Seminar 4

Solvable higher order differential equation

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}) \quad , \quad n \geq 2$$

I Diff. Eq. of the form: $y^{(n)} = f(x)$.

f is continuous fct.

$$y^{(n)} = f(x) \Rightarrow (y^{(n-1)})' = f(x)$$

$$\Rightarrow y^{(n-1)}(x) = \underbrace{\int f(x)\,dx}_{F(x)} + c_1 \, , \; c_1 \in \mathbb{R}.$$

$$\Rightarrow$$

$$y^{(n-1)}(x) = F(x) + c_1$$

$$(y^{(n-2)})' = F(x) + c_1$$

$$y^{(n-2)}(x) = \int (F(x)\,dx + c_1)\,dx + c_2$$

$$y^{(n-2)}(x) = \int F(x)\,dx + c_1 x + c_2 \, , \; c_1, c_2 \in \mathbb{R}$$

$$\vdots$$

Ex 1: Solve the following diff. eqs:

a) $y''' = x + \cos x + \sin x$

b) $y''' = \ln x$

c) $y'' = x \cdot e^x$

d) $y'' = 1 + tg^2 x$

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a) $y''' = x + \cos x + \sin x$

$(y'')' = x + \cos x + \sin x$

$y'' = \int (x + \cos x + \sin x)\,dx + c_1$

$y'' = \frac{x^2}{2} + \sin x - \cos x + c_1$

$(y')' = \frac{x^2}{2} + \sin x - \cos x + c_1 \Rightarrow y' = \int \left(\frac{x^2}{2} + \sin x - \cos x + c_1\right)dx + c_2$

$y' = \frac{x^3}{6} - \cos x - \sin x + c_1 x + c_2$

$y(x) = \int \left(\frac{x^3}{6} - \cos x - \sin x + c_1 x + c_2\right)dx + c_3$

$$\boxed{y(x) = \frac{x^4}{24} - \sin x + \cos x + c_1\frac{x^2}{2} + c_2 x + c_3, \quad c_1, c_2, c_3 \in \mathbb{R}}$$

b) $y''' = \ln x$

$$\overset{(x)'}{1}\cdot \ln x$$

$$(y'')' = \ln x \implies y'' = \int \ln x \, dx + c_1 = \int (x)' \cdot \ln x \, dx + c_1 =$$

$$= x \cdot \ln x - \int \cancel{x} \cdot \frac{1}{\cancel{x}} \, dx + c_1 = x \cdot \ln x - x + c_1 .$$

$$y'' = x \ln x - x + c_1$$

$$(y')' = x \ln x - x + c_1 \implies y' = \int (\overset{(\frac{x^2}{2})'}{x} \ln x - x + c_1) \, dx + c_2 =$$

$$= \frac{x^2}{2} \ln x - \int \frac{x^{\cancel{2}}}{2} \cdot \frac{1}{\cancel{x}} \, dx - \frac{x^2}{2} + c_1 x + c_2$$

$$= \frac{x^2}{2} \ln x - \frac{x^2}{4} - \frac{x^2}{2} + c_1 x + c_2$$

$$y'(x) = \frac{x^2}{2} \ln x - \frac{3}{4} x^2 + c_1 x + c_2$$

$$y(x) = \int \left(\frac{x^2}{2} \ln x - \frac{3}{4} x^2 + c_1 x + c_2 \right) dx + c_3 =$$

$$= \frac{x^3}{6} \ln x - \int \frac{x^{\cancel{3}2}}{6} \cdot \frac{1}{\cancel{x}} \, dx + c_1 \frac{x^2}{2} + c_2 x + c_3$$

$$\boxed{y(x) = \frac{x^3}{6} \ln x - \frac{1}{18} x^3 + c_1 \frac{x^3}{6} + c_2 \frac{x^2}{2} + c_3 , \; c_1, c_2, c_3 \in \mathbb{R}}$$

$\underline{\text{II}}$ Diff. Eq. of the form: $F(x, y^{(k)}, y^{(k+1)}, \dots, y^{(n)}) = 0$, $k \geq 2$

subst $\boxed{y^{(k)} = z}$ $\Rightarrow$ $\boxed{F(x, z, z', \dots, z^{(n-k)}) = 0}$: a diff. eq of order $n-k$.

if. the $n-k$ order diff. eq is <u>solvable</u> and the solution is obtained in <u>explicit</u> form $\boxed{z(x) = \varphi(x, c_1, \dots, c_{n-k})}$.

$\Rightarrow$ $\boxed{y^{(k)} = z}$ $\Rightarrow$ $y^{(k)}(x) = \varphi(x, c_1, \dots, c_{n-k})$ (a k order diff. eq of type $\underline{\text{I}}$)

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$\vdots$

$$\boxed{y(x) = \psi(x, c_1, \dots, c_n)}$$

<u>Ex 2</u>. Solve the diff. eqs.:

a) $y^{(4)} + y''' = 0$

b) $xy''' - 3y'' - 4x^2 = 0$

c) $y' \cdot (1 + y') = a \cdot y''$, $a \in \mathbb{R}^*$

d) $xy'' + y' + x = 0$

a) $y^{(4)} + y''' = 0$

subst $\boxed{z = y'''}$ $\Rightarrow$ $z' + z = 0$ a first order linear homogeneous diff. eq.

$z' = -z \Rightarrow \frac{dz}{dx} = -z \Rightarrow \int \frac{dz}{z} = -\int dx \Rightarrow \ln z = -x + \ln c$

$\Rightarrow \boxed{z(x) = c.e^{-x}, c \in \mathbb{R}.}$

$y''' = z \Rightarrow y''' = c_1 e^{-x} \Rightarrow (y'')' = c_1 e^{-x} \Rightarrow$

$y'' = \int c_1 e^{-x} dx + c_2$

$y'' = -c_1 e^{-x} + c_2$

$(y')' = -c_1 e^{-x} + c_2 \Rightarrow y' = \int(-c_1 e^{-x} + c_2)dx + c_3 =$

$y' = c_1 e^{-x} + c_2 x + c_3 \Rightarrow$

$\Rightarrow y(x) = \int(c_1 e^{-x} + c_2 x + c_3)dx + c_4$

$\boxed{y(x) = -c_1 e^{-x} + c_2 \frac{x^2}{2} + c_3 x + c_4, \quad c_1, c_2, c_3, c_4 \in \mathbb{R}}$

b) $xy''' - 3y'' - 4x^2 = 0$

subst $\boxed{y'' = z}$ $\Rightarrow$ $xz' - 3z - 4x^2 = 0$

$xz' - 3z = 4x^2 \mid : x$

$\boxed{z' - \frac{3}{x} \cdot z = 4 \cdot x}$ first order nonhomogeneous linear diff. eq.

$z' - \frac{3}{x} z = 0 \Rightarrow z' = \frac{3}{x} z \Rightarrow \frac{dz}{dx} = \frac{3}{x} \Rightarrow \int \frac{dz}{z} = \int \frac{3}{x} dx \Rightarrow$

$\Rightarrow \ln z = 3 \cdot \ln x^3 + \ln c$

$z_0(x) = c \cdot x^3, \ c \in \mathbb{R}.$

$z_p(x) = c(x) \cdot x^3$: $\quad z_p' - \frac{3}{x} \cdot z_p = 4 \cdot x$

$c'(x) \cdot x^3 + \cancel{c(x) \cdot 3x^2} - \cancel{\frac{3}{x} \cdot c(x) \cdot x^3}^{2} = 4x$

$c'(x) \cdot x^3 = 4x \Rightarrow c'(x) = \frac{4}{x^2} \Rightarrow c(x) = \int \frac{4}{x^2} dx = -\frac{4}{x}$

$\Rightarrow z_p(x) = c(x) \cdot x^3 = -\frac{4}{x} \cdot x^3 = -4x^2$

the gen. sol. of. diff. eq. in z :

$z = z_0 + z_p \Rightarrow \boxed{z(x) = c \cdot x^3 - 4x^2}$

$$y'' = z \Rightarrow y'' = c_1 x^3 - 4x^2$$

$$y' = \int (c_1 x^3 - 4x^2)\,dx + c_2$$

$$y' = c_1 \frac{x^4}{4} - 4 \cdot \frac{x^3}{3} + c_2$$

$$y = \int \left(c_1 \frac{x^4}{4} - \frac{4}{3} x^3 + c_2\right) dx + c_3$$

$$\Rightarrow \boxed{y(x) = c_1 \frac{x^5}{20} - \frac{1}{3} x^4 + c_2 x + c_3, \quad c_1, c_2, c_3 \in \mathbb{R}}$$

III Diff. Eq. of the form: $F(y, y', \dots, y^{(n)}) = 0$.

$y = y(x)$.

subst $y' = p(y)$

$$y'' = (y')' = (p(y))' = p'(y) \cdot y' = p'(y) \cdot p(y)$$

the order of the diff. eq is reduced by 1.

$$\Rightarrow \dots \Rightarrow G(y, p(y), p'(y), \dots, p^{(n-1)}(y)) = 0$$

if eq. in $p=p(y)$ is solvable and the sol. is obtained in explicit form $\underline{p(y)=\varphi(y, c_1, \dots, c_{n-1})}$ then.

$$y' = p(y) \Rightarrow \left| \underline{y' = \varphi(y, c_1, \dots, c_n)} \right|$$ first order separable var. diff. eq

$\underline{y=y(x)}$

<u>Ex. 3</u>. Solve the diff. eqs.:

a) $\begin{cases} y^2 + (y')^2 - 2y \cdot y'' = 0 \\ y(0) = y'(0) = 1 \end{cases}$

b) $yy'' - 2yy' \ln y = (y')^2$

c) $y'' - \dfrac{(y')^2}{y} - yy' = 0$

a) $y^2 + (y')^2 - 2y\cdot y'' = 0$

$y' = p(y) \Longrightarrow y'' = p'(y)\cdot p(y)$

$y'' = p'(y)\cdot \underbrace{y'}_{p(y)}$

$\Rightarrow y^2 + p^2(y) - 2\cdot y\cdot p'(y)\cdot p(y) = 0$ ← p is the unknown function with respect to y.

$2y\cdot p'\cdot p = y^2 + p^2$

$$p' = \frac{y^2+p^2}{2y\cdot p}$$

$$p' = \frac{y^{\cancel{2}}\left(1+\left(\frac{p}{y}\right)^2\right)}{2\cancel{y}\cdot p}$$

$$\boxed{p' = \frac{1+\left(\frac{p}{y}\right)^2}{2\cdot\frac{p}{y}}}$$

a first order homogeneous eq. in the Euler sens.

$\left(y' = f\left(\frac{y}{x}\right)\right)$

$$\boxed{z = \frac{p}{y}} \qquad z(y) = \frac{p(y)}{y}$$

$$p = z \cdot y$$

$$p'(y) = z + z' \cdot y \Rightarrow$$

$$\Rightarrow \quad z + y \cdot z' = \frac{1+z^2}{2 \cdot z} \Rightarrow y \cdot z' = \frac{1+z^2}{2 \cdot z} - \frac{2z}{2}$$

$$\Rightarrow y \cdot z' = \frac{1+z^2-2z^2}{2z} \Rightarrow y \cdot z' = \frac{1-z^2}{2z}$$

$$\boxed{z' = \frac{1}{y} \cdot \frac{1-z^2}{2z}}$$ separable diff. eq.

$$1 - z^2 = 0 \Rightarrow z^2 = 1$$

$$z = \pm 1$$

$$\boxed{z(y) \equiv \pm 1 \text{ singular sol.}}$$

$$z' = \frac{dz}{dy} \Rightarrow \frac{2z\,dz}{1-z^2} = \frac{1}{y}\,dy \quad \Big| \cdot (-1)$$

$$\int \frac{-2z}{1-z^2}\,dz = \int -\frac{1}{y}\,dy \Rightarrow \ln(1-z^2) = -\ln y + \ln c$$

$1 - z^2 = \frac{c}{y} \Rightarrow \boxed{z^2 = 1 - \frac{c}{y}}$

$z = \frac{p}{y} \rightarrow \left(\frac{p}{y}\right)^2 = 1 - \frac{c}{y} \quad | \cdot y^2$

$$p^2 = y^2 - c \cdot y$$

$$(y')^2 = y^2 - c\,y$$

$y'(0) = 1$ $\quad (y'(0))^2 = y(0)^2 - c \cdot y(0)$

$y(0) = 1$ $\quad 1 = 1 - c \cdot 1 \Rightarrow \boxed{c = 0}$.

$\Rightarrow (y')^2 = y^2 \Rightarrow y' = \pm y$

$\frac{dy}{dx} = \pm y \Rightarrow \frac{dy}{y} = \pm dx \Rightarrow$

$\Rightarrow \ln y = \pm x + \ln c_1$

$y(x) = c_1 e^{\pm x}$

$y(0) = 1 \Rightarrow \quad c_1 = 1 \Rightarrow$ $\boxed{y(x) = e^{\pm x} \text{ an sol. of IVP.}}$