SEMINARY 1

ODEs

1. Solve $y' = 2x(1+y^2)$, $y = y(x)$

$y' = 2x(1+y^2)$

$\frac{dy}{dx} = 2x(1+y^2) \Leftrightarrow \frac{dy}{1+y^2} = 2x\,dx \Rightarrow \int \frac{1}{1+y^2}dy = \int 2x\,dx$

$\arctan y = x^2 + C \Rightarrow y = \tan(x^2 + C$

2. Solve $y' + y\tan x = \frac{1}{\cos x}$, $y = y(x)$

I. $y' + y\tan x = 0$

$\frac{dy}{dx} = -y\tan x \Leftrightarrow \frac{dy}{y} = -\tan x\,dx \Rightarrow \int \frac{1}{y}dy = -\int \tan x\,dx$

$\ln|y| = \ln|\cos x| + \ln|C| \Rightarrow \ln|y| = \ln(\cos x \cdot C) \Rightarrow \boxed{y_0 = \cos x \cdot C}$

II. $y_p = C(x)\cos x$

$\Rightarrow C'(x)\cos x - C\sin x + C(x)\cos x\,\frac{\sin x}{\cos x} = \frac{1}{\cos x}$

$C'(x)\cos x = \frac{1}{\cos x} \Rightarrow C'(x) = \frac{1}{\cos^2 x} \Rightarrow C(x) = \tan x$

$\Rightarrow \boxed{y_p = \frac{\sin x}{\cos x} \cdot \cos x = \sin x}$

III. $y = y_p + y_0$; $y = C\cos x + \sin x$

3. Solve $y'' - y = 0$, $y = y(x)$

$y'' - y = 0$ we attach the characteristic eq.

$r^2 - 1 = 0 \Rightarrow r^2 = 1 \Rightarrow r = \pm 1 \Rightarrow y(x) = C_1 e^{x} + C_2 e^{-x}$

4. Solve $y'' - 5y' + 6y = 6x^2 - 10x + 2$, $y = y(x)$

I. $y'' - 5y' + 6y = 0$ we attach the characteristic eq.

$r^2 - 5r + 6 = 0$ $\quad \Delta = 25 - 4 \cdot 1 \cdot 6 = 25 - 24 = 1 \Rightarrow r_1 = \frac{5+1}{2} = 3$, $r_2 = \frac{5-1}{2} = 4$

$\Rightarrow y_0(x) = C_1 e^{3x} + C_2 e^{2x}$

II. We seek a particular solution in this form

$y_p(x) = Ax^2 + Bx + C$

$y_p'(x) = 2Ax + B$

$y_p''(x) = 2A$

III. We return to the original eq. an substitute y with y_p

$y_p'' - 5y_p' + 6y_p = 6x^2 - 10x + 2$

$\Rightarrow 2A - 10Ax - 5B + 6Ax^2 + 6Bx + 6C = 6x^2 - 10x + 2$

$6x^2A - x(10A - 6B) + 2A - 5B + 6C = 6x^2 - 10x + 2 \ldots$

Fundamental Differential Operators

Let $\Omega \subseteq \mathbb{R}^m$ open (smooth enough)

- **Gradient**: $\nabla : C^1(\Omega) \to C(\Omega, \mathbb{R})$

$$\nabla u = \left(\frac{\partial u}{\partial x_1} \cdots \frac{\partial u}{\partial x_m}\right)$$

- **Divergence**: $\text{div} : C^1(\Omega, \mathbb{R}^m) \to C(\Omega)$

$$\text{div}\, \mathbb{V} := \sum_{j=1}^{m} \frac{\partial v_j}{\partial x_j}, \quad \mathbb{V} = (v_1 \ldots v_m)$$

- **Laplacian**: $\Delta : C^2(\Omega) \to C(\Omega)$

$$\Delta u := \sum_{j=1}^{m} \frac{\partial^2 u}{\partial x_j^2}$$

- **Directional derivatives**: $\frac{\partial}{\partial v} : C^1(\Omega) \to C(\Omega)$

$$\frac{\partial u}{\partial v} = (\nabla u(x), v), \text{ where } v \in \mathbb{R}^m, |v| = 1$$

5. Compute ∇u for $u : \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}$ $\quad u(x,y,z) = x + \ln(y^2+z^2) + x^2yz^2$

$$\nabla u(x,y,z) = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$$

$$\nabla u(x,y,z) = \left(1 + 2xyz^2, \frac{2y}{y^2+z^2} + x^2z^2, \frac{2z^2}{y^2+z^2} + 2x^2yz\right.$$

6. Compute $\text{div}\, \mathbb{V} := \Omega' \subseteq \mathbb{R}^3 \to \Omega \subseteq \mathbb{R}^3$ $\quad \mathbb{V}(x,y,z) = (x+y+z, x^2+z^2, x+y)$

$$\text{div}\, \mathbb{V} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = 1 + 0 + 0 = 1$$

7. Compute Δu for $u : \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}$ $\quad u(x,y,z) = x^3 + y^3 + z^3 + 3xy + 3yz$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$\left.\begin{array}{l} \frac{\partial u}{\partial x} = 3x^2 + 3y; \quad \frac{\partial^2 u}{\partial x^2} = 6x \\ \frac{\partial u}{\partial y} = 3y^2 + 3x + 3z; \quad \frac{\partial^2 u}{\partial y^2} = 6y \\ \frac{\partial u}{\partial z} = 3z^2 + 3y; \quad \frac{\partial^2 u}{\partial z^2} = 6z \end{array}\right\} \Rightarrow \Delta u = 6x + 6y + 6z$$

8. Compute $\frac{\partial u}{\partial v}$ for $v : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$ $\quad u(x,y) = e^x + \ln y + xy$ where $v = \frac{1}{2}(\sqrt{2}, \sqrt{2})$

$$\nabla u(x,y) = \left(e^x + y, \frac{1}{y} + x\right)$$

$$\frac{\partial u}{\partial v} = (\nabla u(x,y), v) = \left(e^x + y, \frac{1}{y} + x\right)\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}(e^x+y) + \sqrt{2}\left(\frac{1}{y}+x\right)}{2} = \frac{\sqrt{2}}{2}\left(e^x + y + \frac{1}{y} + x\right)$$

We will focus on

① Poisson's Eq: $\Delta u = f$, $x \in \Omega$

② Heat Eq: $\frac{\partial u}{\partial t} - a^2 \Delta u = f$, $(x,t) \in \Omega \times (0,\infty)$

③ Wave Eq: $\frac{\partial^2 u}{\partial t^2} - a^2 \Delta u = f$, $(x,t) \in \Omega \times (0,\infty)$

Solutions to some particular eqs:

• Poisson's eq $m = 1$, $\Omega(a,b)$, $-\infty \le a < b < \infty$

$f \in C(\Omega) : u'' = f$, $x \in (a,b)$

$\Rightarrow u(x) = C_1 x + C_2 + \int_{x_0}^{x} \left(\int_{t_0}^{t} f(s)\,ds \right) dt$

$C_1, C_2 \in \mathbb{R}$, $x_0 \in (a,b)$ - fixed

• Laplace's eq in $\Omega \subseteq \mathbb{R}^2$ open:

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, $(x,y) \in \Omega$

solutions: $u(x,y) = C_1 x + C_2 y + C_3$

$u(x,y) = C_1 (x^2 - y^2)$

we seek a solution in the form: $u(x,y) = A(x) B(x)$

substitute and we get : $A''(x) B(y) + A(x) B''(y) = 0$

$\frac{A''(x)}{A(x)} = -\frac{B''(y)}{B(y)} =: \lambda$ (const) $\Rightarrow \begin{cases} A''(x) - \lambda A(x) = 0 \\ B''(y) - \lambda B(y) = 0 \end{cases}$

we solve these eq. for $\lambda = c^2$ and then we solve these eq. for $\lambda = -c^2$, where $c \in \mathbb{R}$

HW: Case 1 Solve $A''(x) - c^2 A(x) = 0$

$B''(y) + c^2 B(y) = 0$

Case 2 Solve $A''(x) + c^2 A(x) = 0$

$B''(y) - c^2 B(y) = 0$

conclusion: $e^{cx} \sin cy$, $e^{cx} \cos cy$, $e^{cy} \sin cx$, $e^{cy} \cos cx$ are all harmonic

! There are more harmonic functions

HW: Solve the Heat eq. $\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = 0$ on $(\alpha, \beta) \times (0,\infty)$ using separation of variables

the sol. is of form: $u(x,t) = A(x) B(t)$

• Homogenous Wave eq. $(m = 1)$

$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0$, $x \in (\alpha, \beta)$, $t \in (0,\infty)$

we rewrite $u_{tt} - a^2 u_{xx} = 0$

add them for x

use the new (change) variables : $\begin{cases} y = x + at & (y = y(x,t)) \\ s = x - at & (s = s(x,t)) \end{cases} \Rightarrow \begin{matrix} x = x(y,s) \\ t = t(y,s) \end{matrix}$

substract them for t

$u(x,t) = u(x(y,s), t(y,s)) = u(y,s) = u(y(x,t), s(x,t))$

$u_x = u_y \cdot y_x + u_s \cdot s_x$; $u_t = u_y \cdot y_t + u_s \cdot s_t$

$$u_{xx} = \frac{\partial}{\partial x}(u_x) = \frac{\partial}{\partial x}(u_y \cdot y_x + u_s \cdot s_x) = \frac{\partial}{\partial x}(u_y) \cdot y_x + u_y \cdot y_{xx} + \frac{\partial}{\partial x}(u_s) \cdot s_x + u_s \cdot s_{xx}$$

$$= (u_{yy} \cdot y_x + u_{ys} \cdot s_x)\, y_x + u_y \cdot y_{xx} + (u_{ss} \cdot s_x + u_{sy} \cdot y_x)\, s_x + u_s \cdot s_{xx}$$

$$u_{xx} = u_{yy} \cdot y_x^2 + u_{ys}\, s_x y_x + u_y \cdot y_{xx} + u_{ss}\, s_x^2 + u_{sy}\, y_x s_x + u_s \cdot s_{xx}$$

$$u_{tt} = u_{yy} \cdot y_t^2 + u_{ys}\, s_t y_t + u_y \cdot y_{tt} + u_{ss}\, s_t^2 + u_{sy}\, y_t s_t + u_s \cdot s_{tt}$$

$$u_{xt} = u_{yy}\, y_x y_t + u_{ys}\, y_x s_t + u_y \cdot y_{xt} + u_{ss}\, s_x s_t + u_{sy}\, y_t s_x + u_s\, s_{xt}$$

solution of our problem:
$$\begin{cases} u_t = a u_y - a u_s \\ u_x = u_y + u_s \\ u_{tt} = a^2(u_{yy} - 2u_{ys} + u_{ss}) \\ u_{xx} = u_{yy} + 2u_{ys} + u_{ss} \end{cases}$$

$\Rightarrow$ we substitute into $u_{tt} - a^2 u_{xx} = 0$ and we get:

$$a^2 u_{yy} - 2a^2 u_{ys} + a^2 u_{ss} - a^2 u_{yy} + 2a^2 u_{ys} - a^2 u_{ss} = 0$$

this is not ok or idk it should be $\Rightarrow u_{ys} = 0$

$$\Rightarrow a^2 u_{ys} = 0 \Rightarrow u_{ys} = 0 \Leftrightarrow \frac{\partial^2 u}{\partial y \partial s} = 0 \Leftrightarrow \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial s}\right) = 0 \Leftrightarrow \frac{\partial u}{\partial s} = \psi(s) \Leftrightarrow u = \underbrace{\int \psi(s)\, ds}_{:= \Psi(s)} + \varphi(y)$$

$$\Leftrightarrow u = \Psi(s) + \varphi(y)$$

$$u(x,t) = \Psi(x - at) + \varphi(x + at)$$

SEMINARY 2

Consider a general 2^{nd} order PDE: $a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} + \Phi(x,y,u,u_x,u_y) = 0$ (*) where $|a|+|b|+|c| \neq 0$

Attach the following characteristic poly: $a\left(\frac{dy}{dx}\right)^2 - b\left(\frac{dy}{dx}\right) + c = 0$

For the polynomial: $\Delta > 0 \Rightarrow$ HIPERBOLIC

$\Delta = 0 \Rightarrow$ PARABOLIC

$\Delta < 0 \Rightarrow$ ELIPTIC

The solutions indicate the change of variables: $\begin{cases} \frac{dy}{dx} = \lambda_1(x,y) \\ \frac{dy}{dx} = \lambda_2(x,y) \end{cases} \Rightarrow \begin{cases} \overbrace{\varphi(x,y)}^{s:=} = c_1 \\ \underbrace{\psi(x,y)}_{t:=} = c_2 \end{cases}$

solutions of the charac. eq.

Using the change of variables one can obtain the following reduced form (canonical forms):

$u_{st} = \Phi_1(s,t,u,u_s,u_t)$ - hyperbolic

$u_{tt} = \Phi_1(s,t,u,u_s,u_t)$ - parabolic

$u_{ss} + u_{tt} = \Phi_1(s,t,u,u_s,u_t)$ - eliptic

1. Obtain the canonical form of the eq.: $u_{xx} - 2u_{xy} - 3u_{yy} + u_y = 0$

from * $\Rightarrow a = 1,\ b = -2,\ c = -3$

charac. eq. $\left(\frac{dy}{dx}\right)^2 + 2\left(\frac{dy}{dx}\right) - 3 = 0 \quad \rightarrow \Delta = 4 + 12 = 16 > 0 \Rightarrow$ hyperbolic

$\Rightarrow \frac{dy}{dx} = \frac{-2 \pm 4}{2} \Rightarrow \frac{dy}{dx} = \begin{cases} 1 \\ -3 \end{cases}$

$\begin{cases} \frac{dy}{dx} = 1 \Rightarrow dy = dx \Rightarrow y = x + c_1,\ c \in \mathbb{R} \Rightarrow y - x = c_1,\ c_1 \in \mathbb{R} \\ \frac{dy}{dx} = -3 \Rightarrow dy = -3dx \Rightarrow y = -3x + c_2,\ c_2 \in \mathbb{R} \Rightarrow y + 3x = c_2,\ c_2 \in \mathbb{R} \end{cases}$

The change of variables is $\begin{cases} s = -x + y \\ t = 3x + y \end{cases}$

We use the formulas: $u_x = u_s \cdot s_x + u_t \cdot t_x$

$u_y = u_t \cdot t_y + u_s \cdot s_y$

$u_{xx} = u_{ss} \cdot s_x^2 + 2u_{st} \cdot t_x s_x + u_s \cdot s_{xx} + u_{tt} t_x^2 + u_t t_{xx}$

$u_{yy} = u_{ss} \cdot s_y^2 + 2u_{st} t_y s_y + u_s \cdot s_{yy} + u_{tt} t_y^2 + u_t t_{yy}$

$u_{xy} = u_{ss} \cdot s_x s_y + u_{st} \cdot s_x t_y + u_s s_{xy} + u_{tt} t_x t_y + u_{ts} s_y t_x + u_t \cdot t_{xy}$

$\begin{cases} s_x = -1 \\ s_y = 1 \\ t_x = 3 \\ t_y = 1 \end{cases} \Rightarrow \begin{matrix} s_{xy} = 0 & t_{xy} = 0 \\ s_{xx} = 0 & t_{xx} = 0 \\ s_{yy} = 0 & t_{yy} = 0 \end{matrix} \Rightarrow$

$u_{xx} = u_{ss} \cdot 1 + 2u_{st}(-3) + u_s \cdot 0 + u_{tt} 9 + u_t \cdot 0$

$u_{yy} = u_{ss} \cdot 1 + 2u_{st} + 0 + u_{tt} + 0$

$u_{xy} = -u_{ss} - u_{st} + 0 - u_{tt} + u_{ts} + 0$

$u_y = u_s + u_t$

... $u_{xx} - 2u_{xy} - 3u_{yy} + u_y = 0$

$(\cancel{u_{ss}} - 6u_{st} + \cancel{9u_{tt}}) - 2(\cancel{u_{ss}} - u_{st} + \cancel{3u_{tt}} + 3u_{st}) - 3(\cancel{u_{ss}} - 2u_{st} + \cancel{u_{tt}}) + u_s + u_t = 0 \Rightarrow -16u_{st} + u_s + u_t = 0$

$\Rightarrow u_{st} = \frac{1}{16}(u_s + u_t)$ hiperbolic $\square$

SEMINARY 3

1. Derive the canonical form of the following eq.:

a) $u_{xx} - 4u_{xy} + 4u_{yy} - u_x = 0$

from * (Sem 2) $\Rightarrow a = 1,\ b = -4,\ c = 4$

charac. eq: $\left(\frac{dy}{dx}\right)^2 + 4\frac{dy}{dx} + 4 = 0 \Rightarrow \Delta = 16 - 16 = 0 \Rightarrow$ PARABOLIC

$$\Rightarrow \frac{dy}{dx} = \frac{-4}{2} = -2$$

$\frac{dy}{dx} = -2 \Rightarrow dy = -2dx \Rightarrow y = -2x + c_1,\ c_1 \in \mathbb{R} \Rightarrow y + 2x = c_1$

The change of variables is $\begin{cases} s = y + 2x \\ t = y \end{cases}$ (you can also choose $t = x$, bc. we have parabolic eq.)

We use the formulas: $u(x,y) = u(s(x,y), t(x,y))$

$$\begin{cases} u_x = u_s \cdot s_x + u_t \cdot t_x = 2u_s + 0 \\ u_y = u_t \cdot t_y + u_s\, s_y = u_t + u_s \\ u_{xx} = u_{ss}\, s_x^2 + 2u_{st} \cdot t_x s_x + u_s \cdot s_{xx} + u_{tt} \cdot t_x^2 + u_t \cdot t_{xx} = 4u_{ss} + 0 + 0 + 0 + 0 \\ u_{yy} = u_{ss}\, s_y^2 + 2u_{st} \cdot t_y s_y + u_s \cdot s_{yy} + u_{tt} \cdot t_y^2 + u_t \cdot t_{yy} = u_{ss} + 2u_{st} + 0 + u_{tt} + 0 \\ u_{xy} = u_{ss}\, s_x s_y + u_{st} \cdot s_x t_y + u_s\, s_{xy} + u_{tt}\, t_x t_y + u_{ts}\, s_y t_x + u_t\, t_{xy} = 2u_{ss} + 2u_{st} + 0 + 0 + 0 + 0 \end{cases}$$

We return to our eq:

$$\Rightarrow 4u_{ss} - 4(2u_{ss} + 2u_{st}) + 4(u_{ss} + 2u_{st} + u_{tt}) - 2u_s = 0$$

$$\Rightarrow \cancel{4u_{ss}} - \cancel{8u_{ss}} - \cancel{8u_{st}} + \cancel{4u_{ss}} + \cancel{8u_{st}} + 4u_{tt} - 2u_s = 0 \Rightarrow 2u_{tt} - u_s = 0 \Rightarrow u_{tt} = \frac{u_s}{2}$$

b) $9u_{xx} - 6u_{xy} + 5u_{yy} - u_x = 0$

from * (Sem 2) $\Rightarrow a = 9,\ b = -6,\ c = 5$

charac eq: $9\left(\frac{dy}{dx}\right)^2 + 6\frac{dy}{dx} + 5 = 0 \Rightarrow \Delta = 36 - 180 = -144 \Rightarrow$ ECLIPTIC

$$\frac{dy}{dx} = \frac{-6 \pm i12}{18} = \begin{cases} \frac{-1+2i}{3} \\ \frac{-1-2i}{3} \end{cases}$$

$$\begin{cases} \frac{dy}{dx} = -\frac{1}{3} + \frac{2}{3}i \Rightarrow y = -\frac{1}{3}x + \frac{2}{3}xi + c_1 \Rightarrow c_1 = -\frac{1}{3}x - y + \frac{2}{3}xi \Rightarrow c_1 = -x - 3y + 2xi \Rightarrow c_1 = x + 3y - 2xi\ (= z_1(x,y)) \\ \frac{dy}{dx} = \frac{-1}{3} - \frac{2}{3}i \Rightarrow y = -\frac{1}{3}x - \frac{2}{3}xi + c_2 \Rightarrow c_2 = -\frac{1}{3}x - y - \frac{2}{3}xi \Rightarrow c_2 = -x - 3y - 2xi \Rightarrow c_2 = x + 3y + 2xi\ (= z_2(x,y)) \end{cases}$$

The change of variables is $\begin{cases} s = x + 3y\ \left(= \frac{z_1 + z_2}{2}\right) \to \text{real part} \\ t = -2x\ \left(= \frac{z_1 - z_2}{2i}\right) \to \text{imaginary part} \end{cases}$

We use the formulas:

$$\begin{cases} u_x = u_s \cdot s_x + u_t \cdot t_x = u_s - 2u_t \\ u_y = u_s \cdot s_y + u_t \cdot t_y = 3u_s + 0 \\ u_{xx} = u_{ss}\, s_x^2 + 2u_{st}\, s_x t_x + u_s\, s_{xx} + u_{tt}\, t_x^2 + u_t \cdot t_{xx} = u_{ss} - 4u_{st} + 0 + 4u_{tt} + 0 \\ u_{yy} = u_{ss}\, s_y^2 + 2u_{st} \cdot t_y s_y + u_s \cdot s_{yy} + u_{tt} \cdot t_y^2 + u_t \cdot t_{yy} = 9u_{ss} + 0 + 0 + 0 + 0 \\ u_{xy} = u_{ss}\, s_x s_y + u_{st} \cdot s_x t_y + u_s\, s_{xy} + u_{tt}\, t_x t_y + u_{ts}\, s_y t_x + u_t\, t_{xy} = 3u_{ss} + 0 + 3u_s + 0 + 0 - 6u_{st} + 0 \end{cases}$$

We return to our eq:

$$9(u_{ss} - 4u_{st} + 4u_{tt}) - 6(3u_{ss} + 3u_s - 6u_{st}) + 5 \cdot 9u_{ss} - u_s + 2u_t = 0 \Rightarrow \ldots \Rightarrow 36u_{ss} + 36u_{tt} + \ldots = 0$$

$$u_{ss} + u_{tt} = \frac{1}{36}(\ldots)$$

do comp at home

2. Find the solutions of the following PDEs:

- $u_{xy}=0 \Rightarrow \frac{\partial}{\partial x}(u_y)=0 \Rightarrow u_y=c(y) \Rightarrow u=\int c(y)\,dy+f(x) \Rightarrow u=f(x)+g(y)$, $\forall f,g\in C^2$

- $u_{xx}-a^2u_{yy}=0$

case I: $a=0 \Rightarrow u_{xx}=0 \Rightarrow \frac{\partial}{\partial x}(u_x)=0 \Rightarrow u_x=c(y) \Rightarrow u=\int c(y)\,dx+c_1(y) \Rightarrow u=c(y)x+c_1(y)$, $\forall c,c_1\in C^2$

case II: $a\neq 0 \Rightarrow u_{xx}-a^2u_{yy}=0$

$$\lambda=\frac{dy}{dx} \Rightarrow \lambda^2-a^2=0 \Leftrightarrow \lambda=\pm a \Rightarrow \frac{dy}{dx}=\pm a$$

$$\begin{cases}\frac{dy}{dx}=a \Rightarrow dy=a\,dx \Rightarrow y=ax+c_1 \Rightarrow c_1=y-ax\\ \frac{dy}{dx}=-a \Rightarrow dy=-a\,dx \Rightarrow y=-ax+c_2 \Rightarrow c_2=y+ax\end{cases}$$

the change of variables: $\begin{cases}s=y-ax\\ t=y+ax\end{cases}$

(we use the formulas: $\begin{cases}u_{xx}=u_{ss}s_x^2+2u_{st}s_xt_x+u_ss_{xx}+u_{tt}t_x^2+u_t\cdot t_{xx} &=a^2u_{ss}-2a^2u_{st}+a^2u_{tt}\\ u_{yy}=u_{ss}s_y^2+2u_{st}\cdot t_ys_y+u_s\cdot s_{yy}+u_{tt}\cdot t_y^2+u_t\cdot t_{yy} &=u_{ss}-2u_{st}+u_{tt}\end{cases}$)

$\Rightarrow 4a^2u_{st}=0 \Rightarrow u_{st}=0 \Rightarrow u=f(s)+g(t) \Rightarrow u=f(y-ax)+g(y+ax)$

- $u_{xy}+au_x=0$

$\frac{\partial}{\partial x}(u_y+au)=0$

$u_y+au=\varphi(y) \overset{\text{ODE}}{\underset{\text{in } y}{\Rightarrow}} u'+au=\varphi(y)$

$u_h=e^{-ay}c(x)$

$u_p=e^{-ay}\int_0^y e^{-at}\varphi(t)\,dt$

$\Rightarrow u(x,y)=e^{-ay}c(x)+\int_0^y e^{(t-y)}\varphi(t)\,dt$

Homework: Canonical form for:
- $4u_{xx}+5u_{xy}+u_{yy}+u_x+u_y=2$
- $u_{xx}+u_{xy}+u_{yy}+u_x=0$
- $4u_{xx}+4u_{xy}+u_{yy}-2u_y=0$

Solve the eq: $u_{xy}+au_x+bu_y+abu=0$

SEMINARY 4

Homework 1: ① Find the canonical forms of: a) $y^2 u_{xx} + 2y u_{xy} + u_{yy} = 0$

b) $x^2 u_{xx} + y^2 u_{yy} = 0$

Hints: a) Case 1: $y = 0$

Case 2: $y \neq 0$ $\begin{cases} s = -x + \frac{y^2}{2} \\ t = y \end{cases} \Rightarrow u_{tt} = -u_s$

b) Case 1: $x = 0$

Case 2: $y = 0$

Case 3: $x \neq 0, y \neq 0 \Rightarrow \begin{cases} s = -\ln x \\ t = -\ln y \end{cases} \Rightarrow u_{ss} + u_{tt} = -u_s - u_t$

LAPLACE'S EQUATION IN POLAR COORDINATES

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$; $r \in [0, \infty)$, $\theta \in [0, 2\pi)$ & $\begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \arctan \frac{y}{x} \end{cases}$

Homework 2: ② Find the Laplacian in polar coord.

Hint: $u_{xx} + u_{yy} = 0$

find: $r_x = (\sqrt{x^2+y^2})'_x = \frac{x}{\sqrt{x^2+y^2}} = \frac{r\cos\theta}{r} = \cos\theta$ ... and so on for the rest and every result is in r & θ

$\Rightarrow \boxed{u_{rr} + \frac{1}{r^2} u_{\theta\theta} + \frac{1}{r} u_r = 0}$ Laplace's eq. in polar coord.

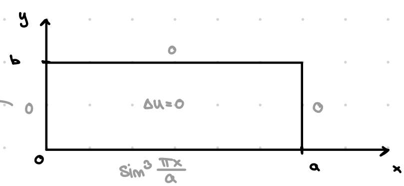
In addition: $\boxed{\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + u_{zz}}$ LAPLACE'S OP. IN CILINDRICAL COORD.

$\boxed{\Delta u = u_{rr} + \frac{2}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + \frac{\operatorname{ctan}\theta}{r^2} u_\theta + \frac{1}{r^2 \sin^2\theta} u_{\varphi\varphi}}$ LAPLACE'S OP. IN SPHERICAL COORD.

LAPLACE'S EQ. ON A RECTAGULAR DOMAIN

1. Solve $\begin{cases} \Delta u(x,y) = 0 \ ; \ 0 < x < a \ , \ 0 < y < b \leftarrow \text{rectangle} \\ u(x,0) = \sin^3 \frac{\pi x}{a} \ , \ u(x,b) = 0 \ ; \ 0 \le x \le a \\ u(0,y) = u(a,y) = 0 \ ; \ 0 \le y \le b \end{cases}$ BC

based on these



if these are considered $\Rightarrow$ null

STEP 1: Seek sol. in the form: $u(x,y) = A(x) \cdot B(y)$

We aply boundary conditions (BC) and we have $B(b) = 0$, $A(0) = 0$, $A(a) = 0$ (we don't consider $A(x) = 0$ & $B(y) = 0$)

$\Delta u = 0 \Leftrightarrow u_{xx} + u_{yy} = 0 \Leftrightarrow A''(x)B(y) + A(x)B''(y) = 0 \Leftrightarrow \frac{A''(x)}{A(x)} = -\frac{B''(y)}{B(y)} = \lambda$ (const.)

$\begin{cases} A''(x) - \lambda A(x) = 0 \\ B''(y) + \lambda B(y) = 0 \end{cases}$

STEP 2: Solve $A''(x) - \lambda A(x) = 0$

CASE 1: $\lambda > 0 \Rightarrow$ charac. eq.: $r^2 - \lambda = 0 \Rightarrow r^2 = \lambda \Rightarrow r = \pm\sqrt{\lambda}$

$A(x) = c_1 e^{\sqrt{\lambda} x} + c_2 e^{-\sqrt{\lambda} x}$; $c_1, c_2 \in \mathbb{R}$

$$\begin{cases} A(0)=0 \\ A(a)=0 \end{cases} \Rightarrow \begin{cases} c_1+c_2=0 \\ c_1 e^{\sqrt{\lambda} a} + c_2 e^{-\sqrt{\lambda} a} = 0 \end{cases} \Rightarrow \begin{cases} c_2 = -c_1 \\ c_1 e^{a\sqrt{\lambda}} - c_1 e^{-a\sqrt{\lambda}} = 0 \Rightarrow c_1 (e^{a\sqrt{\lambda}} - e^{-a\sqrt{\lambda}}) = 0 \end{cases}$$

$c_1 = 0 \Rightarrow c_2 = 0 \Rightarrow A(x) = 0$ (we don't consider / we omit)

$e^{\sqrt{\lambda} a} = e^{-\sqrt{\lambda} a} \Rightarrow \lambda = 0$ (impossible)

CASE 2: $\lambda = 0 \Rightarrow A''(x) = 0 \Rightarrow A(x) = cx + d$; $c, d \in \mathbb{R}$

$$\left.\begin{cases} A(0)=0 \Rightarrow d=0 \\ A(a)=0 \Rightarrow c=0 \end{cases}\right\} \Rightarrow A(x)=0 \text{ (we omit)}$$

CASE 3: $\lambda < 0 \Rightarrow$ charac. eq. : $r^2 - \lambda = 0 \Rightarrow r^2 = \lambda \Rightarrow r^2 = (-1)(-\lambda) \Rightarrow r = \pm i\sqrt{-\lambda}$

$A(x) = c_1 \cos(x\sqrt{-\lambda}) + c_2 \sin(x\sqrt{-\lambda})$, $c_1, c_2 \in \mathbb{R}$

$$\begin{cases} A(0)=0 \Rightarrow c_1=0 \\ A(a)=0 \Rightarrow \underbrace{c_1\cos(a\sqrt{-\lambda})}_{=0} + c_2 \sin(a\sqrt{-\lambda}) = 0 \end{cases} \Rightarrow \begin{cases} c_2 = 0 \Rightarrow A(x)=0 \text{ (we omit)} \\ \sin(a\sqrt{-\lambda}) = 0 \Rightarrow a\sqrt{-\lambda} = k\pi,\ k \in \mathbb{Z} \end{cases}$$

$$\Downarrow \quad \boxed{\lambda = \frac{k^2\pi^2}{a^2}(-1)} \Rightarrow$$

$$\Rightarrow \boxed{A(x) = c_2 \sin\left(\frac{k\pi}{a}x\right)}$$

STEP 3: Solve eq: $B''(y) + \lambda B(y) = 0$

At STEP 2 we saw $\lambda < 0$ then: charac. eq : $r^2 + \lambda = 0 \Leftrightarrow r = \pm\sqrt{-\lambda}$

$B(y) = c_1 e^{y\sqrt{-\lambda}} + c_2 e^{-y\sqrt{-\lambda}}$

$B(b) = 0 \Leftrightarrow c_1 e^{b\sqrt{-\lambda}} + c_2 e^{-b\sqrt{-\lambda}} = 0 \Leftrightarrow -c_1 e^{2b\sqrt{-\lambda}} = c_2$

$$B(y) = c_1 (e^{y\sqrt{-\lambda}} - e^{2b\sqrt{-\lambda}} e^{-y\sqrt{-\lambda}}) = \underbrace{2c_1 e^{b\sqrt{-\lambda}}}_{=c_3} \frac{e^{-b\sqrt{-\lambda}} e^{y\sqrt{-\lambda}} - e^{b\sqrt{-\lambda}} e^{-y\sqrt{-\lambda}}}{2} = c_3 \frac{e^{-\sqrt{-\lambda}(b-y)} - e^{\sqrt{-\lambda}(b-y)}}{2}$$

see complex Analysis for this

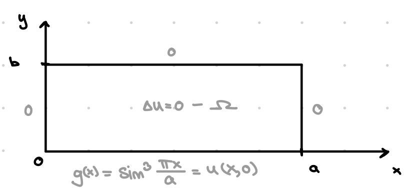
$$\boxed{B(y) = c_3 \operatorname{sh}(\sqrt{-\lambda}(b-y))}$$

STEP 4: $\boxed{u_k(x,y) = A_k(x) B_k(y) = a_k \sin\left(\frac{k\pi}{a}x\right) \operatorname{sh}\left(\frac{k\pi}{a}(b-y)\right)}$

STEP 5: The solution of this Dirichlet problem is $\boxed{u(x,y) = \sum_{k=1}^{\infty} u_k(x,y)}$

if we have $\neq 0$ on the bottom of the rectangle these 2 are always true (don't go and compute everything)

For our problem :



! $\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin(3x)$

We have the boundary conditions: $y=0 : u(x,0) = \sum_{k=1}^{\infty} a_k \sin\left(\frac{k\pi}{a}x\right) \operatorname{sh}\left(\frac{k\pi}{a}b\right) = \sin^3\frac{\pi x}{a}$

$\forall k \in \mathbb{N} \setminus \{1,3\}$ we have $a_k = 0 \Rightarrow k=1 : a_1 \sin\left(\frac{\pi}{a}x\right)\operatorname{sh}\left(\frac{\pi}{a}b\right) = \frac{3}{4}\sin\left(\frac{\pi}{a}x\right) \Rightarrow a_1 = \frac{3}{4\operatorname{sh}\left(\frac{\pi}{a}b\right)}$

$k=3 : a_3 \sin\left(3\frac{\pi}{a}x\right)\operatorname{sh}\left(3\frac{\pi}{a}b\right) = -\frac{1}{4}\sin 3\left(\frac{\pi}{a}x\right) \Rightarrow a_3 = -\frac{1}{4\operatorname{sh}\left(\frac{3\pi}{a}b\right)}$

Finally, $u(x,y) = \frac{3}{4\operatorname{sh}\left(\frac{\pi}{a}b\right)} \sin\left(\frac{\pi}{a}x\right)\operatorname{sh}\left(\frac{\pi}{a}(b-y)\right) - \frac{1}{4\operatorname{sh}\left(\frac{3\pi}{a}b\right)} \sin\left(3\frac{\pi}{a}x\right)\operatorname{sh}\left(3\frac{\pi}{a}(b-y)\right)$

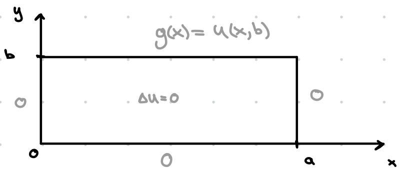
We also have the situations:

II $\Delta u = 0$ in Ω ; $0 \le x \le a$, $0 \le y \le b$

$$\boxed{u(x,y) = \sum_{k=1}^{\infty} u_k(x,y)}$$

$$\boxed{u_k(x,y) = a_k \sin\left(\frac{k\pi y}{b}\right) \operatorname{sh}\left(\frac{k\pi(a-x)}{b}\right)}$$

III $\Delta u = 0$ in Ω ; $0 \le x \le a$, $0 \le y \le b$



$$\boxed{u(x,y) = \sum_{k=1}^{\infty} u_k(x,y)}$$

$$\boxed{u_k(x,y) = a_k \sin\left(\frac{k\pi x}{a}\right) \operatorname{sh}\left(\frac{k\pi y}{a}\right)}$$

IV $\Delta u = 0$ in Ω ; $0 \le x \le a$, $0 \le y \le b$

$$\boxed{u(x,y) = \sum_{k=1}^{\infty} u_k(x,y)}$$

$$\boxed{u_k(x,y) = a_k \sin\left(\frac{k\pi y}{b}\right) \operatorname{sh}\left(\frac{k\pi x}{b}\right)}$$

2. Solve $\Delta u = 0$ on $[0,\pi] \times [0,\pi]$ in cases:

a) $u(0,y) = g(y) = 5\sin 2y - \sin 7y$ & 0 elsewhere on the bd.

b) $u(x,0) = g(x) = \sin x$

$u(0,y) = g(y) = 3\sin 2y$

c) $u(x,0) = g(x) = \sin x$

$u(0,y) = g_1(y) = 3\sin 2y$

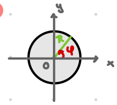
$u(\pi,y) = g_2(y) = -\sin 3y$

Hint: Always decompose the given problem into cases 1,2,3,4 and for the final solution add the sol. in each individual case

SEMINARY 5

LAPLACE'S EQUATIONS ON CIRCULAR DOMAINS

$$\begin{cases} \Delta u = 0 & \text{in } B_R \\ u|_{\rho=R} = g(\varphi) & \text{on } \partial B_R \end{cases}$$



STEP 1: $u(\rho,\varphi) = A(\rho)\,B(\varphi)$

Recall that $\Delta u = 0 \Leftrightarrow u_{xx} + u_{yy} = 0 \Leftrightarrow u_{\rho\rho} + \frac{1}{\rho}u_\rho + \frac{1}{\rho^2}u_{\varphi\varphi} = 0$

Then, by substitution $A''(\rho)B(\varphi) + \frac{1}{\rho}A'(\rho)B(\varphi) + \frac{1}{\rho^2}A(\rho)B(\varphi) = 0$

$$\Rightarrow \left[A''(\rho) + \frac{1}{\rho}A'(\rho)\right]B(\varphi) + \frac{1}{\rho^2}A(\rho)B''(\varphi) = 0 \;\Big|\cdot \rho^2$$

$$\Rightarrow (\rho^2 A''(\rho) + \rho A'(\rho))B(\varphi) + A(\rho)B''(\varphi) = 0$$

$$\Rightarrow \frac{\rho^2 A''(\rho) + \rho A'(\rho)}{A(\rho)} = -\frac{B''(\varphi)}{B(\varphi)} = \lambda \text{ (constant)} \Rightarrow \begin{cases} \rho^2 A''(\rho) + \rho A'(\rho) - \lambda A(\rho) = 0 \\ B''(\varphi) + \lambda B(\varphi) = 0 \end{cases}$$

STEP 2: Solve $B''(\varphi) + \lambda B(\varphi) = 0$

$$r^2 + \lambda = 0 \quad \begin{cases} \lambda > 0: & B(\varphi) = c_1\cos(\varphi\sqrt{\lambda}) + c_2\sin(\varphi\sqrt{\lambda}) \\ \lambda = 0: & B(\varphi) = c_1\varphi + c_2 \\ \lambda < 0: & B(\varphi) = c_1 e^{\varphi\sqrt{-\lambda}} + c_2 e^{-\varphi\sqrt{-\lambda}} \end{cases}$$

$\varphi \in [0, 2\pi) \Rightarrow B(\varphi) = B(\varphi + 2\pi)$ so $\lambda > 0$

Let's denote $k = \sqrt{\lambda} \Rightarrow \lambda = k^2$ so $\boxed{B_k(\varphi) = c_1\cos(k\varphi) + c_2\sin(k\varphi)}$

STEP 3: Let's solve $\rho^2 A''(\rho) + \rho A'(\rho) - \lambda A(\rho) = 0$

Seek the solution in the form $A(\rho) = \rho^\alpha$ so $A''(\rho) = \alpha(\alpha-1)\rho^{\alpha-2}$

The unknown is α and we get α after we verify the eq.: $\rho^2\rho^{\alpha-2}\alpha(\alpha-1) + \rho\cdot\rho^{\alpha-1}\alpha - \lambda\rho^\alpha = 0$

$$\Rightarrow \rho^\alpha((\alpha-1)\alpha + \alpha - \lambda) = 0 \Rightarrow \alpha^2 - \cancel{\alpha} + \cancel{\alpha} - \lambda = 0 \Rightarrow \alpha^2 = \lambda \Rightarrow \alpha = \pm\sqrt{\lambda}$$

Hence $\alpha = \pm k$ and we get $\boxed{A(\rho) = c_k\rho^k + d_k\rho^{-k}}$

! $k = 0 \Rightarrow \rho^2 A''(\rho) + \rho A'(\rho) = 0 \Rightarrow \dots \Rightarrow A(\rho) = c_1\ln\rho + c_2$ (same procedure)

STEP 4: Impose the BC $u|_{\rho=R} = g(\varphi)$ & develop $g(\varphi)$ in Fourier series on $[0, 2\pi]$. We have that:

$$g(\varphi) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k\cos(k\varphi) + b_k\sin(k\varphi)) \;\underbrace{\Big|\cdot\cos(k\varphi)\;\Big|\int_0^{2\pi}d\varphi}$$ we do this to know the coeff

$$\Rightarrow \int_0^{2\pi} g(\varphi)\cos(k\varphi)\,d\varphi = \frac{a_0}{2}\int_0^{2\pi}\cos(k\varphi)\,d\varphi + a_k\int_0^{2\pi}\cos^2(k\varphi)\,d\varphi$$

$$= \frac{a_0}{2}\cdot 0 + a_k\cdot\pi$$

$$\int_0^{2\pi}\cos(k\varphi)\,d\varphi = \frac{\sin(k\varphi)}{k}\Big|_0^{2\pi} = 0$$

$$\int_0^{2\pi}\cos^2(k\varphi)\,d\varphi = \int_0^{2\pi}\frac{1+\cos(2k\varphi)}{2}\,d\varphi = \dots = \pi$$

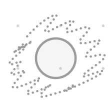
$$\Rightarrow \boxed{a_k = \frac{1}{\pi}\int_0^{2\pi} g(\varphi)\cos(k\varphi)\,d\varphi}$$

To obtain the other ones $\Big|\cdot\sin(k\varphi)\;\Big|\int_0^{2\pi}d\varphi$

$$\boxed{a_0 = \frac{1}{\pi}\int_0^{2\pi} g(\varphi)\,d\varphi} \qquad \boxed{b_k = \frac{1}{\pi}\int_0^{2\pi} g(\varphi)\sin(k\varphi)\,d\varphi}$$

Cases for ρ: CASE 1: $\boxed{\rho < R: u(\rho,\varphi) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k\cos(k\varphi) + b_k\sin(k\varphi))\left(\frac{\rho}{R}\right)^k}$

CASE 2: $\boxed{\rho > R: u(\rho,\varphi) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k\cos(k\varphi) + b_k\sin(k\varphi))\left(\frac{R}{\rho}\right)^k}$



1 Exercises: • Solve $\begin{cases} \Delta u = 0, & \rho < R \\ u|_{\rho=R} = 3\sin\varphi - \sin 3\varphi \end{cases}$

$$u(R,\varphi) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos(k\varphi) + b_k \sin(k\varphi)\right)\left(\frac{\rho}{R}\right)^k \overset{!}{=} 3\sin\varphi - \sin 3\varphi$$

Note that: $a_0 = 0$, $a_k = 0$, $\forall k \in \mathbb{N}^*$, $b_1 = 3$, $b_3 = -1$, $b_k = 0$ $\forall k \in \mathbb{N} \setminus \{1,3\}$ $\Rightarrow u(\rho,\varphi) = 3\sin\varphi \frac{\rho}{R} - \sin 3\varphi \left(\frac{\rho}{R}\right)^3$

Homework: Prove that $b_1 = 3$. Indeed

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} (3\sin\varphi - \sin 3\varphi)\sin\varphi \, d\varphi = \dots = 3$$

• Solve $\begin{cases} \Delta u = 0, & \rho < R \ \& \ \rho > R \\ u|_{\rho=R} = \sin\varphi - \sin 3\varphi \end{cases}$

REMARK: Solve the problem for $\rho < R$ and after that solve for $\rho > R$

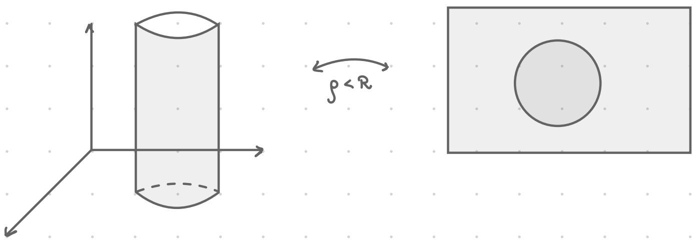
$$u(R,\varphi) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos(k\varphi) + b_k \sin(k\varphi)\right)\left(\frac{\rho}{R}\right)^k \overset{!}{=} \sin\varphi - \sin 3\varphi$$

$$u(R,\varphi) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos(k\varphi) + b_k \sin(k\varphi)\right)\left(\frac{R}{\rho}\right)^k \overset{!}{=} \sin\varphi - \sin 3\varphi$$

... Homework

2. Find the stationary distribution of the temperature $u(\rho,\varphi,z)$ inside an infinite cylinder of radius R in the case of the surface of the cylinder, the temperature is kept at the value

$$u(R,\varphi,z) = A\cos\varphi$$



Heat eq:

$$\underbrace{\frac{\partial u}{\partial t}}_{=0} - a^2 \Delta u = \underbrace{f}_{=0}$$

the temp. doesn't change itself

don't have outside temp.

$\Rightarrow \begin{cases} \Delta u = 0, & \rho < R \\ u(R,\varphi) = A\cos\varphi \end{cases} \Rightarrow u(\rho,\varphi) = \dots = A\cos\varphi \frac{\rho}{R}$

3. (Recap) Prove that there is no solution $u \in C^2(\overline{B})$, $B = B(0,R)$ for the Neumann problem:

$$\begin{cases} \Delta u = 1 & \text{in } B \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial B \end{cases}$$

Recall that: $\int_{\partial B} \frac{\partial u}{\partial \nu} d\sigma = \int_B \Delta u \, dx$ -> C3 (Particular case for G1)

$$0 = \int_B 1 \, dx = \mathrm{Vol}(B) \ \lightning$$

4. Prove that the full version of the MVT for harmonic function holds $u(x) = \frac{m}{\omega_m R^m} \int_{B(x,R)} u(y)\,dy$ (we'll have to add the details)

We know that $u(x) = \frac{1}{\omega_m r^{m-1}} \int_{\partial B(x,r)} u(y)\,d\sigma_y \ \Big| \int_0^R dr$, where $0 < r < R$

$$\Rightarrow u(x)\,\omega_m \int_0^R r^{m-1} dr = \int_0^R \int_{\partial B(x,r)} u(y)\,d\sigma_y\,dr$$

$$u(x)\,\omega_m R^m \frac{1}{m} = \int_{B(x,R)} u(y)\,dy$$

SEMINARY 6

GREEN & FDT

1. Given that $\Delta u = 1$ on $\Omega = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2<R\}$, find $\int_{\partial\Omega} \frac{\partial u}{\partial \nu}\, d\sigma$

$$\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\,d\sigma \overset{G1}{=} \int_\Omega \Delta u\, dx = \int_\Omega 1\, dx = m(\Omega) = \pi R^2$$

$$\Delta u := \sum_{j=1}^{m} \frac{\partial^2 u}{\partial x_j^2}$$

2. Let $u(x,y)=x^2+y^2-xy$ on $\Omega=(0,3)\times(0,3)$. Find $\int_{\partial\Omega}\nabla u\cdot\nu\, d\sigma$

$$\int_{\partial\Omega}\nabla u\cdot\nu\, d\sigma \underset{\text{true}}{=} \int_{\partial\Omega}\frac{\partial u}{\partial\nu}\,d\sigma \overset{FDT}{=} \int_\Omega \operatorname{div}(\nabla u)\,dx = \int_\Omega \Delta u\, dx = \int_0^3\int_0^3 4\,dx\,dy = 4\int_0^3 dx\int_0^3 dy = 4\, x\Big|_0^3\, y\Big|_0^3 = 4\cdot 3\cdot 3 = 36$$

everytime (cannot find where)

THE MAXIMUM PRINCIPLE

STRONG MAXIMUM PRINCIPLE

$\Omega\subset\mathbb{R}^m$ domain, $u\in C^2(\Omega)$ st $\Delta u\geq 0$, $\forall x\in\Omega$

if $\exists\, x_0\in\Omega$ st. $u(x_0)=\sup_\Omega u \Rightarrow u\equiv$ constant on Ω

COROLLARY: $\Omega\subset\mathbb{R}^m$ bounded domain, $u\in C^2(\Omega)\cap C(\overline{\Omega})$ nonconstant on Ω

1) $\Delta u\geq 0$ in $\Omega \Rightarrow u(x) < \max_{\partial\Omega} u$, $\forall x\in\Omega$

2) $\Delta u\leq 0$ in $\Omega \Rightarrow u(x) > \min_{\partial\Omega} u$, $\forall x\in\Omega$

3) $\Delta u = 0$ in $\Omega \Rightarrow \min_{\partial u} u < u(x) < \max_{\partial u} u$, $\forall x\in\Omega$

3. Let $\Omega\subset\mathbb{R}^m$ domain, let $c\in C(\Omega)$ st. $c(x)\leq 0$ in Ω & let $u\in C^2(\Omega)$ st. $\Delta u + c(x)u\geq 0$ in Ω.

Show that if $\sup_\Omega u>0$ & $\exists\, x_0\in\Omega$ st $u(x_0)=\sup_\Omega u \Rightarrow u\equiv$ constant on Ω

Define $\mathcal{M}=\{x\in\Omega \mid u(x)=\sup_\Omega u := m\}$

We show that $\mathcal{M}$ non-empty, closed & open

- $\mathcal{M}$ is non-empty, since $\exists\, x_0\in\Omega$ st $u(x_0)=\sup_\Omega u$
- $\mathcal{M}$ is closed (closedness by sequences + u is continuous) $\to$ see C5
- $\mathcal{M}$ is open

set of neighbourhoods

Assume $x\in\mathcal{M} \Rightarrow \exists\,\Omega'\subset\Omega$ open & connected st. $\Omega'\in\mathcal{V}(x)$ st. $u(y)\geq 0$, $\forall y\in\Omega'$

$\Rightarrow c(y)u(y)\leq 0$ in $\Omega' \Rightarrow \Delta u\geq 0$ in $\Omega' \overset{SMP}{\Longrightarrow} u\equiv$ constant Ω' & $u(x)=m$, $\forall x\in\Omega'$

$\Rightarrow \Omega'\subset\mathcal{M} \Rightarrow \mathcal{M}$ is open

bc. of these

$\mathcal{M}=\Omega \Rightarrow u\equiv$ constant in Ω

4. Let $\Omega\subset\mathbb{R}^m$ bounded domain & $c\in C(\Omega)$ st. $c(x)\leq 0$ in Ω. Show that if $u\in C^2(\Omega)\cap C(\overline{\Omega})$ st.

$\Delta u + c(x)u\geq 0$ in Ω & $u\leq 0$ on $\partial\Omega \Rightarrow u\leq 0$ in $\overline{\Omega}$

Assume by contradiction that $u>0$ in $\overline{\Omega} \Rightarrow \sup_\Omega u = \max_{\overline{\Omega}} u > 0 \overset{Ex\,3}{\Longrightarrow} u\equiv$ const in Ω ↯ $\Rightarrow$ hence $u\leq 0$ in $\overline{\Omega}$

5. Let $\Omega = \{ x \in \mathbb{R}^2 \mid |x| \le 1 \}$. If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ solves $\begin{cases} \Delta u = 0, & |x| < 1 \\ u = 2\ln(2+\sin\varphi) - \sin\varphi, & |x| = 1 \end{cases}$

find the extreme values of u & the points in $\bar{\Omega}$ in which these values are obtained.

By applying the corollary of SMP we have $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$, $\min_{\bar{\Omega}} u = \min_{\partial\Omega} u$

Denote $\sin\varphi = t$, $t \in [-1,1] \Rightarrow f(t) = 2\ln(2+t) - t \Rightarrow f'(t) = -\frac{t}{2+t}$

$f'(t) = 0 \Rightarrow t = 0$

t	-1		0		1
$f'(t)$		$+ + + + + +$	0	$- - - - - -$	
$f(t)$	1		$2\ln 2$		$2\ln 3 - 1$

$f(0) = 2\ln 2$

$f(1) = 2\ln 3 - 1$

$f(-1) = 1$

$\Rightarrow \max_{\bar{\Omega}} = 2\ln 2$; $\min_{\bar{\Omega}} u = 2\ln 3 - 1$ (compare to make sure)

Work the points in which these values are obtained!

6. **(LIOUVILLE'S THEOREM)** If u is harmonic in $\mathbb{R}^n$ and m is bounded below (inferior) or bounded above (superior) $\Rightarrow u$ is constant

PROOF: Assume $u \ge 0$ (if we have sth like $u \ge -1 \Rightarrow u - 1 \ge 0$; $u - 1 := t \Rightarrow t \ge 0$)

Let $x_1, x_2 \in \mathbb{R}^n$ arbitrary and let $r_1, r_2 > 0$ st. $r_1 = r_2 + |x_1 - x_2| \ge r_2$ (make sure that one ball is contained in the other)

$\Rightarrow$ Hence $B_{r_2}(x_2) \subset B_{r_1}(x_1)$

Apply the "full" MVT and obtain $u(x_2) = \frac{n}{\omega_n r_2^n} \int_{B_{r_2}} u(y)\,dy \le \frac{n}{\omega_n r_2^n} \int_{B_{r_1}} u(y)\,dy \overset{\text{Full MVT}}{\underset{\text{for } B_{r_1}}{=\!=}} \left(\frac{r_1}{r_2}\right)^n u(x_1) \Rightarrow u(x_2) \le u(x_1)$

Since x_1, x_2 arbitrary $\Rightarrow u(x_1) \le u(x_2)$ $\Big\} \Rightarrow$

$\Rightarrow u(x_1) = u(x_2)$

7. a) Let $B(0,3) \subseteq \mathbb{R}^3$. Find $u(0)$ if $u\big|_{\partial B(0,3)} = 4$

we use $u(x) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B(x,r)} u(y)\,d\sigma$, $\omega_n = \frac{2\pi}{\Gamma(\frac{n}{2})}$

$u(0) = \frac{1}{\omega_3 \cdot 3^2} \int_{\partial B(0,3)} u\,d\sigma = \frac{4}{\cancel{4\pi}\cdot 9} \int_{\partial B(0,3)} d\sigma = \ldots$ compute this

$\omega_3 = 4\pi$

b) Let $B(0,4) \subseteq \mathbb{R}^3$. Find $\int_{\partial B(0,4)} u(y)\,d\sigma$ if $u(0) = \frac{1}{4}$

we use $u(x) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B(x,r)} u(y)\,d\sigma$, $\omega_n = \frac{2\pi}{\Gamma(\frac{n}{2})}$

Green's Function:

$G: \Omega \times \bar{\Omega} \setminus \Lambda \to \mathbb{R}$

$\Lambda := \{(x,y) \in \Omega \mid x = y\}$

$G(x,y) = -N(x-y) + \Phi(x,y)$, where $\Phi: \Omega \times \bar{\Omega} \to \mathbb{R}$ s.t. $\forall x \in \Omega$, $\Phi(x,\cdot) \in C^2(\bar{\Omega})$ & $\begin{cases} \Delta_y \Phi(x,\cdot) = 0 & \text{in } \Omega \\ \Phi(x,\cdot) = N(x-\cdot) & \text{on } \partial\Omega \end{cases}$

$N(x-y) = \begin{cases} \frac{1}{(2-n)\omega_n}|x-y|^{2-n}, & n \geq 2 \\ \frac{1}{2\pi}\ln|x-y|, & n = 2 \end{cases}$

the fundamental solution of Laplace's eq.

Green's function in the case of "simple" domains:

- Green's function for the disk of radius R in $\mathbb{R}^2$

$G(x,y) = -\frac{1}{2\pi}\ln|x-y| + \frac{1}{2\pi}\ln\frac{|x|}{R}\cdot|\bar{x}-y|$, $x \neq 0$.

↳ see lecture 6.

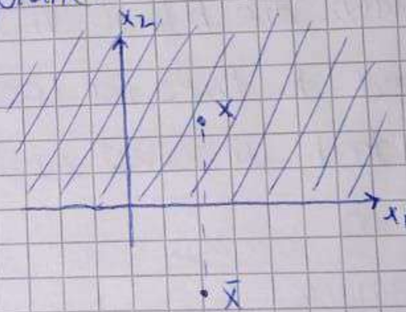
- Green's function for the half-plane.

$x = (x_1, x_2)$, $\bar{x} = (x_1, -x_2)$

$\Omega = \{x \in \mathbb{R}^2 \mid x_2 > 0\}$

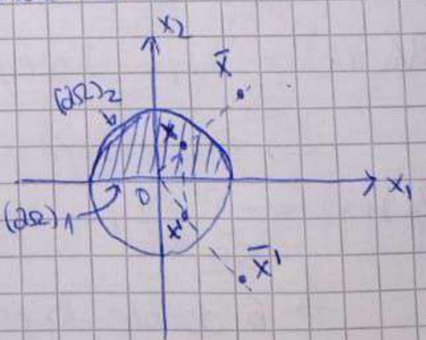
$\partial\Omega = \{x \in \mathbb{R}^2 \mid x_2 = 0\}$

$G(x,y) = -\frac{1}{2\pi}\ln|x-y| + \frac{1}{2\pi}\ln|\bar{x}-y|$



- Green's function for the half-disk

$G(x,y) = -\frac{1}{2\pi}\ln|x-y| + \frac{1}{2\pi}\ln\frac{|x|}{R}|\bar{x}-y| + \frac{1}{2\pi}\ln|x'-y| - \frac{1}{2\pi}\ln\frac{|x'|}{R}|\bar{x}'-y|$.



Potential Theory for Laplace's equation:

Subject :

Date : ....../....../......

Objective: Solve the existence (of a solution) problem for the following BVP's:

(Di): $\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$ (DE): $\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$

(Ni): $\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = h & \text{on } \partial\Omega \end{cases}$ (NE): $\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \\ \frac{\partial u}{\partial \nu} = h & \text{on } \partial\Omega \end{cases}$

where $\Omega \subset \mathbb{R}^n$ open, bounded of "sufficient" regularity.

let $V_f(x) = \int_\Omega N(x-y) \cdot f(y)\, dy$

$\hookrightarrow$ volume potential of density f.

Theorem:

If $f \in C^1(\Omega)$ then $V_f \in C^2(\Omega)$ and $\Delta V_f = f$ in Ω.

Remark:

With the help of V_f ~~BVPs~~ BVP's for Poisson'eq are reduced to BVPs for Laplace's equation.

Indeed, u solves $\Delta u = f$, then $v = u - V_f$ gives us $\Delta v = \Delta u - \Delta V_f = f - f = 0$

Remark:

$V_f \in C^1(\mathbb{R}^n) \Rightarrow \exists V_f, \frac{\partial V_f}{\partial \nu}$ on $\partial\Omega$.

let $\cdot (V_\alpha)(x) := \int_{\partial\Omega} N(x-y)\, \alpha(y)\, d\Gamma_y$

$\hookrightarrow$ single-layer potential of density α.

$\cdot (W_\beta)(x) := -\int_{\partial\Omega} \frac{\partial N}{\partial \nu_y}(x-y) \beta(y)\, d\Gamma_y$

$\hookrightarrow$ double-layer potential of density β.

let $\alpha, \beta \in C(\partial\Omega)$.

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Theorem:

- $\Delta(W_\beta) = 0$ on $\mathbb{R}^n \setminus \partial\Omega$ and $W_\beta|_{\partial\Omega} \in C(\partial\Omega)$.

- The following jump formulas hold.

$\forall x_0 \in \partial\Omega$. $(W_\beta)(x) \xrightarrow[\substack{x \to x_0 \\ x \in \Omega}]{} (W_\beta)(x_0) - \frac{1}{2}\beta(x_0)$

$(W_\beta)(x) \xrightarrow[\substack{x \to x_0 \\ x \in \mathbb{R}^n \setminus \Omega}]{} (W_\beta)(x_0) + \frac{1}{2}\beta(x_0)$

Theorem:

The Dirichlet interior problem (Di) has at least one solutions.

Proof:

Let us seek a solution of the problem:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

in the form: $u = W_\beta$, where $\beta \in C(\partial\Omega)$ unknown.

Since $u|_{\partial\Omega} = g$, by substitution, we get

$\Rightarrow (W_\beta)(x) - \frac{1}{2}\beta(x) = g(x)$

$$-\frac{1}{2}\beta(x) - \int_{\partial\Omega} \frac{\partial N}{\partial \nu_y}(x-y)\beta(y)\,d\Gamma_y = g(x) \quad , \forall x \in \partial\Omega$$

If one shows that in add $-\frac{1}{2}I - T$ is inj $\Rightarrow -\frac{1}{2}I - T$ is an isom.

$$\underbrace{(\underbrace{-\frac{1}{2}I}_{\text{isomorphism}} - \underbrace{T}_{\text{compact}})}_{\text{Fredholm op. of index 0}}\beta = g$$

$\Rightarrow \beta = (-\frac{1}{2}I - T)^{-1} g \Rightarrow u = W((-\frac{1}{2}I - T)^{-1} g)$ ∎

Elliptic Eq ~~and~~ in Divergence Form

let us consider

$$(*) \begin{cases} -\operatorname{div}(A(x)\nabla u) + q_0(x)u = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

On some $\Omega \subseteq \mathbb{R}^n$ open & bounded, where.

$A(x) = [a_{ij}(x)]_{1\le i,j\le n}$, $a_{ij} \in L^\infty(\Omega)$. & $a_{ji} = a_{ij}$.

& $a_0(x) \geq 0$, $\forall x \in \Omega$, $a_0 \in L^\infty(\Omega)$.

It must also satisfy: $\exists \mu > 0$ s.t. $\Rightarrow$ $\sum_{i,j=1}^{n} a_{ij}(x)\xi_i \xi_j \geq \mu |\xi|^2$, $\forall \xi \in \mathbb{R}^n$, $x \in \Omega$.

↳ strong ellipticity cond.

Let us introduce:

$$E(u) = \int_\Omega \left[\frac{1}{2} \left(\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + a_0(x) u^2 \right) - fu \right] dx.$$

$$a(u,v) := \int_\Omega \left(\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + a_0 uv \right) dx.$$

$(C_0^1(\bar{\Omega}), a(\cdot,\cdot))$ - inner product space.

$|u| = \sqrt{a(u,u)}$

∇ $u \in H_0^1(\Omega)$ weak solution of (*) if $a(u,v) = (f,v)_{L^2}$, $\forall v \in H_0^1(\Omega)$

or $E(u) < E(w)$, $\forall w \neq u$.

SEMINARY 8

Let $Lu := -\operatorname{div}(A(x)\nabla u) + a_0(x)u$

$A(x) \in \mathcal{M}_m$ st. $a_{jk} \in L^\infty(\Omega)$, $a_{jk} = a_{kj}$, $a_0 \in L^\infty(\Omega)$, $a_0 \geq 0$

$\sum_{k,j=1}^{m} a_{jk}(x)\xi_j \xi_k \geq \mu |\xi|^2$ ($\exists \mu > 0$ st), $\forall \xi \in \mathbb{R}^m$, $\forall$ almost all $x \in \Omega$

↳ strong ellipticity cond.

$a(u,v) = \int_\Omega \left(\sum_{j,k=1}^{m} a_{jk} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_k} + a_0 uv \right) dx$

On $C_0^1(\bar{\Omega})$, $a(u,v)$ - inner product ; $|u| := a(u,u)^{\frac{1}{2}} := \left\{ \int_\Omega \sum_{j,k=1}^{m} a_{jk} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_k} + a_0 u^2) dx \right\}^{1/2}$

Let's consider the Dirichlet pb: $\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$ (*)

DEF:
- $u \in C^2(\bar{\Omega})$ is a classical sol. of (*) iff it satisfies the identities in (*) pointwise ($f \in C(\bar{\Omega})$)
- $u \in H_0^1(\Omega)$ is a weak sol. (if $f \in L^2(\Omega)$) iff $a(u,v) = (f,v)_{L^2}$, $\forall v \in H_0^1(\Omega)$

$E: H_0^1(\Omega) \to \mathbb{R}$; $E(u) = \frac{1}{2} a(u,u) - (f,u)_{L^2}$ ⇝ energy func.

1. Prove that $a(\cdot,\cdot)^{\frac{1}{2}}$, $|\cdot|_{0,1}$ are equivalent norms on $C_0^1(\bar{\Omega})$

Starting after the end of the sol. of ex.4

We should know: Poincare $\int_\Omega u^2 dx \leq c^2 \int_\Omega |\nabla u|^2 dx$

$a(\cdot,\cdot)^{\frac{1}{2}}$ & $|\cdot|_{0,1}$ are equivalent $\iff \exists c_1, c_2 > 0$ st. $c_1 |u|_{0,1} \leq a(u,u)^{\frac{1}{2}} \leq c_2 |u|_{0,1}$

Cauchy-S... $(\int fg)^2 \leq (\int f)^2 (\int g)^2$

2. Consider the Dirichlet pb: $\begin{cases} -u'' + u = f & \text{in } (0,1) \quad (**) \\ u(0) = u(1) = 0 \end{cases}$

$\to A(x) = 1$ (it's a matrix of one elem.)

$\to a_0 = 1$

(we don't have partial derivatives on the real line) so we have the normal derivative

Write the energy functional for (**) and determine $E'(u;v) = \lim_{t \searrow 0} \frac{E(u+tv) - E(u)}{t}$

Define the weak sol. of (**) and give characterisations of it.

We know that $E(u) = \frac{1}{2} a(u,u) - \underbrace{(f,u)_{L^2}}_{= \int_0^1 fu\,dx}$

$a(u,v) = \int_0^1 (u'v' + uv)\,dx$

$\Rightarrow E(u) = \int_0^1 \left(\frac{1}{2}|u'|^2 + \frac{1}{2}u^2 - fu\right)dx$

Let's determine $E(u+v)$: $E(u+v) = \int_0^1 \left(\frac{1}{2}|(u+tv)'|^2 + \frac{1}{2}(u+tv)^2 - f(u+tv)\right)dx$

$\to$ get the point u do it for $E(u+tv)$

$= \int_0^1 \left(\frac{1}{2}|u'|^2 + \frac{t^2}{2}|v'|^2 + u'v' + \frac{1}{2}u^2 + \frac{t^2}{2}v^2 + tuv - fu - fv\right)dx$

$= E(u) + \frac{1}{2}\int_0^1 |v'|^2 dx + \frac{1}{2}\int_0^1 v^2 dx + \boxed{\int_0^1 (u'v' + uv)\,dx - \int_0^1 fv\,dx}$

$\lim_{t \searrow 0} \frac{E(u+tv) + E(u)}{t} = \lim_{t \searrow 0} \frac{\frac{t^2}{2}\int_0^1 |v'|^2 dx + \frac{t^2}{2}\int_0^1 v^2 dx + \cancel{t}\int_0^1 (u'v' + uv)\,dx - \cancel{t}\int_0^1 fv\,dx}{\cancel{t}} = \int_0^1 (u'v' + uv - fv)\,dx$

Let $f \in L^2(0,1)$, $u \in H_0^1(\Omega)$ weak sol of (**) iff u satisfies $\underbrace{a(u,v) - (f,v)_{L^2}}_{= \int_0^1 (u'v' + uv - fv)dx} = 0 \quad \forall v \in H_0^1(\Omega)$

Let $f \in L^2(0,1)$, $u \in H_0^1(\Omega)$. The following are equivalent: (i) u weak sol of (**)

(ii) u is the absolute strict minimum point of E i.e. $E(u) < E(w)$, $\forall w \in H_0^1(\Omega)$ $w \neq u$

3. Write the energy functional and give characterisations of the weak sol. for:

a) $\begin{cases} -\Delta u + (1 + |x|^2)u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$ b) $\begin{cases} -\Delta u + u = |x| & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$ $a(u,v) = \int_\Omega (\nabla u \nabla v + uv)\,dx$

a) $-\Delta u + su = 1 \;|\cdot v \;|\int \Rightarrow \int -\Delta u\, v = \int \nabla u \nabla v$ (the trick) $\to$ apply Green

$a(u,v) = \int_\Omega (\nabla u \nabla v + (1+|x|^2)uv)\,dx$

$E(u) = \frac{1}{2}a(u,u) - (f,u)_{L^2} = \frac{1}{2}\int_\Omega (\nabla u \nabla u + (1+|x|^2)u \cdot u)\,dx - \int_\Omega u\,dx$

$= \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \frac{(1+|x|^2)}{2}u^2 - u\right)dx$

Proposition: Let $f \in L^2(\Omega)$, $u \in H_0^1(\Omega)$. The following are equivalent: (i) u satisfies: $a(u,v) = (f,v)_{L^2}$ $\forall v \in H_0^1(\Omega)$

(ii) u is the strict absolute min point of the energy functional i.e. $E(u) < E(w)$, $\forall w \in H_0^1(\Omega)$, $w \neq u$

4. Write the Dirichlet problem for the energy functional:

a) $E(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + u^2 + |x|u\right) dx$ b) $E(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}u^2 + |x|u\right) dx$

a) $E(u) = \frac{1}{2} a(u,u) - \underbrace{(f,u)_{L^2}}_{= \int_\Omega f u \, dx}$

$$E(u) = \underbrace{\int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}\cdot 2u^2\right) dx}_{\frac{1}{2}a(u,u)} - \underbrace{\int_\Omega \overline{-|x|}^{=f} u \, dx}_{(f,u)_{L^2}}$$

apply Green in reverse

$a(u,v) = \int_\Omega (\nabla u \nabla v + 2uv)\, dx$

$f = -|x|$

$$\Rightarrow \begin{cases} -\Delta u + 2u = -|x| & \text{in } \Omega \\ u = 0 \end{cases}$$

SEMINARY 9

FOURIER SERIES

$(H, (\cdot,\cdot))$ real Hilbert series

$(\phi_k)_{k\geq 1} \subseteq H$ orth sys iff $(\phi_k, \phi_j) = \begin{cases} 1, & k=j \\ 0, & k\neq j \end{cases}$

$u \in H \longmapsto \sum_{k=1}^{\infty} (u, \phi_k)\phi_k$ - Fourier Series of u w.r.t. $(\phi_k)_{k\geq 1}$

<u>THEOREM:</u> (ϕ_k) orth sys. The following are equivalent: (1) $\forall u \in H: \sum_{k=1}^{\infty} (u, \phi_k)\phi_k = u$

(2) If $(u, \phi_k) = 0$, $\forall k = 1, 2 \dots$ then $u = 0$

(3) $|u|^2 = \sum_{k=1}^{\infty} (u, \phi_k)^2$, $\forall u \in H$

(4) $\forall u \in H$ can be approx. as well as we wish by finite linear comb. of elem. of $(\phi_k)_{k\geq 1}$

DEF: The orth. sys $(\phi_k)_{k\geq 1}$ is called complete if it satisfies one of the equivalent cond. from the Th ↑

1. Prove the orth. and completeness properties for the system:

a) $\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\sin x, \frac{1}{\sqrt{\pi}}\cos x, \frac{1}{\sqrt{\pi}}\sin 2x, \frac{1}{\sqrt{\pi}}\cos 2x, \dots$ in $L^2(-\pi,\pi)$

Check orth. for all possible comb.

$(\phi_1, \phi_1)_{L^2} = \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2\pi}}\,dx = 1$ → inner product

$(\phi_1, \phi_k')_{L^2} = \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{\pi}} \sin kx\,dx \overset{?}{=} 0$ ← we solve this

$(\phi_1, \phi_k'')_{L^2} = \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{\pi}} \cos kx\,dx \overset{?}{=} 0$

$(\phi_k', \phi_k')_{L^2} = \int_{-\pi}^{\pi} \frac{1}{\sqrt{\pi}}\sin kx \cdot \frac{1}{\sqrt{\pi}}\sin kx\,dx \overset{?}{=} 1$ $\qquad \cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$

$(\phi_k'', \phi_k'')_{L^2} = \int_{-\pi}^{\pi} \frac{1}{\sqrt{\pi}}\cos kx \cdot \frac{1}{\sqrt{\pi}}\cos kx\,dx \overset{?}{=} 1$ $\qquad \cos 2x = 2\cos^2 x - 1$

$(\phi_m', \phi_m'')_{L^2} = \int_{-\pi}^{\pi} \frac{1}{\sqrt{\pi}}\sin mx \cdot \frac{1}{\sqrt{\pi}}\cos mx\,dx \overset{?}{=} 0$ $\qquad 2\sin x\cos y = \sin(x+y) + \sin(x-y)$

$(\phi_m', \phi_n')_{L^2} = \int_{-\pi}^{\pi} \frac{1}{\sqrt{\pi}}\sin mx \cdot \frac{1}{\sqrt{\pi}}\sin nx\,dx \overset{?}{=} 0$ $\qquad 2\sin x\sin y = \cos(x-y) - \cos(x+y)$

$(\phi_m', \phi_n')_{L^2} = \int_{-\pi}^{\pi} \frac{1}{\sqrt{\pi}}\cos mx \cdot \frac{1}{\sqrt{\pi}}\cos nx\,dx \overset{?}{=} 0$

we accept this

We accept that (a) is complete (Weierstrass approx th. + $C(-\pi,\pi)$ is dense in $L^2(-\pi,\pi)$ + min. prop. of Fourier coef.)

b) $\frac{1}{\sqrt{\pi}}, \sqrt{\frac{2}{\pi}}\cos x, \sqrt{\frac{2}{\pi}}\cos 2x, \dots$ in $L^2(0,\pi)$

We show (b) is complete

Let $\left(u, \sqrt{\frac{2}{\pi}}\cos kx\right)_{L^2(0,\pi)} = 0 \quad \forall k$ → constants always jump outside

We show that $u = 0$

We know that $u: (0,\pi) \to \mathbb{R}$; let us define $\tilde{u}: (-\pi,\pi) \to \mathbb{R}$, $\tilde{u}(x) = \begin{cases} u(x), & x \in [0,\pi) \\ u(-x), & x \in (-\pi, 0] \end{cases}$ even function

$\Rightarrow (\tilde{u}, \cos kx)_{L^2(-\pi,\pi)} = \int_{-\pi}^{\pi} \tilde{u}(x)\cos(kx)\,dx = 2\int_0^{\pi} \tilde{u}(x)\cos kx\,dx = 2\int_0^{\pi} u(x)\cos kx\,dx = 2(u, \cos kx)_{L^2(0,\pi)} = 0$

$\Rightarrow (\tilde{u}, \cos kx)_{L^2(-\pi,\pi)} = 0, \forall k \xRightarrow{(a)\text{ complete}} \tilde{u} = 0$ on $(-\pi,\pi) \Rightarrow u = 0$ on $(0,\pi)$

c) $\left(\sqrt{\frac{2}{\pi}}\sin kx\right)_{k\geq 1}$ in $L^2(0,\pi)$

2. Develop the functions $1, x, x^2, \sin^2 x$ in Fourier Series w.r.t. systems from the previous problem

1 - w.r.t. (c)

$$\left(1, \sqrt{\tfrac{2}{\pi}}\sin k\pi\right)_{L^2(0,\pi)} = \sqrt{\tfrac{2}{\pi}}\int_0^\pi \sin kx\,dx = -\sqrt{\tfrac{2}{\pi}}\cdot\frac{1}{k}\cos kx\Big|_0^\pi = -\sqrt{\tfrac{2}{\pi}}\cdot\frac{1}{k}(\cos k\pi - 1) = \begin{cases} 0, & k = 2j \\ \frac{2}{k}\sqrt{\frac{2}{\pi}}, & k = 2j-1 \end{cases}$$

$$\Rightarrow 1 = \sum_{j=1}^{\infty} \frac{2\sqrt{2}}{\sqrt{\pi}}\cdot\frac{1}{2j-1}\cdot\sqrt{\tfrac{2}{\pi}}\sin((2j-1)x)$$

$$1 = \sum_{j=1}^{\infty} \frac{4}{(2j-1)\pi}\sin((2j-1)x)$$

x - w.r.t. (c)

$$\left(x, \sqrt{\tfrac{2}{\pi}}\sin kx\right)_{L^2(0,\pi)} = \sqrt{\tfrac{2}{\pi}}\int_0^\pi x\sin kx\,dx = \sqrt{\tfrac{2}{\pi}}\left(-\frac{x}{k}\cos kx\Big|_0^\pi + \int_0^\pi \frac{\cos kx}{k}\,dx\right) = \sqrt{\tfrac{2}{\pi}}\left(-\frac{\pi}{k}\cos k\pi + \frac{1}{k}\cdot\underbrace{\frac{\sin kx}{k}\Big|_0^\pi}_{=0}\right) = \sqrt{\tfrac{2}{\pi}}\left(-\frac{\pi}{k}(-1)^k\right)$$

$$\Rightarrow x = \sum_{k=1}^{\infty}\sqrt{\tfrac{2}{\pi}}\left(-\frac{\pi}{k}\right)(-1)^k\sqrt{\tfrac{2}{\pi}}\sin kx$$

$$x = 2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sin(kx)$$

x^2 - w.r.t (a)

Note that w.r.t. (a) we have $f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k\sin kx + b_k\cos kx)$

$$a_0 = \int_{-\pi}^{\pi} f(x)\cdot\frac{1}{\pi}\,dx, \quad a_0 = \frac{2}{3}\pi^2$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx, \quad a_k = 0$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx, \quad b_k = \frac{4(-1)^k}{k^2}$$

$$\Rightarrow x^2 = \frac{2}{3}\pi^2 + 4\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\cos(kx)$$

SEMINARY 10

1. Let $\Omega \subset \mathbb{R}^m$ bounded & open. It is known that $(\phi_k)_{m\geq 1}$ is orth. and complete in $L^2(\Omega)$. Show that $\left(\frac{1}{\sqrt{\lambda_k}}\phi_k\right)_{k\geq 1}$ is orth. & complete in $H_0^1(\Omega)$

Remark: $(\phi_k, v)_{H_0^1} = \lambda_k (\phi_k, v)_{L^2}$

I ORTH

$$\left(\frac{1}{\sqrt{\lambda_k}}\phi_k, \frac{1}{\sqrt{\lambda_m}}\phi_m\right)_{H_0^1} = \frac{1}{\sqrt{\lambda_k \lambda_m}}(\phi_k, \phi_m)_{H_0^1} = \frac{\lambda_k}{\sqrt{\lambda_k \lambda_m}}(\phi_k, \phi_m)_{L^2} = \sqrt{\frac{\lambda_k}{\lambda_m}}(\phi_k, \phi_m)_{L^2} = \begin{cases} 0, & k \neq m \\ 1, & k = m \end{cases}$$

II COMPLETE

$$\left(u, \frac{1}{\sqrt{\lambda_k}}\phi_k\right)_{H_0^1} = 0 \ \forall k \overset{?}{\Rightarrow} u = 0$$

$$0 = \left(u, \frac{1}{\sqrt{\lambda_k}}\phi_k\right)_{H_0^1} = \lambda_k\left(u, \frac{1}{\sqrt{\lambda_k}}\phi_k\right)_{L^2} = \frac{\lambda_k}{\sqrt{\lambda_k}}(u, \phi_k)_{L^2} \Rightarrow (u, \phi_k)_{L^2} = 0 \ \forall k \underset{\text{in } L^2}{\overset{(\phi_k) \text{ comp}}{\Longrightarrow}} u = 0$$

$\left(\frac{1}{\sqrt{\lambda_k}}\phi_k\right)_{k\geq 1}$ complete in $H_0^1(\Omega)$

2. Solve the eigenvalue pb. $\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$ in cases: a) $\Omega = (0, a)$
b) $\Omega = (0, a) \times (0, b)$

a) $\begin{cases} -u'' = \lambda u & \text{in } (0, a) \\ u(0) = u(a) = 0 \end{cases}$

$u'' + \lambda u = 0 \Rightarrow r^2 + \lambda = 0 \Rightarrow r^2 = \underbrace{-\lambda}_{<0} \Rightarrow r = \pm i\sqrt{\lambda} \Rightarrow u(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x)$

$u(0) = C_1 = 0$

$u(a) = C_2 \sin(\sqrt{\lambda}a) = 0 \Rightarrow$
- $C_2 = 0$ imp. (too boring)
- $\sin(\sqrt{\lambda}a) = 0 \Rightarrow \sqrt{\lambda}a = k\pi \Rightarrow \boxed{\lambda_k = \left(\frac{k\pi}{a}\right)^2}$

We need to find ϕ_k

$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$

$u_k(x) = C_2 \sin\left(\frac{k\pi}{a}x\right) \ (= \phi_k)$

$(\phi_k, \phi_k)_{L^2(0,a)} = 1 \Rightarrow \int_0^a C_2^2 \sin^2\left(\frac{k\pi}{a}x\right)dx = 1 \Rightarrow C_2^2 \int_0^a \frac{1}{2}\left(1 - \cos 2\frac{k\pi}{a}x\right)dx = 1 \Rightarrow$

$\Rightarrow \frac{C_2^2}{2}\left[x\Big|_0^a - \left(-\frac{\sin 2\frac{k\pi}{a}x}{2\frac{k\pi}{a}}\Big|_0^a\right)\right] = 1 \Rightarrow C_2^2 \cdot \frac{a}{2} = 1 \Rightarrow C_2 = \sqrt{\frac{2}{a}} \Rightarrow \boxed{\phi_k = \sqrt{\frac{2}{a}}\sin\frac{k\pi}{a}x}$

b) $\begin{cases} -(u_{xx} + u_{yy}) = \lambda u & \text{in } (0, a) \times (0, b) \\ u|_{\partial\Omega} = 0 \end{cases}$

Seek the sol. in the form: $u(x, y) = A(x)B(y)$

$A(0) = A(a) = 0$

$B(0) = B(b) = 0$

$-A''(x)B(y) - A(x)B''(y) = \lambda A(x)B(y) \ \Big| : A(x)B(y) \Rightarrow -\frac{A''(x)}{A(x)} - \frac{B''(y)}{B(y)} = \lambda \Rightarrow -\frac{A''(x)}{A(x)} = \frac{B''(y)}{B(y)} + \lambda =: \mu \Rightarrow$

$\Rightarrow \begin{cases} A''(x) + \mu A(x) = 0 & , A(0) = A(a) = 0 \\ B''(y) - (\mu - \lambda)B(y) = 0 & , B(0) = B(b) = 0 \end{cases}$

For the first one: $\mu_k = \left(\frac{k\pi}{a}\right)^2$, $A_k(x) \ (= \phi_k(x)) = \sqrt{\frac{2}{a}}\sin\frac{k\pi x}{a}$

For the second one: $\Rightarrow \begin{cases} -B''(y) = (\lambda - \overbrace{\mu}^{\mu_k})B(y), & 0 < y < b \\ B(0) = B(b) = 0 \end{cases} \Rightarrow \lambda - \mu_k = \frac{j\pi}{b}$, $B_j(y) = \sqrt{\frac{2}{b}}\sin\frac{j\pi y}{b} \Rightarrow \boxed{\lambda_{kj} = \left(\frac{k\pi}{a}\right)^2 + \left(\frac{j\pi}{b}\right)^2}$

$\Rightarrow \boxed{\phi_{kj} = \frac{2}{\sqrt{ab}}\sin\frac{k\pi x}{a}\sin\frac{j\pi y}{b}}$

$$(u(x,y), \phi_{kj})_{((0,a)\times(0,b))} = \int_0^a\int_0^b u(x,y) \cdot \frac{2}{\sqrt{ab}}\sin\frac{k\pi x}{a}\sin\frac{j\pi y}{b}\,dxdy$$

3. Using λ & ϕ of the Dirichlet pb:

Solve $\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$ if a) $\Omega = (0,\pi)$, $f(x) = \sin x + 2\sin 2x$ c) $\Omega = (0,\pi)^2$, $f(x,y) = \sin x \sin 3y$

b) $\Omega = (0,\pi)$, $f(x) = 1$ d) $\Omega = (0,\pi)^2$, $f(x,y) = xy$

a) $\begin{cases} -u'' = f & \text{in } (0,\pi), \quad f \in L^2(0,\pi) \\ u(0) = u(\pi) = 0 \end{cases}$

$$u = \sum_{k=1}^{\infty} \frac{(f,\phi_k)_{L^2(0,\pi)}}{\lambda_k}\phi_k = \sum_{k=1}^{\infty} \underbrace{\frac{(f,\phi_k)_{L^2(0,\pi)}}{\lambda_k}\sqrt{\frac{2}{\pi}}}_{:= C_k} \sin(kx) \Rightarrow C_k = \sqrt{\frac{2}{\pi}}\,\frac{\cancel{\pi^2}}{k^2\cancel{\pi^2}} \int_0^\pi f(x)\sqrt{\frac{2}{\pi}}\sin(kx)\,dx \Rightarrow$$

$\left.\begin{array}{l} \Rightarrow C_k = \dfrac{2}{k^2\pi}\displaystyle\int_0^\pi f(x)\sin kx\,dx \\ f(x) = \underbrace{\sin x}_{:= f_1} + \underbrace{2\sin 2x}_{:= f_2} \end{array}\right\}$ $\Rightarrow$ we split the sol. in two

for $f_1(x)$ we have: $C_1 = \frac{2}{\pi}\int_0^\pi \sin^2 x\,dx = \ldots = 1$ so $u_1(x) = \sin x$

$C_k = \frac{2}{k^2\pi}\int_0^\pi \sin x \sin kx\,dx = 0$ if $k \neq 1$

for $f_2(x)$ we have: $\left.\begin{array}{l} C_k = \dfrac{2}{k^2\pi}\displaystyle\int_0^\pi 2\sin 2x \sin kx\,dx = \begin{cases} 0 & \text{if } k \neq 2 \\ ? & \text{if } k = 2 \end{cases} \\ C_2 = \dfrac{1}{2\pi}\displaystyle\int_0^\pi 2\sin 2x \sin 2x\,dx = \ldots = \dfrac{1}{2} \end{array}\right\}$ $\Rightarrow$ so $u_2(x) = \frac{1}{2}\sin(2x)$

The sol. of the problem is: $u(x) = u_1(x) + u_2(x) = \sin x + \frac{1}{2}\sin(2x)$

c) $\begin{cases} -\Delta u = f & \text{in } (0,\pi)\times(0,\pi) \\ u = 0 & \text{on } \partial\Omega \end{cases}$

$$u = \sum_{k,j=1}^{\infty} \frac{(f,\phi_{kj})_{L^2}}{\lambda_{kj}}\phi_{kj} = \sum_{k,j=1}^{\infty} \underbrace{\frac{(f,\phi_{kj})_{L^2}}{\lambda_{kj}}\cdot\frac{2}{\pi}}_{:= C_{kj}} \sin(kx)\sin(jy)$$

$$C_{kj} = \frac{2}{\pi}\cdot\frac{1}{k^2+j^2}\int_0^\pi\int_0^\pi \underbrace{f(x,y)}_{= \sin x \sin 3y}\frac{2}{\pi}\sin(kx)\sin(jy)\,dx\,dy = \begin{cases} 0 & \text{if } k \neq 1 \text{ or } j \neq 3 \\ ? & \text{if } k = 1 \ \& \ j = 3 \end{cases}$$

$$C_{13} = \frac{1}{1^2+3^2}\int_0^\pi\int_0^\pi \underbrace{\frac{2}{\pi}\sin x \sin 3y}_{\phi_{13}}\,\underbrace{\frac{2}{\pi}\sin x \sin 3y}_{\phi_{13}}\,dx\,dy = \frac{1}{10}\underbrace{(\phi_{13},\phi_{13})_{L^2}}_{=1} \Rightarrow C_{13} = \frac{1}{10}$$

$\Rightarrow u = \frac{1}{10}\sin x \sin 3y$

SEMINARY 11

THE CAUCHY-DIRICHLET PROB FOR THE HEAT EQ

1 Solve the BVP $\begin{cases} \frac{\partial u}{\partial t} - \Delta u = f(x,t), & 0<x<\pi, \ t>0 \\ u(x,0) = g_0(x), & 0<x<\pi \\ u(0,t) = u(\pi,t) = 0, & t>0 \end{cases}$ in the following cases: a) $f=0$, $g_0 = A\sin 3x$ b) $f=0$, $g_0 = x(\pi - x)$ c) $f = t\sin x$, $g_0 = 0$ d) $f = \sin 2x$, $g_0 = \sin^3 x$

Theory: $u = \sum_{k=1}^{\infty} u_k(t)\phi_k(x)$ where $\phi_k : \begin{cases} -\Delta\phi_k = \lambda_k\phi_k & \text{in } \Omega \\ \phi_k = 0 & \text{on } \partial\Omega \end{cases}$ and on $(0,a)$ we know that $\lambda_k = \left(\frac{k\pi}{a}\right)^2$, $\phi_k = \sqrt{\frac{2}{a}}\sin\frac{k\pi x}{a}$

To find u_k we solve the Cauchy pb. $\begin{cases} u_k'(t) + \lambda_k u_k(t) = f_k(t) \\ u_k(0) = g_0^k \end{cases}$ $k = 1, 2, \dots$ where $f_k = (f(\cdot, t), \phi_k)_{L^2}$, $g_0^k = (g_0, \phi_k)_{L^2}$

a) $f_k = 0, \ \forall k$

$g_0^k = 0, \ \forall k, \ k \neq 3 \Rightarrow g_0^3 = \left(A\sin 3x, \sqrt{\frac{2}{\pi}}\sin 3x\right)_{L^2} = A\sqrt{\frac{\pi}{2}}\underbrace{\left(\overbrace{\sqrt{\frac{2}{\pi}}\sin 3x}^{\phi_3}, \overbrace{\sqrt{\frac{2}{\pi}}\sin 3x}^{\phi_3}\right)_{L^2}}_{=1} = A\sqrt{\frac{\pi}{2}}$

So $u = u_3(t)\phi_3(x)$ u_3 is the sol of the IVP:

$\left.\begin{cases} u' + 9u = 0 \\ u(0) = A\sqrt{\frac{\pi}{2}} \end{cases} \Rightarrow \begin{matrix} \frac{du}{dt} = -9u \Rightarrow \dots \Rightarrow u = Ce^{-9t} \\ u(0) = A\sqrt{\frac{\pi}{2}} \end{matrix}\right\} \Rightarrow u_3 = A\sqrt{\frac{\pi}{2}}e^{-9t}$

Finally, our sol. to our BVP is: $u = A\sqrt{\frac{\pi}{2}}e^{-9t}\cdot\sqrt{\frac{2}{\pi}}\sin 3x \Rightarrow \boxed{u = Ae^{-9t}\sin 3x}$

c) $g_0^k = 0, \ \forall k$

$f_k = 0, \ \forall k, \ k \neq 1 \Rightarrow f_1 = (f, \phi_1)_{L^2} = \left(t\sin x, \sqrt{\frac{2}{\pi}}\sin x\right)_{L^2} = t\sqrt{\frac{\pi}{2}}\overbrace{\left(\sqrt{\frac{2}{\pi}}\sin x, \sqrt{\frac{2}{\pi}}\sin x\right)_{L^2}}^{=1} = t\sqrt{\frac{\pi}{2}}$

$u = u_1(t)\phi_1(x)$; u_1 is the sol. of the IVP: $\begin{cases} u' + u = t\sqrt{\frac{\pi}{2}} \\ u(0) = 0 \end{cases}$

For $\left.\begin{matrix} u' + u = 0 \Rightarrow u_h = Ce^{-t} \\ u_p = at + b \Rightarrow \dots \Rightarrow a = \sqrt{\frac{\pi}{2}}, \ b = -\sqrt{\frac{\pi}{2}} \end{matrix}\right\}$ $\Rightarrow u = u_h + u_p$, $u(0) = 0 \Rightarrow \dots \Rightarrow C = \dots$

$\Rightarrow u_1 = (e^{-t} + t - 1)\sqrt{\frac{\pi}{2}} \Rightarrow \boxed{u = (e^{-t} + t - 1)\sin x}$

d) $\boxed{\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x} = g_0$

$f_k = 0, \ \forall k, \ k \neq 2$

$g_0^k = 0, \ \forall k, \ k \neq 1, 3$

$f_2 = \dots = \sqrt{\frac{\pi}{2}}$

$g_0^1 = \dots = \frac{3}{4}\sqrt{\frac{\pi}{2}}$

$g_0^3 = \dots = \frac{1}{4}\sqrt{\frac{\pi}{2}}$

$u = u_1\phi_1 + u_2\phi_2 + u_3\phi_3$

$\left.\begin{matrix} k=1 \begin{cases} u' + u = 0 \\ u(0) = \frac{3}{4}\sqrt{\frac{\pi}{2}} \end{cases} & k=2 \begin{cases} u' + 4u = \sqrt{\frac{\pi}{2}} \\ u(0) = 0 \end{cases} & k=3 \begin{cases} u' + 9u = 0 \\ u(0) = \frac{1}{4}\sqrt{\frac{\pi}{2}} \end{cases} \end{matrix}\right\} \Rightarrow \dots \Rightarrow$

$$\boxed{u = \frac{3}{4}e^{-t}\sin x + \frac{1}{4}(1 - e^{-4t})\sin 2x - \frac{1}{4}e^{-9t}\sin 3x}$$

b) $g_0 = x(\pi - x)$, $f = 0$

$f_k = 0, \forall k$

$g_0^k = \left(x(\pi - x), \sqrt{\frac{2}{\pi}} \sin kx \right)_{L^2(0,\pi)} = \sqrt{\frac{2}{\pi}} \int_0^\pi x(\pi - x) \sin kx \, dx$

$I_1 = \pi \int_0^\pi x \sin kx \, dx = \pi \cdot \frac{1}{k} x(-\cos xk) \Big|_0^\pi + \pi \cdot \frac{1}{k} \int_0^\pi \cos kx \, dx = \frac{\pi}{k} (-1)^{k+1}$

$I_2 = \int_0^\pi x^2 \sin kx \, dx = \ldots = (-1)^{k+1} \frac{\pi^2}{k} + \frac{2}{k^2} (-1)^k = 0$

$g_0^k = \sqrt{\frac{2}{\pi}} \left[\frac{\pi^2}{k} (-1)^{k+1} - \frac{\pi^2}{k} (-1)^{k+1} - \frac{2}{k^3} ((-1)^k - 1) \right] = \begin{cases} 0, & k = 2p \\ \frac{4}{(2p-1)^3} \sqrt{\frac{2}{\pi}}, & k = 2p-1 \end{cases}$

$u_{2p} = 0$

In order to find u_{2p-1}, we solve : $\begin{cases} u' + (2p-1)^2 u = 0 \\ u(0) = g_0^k \end{cases}$ $\Rightarrow \ldots \Rightarrow u_{2p-1} = \sqrt{\frac{2}{\pi}} \cdot \frac{4}{(2p-1)^3} e^{-(2p-1)^2 t}$

$\Rightarrow \boxed{u = 4 \sum_{p=1}^{\infty} \frac{1}{(2p-1)^3} e^{-(2p-1)^2 t} \sin((2p-1)x)}$

THE CAUCHY-DIRICHLET PROB FOR THE WAVE EQ.

1. Solve the BVP $\begin{cases} u_{tt} - u_{xx} = 0, & 0 < x < l, \ t > 0 \\ u(x,0) = g_0(x) & , 0 < x < l \\ u_t(x,0) = g_1(x) & \\ u(0,t) = u(l,t) = 0 & , t > 0 \end{cases}$ in the cases : a) $g_0 = A \sin \frac{\pi x}{l}$, $g_1 = 0$

b) $g_0 = x(l - x)$, $g_1 = 0$

Theory: $u = \sum_{k=1}^{\infty} u_k(t) \phi_k(x)$ where : $-\phi_k''(x) = \lambda_k \phi_k(x)$ and on $(0,l)$: $\lambda_k \left(\frac{k\pi}{l}\right)^2$, $\phi_k = \sqrt{\frac{2}{l}} \sin \frac{k\pi x}{l}$

$u_k(t) = g_0^k \cos(\sqrt{\lambda_k} t) + \frac{g_1^k}{\sqrt{\lambda_k}} \sin(\sqrt{\lambda_k} t)$; $g_0^k = (g_0, \phi_k)_{L^2}$; $g_1^k = (g_1, \phi_k)_{L^2}$

a) $g_1^k = 0, \forall k$

$g_0^k = 0, \forall k, k \neq 1 \Rightarrow g_0^k = \left(A \sin \frac{\pi x}{l}, \sqrt{\frac{2}{l}} \sin \frac{\pi x}{l} \right)_{L^2} = \ldots = A \sqrt{\frac{l}{2}}$ $\Bigg\} \Rightarrow u = u_1(t) \phi_1(x)$

$u_1 = A \sqrt{\frac{l}{2}} \cos \frac{\pi}{l} t \Rightarrow u = A \cancel{\sqrt{\frac{l}{2}}} \cos \frac{\pi}{l} t \cdot \cancel{\sqrt{\frac{2}{l}}} \sin \frac{\pi}{l} x \Rightarrow \boxed{u = A \cos \frac{\pi}{l} t \cdot \sin \frac{\pi}{l} x}$

2. Solve the BVP $\begin{cases} u_{tt} - a^2 u_{xx} = f(x,t), & 0 < x < l, \ t > 0 \\ u(x,0) = g_0(x), \ u_t(x,0) = g_1(x), & 0 < x < l \\ u(0,t) = u(l,t) = 0, & t > 0 \end{cases}$ in the cases: a) $f = A \sin\left(\frac{\pi}{l} x\right)$, $g_0 = B \sin \frac{2\pi}{l} x$, $g_1 = 0$

b) $f = A$, $g_0 = g_1 = 0$

Theory: $u = \sum_{k=1}^{\infty} u_k(t) \phi_k(t)$; λ_k, ϕ_k as in pb. 1 ; g_0^k, g_1^k as in pb. 1 ; $f_k = (f(\cdot,t), \phi_k)_{L^2}$

In order to find $u_k(t)$ we need to solve the IVP : $\begin{cases} u_k''(t) + a^2 \lambda_k u_k(t) = f_k(t) \\ u_k(0) = g_0^k \\ u_k'(0) = g_1^k \end{cases}$

a) $g_1^k = 0, \forall k$

$g_0^k = 0, \forall k, k \neq 2 \Rightarrow g_0^2 = \ldots$ $\Bigg\} \Rightarrow u = u_1(t) \phi_1(x) + u_2(t) \phi_2(x)$

$f_k = 0, \forall k, k \neq 1 \Rightarrow f_1 = \ldots$

$$k=1 \quad \begin{cases} u_1''(t) + a^2\left(\frac{\pi}{\ell}\right)^2 u_1(t) = f_1 \\ u_1(0) = 0 \\ u_1'(0) = 0 \end{cases}$$

$$k=2 \quad \begin{cases} u_2''(t) + a^2\left(\frac{2\pi}{\ell}\right)^2 u_2(t) = 0 \\ u_2(0) = g_0^2 \\ u_2'(0) = 0 \end{cases}$$

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The Fourier Transform

- $f \in L^n(\mathbb{R}^n)$

  $$T[f](y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-ixy}\, dx$$

- Heaviside's function

  $$H(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases}$$

- $h(x) = e^{-1/2 x^2}$, $T[h] = h$
- $T[f(x-x_0)](y) = e^{-ix_0 y}\, T[f](y)$
- $T[f(\lambda x)](y) = \frac{1}{|\lambda|} T[f](y/\lambda)$, $\lambda \neq 0$.

Exercise 1:

$$T[e^{-1/2(x-1)^2}](y) = T[h(x-1)](y) =$$
$$= e^{-i \cdot 1 \cdot y}\, T[h](y)$$
$$= e^{-iy} h(y) = e^{-iy} e^{-1/2 y^2} = e^{-iy - 1/2 y^2}$$

Exercise 2:

$T[e^{-x^2}](y) =$ Hw / ex făcut la curs.

Exercise 3:

$$T[e^{-4(x-2)^2}](y) = T[e^{-1/2 \cdot 8 (x-2)^2}](y) =$$
$$= T[e^{-1/2 (2\sqrt{2})^2 (x-2)^2}](y) =$$
$$= T[e^{-1/2 (2\sqrt{2}(x-2))^2}](y) =$$
$$= T[h(2\sqrt{2}(x-2))](y) =$$
$$= \frac{1}{2\sqrt{2}} T[h(x-2)](y/2\sqrt{2}) =$$
$$= \frac{1}{2\sqrt{2}} e^{-i2y}\, T[h](y/2\sqrt{2}) = \frac{1}{2\sqrt{2}} e^{-i2y} \cdot h(y/2\sqrt{2}) =$$

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$$= \frac{1}{2\sqrt{2}} e^{-i2y} \cdot e^{-1/2 \cdot \frac{y^2}{8}} =$$

$$= \frac{1}{2\sqrt{2}} e^{-i2y - \frac{1}{16}y^2}$$

Exercise 4:

Find the Fourier Transform of $H(R-|x|)$ for $R>0$.

Solution:

$$H(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases}$$

$$H(R-|x|) = \begin{cases} 0, & R-|x| \leq 0 \\ 1, & R-|x| > 0 \end{cases} = \begin{cases} 0, & R \leq |x| \\ 1, & R > |x| \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} 0, & |x| \geq R \\ 1, & |x| < R \end{cases} \Rightarrow \begin{cases} 0, & x \notin (-R,R) \\ 1, & x \in (-R,R) \end{cases}$$

$$\mathcal{F}[H(R-|x|)](y) = \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} e^{-ixy}\, dx = \frac{1}{\sqrt{2\pi}} \left. \frac{e^{-ixy}}{-iy} \right|_{-R}^{R} =$$

$e^{it} = \cos t + i \sin t$

$$= -\frac{1(e^{-iRy} - e^{+iRy})}{\sqrt{2\pi}(iy)} =$$

$$= -\frac{1}{\sqrt{2\pi}(iy)} (\cancel{\cos(-Ry)} + i\sin(-Ry) - \cancel{\cos(Ry)} - i\sin(Ry))$$

$$= +\frac{2i\sin(Ry)}{\sqrt{2\pi}(iy)} = \frac{\sqrt{2}}{\sqrt{\pi}} \frac{\sin(Ry)}{y}$$

Exercise 5:

Let $u(x) = H(x)e^{-x}$. Find the Fourier Transforms of:

$u(x)$, $u(-x)$, $u(x-x_0)$, $e^{ixx_0} u(x)$, $u(x)\sin x$.

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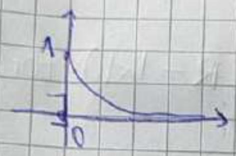
Solution:

$$u(x) = \begin{cases} 0 & , x \leq 0 \\ e^{-x} & , x > 0 \end{cases}$$

(some ideas):



$$T[u](y) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-x} e^{-ixy} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-x(1+iy)} dx =$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{e^{-x(1+iy)}}{-(1+iy)} \Big|_0^\infty = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{1+iy}$$

$u(-x) = u(-1 \cdot x)$ [struck out]

Exercise 6:

Recall that $T[f*g] = \sqrt{2\pi}\, T[f]T[g]$ and $f*g = \sqrt{2\pi}\, T^{-1}[T[f]T[g]]$. Find $f*g$ if $T[f](y)T[g](y) = 4e^{-1/2 y^2}$

Solution:

$$(f*g)(y) = \sqrt{2\pi}\, T^{-1}[4h](y) =$$

$$\boxed{T^{-1}(h) = h}$$

$$= 4\sqrt{2\pi}\, T^{-1}[h](y) =$$

$$= 4\sqrt{2\pi}\, h(y) = \text{substitution} \ldots$$

Recall that:

$$\partial^\beta T[f](y) = T[(-ix)^\beta f(x)](y)$$

$$T[\partial^\beta f](y) = (iy)^\beta T[f](y)$$

$$\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n, \ f \in \mathcal{S}, \ y \in \mathbb{R}^n$$

Exercise 7:

let $f \in \mathcal{S}$, $T[f](y) = (y^2+1)e^{-1/2 y^2}$

Solve $u'' - u = f$ in $\mathcal{S}$

Solution:

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$T[u'' - u] = T[f]$, T linear

$T[u''] - T[u] = T[f]$

$(iy)^2 T[u](y) - T[u](y) = T[f](y)$

$(-y^2 - 1) T[u](y) = T(f)(y)$

$\cancel{(-y^2-1)} T[u](y) = \cancel{(y^2+1)} e^{-1/2 y^2}$

$-T[u](y) = e^{-1/2 y^2}$

$-T[u](y) = T[h](y)$

$T[u](y) = -T[h](y) = T[-h](y)$

$\xrightarrow{T\,inv.} u(y) = -h(y)$