COURSE 4

The Hermite interpolation formula is

$$f = H_n f + R_n f,$$

where $R_n f$ denotes the remainder term (the error).

Theorem 1 *If $f \in C^n[\alpha, \beta]$ and $f^{(n)}$ is derivable on (α, β), with $\alpha = \min\{x, x_0, ..., x_m\}$ and $\beta = \max\{x, x_0, ..., x_m\}$, then there exists $\xi \in (\alpha, \beta)$ such that*

$$(R_n f)(x) = \frac{u(x)}{(n+1)!} f^{(n+1)}(\xi). \tag{1}$$

Proof. Consider

$$F(z) = \begin{vmatrix} u(z) & (R_n f)(z) \\ u(x) & (R_n f)(x) \end{vmatrix}.$$

$F \in C^n[\alpha, \beta]$ and there exists $F^{(n+1)}$ on (α, β).

We have

$$F(x) = 0, \quad F^{(j)}(x_k) = 0, \qquad k = 0, ..., m; \;\; j = 0, ..., r_k;$$

because

$$u(x) = \prod_{k=0}^{m} (x - x_k)^{r_k + 1} \Rightarrow u^{(j)}(x_k) = 0, \;\; j = 0, ..., r_k$$

and

$$(R_m f)^{(j)}(x_k) = f^{(j)}(x_k) - (H_n f)^{(j)}(x_k) = f^{(j)}(x_k) - f^{(j)}(x_k) = 0.$$

So, F and its derivatives have $n + 2$ distinct zeros in (α, β). Applying successively Rolle's theorem it follows that F' has at least $n + 1$ zeros in $(\alpha, \beta) \Rightarrow ... \Rightarrow F^{(n+1)}$ has at least one zero $\xi \in (\alpha, \beta)$, $F^{(n+1)}(\xi) = 0$.

We have

$$F^{(n+1)}(z) = \begin{vmatrix} u^{(n+1)}(z) & (R_n f)^{(n+1)}(z) \\ u(x) & (R_n f)(x) \end{vmatrix},$$

with $u(z) = \prod_{k=0}^{m}(z-z_k)^{r_k+1} \in \mathbb{P}_{n+1} \Rightarrow u^{(n+1)}(z) = (n+1)!$, and $(R_n f)^{(n+1)}(z) = f^{(n+1)}(z) - (H_n f)^{(n+1)}(z) = f^{(n+1)}(z)$ (as, $H_n f \in \mathbb{P}_n$). We get

$$F^{(n+1)}(\xi) = \begin{vmatrix} (n+1)! & f^{(n+1)}(\xi) \\ u(x) & (R_n f)(x) \end{vmatrix} = 0,$$

whence it follows (1). ■

Corollary 2 *If $f \in C^{n+1}[a,b]$ then*

$$|(R_n f)(x)| \leq \frac{|u(x)|}{(n+1)!} \left\| f^{(n+1)} \right\|_{\infty}, \qquad x \in [a,b]$$

where $\|\cdot\|_{\infty}$ denotes the uniform norm ($\|f\|_{\infty} = \sup_{x\in[a,b]} |f(x)|$).

Remark 3 *In case of $m = 0$, i.e., $n = r_0$, (HIP) becomes* **Taylor interpolation problem**. *Taylor interpolation polynomial is*

$$(T_n f)(x) = \sum_{j=0}^{n} \frac{(x-x_0)^j}{j!} f^{(j)}(x_0).$$

Hermite interpolation with double nodes

Example 4 *In the following table there are some data regarding a moving car. We may estimate the position (and the speed) of the car when the time is* $t = 10$ *using Hermite interpolation.*

Time	0	3	5	8	13
Distance	0	225	383	623	993
Speed	75	77	80	74	72

Consider $f : [a, b] \to \mathbb{R}$, $x_0, x_1, ..., x_m \in [a, b]$

and the values $f(x_0), f(x_1), ..., f(x_m), f'(x_0), f'(x_1), ..., f'(x_m)$.

The Hermite interpolation polynomial with double nodes, H_{2m+1}, satisfies the interpolation properties:

$$H_{2m+1}(x_i) = f(x_i), \;\; i = \overline{0, m},$$
$$H'_{2m+1}(x_i) = f'(x_i), \;\; i = \overline{0, m}.$$

It is a polynomial of $n = 2m + 1$ degree.

For computation: use Lagrange polynomial written in Newton form, with divided differences table having each node x_i written twice.

Consider $z_0 = x_0$, $z_1 = x_0$, $z_2 = x_1$, $z_3 = x_1$, $\dots$, $z_{2m} = x_m$, $z_{2m+1} = x_m$.

Form divided differences table: each node appear twice, in the first column write the values of f for each node twice; in the second column, at the odd positions put the values of the derivatives of f; the other elements are computed using the rule from divided differences.

We obtain the following table:

z_0	$f(z_0)$	$(\mathcal{D}^1 f)(z_0) = f'(x_0)$	$(\mathcal{D}^2 f)(z_0)$		$(\mathcal{D}^{2m} f)(z_0)$	$(\mathcal{D}^{2m+1} f)(z_0)$
z_1	$f(z_1)$	$(\mathcal{D}^1 f)(z_1)$	$\vdots$		$(\mathcal{D}^{2m} f)(z_1)$	
z_2	$f(z_2)$	$(\mathcal{D}^1 f)(z_2) = f'(x_1)$				
z_3	$f(z_3)$	$\vdots$				
$\vdots$	$\vdots$	$(\mathcal{D}^1 f)(z_{2m-1})$	$(\mathcal{D}^2 f)(z_{2m-1})$	$\ddots$		
z_{2m}	$f(z_{2m})$	$(\mathcal{D}^1 f)(z_{2m}) = f'(x_m)$		$\dots$		
z_{2m+1}	$f(z_{2m+1})$			$\dots$		

Newton interpolation polynomial for the nodes $x_0, ..., x_n$ is

$$(N_n f)(x) = f(x_0) + \sum_{i=1}^{n} (x - x_0)...(x - x_{i-1})(\mathcal{D}^i f)(x_0),$$

and similarly, Hermite interpolation polynomial is

$$(H_{2m+1} f)(x) = f(z_0) + \sum_{i=1}^{2m+1} (x - z_0)...(x - z_{i-1})(\mathcal{D}^i f)(z_0),$$

where $(\mathcal{D}^i f)(z_0)$, $i = 1, ..., 2m+1$ are the elements from the first line and columns $2, ..., 2m+1$.

Example 5 *Consider the double nodes* $x_0 = -1$ *and* $x_1 = 1$, *and* $f(-1) = -3, f'(-1) = 10, f(1) = 1, f'(1) = 2$. *Find the Hermite interpolation polynomial, that approximates the function* f, *in both forms, using the classical formula and using divided differences.*

Sol.

We present here the method with divided differences. We have $m = 1, r_0 = r_1 = 1 \Rightarrow n = 3$

$z_0 = -1$	$f(-1) = -3$	$f'(-1) = 10$	$\frac{\frac{f(1)-f(-1)}{2} - f'(-1)}{z_2 - z_0} = -4$	$\frac{0-(-4)}{z_3 - z_0} = 2$
$z_1 = -1$	$f(-1) = -3$	$\frac{f(1)-f(-1)}{z_2 - z_1} = 2$	$\frac{f'(1) - \frac{f(1)-f(-1)}{2}}{z_3 - z_1} = 0$	
$z_2 = 1$	$f(1) = 1$	$f'(1) = 2$		
$z_3 = 1$	$f(1) = 1$			

The Hermite interpolation polynomial is

$$
\begin{aligned}
(H_3 f)(x) &= f(z_0) + \sum_{i=1}^{3} (x - z_0)...(x - z_{i-1})(\mathcal{D}^i f)(z_0) \\
&= f(z_0) + (x - z_0)(\mathcal{D}^1 f)(z_0) + (x - z_0)(x - z_1)(\mathcal{D}^2 f)(z_0) \\
&+ (x - z_0)(x - z_1)(x - z_2)(\mathcal{D}^3 f)(z_0)
\end{aligned}
$$

i.e.,

$$\begin{aligned}
(H_3f)(x) =& f(-1) + (x+1)f'(-1) + (x+1)^2\tfrac{f(1)-f(-1)-2f'(-1)}{4} \\
& + (x+1)^2(x-1)\tfrac{2f'(1)-f(1)+f(-1)}{4} \\
=& -3 + 10(x+1) - 4(x+1)^2 + 2(x+1)^2(x-1) \\
=& 2x^3 - 2x^2 + 1.
\end{aligned}$$

2.4. Birkhoff interpolation

Let $x_k \in [a,b]$, $k = 0,1,...,m$, $x_i \neq x_j$ for $i \neq j, r_k \in \mathbb{N}$ and $I_k \subset \{0,1,...,r_k\}$, $k = 0,1,...,m$, $f : [a,b] \to \mathbb{R}$ s.t. $\exists f^{(j)}(x_k)$, $k = 0,...,m$, $j \in I_k$, and denote $n = |I_0| + ... + |I_m| - 1$, where $|I_k|$ is the cardinal of the set I_k.

The Birkhoff interpolation problem (BIP) consists in determining the polynomial P of the smallest degree such that

$$P^{(j)}(x_k) = f^{(j)}(x_k), \quad k = 0,...,m; \; j \in I_k.$$

Remark 6 *If $I_k = \{0,1,...,r_k\}$, $k = 0,...,m$, then (BIP) reduces to a (HIP). Birkhoff interpolation is also called* lacunary Hermite interpolation.

In order to check if (BIP) has an unique solution, we consider the polynomial $P(x) = a_n x^n + ... + a_0$ and the $(n+1) \times (n+1)$ linear system

$$P^{(j)}(x_k) = f^{(j)}(x_k), \quad k = 0,...,m; \; j \in I_k, \tag{2}$$

having as unknowns the coefficients of the polynomial. If the determinant of the system (2) is nonzero than (BIP) has an unique solution.

Definition 7 *A solution of (BIP), if exists, is called* **Birkhoff interpolation polynomial**, *denoted by* $B_n f$.

Birkhoff interpolation polynomial is given by

$$(B_n f)(x) = \sum_{k=0}^{m} \sum_{j \in I_k} b_{kj}(x) f^{(j)}(x_k), \tag{3}$$

where $b_{kj}(x)$ denote the Birkhoff fundamental interpolation polynomials. They fulfill relations:

$$\begin{aligned} & b_{kj}^{(p)}(x_\nu) = 0,\ \nu \neq k, \quad p \in I_\nu \\ & b_{kj}^{(p)}(x_k) = \delta_{jp},\ p \in I_k, \quad \text{for } j \in I_k \text{ and } \nu, k = 0, 1, ..., m, \end{aligned} \tag{4}$$

with $\delta_{jp} = \begin{cases} 1, & j = p \\ 0, & j \neq p. \end{cases}$

Remark 8 *Because of the gaps of the interpolation conditions, it is hard to find an explicit expression for* b_{kj}, $k = 0, ..., m;\ j \in I_k$. *They are found using relations (4).*

Birkhoff interpolation formula is

$$f = B_n f + R_n f,$$

where $R_n f$ denotes the remainder term.

Example 9 *Let* $f \in C^2[0,1]$, *the nodes* $x_0 = 0$, $x_1 = 1$ *and we suppose that we know* $f(0) = 1$ *and* $f'(1) = \frac{1}{2}$. *Find the corresponding interpolation formula.*

Sol. *We have* $m = 1,\ I_0 = \{0\},\ I_1 = \{1\}$, *so* $n = 1 + 1 - 1 = 1$.

We check if there exists a solution of the problem.

Consider $P(x) = a_1 x + a_0 \in \mathbb{P}_1$ *and the system*

$$\begin{cases} P(0) = f(0) \\ P'(1) = f'(1) \end{cases} \Longleftrightarrow \begin{cases} a_0 = f(0) \\ a_1 = f'(1) \end{cases}.$$

The determinat of the system is

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0,$$

so the problem has an unique solution.

The Birkhoff polynomial is

$$(B_1 f)(x) = b_{00}(x) f(0) + b_{11}(x) f'(1) \in \mathbb{P}_1.$$

We have $b_{00}(x) = ax + b \in \mathbb{P}_1$ *and*

$$\begin{cases} b_{00}(x_0) = 1 \\ b'_{00}(x_1) = 0 \end{cases} \Longleftrightarrow \begin{cases} b_{00}(0) = 1 \\ b'_{00}(1) = 0 \end{cases} \Leftrightarrow \begin{cases} b = 1 \\ a = 0 \end{cases},$$

whence

$$b_{00}(x) = 1.$$

For $b_{11}(x) = cx + d \in \mathbb{P}_1$ *we have*

$$\begin{cases} b_{11}(x_0) = 0 \\ b'_{11}(x_1) = 1 \end{cases} \Longleftrightarrow \begin{cases} b_{11}(0) = 0 \\ b'_{11}(1) = 1 \end{cases} \Leftrightarrow \begin{cases} d = 0 \\ c = 1 \end{cases}$$

whence

$$b_{11}(x) = x.$$

So,

$$(B_1 f)(x) = f(0) + x f'(1) = 1 + \frac{1}{2}x.$$