THE UNIVERSITY OF TEXAS AT AUSTIN
McCOMBS SCHOOL OF BUSINESS

Probability Review 1

Lecture 2

STA 371G

Probability Theory

The Concept of Probability

What is common among the following?

- Outcome of rolling a die
- S&P500 index at the and of January
- Number of iPhone 7s to be sold over the next year
- Number of unique visitors to Amazon.com over the next week
- Lifetime of your MacBook Air

We cannot predict any of these with certainty.
Yet, we can model them using probability theory and study the values they might take, associated probabilities etc.

Probability Theory

Definitions

Definition

A random variable expresses the outcome of a random experiment as a number. It is denoted by an uppercase letter.

- Random experiment → Rolling a die
- Random variable → X : The outcome when a die is rolled
- Random variable →

$$Y : \begin{cases} 1, & \text{if outcome is odd number,} \\ 2, & \text{if outcome is even number} \end{cases}$$

Probability Theory

Definitions

Definition

A discrete random variable is a random variable with a finite (or countably infinite) range.
A continuous random variable is a random variable with an interval (either finite or infinite) of real numbers for its range.

Examples

- X : Number of stocks on NYSE whose price change today (discrete)
- Y : Average price change of the stocks on NYSE (continuous)

Probability Theory

Exercise

Discrete or continuous?

- Number of iPhone 7s to be sold over the next year
- Lifetime of your MacBook Air
- Number of unique visitors to Amazon.com over the next week
- S&P500 index at the and of January

Probability Theory

Definitions

Definition

Probability is the measure of the likelihood that a particular outcome (or set of outcomes) will be observed.

Probability is a number always between 0 and 1.

0 implies impossibility, 1 implies certainty.

Examples

- X : S&P500 at the end of 2017, $P(X > 2270) = 0.85$
- Y : Lifetime of your MacBook, $P(Y > 15 \text{ years}) = 0.05$

Probability Distributions

So, we have defined a random variable. How do we know what the probabilities are?

For example, what is the probability that your MacBook will break down after 5 years but before 7 years? That is, $P(5 < X < 7) = ?$

Definition

The probability distribution of a random variable X is a description of the probabilities associated with the possible values of X.

Discrete random variable ⟶ Probability Mass Function (p.m.f.)
Continuous random variable ⟶ Probability Density Function (p.d.f.)

Probability Distributions

Discrete Random Variables

Example

X: The outcome when you roll n-sided fair die.

Since this is a fair die, the corresponding probability mass function:

$$f(x) = \begin{cases} \frac{1}{n} & x = 1, \ldots, n, \\ 0 & \text{otherwise.} \end{cases}$$

- $f(2) = P(X = 2)$, which is the probability of observing a “2.” This interpretation will not hold for continuous random variables.
- Sum of probabilities is always 1. $(n \times \frac{1}{n})$.
- This is an example of Discrete Uniform Distribution.

Probability Distributions

Continous Random Variables

Example

Y : Lifetime of your MacBook (in years)

Let's assume Y has a Continuous Uniform Distribution with a maximum of 20 years. Its probability distribution is then given by the following probability density function:

$$f(y) = \begin{cases} \frac{1}{20} & 0 \leq y \leq 20, \\ 0 & \text{otherwise.} \end{cases}$$

What is $P(Y = 5) = ?$ or $P(Y = 5.5) = ?$ or $P(Y = 5.551234123) = ?$ They are all 0.

Probability Distributions

Continous Random Variables

Warning!

For a continuous random variable, $P(Y = a)$ is always zero, regardless of a.

For this reason, for continuous random variables we ask questions like "$P(a \leq Y \leq b) = ?$"

And we take integrals to find such probabilities.

Probability Distributions

Continous Random Variables

Example

Y : Lifetime of your MacBook (in years)

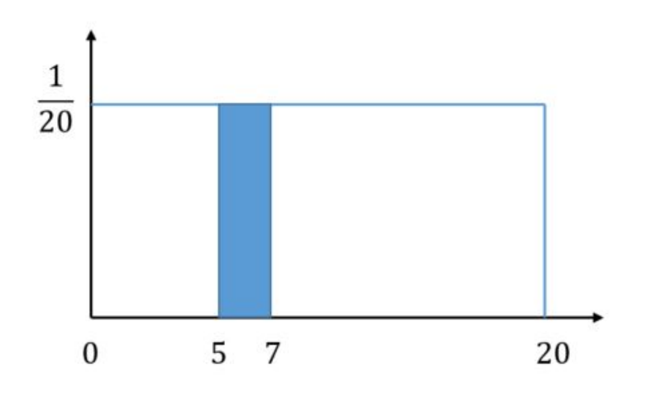
$$f(y) = \begin{cases} \frac{1}{20} & 0 \leq y \leq 20, \\ 0 & \text{otherwise.} \end{cases}$$

What is $P(5 < Y < 7) =?$

$$P(5 < Y < 7) = \int_5^7 \frac{1}{20} dy = \frac{y}{20}|_5^7 = \frac{7}{20} - \frac{5}{20} = \frac{1}{10}$$

In general, $P(a \leq Y \leq b) = \int_a^b f(y)dy$.

Probability Distributions



Mean, Variance and Standard Deviation

Definition

Mean or Expected Value of a random variable *X* is a measure of the center of its probability distribution. It is a weighted average of all possible values *X* can take, where the weights are the corresponding probabilities.

Discrete random variable *X*

$$\mu_X = E[X] = \sum_x x f(x)$$

Continuous random variable *Y*

$$\mu_Y = E[Y] = \int_y y f(y) dy$$

Mean, Variance and Standard Deviation

Definition

Variance of a random variable X is a measure of the dispersion, or variability in its distribution. Standard Deviation of X is the square root of its variance.

Discrete random variable X

$$\sigma_X^2 = Var(X) = E[\,(X - \mu_X)^2] = \sum_x (x - \mu_x)^2 f(x)$$

Continuous random variable Y

$$\sigma_Y^2 = Var(Y) = E[\,(Y - \mu_Y)^2] = \int_y (y - \mu_y)^2 f(y) dy$$

Law of Large Numbers

Definition

For a random variable X, the average of $X_1, X_2, \ldots, X_n$ gets very close to the expected value of X ($E[X]$) for large n.

Example

A die is rolled $n = 4$ times: $x_1 = 4$, $x_2 = 6$, $x_3 = 1$, $x_4 = 1$. The average is

$$\frac{x_1 + x_2 + x_3 + x_4}{4} = \frac{4 + 6 + 1 + 1}{4} = 3$$

For large n, the average will be around 3.5; because $E[X] = 3.5$.

Law of Large Numbers

R Exercise

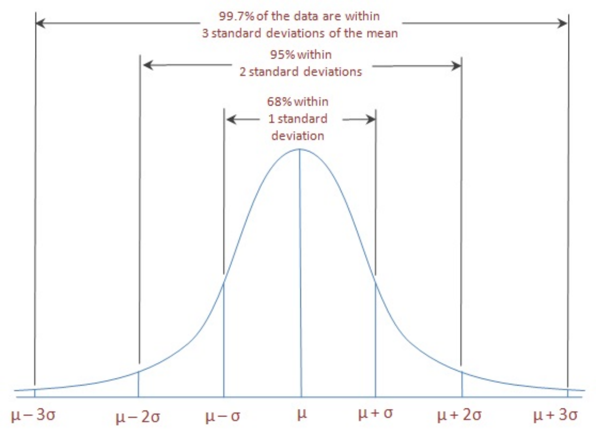
Go to R Studio...

```
# Generate a random number in [0,1]
runif(1)
# Generate a random number in [1,7]
runif(1, min=1, max=7)
# Floor it down to simulate a die
floor(runif(1, min=1, max=7))
# Simulate 3 dice
floor(runif(3, min=1, max=7))
# Take the average
mean(floor(runif(3, min=1, max=7)))
# Let's increase the number of dice
mean(floor(runif(10, min=1, max=7)))
```

Normal Distribution a.k.a. the Bell Curve

A very common continuous probability distribution.

The mean and variance together uniquely define the distribution: $N(\mu, \sigma^2)$



Central Limit Theorem

Definition

The average of large number of independent random variables will be approximately normally distributed.

Example

$$\text{a student's exam grade} = \frac{\text{preparedness} + \text{focus} + \text{IQ} + \ldots}{\text{\# of factors}}$$

Distribution of the exam grades then tend to be normal...

Central Limit Theorem

R Exercise

X : The price of a house. Assume X is uniform in $[100, 400]$ ($K).

Y : Average house price in a zip code

We will simulate z number of zip codes, each containing n houses.

```r
# Simulating a zip code with 3 houses
runif(3, min=100, max=400)
# Repeat this for 5 zip codes.
house_prices <- t(replicate(5, runif(3, min=100, max=400) ))
# Find the average house price in each zip code
avg_house_prices <- rowMeans(house_prices)
# See what you got
hist(avg_house_prices)
# Increase n and z and try again!
```