135A: Discussion Week 4 Manik Seth

Names (of all group members):

NEW RULE: If you don't come to discussion you must solve, or turn in a good approach at solving ALL problems. If you come to discussion and don't finish all problems, then you can tell me and mark it on the sheet of paper I'll bring along.

Problem 1: Can an event be independent "of itself"?

Can an event be independent "of itself", i.e. if A is an event, can A and A be independent? Explain your answer. (Hint: Write down the definition of being independent and look what it means.)

$P(A|A) = \frac{P(A \cap A)}{P(A)} = P(A)P(A)$ can be independent only if $P(A)=1$

Problem 2: Random Variables – Sum of two dice

Recall, a random-variable is a function $X : \Omega \to \mathbb{R}$. We write $\mathbb{P}(X = x) := \mathbb{P}(\{\omega \in \Omega : X(\omega) = x\})$. In words: the probability that a random variable has some value x, is the probability that an outcome occurs, whose value (under the random variable) is x.

The set-up of this problem is to roll two fair dice. Abstractly, let $\Omega = \{1, \ldots, 6\}^2$ (all tuples with entries in $1, \ldots, 6$), and define the probability $\mathbb{P}(\{\omega\}) = \frac{1}{36}$ (and the set of events is $\mathcal{P}(\Omega)$ which means all subset of Ω).

(a) Let Z be a random variable defined as $Z((i, j)) := i + j$. Determine $\mathbb{P}(Z = 4)$.

$(1,3)$ $(3,1)$
$(2,2)$

$P(Z=4) = P(i+j$

$3P(\{w\}) = \boxed{\frac{3}{36}}$

(b) **Fact:** We can add random variables. If X, Y are random variables, then $X+Y$ is a "new" random variable. Now let X, Y be random variables defined as $X((i,j)) = i$ and $Y((i,j)) := j$. Determine $\mathbb{P}(X+Y=4)$ (by hand!).

$(1,3)$ (i,j)

$(1,3)$ $X=1$ $Y=3$

$(2,2)$ $X=2$ $Y=2$

$(3,1)$ $X=3$ $Y=1$

$[P(X=1)\cdot P(Y=3)]+[P(X=2)\cdot P(Y=2)]+[P(X=3)\cdot P(Y=1)]$

$= \frac{1}{36}+\frac{1}{36}+\frac{1}{36} = \boxed{\frac{3}{36}}$

Problem 3: Expected Value: Sum of two dice

(a) Let Z be the random variable, as in the set-up of problem 2(a). For $i \in \{2,\dots,6\}$ how do $\mathbb{P}(Z=i)$ and $\mathbb{P}(Z=12-i)$ relate? Next, determine $\mathbb{P}(X=2), \mathbb{P}(X=3), \mathbb{P}(X=4), \mathbb{P}(X=5), \mathbb{P}(X=6)$ and $\mathbb{P}(X=7)$. Lastly, what is $\mathbb{P}(X=x)$ for any $x \in \mathbb{R} \setminus \{2,\dots,12\}$?

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No x, all Z

$P(X=x)=0$

Equal, when top 2 side $=i$, underside $=14-i$

$P(Z=2) = 2P(\{w\}) = \frac{1}{36}$

$P(Z=3) = 2[2P(w)] = \frac{2}{36}$

$P(Z=4) = \frac{3}{36}$

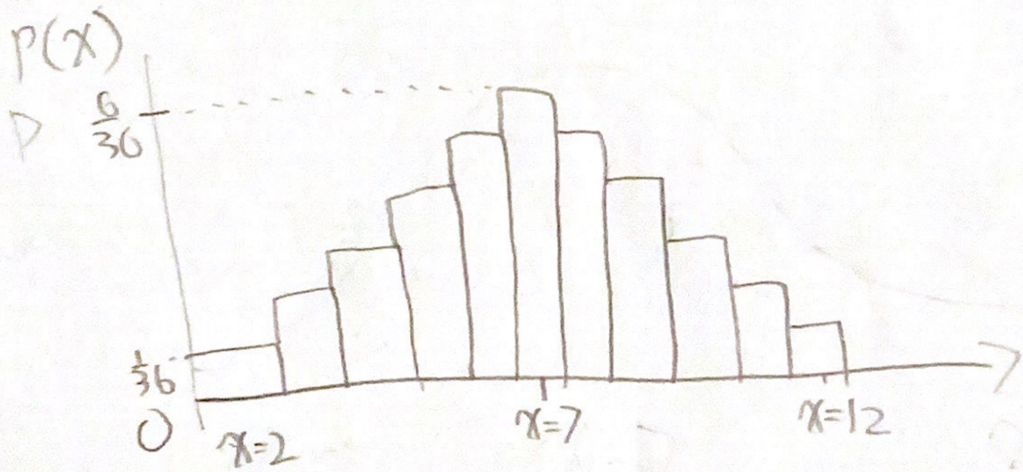
$P(Z=5) = 4P(w) = \frac{4}{36}$

$P(Z=6) = 5P(w) = \frac{5}{36}$

$P(Z=7) = 6(P(w)) = \frac{6}{36}$

(b) Associated to a random variable X, comes a *probability mass function* (p.m.f.) denoted p_X. It is "the distribution of probabilities" amongst the values that a random variable can take. Concretely we define $p_X(x) := \mathbb{P}(X=x) = \mathbb{P}(\{\omega \in \Omega : X(\omega) = x\})$.

Exercise: Draw the graph of p_Z, for Z as in part (a). (Draw the graph, as if the domain of p_Z is all of $\mathbb{R}$.)



Recall that the expected value of a random variable is

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot \mathbb{P}(\{\omega\}) = \sum_{x \in X(\Omega)} x \cdot \mathbb{P}(X = x).$$

(In lecture you used the second definition, but one can show that these two sums are equal – think about it!)

(c) Determine $\mathbb{E}[Z]$ (You already did most of the work in part (a).)

$$E[Z] = 2\cdot\frac{1}{36} + \left(3\cdot\frac{2}{36}\right) + \left(4\cdot\frac{3}{36}\right) + \left(5\cdot\frac{4}{36}\right) + \left(6\cdot\frac{5}{36}\right) + \left(7\cdot\frac{6}{36}\right) + \left(8\cdot\frac{5}{36}\right) + \left(9\cdot\frac{4}{36}\right) + \left(10\cdot\frac{3}{36}\right) + \left(11\cdot\frac{2}{36}\right) + \left(12\cdot\frac{1}{36}\right)$$

$$E[X] = 7$$

(d) Let X, Y be as in 2(b). Determine $\mathbb{E}[X], \mathbb{E}[Y]$ and $\mathbb{E}[X+Y]$.

$$E[X] = \left(1\cdot\frac{1}{6}\right) + \left(2\cdot\frac{1}{6}\right) + \left(3\cdot\frac{1}{6}\right) + \left(4\cdot\frac{1}{6}\right) + \left(5\cdot\frac{1}{6}\right) + \left(6\cdot\frac{1}{6}\right) = E[Y]$$

$$\boxed{\begin{aligned} &E[X] = E[Y] = \frac{21}{6} \\ &E[X+Y] = 7 \end{aligned}}$$

(e) OPTIONAL: Show that $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$. This property is called "linearity". (Hint: Use the "first" definition of the expected value, the first one that I wrote before problem (c).