis step, inductive step and inductive hypothesis. Read Section 0.2 from *An Introduction to Non-classical Logic*. Note also that Priest's notation for the material conditional ($\rightarrow$) is "$\supset$" and that his notation for the biconditional ($\leftrightarrow$) is "$\equiv$". The structure of inductive proofs on formulas is as follows:

Basis step: A proof that all propositional letters have the property E.

Inductive hypothesis: Take arbitrary formulas P and Q and suppose that they have property E.

Inductive step: For each logical operator $\otimes$ in the language, a proof that $P \otimes Q$ has property E follows from the inductive hypothesis.

What the atomic formulas are and what the logical operators are determines the structure of the proof. Here are some additional exercises concerning inductive proofs on formulas.

Exercise 9. *Consider the language of propositional logic without negation, i.e. the negation-free well-formed formulas, abbreviated as* neg-free-wffs.

i Each propositional letter p is a neg-free-wff.

ii If A and B are neg-free-wffs, then so are $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$ and $(A \leftrightarrow B)$.

iii Nothing is a neg-free-wff unless it is generated by finitely many repeated applications of i and ii.

Also consider the valuation v that assigns truth value 1 to all propositional letters.

Show by induction that $v(A) = 1$ for all neg-free-wffs.

Another useful way in which induction is often used is in an *inductive definition of an operator* on members of an inductively defined set – for example, the length of a formula of propositional logic:

Exercise 10. *The length of a formula P of propositional logic, $l(P)$ without counting its parentheses, can be defined as follows:*

$$\begin{array}{lcl} l(P) & = & 1 \textit{ if } P \textit{ is an atomic wff} \\ l(\neg P) & = & 1 + l(P) \\ l((P \wedge Q)) & = & l(P) + 1 + l(Q) \\ l((P \vee Q)) & = & l(P) + 1 + l(Q) \\ l((P \rightarrow Q)) & = & l(P) + 1 + l(Q) \\ l((P \leftrightarrow Q)) & = & l(P) + 1 + l(Q) \end{array}$$

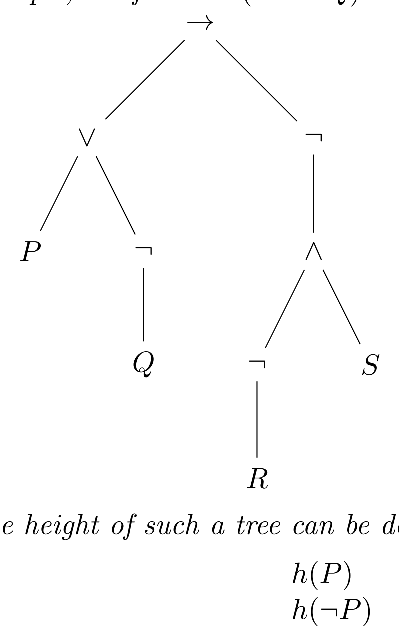
The set of atomic wffs occurring in a formula P of propositional logic, $a(P)$, can be defined as follows:

$$\begin{array}{lcl} a(P) & = & \{P\} \textit{ if } P \textit{ is an atomic wff} \\ a(\neg P) & = & a(P) \\ a((P \wedge Q)) & = & a(P) \cup a(Q) \\ a((P \vee Q)) & = & a(P) \cup a(Q) \\ a((P \rightarrow Q)) & = & a(P) \cup a(Q) \\ a((P \leftrightarrow Q)) & = & a(P) \cup a(Q) \end{array}$$

Let $\#(S)$ denote the number of elements of set S. Show by induction that for every wff P:

$$\#a(P) \leq l(P)$$

Exercise 11. *One can represent a formula of propositional logic as a kind of tree. For example, the formula $(P \vee \neg Q) \rightarrow \neg(\neg R \wedge S)$ can be represented as the following tree:*



The height of such a tree can be defined as follows:

$$\begin{array}{lcl} h(P) & = & 0 \textit{ if } P \textit{ is an atomic wff} \\ h(\neg P) & = & 1 + h(P) \\ h((P \wedge Q)) & = & 1 + \max(h(P), h(Q)) \\ h((P \vee Q)) & = & 1 + \max(h(P), h(Q)) \\ h((P \rightarrow Q)) & = & 1 + \max(h(P), h(Q)) \\ h((P \leftrightarrow Q)) & = & 1 + \max(h(P), h(Q)) \end{array}$$

Show by induction that for every wff P:

$$h(P) < l(P)$$

Exercise 12. *Show by induction that for every wff P:*

$$\#(a(P)) \leq 2^{h(P)}$$

Exercise 13. *Consider the language of propositional logic without negation, implication (conditional) and bi-implication (biconditional), i.e. the* neg-imp-free-wffs.

i *Each propositional letter p is a neg-imp-free-wff.*

ii *If A and B are neg-imp-free-wffs, then so are $(A \wedge B)$ and $(A \vee B)$.*

iii *Nothing is a neg-imp-free-wff unless it is generated by finitely many repeated applications of i and ii.*

If two valuations v and v' are given, let us say that $v \prec v'$ iff for all propositional letters p, if $v(p) = 1$, then $v'(p) = 1$.

Show by induction that if $v \prec v'$, then for all neg-imp-free-wffs A: if $v(A) = 1$, then $v'(A) = 1$.

Exercise 14. *A formula has as subformulas itself and all formulas that appear in its construction. For example, the formula $(p \wedge q) \vee (p \to \neg r)$ has as subformulas itself, $(p \wedge q)$, $(p \to \neg r)$, the first p, q, the second p, $\neg r$, and r; altogether 8 subformulas.*

1. *Define by induction the operator subf that gives for each formula P the number of its subformulas.*

2. *What can you conclude when you compare a formula's length (see Exercise 10) to its number of subformulas? Prove by induction.*

Exercise 15. *A connective c_1 in a formula* **governs** *a connective c_2 in that formula if and only if c_2 occurs in the scope of c_1. For example, in the sentence $(p \wedge q) \vee (p \to \neg r)$, the $\vee$ governs the $\wedge$, the $\to$ and the $\neg$. The connective-depth of a formula P is the maximal length of a sequence ("nest") of connectives of which each one governs the next one in P. For the example, connective-depth$((p \wedge q) \vee (p \to \neg r))$ = 3.*

Define the operator connective-depth by induction on formulas.

Exercise 16. *Let $\mathscr{L}_{\wedge,\vee}$ be a restricted language of propositional logic based on only operators $\vee$ and $\wedge$. This language is defined inductively as follows:*

i *Each propositional letter p is a formula of $\mathscr{L}$.*

ii *If A and B are formulas of $\mathscr{L}_{\wedge,\vee}$, then so are $(A \wedge B)$ and $(A \vee B)$.*

iii *Nothing is a formula of $\mathscr{L}_{\wedge,\vee}$ unless it is generated by finitely many repeated applications of i, ii and iii.*

1. *Give an inductive definition of $v(A)$, the truth value of formula A in the language $\mathscr{L}_{\wedge,\vee}$ under valuation v.*

2. *Let valuation v_1 be given such that $v_1(p) = 1$ for all propositional letters p. Prove by induction that $v_1(A) = 1$ for all formulas A in $\mathscr{L}_{\wedge,\vee}$.*

3. *Is $\{\wedge, \vee\}$ functionally complete, i.e. is it the case that every formula of propositional logic is equivalent to a formula in $\mathscr{L}_{\wedge,\vee}$?*

6 Answer to exercise

Answer to exercise 10. We have to show by induction that for every wff P: $\#a(P) \leq l(P)$.

Basic step Take an arbitrary propositional letter p. Clearly $a(p) = \{p\}$ and so $\#a(p) = 1$. Also $l(p) = 1$. Therefore $\#a(p) \leq l(p)$.

Inductive hypothesis Take two arbitrary wffs A and B. Suppose that $\#a(A) \leq l(A)$ and $\#a(B) \leq l(B)$.

Inductive step

negation Consider the wff $\neg A$. By definition, $a(\neg A) = a(A)$ and so $\#a(\neg A) = \#a(A)$. By the inductive hypothesis, it is the case that $\#a(A) \leq l(A)$. Since $l(\neg A) = 1 + l(A)$, it follows that $\#a(\neg A) \leq l(\neg A)$.

conjunction Consider the wff $(A \wedge B)$. By definition, $a(A \wedge B) = a(A) \cup a(B)$ and so $\#a(A \wedge B) \leq \#a(A) + \#a(B)$. Since $l(A \wedge B) = 1 + l(A) + l(B)$, it follows from the inductive hypothesis that $\#a(A \wedge B) \leq l(A \wedge B)$.

The cases for disjunction, conditional and biconditional are analogous to the case for conjunction.

Therefore, by induction, for every wff P, it is the case that $\#a(P) \leq l(P)$.

References

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