MATHEMATICAL LOGIC

Lecture 1

09.10.2023

Organization

- manual in the Files section
- http://math.ubbcluj.ro/~marcus
- exam – in the exam session Jan – Feb 2023
  - duration 2 hours; 4 questions, 1–10.
    + bonus points at the seminar
- attendance: 75% of the seminars (9 seminars) (rule of UBB)
- prerequisites: logic, sets, functions 9th grade

Introduction

- we formally analyse mathematical proofs
- we apply mathematical methods to logic
- This course is <u>not</u> – philosophical logic → Fac. of History and Philosophy
  - computational logic → C.S. dept.
- we mostly do set theory

Chapter 1 Propositional logic

"Naively", a proposition (sentence) is a statement which is known to be <u>true</u> or <u>false</u>. We may also form composite sentences, by using words like or, and, not, ...

Aristotle — creator of formal logic - Organon

The above point of view is not satisfying.

We need to introduce a formal language.

A. Formulas

Def. The language of Propositional logic consists of:

1) symbols a) parentheses (,)

b) connectives

$\neg$ not, non (negation), $\vee$ or (disjunction), $\wedge$ and (conjunction), $\to$ if...then (implication) (conditional) $\supset$, $\leftrightarrow$ if and only if biconditional (equivalence) $\equiv$

c) atoms (atomic formulas) $a, b, c, x_1, x_2, p, q, \ldots$

(we fix an alphabet $\mathcal{A}$)

2) Formulas — are obtained recursively (by induction) as follows:

a) atomic formulas are formulas

b) if A and B are formulas, then:

$(\neg A)$, $(A \vee B)$, $(A \wedge B)$, $(A \to B)$, $(A \leftrightarrow B)$

are also formulas.

Examples 1) $\underbrace{(\underbrace{(\underbrace{(\neg p)} \to q)} \vee \underbrace{(\underbrace{(p \vee \underbrace{(\neg r)})} \leftrightarrow q)})}$

we recognize subformulas

Remark In practice, we omit some parentheses:

- we assign a priority order to connectives

  I $\neg$

  II $\vee$, $\wedge$

  III $\rightarrow$ $\longleftrightarrow$

- we omit the external parentheses

Then, the above formula becomes:

$$(\neg p \rightarrow q) \vee ((p \vee \neg r) \longleftrightarrow q)$$

2) $(p \rightarrow \vee r)\neg\, p q$ is not a formula

3) $((\neg p))$ <u>not</u> a formula; if A is a formula, then (A) is <u>not</u> a formula!

B. Interpretation of Formulas. Truth values

We assign a <u>truth value</u> to each formula as follows:

- we assume we are given a function.

$$\underset{\text{alphabet}}{v : \mathcal{A}} \longrightarrow \{0, 1\}$$

0 = false, 1 = true

- the truth value of a composite formula is given by the following <u>truth tables</u>;

A	$\neg A$
0	1
1	0

A	B	$A \vee B$	$A \wedge B$	$A \rightarrow B$ (logic cond.)	$A \longleftrightarrow B$
0	0	0	0	1	1
0	1	1	0	1	0
1	0	1	0	0	0
1	1	1	1	1	1

Relations between formulas

Def Let A, B be formulas.

a). We say that A implies B, or that B is a consequence of A if $v(A \to B) = 1$

not. $A \Longrightarrow B$

b) We say that A and B are equivalent if $v(A \leftrightarrow B) = 1$. not $A \Longleftrightarrow B$

Def The formula A is called a:

a) tautology if $v(A) = 1$ for any interpretation $v : \mathcal{A} \to \{0, 1\}$

b) contradiction if $v(A) = 0$ for any interpretation

c) satisfiable formula if there is at least one interpretation for which $v(A) = 1$

C. The decision problem

Given a formula A, decide whether A is a taut, contrad, or is satisfiable.

Methods for solving the decision problem:

1). By using truth tables

2) By using normal forms

Def a). We say that A has disjunctive normal form (DNF) if $A = A_1 \vee \dots \vee A_n$ is a disjunction of <u>elementary conjunctions</u>, i.e. (identity that is) $A_j = B_1 \wedge \dots \wedge B_m$, where each B_k is an atom or a negation of an atom

b) We say that A has <u>conjunctive normal form</u> (CNF) if $A = A_1 \wedge \dots \wedge A_n$ is a conjunction of <u>elementary disjunctions</u>, i.e. $A_j = B_1 \vee \dots \vee B_m$, where each B_k is an atom or a negation of an atom.

<u>Theorem</u> For any formula A, there is a formula B in DNF (or CNF) such that $A \iff B$

<u>Remark</u>. B is obtained by using the following <u>fundamental tautologies</u>:

1). $A \leftrightarrow B \iff (A \to B) \wedge (B \to A)$ (law of equivalence)

2) $A \to B \iff \neg A \vee B$ (law of implication)

3) $\neg(A \vee B) \iff \neg A \wedge \neg B$ (De Morgan laws)
$\neg(A \wedge B) \iff \neg A \vee \neg B$

4) $\neg\neg A \iff A$ (law of double negation)

5). $A \vee \neg A \iff 1$ (law of excluded middle)
tertium non datur

6) $A \wedge \neg A \iff 0$ (law of non contradiction)

7). $A \underset{\wedge}{\vee} B \iff B \underset{\wedge}{\vee} A$ (commutativity)

- $A \underset{\wedge}{\vee} (B \underset{\wedge}{\vee} C) \Leftrightarrow (A \underset{\wedge}{\vee} B) \underset{\wedge}{\vee} C$ (associativity)

- $A \underset{\wedge}{\vee} (B \underset{\vee}{\wedge} C) \Leftrightarrow (A \underset{\wedge}{\vee} B) \underset{\vee}{\wedge} (A \underset{\wedge}{\vee} C)$ (distributivity)

- $A \vee (A \wedge B) \Leftrightarrow A$
  $A \wedge (A \vee B) \Leftrightarrow A$ (absorption)

- $A \vee A \Leftrightarrow A$
  $A \wedge A \Leftrightarrow A$ (idempotence)

Remark normal forms are useful because we can easily analyze their truth values:

- $A_1 \vee \dots \vee A_n$ – true if at least one of the term is true
  – false if all the terms are false

- $A_1 \wedge \dots \wedge A_n$ – true if all the terms are true
  – false if at least one term is false

3) Formal deduction

We start with some formulas, called axioms and we obtain new formulas by using inference rules, such as:

Modus Ponens (MP):

$$\frac{A,\ A \to B}{B}$$

← premises

← conclusion

Remark The notation $\dfrac{A_1, \dots, A_n}{B}$ means:

$A_1 \wedge \dots \wedge A_n \Longrightarrow B$, i.e. the formula

$A_1 \wedge \dots \wedge A_n \longrightarrow B$ is a tautology.

Example check that (MP) is a valid inference rule.

Solution we consider the formula:

$C = (A \wedge (A \to B)) \to B$, and we check that this formula is a tautology;

- 1st method (truth tables)

A	B	$A \to B$	$A \wedge (A \to B)$	C
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

hence C is a tautology

- 2nd method (normal forms)

$$C = (A \wedge (A \to B)) \to B \underset{\text{l. impl.}}{\Longleftrightarrow} \neg(A \wedge (\neg A \vee B)) \vee B$$

$$\underset{\text{De Morg}}{\Longleftrightarrow} \neg A \vee \neg(\neg A \vee B) \vee B$$

$$\underset{\text{De Morg}}{\Longleftrightarrow} \neg A \vee (\neg\neg A \wedge \neg B) \vee B$$

$$\Longleftrightarrow \neg A \vee (A \wedge \neg B) \vee B \quad (DNF)$$

- if $A = 0 \Rightarrow \neg A = 1 \Rightarrow C = 1$
- if $B = 1 \Rightarrow C = 1$
- assume $A = 1$, $B = 0$. then $A \wedge \neg B = 1$, hence $C = 1$

hence C is tautology !

we continue the calculation to obtain a (CNF) from the above (DNF):

$$C \overset{\text{comm}}{\underset{\text{distrib.}}{\iff}} (\neg A \vee B \vee A) \wedge (\neg A \vee B \vee \neg B) \quad (CNF)$$

$$\iff (1 \vee B) \wedge (\neg A \vee 1)$$

$$\iff 1 \wedge 1 \iff 1$$

so C is a taut.

Homework ex 1 – 13 (manual)