Complete lattices (continued)

Def. 1) The ordered set $(A, \leq)$ is called a lattice if $\forall x, y \in A$

$$\exists \inf \{x, y\}, \quad \exists \sup \{x, y\}.$$

2) $(A, \leq)$ is called a complete lattice if $\forall B \subseteq A$

$$\exists \inf B \quad \text{and} \quad \exists \sup B.$$

Theorem (characterization of complete lattices)

Let $(A, \leq)$ be an ordered set. The following statements are equivalent

(i) A is a complete lattice

(ii) $\forall B \subseteq A \quad \exists \inf B$.

(iii) $\forall B \subseteq A \quad \exists \sup B$.

Exple

$\sup \{c, d\} = d$.
$\sup \{a, b\} = c$
$\sup \{a, c\} = c$
$\nexists \inf \{a, b\}$
$\nexists \sup \emptyset$ being $\nexists \min A$

Proof. (i) $\Rightarrow$ (ii), (iii) by def

(ii) $\Rightarrow$ (i) Let $B \subseteq A$. We have to show that $\exists \sup B$.

Let $C = \{ y \in A \mid \forall x \in B \quad x \leq y \}$ the set of all upper bounds for B.

By the assumption (ii) $\exists \inf C =: a$.

We prove that $a = \sup B$.

• we show that a as an upper bound of B, ie $x \leq a \ \forall x \in B$.
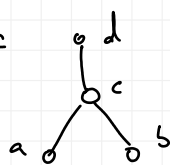
Indeed, let $x \in B$; then $\forall y \in C$ we have $x \leq y$. This means that x is a min/bound for C, but $a = \inf C$ hence $x \leq a$.

• we show that a is the least upper bound for B.

Let $a' \in A$ be another upper bound for B, ie $a' \in C$.

But $a = \inf C$, hence $a \leq a'$.

$(|i'i| \implies)(i)$ Homework ‼

Conclusion: L a complete lattice.

$$\sup B = \inf \{ y \in A \mid y \text{ is a majorant of } B \}$$

$$\inf B = \sup \{ y \in A \mid y \text{ is a lower bound for } B \}$$
$$(\text{minorant})$$

Example 1). $(\mathcal{P}(M), \subseteq)$ is a complete lattice,

where $\inf (X_i)_{i \in I} = \bigcap_{i \in I} X_i$

$$\sup (X_i)_{i \in I} = \bigcup_{i \in I} X_i \qquad \forall \text{ subsets}$$
$$X_i \subseteq M.$$
$$i \in I.$$
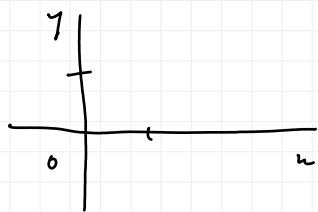
1). Let $(G, \cdot)$ be a group.

Denote $\mathcal{S}(G) = \{ H \mid H \text{ subgroup of } G \}$

Let $(H_i)_{i \in I}$ be a family of subgroups of G

then $\inf (H_i)_{i \in I} = \bigcap_{i \in I} H_i$ because the intersection of subgroups is a subgroup

Remark the union of subgroups is, in general, not a subgroup

$(\mathbb{R} \times \mathbb{R}, +)$ grp

$H_1 = Ox = \{ (x, 0) \mid x \in \mathbb{R} \}$ subgrp of $\mathbb{R} \times \mathbb{R}$

$H_2 = Oy = \{ (0, y) \mid y \in \mathbb{R} \}$ subgrp of $\mathbb{R} \times \mathbb{R}$

but $H_1 \cup H_2$ is not a subgrp. Let $a = (1, 0)$, $b = (0, 1)$
$$\underset{H_1}{\cap} \qquad \underset{H_2}{\cap}$$

$a, b \in H_1 \cup H_2$. but $a + b = (1, 1) \notin H_1 \cup H_2$

According to the theorem:

$$\text{Sp}\left(H_i\right)_{i \in I} = \bigcap_{\substack{H \leq G \\ H \supseteq H_i, \forall i \in I}} H$$

Well-ordered sets

<u>Def.</u> Let $(A, \leq)$ be an ordered set. We say that A is well-ordered, if $\forall \emptyset \neq B \subseteq A \quad \exists \min B$.

<u>Examples</u> 1) $(\mathbb{N}, \leq)$ is well-ordered (we'll see!)

(Note that $\exists \inf \emptyset = \max(\mathbb{N}, \leq)$)

2). If A is well-ordered, then A is totally ordered, because

$\forall x, y \in A \quad \exists \min \{x, y\}$.

The converse is not true.

e.g. $(\mathbb{R}, \leq)$ is totally ordered, but not well-ordered

because, for instance, $\nexists \min (a, b)$. , or

$\nexists \min \left\{ \frac{1}{n} \mid n \in \mathbb{N}^* \right\}$

$\frac{1}{n} > \frac{1}{n+1}$ but it has inf equal to 0.

3) Any finite totally ordered set is well-ordered (by induction)

$(\forall a > 0 \ \exists n \ \text{s.t.} \ \frac{1}{n} < a)$

4). $(\mathbb{N}, |)$, $(\mathcal{P}(M), \subseteq)$ where $|M| \geq 2$ are not totally ordered, hence they are not well-ordered.

5) $\emptyset$ is well-ordered, by def

6) Let $A = \bigcup_{i \in \mathbb{N}} \mathbb{N} \times \{i\}$ $\Leftarrow$ see the diagram. it is well-ordered

(left margin, top to bottom)
$(3,4)$
$(2,4)$
$(1,4)$
$(0,4)$
$\vdots$
$(3,1)$
$(2,1)$
$(1,1)$
$(0,1)$
$\vdots$
$(2,0)$
$(1,0)$
$(0,0)$

Theorem (Characterization of well-ordered sets)

Let $(A, \leq)$ be a non-empty ordered set. The following statements are equivalent:

(i) A is well-ordered (A satisfies the minimum condition)

(ii) (inductive condition) A is totally ordered, $\exists \min A =: a_0$, and for any subset $B \subseteq A$ which satisfies:

$1^\circ \qquad a_0 \in B$.

$2^\circ \qquad \forall y \in A, \text{ if } \{x \in A \mid x < y\} \subseteq B.$
$\qquad\qquad \text{then } y \in B.$

we have $B = A$.

Corollary (The principle of (strong) complete induction).

Let $(A, \leq)$ be a non-empty well-ordered set, and let P be a predicate on A. Assume that:

$1^\circ \quad P(a_0)$ is true, where $a_0 = \min A$

$2^\circ \quad \forall y \in A, \text{ if } P(x) \text{ is true } \forall x < y, \text{ then } P(y) \text{ is the}$

Then $P(a)$ is true for any $a \in A$.

Proof (cor) Let $B = \{a \in A \mid P(a) \text{ is true}\}$

By assumption, B satisfies conditions $1^\circ, 2^\circ$ of the theorem. It follows that $B = A$

Proof (thm) $(ii) \Rightarrow (i)$ Let $\emptyset \neq B \subseteq A$ be a non-empty subset of A; we have to show that $\exists \min B$.

Assume, by contradiction, that $\nexists \min B$. We consider the subset $A \setminus B = \complement_A(B)$

- We have that $a_0 \in A \setminus B$, because otherwise, if $a_0 \in B$ then a_0 would be the smallest of B.

  Here $A \setminus B$ satisfies $1°$

- Let $y \in A$ s.t. $\forall x \in A$, $x < y$ we have $x \in A \setminus B$.

  If $y \in B$, then we would get that $y = \min B$, impossible

  It follows that $y \in A \setminus B$.

  This means that $A \setminus B$ satisfies cond $2°$.

By assumption (ii), we get $A \setminus B = A$. Then $B = \emptyset$, contrd.

___(i) $\Rightarrow$ (ii)___ Assume that A is well-ordered. Then $\exists a_0 = \min A$

and is totally ordered. $_{\neq \emptyset}$

Let $B \subseteq A$ satisfying conditions $1°$, $2°$.

We have to show that $B = A$.

Assume, by contradiction, that $B \neq A$, ie $A \setminus B = \complement_A B \neq \emptyset$ (1)

By (1), $\exists a_i = \min(A \setminus B)$, hence $\underline{\underline{a \notin B}}$.

Let $x \in A$ s.t $x < a$, hence $x \notin A \setminus B$, here $x \in B$

By cond $2°$ on B, we deduce that $\underline{\underline{a \in B}}$ contradiction.
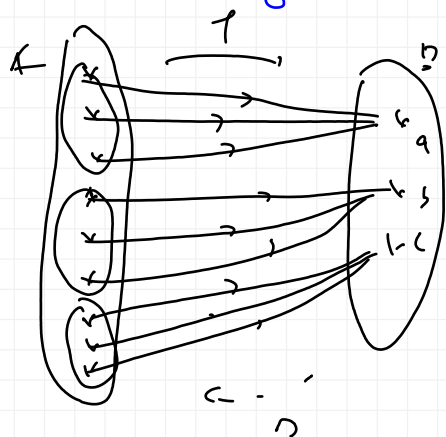
It follows that $B = A$.

The Axiom of Choice

(AC) Let $I \neq \emptyset$ be an index set, let $(A_i)_{i \in I}$ be a

family of non-empty sets such that $A_i \cap A_j = \emptyset$ for $i \neq j$

Then there is $\underset{\underset{\text{a set}}{\wedge}}{X}$ s.t. $\forall i \in I$ $|X \cap A_i| = 1$

Statements equivalent to the axiom of choice

1). Every surjective function has a right inverse (section)

$$f \circ s = 1_B.$$

$s(c) = ?$ we choose an element

from $f^{-1}(c) = \{ x \in A \mid f(x) = c \}$

Ernst Zermelo

2). Zermelo's well-ordering principle

On any set A there is an order relation $\leq$
such that $(A, \leq)$ is well-ordered.

3) (Zorn's Lemma.)   Max Zorn

Let A be a non-empty ordered set. Assume that
any totally ordered subset B of A has a
majorant in A.
Then in A there exist maximal elements.

Application   any vector space has a basis
                                              "
                      a maximal linearly independent subset

Homework.   ex. $73^*, 74^*, 75^*$