Seminar 2: Def 1. Let X be a set. A family $\mathcal{A} \subseteq \mathcal{P}(X)$ is called a σ-algebra on X if:

(A1) $\emptyset \in \mathcal{A}$

(A2) $A \in \mathcal{A} \Rightarrow CA \in \mathcal{A}$

(A3) $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$

Property 1. Let $\mathcal{A}$ be a σ-algebra on a set X. Then

(i) $X \in \mathcal{A}$

(ii) $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A} \Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{A}$

(iii) $A_1, \dots, A_N \in \mathcal{A} \Rightarrow \bigcup_{n=1}^{N} A_n \in \mathcal{A}, \bigcap_{n=1}^{N} A_n \in \mathcal{A}$

(iv) $A, B \in \mathcal{A} \Rightarrow A \setminus B \in \mathcal{A}$

Ex 1. Which of the following families of sets in $\mathbb{R}$ are σ-algebras on $\mathbb{R}$?

a) $\mathcal{A} = \{A \subseteq \mathbb{R} \mid A \text{ is finite}\}$

Sol. $\mathbb{R} \notin \mathcal{A} \Rightarrow \mathcal{A}$ is not a σ-algebra on $\mathbb{R}$.

b) $\mathcal{A} = \{A \subseteq \mathbb{R} \mid A \text{ is finite or } CA \text{ is finite}\}$

Sol: Let $A_n = \{n\}$, $n \in \mathbb{N} \Rightarrow \forall n \in \mathbb{N}$, $A_n \in \mathcal{A}$

$\bigcup_{n=1}^{\infty} A_n = \mathbb{N} \notin \mathcal{A} \Rightarrow \mathcal{A}$ is not a σ-algebra

c) $\mathcal{A} = \{A \subseteq \mathbb{R} \mid A \text{ is at most countable or } CA \text{ is at most countable}\}$

Sol: (A1) $\emptyset \in \mathcal{A}$

(A2) Let $A \in \mathcal{A}$

Case 1: A is at most countable, $A = C(CA) \Rightarrow CA \in \mathcal{A}$

Case 2: A is not at most countable $\Rightarrow CA$ is at most countable $\Rightarrow A \in \mathcal{A}$

(A3) Let $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$

Case 1: A is at most countable, $A = C(CA) \Longrightarrow$
$CA \in \mathcal{A}$

Case 2: A is not at most countable $\Rightarrow CA$ is at most countable $\Rightarrow A \in \mathcal{A}$

(A3) Let $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$

Case 1: $\forall n \in \mathbb{N}$, A_n is at most countable $\Longrightarrow \bigcup_{n=1}^{\infty} A_n$ is at most countable, as a countable union of countable sets.

Case 2: $\exists n_0 \in \mathbb{N}$ s.t. A_{n_0} is not at most countable $\Longrightarrow$

$\Longrightarrow CA_{n_0}$ is at most countable.

$$C\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcap_{n=1}^{\infty} CA_n \subseteq CA_{n_0} \Longrightarrow C\left(\bigcup_{n=1}^{\infty} A_n\right) \text{ is}$$

at most countable $\Longrightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

So, $\mathcal{A}$ is a σ-algebra on $\mathbb{R}$.

is not at most countable $\Rightarrow$
table $\Rightarrow C\left(\bigcup_{n=1}^{\infty} A_n\right)$ is
$\subseteq CA_{n_0}$

$A_n \in \mathcal{A}$.

$\mathbb{R}$.

of two σ-algebras on a set X
gebra on X

following two σ-algebras on $\mathbb{R}$:

$\mathcal{A}_2 = \{\phi, \mathbb{R}, \{1\}, \mathbb{R}\setminus\{1\}\}$.

$1\} \notin \mathcal{A}_1 \cup \mathcal{A}_2 \Rightarrow \mathcal{A}_1 \cup \mathcal{A}_2$ is not

<u>Def 2</u>: Let X be a set and $\mathcal{E} \subseteq \mathcal{P}(X)$. The family

$$\sigma(\mathcal{E}) = \bigcap \{\mathcal{A} \mid \mathcal{A} \text{ - is a } \sigma\text{-algebra on } X,\ \mathcal{E} \subseteq \mathcal{A}\}$$ is called

the σ-algebra generated by the family $\mathcal{E}$ (is the smallest σ-algebra that contains the family $\mathcal{E}$).

Let $X = \mathbb{R}^n$ and $\mathcal{E}$ be the family of all open subsets of $\mathbb{R}$, then $\sigma(\mathcal{E}) =: \mathcal{B}(\mathbb{R}^n)$ is called the family of all Borel sets in $\mathbb{R}^n$.

<u>Prop</u>. $\mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \mathcal{P}(X) \Rightarrow \sigma(\mathcal{E}_1) \subseteq \sigma(\mathcal{E}_2)$

A is finite}

, $A_n \in \mathcal{A}$

σ-algebra

ntable or CA is at …

cou…

le, $A = C(CA)$

table $\Rightarrow$ CA is a…

countable $\Rightarrow$ (

ntable union

Case 2: $\exists n_0 \in \mathbb{N}$ s.t. A_{n_0} is not at most countable $\Rightarrow$

$\Rightarrow$ CA_{n_0} is at most countable $\Rightarrow$ $C\left(\bigcup_{n=1}^{\infty} A_n\right)$ is

$C\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcap_{n=1}^{\infty} CA_n \subseteq CA_{n_0}$

…t countable $\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

… is a σ-algebra on $\mathbb{R}$.

…w that the union of two σ-algebras on a set X

…not necessarily a σ-algebra on X

Let us consider the following two σ-algebras on $\mathbb{R}$:

…$\emptyset, \mathbb{R}, \{0\}, \mathbb{R}\setminus\{0\}\}$, $\mathcal{A}_2 = \{\emptyset, \mathbb{R}, \{1\}, \mathbb{R}\setminus\{1\}\}$.

… $\mathcal{A}_1 \cup \mathcal{A}_2$, but $\{0,1\} \notin \mathcal{A}_1 \cup \mathcal{A}_2 \Rightarrow \mathcal{A}_1 \cup \mathcal{A}_2$ is not

<u>Def 2</u>: Let X be a set and $\mathcal{E} \subseteq \mathcal{P}(X)$…

$\sigma(\mathcal{E}) = \bigcap\{\mathcal{A} \mid \mathcal{A}$ - is a σ-algebra …

the σ-algebra generated by the fam…

σ-algebra that contains the famil…

Let $X = \mathbb{R}^n$ and $\mathcal{E}$ be the famil…

of $\mathbb{R}$, then $\sigma(\mathcal{E}) =: \mathcal{B}(\mathbb{R}^n)$ is c…

all Borel sets in $\mathbb{R}^n$.

<u>Prop</u>. $\mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \mathcal{P}(X) \Rightarrow \sigma(\mathcal{E}_1) \subseteq \sigma(\mathcal{E}_2)$

Case 2: $\exists n_0 \in \mathbb{N}$ s.t. A_{n_0} is not at most countable $\Rightarrow$

$\Rightarrow CA_{n_0}$ is at most countable.

$$C\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcap_{n=1}^{\infty} CA_n \subseteq CA_{n_0} \Rightarrow C\left(\bigcup_{n=1}^{\infty} A_n\right) \text{ is}$$

at most countable $\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

So $\mathcal{A}$ is a σ-algebra on $\mathbb{R}$.

Ex 2: Show that the union of two σ-algebras on a set X is not necessarily a σ-algebra on X.

Sol: Let us consider the following two σ-algebras on $\mathbb{R}$:

$\mathcal{A}_1 = \{\emptyset, \mathbb{R}, \{0\}, \mathbb{R}\setminus\{0\}\}$, $\mathcal{A}_2 = \{\emptyset, \mathbb{R}, \{1\}, \mathbb{R}\setminus\{1\}\}$.

$\{0\}, \{1\} \in \mathcal{A}_1 \cup \mathcal{A}_2$, but $\{0,1\} \notin \mathcal{A}_1 \cup \mathcal{A}_2 \Rightarrow \mathcal{A}_1 \cup \mathcal{A}_2$ is not a σ-algebra on $\mathbb{R}$.

most countable $\Rightarrow$

$\Rightarrow C\left(\bigcup_{n=1}^{\infty} A_n\right)$ is

alg

X

Def 2: Let X be a set and $\mathcal{E} \subseteq \mathcal{P}(X)$. The family
$\sigma(\mathcal{E}) = \bigcap \{ \mathcal{A} \mid \mathcal{A}$ - is a σ-algebra on X, $\mathcal{E} \subseteq \mathcal{A} \}$ is called the σ-algebra generated by the family $\mathcal{E}$ (is the smallest σ-algebra that contains the family $\mathcal{E}$).

Let $X = \mathbb{R}^n$ and $\mathcal{E}$ be the family of all open subsets of $\mathbb{R}$, then $\sigma(\mathcal{E}) =: \mathcal{B}(\mathbb{R}^n)$ is called the family of all Borel sets in $\mathbb{R}^n$.

Prop. $\mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \mathcal{P}(X) \Rightarrow \sigma(\mathcal{E}_1) \subseteq \sigma(\mathcal{E}_2)$

$\mathcal{E}$ is a σ-algebra $\Rightarrow \sigma(\mathcal{E}) = \mathcal{E}$

Remark 1. $\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{F})$, where $\mathcal{F}$ is the family of all closed subsets of $\mathbb{R}^n$

$\mathcal{H} = \{ [a_1, b_1] \times \dots \times [a_n, b_n] \mid a_i, b_i \in \mathbb{R}, a_i \leq b_i, i = \overline{1,n} \}$

Def 2: Let X be a set and $\mathcal{E} \subseteq \mathcal{P}(X)$ The family

$$\sigma(\mathcal{E}) = \bigcap \{\mathcal{A} \mid \mathcal{A} \text{ - is a } \sigma\text{- algebra on } X, \mathcal{E} \subseteq \mathcal{A}\}$$

the σ-algebra generated by the family $\mathcal{E}$ (is the σ-algebra that contains the family $\mathcal{E}$).

Let $X = \mathbb{R}^m$ and $\mathcal{E}$ be the family of all open of $\mathbb{R}$, then $\sigma(\mathcal{E}) =: \mathcal{B}(\mathbb{R}^m)$ is called the fam all Borel sets in $\mathbb{R}^m$.

Prop. $\mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \mathcal{P}(X) \Rightarrow \sigma(\mathcal{E}_1) \subseteq \sigma(\mathcal{E}_2)$

$\mathcal{E}$ is a σ-algebra $\Rightarrow \sigma(\mathcal{E}) = \mathcal{E}$

Remark 1: $\mathcal{B}(\mathbb{R}^m) = \sigma(\mathcal{F})$, where $\mathcal{F}$ is the family of all subsets of $\mathbb{R}^m$

$$\mathcal{H} = \{[a_1, b_1] \times \ldots \times [a_m, b_m] \mid a_i, b_i \in \mathbb{R}, a_i \leq b_i, i = \overline{1,m}\}$$

$$\mathcal{H}' = \{[a_1, b_1] \times \ldots \times [a_m, b_m] \mid a_i, b_i \in \mathbb{Q}, a_i \leq b_i, i = \overline{1,m}\}$$

Ex3. Show that a) $\sigma(\mathcal{H}') = \sigma(\mathcal{H}) = \mathcal{B}(\mathbb{R}^m)$

Sol. Let us denote by $\mathcal{E}$ and $\mathcal{F}$ the families of all open and closed subsets of $\mathbb{R}^m$, respectively.

$$\mathcal{H}' \subseteq \mathcal{H} \subseteq \mathcal{F} \Rightarrow \sigma(\mathcal{H}') \subseteq \sigma(\mathcal{H}) \subseteq \sigma(\mathcal{F}) = \mathcal{B}(\mathbb{R}^m) \quad (*)$$

Let G be a non-empty open subset of $\mathbb{R}^m$ $\xRightarrow[\text{Sem 1}]{\text{Ex 1}}$

$$\Rightarrow \exists (H_n)_{n\in\mathbb{N}} \subseteq \mathcal{H}' \text{ s.t. } G = \bigcup_{n=1}^{\infty} \underbrace{H_n}_{\in \sigma(\mathcal{H}')} \in \sigma(\mathcal{H}')$$

$$\Rightarrow \mathcal{E} \subseteq \sigma(\mathcal{H}') \Rightarrow \sigma(\mathcal{E}) = \mathcal{B}(\mathbb{R}^m) \subseteq \sigma(\mathcal{H}') \quad \emptyset \in \sigma(\mathcal{H}') \stackrel{(*)}{\Rightarrow}$$

$$\Rightarrow \sigma(\mathcal{H}') = \sigma(\mathcal{H}) = \mathcal{B}(\mathbb{R}^m)$$

b) $\mathcal{B}(\mathbb{R})$ is generated by the following families of sets in $\mathbb{R}$:

(i) $\mathcal{I} = \{(-\infty, q) \mid q \in \mathbb{Q}\}$

(ii) $\mathcal{J} = \{(-\infty, q] \mid q \in \mathbb{Q}\}$

(iii) $\mathcal{I} = \{(q, \infty) \mid q \in \mathbb{Q}\}$

(iv) $\mathcal{I} = \{[q, \infty) \mid q \in \mathbb{Q}\}$ Homework

<u>Ex3</u>: Show that a) $\sigma(\mathcal{K}') = \sigma(\mathcal{K}) = \mathcal{B}(\mathbb{R}^m)$

<u>Sol</u>: Let us denote by $\mathcal{E}$ and $\mathcal{F}$ the families of all open and closed subsets of $\mathbb{R}^m$, respectively.

$$\mathcal{K}' \subseteq \mathcal{K} \subseteq \mathcal{F} \Rightarrow \sigma(\mathcal{K}') \subseteq \sigma(\mathcal{K}) \subseteq \sigma(\mathcal{F}) = \mathcal{B}(\mathbb{R}^m)$$

b) $\mathcal{A} = \{A \subseteq$

<u>Sol</u>: Let $A_n =$

$\bigcup_{n=1}^{\infty} A_n = \mathbb{N}$

c) $\mathcal{A} = \{A$

<u>Sol</u>: (A1) $\emptyset$

(A2) L

Case 1

$CA \in$

Case 2

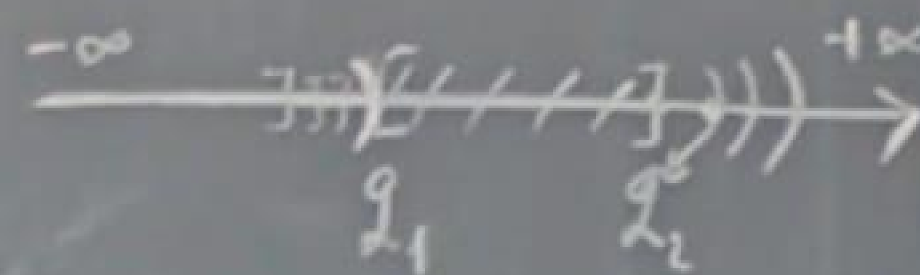
countable

(A3) Let

Case 1

In this case, $\mathcal{H}' \subseteq \sigma(\mathcal{I}) \Rightarrow \sigma(\mathcal{H}') = \mathcal{B}(\mathbb{R}) \subseteq \sigma(\mathcal{I}) \subseteq \mathcal{B}(\mathbb{R})$

$\Rightarrow \sigma(\mathcal{I}) = \mathcal{B}(\mathbb{R})$.



(i) Let $q_1, q_2 \in \mathbb{Q}$, $q_1 \le q_2$,

$[q_1, q_2] = (-\infty, q_2] \setminus (-\infty, q_1)$

$(-\infty, q_2] = \bigcap_{n=1}^{\infty} \underbrace{\left(-\infty, \underbrace{q_2 + \frac{1}{n}}_{\in \mathbb{Q}}\right)}_{\in \mathcal{I}} \in \sigma(\mathcal{I})$; $(-\infty, q_1) \in \mathcal{I} \Rightarrow$

$\Rightarrow [q_1, q_2] \in \sigma(\mathcal{I})$

$\Rightarrow \sigma(\mathcal{I}) = \mathcal{B}(\mathbb{R})$

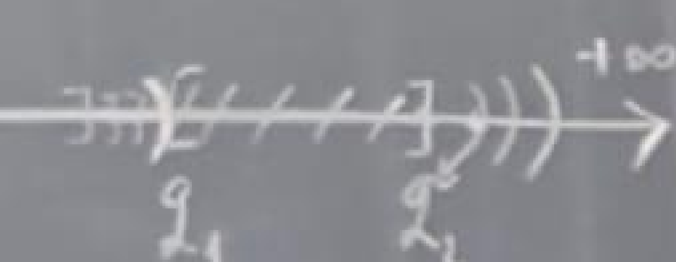
(ii) Let $q_1, q_2 \in \mathbb{Q}$, $q_1 \le q_2$

$[q_1, q_2] = (-\infty, q_2] \setminus (-\infty, q_1)$

$(-\infty, q_2] \in \mathcal{I}$, $(-\infty, q_1) = \bigcup_{n=1}^{\infty} \underbrace{\left(-\infty, \overbrace{q_1 - \frac{1}{n}}^{\in \mathbb{Q}}\right]}_{\in \mathcal{I}} \in \sigma(\mathcal{I})$

$\Rightarrow [q_1, q_2] \in \sigma(\mathcal{I}) \Rightarrow \sigma(\mathcal{I}) = \mathcal{B}(\mathbb{R})$.

$B(\mathbb{R}) \subseteq \sigma(\mathcal{T}) \subseteq B(\mathbb{R})$

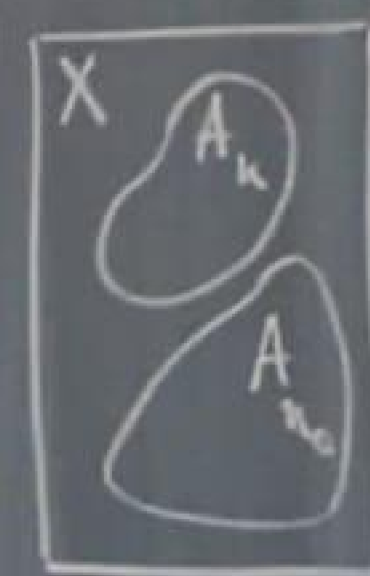


; $(-\infty, q_1) \in \mathcal{T} \Rightarrow$

$\underbrace{\overbrace{[q_1 - \frac{1}{n}]}^{\in \mathbb{Q}}}_{\in \mathcal{T}} \in \sigma(\mathcal{T})$

Ex 4: Let us consider the σ-algebra $\mathcal{A} = \{A \subseteq \mathbb{R} \mid A$ is at most countable or CA is at most countable$\}$ and $\mathcal{E} = \{\{x\} \mid x \in \mathbb{R}\}$. Show that $\sigma(\mathcal{E}) = \mathcal{A}$.

Sol.: ($\forall x \in \mathbb{R}$, $\{x\}$ is finite, so at most countable $\Rightarrow$ $\Rightarrow \{x\} \in \mathcal{A}$) $\Rightarrow \mathcal{E} \subseteq \mathcal{A} \Rightarrow \sigma(\mathcal{E}) \subseteq \mathcal{A}$. (**)



Let $A \in \mathcal{A}$. Case 1: A is at most countable $\Rightarrow$

$\Rightarrow A = \bigcup_{a \in A} \underbrace{\{a\}}_{\in \mathcal{E}} \in \sigma(\mathcal{E})$

Case 2: CA is at most countable $\Rightarrow CA \in \sigma(\mathcal{E}) \Rightarrow$

$\Rightarrow A = C(CA) \in \sigma(\mathcal{E})$

$\Rightarrow \mathcal{A} \subseteq \sigma(\mathcal{E}) \overset{(**)}{\Longrightarrow} \sigma(\mathcal{E}) = \mathcal{A}$

Def 3: Let $\mathcal{A}$ be a σ-algebra on a set X. A measure on $\mathcal{A}$ is a function $\mu : \mathcal{A} \to [0, \infty]$ such that:

(i) $\mu(\emptyset) = 0$

(ii) (σ-additivity) if $(A_n)_{n \in \mathbb{N}}$ is a family of pairwise disjoint sets, then $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$.

Ex 5: Let $\mathcal{A}$ be the σ-algebra f
consider the function $\mu : \mathcal{A} \to [$
Show that μ is a measure on $\mathcal{A}$

Mention that μ is well-def
CA are at most countable (in th
at most countable).

Sol: (i) $\mu(\emptyset) = 0$, (ii) Let $(A_n$
pairwise disjoint sets.

Case 1: $\forall n \in \mathbb{N}$, A_n is at most
countable $\Rightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = 0 = \sum$

Case 2: $\exists n_0 \in \mathbb{N}$ s.t. A_{n_0} is not at

$C\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcap_{n=1}^{\infty} CA_n \subseteq CA_{n_0} \to C($

$\Rightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = 1$

Let $n \in \mathbb{N} \setminus \{n_0\} \Rightarrow A_n \cap A_{n_0} = \emptyset \Rightarrow$
countable $\Rightarrow \mu(A_n) = 0$

$\Rightarrow \sum_{n=1}^{\infty} \mu(A_n) = \mu(A_{n_0}) = 1 = \mu($

-algebra $\mathcal{A}=\{A\subseteq\mathbb{R} \mid A$ is

t most countable$\}$ and

that $\sigma(\mathcal{E})=\mathcal{A}$.

te, so at most countable $\Rightarrow$

$A \Rightarrow \sigma(\mathcal{E})\subseteq\mathcal{A}$. (**)

t most countable $\Rightarrow$

$\sigma(\mathcal{E})$.

ntable $\Rightarrow CA\in\sigma(\mathcal{E}) \Rightarrow$

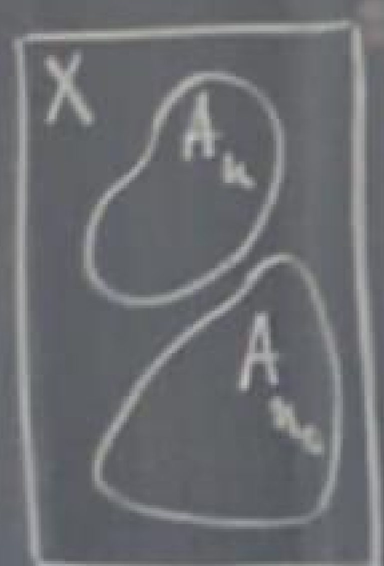
$) = \mathcal{A}$.

on a set X. A measure on $\mathcal{A}$ is

] such that:

$n)_{n\in\mathbb{N}}$ is a family of pairwise

$\bigcup\limits_{n=1}^{\infty} A_n) = \sum\limits_{n=1}^{\infty}\mu(A_n)$.



Ex 5: Let $\mathcal{A}$ be the σ-algebra from Ex 4 and let us consider the function $\mu:\mathcal{A}\to[0,\infty]$, $\mu(A)=\begin{cases}0, & A \text{ is at most countable}\\ 1, & CA \text{ is at most countable}\end{cases}$

Show that μ is a measure on $\mathcal{A}$.

Mention that μ is well-defined, $\nexists A\in\mathcal{A}$ s.t. A and CA are at most countable (in this case $\mathbb{R}=A\cup CA$ would be at most countable).

Sol: (i) $\mu(\emptyset)=0$, (ii) Let $(A_n)_{n\in\mathbb{N}}\subseteq\mathcal{A}$ be a family of pairwise disjoint sets.

$\mu(A_n)=0$

Case 1: $\forall n\in\mathbb{N}$, A_n is at most countable $\Rightarrow \bigcup\limits_{n=1}^{\infty} A_n$ is at most countable $\Rightarrow \mu\left(\bigcup\limits_{n=1}^{\infty} A_n\right)=0=\sum\limits_{n=1}^{\infty}\mu(A_n)$

$\mu(A_{n_0})=1$

Case 2: $\exists n_0\in\mathbb{N}$ s.t. A_{n_0} is not at most countable $\Rightarrow CA_{n_0}$ is at most countable

$C\left(\bigcup\limits_{n=1}^{\infty} A_n\right)=\bigcap\limits_{n=1}^{\infty} CA_n\subseteq CA_{n_0}\to C\left(\bigcup\limits_{n=1}^{\infty} A_n\right)$ is at most countable $\Rightarrow$

$\Rightarrow \mu\left(\bigcup\limits_{n=1}^{\infty} A_n\right)=1$

Let $n\in\mathbb{N}\setminus\{n_0\} \Rightarrow A_n\cap A_{n_0}=\emptyset \Rightarrow A_n\subseteq CA_{n_0} \Rightarrow A_n$ is at most countable $\Rightarrow \mu(A_n)=0$

$\Rightarrow \sum\limits_{n=1}^{\infty}\mu(A_n)=\mu(A_{n_0})=1=\mu\left(\bigcup\limits_{n=1}^{\infty} A_n\right)$.