Seminar 10 and 11 - 2025

Theoretical aspects

$\triangleright$ Let X be a random variable with the expectation $E(X)$ and variance $V(X)$. Assume $a > 0$. Then the following inequalities are true:

(1) **Markov's inequality**: $P\Big(|X| \geq a\Big) \leq \dfrac{1}{a} E|X|.$

(2) **Chebyshev's inequality**: $P\Big(|X - E(X)| \geq a\Big) \leq \dfrac{1}{a^2} V(X).$

A sequence $(X_n)_{n\in\mathbb{N}}$ of random variables converges in probability to a random variable X, denoted by $X_n \xrightarrow{\mathbb{P}} X$, if

$$\lim_{n\to\infty} \mathbb{P}\Big(|X_n - X| \leq \varepsilon\Big) = 1 \quad \text{for every} \quad \varepsilon > 0\,.$$

A sequence $(X_n)_{n\geq 1}$ of random variables converges in mean square to a random variable X if

$$\lim_{n\to\infty} \mathbb{E}[|X_n - X|^2] = 0.$$

This convergence is denoted by $X_n \xrightarrow{L^2} X$.

A sequence $(X_n)_{n\geq 1}$ of random variables converges in distribution to a random variable X if

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$$

in each continuity point x of F_X. This convergence is denoted by $X_n \xrightarrow{d} X$.

1. A program returns a value according to a random variable X with $E(X) = m \in \mathbb{R}$ and $V(X) = \sigma^2$, $\sigma > 0$. Prove that X takes values in the interval $(m - 3\sigma, m + 3\sigma)$ with more than 88% probability.

A: By Chebyshev's inequality, $P(|X - E(X)| \geq 3\sigma) \leq \frac{1}{9\sigma^2} V(X)$, so $1 - P(|X - m| < 3\sigma) \leq \frac{1}{9}$. Hence, $P(-3\sigma < X - m < 3\sigma) \geq \frac{8}{9} = 0.(8) > 0.88$.

2. The number of items produced in a factory during a day is a random variable with mean 50. If we consider a day, which event is more likely: E_1:"the production is more than 100 items in this day" or E_2:"the production is at most 100 items in this day"?

A: First, note that $P(E_1) = P(X > 100) = P(X \geq 101)$, while $P(E_2) = P(X \leq 100)$. Since $X = |X|$, by Markov's inequality, $P(E_1) = P(X \geq 101) \leq \frac{1}{101} E(X) = \frac{50}{101} < \frac{1}{2}$. Since $E_2 = \bar{E}_1$, we get $P(E_2) > \frac{1}{2}$, hence E_2 is more likely.

3. Let $(X_n)_{n\in\mathbb{N}^*}$ be a sequence of independent random variables with $Unif[a,b]$ distribution, where $a < b$. Define for each $n \in \mathbb{N}^*$

$$Y_n = \max\{X_1, \ldots, X_n\} \;\text{ and }\; Z_n = \min\{X_1, \ldots, X_n\}.$$

Prove that $Y_n \xrightarrow{P} b$ and $Z_n \xrightarrow{P} a$.

A: From the definition of the uniform distribution, for each $n \in \mathbb{N}^*$, the distribution function of X_n is given by

$$F_{X_n}(x) = P(X_n \le x) \quad = \quad \begin{cases} 0, & \text{if } x < a \\ \dfrac{x-a}{b-a}, & \text{if } a \le x < b \\ 1, & \text{if } b \le x\ . \end{cases}$$

Then, by the independence of the random variables in $(X_n)_{n\in\mathbb{N}^*}$, we have for every $x \in \mathbb{R}$

$$P(Y_n \le x) = P(\max\{X_1, \dots, X_n\} \le x) = P\Big(\bigcap_{k=1}^{n}\{X_k \le x\}\Big) = \Big(F_{X_1}(x)\Big)^n.$$

For $x = b$ we get $P(Y_n \le b) = 1$. Moreover, we have

$$\begin{aligned} &P(|Y_n - b| > \varepsilon) = P(b - Y_n > \varepsilon) = P(Y_n < b - \varepsilon) \\ &= \begin{cases} \left(1 - \frac{\varepsilon}{b-a}\right)^n, & \text{if } 0 < \varepsilon \le b-a \\ 0, & \text{if } b - a < \varepsilon\ . \end{cases} \end{aligned}$$

Therefore, $\lim\limits_{n\to\infty} P(|Y_n - b| > \varepsilon) = 0$ for each $\varepsilon > 0$. Observe that

$$Z_n = \min\{X_1, \dots, X_n\} = -\max\{-X_1, \dots, -X_n\}$$

with $-X_n \sim Unif[-b, -a]$ for each $n \in \mathbb{N}^*$. We apply the above result for the sequence $(-X_n)_{n\in\mathbb{N}^*}$ and we obtain $-Z_n \xrightarrow{P} -a$. It follows that $Z_n \xrightarrow{P} a$.

4. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of Bernoulli random variables. Prove that $X_n \xrightarrow{P} 0$ if and only if $X_n \xrightarrow{L^2} 0$.
A: Let $p_n \in (0,1)$ be such that $X_n \sim Bernoulli(p_n)$, $n \in \mathbb{N}$. From Example 32 in the course we have $X_n \xrightarrow{P} 0 \iff \lim_{n\to\infty} p_n = 0$. Since $E(X_n^2) = 1^2 \cdot p_n + 0^2 \cdot (1 - p_n) = p_n$, $X_n \xrightarrow{L^2} 0 \iff \lim_{n\to\infty} p_n = 0$.

5. Let $\lambda > 0$. A calling center has the following property, for every $n \in \mathbb{N}$, $n \ge 100$, during an hour interval $(0, 1]$: the calls arrive independently with at most one call in each time subinterval $\left(\frac{i}{n}, \frac{i+1}{n}\right]$, one call has probability $\frac{\lambda}{n}$ to occur, $i = \overline{0, n-1}$. Let's denote by X_n the corresponding total number of calls. Prove that $X_n \xrightarrow{d} X$, where $X \sim Poiss(\lambda)$.
A: Let $k \in \mathbb{N}$. For every $n \in \mathbb{N}$, $n \ge 100$, $n \ge k$, we have $X_n \sim Bino(n, \frac{\lambda}{n})$ and thus

$$P(X_n = k) = C_n^k \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} = \frac{\left(1 - \frac{\lambda}{n}\right)^n}{\left(1 - \frac{\lambda}{n}\right)^k} \frac{\lambda^k}{k!} \frac{n}{n} \frac{n-1}{n} \cdots \frac{n-k+1}{n}.$$

Hence, $\lim_{n\to\infty} P(X_n = k) = \frac{\lambda^k}{k!} \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^n = \frac{\lambda^k}{k!} e^{-\lambda}$. Let F be the cumulative distribution function of X. For $x \in \mathbb{R}$, $F_{X_n}(x) = \sum_{k\le x} P(X_n = k) \to F(x) = \sum_{k \le x} \frac{\lambda^k}{k!} e^{-\lambda}$, as $n \to \infty$. So, $X_n \xrightarrow{d} X$.

6. Let $(X_n)_{n\in\mathbb{N}^*}$ be a sequence of independent random variables with $Unif[0, 1]$ distribution. Define for each $n \in \mathbb{N}^*$

$$Y_n = \max\{X_1, \dots, X_n\} \ \text{ and } \ Z_n = \min\{X_1, \dots, X_n\}.$$

Prove that $Y_n \xrightarrow{L^2} 1$ and $Z_n \xrightarrow{L^2} 0$.

A: Let $F(x) = \begin{cases} 0, & x < 0 \\ x, & x \in [0,1] \\ 1, & x > 1 \end{cases}$ be the cumulative distribution function of $Unif[0,1]$.

As in the solution of problem 3,

$$F_{Y_n}(x) = P(Y_n \le x) = P(\bigcap_{k=1}^{n} X_k \le x) = (F(x))^n, \quad x \in \mathbb{R}.$$

So, for $n \in \mathbb{N}^*$, $f_{Y_n}(x) = \begin{cases} nx^{n-1}, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$ is a density function for Y_n. Hence, $E((Y_n - 1)^2) = E(Y_n^2) - 2E(Y_n) + 1 = \int_0^1 nx^{n+1}dx - 2\int_0^1 nx^n dx + 1 = \frac{n}{n+2} - \frac{2n}{n+1} + 1 \to 0$, $n \to \infty$, and thus $Y_n \xrightarrow{L^2} 1$.

Next,

$$F_{Z_n}(x) = P(Z_n \le x) = 1 - P(\bigcap_{k=1}^{n} X_k > x) = 1 - (1 - F(x))^n, \quad x \in \mathbb{R}.$$

Hence, for $n \in \mathbb{N}^*$, $f_{Z_n}(x) = \begin{cases} n(1-x)^{n-1}, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$ is a density function for Z_n. Therefore, $E(Z_n^2) = \int_0^1 nx^2(1-x)^{n-1}dx = n\left(-x^2\frac{(1-x)^n}{n}\Big|_0^1 - \frac{2}{n}x\frac{(1-x)^{n+1}}{n+1}\Big|_0^1 - \frac{2}{n(n+1)}\frac{(1-x)^{n+2}}{n+2}\Big|_0^1\right) = \frac{2}{(n+1)(n+2)} \to 0$, $n \to \infty$, and thus $Z_n \xrightarrow{L^2} 0$.

7*. Consider a sequence of distinct coins such that the probability of getting a head with the nth coin is $\frac{1}{n}$, $n \in \mathbb{N}^*$. Let X_n be 1, if the toss of the nth coin shows a head, and 0, otherwise. Do we have $X_n \xrightarrow{a.s.} 0$?

A: Let's compute $P(\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = 0\})$.

For every $\omega \in \Omega$, since $X_n(\omega) \in \{0,1\}$, $\lim_{n\to\infty} X_n(\omega) = 0$ if and only if $\exists m \in \mathbb{N}^*$ such that $X_n(\omega) = 0, \forall n \ge m$.

For $m \in \mathbb{N}^*$, let $A_m = \{\omega \in \Omega : X_n(\omega) = 0, \forall n \ge m\}$. So,

$$P(\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = 0\}) = P(\bigcup_{m=1}^{\infty} A_m) = \lim_{m\to\infty} P(A_m),$$

in view of Theorem 4 from the course, because $(A_m)_{m\in\mathbb{N}^*}$ is an increasing sequence of events.

Next, let $m \in \mathbb{N}^*$. Let $B_k = \bigcap_{n=m}^{k}(X_n = 0)$, $k \ge m$. Since $(B_k)_{k\ge m}$ is a decreasing sequence of events, by Theorem 4 from the course,

$$\begin{aligned} P(A_m) = P(\bigcap_{n=m}^{\infty}(X_n = 0)) &= \lim_{k\to\infty} P(B_k) = \lim_{k\to\infty} P\left(\bigcap_{n=m}^{k}(X_n = 0)\right) \\ &= \lim_{k\to\infty} P(X_m = 0, X_{m+1} = 0, \dots, X_k = 0) \\ &= \lim_{k\to\infty} P(X_m = 0)P(X_{m+1} = 0)\dots P(X_k = 0) \\ &= \lim_{k\to\infty} \frac{m-1}{m}\frac{m}{m+1}\dots\frac{k-1}{k} = \lim_{k\to\infty}\frac{m-1}{k} = 0, \end{aligned}$$

where we use the independence of the random variables. Hence, $P(\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega)\}) = 0$ and thus we don't have $X_n \xrightarrow{a.s.} 0$.

8. Let $(X_n)_n$, be a sequence of random variables such that for each $n \in \mathbb{N}^*$: $X_n \sim Exp(n)$, i.e, X_n has the following density function

$$f_{X_n}(t) = \begin{cases} 0, & \text{if } \ t \le 0 \\ ne^{-nt}, & \text{if } \ t > 0\,. \end{cases}$$

(a) Prove that $X_n \xrightarrow{P} 0$.
(b) Consider $Y_n = nX_n$, for each $n \in \mathbb{N}^*$. Prove that $(Y_n)_n$ does not converge in probability to 0.
(c) Write the cumulative distribution function (cdf) of $Z_n = \frac{1}{\sqrt{n}} Y_n$, $n \in \mathbb{N}^*$. Does $(Z_n)_n$ converge in probability to 0?

A: (a) We have to prove that

$$\lim_{n\to\infty} \mathbb{P}\Big(|X_n| \le \varepsilon\Big) = 1 \quad \text{for every} \quad \varepsilon > 0\,.$$

We compute

$$\mathbb{P}\Big(|X_n| \le \varepsilon\Big) = \Big(-\varepsilon \le X_n \le \varepsilon\Big) = \int_{-\varepsilon}^{\varepsilon} f_{X_n}(t)dt = \int_0^{\varepsilon} ne^{-nt}dt = -e^{-nt}\Big|_0^{\varepsilon} = 1 - e^{-n\varepsilon}.$$

Therefore

$$\lim_{n\to\infty} \mathbb{P}\Big(|X_n| \le \varepsilon\Big) = 1 \quad \text{for every} \quad \varepsilon > 0\,,$$

hence $X_n \xrightarrow{\mathbb{P}} 0$.
(b) We have

$$\mathbb{P}\Big(|Y_n| \le \varepsilon\Big) = \mathbb{P}\Big(|nX_n| \le \varepsilon\Big) = \Big(-\frac{\varepsilon}{n} \le X_n \le \frac{\varepsilon}{n}\Big) = \int_{-\frac{\varepsilon}{n}}^{\frac{\varepsilon}{n}} f_{X_n}(t)dt = \int_0^{\frac{\varepsilon}{n}} ne^{-nt}dt = -e^{-nt}\Big|_0^{\frac{\varepsilon}{n}} = 1 - e^{-\varepsilon}.$$

$$\Rightarrow \lim_{n\to\infty} \mathbb{P}\Big(|X_n| \le \varepsilon\Big) = 1 - e^{-\varepsilon} \ne 1 \quad \text{for every} \quad \varepsilon > 0\,,$$

hence $(Y_n)_n$ does not converge in probability to 0.
(c) We have $Z_n = \frac{1}{\sqrt{n}} Y_n = \sqrt{n} X_n$, $n \in \mathbb{N}^*$. We write the cdf of Z_n and $z \in \mathbb{R}$

$$\begin{aligned} F_{Z_n}(z) = P(\sqrt{n}X_n \le z) = F_{X_n}\left(\frac{z}{\sqrt{n}}\right) &= \int_{-\infty}^{\frac{z}{\sqrt{n}}} f_{X_n}(t)dt \\ &= \begin{cases} \displaystyle\int_{-\infty}^{\frac{z}{\sqrt{n}}} 0dt = 0, & \text{if } \ z \le 0 \\ \displaystyle\int_0^{\frac{z}{\sqrt{n}}} ne^{-nt}dt = -e^{-nt}\Big|_0^{\frac{z}{\sqrt{n}}} = 1 - e^{-\sqrt{n}z}, & \text{if } \ z > 0\,. \end{cases} \end{aligned}$$

It holds

$$\mathbb{P}\Big(|Z_n| \le \varepsilon\Big) = F_{Z_n}(\varepsilon) - F_{Z_n}(-\varepsilon) = 1 - e^{-\sqrt{n}\varepsilon} \Longrightarrow \lim_{n\to\infty} \mathbb{P}\Big(|Z_n| \le \varepsilon\Big) = 1 \quad \text{for every} \quad \varepsilon > 0\,.$$

Therefore, $(Z_n)_n$ converges in probability to 0.