Lecture 5

March 24, 2022

Remark. $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$, $a \in A \cap A'$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

2.1. Definition. Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}^m$, $a \in A \cap A'$. The derivative of f at a is defined by

$$f'(a) := \lim_{x \to a} \frac{1}{x - a}\left[f(x) - f(a)\right]$$

provided that the limit in the right side exists in $\mathbb{R}^m$.

$$f'(a) \in \mathbb{R}^m$$

∂ = curly d

3. Partial derivatives

3.1. Definition. Let $A \subseteq \mathbb{R}^2$, $(a, b) \in \operatorname{int} A$, $f = f(x, y) : A \to \mathbb{R}$.

The partial derivatives of f at (a, b) are defined by

$$\frac{\partial f}{\partial x}(a, b) = \lim_{x \to a} \frac{f(x, b) - f(a, b)}{x - a} \qquad \frac{\partial f}{\partial y}(a, b) = \lim_{y \to b} \frac{f(a, y) - f(a, b)}{y - b},$$

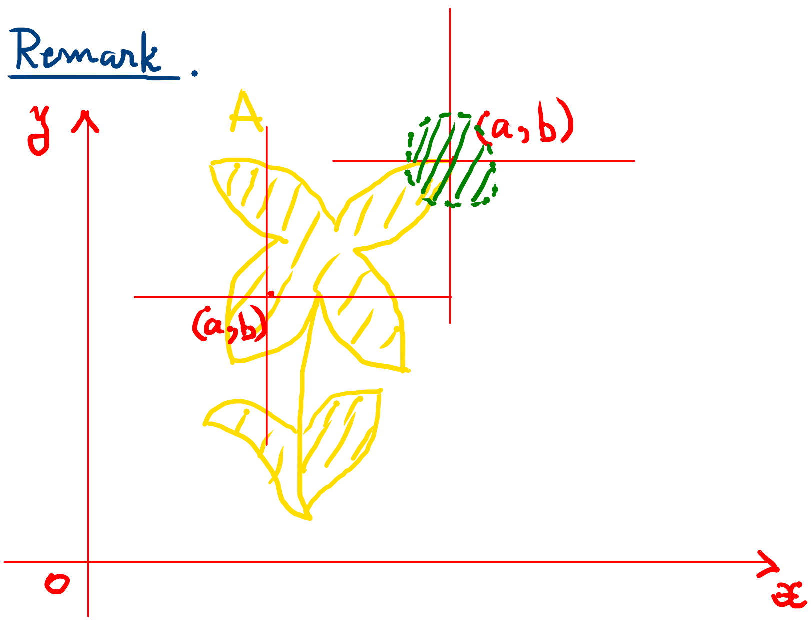
provided that the above limits exist.

Example. $f(x,y) = \arctan \frac{x}{y}$ $\quad f: \mathbb{R} \times (0,\infty) \to \mathbb{R}$

$$\frac{\partial f}{\partial x}(x,y) = \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{1}{y} = \frac{y^2}{x^2+y^2} \cdot \frac{1}{y} = \frac{y}{x^2+y^2}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{1}{1+\frac{x^2}{y^2}} \cdot x \cdot \left(-\frac{1}{y^2}\right) = \frac{\cancel{y^2}}{x^2+y^2} \cdot \frac{-x}{\cancel{y^2}} = -\frac{x}{x^2+y^2}.$$

Remark.



Let $A \subseteq \mathbb{R}^3$, $(a,b,c) \in \operatorname{int} A$,

$f = f(x,y,z): A \to \mathbb{R}$

$$\frac{\partial f}{\partial x}(a,b,c) = \lim_{x \to a} \frac{f(x,b,c) - f(a,b,c)}{x-a}$$

$$\frac{\partial f}{\partial y}(a,b,c) =$$

$$\frac{\partial f}{\partial z}(a,b,c) =$$

Let $A \subseteq \mathbb{R}^n$, $a = (a_1, \dots, a_n) \in \operatorname{int} A$, $f : A \to \mathbb{R}$, $j \in \{1, \dots, n\}$. The partial derivative of f wrto x_j at a is defined by

$$\frac{\partial f}{\partial x_j}(a) = \lim_{x_j \to a_j} \frac{f(a_1, \dots, a_{j-1}, x_j, a_{j+1}, \dots, a_n) - f(a_1, \dots, a_{j-1}, a_j, a_{j+1}, \dots, a_n)}{x_j - a_j} \in \mathbb{R},$$

provided that the limit exists.

<u>3.2. Definition</u>. Let $A \subseteq \mathbb{R}^n$, $a = (a_1, \dots, a_n) \in \operatorname{int} A$, $f : A \to \mathbb{R}^m$, $j \in \{1, \dots, n\}$. The partial derivative of wrto x_j at a is defined by

$$\frac{\partial f}{\partial x_j}(a) = \lim_{x_j \to a_j} \frac{1}{x_j - a_j} \left[f(a_1, \dots, a_{j-1}, x_j, a_{j+1}, \dots, a_n) - f(a_1, \dots, a_{j-1}, a_j, a_{j+1}, \dots, a_n) \right] \in \mathbb{R}^m$$

provided that the limit exists.

<u>3.3. Theorem</u> Let $A \subseteq \mathbb{R}^n$, $a \in \operatorname{int} A$, $f = (f_1, \dots, f_m) : A \to \mathbb{R}^m$, $j \in \{1, \dots, n\}$ Then:

1° If f is partially differentiable at a wrto $x_j \Rightarrow f_1, \dots, f_m$ are partially diff. at a wrto x_j and

$$\frac{\partial f}{\partial x_j}(a) = \left(\frac{\partial f_1}{\partial x_j}(a), \dots, \frac{\partial f_m}{\partial x_j}(a) \right). \qquad (1)$$

2° If $f_1, \dots, f_m$ are partially diff. at a wrto $x_j \Rightarrow f$ is partially diff. at a wrto x_j and (1) holds.

3.4. Geometric meaning of partial derivatives

$z=f(x,y)$

Let $F:\mathbb{R}^2\to\mathbb{R}$ be a function. The equation

$$F(x,y)=0$$

is the implicit equation of a plane curve Γ. Let $(x_0,y_0)\in\Gamma \iff F(x_0,y_0)=0$

Then $\vec{N}\left(\frac{\partial F}{\partial x}(x_0,y_0),\ \frac{\partial F}{\partial y}(x_0,y_0)\right)=\frac{\partial F}{\partial x}(x_0,y_0)\,\vec{i}+\frac{\partial F}{\partial y}(x_0,y_0)\,\vec{j}$

is a normal vector to Γ at (x_0,y_0) (i.e. $\vec{N}$ is $\perp$ to the tangent line to Γ at (x_0,y_0))

Ex $\mathcal{E}:\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $\qquad (x_0,y_0)\in\mathcal{E}$

$\left\{\begin{array}{l} t:\ Ax+By+C=0 \\ \vec{n}(A,B)\perp t \end{array}\right.$

$t=$ tangent line to $\mathcal{E}$ at (x_0,y_0)

$t:\ \frac{x x_0}{a^2}+\frac{y y_0}{b^2}=1 \iff \frac{x_0}{a^2}\cdot x+\frac{y_0}{b^2}\cdot y-1=0$

$\Rightarrow \vec{n}\left(\frac{x_0}{a^2},\ \frac{y_0}{b^2}\right)$ is a normal vector to t, hence to $\mathcal{E}$.

$\mathcal{E}:\ F(x,y)=0$ where $F(x,y)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$

$\frac{\partial F}{\partial x}(x,y)=\frac{2x}{a^2}$

$\frac{\partial F}{\partial y}(x,y)=\frac{2y}{b^2}$

$\vec{N}\left(\frac{2x_0}{a^2},\ \frac{2y_0}{b^2}\right)\perp t$ because $\vec{N}=2\vec{n}$

$y = f(x)$ $\quad f: D \subseteq \mathbb{R} \to \mathbb{R}$, $\quad x_0 \in D$

t = tangent line to G_f at $(x_0, f(x_0))$

$t: \quad y - f(x_0) = f'(x_0)(x - x_0)$

$f'(x_0) \cdot x - y + f(x_0) - x_0 f'(x_0) = 0$

$\vec{n}(f'(x_0), -1)$ is normal to t

$\underbrace{f(x) - y}_{F(x,y)} = 0$

$$\vec{N}\left(\frac{\partial F}{\partial x}(x_0, y_0), \frac{\partial F}{\partial y}(x_0, y_0)\right) = \vec{N}(f'(x_0), -1)$$

$F: \mathbb{R}^3 \to \mathbb{R}$, $\quad F(x,y,z) = 0$ is the implicit equation of a surface Σ.

$\mathcal{E}: \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad$ ellipsoid

Let $(x_0, y_0, z_0) \in \Sigma \iff F(x_0, y_0, z_0) = 0$

Then $\vec{N}\left(\frac{\partial F}{\partial x}(x_0, y_0, z_0), \frac{\partial F}{\partial y}(x_0, y_0, z_0), \frac{\partial F}{\partial z}(x_0, y_0, z_0)\right)$ is a normal vector to the surface Σ at (x_0, y_0, z_0) i.e., $\vec{N} \perp \pi$, where π is the tangent plane to Σ at (x_0, y_0, z_0).

3.5. Definition. Let $A \subseteq \mathbb{R}^n$, $a \in \operatorname{int} A$, $f = (f_1, \dots, f_m) : A \to \mathbb{R}^m$. Assume that all partial derivatives of f exist at a. Then we define

$$J(f)(a) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \frac{\partial f_1}{\partial x_2}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \frac{\partial f_2}{\partial x_1}(a) & \frac{\partial f_2}{\partial x_2}(a) & \cdots & \frac{\partial f_2}{\partial x_n}(a) \\ \vdots & & & \\ \frac{\partial f_m}{\partial x_1}(a) & \frac{\partial f_m}{\partial x_2}(a) & \cdots & \frac{\partial f_m}{\partial x_n}(a) \end{pmatrix} \in \mathcal{M}_{m \times n}(\mathbb{R})$$

the Jacobi matrix of f at the point a

$\underline{m=1}$ $\quad f : A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ $\quad J(f)(a) = \left(\frac{\partial f}{\partial x_1}(a) \quad \frac{\partial f}{\partial x_2}(a) \quad \cdots \quad \frac{\partial f}{\partial x_n}(a) \right) \in \mathcal{M}_{1 \times n}(\mathbb{R})$

We consider the vector $\nabla f(a) := \left(\frac{\partial f}{\partial x_1}(a), \frac{\partial f}{\partial x_2}(a), \dots, \frac{\partial f}{\partial x_n}(a) \right) \in \mathbb{R}^n$

the gradient of f at a

(∇ = nabla)

3.6. Example of a partially differentiable function that is discontinuous.

Let $f: \mathbb{R}^2 \to \mathbb{R}$, $\quad f(x,y) = \begin{cases} \dfrac{x^2 y}{x^6+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

Then f is partially differentiable at $(0,0)$, but it is discontinuous at this point.

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x \to 0} \frac{\overbrace{f(x,0)}^{=0} - \overbrace{f(0,0)}^{=0}}{x-0} = \lim_{x\to 0} \frac{0}{x} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y \to 0} \frac{\overbrace{f(0,y)}^{=0} - \overbrace{f(0,0)}^{=0}}{y-0} = \lim_{y\to 0} \frac{0}{y} = 0$$

$$\lim_{k\to\infty}\left(\frac{1}{k}, \frac{1}{k^3}\right) = (0,0) \qquad \lim_{k\to\infty} f\left(\frac{1}{k}, \frac{1}{k^3}\right) = \lim_{k\to\infty} \frac{\frac{1}{k^2}\cdot\frac{1}{k^3}}{\frac{1}{k^6}+\frac{1}{k^6}} = \lim_{k\to\infty} \frac{1}{\cancel{k^5}} \cdot \frac{\cancel{k^6}}{2} = \infty \neq 0 = f(0,0)$$

$\Longrightarrow$ f is not continuous at $(0,0)$.

characterization of continuity by means of sequences

Remark. $f : A \subseteq \mathbb{R} \to \mathbb{R}$, $a \in A \cap A'$

f is differentiable at a $\iff \exists \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a} =: c \in \mathbb{R}$

$\iff \exists\, c \in \mathbb{R}$ s.t. $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a} = c$

$\iff \exists\, c \in \mathbb{R}$ s.t. $\lim\limits_{x \to a} \dfrac{f(x) - f(a) - c(x-a)}{x - a} = 0$

$\iff \exists\, \varphi \in L(\mathbb{R}, \mathbb{R})$ s.t. $\lim\limits_{x \to a} \dfrac{f(x) - f(a) - \varphi(x-a)}{x - a} = 0$

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$\varphi \in L(\mathbb{R}, \mathbb{R}) \iff \exists\, c \in \mathbb{R}$ s.t. $\varphi(x) = cx \quad \forall x \in \mathbb{R}$

4. Differentiability of vector functions of a vector variable

4.1. Lemma Let $A \subseteq \mathbb{R}^n$, $a \in \operatorname{int} A$, $f : A \to \mathbb{R}^m$, and let $\varphi_1, \varphi_2 \in L(\mathbb{R}^n, \mathbb{R}^m)$ s.t.

$$\lim_{x \to a} \frac{1}{\|x - a\|} \left[f(x) - f(a) - \varphi_i(x - a) \right] = 0_m \qquad \text{for } i = 1, 2.$$

Then $\varphi_1 = \varphi_2$.

4.2. Definition. Lemma 4.1 enables us to give the following definition: let $A \subseteq \mathbb{R}^n$, $a \in \operatorname{int} A$, $f: A \to \mathbb{R}^m$. Then f is said to be (Fréchet) differentiable at a if $\exists\, \varphi \in L(\mathbb{R}^n, \mathbb{R}^m)$ s.t.

$$\lim_{x \to a} \frac{1}{\|x-a\|} \left[f(x) - f(a) - \varphi(x-a) \right] = 0_m. \qquad (1)$$

$$\boxed{L4.1}$$

If f is differentiable at a $\Rightarrow$ the linear mapping φ satisfying (1) is unique.
This unique linear mapping is called the (Fréchet) differential of f at a, and it will be denoted by $df(a)$

$$df(a) \in L(\mathbb{R}^n, \mathbb{R}^m)$$

$$\forall h \in \mathbb{R}^n \qquad df(a)(h) \in \mathbb{R}^m$$

We have

$$\lim_{x \to a} \frac{1}{\|x-a\|} \left[f(x) - f(a) - df(a)(x-a) \right] = 0_m.$$

4.3. Proposition. Let $A \subseteq \mathbb{R}^n$, $a \in \operatorname{int} A$, $f: A \to \mathbb{R}^m$. Then the following assertions are true.

1° If f is differentiable at a $\Rightarrow$ $\exists\, \omega: A \to \mathbb{R}^m$ s.t.

$$\lim_{x \to a} \omega(x) = 0_m$$

$$f(x) = f(a) + df(a)(x-a) + \|x-a\| \cdot \omega(x) \qquad \forall x \in A.$$

2° If $\exists\, \varphi \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $\exists\, \omega: A \to \mathbb{R}^m$ s.t.

$$\lim_{x \to a} \omega(x) = 0_m$$

$$f(x) = f(a) + \varphi(x-a) + \|x-a\|\, \omega(x) \qquad \forall x \in A$$

$\Rightarrow$ f is differentiable at a and $df(a) = \varphi$.

4.4. Theorem Let $A \subseteq \mathbb{R}^n$, $a \in \text{int}\, A$, $f : A \to \mathbb{R}^m$. If f is differentiable at $a \Rightarrow$

$\Rightarrow$ f is continuous at a.

Proof. f is differentiable at $a \overset{\boxed{P4.3}}{\Longrightarrow} \exists\, \omega : A \to \mathbb{R}$ s.t.

$$\lim_{x \to a} \omega(x) = 0_m$$

$$f(x) = f(a) + df(a)(x-a) + \|x-a\| \cdot \omega(x) \qquad \forall x \in A$$

Without lose in generality we may assume that $\omega(a) = 0_m$, i.e., ω is continuous at a.

$$0 \leq \|f(x) - f(a)\| = \|df(a)(x-a) + \|x-a\| \cdot \omega(x)\|$$

$$\leq \|df(a)(x-a)\| + \|x-a\| \cdot \|\omega(x)\|$$

$$\leq \|df(a)\| \cdot \|x-a\| + \|x-a\| \cdot \|\omega(x)\|$$

$$\|\varphi(x)\| \leq \|\varphi\| \cdot \|x\|$$

see Lecture 4

$$0 \xleftarrow{x \to a}$$

$$\Rightarrow \lim_{x \to a} \|f(x) - f(a)\| = 0 \Longrightarrow \lim_{x \to a} f(x) = f(a) \Longrightarrow f \text{ is continuous at } a$$

4.5. Theorem. Let $A \subseteq \mathbb{R}^n$, $a \in \operatorname{int} A$, $f = (f_1, \dots, f_m) : A \to \mathbb{R}^m$. Then:

1° If f is differentiable at a $\Rightarrow$ $f_1, \dots, f_m$ are all differentiable at a and

$$\underbrace{df(a)}_{L(\mathbb{R}^n, \mathbb{R}^m)} = \left(df_1(a), \dots, df_m(a) \right). \qquad (1)$$

2° If $f_1, \dots, f_m$ are all differentiable at a $\Rightarrow$ f is diff. at a and (1) holds.

4.6. Theorem. Let $A \subseteq \mathbb{R}^n$, $a \in \operatorname{int} A$, $f : A \to \mathbb{R}^m$ s.t. f is Fréchet differentiable at a. Then:

1° f is partially differentiable at a and $\frac{\partial f}{\partial x_j}(a) = df(a)(e_j)$ $\forall\, j = \overline{1,n}$.

2° $[df(a)] = J(f)(a)$.

3° $\forall\, h = (h_1, \dots, h_n) \in \mathbb{R}^n$:

$$df(a)(h) = h_1 \frac{\partial f}{\partial x_1}(a) + \dots + h_n \frac{\partial f}{\partial x_n}(a).$$

$A \subseteq \mathbb{R}^n$, $a = (a_1, \dots, a_n) \in \text{int} A$, $f : A \to \mathbb{R}$ s.t. f is partially diff. at a

$$f \text{ is differentiable at } a \iff \ell = \lim_{(h_1,\dots,h_n)\to 0_n} \frac{f(a_1+h_1,\dots,a_n+h_n) - f(a) - \sum_{j=1}^{n} h_j \frac{\partial f}{\partial x_j}(a)}{\sqrt{h_1^2+\dots+h_n^2}} = 0$$

Algorithm to check the differentiability of f at a

Step I Check the existence of the partial derivatives at a

- If at least one partial derivative does not exist $\Rightarrow$

  $\Rightarrow$ f is not diff. at a STOP

- If all partial derivatives at a exist, then go to Step II

Step II Check the existence of the limit ℓ

- If $\ell = 0 \Rightarrow f$ is differentiable at a

- Otherwise $\Rightarrow f$ is not differentiable at a.