THE UNIVERSITY OF TEXAS AT AUSTIN
McCOMBS SCHOOL OF BUSINESS

Probability Review 1

Lecture 2

STA 371G

Probability Theory

The Concept of Probability

What is common among the following?

Probability Theory

The Concept of Probability

What is common among the following?

- Outcome of rolling a die
- S&P500 index at the and of January
- Number of iPhone 7s to be sold over the next year
- Number of unique visitors to Amazon.com over the next week
- Lifetime of your MacBook Air

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Yet, we can model them using probability theory and study the values they might take, associated probabilities etc.

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- Random variable $\longrightarrow$

$$Y : \begin{cases} 1, & \text{if outcome is odd number,} \\ 2, & \text{if outcome is even number} \end{cases}$$

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- X : Number of stocks on NYSE whose price change today (discrete)
- Y : Average price change of the stocks on NYSE (continuous)

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Discrete or continuous?

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Examples

- X : S&P500 at the end of 2017, $P(X > 2270) = 0.85$
- Y : Lifetime of your MacBook, $P(Y > 15 \text{ years}) = 0.05$

Probability Distributions

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Continuous random variable $\longrightarrow$ Probability Density Function (p.d.f.)

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Discrete Random Variables

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Since this is a fair die, the corresponding probability mass function:

$$f(x) = \begin{cases} \frac{1}{n} & x = 1, \ldots, n, \\ 0 & \text{otherwise.} \end{cases}$$

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- This is an example of Discrete Uniform Distribution.

Probability Distributions

Continous Random Variables

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Y : Lifetime of your MacBook (in years)

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Let's assume Y has a Continuous Uniform Distribution with a maximum of 20 years. Its probability distribution is then given by the following probability density function:

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What is $P(Y = 5) = ?$ or $P(Y = 5.5) = ?$ or $P(Y = 5.551234123) = ?$

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They are all 0.

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And we take integrals to find such probabilities.

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$$P(5 < Y < 7) = \int_5^7 \frac{1}{20} dy = \frac{y}{20} \Big|_5^7 = \frac{7}{20} - \frac{5}{20} = \frac{1}{10}$$

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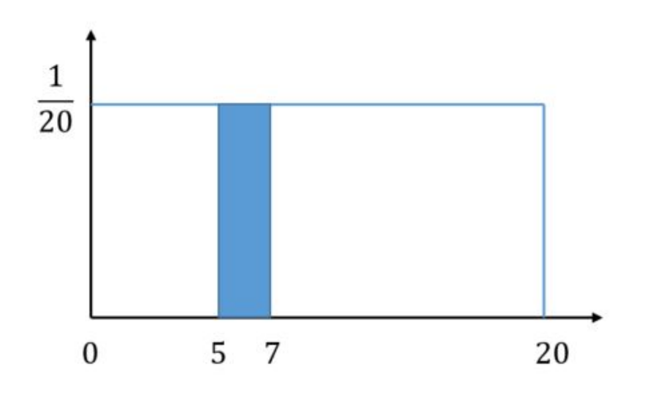
In general, $P(a \leq Y \leq b) = \int_a^b f(y) dy$.

Probability Distributions



Mean, Variance and Standard Deviation

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Mean or Expected Value of a random variable X is a measure of the center of its probability distribution. It is a weighted average of all possible values X can take, where the weights are the corresponding probabilities.

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Discrete random variable X

$$\mu_X = E[X] = \sum_x x f(x)$$

Continuous random variable Y

$$\mu_Y = E[Y] = \int_y y f(y) dy$$

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Law of Large Numbers

Definition

For a random variable X, the average of $X_1, X_2, \ldots, X_n$ gets very close to the expected value of X ($E[X]$) for large n.

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A die is rolled $n = 4$ times: $x_1 = 4$, $x_2 = 6$, $x_3 = 1$, $x_4 = 1$. The average is

$$\frac{x_1 + x_2 + x_3 + x_4}{4} = \frac{4 + 6 + 1 + 1}{4} = 3$$

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$$\frac{x_1 + x_2 + x_3 + x_4}{4} = \frac{4 + 6 + 1 + 1}{4} = 3$$

For large n, the average will be around 3.5; because $E[X] = 3.5$.

Law of Large Numbers

R Exercise

Go to R Studio...

```
# Generate a random number in [0,1]
runif(1)
# Generate a random number in [1,7]
runif(1, min=1, max=7)
# Floor it down to simulate a die
floor(runif(1, min=1, max=7))
# Simulate 3 dice
floor(runif(3, min=1, max=7))
# Take the average
mean(floor(runif(3, min=1, max=7)))
# Let's increase the number of dice
mean(floor(runif(10, min=1, max=7)))
```

Normal Distribution a.k.a. the Bell Curve

A very common continuous probability distribution.

The mean and variance together uniquely define the distribution:

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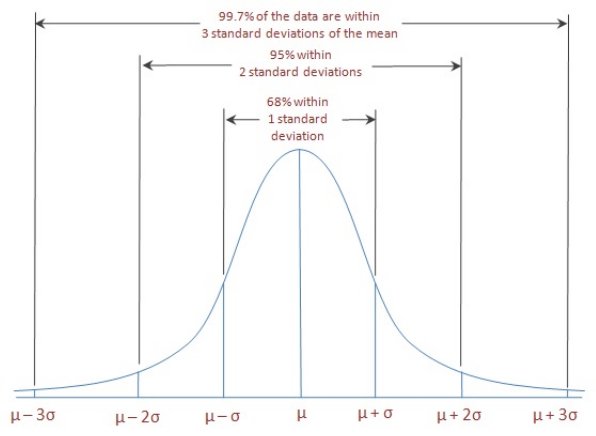

Central Limit Theorem

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Distribution of the exam grades then tend to be normal...

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X : The price of a house. Assume X is uniform in $[100, 400]$ ($K).

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We will simulate z number of zip codes, each containing n houses.

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```r
# Simulating a zip code with 3 houses
runif(3, min=100, max=400)
# Repeat this for 5 zip codes.
house_prices <- t(replicate(5, runif(3, min=100, max=400) ))
# Find the average house price in each zip code
avg_house_prices <- rowMeans(house_prices)
# See what you got
hist(avg_house_prices)
# Increase n and z and try again!
```