Seminar 8

Linear differential equations

(1) $y^{(n)}+a_1(x)y^{(n-1)}+\dots+a_{n-1}(x)\cdot y'+a_n(x)\cdot y=f(x)$.

$a_1,\dots,a_n,f\in C(I)$

$f\not\equiv 0 \Rightarrow$ nonhomogeneous linear diff. eq.

$f\equiv 0 \Rightarrow$ homogeneous linear diff. eq.

(2) $y^{(n)}+a_1(x)y^{(n-1)}+\dots+a_{n-1}(x)\cdot y'+a_n(x)\cdot y=0$

$L: C^n(I)\to C(I)$

$y\mapsto Ly=y^{(n)}+a_1y^{(n-1)}+\dots+a_n\cdot y$

L is a linear operator

$(1)\Leftrightarrow Ly=f$.

$$\boxed{S=\ker L+\{y_p\}}$$

$\ker L=\{y \mid Ly=0\}$ sol. set of (2)

y_p is a partic. sol. of (1).

Theorem. $\dim \ker L = n$, $\ker L$ is a linear subspace of $C(I)$.

$\exists \{y_1, \dots, y_n\} \subset \ker L$ a basis in $\ker L \iff$

$\iff$ $\{y_1, \dots, y_n\} \subset \ker L$ and $\{y_1, \dots, y_n\}$ is a linear independent system.

$\iff \{y_1, \dots, y_n\} \subset \ker L$ and $W(x; y_1, \dots, y_n) \neq 0$

$$W(x; y_1, \dots, y_n) = \begin{vmatrix} y_1 & \dots & y_n \\ y_1' & & y_n' \\ \vdots & & \vdots \\ y_1^{(n-1)} & & y_n^{(n-1)} \end{vmatrix}$$ the wronskian.

Theorem.

a) If $y_1, \dots, y_n \in C^n(I)$ are linearly dependent $\underset{\not\Leftarrow}{\Rightarrow}$

$\Rightarrow$ $W(x; y_1, \dots, y_n) \equiv 0$

b) If $y_1, \dots, y_n \in \ker L$ are linearly independent $\Rightarrow$

$\underset{\Leftarrow}{\Rightarrow}$ $W(x; y_1, \dots, y_n) \neq 0$, $\forall x \in I$.

Ex. 1. $W(x; y_1, \dots, y_n) \equiv 0$ on $I = [a,b] \Rightarrow$

Are the functions $y_1, \dots, y_n$ linearly dependent?

Answer is No.

$I = [-1,1]$

$n = 2$.

$$y_1(x) = \begin{cases} x^2, & x \in [-1,0) \\ 0, & x \in [0,1] \end{cases}$$

$$y_2(x) = \begin{cases} 0, & x \in [-1,0) \\ x^2, & x \in [0,1] \end{cases}$$

$y_1, y_2 \in C^1(I)$

$$W(x; y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{cases} \begin{vmatrix} x^2 & 0 \\ 2x & 0 \end{vmatrix}, & x \in [-1,0) \\ \begin{vmatrix} 0 & x^2 \\ 0 & 2x \end{vmatrix}, & x \in [0,1] \end{cases} = 0$$

$\Rightarrow W(x; y_1, y_2) \equiv 0$ on $[-1,1]$.

$\{y_1, y_2\}$ is a linear independent system of functions.

$$c_1 y_1 + c_2 y_2 = 0 \overset{?}{\Rightarrow} c_1 = c_2 = 0$$

$c_1 y_1 + c_2 y_2 = 0$ $\qquad$ $c_1 \cdot y_1(x) + c_2 \cdot y_2(x) = 0 \quad \forall x \in [-1,1]$

if $x \in [-1, 0)$ then: $c_1 . x^2 + c_2 \cdot 0 = 0 \Rightarrow c_1 x^2 = 0 \quad \forall x \in [-1, 0]$

$\Rightarrow \boxed{c_1 = 0}$

if $x \in [0,1]$ then $c_2 \cdot y_2(x) = 0$

$c_2 x^2 = 0 \Rightarrow c_2 = 0$

$\Rightarrow$ y_1, y_2 are linearly independent.

Ex. 2. Find the linear homogeneous diff. eq which has as a fundam. syst. of solutions the functions:

$y_1(x) = e^x . \cos 3x$, $y_2(x) = e^x . \sin 3x$.

$y \in \ker L$ $\qquad$ if $\{y_1, y_2\}$ is a fundam. syst. of sol $\Rightarrow$

$\dim \ker L = 2$, $\Rightarrow$ the diff. eq. has the order 2.

$$\boxed{y'' + a_1(x) . y' + a_2(x) . y = 0}$$

if $y \in \ker L \Rightarrow \{y, y_1, y_2\}$ is a linear dependent syst. of functions.

$$\Rightarrow W(x; y, y_1, y_2) \equiv 0$$

$$\underset{=}{y}$$

$$\begin{vmatrix} y & y_1 & y_2 \\ y' & y_1' & y_2' \\ y'' & y_1'' & y_2'' \end{vmatrix} = 0, \quad \forall x \in I.$$

$$\begin{vmatrix} y & e^x \cos 3x & e^x \sin 3x \\ y' & e^x \cos 3x - 3e^x \sin 3x & e^x \sin 3x + 3e^x \cos 3x \\ y'' & e^x \cos 3x - 6e^x \sin 3x + 9e^x \cos 3x & e^x \sin 3x + 6e^x \cos 3x - 9e^x \sin 3x \end{vmatrix} = 0$$

$$\Rightarrow \ldots \Rightarrow \boxed{y'' - 2y' + 10y = 0}$$

Ex 3. Consider the diff. eq.

$$Ly = 0$$

where $Ly = y'' + a_1 \cdot y' + a_2 \cdot y$, $a_1, a_2 \in C(I)$.

Make the change of variables $y = y_1 \cdot z$ where y_1 is given, $y_1 \in C^2(I)$. Conclusion!

$$y'' + a_1 \cdot y' + a_2 y = 0.$$

$y = y(x)$ is the unknown function

$y(x) \longmapsto z(x)$ $\boxed{y(x) = y_1(x) \cdot z(x).}$ where $y_1 \in C^2(I)$ is given.

$y' = y_1' \cdot z + y_1 \cdot z' \,/ \cdot a_1$

$y'' = y_1'' \cdot z + y_1' \cdot z' + y_1' \cdot z' + y_1 \cdot z'' = y_1'' \cdot z + 2 \cdot y_1' \cdot z' + y_1 \cdot z''$

$\Rightarrow$ $y_1'' \cdot z + 2 \cdot y_1' \cdot z' + y_1 z''$

$+ a_1 y_1' \cdot z + a_1 \cdot y_1 \cdot z' +$ $\Rightarrow$

$+ a_2 y_1 \cdot z = 0$

$$y_1 \cdot z'' + z' \cdot (2 \cdot y_1' + a_1 \cdot y_1) + z \cdot \underbrace{(y_1'' + a_1 \cdot y_1' + a_2 \cdot y_1)}_{Ly_1 = 0} = 0$$

if y_1 is a solution of the eq.

$$y'' + a_1 y' + a_2 y = 0$$

then $Ly_1 = 0$ and

$$y_1 z'' + (2y_1' + a_1 \cdot y_1) \cdot z' = 0$$

second order diff. eq. of the form:

$$F(z'', z') = 0$$

$$z' = u \Rightarrow F(u', u) = 0$$

$\Rightarrow$ first order diff. eq.

$z' = u$

$$\Rightarrow \boxed{y_1 u' + (2y_1 + a_1 y_1) \cdot u = 0}$$

a first order homogeneous linear eq. (solvable one).

<u>Conclusion</u>: The minimal requirement to solve a homog. linear second order diff. eq is to find one solution.

<u>Ex4</u>. Solve the following diff. eq.:

a) $xy'' - (2x+1)y' + 2.y = 0$ knowing that admits a sol. of the form $y_1(x) = e^{\alpha x}$

b) $xy'' - (x+3).y' + 2y = 0$ knowing that it admits a solution polynomial of degree 2. (homework).

---

a) $y_1(x) = e^{\alpha x}$ $\quad \alpha = ?$

$y_1' = \alpha . e^{\alpha x}$

$y_1'' = \alpha^2 e^{\alpha x}$

$xy_1'' - (2x+1)y_1' + 2.y_1 = 0$

$x.\alpha^2 \cancel{e^{\alpha x}} - (2x+1).\alpha.\cancel{e^{\alpha x}} + 2\cancel{e^{\alpha x}} = 0$

$\alpha^2 x - (2x+1)\alpha + 2 = 0$

$\underbrace{\alpha^2 x} - \underbrace{2\alpha x} - \alpha + 2 = 0$

$\alpha x(\alpha - 2) - (\alpha - 2) = 0$

$(\alpha - 2).(\alpha x - 1) = 0 \ \forall x.$

$\alpha - 2 = 0 \Rightarrow \boxed{\alpha = 2}$

$\Rightarrow \boxed{y_1(x) = e^{2x}}$

$y = y_1 \cdot z \quad \to \boxed{y = e^{2x} \cdot z} \quad | \cdot 2$

$y' = 2e^{2x} \cdot z + e^{2x} \cdot z' \quad | \cdot -(2x+1)$

$y'' = 4 \cdot e^{2x} \cdot z + 2e^{2x} \cdot z' + 2 \cdot e^{2x} \cdot z' + e^{2x} \cdot z''$

$y'' = 4e^{2x} \cdot z + 4e^{2x} \cdot z' + e^{2x} \cdot z'' \quad | \cdot x$

$xy'' - (2x+1)y' + 2y = 0$

$4x\cancel{e^{2x}} \cdot z + 4x\cancel{e^{2x}} \cdot z' + \underline{x\cancel{e^{2x}} \cdot z''}$

$-2(2x+1)\cancel{e^{2x}} \cdot z - (2x+1)\cancel{e^{2x}} \cdot z' + 2\cancel{e^{2x}} z = 0 \quad | : e^{2x}$

$\Rightarrow xz'' + z' \cdot (4x - 2x - 1) + z \cdot (\cancel{-4x} - \cancel{2} \ \cancel{+4x} + \cancel{2}) = 0$

$\Rightarrow \boxed{xz'' + (2x-1) \cdot z' = 0}$

$z' = u \quad \Rightarrow \quad xu' + (2x-1)u = 0$

$u' = -\frac{(2x-1)}{x} \cdot u \qquad u' = \frac{du}{dx}$

$\frac{u'}{u} = -2 + \frac{1}{x}$

$$\frac{1}{u}\cdot\frac{du}{dx} = -2+\frac{1}{x}$$

$$\int \frac{du}{u} = \int\left(-2+\frac{1}{x}\right)\cdot dx$$

$$\ln u = -2x+\ln x+\ln c$$

$$u(x) = c\cdot e^{-2x+\ln x} \Rightarrow \boxed{u(x) = c\cdot x\cdot e^{-2x}}$$

$$z' = u \Rightarrow z' = c_1 x\cdot e^{-2x}$$

$$z(x) = \int c_1 x \underbrace{e^{-2x}}_{\left(\frac{e^{-2x}}{(-2)}\right)'} dx + c_2$$

$$\Rightarrow z(x) = c_1\left(-\frac{x}{2}e^{-2x}+\frac{1}{2}\int e^{-2x}dx\right)+c_2$$

$$\boxed{z(x) = c_1\left(-\frac{x}{2}e^{-2x}-\frac{1}{4}e^{-2x}\right)+c_2}$$

$$\Rightarrow y = e^{2x}\cdot z \Rightarrow y(x) = c_1\left(-\frac{x}{2}\cancel{e^{-2x}}-\frac{1}{4}\cdot\cancel{e^{-2x}}\right)\cdot\cancel{e^{2x}}+c_2e^{2x}$$

$$\boxed{y(x) = c_1\cdot\left(-\frac{x}{2}-\frac{1}{4}\right)+c_2e^{2x},\ c_1, c_2\in\mathbb{R}}$$

Ex 5. Solve the diff. eq.:

$$(2x+1)y'' + 4x \cdot y' - 4y = (2x+1)^2$$

knowing that the homogeneous eq. has as solutions the functions $y_1(x) = x$, $y_2(x) = e^{-2x}$.

---

the gen. sol. $y = y_0 + y_p$ where

y_0 is the gen. sol. of the homog. eq.

$$\boxed{(2x+1)y'' + 4xy' - 4y = 0}$$

y_p is a partic. sol. of the nonhomog. eq.

y_1, y_2 are sol. of. the homog. eq.

we check if $\{y_1, y_2\}$ is linear indep. syst. of functions.

$\Leftrightarrow W(x; y_1, y_2) \neq 0$.

$$W(x; y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \overset{?}{\neq} 0$$

$$W(x, y_1, y_2) = \begin{vmatrix} x & e^{-2x} \\ 1 & -2e^{-2x} \end{vmatrix} = -2x e^{-2x} - e^{-2x} =$$

$$= -\underbrace{e^{-2x}}_{>0}\underbrace{(2x+1)}_{\neq 0 \text{ if } x \neq -\frac{1}{2}.}$$

$$\underbrace{\phantom{e^{-2x}}}_{<0}$$

Remark: We solve our eq. on interval which does not contain $x = -\frac{1}{2}$

$\Rightarrow W(x, y_1, y_2) \neq 0 \Rightarrow \{y_1, y_2\}$ is a fundam. system of. solutions.

$\Rightarrow y_0 = c_1 y_1 + c_2 y_2$

$$\boxed{y_0(x) = c_1 x + c_2 e^{-2x}, \quad c_1, c_2 \in \mathbb{R}}$$

$y_p(x) = ?$

$$y_p(x) = c_1(x)\cdot x + c_2(x)\cdot e^{-2x} \quad /(-4)$$

$$(2x+1)\,y_p'' + 4x\cdot y_p' - 4y_p = (2x+1)^2$$

$$y_p'(x) = \underline{c_1'\cdot x} + c_1 + \underline{c_2'\cdot e^{-2x}} - 2c_2\cdot e^{-2x}$$

we impose the cond. that: $\boxed{c_1'x + c_2'e^{-2x} = 0}$

$$\Rightarrow y_p'(x) = c_1 - 2c_2 e^{-2x} \quad /\cdot 4x$$

$$y_p''(x) = c_1' - 2c_2'\cdot e^{-2x} + 4c_2 e^{-2x} \quad /\cdot 2x+1$$

$$\underline{(2x+1)c_1'} - 2\,\underline{(2x+1)c_2'}\cdot e^{-2x} + 4(2x+1)c_2 e^{-2x}$$

$$\cancel{+4xc_1} - 8c_2 x e^{-2x} - \cancel{4c_1x} - 4c_2 e^{-2x} = (2x+1)^2$$

$$(2x+1)c_1' - (4x+2)c_2' e^{-2x} + c_1(4x-4x) +$$

$$+ c_2\left(\underbrace{8x+4-8x}_{=0} - 4\right)e^{-2x} = (2x+1)^2 \Rightarrow$$

$$(\cancel{2x+1})c_1' - 2(\cancel{2x+1})c_2' e^{-2x} = (2x+1)^{\cancel{2}} \quad \Big| : (2x+1)$$

$$\begin{cases} c_1' - 2c_2' \cdot e^{-2x} = 2x+1 \\ c_1' x + c_2' e^{-2x} = 0 \quad | \cdot 2 \end{cases} \Rightarrow c_1', c_2' \xRightarrow{\int} c_1, c_2 \Rightarrow y_p(x).$$

(homework).

$$\begin{cases} c_1' - 2c_2' e^{-2x} = 2x+1 \\ 2c_1' + 2c_2' e^{-2x} = 0 \end{cases}$$

$$3c_1' \quad / \quad = 2x+1 \quad \Rightarrow c_1'(x) = \frac{2}{3}x + \frac{1}{3}.$$

$$c_2' e^{-2x} = -c_1' x \Rightarrow c_2' = -c_1' \cdot x \cdot e^{2x} = \left(-\frac{2}{3}x^2 - \frac{1}{3}x\right)e^{2x}$$

$$\begin{cases} c_1'(x) = \frac{2}{3}x + 1 \\ c_2'(x) = \left(-\frac{2}{3}x^2 - \frac{1}{3}x\right)e^{2x} \end{cases} \xRightarrow{\int} \begin{cases} c_1(x) = \ldots \\ c_2(x) = \ldots \end{cases}$$