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Geometría I

1º Grado en Matemáticas

Facultad de Ciencias
Universidad de Granada

Examen de Geometría I

Convocatoria Extraordinaria - 4 de febrero de 2020

1. Razónese si son verdaderas o falsas las afirmaciones siguientes:

   a) **[2 puntos]** Sea $V(K)$ un espacio vectorial de dimensión $2n+1$ con $n \in \mathbb{N}$ y sea $f : V \to V$ un endomorfismo tal que $f \circ f = 0$. Entonces, $\dim(\mathrm{Im}(f)) \leq n$.

   b) **[1.5 puntos]** Sea $V(\mathbb{C})$ un espacio vectorial complejo. Entonces $V(\mathbb{C})$ es finitamente generado si y solo si V es finitamente generado al considerarlo como espacio vectorial sobre $\mathbb{R}$.

2. **[3 puntos]** Sean $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ y $C = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, y sea f_A el endomorfismo de $\mathbb{R}^2(\mathbb{R})$ cuya matriz en la base usual es A. ¿Son A y C equivalentes? En caso afirmativo, encontrar matrices regulares P y Q de orden 2×2 tales que $C = Q^{-1}AP$. Además, obtener bases $B, \bar{B}$ de $\mathbb{R}^2$ tales que la matriz de f_A en esas bases sea C. ¿Son A y C semejantes?

3. En $M_2(\mathbb{R})$ se considera el subespacio $S_2(\mathbb{R})$ formado por las matrices simétricas y

$$U_\lambda = L\{\begin{pmatrix} 1 & -1 \\ \lambda+1 & 2 \end{pmatrix}, \begin{pmatrix} \lambda & -\lambda \\ 2 & \lambda-2 \end{pmatrix}\}.$$

   a) **[2 puntos]** Calcular, para cada $\lambda \in \mathbb{R}$, una base de U_λ, $U_\lambda \cap S_2(\mathbb{R})$, $U_\lambda + S_2(\mathbb{R})$.

   b) **[1.5 puntos]** Para $\lambda = 1$, ampliar la base de U_1 obtenida en el apartado anterior hasta una base B de $M_2(\mathbb{R})$. Obtener la base dual de B. Calcular el anulador de U_1.

Duración: 3 horas. El 3 aparte.

) Febrero 2020: $A=\begin{pmatrix}1&0\\1&0\end{pmatrix}$, $C=\begin{pmatrix}1&2\\2&4\end{pmatrix}$ - $f_A:\mathbb{R}^2(\mathbb{R})\longrightarrow\mathbb{R}^2(\mathbb{R})$. ¿A y C equivalentes?

$A = M(f_A, B_u)$ Encontrar matrices regulares P, Q : $C=Q^{-1}\cdot A\cdot P$

Obtener bases $B,\bar{B}$: $M(f_A, B\rightarrow\bar{B})=C$ ¿A y C semejantes?

$rg(A)=1=rg(C)\Rightarrow$ A y C son equivalentes.

$$\underbrace{M(f_A,\bar{B}\leftarrow B)}_{C}=\underbrace{M(I,\bar{B}\leftarrow B_u)}_{Q^{-1}}\cdot\underbrace{M(f_A,B_u)}_{A}\cdot\underbrace{M(I,B_u\leftarrow B)}_{P}$$

Sea $Q_A\in Gl_2(\mathbb{R})$, Sea B' base $\mathbb{R}^2$.

$$M(f_A, B'\leftarrow B_u)=M(I,B'\leftarrow B_u)\cdot M(f_A,B_u)\cdot I$$

$M(f_A, B'\leftarrow B_u)=\begin{pmatrix}1&0\\0&0\end{pmatrix}=A'=C'$, $M(I,B'\leftarrow B_u)=Q_A^{-1}$, $M(f_A,B_u)=\begin{pmatrix}1&0\\1&0\end{pmatrix}=A$, $I=P_A$

$\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}1&0\\1&0\end{pmatrix}=\begin{pmatrix}1&0\\0&0\end{pmatrix}\Leftrightarrow \begin{matrix}a+b=1\\c+d=0\end{matrix}\Big\}\overset{ej.}{\Rightarrow} Q_A^{-1}=\begin{pmatrix}1&0\\1&-1\end{pmatrix}\Rightarrow Q_A=\begin{pmatrix}1&0\\1&-1\end{pmatrix}$

$\Downarrow$

$B'=\{(1,1),(0,-1)\}$

$$Q_A\cdot A'\cdot P_A^{-1}=A$$

$C'=Q_C^{-1}\cdot C\cdot P_C \Leftrightarrow \begin{pmatrix}1&0\\0&0\end{pmatrix}=Q_C^{-1}\cdot\begin{pmatrix}1&2\\2&4\end{pmatrix}\cdot P_C$, $C'' = C'$, $\begin{pmatrix}1&2\\2&4\end{pmatrix}=M(f_C,B_u)$

$\Updownarrow$

$C=Q_C\,C'\,P_C^{-1}$

$\ker(f_C)=L((2,-1))\Rightarrow B''=\{(1,0),(2,-1)\}$

$f_C(1,0)=(1,2)\Rightarrow B'''=\{(1,2),(1,4)\}$

$C'=M(f_C,B'''\leftarrow B'')$, $Q_C=M(I,B'''\rightarrow B_u)=\begin{pmatrix}1&1\\2&4\end{pmatrix}$, $P_C=M(I,B''\rightarrow B_u)=\begin{pmatrix}1&2\\0&-1\end{pmatrix}$

$$Q_C\cdot C'\cdot P_C^{-1}=Q^{-1}\cdot Q_A\cdot A'\cdot P_A^{-1}\cdot P\Leftrightarrow C'=\underbrace{Q_C^{-1}\cdot Q^{-1}}_{Q_A^{-1}}\cdot A\cdot\underbrace{P\cdot P_C}_{P_A=I}$$

$$Q_C^{-1}\cdot Q^{-1}=Q_A^{-1}\Leftrightarrow Q=Q_A\cdot Q_C^{-1}=\begin{pmatrix}1&0\\1&-1\end{pmatrix}\cdot\begin{pmatrix}2&-1/2\\-1&1/2\end{pmatrix}=\boxed{\begin{pmatrix}2&-1/2\\3&-1\end{pmatrix}=Q}$$

$$P\cdot P_C=I\Leftrightarrow P=P_C^{-1}=\boxed{\begin{pmatrix}1&2\\0&-1\end{pmatrix}=P}$$

$$Q^{-1}=\begin{pmatrix}2&-1\\6&-4\end{pmatrix}$$

$$Q^{-1}\cdot A\cdot P=\begin{pmatrix}2&-1\\6&-4\end{pmatrix}\cdot\begin{pmatrix}1&0\\1&0\end{pmatrix}\cdot\begin{pmatrix}1&2\\0&-1\end{pmatrix}=\begin{pmatrix}1&0\\2&0\end{pmatrix}\begin{pmatrix}1&2\\0&-1\end{pmatrix}=\begin{pmatrix}1&2\\2&4\end{pmatrix}\ \checkmark$$

$Q=M(I,\bar{B}\rightarrow B_u)\Rightarrow\bar{B}=\{(2,3),(-\frac{1}{2},-1)\}$, $P=M(I,B\rightarrow B_u)\Rightarrow B=\{(1,0),(2,-1)\}$

A y C no son semejantes, ya que traza(A) = 1+0 = 1, mientras que traza (C) = 1+4 = 5, luego las trazas no coinciden (condición necesaria para que sean semejantes)

Escaneado con CamScanner

3) Febrero 2020.

$V = M_2(\mathbb{R})$, subespacio $S_2(\mathbb{R})$, subespacio $U_\lambda = L\left(\underbrace{\begin{pmatrix} 1 & -1 \\ \lambda+1 & 2 \end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix} \lambda & -\lambda \\ 2 & \lambda-2 \end{pmatrix}}_{v_2}\right)$ $\quad B_{S_2(\mathbb{R})} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$

a) Para cada λ, una base de U_λ, $U_\lambda \cap S_2(\mathbb{R})$, $U_\lambda + S_2(\mathbb{R})$.

¿$\dim U_\lambda$? $\quad \dim U_\lambda = 1 \iff v_1$ y v_2 son l.d. $\iff \exists a, b \neq 0$: $a\begin{pmatrix} 1 & -1 \\ \lambda+1 & 2 \end{pmatrix} + b\begin{pmatrix} \lambda & -\lambda \\ 2 & \lambda-2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} a+b\lambda & -a-b\lambda \\ a\lambda+a+2b & 2a+b\lambda-2b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} a+b\lambda=0 \\ -a-b\lambda=0 \\ a\lambda+a+2b=0 \\ 2a+b\lambda-2b=0 \end{cases} \Rightarrow a=-b\lambda \Rightarrow \begin{cases} -b\lambda^2-b\lambda+2b=0 \\ -b\lambda-2b=0 \iff \boxed{\lambda=-2} \end{cases}$$

$\lambda = -2$:

$\dim(U_{-2}) = 1 \qquad B_{U_{-2}} = \left\{ \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \right\} \qquad w \in U_{-2} \cap S_2(\mathbb{R}) \iff w \in U_{-2} \wedge w \in S_2(\mathbb{R})$

$$w = a\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = b\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + d\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \iff \begin{pmatrix} a-b & -a-d \\ -a-d & 2a-c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \iff \begin{cases} a-b=0 \\ -a-d=0 \\ 2a-c=0 \end{cases}$$

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 \\ 2 & 0 & -1 & 0 & 0 \end{array}\right) \xrightarrow[F_3=F_3-2F_1]{F_2=F_2+F_1} \left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 2 & -1 & 0 & 0 \end{array}\right) \xrightarrow{F_3=F_3+2F_2} \left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & -2 & 0 \end{array}\right) \begin{cases} a=-\mu \\ b=-\mu \\ c=-2\mu \\ d=\mu \end{cases}$$

depende de un parámetro $\Rightarrow \dim(U_{-2} \cap S_2(\mathbb{R})) = 1$

$B_{U_{-2} \cap S_2(\mathbb{R})} = \left\{ \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix} \right\}$

$\dim(U_{-2} + S_2(\mathbb{R})) = \dim(U_{-2}) + \dim(S_2(\mathbb{R})) - \dim(U_{-2} \cap S_2(\mathbb{R})) = \cancel{1}+3-\cancel{1} = 3$

$$a\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} + b\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \iff \begin{cases} a+b=0 & \Rightarrow b=0 \\ -a=0 & \Rightarrow a=0 \\ -a=0 \\ 2a+c=0 & \Rightarrow c=0 \end{cases}$$

$\Rightarrow \underset{\in U_{-2}}{\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}}, \underset{\in S_2(\mathbb{R})}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}$ y $\underset{\in S_2(\mathbb{R})}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}$ son l.i.

$\Rightarrow B_{S_2(\mathbb{R})+U_{-2}} = \left\{ \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

$\lambda \neq -2$:

$\dim(U_\lambda) = 2$, $\quad B_{U_\lambda} = \left\{ \begin{pmatrix} 1 & -1 \\ \lambda+1 & 2 \end{pmatrix}, \begin{pmatrix} \lambda & -\lambda \\ 2 & \lambda-2 \end{pmatrix} \right\}$; $\quad w = a\begin{pmatrix} 1 & -1 \\ \lambda+1 & 2 \end{pmatrix} + b\begin{pmatrix} \lambda & -\lambda \\ 2 & \lambda-2 \end{pmatrix} = c\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + e\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

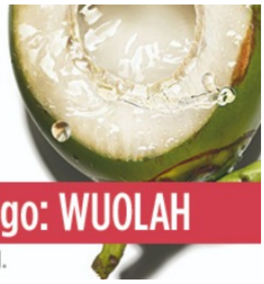
$$\iff \begin{pmatrix} a+b\lambda-c & -a-b\lambda-e \\ a(\lambda+1)+2b-e & 2a+(\lambda-2)b-d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \iff \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \left(\begin{array}{ccccc|c} 1 & \lambda & -1 & 0 & 0 & 0 \\ -1 & -\lambda & 0 & 0 & -1 & 0 \\ \lambda+1 & 2 & 0 & 0 & -1 & 0 \\ 2 & \lambda-2 & 0 & -1 & 0 & 0 \end{array}\right) \end{array} \xrightarrow{C_3 \leftrightarrow C_1}$$

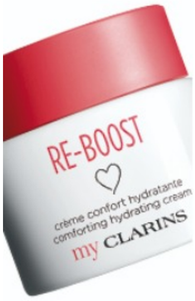
$$\begin{array}{c} \begin{array}{ccccc} c & b & a & d & e \end{array} \\ \left(\begin{array}{ccccc|c} -1 & \lambda & 1 & 0 & 0 & 0 \\ 0 & -\lambda & -1 & 0 & -1 & 0 \\ 0 & 2 & \lambda+1 & 0 & -1 & 0 \\ 0 & \lambda-2 & 2 & -1 & 0 & 0 \end{array}\right) \end{array} \xrightarrow{C_5 \leftrightarrow C_2} \begin{array}{c} \begin{array}{ccccc} c & e & a & d & b \end{array} \\ \left(\begin{array}{ccccc|c} -1 & 0 & 1 & 0 & \lambda & 0 \\ 0 & -1 & -1 & 0 & -\lambda & 0 \\ 0 & -1 & \lambda+1 & 0 & 2 & 0 \\ 0 & 0 & 2 & -1 & \lambda-2 & 0 \end{array}\right) \end{array} \xrightarrow{F_3=F_3-F_2} \left(\begin{array}{ccccc|c} -1 & 0 & 1 & 0 & \lambda & 0 \\ 0 & -1 & -1 & 0 & -\lambda & 0 \\ 0 & 0 & \lambda+2 & 0 & 2+\lambda & 0 \\ 0 & 0 & 2 & -1 & \lambda-2 & 0 \end{array}\right) \xrightarrow[\lambda+2\neq 0]{F_4=F_4-\frac{2}{\lambda+2}F_3}$$

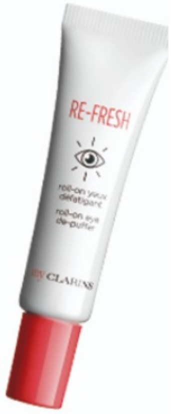
$$\begin{array}{c} \begin{array}{ccccc} c & e & a & d & b \end{array} \\ \left(\begin{array}{ccccc|c} -1 & 0 & 1 & 0 & \lambda & 0 \\ 0 & -1 & -1 & 0 & -\lambda & 0 \\ 0 & 0 & \lambda+2 & 0 & \lambda+2 & 0 \\ 0 & 0 & 0 & -1 & \lambda-4 & 0 \end{array}\right) \end{array} \begin{cases} a=-\mu \\ b=\mu \\ c=(\lambda-1)\mu \\ d=(\lambda-4)\mu \\ e=(1-\lambda)\mu \end{cases} \Rightarrow \begin{pmatrix} -\mu+\lambda\mu & \mu-\lambda\mu \\ -\mu\lambda+\mu & \mu\lambda-4\mu \end{pmatrix} \in U_\lambda \cap S_2(\mathbb{R}), \ \forall \mu \in \mathbb{R}$$

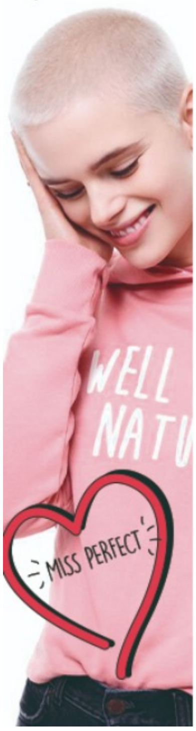
$B_{U_\lambda \cap S_2(\mathbb{R})} = \left\{ \begin{pmatrix} \lambda-1 & 1-\lambda \\ 1-\lambda & \lambda-4 \end{pmatrix} \right\}$

$\dim(U_\lambda + S_2(\mathbb{R})) = \dim(U_\lambda) + \dim(S_2(\mathbb{R})) - \dim(U_\lambda \cap S_2(\mathbb{R}))$
$= 2+3-1 = 4 \Rightarrow B_{U_\lambda + S_2(\mathbb{R})} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$









b) Para $\lambda=1$, ampliar la base de U_1 a una base B de $M_2(\mathbb{R})$. Obtener base dual de B.

Calcular anulador de U_1.

$$B_{U_1}=\left\{\begin{pmatrix}1&-1\\2&2\end{pmatrix},\begin{pmatrix}1&-1\\2&-1\end{pmatrix}\right\}$$

$$a\begin{pmatrix}1&-1\\2&2\end{pmatrix}+b\begin{pmatrix}1&-1\\2&-1\end{pmatrix}+c\begin{pmatrix}1&0\\0&0\end{pmatrix}+d\begin{pmatrix}0&0\\1&0\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}\Rightarrow\begin{pmatrix}a+b+c&-a-b\\2a+2b+d&2a-b\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

$$\left(\begin{array}{cccc|c}1&1&1&0&0\\-1&-1&0&0&0\\2&2&0&1&0\\2&-1&0&0&0\end{array}\right)\xrightarrow[\substack{F_3=F_3-2F_1\\F_4=F_4-2F_1}]{F_2=F_2+F_1}\left(\begin{array}{cccc|c}1&1&1&0&0\\0&0&1&0&0\\0&0&-2&1&0\\0&-3&-2&0&0\end{array}\right)\xrightarrow{F_4\leftrightarrow F_2}\left(\begin{array}{cccc|c}1&1&1&0&0\\0&-3&-2&0&0\\0&0&-2&1&0\\0&0&1&0&0\end{array}\right)\Rightarrow\begin{cases}a=0\\b=0\\c=0\\d=0\end{cases}$$

$\Rightarrow$ los vectores son l.i. $\Rightarrow B=\left\{\begin{pmatrix}1&-1\\2&2\end{pmatrix},\begin{pmatrix}1&-1\\2&-1\end{pmatrix},\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix}\right\}$, $B^*=\{\phi^1,\phi^2,\phi^3,\phi^4\}$

$M(\phi^1,B)=(1\ 0\ 0\ 0)$

$M(\phi^2,B)=(0\ 1\ 0\ 0)$

$M(\phi^3,B)=(0\ 0\ 1\ 0)$

$M(\phi^4,B)=(0\ 0\ 0\ 1)$

Sea $B_u=\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$

$$M(I,B_u\leftarrow B)=\begin{pmatrix}1&1&1&0\\-1&-1&0&0\\2&2&0&1\\2&-1&0&0\end{pmatrix},\quad M(I,B\leftarrow B_u)=M(I,B_u\leftarrow B)^{-1}$$

$$M(I,B\leftarrow B_u)=-\frac{1}{3}\begin{pmatrix}0&1&0&-1\\0&2&0&1\\-3&-3&0&0\\0&-6&-3&0\end{pmatrix}$$

$$\Rightarrow M(\phi^i,B_u)=M(\phi^i,B)\cdot M(I,B\leftarrow B_u)=M(\phi^i,B)\begin{pmatrix}0&-1/3&0&1/3\\0&-2/3&0&-1/3\\1&1&0&0\\0&2&1&0\end{pmatrix}$$

$M(\phi^1,B_u)=(0\ \ -\frac{1}{3}\ \ 0\ \ \frac{1}{3})$, $M(\phi^2,B_u)=(0\ \ -\frac{2}{3}\ \ 0\ \ -\frac{1}{3})$, $M(\phi^3,B_u)=(1\ 1\ 0\ 0)$

$M(\phi^4,B_u)=(0\ 2\ 1\ 0)$; Así dada una matriz cualquiera $\begin{pmatrix}a&b\\c&d\end{pmatrix}\Rightarrow$

$$\phi^1\begin{pmatrix}a&b\\c&d\end{pmatrix}=\frac{-b+d}{3},\quad\phi^2\begin{pmatrix}a&b\\c&d\end{pmatrix}=\frac{-2b-d}{3},\quad\phi^3\begin{pmatrix}a&b\\c&d\end{pmatrix}=a+b,\quad\phi^4\begin{pmatrix}a&b\\c&d\end{pmatrix}=2b+c$$

$$An(U_1)=L(\phi^3,\phi^4)\quad\left(\text{ya que }\phi^3\begin{pmatrix}1&-1\\2&2\end{pmatrix}=\phi^3\begin{pmatrix}1&-1\\2&-1\end{pmatrix}=\phi^4\begin{pmatrix}1&-1\\2&2\end{pmatrix}=\phi^4\begin{pmatrix}1&-1\\2&-1\end{pmatrix}=0\right)$$

1) Febrero 2020: V ó F

a) Sea $V(k)$ e.v. dim $2n+1$, $n\in\mathbb{N}$, $f:V\to V$, $f\circ f=0\Rightarrow \overset{r(f)}{\dim(\operatorname{Im}(f))}\leq n$

$f\circ f=0\Rightarrow f(f(v))=0,\forall v\in V\Rightarrow \operatorname{Im}(f)\subset\ker(f)\Rightarrow r(f)\leq n(f)\Rightarrow 2n+1=r(f)+n(f)\leq 2n(f)$

$\Leftrightarrow n(f)\geq n+\frac{1}{2}>n\Rightarrow 2n+1=r(f)+n(f)>r(f)+n\Leftrightarrow n+1>r(f)\Leftrightarrow n+1>\boxed{n\geq r(f)}$

Verdadera.