Lecture 4

Initial value problems. The Existence and Uniqueness Theorems (II)

(1) $\begin{cases} y' = f(x,y) \\ y(x_0) = y^0 \end{cases}$ $f : [x_0-a, x_0+a] \times \mathbb{R} \to \mathbb{R}$, $a>0$

(2) $y^0 \in \mathbb{R}$.

$(1)+(2) \Leftrightarrow (3)$ $\quad y(x) = y^0 + \int_{x_0}^{x} f(s, y(s))\, ds$. Volterra integral equation.

Theorem 1. (The $\exists!$ Theorem in the space).

Let's consider the IVP (1)+(2) and suppose that:

(i) $f \in C([x_0-a, x_0+a] \times \mathbb{R}, \mathbb{R})$

(ii) f is Lipschitz with respect to the second variable on $[x_0-a, x_0+a] \times \mathbb{R}$, i.e.

$\exists L_f > 0$ s.t. $|f(x,u) - f(x,v)| \le L_f \cdot |u-v|$, $\forall x \in [x_0-a, x_0+a]$, $\forall u, v \in \mathbb{R}$

Then:

(a) The IVP (1)+(2) has an unique sol $y^* \in C[x_0-a, x_0+a]$

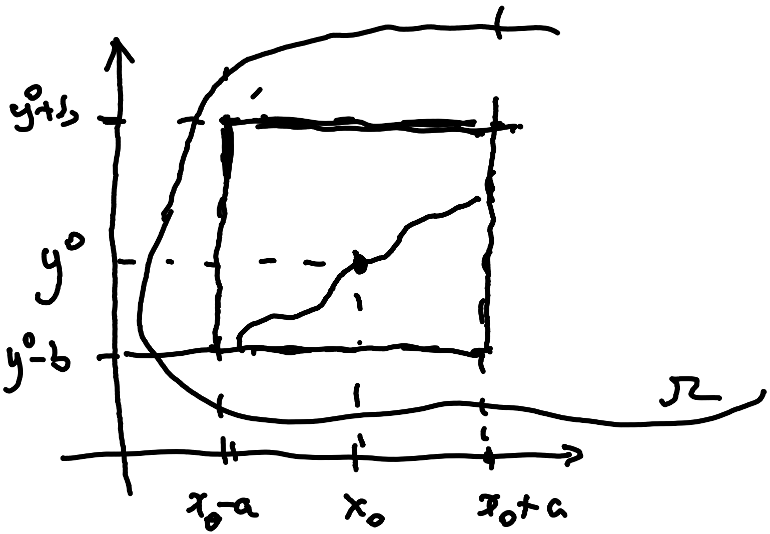
(b) The unique sol. y^* can be obtained from any successive approximation seq. starting from any $y_0 \in C[x_0-a, x_0+a]$

(c) error estimation

<u>Remark</u>. Suppose $f(x,\cdot) \in C^1(\mathbb{R})$

If. $\left|\frac{\partial f}{\partial y}(x,y)\right| \leq M$, $\forall x \in [x_0-a, x_0+a]$, <u>$\forall y \in \mathbb{R}$</u> $\Rightarrow$

$\Rightarrow$ f is lipschitz with respect to the second variable y and $L_f = M$.



Suppose that $f : \Omega \to \mathbb{R}$, $\Omega \subseteq \mathbb{R}^2$, $(x_0, y^0) \in \Omega$

Ω a domain (open set + connex set)

$a, b > 0 \Rightarrow$

$\Rightarrow \overline{D} = [x_0-a, x_0+a] \times [y^0-b, y^0+b]$

$\overline{D} \subseteq \Omega$

$X = C([x_0-a, x_0+a], \mathbb{R})$ $\quad$ $(X, \|\cdot\|_c)$ Banach space.

$\|\cdot\|_c$ $\quad$ $\|y\|_c = \max\limits_{x \in [x_0-a, x_0+a]} |y(x)|$

$\overline{B}(y^0, b) = \{ y \in C[x_0 - a, x_0 + a] \mid \|y - y^0\|_C \leq b \} \subseteq X.$

the closed ball centred in y^0 with radius b.

$$\left.\begin{array}{l} \overline{B}(y^0,b) \text{ is a closed subset of } X \\ X \text{ is a Banach space} \end{array}\right\} \Rightarrow \begin{array}{l} (\overline{B}(y^0,b), \|\cdot\|_C) \\ \text{is a Banach space} \end{array}$$

$$(1)+(2) \Leftrightarrow y(x) = \underbrace{y^0 + \int_{x_0}^{x} f(s, y(s))\,ds}_{A(y)(x)} \quad (3)$$

$A : X \to X$

A is well defined if $\boxed{f \in C(\Omega, \mathbb{R})}$ continuity condition

$(3) \Leftrightarrow \quad y = A(y)$ a fixed point problem.

$A : \overline{B}(y^0, b) \underset{?}{\longrightarrow} X$

$A(\overline{B}(y^0,b)) \overset{?}{\subseteq} \overline{B}(y^0, b)$

$y \in \overline{B}(y^0, b) \overset{?}{\Longrightarrow} A(y) \in \overline{B}(y^0, b)$

Let $y \in \overline{B}(y^0, b) \Rightarrow \|y - y^0\|_C \le b \Leftrightarrow \max_{x \in [x_0 - a, x_0 + a]} |y(x) - y^0| \le b$

$\Leftrightarrow |y(x) - y^0| \le b \ , \ \forall x \in [x_0 - a, x_0 + a]$.

$$|A(y)(x) - y^0| = \left| \int_{x_0}^{x} f(s, y(s))\, ds \right| \le$$

$$\le \left| \int_{x_0}^{x} |f(s, y(s))|\, ds \right| \le$$

$$\left(\left.\begin{matrix} f \in C(\Omega, \mathbb{R}) \\ \overline{D} \subseteq \Omega \text{ compact subset} \end{matrix}\right\} \Rightarrow f \in C(\overline{D}, \mathbb{R}) \atop \Rightarrow f \text{ is bounded on } \overline{D} \Rightarrow \exists M_f > 0 \text{ s.t. } |f(x,u)| \le M_f \ \forall (x,u) \in \overline{D} \right)$$

$$\le \left| \int_{x_0}^{x} M_f\, ds \right| = M_f \cdot \left| \int_{x_0}^{x} ds \right| = M_f \cdot |x - x_0| \le M_f \cdot a$$

$$|A(y)(x) - y^0| \le M_f \cdot a \qquad , \ \forall x \in [x_0 - a, x_0 + a]$$

$$\Rightarrow \|A(y) - y^0\|_C \le M_f \cdot a \le b$$

we impose the condition $\boxed{M_f \cdot a \leq b}$ the invariance condition

$\Rightarrow \|A(y) - y^0\|_C \leq M_f \cdot a \leq b \Rightarrow A(y) \in \overline{B}(y^0, b)$.

$\Rightarrow A : \overline{B}(y^0, b) \to \overline{B}(y^0, b)$

using the same technique from the proof of Th. 1

$\Rightarrow$ A is contraction with $L_A = \frac{L_f}{L_f + 1}$ $\left(L_A = \frac{L_f}{b} \right)$

we need to suppose that f is a lipschitz function with respect to the second variable on $\overline{D}$, i.e.

$$\boxed{\exists L_f > 0 \text{ s.t } |f(x,u) - f(x,v)| \leq L_f \cdot |u - v|, \ \forall (x,u), (x,v) \in \overline{D}}$$

Theorem 2 (The $\exists!$ Theorem in the ball)

Let consider the IVP (1)+(2), where $f:\Omega\to\mathbb{R}$, $\Omega\subseteq\mathbb{R}^2$ domain (open and connex set). Suppose that:

(i) $f\in C(\Omega,\mathbb{R})$ (continuity condition)

(ii) let $a,b>0$, $D=[x_0-a,x_0+a]\times[y^0-b,y^0+b]\subseteq\Omega$

f is lipschitz with respect to the second variable on D, i.e.

$\exists L_f>0$ s.t. $|f(x,u)-f(x,v)|\le L_f|u-v|$, $\forall (x,u),(x,v)\in D$

Then:

(a) the IVP (1)+(2) has a unique solution

$$y^*\in C([x_0-h,x_0+h],[y^0-b,y^0+b])$$

where $h=\min\{a,\frac{b}{M_f}\}$, $M_f=\max\limits_{(x,u)\in\overline{D}}|f(x,u)|$

(b) the unique sol. y^* can be obtained from any succesive approximation sequence starting from any $y_0\in\overline{B}(y^0,b)$

$$A^n(y_0)\to y^*,\ \forall y_0\in\overline{B}(y^0,b)$$

$$\Updownarrow$$

$$y_0\in C([x_0-h,x_0+h],[y^0-b,y^0+b])$$

(c) we have the error estimate from $\exists!$ Th in the space

Examples

1) $\begin{cases} y' = x.\cos(y(x)) \\ y(0) = 0 \end{cases}$

$x_0 = 0$, $y^0 = 0$ $\quad I = [x_0 - a, x_0 + a] = [-a, a]$, $a > 0$

$f(x,y) = x.\cos(y)$, $f : [-a,a] \times \mathbb{R} \to \mathbb{R}$.

$$\left| \frac{\partial f}{\partial y}(x,y) \right| = \left| x.(-\sin(y)) \right| = |x| . |\sin(y)| \leq$$

$$\leq a . 1 = a.$$

$\Rightarrow \frac{\partial f}{\partial y}$ is bounded on $[-a,a] \times \mathbb{R}$ $\Rightarrow$

$\Rightarrow f$ is lipschitz with respect to y on $[-a,a] \times \mathbb{R}$

with $L_f = a$

f is cont.

$\Longrightarrow$ ivp has an unique sol. $y^* \in C[-a,a]$

Th. 1

($\exists$! Th in space)

$$A^n(y_0) \longrightarrow y^*, \quad \forall y_0 \in C[-a,a].$$

$$(\text{ivp}) \iff y(x) = y^0 + \int_{x_0}^{x} f(s, y(s))\,ds$$

$$y(x) = \underbrace{\int_0^x s \cdot \cos(y(s))\,ds}_{A(y)(x)}.$$

the equivalent Volterra integral eq.

the succesive approx. seq. :

$$y_{n+1} = A(y_n)$$

$$y_{n+1}(x) = \int_0^x s \cdot \cos(y_n(s))\,ds.,$$

the starting function $y_0 \in C[-a,a]$

if we choose $y_0(x) \equiv 0$

$$y_1(x) = \int_0^x s \cdot \cos(y_0(s))\, ds = \int_0^x s \cdot \underbrace{\cos(0)}_{=1}\, ds =$$

$$= \int_0^x s\, ds = \frac{s^2}{2}\Big|_0^x = \frac{x^2}{2}$$

$$y_2(x) = \int_0^x s \cdot \cos(y_1(s))\, ds = \int_0^x s \cdot \cos\left(\frac{s^2}{2}\right) ds.$$

$\sim \ldots\ldots$

$\vdots$

$$y_n(x) \to y^*(x)$$

2) $\begin{cases} y' = 2x^2 + 3 \cdot y^4 \\ y(0) = 0 \end{cases}$

$x_0 = 0$, $y^0 = 0$

$f(x,y) = 2x^2 + 3y^4$, $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ cont. function ($\underline{\underline{f \in C^1}}$)

$a > 0$ $f: [-a, a] \times \mathbb{R} \to \mathbb{R}$

$$\left|\frac{\partial f}{\partial y}(x,y)\right| = \left|12 \cdot y^3\right| = 12 \cdot |y|^3 \xrightarrow[y \to \pm\infty]{} +\infty$$

$\Rightarrow$ $\frac{\partial f}{\partial y}$ is not bounded on $[-a,a] \times \mathbb{R}$.

$\Rightarrow$ $\frac{\partial f}{\partial y}$ is not lipschitz with respect to y on $[-a,a] \times \mathbb{R}$

$\Rightarrow$ we cannot apply Th. 1.

$a, b > 0$: $\overline{D} = [-a,a] \times [-b,b]$, $\overline{D} \subseteq \mathbb{R}^2$

f is cont on $\mathbb{R}^2 \Rightarrow f$ is cont on $\overline{D}$

$$\left|\frac{\partial f}{\partial y}(x,y)\right| = 12 \cdot |y|^3 \leq 12 \cdot b^3, \quad \forall (x,y) \in \overline{D}$$

$\Rightarrow$ f is lipschitz with respect to y on $\overline{D}$

$\underset{\text{Th. 2}}{\Rightarrow}$ $\exists!\ y^* \in C([-h,h],[-b,b])$

where $h = \min\left\{a, \frac{b}{M_f}\right\}$,

$$M_f = \max_{(x,u) \in \overline{D}} |f(x,u)|.$$

for $a=1$, $b=1$ $\Rightarrow \overline{D} = [-1,1] \times [-1,1]$

$\Rightarrow \exists!\, y^* \in C\left([-h,h],[-1,1]\right)$

where $h = \min\left\{1, \frac{1}{M_f}\right\}$

$$M_f = \max_{(x,u)\in\overline{D}} |f(x,u)|$$

$$|f(x,u)| = |2x^2 + 3 \cdot y^4| \le 2 \cdot \underbrace{|x|^2}_{\le 1} + 3 \cdot \underbrace{|y|^4}_{\le 1} \le$$

$$\le 2+3 = 5$$

$$h = \min\left\{1, \frac{1}{5}\right\} = \frac{1}{5}$$

$$y^* \in C\left(\left[-\frac{1}{5}, \frac{1}{5}\right], [-1,1]\right)$$