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Geometría I (Examen final)
10 de enero de 2019

1. Discutir y resolver el siguiente sistema de ecuaciones sobre $\mathbb{R}$ en función de los valores $a, b, c \in \mathbb{R}$.

$$\begin{aligned} x - y + z + t &= 0, \\ x + y + z + t &= a, \\ x - by + cz + t &= 0. \end{aligned}$$

2. Sea V un espacio vectorial sobre $\mathbb{K}$ de dimensión finita y U un subespacio vectorial de V. Definir el anulador de U. Probar que es un subespacio vectorial del espacio dual V^* y que su dimensión es igual a $\dim(V) - \dim(U)$.

3. Sea $\mathcal{M}_2(\mathbb{R})$ el espacio vectorial de las matrices cuadradas de orden dos con coeficientes en $\mathbb{R}$. Se considera la aplicación $f\colon \mathcal{M}_2(\mathbb{R}) \to \mathcal{M}_2(\mathbb{R})$ definida por:

$$f(X) = AX - XA,$$

donde

$$A = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}.$$

   a) Probar que f es una aplicación lineal.
   b) Hallar la matriz de f en la base:

$$B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

   c) Calcular una base del núcleo y una base de la imagen de f.

4. En $\mathbb{R}^3$ se consideran las formas lineales $\psi_1, \psi_2, \psi_3 : \mathbb{R}^3 \to \mathbb{R}$ definidas por:

$$\psi_1(x,y,z) = x + z, \qquad \psi_2(x,y,z) = x + y + 2z, \qquad \psi_3(x,y,z) = y.$$

   a) Expresar ψ_1, ψ_2 y ψ_3 en términos de la base $B_0^* = \{\omega_1, \omega_2, \omega_3\}$ dual de la base usual B_0.
   b) Comprobar que $B^* = \{\psi_1, \psi_2, \psi_3\}$ es una base de $(\mathbb{R}^3)^*$.
   c) Calcular la base B de $\mathbb{R}^3$ cuya base dual es B^*.

Segundo parcial: 2, 3, 4.
Toda la asignatura: 1, 2, 3.

1) $\left(\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & a \\ 1 & -b & c & 1 & 0 \end{array}\right) \underset{F_3=F_3-F_1}{\overset{F_2=F_2-F_1}{\sim}} \left(\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 & a \\ 0 & 1-b & c-1 & 0 & 0 \end{array}\right) \overset{F_2 \leftrightarrow F_3}{\sim} \begin{array}{c} \begin{array}{cccc} x & y & z & t \end{array} \\ \left(\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 0 \\ 0 & 1-b & c-1 & 0 & 0 \\ 0 & 2 & 0 & 0 & a \end{array}\right) \end{array} \overset{C_2 \leftrightarrow C_4}{\sim}$

$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & c-1 & 1-b & 0 \\ 0 & 0 & 0 & 2 & a \end{array}\right) \qquad 2y=a \Rightarrow y=\frac{a}{2}$

Si $c=1 \wedge b=1$:

$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & a \end{array}\right) \quad x-y+z+t=0 \Leftrightarrow \quad x \quad +z+t=\frac{a}{2} \Rightarrow \begin{cases} x=\frac{a}{2}-\lambda-\mu \\ z=\lambda \\ t=\mu \end{cases} \Rightarrow (x,y,z,t)=(0,\frac{a}{2},0,0)+L((-1,0,1,0),(-1,0,0,1))$ Si $a\neq 0$

Si $a=0$:

$(x,y,z,t)=L((-1,0,1,0),(-1,0,0,1))$

SCI

Si $c=1 \wedge b\neq 1$:

$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1-b & 0 \\ 0 & 0 & 0 & 2 & a \end{array}\right) \quad (1-b)y=0 \Leftrightarrow y=\frac{0}{(1-b)}=0 \quad .$ Si $a\neq 0 \Rightarrow y=\frac{a}{2}\neq 0$ $\Leftrightarrow$ SI

Si $a=0 \Rightarrow y=0$

$x+z+t=0 \Leftrightarrow (x,z,t)=(-\lambda-\mu,\lambda,\mu)=L((-1,1,0),(-1,0,1))$ SCI

$(x,y,z,t)=L((-1,0,1,0),(-1,0,0,1))$

Si $c\neq 1 \wedge b=1$:

$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & c-1 & 0 & 0 \\ 0 & 0 & 0 & 2 & a \end{array}\right) \quad \begin{array}{l} (c-1)z=0 \Leftrightarrow z=0 \\ y=\frac{a}{2} \end{array} \qquad x+t=\frac{a}{2} \Leftrightarrow x=\frac{a}{2}-t \Rightarrow (x,y,z,t)=(\frac{a}{2},0,0,0)+L((-1,0,0,1))$

SCI

Si $c\neq 1 \wedge b\neq 1$:

$y=\frac{a}{2}, \quad (c-1)z+\frac{(1-b)a}{2}=0 \Leftrightarrow z=\frac{-(1-b)a}{2(c-1)}, \quad x-y+z+t=0 \Leftrightarrow x+t=y-z=\frac{a}{2}+\frac{(1-b)a}{2(c-1)}$

$\begin{cases} x=\frac{a}{2}+\frac{(1-b)a}{2(c-1)}-\lambda \\ t=\lambda \end{cases} \Rightarrow (x,y,z,t)=\left(\frac{a}{2}+\frac{(1-b)a}{2(c-1)}, \frac{a}{2}, \frac{-(1-b)a}{2(c-1)}, 0\right)+L((-1,0,0,1))$ SCI

2) $V(\mathbb{K})$ e.v. $\dim(V)=n$, $U\leq V$ ¿Qué es el anulador de U? Probar que $an(U)\leq V^*$, y que $\dim(an(U))=\dim(V)-\dim(U)$.

$an(U)=\{\varphi\in V^* : \varphi(u)=0, \ \forall u\in U\}$ Sean $\{\phi^1,\phi^2\}\in an(U)$. Sean $k^1,k^2\in\mathbb{K}$. $(k^1\phi^1+k^2\phi^2)(u)=$

$k^1\phi^1(u)+k^2\phi^2(u)=k^1\cdot 0+k^2\cdot 0=0 \Rightarrow$ s.e.v.

Sea $\dim(U)=m\leq n=\dim(V)$, Una base $B=\{\overbrace{v_1,\dots,v_m}^{\text{Base de } U},v_{m+1},\dots,v_n\}$ de V. Sea $B^*=\{\phi^1,\dots,\phi^m,\phi^{m+1},\dots,\phi^n\}$

Base dual de B. $\Rightarrow \phi^i(v_j)=0 \quad \left.\begin{array}{l} i=\{m+1,\dots,n\} \\ j=\{1,\dots,m\} \end{array}\right\} \Rightarrow \dim(an(U))=n-m=\dim(V)-\dim(U)$

$f: M_2(\mathbb{R}) \to M_2(\mathbb{R})$, $f(X) = AX - XA$, $A = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$ $X_1 = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$, $X_2 = \begin{pmatrix} x' & y' \\ z' & t' \end{pmatrix}$

a) Probar que f es aplicación lineal.

f lineal si, dados $u, v \in V$, $a, b \in \mathbb{R} \Rightarrow f(au + bv) = af(u) + bf(v)$

$$f(aX_1 + bX_2) = A(aX_1 + bX_2) - (aX_1 + bX_2)A = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} ax+bx' & ay+by' \\ az+bz' & at+bt' \end{pmatrix} - \begin{pmatrix} ax+bx' & ay+by' \\ az+bz' & at+bt' \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 2ax+2bx'+az+bz' & 2ay+2by'+at+bt' \\ ax+bx'-2az-2bz' & ay+by'-2at-2bt' \end{pmatrix} - \begin{pmatrix} 2ax+2bx'+ay+by' & ax+bx'-2ay-2by' \\ 2az+2bz'+at+bt' & az+bz'-2at-2bt' \end{pmatrix}$$

$$= \begin{pmatrix} az-ay+bz'-by' & 4ay+4by'+at-ax+bt'-bx' \\ ax-at+bx'-bt'-4az-4bz' & ay-az+by'-bz' \end{pmatrix}$$

$$af(X_1) + bf(X_2) = a(AX_1 - X_1A) + b(AX_2 - X_2A) = a\left(\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x & y \\ z & t \end{pmatrix} - \begin{pmatrix} x & y \\ z & t \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}\right) + b\left(\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x' & y' \\ z' & t' \end{pmatrix} - \begin{pmatrix} x' & y' \\ z' & t' \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}\right)$$

$$= a\left(\begin{pmatrix} 2x+z & 2y+t \\ x-2z & y-2t \end{pmatrix} - \begin{pmatrix} 2x+y & x-2y \\ 2z+t & z-2t \end{pmatrix}\right) + b\left(\begin{pmatrix} 2x'+z' & 2y'+t' \\ x'-2z' & y'-2t' \end{pmatrix} - \begin{pmatrix} 2x'+y' & x'-2y' \\ 2z'+t' & z'-2t' \end{pmatrix}\right) =$$

$$a\begin{pmatrix} z-y & 4y+t-x \\ -4z+x-t & y-z \end{pmatrix} + b\begin{pmatrix} z'-y' & 4y'+t'-x' \\ -4z'+x'-t' & y'-z' \end{pmatrix} = \begin{pmatrix} az-ay+bz'-by' & 4ay+at-ax+4by'+bt'-bx' \\ -4az+ax-at-4bz'+bx'-bt' & ay-az+by'-bz' \end{pmatrix}$$

Hemos probado que $f(aX_1 + bX_2) = af(X_1) + bf(X_2) \Leftrightarrow f$ es lineal.

b) $B_u = \left\{ \underset{e_1}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}, \underset{e_2}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \underset{e_3}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}, \underset{e_4}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} \right\}$, ¿$M(f, B_u)$?

$$f\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = Ae_1 - e_1A = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 0e_1 - 1e_2 + 1e_3 + 0e_4$$

$$f\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = Ae_2 - e_2A = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ 0 & 1 \end{pmatrix} = -1e_1 + 4e_2 + 0e_3 + 1e_4$$

$$f\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = Ae_3 - e_3A = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -4 & -1 \end{pmatrix} = 1e_1 + 0e_2 - 4e_3 - 1e_4$$

$$f\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = Ae_4 - e_4A = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 0e_1 + 1e_2 - 1e_3 + 0e_4$$

$$\Rightarrow M(f, B_u) = \begin{pmatrix} 0 & -1 & 1 & 0 \\ -1 & 4 & 0 & 1 \\ 1 & 0 & -4 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$

c) ¿Base de $\ker(f)$, Base $\operatorname{Im}(f)$? Sea $X = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$

$\dim(M_2(\mathbb{R})) = 4 = r(f) + \overset{2}{\overset{\shortparallel}{n(f)}} \Leftrightarrow r(f) = 2$ $f(e_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $f(e_2) = \begin{pmatrix} -1 & 4 \\ 0 & 1 \end{pmatrix}$ l.i. $\Rightarrow B_{\operatorname{Im}(f)} = \left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 4 \\ 0 & 1 \end{pmatrix} \right\}$

$$M(f, B_u) \cdot C_{B_u}(X) = \begin{pmatrix} 0 & -1 & 1 & 0 \\ -1 & 4 & 0 & 1 \\ 1 & 0 & -4 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \left(\begin{array}{cccc|c} 0 & -1 & 1 & 0 & 0 \\ -1 & 4 & 0 & 1 & 0 \\ 1 & 0 & -4 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{array}\right) \underset{F_3 \leftrightarrow F_1}{\sim} \left(\begin{array}{cccc|c} 1 & 0 & -4 & -1 & 0 \\ -1 & 4 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{array}\right) \underset{F_2 = F_2 + F_1}{\sim}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -4 & -1 & 0 \\ 0 & 4 & -4 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{array}\right) \underset{F_4 = F_4 - \frac{1}{4}F_2}{\overset{F_3 = F_3 + \frac{1}{4}F_2}{\sim}} \left(\begin{array}{cccc|c} 1 & 0 & -4 & -1 & 0 \\ 0 & 4 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{matrix} z = \lambda \\ t = \mu \end{matrix} \Rightarrow \begin{matrix} x = 4\lambda + \mu \\ y = \lambda \end{matrix}$$

$\Rightarrow (x, y, z, t) = L((4, 1, 1, 0), (1, 0, 0, 1))_{B_u}$ Base $\ker(f)$

$B_{\ker f} = \left\{ \begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

4) $\psi^1, \psi^2, \psi^3 : \mathbb{R}^3 \to \mathbb{R}$; $\psi^1(x,y,z) = x+z$, $\psi^2(x,y,z) = x+y+2z$, $\psi^3(x,y,z) = y$

a) ψ^1, ψ^2, ψ^3 en términos de $B_u^* = \{w^1, w^2, w^3\}$

$\psi^1(x,y,z) = x+z = 1\cdot w^1(x,y,z) + 0\cdot w^2(x,y,z) + 1\cdot w^3(x,y,z) \Rightarrow C(\psi^1)_{B_u^*} = \begin{pmatrix}1\\0\\1\end{pmatrix}$

$\psi^2(x,y,z) = x+y+2z = 1\cdot w^1(x,y,z) + 1\cdot w^2(x,y,z) + 2\cdot w^3(x,y,z) \Rightarrow C(\psi^2)_{B_u^*} = \begin{pmatrix}1\\1\\2\end{pmatrix}$

$\psi^3(x,y,z) = y = 0\cdot w^1(x,y,z) + 1\cdot w^2(x,y,z) + 0\cdot w^3(x,y,z) \Rightarrow C(\psi^3)_{B_u^*} = \begin{pmatrix}0\\1\\0\end{pmatrix}$

b) Probar que $B^* = \{\psi^1, \psi^2, \psi^3\}$ es una base de $(\mathbb{R}^3)^*$. Sean $b^1, b^2, b^3 \in \mathbb{K}$ ($\mathbb{R}$ en este caso)

$(b^1\psi^1 + b^2\psi^2 + b^3\psi^3)(x,y,z) = 0 \Leftrightarrow b^1\cdot\psi^1(x,y,z) + b^2\cdot\psi^2(x,y,z) + b^3\cdot\psi^3(x,y,z) =$

$b^1(x+z) + b^2(x+y+2z) + b^3(y) = x(b^1+b^2) + y(b^2+b^3) + z(b^1+2b^2) = 0$

$\Leftrightarrow \begin{cases} b_1+b_2 = 0 \\ b_2+b_3 = 0 \\ b_1+2b_2 = 0 \end{cases} \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 \end{array}\right) \underset{\sim}{F_3 = F_3 - F_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \Rightarrow \boxed{b^2=0} \Rightarrow \boxed{b^3=0}$

$\Rightarrow \boxed{b^1=0}$

Luego por definición son l.i., y como $\dim(B^*) = \dim(\mathbb{R}^{3*}) = 3 \Rightarrow B^*$ es una base de $\mathbb{R}^{3*}$

c) Calcular la base B de $\mathbb{R}^3$ cuya base dual es B^*.

Sea $B = \{v_1, v_2, v_3\}$ la base.

$M(\psi^1, \{1\} \leftarrow B) = (1\ 0\ 0)$, $M(\psi^2, \{1\} \leftarrow B) = (0\ 1\ 0)$, $M(\psi^3, \{1\} \leftarrow B) = (0\ 0\ 1)$

Sea $\begin{cases} v_1 = (a^1, a^2, a^3) \\ v_2 = (b^1, b^2, b^3) \\ v_3 = (c^1, c^2, c^3) \end{cases}$

(i) $\begin{cases} \psi^1(a^1,a^2,a^3) = a^1+a^3 = 1 \\ \psi^1(b^1,b^2,b^3) = b^1+b^3 = 0 \\ \psi^1(c^1,c^2,c^3) = c^1+c^3 = 0 \end{cases}$

(ii) $\begin{cases} \psi^2(a^1,a^2,a^3) = a^1+a^2+2a^3 = 0 \\ \psi^2(b^1,b^2,b^3) = b^1+b^2+2b^3 = 1 \\ \psi^2(c^1,c^2,c^3) = c^1+c^2+2c^3 = 0 \end{cases}$

(iii) $\begin{cases} \psi^3(a^1,a^2,a^3) = a^2 = 0 \\ \psi^3(b^1,b^2,b^3) = b^2 = 0 \\ \psi^3(c^1,c^2,c^3) = c^2 = 1 \end{cases}$

(i) $\Downarrow$ (ii)

$\begin{cases} a^1+2a^3 = 0 \\ a^1+a^3 = 1 \end{cases}$

$\begin{cases} b^1+2b^3 = 1 \\ b^1+b^3 = 0 \end{cases}$

$\begin{cases} c^1+2c^3 = -1 \\ c^1+c^3 = 0 \end{cases}$

$a^1 = 2,\ a^2 = 0,\ a^3 = -1$

$b^1 = -1,\ b^2 = 0,\ b^3 = 1$

$c^1 = 1,\ c^2 = 1,\ c^3 = -1$

$\begin{cases} v_1 = (2,0,-1) \\ v_2 = (-1,0,1) \\ v_3 = (1,1,-1) \end{cases} \Leftarrow$