Lecture 6

Population models for single specie

1) Exponential growth (Malthus 1798)

$N(t)$ — the pop. size at the moment $t>0$

N_0 — initial pop. size at initial moment $t=0$.

$$\boxed{N(0)=N_0}\quad , N_0>0$$

$\frac{N'}{N}$ — per capita growth rate.

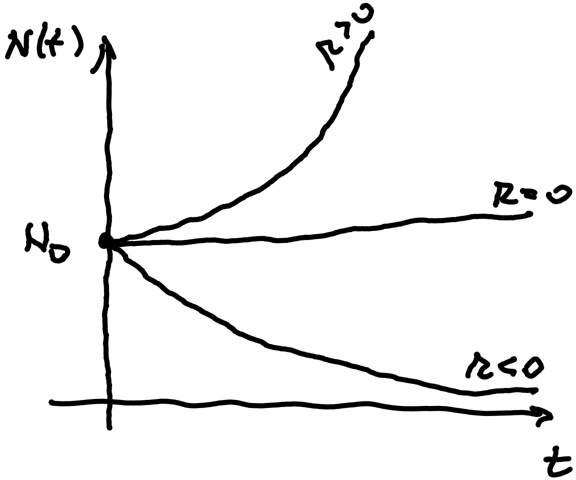
Malthus supposed that $\frac{N'}{N} = r = \text{const}$.

$$\boxed{r=b-d}$$

b — per capita birth rate
d — per capita death rate

$$\begin{cases} N' = rN \\ N(0)=N_0 \end{cases}$$

model solution: $\boxed{N(t) = N_0 \cdot e^{rt}}$



if $r>0 \Rightarrow N(t) \xrightarrow[t\to+\infty]{} +\infty$
pop growth unbounded

if $r=0 \Rightarrow N(t) \equiv N_0$
pop. remains constant in time

if $r<0 \Rightarrow N(t) \xrightarrow[t\to+\infty]{} 0$
pop disapears in time.

2) The logistic model (Verhulst (1838), Pearl - Reed (1920))

- the environment factors and the competition for the resources limit the pop. growth.
- the per capita growth rate depends on the pop. size

$$r = r(N) \Rightarrow \boxed{\frac{N'}{N} = r(N)}$$

$$r(N) = ?$$

- per capita growth rate decreases when the pop. size increases

  $\Rightarrow r(N)$ should be a decreasing function

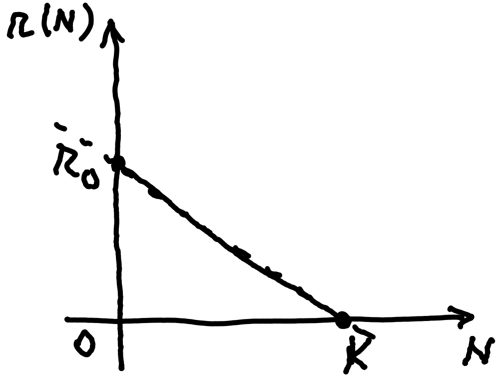
- K - the carrying capacity constant =

  = the maximum pop. that environment can support.

  $N(t) \to K \Rightarrow r(N) \to 0$

- when the pop. size is small (with respect to K) then the competition phenomenon for resources can be neglected, so in this case pop. grows according to the Malthus model

  $N(t) \to 0 \Rightarrow r(N) \to r_0$ intrinsic growt rate



$N(t) \to K \Rightarrow r(N) \to 0$

$N(t) \to 0 \Rightarrow r(N) \to r_0$

the function $r(N)$ should interpolate the points $(0, r_0)$ and $(K, 0)$ using linear interpolation

$$\boxed{r(N) = r_0 \left(1 - \frac{N}{K}\right)}$$

$$\begin{cases} N' = r_0 N \left(1 - \frac{N}{K}\right) \\ N(0) = N_0 \end{cases}$$ the logistic model

$N' = r_0 N \left(1 - \frac{N}{K}\right)$ separable diff. eq.

$N \equiv 0$
$N \equiv K$ are singular solutions

$$\frac{dN}{dt} = r_0 \cdot N \cdot \frac{K-N}{K} \Rightarrow \int \frac{K}{N(K-N)} dN = \int r_0 dt$$

$$\frac{K}{N(K-N)} = \frac{A}{N} + \frac{B}{K-N} \Rightarrow K = A.(K-N) + B.N$$

$N=0 \rightarrow K = A.K \rightarrow A = 1$

$N=K \rightarrow K = B.K \Rightarrow B = 1$

$$\frac{K}{N(K-N)} = \frac{1}{N} + \frac{1}{K-N}$$

$$\Rightarrow \int \left(\frac{1}{N} + \frac{1}{K-N} \right) dN = \int r_0 . dt$$

$$\ln N - \ln(K-N) = r_0 t + \ln c$$

$$\ln \frac{N}{K-N} = r_0 t + \ln c$$

$$\frac{N}{K-N} = c . e^{r_0 t}, c \in R$$

$$\Rightarrow N = c . e^{r_0 t} . K - c e^{r_0 t} . N$$

...

$$N\left(1+c.e^{r_0 t}\right) = c.e^{r_0 t}.K$$

$$\boxed{N(t) = \frac{c.e^{r_0 t}.K}{1+ce^{r_0 t}}, \ c \in \mathbb{R}}$$ the gen. sol.

$$N(0)=N_0 \Rightarrow \frac{c.K}{1+c} = N_0 \Rightarrow cK = N_0 + cN_0$$

$$\Rightarrow c(K-N_0) = N_0 \Rightarrow \boxed{c = \frac{N_0}{K-N_0}}$$

$$\Rightarrow N(t) = \frac{\frac{N_0}{K-N_0}.e^{r_0 t}.K}{1+\frac{N_0}{K-N_0}.e^{r_0 t}} \Rightarrow$$

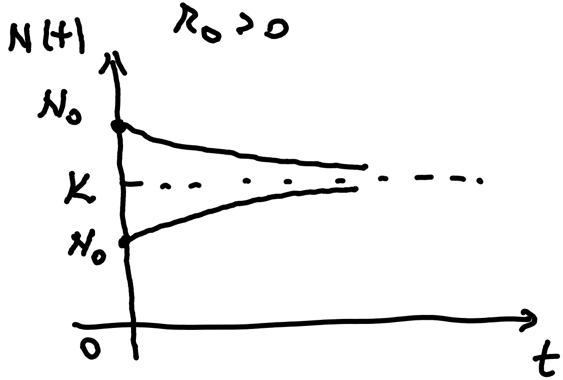
$$\boxed{N(t) = \frac{N_0 e^{r_0 t}.K}{K-N_0+N_0 e^{r_0 t}} = \frac{K.N_0}{(K-N_0)e^{-r_0 t}+N_0}}$$

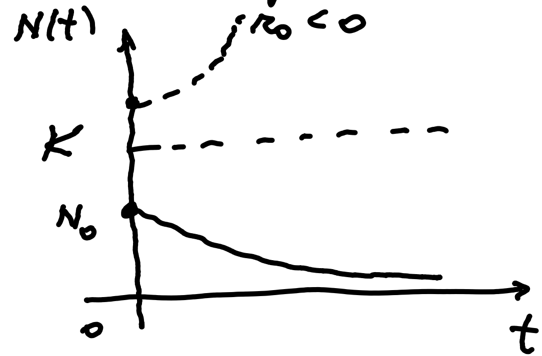
the model solution

- if $r_0 > 0 \Rightarrow N(t) \xrightarrow[t \to +\infty]{} K$ the pop. will tend to the carrying const. capacity

- if $r_0 = 0 \Rightarrow N(t) \equiv N_0$ the pop. remains constant in time

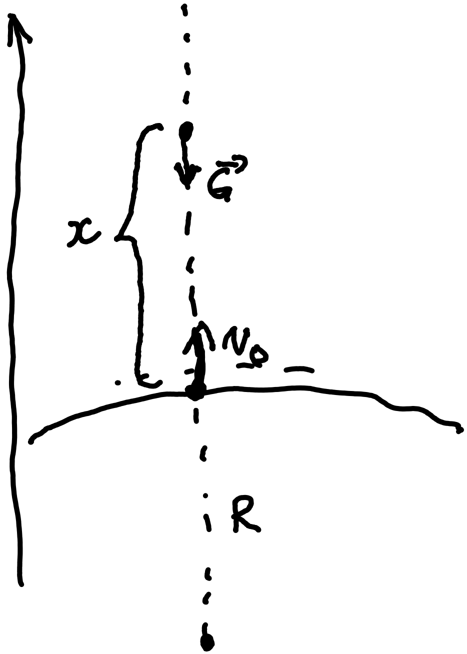
- if $r_0 < 0$ and $N_0 < K \Rightarrow N(t) \xrightarrow[t \to +\infty]{} 0$ the pop. will deseappear in time.





The vertical throwing. Escape velocity

Problem: An object of a constant mass is projected away from the earth in a direction perpendicular to the earth surface with the initial velocity v_0.
Assuming that there is no air resistance but taking into consideration the variation of the earth's gravitational field, find the expression of the velocity with the respect to the distance from the earth' surface.



x – the distance from the object to the surface

$v(x) = ?$

<u>Law:</u> The gravitational force acting on an object is inversely proportional to the square of the distance from the object to the earth's center.

$$\boxed{G(x) = -\frac{k}{(x+R)^2}}$$

„–" sign means that $G(x)$ is directed in the negative sens of x direction

at the earth' surface

$x = 0 \Rightarrow G(0) = -mg \Rightarrow -\frac{k}{R^2} = -mg \Rightarrow k = mgR^2$

$$\Rightarrow \boxed{G(x) = -mg \cdot \left(\frac{R}{x+R}\right)^2}$$

Newton law: $\boxed{m \cdot a = F}$ $\quad F = G$

$x = x(t)$ — the object position at the time t

$v(t) = x'(t)$

$a(t) = v'(t) = x''(t) \quad \Rightarrow \quad \boxed{m \cdot x''(t) = G}$

$$a(t) = v'(t) = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = v'(x) \cdot x'(t) = v'(x) \cdot v$$

$$\Rightarrow \cancel{m} \cdot v'(x) \cdot v = -\frac{\cancel{m} g R^2}{(x+R)^2}$$

$$\Rightarrow \begin{cases} v'(x) \cdot v(x) = -\dfrac{gR^2}{(x+R)^2} & \text{separable diff. eq.} \\ v(0) = v_0 \end{cases}$$

$$\underset{\frac{dv}{dx}}{v'} \cdot v = -\frac{gR^2}{(x+R)^2} \Rightarrow v \cdot dv = -\frac{gR^2}{(x+R)^2} \cdot dx \;\Big|\cdot 2$$

$$\int 2v \cdot dv = \int -\frac{2gR^2}{(x+R)^2}\, dx$$

$$\boxed{v^2 = \frac{2gR^2}{x+R} + c \quad , c \in \mathbb{R}}$$ the gen. sol.

$$v(0) = v_0 \Rightarrow v_0^2 = \frac{2gR^{\cancel{2}}}{\cancel{R}} + c \Rightarrow \boxed{c = v_0^2 - 2gR}$$

$\Rightarrow$ the model solution

$$v^2 = \frac{2gR^2}{x+R} + v_0^2 - 2gR$$

$$\boxed{v(x) = \pm\sqrt{\frac{2gR^2}{x+R} + v_0^2 - 2gR}}$$

Maximal altitude

h – maximal altitude

$\Rightarrow v(h) = 0$ $\qquad h(v_0) = ?$

$$\Rightarrow \boxed{\frac{2gR^2}{h+R} + v_0^2 - 2gR = 0}$$

$$\Rightarrow \frac{2gR^2}{h+R} = 2gR - v_0^2 \Rightarrow h + R = \frac{2gR^2}{2gR - v_0^2}$$

$$\Rightarrow h = \frac{2gR^2}{2gR - v_0^2} - R = \frac{2gR^2 - 2gR^2 + Rv_0^2}{2gR - v_0^2}$$

$$\Rightarrow \boxed{h(v_0) = \frac{Rv_0^2}{2gR - v_0^2}}$$

Escape velocity

v_e - escape velocity $\Leftrightarrow$ v_0 such that the object will not return to the earth

$\Rightarrow h \to +\infty$

$$\Rightarrow v_e = \lim_{h \to +\infty} v_0(h)$$

$$h = \frac{R v_0^2}{2gR - v_0^2} \Rightarrow \dots \Rightarrow$$

$$v_0^2 = \frac{2gRh}{h+R}$$

$$\Rightarrow \boxed{v_0(h) = \sqrt{\frac{2gRh}{h+R}}}$$

$$v_e = \lim_{h \to \infty} \sqrt{\frac{2gRh}{h+R}} = \sqrt{2gR} \simeq 11.1 \ \frac{km}{s}$$