Seminar 11

1. Are the following sets (left, right, two-sided) ideals of the ring $M_2(\mathbb{R})$:

(i) $\mathcal{A} = \left\{ \begin{pmatrix} a & a \\ 0 & 0 \end{pmatrix} \bigm| a \in \mathbb{R} \right\}$;

(ii) $\mathcal{B} = \left\{ \begin{pmatrix} a & a \\ 0 & b \end{pmatrix} \bigm| a, b \in \mathbb{R} \right\}$;

(iii) $\mathcal{C} = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \bigm| a, b \in \mathbb{R} \right\}$?

2. In an arbitrary ring, is the intersection of a left ideal and a right ideal a two-sided ideal?

3. Are the following sets ideals of the ring $\mathbb{R}[X]$:
(i) $A = \{f \in \mathbb{R}[X] \mid \text{the free term of } f \text{ is } 0\}$;
(ii) $B = \{f \in \mathbb{R}[X] \mid \text{the free term of } f \text{ is } 1\}$;
(iii) $C = \{f \in \mathbb{R}[X] \mid \text{the coefficient of the term of degree 1 of } f \text{ is } 0\}$?

4. Let $R = \{\frac{m}{n} \in \mathbb{Q} \mid n \text{ is odd}\}$ and $U = \{\frac{m}{n} \in R \mid m \text{ is even}\}$. Show that R is a subring of the field $\mathbb{Q}$ and U is an ideal of R.

5. Let R be a ring and $a \in R$. Show that $Ra = \{ra \mid r \in R\}$ is a left ideal of R and $aR = \{ar \mid r \in R\}$ is a right ideal of R.

6. Let R be a ring and

$$Ann(R) = \{a \in R \mid \forall x \in R, ax = 0 = xa\}.$$

Show that $Ann(R)$ is an ideal of R, called the *annihilator* of R.

7. Determine the ideals of the ring $\mathbb{Z}_8$, and draw the Hasse diagram of its ideal lattice.

8. Determine the ideals of the ring $\mathbb{Z}_{12}$ and draw the Hasse diagram of its ideal lattice.