Quantitative Risk Engine: Volatility Forecasting & VaR Backtesting

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Quantitative Risk Modeling

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Abstract

This document outlines the mathematical framework and methodology used in the accompanying Python project. The objective is to estimate the 1-day ahead **Value-at-Risk (VaR)** for the S&P 500 (SPY) using two distinct econometric approaches: an industry-standard **EWMA** baseline and a conditional **GARCH(1,1)** model with Student-t innovations. The predictive power of these models is validated through rigorous statistical backtesting (Kupiec and Christoffersen tests).

1 Project Overview

Financial time-series are characterized by *heteroskedasticity*—volatility is not constant but clusters over time. This project addresses the challenge of risk estimation by comparing a reactive baseline model against a mean-reverting econometric model. The core goal is to determine which approach provides better calibration for capital allocation during periods of market stress.

2 Mathematical Framework

2.1 Data Preprocessing

Financial asset prices P_t are non-stationary. To perform valid statistical inference, prices are transformed into daily simple returns r_t. This ensures the dataset exhibits mean-reverting properties essential for GARCH modeling.

$$r_t = \frac{P_t - P_{t-1}}{P_{t-1}} \tag{1}$$

2.2 Model 1: EWMA (RiskMetrics Baseline)

The Exponentially Weighted Moving Average (EWMA) serves as the benchmark. It uses a recursive formulation that assigns exponentially decaying weights to past squared returns. It is highly reactive to shocks but lacks a long-term mean reversion component.

$$\sigma_t^2 = \lambda \sigma_{t-1}^2 + (1 - \lambda) r_{t-1}^2 \tag{2}$$

Where $\lambda = 0.94$ is the decay factor, consistent with the J.P. Morgan RiskMetrics standard for daily data.

2.3 Model 2: GARCH(1,1) with Student-t Innovations

To capture the "memory" of volatility and mean reversion, a Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model is employed. The conditional variance is defined as:

$$\sigma_t^2 = \omega + \alpha r_{t-1}^2 + \beta \sigma_{t-1}^2 \tag{3}$$

Distribution Assumption: During the Maximum Likelihood Estimation (MLE) of parameters (ω, α, β), the model assumes the residuals follow a **Student-t distribution** rather than a Normal distribution. This accounts for the *leptokurtosis* (fat tails) observed in financial markets, ensuring parameters are robust to extreme outliers.

2.4 Value-at-Risk (VaR) Implementation

The 1-day ahead Value-at-Risk at the 95% confidence level ($\alpha = 0.05$) is computed as:

$$VaR_{t+1}^{\alpha} = -\sigma_{t+1} \cdot \Phi^{-1}(1-\alpha) \tag{4}$$

Where Φ^{-1} is the inverse Cumulative Distribution Function (CDF) of the **Standard Normal Distribution** (≈ 1.645).

Methodological Note: Although the GARCH model is fitted using a Student-t likelihood to accurately estimate the volatility dynamics (σ_t), the VaR threshold calculation utilizes the Normal approximation. This design choice isolates the *volatility process* as the primary variable of comparison between EWMA and GARCH, ensuring a direct assessment of volatility forecasting performance without conflating results with tail-distribution assumptions.

3 Backtesting Framework

A risk model is validated by comparing predicted VaR against realized returns. An indicator variable I_t (Hit) is defined as:

$$I_t = \begin{cases} 1 & \text{if } r_t < -VaR_t \quad \text{(Violation)} \\ 0 & \text{if } r_t \geq -VaR_t \end{cases} \tag{5}$$

3.1 Kupiec Test (Unconditional Coverage)

This test evaluates the **frequency** of exceptions. The Null Hypothesis H_0 states that the observed failure rate $\hat{p}$ equals the expected failure rate p (5%).

- Violations significantly > 5%: The model underestimates risk.
- Violations significantly < 5%: The model overestimates risk (inefficient capital usage).

3.2 Christoffersen Test (Independence)

This test evaluates the **clustering** of exceptions. The Null Hypothesis H_0 states that violations are independent over time. A robust risk model should adjust its volatility forecast immediately after a shock; therefore, consecutive violations ($I_t = 1$ followed by $I_{t+1} = 1$) indicate a failure to adapt to market regimes.

4 Results Analysis

The analysis performed in the notebook yields the following insights:

1. **Volatility Dynamics:** The GARCH model exhibits sharper peaks during stress periods (e.g., 2020), reflecting the mean-reverting nature of volatility, whereas EWMA provides a smoother but slower adaptation.

2. **Robustness:** By incorporating Student-t innovations in the estimation phase, the GARCH model parameters are less sensitive to noise, resulting in a more stable conditional volatility forecast compared to the raw historical data.

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