Chapter 5. Ordered sets

Order relations

Def. An homogeneous relation $\rho = (A, A, R)$ is called an order relation if ρ is:

pre-order { (R) reflexive ($\forall x \in A \quad x \rho x$) i.e. $1_A \subseteq \rho$
(T) transitive ($\forall x, y, z \in A \quad x\rho y, y \rho z \implies x \rho z$). ($\rho \circ \rho \subseteq \rho$)

(A) antisymmetric ($\forall x, y \in A \quad x \rho y, y \rho x \implies x = y$)
i.e. $\rho \cap \rho^{-1} \subseteq 1_A$, in fact $\rho \cap \rho^{-1} = 1_A$

Rem an rel. which is R, T, S, A $\iff \rho = 1_A$

- If, in addition, ρ satisfies:
$\forall x, y \in A$ we have $x \rho y$ or $y \rho x$
(i.e. any two elements are comparable, ie $\rho \cup \rho^{-1} = A \times A$)
then we say that ρ is a total order

Terminology the pair (A, ρ) is called an ordered set (a totally ordered set) poset

(in many books: – order $\longrightarrow$ partial order
– total order $\longrightarrow$ order

Notations: for an order relation ρ: $\le$, $\subseteq$, $\preccurlyeq$, ...
$< = \lneq = \rho \setminus 1_A$
strict order

Examples 1). $(\mathbb{N}, \leq)$, $(\mathbb{Z}, \leq)$, $(\mathbb{Q}, \leq)$, $(\mathbb{R}, \leq)$

these are totally ordered sets.

2). $(\mathbb{Z}, |)$. $\left(a | b \overset{def}{\iff} \exists x \in \mathbb{Z} \text{ s.t. } b = ax \right)$.

we know that „|" on $\mathbb{Z}$ is R, T, $\cancel{S}$

$\boxed{\neg(p \to q) \iff p \wedge \neg q}$

(A) we study antisymmetry:

$\forall a, b \in \mathbb{Z}$, $a|b$, $b|a \overset{?}{\implies} a = b$

not in general! e.g. $3|-3$, $-3|3$, $3 \neq -3$

hence $(\mathbb{Z}, |)$ is a preordered set.

2') $(\mathbb{N}, |)$. it is an ordered set.

is it totally ordered?

$\forall x, y \in \mathbb{N}$ do we have $x|y$ or $y|x$?

no! e.g. $2 \nmid 3$, $3 \nmid 2$

3). Let M be any set and consider $(\mathcal{P}(M), \subseteq)$

we know that this is an ordered set.

To discuss total order, we consider cases

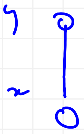
- $\mathcal{P}(\emptyset) = \{\emptyset\}$ totally ordered
- Let $M = \{1\}$, $\mathcal{P}(M) = \{\emptyset, M\}$ is totally ordered.
- Assume $|M| \geq 2$ say, $x, y \in M$, $x \neq y$.

$\{x\}$, $\{y\}$ are not comparable

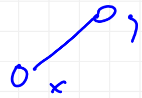
hence $(\mathcal{P}(M), \subseteq)$ is not totally ordered

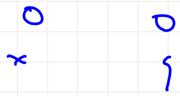
Hasse diagrams

(are used to visualize ordered sets with few elements)

conventions: 

means $x < y$, and there are no elements between x and y

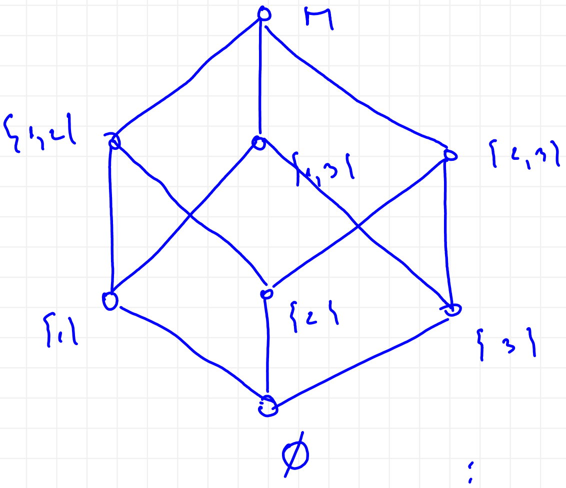
variant: 



means that x, y are incomparable.


makes no sense.

Example. 1) Let $M = \{1,2,3\}$. Draw the Hasse diagram of $(\mathcal{P}(M), \subseteq)$. We have $\mathcal{P}(M) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, M\}$



2). $(\mathbb{N}, \leq)$

totally ordered (also called chain)

Morphisms between ordered sets

Def Let (A, ρ), (B, σ) be two ordered sets, and let $f: A \to B$ be a function:

a). f is increasing (morphism of ordered sets). if

$$\forall x, x' \in A \quad x \rho x' \implies f(x) \sigma f(x')$$

a') f is strictly increasing if

$$\forall x, x' \in A \quad x \rho x', \ x \neq x' \implies f(x) \sigma f(x'), \ f(x) \neq f(x')$$

$(x < x')$ $\qquad f(x) < f(x')$

(i.e. f is increasing and injective)

b) f is decreasing (antimorphism). if

$$\forall x, x' \in A \quad x \rho x' \implies f(x') \sigma f(x)$$

b') f is strictly decreasing if

$$\forall x, x' \in A, \ x \rho x', \ x \neq x' \implies f(x') \sigma f(x), \ f(x) \neq f(x')$$

i.e. f is decreasing and injective

c). f is an isomorphism of ordered sets if

f is increasing, bijective, and f^{-1} is increasing

c') f is an anti-isomorphism of ordered sets if

f is decreasing, bijective and f^{-1} is decreasing

Example. Consider $1_{\mathbb{N}^*} : (\mathbb{N}^*, |) \longrightarrow (\mathbb{N}^*, \leq)$, $1_{\mathbb{N}^*}(x) = x$ $\forall x \in \mathbb{N}^*$

- this is bijective, $1_{\mathbb{N}^*}^{-1} = 1_{\mathbb{N}^*}$
- increasing? $x | y \overset{?}{\implies} x \leq y$ true! (morphism)
- $1_{\mathbb{N}^*} : (\mathbb{N}^*, \leq) \to (\mathbb{N}^*, |)$ is increasing?

$x \leq y \overset{?}{\implies} x | y$ no! in gen. e.g. $2 \leq 3$ and $2 \nmid 3$.

Special elements in ordered sets

Def Let $(A, \le)$ be an ordered set.

a) The elem't $a \in A$ is called *the least* (minimum) elem. of A (not. $a = \min A$) if $\forall x \in A \quad a \le x$.

a') $a \in A$ is *the largest elem't* (maximum) of A if $\forall x \in A \quad x \le a$

Rem $\min A$, $\max A$ do not always exist, but when they exist, they are unique!

b). The elem $a \in A$ is a *minimal element* if there are no strictly smaller elements:

i.e. $\forall x \in A \quad x \le a \Longrightarrow x = a$

b') the elem. $a \in A$ is a *maximal element* if there are no strictly larger elements:

i.e. $\forall x \in A \quad a \le x \Longrightarrow x = a$.

Examples 1). In $(\mathbb{N}, \le)$ $\quad \min(\mathbb{N}, \le) = 0$

$\nexists \max(\mathbb{N}, \le)$

2). $(\mathcal{P}(M), \subseteq)$ $\quad \min \mathcal{P}(M) = \emptyset$, $\max \mathcal{P}(M) = M$.

Now, let $(\mathcal{P}(M) \setminus \{\emptyset, M\}, \subseteq)$. Then $\nexists \min$, $\nexists \max$

- every subset with 1 elem, $\{x\}$, is minimal
- sets of the form $M \setminus \{x\}$ are maximal

3). $(\mathbb{N}, |)$ $\quad \min(\mathbb{N}, |) = 1$, bec. $1 | a \ \forall a \in \mathbb{N}$

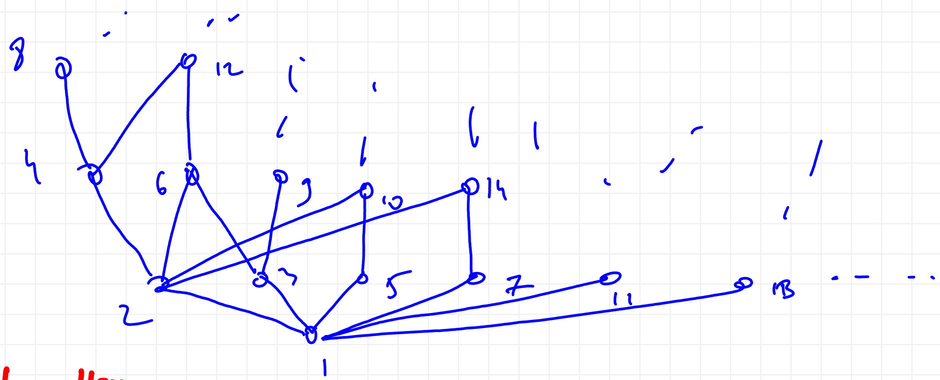
$\max(\mathbb{N}, |) = 0$, bec. $a | 0 \ \forall a \in \mathbb{N}$

Consider $(\mathbb{N} \setminus \{0,1\}, \mid)$.

$0 \mid 0$ is true
$0 : 0$ makes no sense

- $\nexists$ min, $\nexists$ max.
- minimal elements are the prime numbers
- $x \mid 2x$ $\forall x \in \mathbb{N}$, so $\nexists$ maximal element

Hasse diagram:



Lattices

Def Let $(A, \leq)$ be an ordered set and $B \subseteq A$.

a) the elem $a \in A$ is a <u>minorant</u> (<u>lower bound</u>) for B

if $\forall x \in B$ $a \leq x$

a') $a \in A$ is a <u>majorant</u> (<u>upper bound</u>) for B

if $\forall x \in B$ $x \leq a$.

b). $a \in A$ is the infimum of B ($\inf B$) if

a is the largest minorant for B.

i.e. $\begin{cases} a \leq x & \forall x \in B \\ \forall a' \in A \ a' \leq x \ \forall x \in B \Rightarrow a' \leq a \end{cases}$

b') $a \in A$ is the <u>supremum of B</u> ($\sup. B$) if

a is the least majorant for B.

i.e. $\begin{cases} x \leq a & \forall x \in B \\ \forall a' \in A \ x \leq a' \ \forall x \in B \Rightarrow a \leq a' \end{cases}$

Rm inf B, sup B do not always exist, but when they exist, they are unique

Def c). the ordered set $(A, \le)$ is called a lattice if $\forall x, y \in A$ $\exists \inf\{x,y\}$ and $\exists \sup\{x,y\}$

Rm by induction, every finite subset will have inf, sup

d). $(A, \le)$ is called a complete lattice if $\forall B \subseteq A$ has $\inf B$, $\sup B$

Examples 1) Let $(A, \le)$ be totally ordered.

then: $\inf\{x,y\} = \min\{x,y\}$

$\sup\{x,y\} = \max\{x,y\}$

Hence any totally ordered set is a lattice.

- $(\mathbb{R}, \le)$ is totally ordered, but it is not a complete lattice, because $\nexists \inf(-\infty, a)$, $\nexists \sup(a, +\infty)$.
- $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is a complete lattice.

Rm every finite lattice is complete

2). Let $(\mathbb{N}, |)$. Let $a, b \in \mathbb{N}$.

- $\inf\{a,b\} = \gcd(a,b) = d \overset{not}{=} (a,b)$

by def $\begin{cases} d \mid a, \ d \mid b \\ d' \mid a, \ d' \mid b \Longrightarrow d' \mid d \end{cases}$

- $\sup\{a,b\} = \mathrm{lcm}(a,b) = m \overset{not}{=} [a,b]$

by def: $\begin{cases} a \mid m, \ b \mid m \\ a \mid m', \ b \mid m' \Longrightarrow m \mid m' \end{cases}$

3). $(\mathcal{P}(M), \subseteq)$.

e.g. let $X = \{1,2\}$, $Y = \{2,3\}$

$\inf\{X,Y\} = \{2\} = X \cap Y$

$\sup\{X,Y\} = M = X \cup Y$

In general, let $\mathcal{B} \subseteq \mathcal{P}(M)$. Then,

$$\inf \mathcal{B} = \bigcap_{X \in \mathcal{B}} X = \{x \in M \mid \forall X \in \mathcal{B},\ x \in X\}$$

$$\sup \mathcal{B} = \bigcup_{X \in \mathcal{B}} X = \{x \in M \mid \exists X \in \mathcal{B},\ x \in X\}.$$

conclusion: $(\mathcal{P}(M), \subseteq)$ is a complete lattice.

4). let $(A, \leq)$ be an ordered set.

$\inf \emptyset = \max A$ (because $\forall a \in A$ is minorant for $\emptyset$) (if it exists)

$\sup \emptyset = \min A$ (if it exists; because $\forall a \in A$ is majorant for $\emptyset$)

$\boxed{\text{Homework : } 65 - 72}$