Laboratory 4: Mathematical models given by differential equations

Exercise 1 *Let's consider the model of radioactive decay:*

$$\begin{cases} R'(t) & = -k \cdot R(t) \\ R(0) & = R_0 \end{cases} .$$

(a) Find the IVP solution;

(b) $T_{1/2}$*−the half-life of radioactive substance is the length of time it takes the substance to decay to half of its original amount. Find the relation between* k*, the decay constant, and* $T_{1/2}$*, the half-life;*

(c) Find the decay constant for a radioactive substance for the given half-life value

(i) $T_{1/2} = 5730$ *years for* C^{14}

(ii) $T_{1/2} = 4,468 \cdot 10^9$ *years for* U^{238}

(iii) $T_{1/2} = 706 \cdot 10^6$ *years for* U^{235}

(d) In two years $3\ g$ *of radioisotope decay to* $0,9\ g$*. Find the half-life and the decay constant.*

(e) (Carbon dating of Shroud from Turin) In 1988 three independent dating tests revealed that the quantity of C^{14} *in the shroud was between* 91.57% *and* 93.021%*. Using the decay constant for* C^{14} *found it in the previous exercise determine when shroud was made.*

Exercise 2 *Let's consider the thermal cooling model:*

$$\begin{cases} T'(t) & = -k \cdot (T(t) - T_A) \\ T(0) & = T_0 \end{cases} .$$

(a) Find the IVP solution;

(b) Plot some solutions for different values of T_0*;*

(c) Suppose that in the case of a crime the victim body was discovered at 11.00 o'clock. The medical examiner at 11.30 *and measures the victim body temperature and he gets* $34.22°C$*. An hour later, he takes, again, the body temperature and he gets* $34.11°C$*. Supposing that the room temperature is* $21°C$ *estimate the time of the death.*

Exercise 3 *Let us consider two mathematical models of one population growth:*

$$\begin{cases} x'(t) & = r \cdot x(t) \\ x(0) & = x_0 \end{cases} \quad \textit{Malthus' model}$$

$$\begin{cases} x'(t) & = r_0 \cdot x(t) \left[1 - \dfrac{x(t)}{K}\right] \\ x(0) & = x_0 \end{cases} \quad \textit{Verhulst's model}$$

We know the USA population size:

1820 :	$9.6 \cdot 10^6$
1830 :	$12.9 \cdot 10^6$
1840 :	$17.1 \cdot 10^6$
1850 :	$23.2 \cdot 10^6$
1860 :	$31.4 \cdot 10^6$
1870 :	$38.6 \cdot 10^6$
1880 :	$50.2 \cdot 10^6$
1890 :	$62.9 \cdot 10^6$
1900 :	$76 \cdot 10^6$
1910 :	$92 \cdot 10^6$
1920 :	$106.5 \cdot 10^6$
1930 :	$123.2 \cdot 10^6$

(a) *Find the solution of these two models;*

(b) *Draw some solutions;*

(c) *Using the dates of USA population, find the corresponding parameters of the models;*

(d) *For the calculated parameters compare the given data with the obtained values from these models.*

Exercise 4 *Let's considered the model of vertical throwing that describes the dependence of the velocity on the distance from the surface of the earth:*

$$\begin{cases} v\,(x)\,v'\,(x) &= -\dfrac{gR^2}{(x+R)^2} \\ v(0) &= v_0 \end{cases}$$

where x is the the distance from the surface of the earth, R is the earth radius.

(a) *Find the IVP solution;*

(b) *For the initial velocity $v_0 = 50\frac{m}{s}$ find the the body velocity at the height of $75\ m$ (it will take $R = 6371\ km$, $g = 9.81\frac{m}{s^2}$);*

(c) *Determine the maximum altitude at which the body reaches for the data from point (b);*

(d) Find the escape velocity for each earth location if you know

$R_e = 6,378137 \cdot 10^6\ km$ is the the equatorial earth radius

$R_p = 6,356752 \cdot 10^6\ km$ is the the polar earth radius

$R_m = 6,372797 \cdot 10^6\ km$ is the the average earth radius