Seminar 12

22.05.2025

$\boxed{1}$ Compute $I = \int\limits_{\gamma} \frac{x^2+y^2}{z}\, ds$, where $\gamma : [1,2] \to \mathbb{R}^3$ is

↳ 1^{st} Line int.

the parametrized path defined by $\gamma(t) = (t\cdot \sin t, t\cdot\cos t, t)$.

Sol: $I = \int\limits_1^2 \frac{t^2\cdot\sin^2 t + t^2\cos^2 t}{t} \cdot \|\gamma'(t)\|\, dt$

$= \int\limits_1^2 t\cdot \|\gamma'(t)\|\, dt$

$\gamma'(t) = (\sin t + t\cos t, \cos t - t\sin t, 1)$

$\|\gamma'(t)\|^2 = \sin^2 t + 2t\cancel{\sin t\cos t} + t^2\cos^2 t - \underbrace{2t\cancel{\sin t \cos t}}_{} + t^2\sin^2 t + 1$ (with $\cos^2 t$ written above)

$= t^2 + 2$

$I = \int\limits_1^2 t\cdot\sqrt{t^2+2}\, dt$

$\sqrt{t^2+2} = u$

$t^2+2 = u^2$

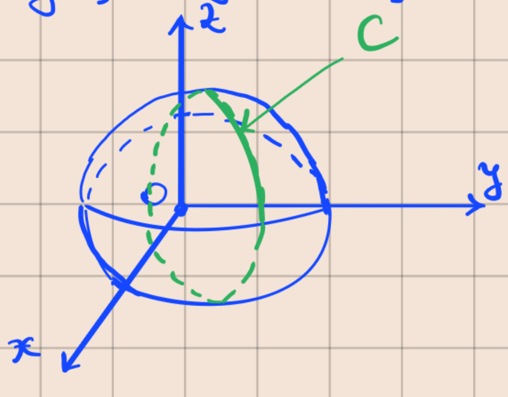
$2u\,du = 2t\,dt$

$u\,du = t\,dt$

$= \int\limits_{\sqrt3}^{\sqrt6} u\cdot u\, du$

$= \int\limits_{\sqrt3}^{\sqrt6} u^2\, du = \frac{u^3}{3}\Big|_{\sqrt3}^{\sqrt6} = \frac{6\cdot\sqrt6 - 3\sqrt3}{3} = 2\sqrt6 - \sqrt3$

$\boxed{2}$ Compute $I = \int\limits_{\gamma} \frac{z^2}{x^2+y^2+1}\, ds$ when γ is the parametrized path whose image is $C = \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2 = 1, x = y, x,y,z \geq 0\}$



We only need the cont. green line, bc. $x,y,z \geq 0$

A parametrization of C can be obtained by using SPHERICAL COORDINATES ($\rho = 1$)

$$\begin{cases} x = \sin\varphi \cdot \cos\theta \\ y = \sin\varphi \sin\theta \\ z = \cos\varphi \end{cases} \qquad x = y \Leftrightarrow \cos\theta = \sin\theta \Rightarrow \boxed{\theta = \frac{\pi}{4}}$$

$\varphi \in [0; \frac{\pi}{2}]$

$\theta \in [0; \frac{\pi}{2}]$

$$\Rightarrow \begin{cases} x = \frac{\sqrt{2}}{2} \cdot \sin\varphi \\ y = \frac{\sqrt{2}}{2} \cdot \sin\varphi \\ z = \cos\varphi \end{cases} \qquad \gamma : [0; \tfrac{\pi}{2}] \to \mathbb{R}^3, \quad \gamma(\varphi) = (\tfrac{\sqrt{2}}{2}\sin\varphi, \tfrac{\sqrt{2}}{2}\sin\varphi, \cos\varphi$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^2\varphi}{\sin^2\varphi + 1} \cdot \|\gamma'(\varphi)\| \, d\varphi$$

$$\gamma'(\varphi) = \left(\frac{\sqrt{2}}{2}\cos\varphi, \frac{\sqrt{2}}{2}\cos\varphi, -\sin\varphi\right)$$

$$\|\gamma'(\varphi)\|^2 = \frac{1}{2}\cos^2\varphi + \frac{1}{2}\cos^2\varphi + \sin^2\varphi = 1$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos^2\varphi}{\sin^2\varphi + 1} \, d\varphi$$

$$\operatorname{tg}\varphi = t \Rightarrow \frac{\sin^2\varphi}{\cos^2\varphi} = t^2 \Rightarrow \frac{\sin^2\varphi}{\sin^2\varphi + \cos\varphi} = \frac{t^2}{1+t^2}$$

$$\Rightarrow \sin^2\varphi = \frac{t^2}{t^2+1}$$

$$\cos^2\varphi = \frac{1}{t^2+1}$$

$$I = \int_0^{\infty} \frac{\frac{1}{t^2+1}}{\frac{t^2}{t^2+1} + 1} \cdot \frac{1}{t^2+1} \, dt = \int_0^{\infty} \frac{1}{\cancel{t^2+1}} \cdot \frac{\cancel{t^2+1}}{2t^2+1} \cdot \frac{1}{t^2+1} \, dt$$

$$= \int_0^{\infty} \frac{1}{(t^2+1)(2t^2+1)} \, dt = \int_0^{\infty} \frac{2(t^2+1) - (2t^2+1)}{(t^2+1)(2t^2+1)} \, dt = \ldots$$

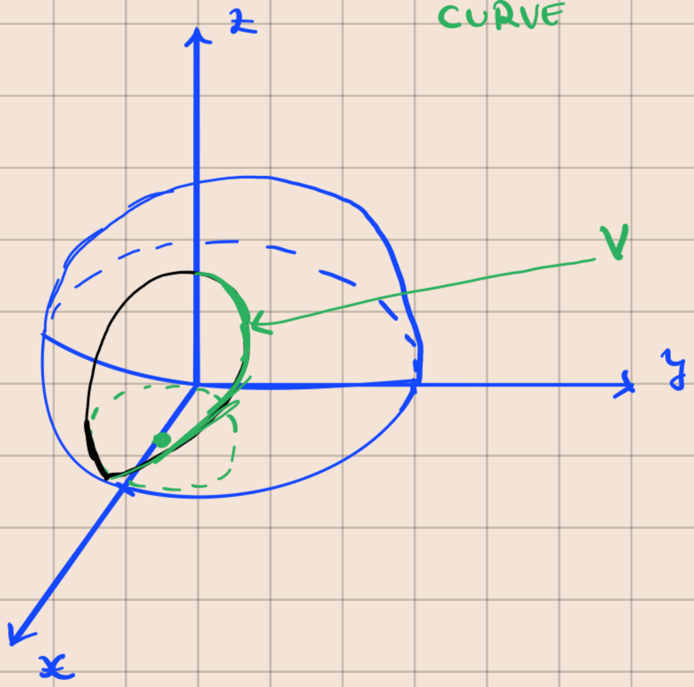
Obs: Pt. descomp în fr. simple, trb notat $t^2 = u$ ca să nu dai în sistem de 4x4

$\boxed{3}$ Compute $I = \int_\gamma y \, ds$ if γ is the parametrized path whose image is $V = \{(x, y, z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1,$ $x^2+y^2 = x, \ y, z \geq 0$

VIVIANI'S CURVE

$$x^2+y^2 = x$$
$$x^2 - x + \frac{1}{4} + y^2 = \frac{1}{4}$$
$$\left(x - \frac{1}{2}\right)^2 + y = \left(\frac{1}{2}\right)^2$$

Sol:



A parametrization of V can be obtained by using SPHERICAL COORDINATES ($\rho = 1$)

$$\begin{cases} x = \sin\varphi \cos\theta \\ y = \sin\varphi \sin\theta \\ z = \cos\varphi \end{cases}$$

$$x^2+y^2 - x \Leftrightarrow \sin^2\varphi = \cancel{\sin\varphi} \cdot \cos\theta$$
$$\Rightarrow \sin\varphi = \cos\theta$$
$$\Downarrow$$
$$\cos\varphi = \sin\theta$$

$$\begin{cases} x = \sin^2\varphi \\ y = \sin\varphi \cdot \cos\varphi \\ z = \cos\varphi \end{cases} \qquad \varphi \in [0; \tfrac{\pi}{2}]$$

$$I = \int_0^{\frac{\pi}{2}} \sin\varphi \cdot \cos\varphi \cdot \|\gamma'(\varphi)\| \, d\varphi$$

$$\gamma'(\varphi) = (2\sin\varphi\cos\varphi, \cos^2\varphi - \sin^2\varphi, -\sin\varphi)$$
$$= (\sin 2\varphi, \cos 2\varphi, -\sin\varphi)$$

$$\|\gamma'(\varphi)\| = \sqrt{1+\sin^2\varphi}$$
$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \sin\varphi \cdot \cos\varphi \cdot \sqrt{1+\sin^2\varphi} \, d\varphi$$

$\sqrt{1+\sin^2\varphi} = t \iff 1+\sin^2\varphi = t^2$

$\cancel{2}\sin\varphi\cos\varphi\, d\varphi = \cancel{2}t\, dt$

$$I = \int_1^{\sqrt{2}} t\cdot t\, dt = \int_1^{\sqrt{2}} t^2\, dt = \frac{t^3}{3}\Big|_1^{\sqrt{2}} = \frac{2\sqrt{2}-1}{3}$$

$\boxed{4}$ $I = \int_\gamma \vec{F}\cdot d\vec{r}$ if $\vec{F}$ is the vector field defined by

$\vec{F}(x,y) = (x-y)\cdot\vec{i} - (x+y)\cdot\vec{j}$ and $\gamma : [0;\frac{\pi}{4}] \to \mathbb{R}^2$ is the parametrized path defined by $\gamma(t) = (e^t\cdot\cos t, e^t\cdot\sin t)$

$$\int_\gamma \vec{F}\cdot d\vec{r} = \int_\gamma (x-y)\,dx - (x+y)\,dy$$

$$= \int_0^{\frac{\pi}{4}} (e^t\cos t - e^t\sin t)(e^t\cos t)'\,dt - \int_0^{\frac{\pi}{4}} (e^t\cos t + e^t\sin t)\cdot(e^t\sin t)'\,dt$$

$$= \int_0^{\frac{\pi}{4}} e^t(\cos t - \sin t)\cdot e^t(\cos t - \sin t)\,dt - \int_0^{\frac{\pi}{4}} e^t(\cos t + \sin t)\cdot e^t(\sin t + \cos t)\,dt$$

$$= \int_0^{\frac{\pi}{4}} e^{2t}\left[(\cos t - \sin t)^2 - (\cos t + \sin t)^2\right]dt$$

$$= \int_0^{\frac{\pi}{4}} e^{2t}\left[(\cancel{\cos t} - \sin t - \cancel{\cos t} - \sin t)(\cos t - \cancel{\sin t} + \cos t + \cancel{\sin t})\right]dt$$

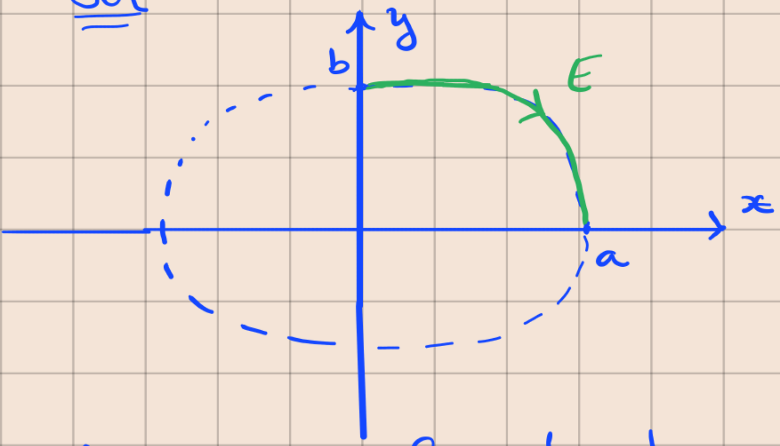
$$= \int_0^{\frac{\pi}{4}} e^{2t}\cdot(-4)\sin t\cos t = -2\int_0^{\frac{\pi}{4}} e^{2t}\cdot\sin 2t\,dt$$

$2t = u \Rightarrow 2dt = du$

$$I = -\int_0^{\frac{\pi}{2}} e^u\cdot\sin u\,du = \ldots$$

5 Compute $I = \int_\gamma \frac{1}{x+a}\,dy$ if γ is the param. path whose image is the arch of ellipse $E = \{(x,y)\in\mathbb{R}^2 \mid \frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ x,y\geq 0\}$ traced clockwise.

Sol:



$$\bar{\gamma}: \begin{cases} x = a\cdot\cos t \\ y = b\cdot\sin t \end{cases} \quad t\in[0;\tfrac{\pi}{2}]$$

$$\int_\gamma \frac{1}{x+a}\,dy = -\int_{\bar\gamma}\frac{1}{x+a}\,dy = -\int_0^{\frac{\pi}{2}}\frac{1}{a\cos t+a}\,b\cdot\cos t\,dt$$

$$= -\frac{b}{a}\int_0^{\frac{\pi}{2}}\frac{\cos t}{\cos t+1}\,dt \qquad tg\frac{t}{2}=u \Rightarrow \cos t=\frac{1-u^2}{1+u^2}$$

$$\left(\sin t = \frac{2u}{1+u^2}\right)$$

$$= -\frac{b}{a}\int_0^1 \frac{\frac{1-u^2}{1+u^2}}{\frac{1-u^2}{1+u^2}+1}\cdot 2\,\frac{du}{1+u^2}$$

$$= -\frac{b}{a}\int_0^1 \frac{1-u^2}{\cancel{1+u^2}}\cdot\frac{\cancel{1+u^2}}{\cancel{2}}\cdot\cancel{2}\cdot\frac{1}{1+u^2}\,du = -\frac{b}{a}\int_0^1\frac{-1-u^2+2}{1+u^2}\,du$$

$$= -\frac{b}{a}\left(-u\Big|_0^1 + 2\cdot\operatorname{arctg} u\Big|_0^1\right) = -\frac{b}{a}\left(-1+\frac{\pi}{2}\right)$$

6 Compute $I = \int_\gamma \vec{F}\cdot d\vec{r}$ if $\vec{F}$ is the vector field defined by $\vec{F}(x,y,z) = y\cdot\vec{i} - x\cdot\vec{j} + (x^2+y^2+z^2)\vec{k}$ and $\gamma:[0;\pi]\to\mathbb{R}^3$ is the param. path defined by $\gamma(t) = (\sin t - t\cos t,\ \cos t + t\sin t,\ 1+t)$

Sol: $I = \int_\gamma \vec{F}\cdot d\vec{r} = \int_\gamma y\,dx - x\,dy + (x^2+y^2+z^2)\,dz$

$$= \int_0^\pi (\cos t + t\sin t)\cdot(\sin t - t\cos t)'\,dt - \int_0^\pi (\sin t - t\cos t)\cdot(\cos t + t\sin t)'\,dt$$

$$+\int_0^\pi \left[(\sin t - t\cos t)^2 + (\cos t + t\sin t)^2 + (t+1)^2\right]\cdot (t+1)'\,dt$$

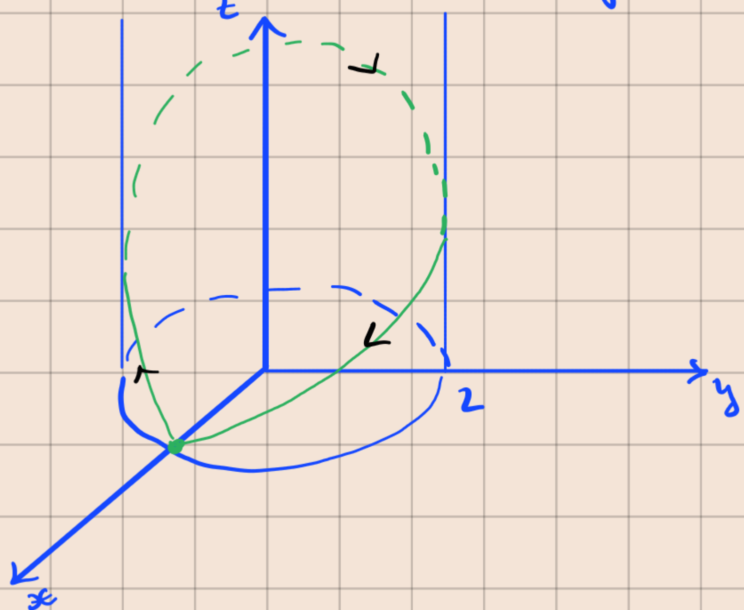
$$= \int_0^\pi (\cos t + t\sin t)\cdot(\cancel{\cos t} - \cancel{\cos t} + t\sin t)\,dt$$
$$- \int_0^\pi (\sin t - t\cos t)\cdot(-\cancel{\sin t} + \cancel{\sin t} + t\cos t)\,dt$$
$$+ \int_0^\pi \Big(\underbrace{\sin^2 t}_{} - \cancel{2t\sin t\cos t} + \underbrace{t^2\cos^2 t}_{} + \underbrace{\cos^2 t}_{1} + \cancel{2t\sin t\cos t} + \underbrace{t^2\sin^2 t}_{t^2\cdot 1} + t^2 + 2t + 1\Big)\,dt$$

$$= \int_0^\pi \cancel{t\sin t\cos t} + t^2\sin^2 t - \cancel{t\sin t\cos t} + t^2\cos^2 t + 2 + 2t^2 + 2t\,dt$$

$$= \int_0^\pi 3t^2 + 2t + 2\,dt = t^3\Big|_0^\pi + t^2\Big|_0^\pi + 2t\Big|_0^\pi$$
$$= \pi^3 + \pi^2 + 2\pi$$

$\boxed{7}$ Let $\vec{F}$ be the vector field in $\mathbb{R}^3$ defined by $\vec{F} = (x^2+y)\,\vec{i} + (y^2+z)\,\vec{j} + (z^2+x)\,\vec{k}$. Compute the work done by $\vec{F}$ in moving a material point along the param. path γ whose im. is $E = \{(x, y, z) \in \mathbb{R}^3 \mid x^2+y^2=4,\ x+z=2\}$ traced counter clockwise for an observer placed at the origin.

Sol: $W = \int_\gamma \vec{F}\cdot d\vec{r} = \int_\gamma (x^2+y)\,dx + (y^2+z)\,dy + (z^2+x)\,dz$



A parametrization of E can be obtained by using CYLINDRICAL COORDINATES. ($\rho = 2$)

$$\bar{\gamma}: \begin{cases} x = 2\cdot\cos\theta \\ y = 2\cdot\sin\theta \\ z = z = 2 - x = 2 - 2\cos\theta \end{cases} , \theta \in [0; 2\pi]$$

$$W = \int_{\bar{\gamma}} (x^2+y)\,dx + (y^2+z)\,dy + (z^2+x)\,dz$$

$$= -\int_0^{2\pi} (4\cos^2\theta + 2\sin\theta)(-2\sin\theta)\,d\theta - \int_0^{2\pi} (4\sin^2\theta + 2 - 2\cos\theta)(2\cos\theta)\,d\theta$$

$$- \int_0^{2\pi} (4 - 8\cos\theta + 4\cos^2\theta + 2\cos\theta)\cdot(2\sin\theta)\,d\theta$$

$$= \int_0^{2\pi} \cancel{8\cos^2\theta\,\sin\theta}\,d\theta + \underline{\int_0^{2\pi} 4\sin^2\theta\,d\theta} - \overbrace{\int_0^{2\pi} 8\sin^2\theta\cos\theta\,d\theta}^{-8\frac{\sin^3\theta}{3}\Big|_0^{2\pi} = 0}$$

$$-4\int_0^{2\pi} \cancel{\cos\theta}\,d\theta + \underline{4\int_0^{2\pi} \cos^2\theta\,d\theta} - 8\int_0^{2\pi} \cancel{\sin\theta\,d\theta} + \underbrace{6\int_0^{2\pi} \cos\theta\sin\theta\,d\theta}_{6\frac{\sin^2\theta}{2}\Big|_0^{2\pi} = 0}$$

$$- \cancel{8\int_0^{2\pi} \cos^2\theta\cdot\sin\theta\,d\theta}$$

$$= 8\pi$$