Lecture 9

Systems of linear diff. equations

$$\begin{cases} y_1' = a_{11}(x)\cdot y_1(x) + \ldots + a_{1n}(x)\cdot y_n(x) + b_1(x) \\ \vdots \\ y_n' = a_{n1}(x)\cdot y_1(x) + \ldots + a_{nn}(x)\cdot y_n(x) + b_n(x) \end{cases}$$

a_{ij}, b_i are cont. functions.

$$\underline{Y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$\Rightarrow$ $\boxed{\underline{Y}' = A\cdot\underline{Y} + B}$ the vectorial form of the system.

$A \in C(I, \mathcal{M}_{nn}(\mathbb{R}))$

$B \in C(I, \mathbb{R}^n)$

(1) $Y' - AY = B$ the nonhomogeneous system

(2) $Y' - AY = 0$ the homogeneous system.

Theorem 1 The IVP:

$$\begin{cases} Y' - AY = B \\ Y(a) = \eta, \quad \eta \in \mathbb{R}^n, \quad a \in I \end{cases}$$

has an unique solution.

$Y(\cdot\,; a, \eta)$ — the unique sol. of IVP.

The homogeneous case

(2) $Y' - AY = 0$

$$S_0 = \{ Y \in C^1(I, \mathbb{R}^n) \mid Y \text{ sol. of } (2) \}$$

$$L : C^1(I, \mathbb{R}^n) \to C(I, \mathbb{R}^n)$$

$$Y \mapsto LY$$

$$LY = Y' - AY$$

op. L is a linear op.

$S_0 = \ker L$

<u>Theorem 2</u> S_0 is a linear subspace of the linear space $C^1(I, \mathbb{R}^n)$ with $\dim S_0 = n$.

<u>Proof</u>. $\left.\begin{array}{l} S_0 = \ker L \\ L \text{ linear op} \end{array}\right\} \Rightarrow S_0$ is a linear subspace of $C^1(I, \mathbb{R}^n)$

$\dim S_0 \overset{?}{=} n$

$$\varphi : \mathbb{R}^n \to S_0 \qquad \left\{\begin{array}{l} Y' - AY = 0 \\ Y(a) = \eta \end{array}\right.$$
$$\eta \longmapsto Y(\cdot\,; a, \eta)$$

From Th 1 $\Rightarrow$ φ is bijective.

φ is a linear isomorphism of linear spaces

$$\varphi(\lambda_1 \eta^1 + \lambda_2 \eta^2) \overset{?}{=} \lambda_1 \varphi(\eta^1) + \lambda_2 \varphi(\eta^2), \quad \lambda_1, \lambda_2 \in \mathbb{R}$$
$$\eta^1, \eta^2 \in \mathbb{R}^n.$$

$\varphi(\lambda_1 r^1 + \lambda_2 r^2)$ is the sol $Y(\cdot; a, \lambda_1 r^1 + \lambda_2 r^2)$

$\varphi(r^1)$ is the sol $Y(\cdot; a, r^1)$

$\varphi(r^2)$ is the sol $Y(\cdot; a, r^2)$

$$\left.\begin{array}{l} V = \lambda_1 Y(\cdot; a, r^1) + \lambda_2 \cdot Y(\cdot; a, r^2) \\ S_0 \text{ is a linear subspace} \end{array}\right\} \Rightarrow V \in S_0$$

$$V(a) = \lambda_1 \cdot Y(a; a, r^1) + \lambda_2 \cdot Y(a; a, r^2) =$$

$$= \lambda_1 \cdot r^1 + \lambda_2 \cdot r^2$$

V is a sol. of the IVP $\begin{cases} LY = 0 \\ Y(a) = \lambda_1 r^1 + \lambda_2 r^2 \end{cases}$

$$\Rightarrow V = Y(\cdot; a, \lambda_1 r^1 + \lambda_2 r^2)$$

Th 1

$$\Rightarrow \lambda_1 \underbrace{Y(\cdot; a, r^1)}_{\varphi(r^1)} + \lambda_1 \cdot \underbrace{Y(\cdot; a, r^2)}_{\varphi(r^2)} = \underbrace{Y(\cdot; a, \lambda_1 r^1 + \lambda_2 r^2)}_{\varphi(\lambda_1 r^1 + \lambda_2 \cdot r^2)}$$

$\dim S_0 = n \Rightarrow \exists \{Y^1, Y^2, \dots, Y^n\} \subset S_0$ a basis

$\forall Y \in S_0 \quad \exists c_1, \dots, c_n \in \mathbb{R}$ such that

$$Y = c_1 \cdot Y^1 + c_2 Y^2 + \dots + c_n \cdot Y^n$$

we denote by $U = (Y^1 \; Y^2 \dots Y^n)$

$$\Rightarrow Y = U \cdot \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

$\{Y^1, Y^2, \dots, Y^n\}$ a basis in $S_0 \Leftrightarrow \{Y^1, Y^2, \dots, Y^n\}$ fundam. system of sol.

the matrix $U = (Y^1 \; Y^2 \dots Y^n)$ is called the fundam matrix of solutions.

To solve the system (2) means to find a fundam. matrix of sol.

$\{Y^1, Y^2, \dots, Y^n\} \subset S_0$ is a basis in S_0 $\Leftrightarrow$

$\Leftrightarrow$ $\{Y^1, \dots, Y^n\}$ is linearly indep. system of funct.

<u>Def</u>.

a) $Y^1, \dots, Y^n$ are linearly dependent $\Leftrightarrow$ $\exists\, (c_1, \dots, c_n) \neq (0, \dots, 0)$ such that $c_1 Y^1 + \dots + c_n Y^n = 0$

b) $Y^1, \dots, Y^n$ are linearly independent $\Leftrightarrow$

$\Leftrightarrow$ $c_1 Y^1 + \dots + c_n Y^n = 0 \Rightarrow c_1 = \dots = c_n = 0$

$$W(x; Y^1, \dots, Y^n) = \begin{vmatrix} y_1^1 & \dots & y_1^n \\ y_2^1 & & y_2^n \\ \vdots & & \vdots \\ y_n^1 & \dots & y_n^n \end{vmatrix}$$

the wronskian of $Y^1, \dots, Y^n$

$$Y^i = \begin{pmatrix} y_1^i \\ \vdots \\ y_n^i \end{pmatrix} \quad i = 1, \dots, n.$$

Theorem 3

a) If $Y^1, \dots, Y^n \in C(I, \mathbb{R}^n)$ are linearly dependent $\Rightarrow$
$\Rightarrow W(\cdot; Y^1, \dots, Y^n) \equiv 0$ on I.

b) If $Y^1, \dots, Y^n \in S_0$ are linearly independent $\Rightarrow$
$\Rightarrow W(x; Y^1, \dots, Y^n) \neq 0$, $\forall x \in I$.

Proof. a) $Y^1, \dots, Y^n$ are linearly dependent $\Rightarrow$
$\Rightarrow$ one function can be obtained as a linear combination of other $n-1$ functions.
$\Rightarrow$ one column of W is a linear combination of othe $n-1$ columns.
$\Rightarrow W(\cdot; Y^1, \dots, Y^n) \equiv 0$ on I.

b) suppose that $\exists\, x_0 \in I$ such that $W(x_0; Y^1, \dots, Y^n) = 0$

we consider the system:

$$\underbrace{\begin{pmatrix} \underline{Y}^1(x_0) & \underline{Y}^2(x_0) & \dots & \underline{Y}^n(x_0) \end{pmatrix}}_{A} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = 0 \quad (3)$$

$$\det A = W(x_0; \underline{Y}^1, \dots, \underline{Y}^n) = 0 \Rightarrow$$

$\Rightarrow \exists \begin{pmatrix} \tilde{c}_1 \\ \tilde{c}_2 \\ \vdots \\ \tilde{c}_n \end{pmatrix} \neq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ sol. of the system (3)

we consider the function

$$\left.\begin{matrix} \tilde{\underline{Y}} = \begin{pmatrix} \underline{Y}^1 & \dots & \underline{Y}^n \end{pmatrix} \begin{pmatrix} \tilde{c}_1 \\ \vdots \\ \tilde{c}_n \end{pmatrix} \\ \underline{Y}^1, \dots, \underline{Y}^n \in S_0 \end{matrix}\right\} \Rightarrow \tilde{\underline{Y}} \in S_0$$

$\tilde{\underline{Y}}$ is a sol. of IVP $\begin{cases} L\underline{Y} = 0 \\ \underline{Y}(x_0) = \tilde{c}_1 \underline{Y}^1(x_0) + \dots + \tilde{c}_n \underline{Y}^n(x_0) = 0 \end{cases}$

$\underset{\text{Th. 1}}{\Rightarrow} \tilde{\underline{Y}} \equiv 0 \Rightarrow \tilde{c}_1 \underline{Y}^1 + \dots + \tilde{c}_n \underline{Y}^n = 0 \Rightarrow \underline{Y}^1, \dots, \underline{Y}^n$ are linearly dep. $\Rightarrow$ contrad.

In S_0 we have the following possibilities:

- if $Y^1, \dots, Y^n \in S_0$ are linearly dependent $\Rightarrow$

$$\Rightarrow W(x; Y^1, \dots, Y^n) = 0, \quad \forall x \in I.$$

- if $Y^1, \dots, Y^n \in S_0$ are linearly indep. $\Rightarrow$

$$\Rightarrow W(x; Y^1, \dots, Y^n) \neq 0, \quad \forall x \in I.$$

<u>Theorem 4</u> (The wronskian criterion)

$\{Y^1, \dots, Y^n\} \subset S_0$ is a fundam. system of solutions for (2)

$\Leftrightarrow$ $\exists x_0 \in I$ such that $W(x_0; Y^1, \dots, Y^n) \neq 0$

The nonhomogeneous case

(1) $\quad Y' - AY = B \qquad A \in C(I, \mathcal{M}_n(\mathbb{R}))$

$B \in C(I, \mathbb{R}^n)$

(1) $\Leftrightarrow LY = B$

$S' = \{ Y \in C^1(I, \mathbb{R}^n) \mid Y \text{ sol. of } (1) \}$

$$\boxed{S' = S_0 + \{Y^p\}}$$

where S_0 is a the sol. set of (2) $LY = 0$

Y^p is a particular sol. of (1)

if U is a fundam. matrix of sol. $\Rightarrow$

$\Rightarrow$ the gen. sol. of (1)

$$\boxed{Y = U \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} + Y^p, \quad c_1, \dots, c_n \in \mathbb{R}}$$

$\underline{y}^p = ?$

<u>The variation</u> of <u>the constants method</u>

we try to find $\underline{y}^p(x) = U(x) \cdot \begin{pmatrix} \varphi_1(x) \\ \vdots \\ \varphi_n(x) \end{pmatrix}$

$$(\underline{y}^p)' - A \cdot \underline{y}^p = B$$

$$\left(U(x) \cdot \begin{pmatrix} \varphi_1(x) \\ \vdots \\ \varphi_n(x) \end{pmatrix} \right)' - A \cdot U(x) \cdot \begin{pmatrix} \varphi_1(x) \\ \vdots \\ \varphi_n(x) \end{pmatrix} = B(x)$$

<u>we have :</u>

1. $\left(U \cdot \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{pmatrix} \right)' = U' \cdot \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{pmatrix} + U \cdot \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{pmatrix}'$

2. U fundam. matrix of sol. $\Rightarrow U' - A \cdot U = 0$

$$\underline{U' \cdot \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{pmatrix}} + U \cdot \begin{pmatrix} \varphi_1' \\ \vdots \\ \varphi_n' \end{pmatrix} - \underline{A \cdot U \cdot \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{pmatrix}} = B$$

$$\underbrace{(U' - AU)}_{=0}\begin{pmatrix}\varphi_1\\ \vdots\\ \varphi_n\end{pmatrix} + U.\begin{pmatrix}\varphi_1'\\ \vdots\\ \varphi_n'\end{pmatrix} = B$$

$$\underset{U^{-1}}{\Rightarrow} \Big/ \quad U.\begin{pmatrix}\varphi_1'\\ \vdots\\ \varphi_n'\end{pmatrix} = B$$

$\det U(x) = W(x; Y^1, \dots, Y^n) \neq 0, \ \forall x \in I.$

$$\Rightarrow \begin{pmatrix}\varphi_1'\\ \vdots\\ \varphi_n'\end{pmatrix} = U^{-1}.B$$

$$\Rightarrow \begin{pmatrix}\varphi_1(x)\\ \vdots\\ \varphi_n(x)\end{pmatrix} = \int_{x_0}^{x} U^{-1}(s).B(s)\,ds$$

$$\Rightarrow Y^p(x) = U.\begin{pmatrix}\varphi_1\\ \vdots\\ \varphi_n\end{pmatrix} \Rightarrow \boxed{Y^p(x) = U(x).\int_{x_0}^{x} U^{-1}(s).B(s)\,ds.}$$

$$\boxed{Y = Y^o + Y^p}$$