Seminar 5- Calculus 2

Thursday, March 27, 2025 8:01 AM

① Let $A \subseteq \mathbb{R}^m$, $a \in \operatorname{int} A$, $f: A \to \mathbb{R}$ s.t. all partial derivatives of f exist at a. Prove that f is diff at a $\Leftrightarrow$

$$\lim_{x\to a} \frac{f(x) - f(a) - \sum_{j=1}^{m} (x_j - a_j)\frac{\partial f}{\partial x_j}(a)}{\|x-a\|} = 0 \quad (1)$$

[$\Rightarrow$] Assume that f is diff. at a $\Rightarrow \lim_{x\to a} \frac{f(x) - f(a) - df(a)(x-a)}{\|x-a\|} = 0$

But $df(a)(x-a) = \sum_{j=1}^{m} (x_j - a_j)\frac{\partial f}{\partial x_j}(a)$

$\Downarrow$ (1) holds

[$\Leftarrow$] Assume that (1) holds

Let $\varphi: \mathbb{R}^m \to \mathbb{R}$

$\varphi(h) = \sum_{j=1}^{m} h_j \frac{\partial f}{\partial x_j}(a) \qquad \forall h = (h_1, \dots, h_m)$

Obs., $\varphi \in L(\mathbb{R}^m, \mathbb{R})$

$(1) \Leftrightarrow \lim_{x\to a} \frac{f(x) - f(a) - \varphi(x-a)}{\|x-a\|} = 0$

$\Rightarrow f$ is diff. at a and $df(a) = \varphi$

Rem. Based on ① we have the following alg. to study the differentiability of a function $f: A \to \mathbb{R}$ at some point $a \in \operatorname{int} A$

1°) Check the existance of all partial derivatives (p.d.) of f at a:

- If at least one p.d. does NOT exist at a then f is NOT diff. at a $\Rightarrow$ STOP
- If all partial der. exist, then go to 2°)

2°) Study the limit $l = \lim_{x\to a} \frac{f(x) - f(a) - \sum_{j=1}^{m} (x_j - a_j)\frac{\partial f}{\partial x_j}(a)}{\|x-a\|}$

$h = x - a$ → change of var.

$$l = \lim_{\substack{h_1\to 0\\ \vdots \\ h_m \to 0}} \frac{f(a_1+h_1, \dots, a_m+h_m) - f(a_1, \dots, a_m) - \sum_{j=1}^{m} h_j \cdot \frac{\partial f}{\partial x_j}(a)}{\sqrt{h_1^2 + \dots + h_m^2}}$$

- If $l = 0 \Rightarrow f$ is diff. at a
- Otherwise $\Rightarrow f$ is NOT diff at a ($\nexists$ or $l \neq 0$)

lat exam

② Study the differentiability of $f: \mathbb{R}^2 \to \mathbb{R}$,

$$f(x,y) = \begin{cases} \frac{x^3 - y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

$\frac{\partial f}{\partial x}(x,y) = \frac{3x^2(x^2+y^2) - (x^3-y^3)\cdot 2x}{(x^2+y^2)^2} = \frac{x^4 + 3x^2y^2 + 2xy^3}{(x^2+y^2)^2}$

$\frac{\partial f}{\partial y}(x,y) = \frac{-3y^2(x^2+y^2) - (x^3-y^3)2y}{(x^2+y^2)^2} = \frac{-y^4 - 3x^2y^2 - 2x^3y}{(x^2+y^2)^2}$

$\forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$\Rightarrow \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$

$\mathbb{R}^2 \setminus \{(0,0)\}$ is an open set

$\Rightarrow f$ is differ. on $\mathbb{R}^2 \setminus \{(0,0)\}$

It remains to study the differ of f at $(0,0)$

$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x - 0} = \lim_{x\to 0} \frac{x}{x} = 1$

$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0} \frac{f(0,y) - f(0,0)}{y - 0} = \lim_{y\to 0} \frac{-y}{y} = -1$

We study the limit:

$$l = \lim_{\substack{h_1\to 0\\ h_2\to 0}} \frac{f(h_1,h_2) - f(0,0) - h_1 \cdot \frac{\partial f}{\partial x}(0,0) - h_2 \cdot \frac{\partial f}{\partial y}(0,0)}{\sqrt{h_1^2 + h_2^2}}$$

$$= \lim_{\substack{h_1\to 0\\ h_2\to 0}} \frac{\frac{h_1^3 - h_2^3}{h_1^2+h_2^2} - 0 - \underset{h_1^2+h_2^2}{h_1} + \underset{h_1^2+h_2^2}{h_2}}{\sqrt{h_1^2+h_2^2}} = \lim_{\substack{h_1\to 0\\ h_2\to 0}} \frac{\cancel{h_1^3} - \cancel{h_2^3} - \cancel{h_1^3} - h_1h_2^2 + h_1^2h_2 + \cancel{h_2^3}}{(h_1^2+h_2^2)\sqrt{h_1^2+h_2^2}}$$

$$= \lim_{\substack{h_1\to 0\\ h_2\to 0}} \underbrace{\frac{-h_1h_2^2 + h_1^2h_2}{(h_1^2+h_2^2)\sqrt{h_1^2+h_2^2}}}_{\omega(h_1,h_2)} \quad \begin{matrix}\to \text{degree } 3 \\ \to \text{degree } 3\end{matrix} \qquad \nexists$$

$\lim_{k\to\infty} \left(\frac{1}{k}, \frac{1}{k}\right) = (0,0) \qquad \lim_{k\to\infty} \omega\left(\frac{1}{k}, \frac{1}{k}\right) = \underline{\underline{0}}$

$\lim_{k\to\infty} \left(\frac{1}{k}, -\frac{1}{k}\right) = (0,0) \qquad \lim_{k\to\infty} \omega\left(\frac{1}{k}, -\frac{1}{k}\right) = \lim_{k\to\infty} \frac{\frac{\cancel{2}}{k^{\cancel{3}}}}{\frac{\cancel{2}}{\cancel{k^2}} \cdot \sqrt{\frac{2}{k^2}}} = \underline{\underline{\frac{1}{\sqrt{2}}}}$

$\Rightarrow \nexists\, l \Rightarrow f$ is NOT diff. at $(0,0)$

Answer: f is diff. on $\mathbb{R}^2 \setminus \{(0,0)\}$ but not at $(0,0)$

③ Study the diff. of $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \sqrt[3]{x^3+y^3}$

$$\frac{\partial f}{\partial x}(x,y) = \frac{1}{\cancel{3}\sqrt[3]{x^3+y^3}} \cdot \cancel{3}x^2 = \frac{x^2}{\sqrt{x^3+y^3}}$$

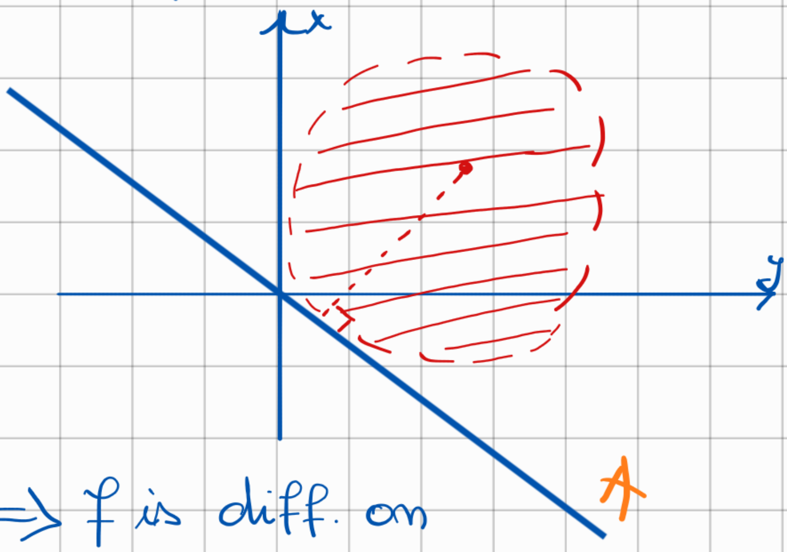
$$\frac{\partial f}{\partial y}(x,y) = \frac{y^2}{\sqrt{x^3+y^3}}$$

$(x^3+y^3)^{\frac{1}{3}}$

$\forall (x,y) \in \mathbb{R}^2$ with $x \neq -y$

Let $A = \{(x,-x) \mid x \in \mathbb{R}\}$

↳ second bisector



$\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous on $\mathbb{R}^2 \setminus A$

$\mathbb{R}^2 \setminus A$ is open

$\Rightarrow$ f is diff. on $\mathbb{R}^2 \setminus A$

It remains to study the differ. of f at points in A, i.e. at points $(a,-a)$

$$\frac{\partial f}{\partial x}(a,-a) = \lim_{x\to a} \frac{f(x,-a) - f(a,-a)}{x-a} = \lim_{x\to a} \frac{\sqrt[3]{x^3 \cancel{-a^3} \cancel{+a^3} - a^3}}{x-a}$$

$$= \lim_{x\to a} \frac{\sqrt[3]{x^3-a^3}}{x-a} = \lim_{x\to a} \sqrt[3]{\frac{\cancel{(x-a)}(x^2+xa+a^2)}{(x-a)^{\cancel{3}\,2}}} = \lim_{x\to a} \sqrt[3]{\frac{3a^2}{0_+}}$$

$= \infty$ if $a \neq 0$

If $a \neq 0 \Rightarrow \frac{\partial f}{\partial x}(a,-a)$ does NOT exist $\Rightarrow$ f is not diff. at $(a,-a)$

Next we check the diff of f at $(0,0)$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x-0} = \lim_{x\to 0} \frac{\sqrt[3]{x^3 - 0}}{x-a}$$

$$= \lim_{x\to 0} \frac{x}{x} = 1$$

$$\frac{\partial f}{\partial y}(0,0) = 1$$

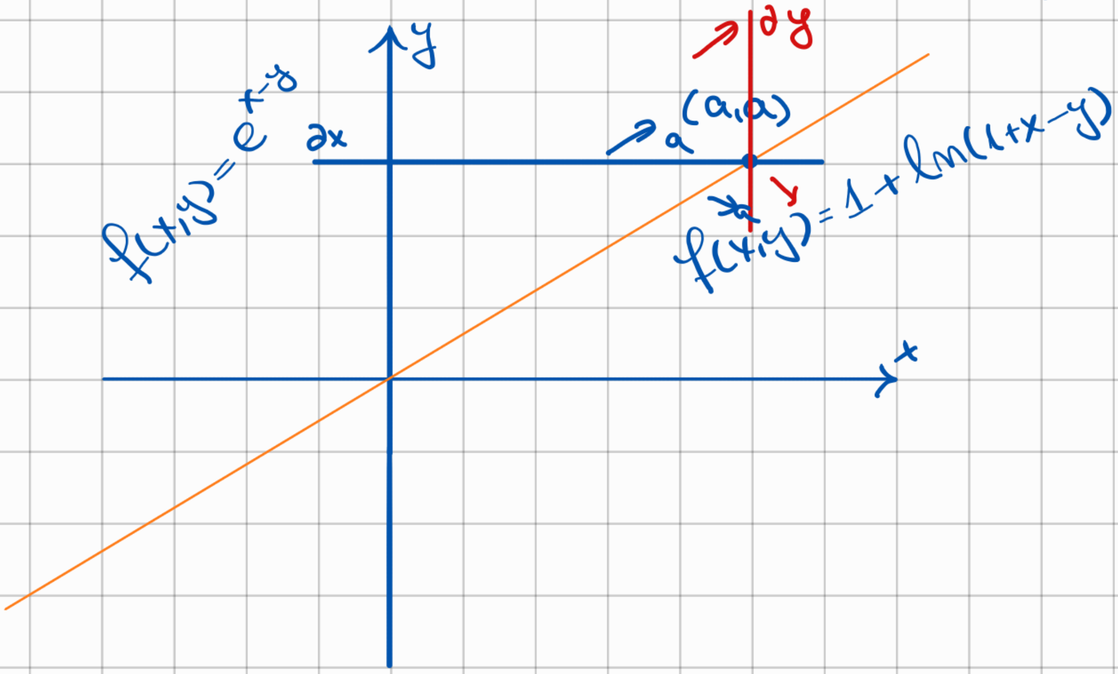
$$l = \lim_{\substack{h_1 \to 0 \\ h_2 \to 0}} \frac{f(h_1,h_2) - f(0,0) - h_1 \frac{\partial f}{\partial x}(0,0) - h_2 \frac{\partial f}{\partial y}(0,0)}{\sqrt{h_1^2 + h_2^2}}$$

$$= \lim_{\substack{h_1 \to 0 \\ h_2 \to 0}} \underbrace{\frac{\sqrt[3]{h_1^3 + h_2^3} - 0 - h_1 - h_2}{\sqrt{h_1^2 + h_2^2}}}_{\omega(h_1,h_2)} \quad \begin{matrix} \to \text{degree } 1 \\ \to \text{degree } 1 \end{matrix}$$

$\Rightarrow \nexists\, l$ (Homework $\to$ Proof) $\left(\frac{1}{k}, \frac{2}{k}\right)$ and $\left(\frac{1}{k}, 0\right)$

<u>Answer:</u> f is diff. at $\mathbb{R}^2 \setminus A$ but not on A

④ Study the diff. of $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} e^{x-y}, & x \le y \\ 1 + \ln(1+x-y), & x > y \end{cases}$



Let $A = \{(x,x) \mid x \in \mathbb{R}\}$

$B = \{(x,y) \mid x < y\}$

$C = \{(x,y) \mid x > y\}$

$\forall (x,y) \in B$ we have:

$$\frac{\partial f}{\partial x}(x,y) = e^{x-y}$$

$$\frac{\partial f}{\partial y}(x,y) = -e^{x-y}$$

$\left.\begin{matrix} \text{cont. on } B \\ B \text{ is open} \end{matrix}\right\} \Rightarrow f$ is diff. on B

analogusly, f is diff. on C

It remains to study the diff. of f at points of the form $(a,a) \in A$

$$\frac{\partial f}{\partial x}(a,a) = \lim_{x \to a} \frac{f(x,a) - f(a,a)}{x-a}$$

$$\lim_{x \nearrow a} \frac{f(x,a) - f(a,a)}{x-a} = \lim_{x \nearrow a} \frac{e^{x-a} - 1}{x-a} = 1$$

$$\lim_{x \swarrow a} \frac{f(x,a) - f(a,a)}{x-a} = \lim_{x \swarrow a} \frac{\cancel{1} + \ln(1+x-a) - \cancel{1}}{x-a} = 1$$

$$\Rightarrow \frac{\partial f}{\partial x}(a,a) = 1$$

(It decreases $\Rightarrow$ other formula for $f(x,a)$)

$$\frac{\partial f}{\partial y}(a,a) = \lim_{y \to a} \frac{f(y,a) - f(a,a)}{y-a}$$

$$\lim_{y \nearrow a} \frac{f(a,y) - f(a,a)}{y-a} = \lim_{y \nearrow a} \frac{1 + \ln(1+a-y) - 1}{y-a} = -1$$

$$\lim_{y \searrow a} \frac{f(a,y) - f(a,a)}{y-a} = \lim_{y \searrow a} \frac{e^{a-y} - 1}{y-a} = -1$$

$$\Rightarrow \frac{\partial f}{\partial y}(a,a) = -1$$

Next we check $\exists\, l$

$$l = \lim_{(h_1,h_2) \to (0,0)} \frac{f(a+h_1, a+h_2) - f(a,a) - h_1 \cdot \frac{\partial f}{\partial x}(a,a) - h_2 \frac{\partial f}{\partial y}(a,a)}{\sqrt{h_1^2 + h_2^2}}$$

$$= \lim_{(h_1,h_2)\to(0,0)} \underbrace{\frac{f(a+h_1,a+h_2)-1-h_1+h_2}{\sqrt{h_1^2+h_2^2}}}_{w(h_1,h_2)}$$

$$\lim_{t\to 0}\frac{e^t-1-t}{t^2}=-\frac{1}{2}$$

$$\underset{(h_1\le h_2)}{w_1(h_1,h_2)} = \frac{e^{h_1-h_2}-1-(h_1-h_2)}{\sqrt{h_1^2+h_2^2}}$$

$$w_1(h_1,h_2) = \frac{e^{h_1-h_2}-1-(h_1-h_2)}{(h_1-h_2)^2}\cdot\frac{h_1^2-2h_1h_2+h_2^2}{\sqrt{h_1^2+h_2^2}}$$

$$= \underbrace{\frac{e^{h_1}-1-(h_1-h_2)}{(h_1-h_2)^2}}_{(h_1,h_2)\to 0 \;\to\; \frac{1}{2}} \cdot \underbrace{\left(\underbrace{\underbrace{\frac{h_1}{\sqrt{h_1^2+h_2^2}}}_{\in[-1,1]}\cdot h_1}_{h_1\cdot\ldots=0} - 2\underbrace{\frac{h_1}{\sqrt{h_1^2+h_2^2}}\cdot h_2}_{0} + \underbrace{\frac{h_2}{\sqrt{h_1^2+h_2^2}}\cdot h_2}_{0}\right)}_{(h_1,h_2)\to 0 \;\to\; 0}$$

$$\lim_{(h_1,h_2)\to 0} w_1(h_1,h_2) = 0 \quad (1)$$

analogously, $\lim_{(h_1,h_2)\to(0,0)} w_2(h_1,h_2) \overset{(2)}{=} 0$,

where $w_2(h_1,h_2) = \dfrac{\cancel{1}\ln(1+h_1-h_2)-\cancel{1}-(h_1-h_2)}{\sqrt{h_1^2+h_2^2}}$

$$\lim_{t\to 0}\frac{\ln(1+t)-t}{t^2}=-\frac{1}{2}$$

By (1)(2) $\Rightarrow \lim_{(h_1,h_2)\to(0,0)} w(h_1,h_2)=0 \Rightarrow f$ is diff. at (a,a)

So, f is diff at $\mathbb{R}^2$.

⑤ Let $f = f(u,v) : \mathbb{R}^2 \to \mathbb{R}^2$ be a diff. fct. on $\mathbb{R}^2$ and let $F : \mathbb{R}^3 \to \mathbb{R}$ be def. by

$F(x,y,z) = f(-3x+y^2-2z^2,\ 2x^3-z^3)$. Det the first order partial der. of F in terms of the 1st ord. p.s. of f

Solution: $F = f \circ g$ where $g(x,y,z) = (\underbrace{-3x+y^2-2z^2}_{u(x,y,z)},\ \underbrace{2x^3-z^3}_{v(x,y,z)})$

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$\frac{\partial F}{\partial z} =$$

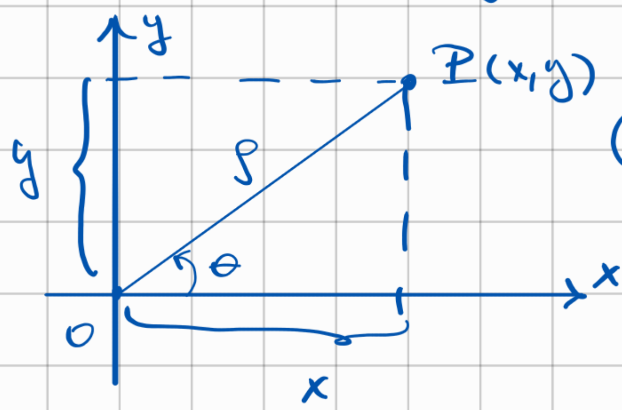
(h.w.)

$$\frac{\partial F}{\partial x}(x,y,z) = \frac{\partial f}{\partial u}(g(x,y,z))\cdot \underbrace{\frac{\partial u}{\partial x}(x,y,z)}_{-3} + \frac{\partial f}{\partial v}(g(x,y,z))\cdot \underbrace{\frac{\partial v}{\partial x}(x,y,z)}_{6x^2}$$

$$\Rightarrow \frac{\partial F}{\partial x}(x,y,z) = -3\frac{\partial f}{\partial u}(-3x+y^2-2z^2,\ 2x^2-z^3) + 6x^2\frac{\partial f}{\partial v}(\text{— '' —})$$

⑥ Using polar coordinates, det. all differ. fct. $f : (0;\infty)^2 \to \mathbb{R}$ satisfying

$$x\cdot\frac{\partial f}{\partial x}(x,y) + y\cdot\frac{\partial f}{\partial y}(x,y) = \frac{x}{\sqrt{x^2+y^2}} \quad \forall (x,y)\in(0;\infty)^2$$



$P(x,y)$ x,y – cart. coord

(ρ,θ) – polar coord

$$\boxed{\begin{aligned} x &= \rho\cos\theta \\ y &= \rho\sin\theta \end{aligned}}$$

Consider $F(\rho,\theta) = f(\rho\cos\theta, \rho\sin\theta)$

$F:(0;\infty)\times(0;\frac{\pi}{2}) \to \mathbb{R}$

$F = f\circ g$ where $g(\rho,\theta) = (\underbrace{\rho\cos\theta}_{x(\rho,\theta)}, \underbrace{\rho\sin\theta}_{y(\rho,\theta)})$

$$\frac{\partial F}{\partial \rho} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \rho} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \rho}$$

$$\frac{\partial F}{\partial \rho}(\rho,\theta) = \frac{\partial f}{\partial x}(g(\rho,\theta))\cdot\underbrace{\frac{\partial x}{\partial \rho}(\rho,\theta)}_{\cos\theta} + \frac{\partial f}{\partial y}(g(\rho,\theta))\cdot\underbrace{\frac{\partial y}{\partial \rho}(\rho,\theta)}_{\sin\theta}$$

$$\frac{\partial F}{\partial \rho}(\rho,\theta) = \cos\theta\cdot\frac{\partial f}{\partial x}(\rho\cos\theta,\rho\sin\theta) + \sin\theta\cdot\frac{\partial f}{\partial y}(\text{—}\shortparallel\text{—}) \quad \Big| \cdot\rho$$

$$\rho\cdot\frac{\partial F}{\partial \rho}(\rho,\theta) = \rho\cos\theta\cdot\frac{\partial f}{\partial x}(\rho\cos\theta,\rho\sin\theta) + \rho\sin\theta\,\frac{\partial f}{\partial y}(\text{—}\shortparallel\text{—})$$

$$= \frac{\rho\cos\theta}{\cancel{\sqrt{\rho^2}}} = \cos\theta$$

$$\Rightarrow \rho\,\frac{\partial F}{\partial \rho}(\rho,\theta) = \cos\theta \;\Big|:\rho \iff \frac{\partial F}{\partial \rho}(\rho,\theta) = \frac{\cos\theta}{\rho}$$

$$F(\rho,\theta) = \int\frac{\cos\theta}{\rho}\,d\rho = \cos\theta\int\frac{1}{\rho}\,d\rho$$

$$= \cos\theta\cdot\ln\rho + h(\theta)$$

↳ fct. care dep. de θ, pt că derivata e 0.

Nu mai punem $+C$

$F = f\circ g \;|\circ g^{-1}()$ to the right

$f = F\circ g^{-1}$

$g(\rho,\theta) = (\underbrace{\rho\cos\theta}_{x}, \underbrace{\rho\sin\theta}_{y})$

$$g^{-1}(x,y) = \big(\underset{\substack{\|\\ r}}{\sqrt{x^2+y^2}}, \underset{\substack{\|\\ \theta}}{\operatorname{arctg}\frac{y}{x}}\big)$$

$$f(x,y) = F(g^{-1}(x,y)) = F\left(\sqrt{x^2+y^2}, \operatorname{arctg}\frac{y}{x}\right)$$

$$= \cos\left(\operatorname{arctg}\frac{y}{x}\right)\cdot \ln\left(\sqrt{x^2+y^2}\right) + h\left(\operatorname{arctg}\frac{y}{x}\right)$$

$$= \frac{x}{2\sqrt{x^2+y^2}}\cdot \ln(x^2+y^2) + h\left(\operatorname{arctg}\frac{y}{x}\right)$$

where $h : (0, \frac{\pi}{2}) \to \mathbb{R}$ is an arbitrary differen. function