# Lecture 3

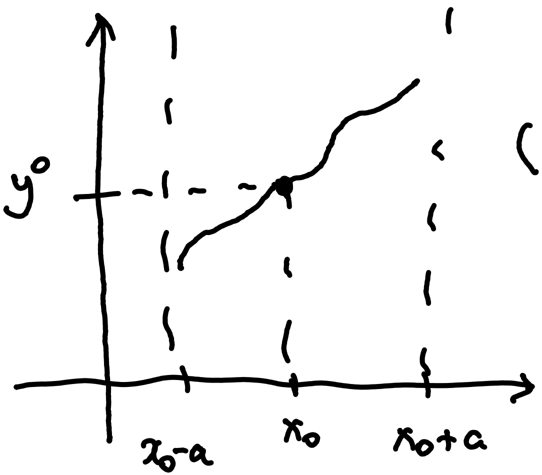
## The Cauchy problem (The initial Value Probl.)

## The Existence and Uniqueness Theorems

$$(1)\ \begin{cases} y' = f(x,y) \\ y(x_0) = y^0 \end{cases}\ (2)$$

$f : [x_0-a, x_0+a] \times \mathbb{R} \to \mathbb{R}$, $a > 0$

$y^0 \in \mathbb{R}$




**Lemma** The probl. $(1)+(2) \Leftrightarrow (3)$

$$(3)\quad y(x) = y^0 + \int_{x_0}^{x} f(s, y(s))\, ds.$$

(Volterra integral equation)

$y \in C[x_0-a, x_0+a]$

$y \longmapsto A(y)$

$$A(y)(x) = y^0 + \int_{x_0}^{x} f(s, y(s))\, ds.$$

$\forall x \in [x_0-a, x_0+a]$

(3) $\Longleftrightarrow$ find $y \in C[x_0-a, x_0+a]$ such that

$$y(x) = A(y)(x), \quad \forall x \in [x_0-a, x_0+a]$$

$$y = A(y)$$

$\Longleftrightarrow$ the solution of (3) is a fixed point of operator $A$.

$$A : C[x_0-a, x_0+a] \xrightarrow{?} C[x_0-a, x_0+a]$$

if $\underline{y \in C[x_0-a, x_0+a]}$ $\overset{?}{\Longrightarrow}$ $A(y) \in C[x_0-a, x_0+a]$

impose the condition:

$$\boxed{f \in C([x_0-a, x_0+a] \times \mathbb{R}, \mathbb{R})}$$

the continuity condition $\Longrightarrow$

$\Longrightarrow$ $A(y) \in C[x_0-a, x_0+a]$.

$X = (C[x_0-a, x_0+a], \|\cdot\|)$ a Banach space.

$\|\cdot\|_c$ – Cebâșkev norm )

$$\|y\|_c = \max_{x \in [x_0-a, x_0+a]} |y(x)|$$

$\|\cdot\|_B$ – the Bielecki norm

$$\|y\|_B = \max_{x \in [x_0-a, x_0+a]} |y(x)| \cdot e^{-\tau |x-x_0|}, \quad \tau > 0$$

$(X, \|\cdot\|_B)$ a Banach space, $\|\cdot\|_C \sim \|\cdot\|_B$

contraction condition: $\exists \alpha \in (0,1)$

$$\|A(y) - A(z)\| \leq \alpha \|y - z\|, \quad \forall y, z \in X. \; ?$$

Let's take $y, z \in X$.

$$|A(y)(x) - A(z)(x)| = \left| \cancel{y^0} + \int_{x_0}^{x} f(s, y(s))\, ds - \cancel{y^0} - \int_{x_0}^{x} f(s, z(s))\, ds \right|$$

$$= \left| \int_{x_0}^{x} \big( f(s, y(s)) - f(s, z(s)) \big) ds \right| \leq$$

$$\leq \left| \int_{x_0}^{x} \left| f(s, y(s)) - f(s, z(s)) \right| ds \right| \leq$$

impose the condition:

$\exists L_f > 0 : \quad |f(s,u) - f(s,v)| \leq L_f \cdot |u - v| \, , \quad \forall s \in [x_0 - a, x_0 + a]$, $\forall u, v \in \mathbb{R}$.

$$\leq \left| \int_{x_0}^{x} L_f |y(s) - z(s)| \, ds \right| =$$

$$= \left| \int_{x_0}^{x} L_f \underbrace{|y(s) - z(s)| \cdot e^{-\tau |s - x_0|}}_{\leq \|y - z\|_B} \cdot e^{\tau |s - x_0|} \, ds \right| \leq$$

$$\leq L_f \cdot \|y - z\|_B \cdot \left| \int_{x_0}^{x} e^{\tau |s - x_0|} \, ds \right| \leq$$

$$\boxed{\left| \int_{x_0}^{x} e^{\tau |s - x_0|} \, ds \right| \overset{?}{\leq} \frac{1}{\tau} e^{\tau |x - x_0|}}$$

if $x \geq x_0 \Rightarrow x \geq s \geq x_0 \Rightarrow |s-x_0| = s-x_0$

$$\Rightarrow \left| \int_{x_0}^{x} e^{\tau|s-x_0|} ds \right| = \left| \int_{x_0}^{x} e^{\tau(s-x_0)} ds \right| =$$

$$= \left| \frac{1}{\tau} e^{\tau(s-x_0)} \Big|_{x_0}^{x} \right| = \frac{1}{\tau} \left| e^{\tau(x-x_0)} - 1 \right| =$$

$$= \frac{1}{\tau} \left( e^{\tau(x-x_0)} - 1 \right) \leq \frac{1}{\tau} e^{\tau(x-x_0)}$$

if $x \leq x_0 \Rightarrow x \leq s \leq x_0 \Rightarrow |s-x_0| = x_0-s$

$$\Rightarrow \left| \int_{x_0}^{x} e^{\tau|s-x_0|} ds \right| = \left| \int_{x_0}^{x} e^{\tau(x_0-s)} ds \right| =$$

$$= \left| \left(-\frac{1}{\tau}\right) \cdot e^{\tau(x_0-s)} \Big|_{x_0}^{x} \right| = \frac{1}{\tau} \left| \underbrace{e^{\tau(x_0-x)}}_{\geq 1} - 1 \right| =$$

$$= \frac{1}{\tau} \left( e^{\tau(x_0-x)} - 1 \right) \leq \frac{1}{\tau} e^{\tau(x_0-x)} = \frac{1}{\tau} e^{\tau|x-x_0|}$$

$$\leq L_f \cdot \|y-z\|_B \cdot \frac{1}{\tau} e^{\tau|x-x_0|}$$

$$|A(y)(x) - A(z)(x)| \leq L_f \cdot \|y-z\|_B \cdot \frac{1}{\tau} \cdot e^{\tau|x-x_0|}$$

$$\Rightarrow |A(y)(x) - A(z)(x)| \cdot e^{-\tau|x-x_0|} \leq \frac{L_f}{\tau} \cdot \|y-z\|_B, \forall x \in [x_0-a, x_0+a]$$

$$\underset{\max\limits_{x\in[x_0-a,x_0+a]}}{\Rightarrow} \|A(y)-A(z)\|_B \leq \frac{L_f}{\tau} \cdot \|y-z\|_B, \forall y,z \in X.$$

$\Rightarrow$ A is a lipschitz operator with $L_A = \frac{L_f}{\tau}$

$\underline{\tau > 0}$

$L_A < 1 \Rightarrow \tau > L_f$.

we choose $\tau = L_f + 1 \Rightarrow L_A = \frac{L_f}{L_f+1} < 1$

$\Rightarrow$ A is a contraction $\underset{\text{contraction principle}}{\Rightarrow}$

Theorem 1 (The existence and uniqueness th. in the space)

Let's consider the ivp (1)+(2). Suppose that:

(i) $f \in C([x_0-a, x_0+a] \times \mathbb{R}, \mathbb{R})$ (cont. cond.)

(ii) $\exists L_f > 0$ s.t. $|f(s,u) - f(s,v)| \leq L_f \cdot |u-v|$, (Lipschitz cond). $\forall s \in [x_0-a, x_0+a]$, $u, v \in \mathbb{R}$.

Then:

(a) ivp (1)+(2) has a unique sol. $y^* \in C[x_0-a, x_0+a]$

(b) the unique sol. $y^*$ can be obtained from any succesive approximation sequence starting from any point $y_0 \in C[x_0-a, x_0+a]$

(c) we have the estimation

$$\| A^n(y_0) - y^* \|_B \leq \frac{L_A^n}{1 - L_A} \| A(y_0) - y_0 \|_B, \forall y_0 \in C[x_0-a, x_0+a]$$

where $L_A = \frac{L_f}{L_f + 1}$

The succesive approximation sequence

$y_0 \in C[x_0-a, x_0+a]$

$$y_n = A^n(y_0) \qquad \boxed{y_{n+1} = A(y_n)}$$

$$\boxed{y_{n+1}(x) = y^0 + \int_{x_0}^{x} f(s, y_n(s))\, ds}$$

<u>Remark</u>

1) $u \in C^1(\mathbb{R})$, if $\exists\, M>0$ such that $|u'(x)| \le M$, $\forall x \in \mathbb{R}$ then $u$ is lipschitz with $L_u = M$

2) $f \in C^1([x_0-a, x_0+a] \times \mathbb{R}, \mathbb{R})$, $f = f(x,y)$

if $\exists\, M>0$ such that $\left|\frac{\partial f}{\partial y}(x,y)\right| \le M$, $\forall x \in [x_0-a, x_0+a]$, $\forall y \in \mathbb{R}$

$\Rightarrow$ $f$ is lipschitz with respect to the second variable and $L_f = M$

Example

$$\begin{cases} y' = \sqrt{y} \\ y(0) = 0 \end{cases}$$

$y(x) \equiv 0$, $y(x) = \frac{x^2}{4}$ are sols of IVP.

$x_0 = 0$, $y^0 = 0$

$f(x,y) = \sqrt{y}$

$f : [-a,a] \times [0,+\infty) \to \mathbb{R}$

$$\left| \frac{\partial f}{\partial y}(x,y) \right| = \left| \frac{1}{2\sqrt{y}} \right| \xrightarrow[y \to 0]{} +\infty$$

$\Rightarrow$ $\frac{\partial f}{\partial y}$ is not bounded on $[-a,a] \times [0,+\infty)$

$\Rightarrow$ $f$ does not satisfy the Lipschitz cond with respect $y$ variable.

## The continuous data dependence property

$$\begin{cases} y' = f(x,y) & (1) \\ y(x_0) = y^0 & (2) \end{cases} \quad \text{and} \quad \begin{cases} z' = g(x,z) & (4) \\ z(x_0) = z^0 & (5) \end{cases}$$

$$\boxed{f, g \in C([x_0 - a, x_0 + a] \times \mathbb{R}, \mathbb{R}) \quad , \quad y^0, z^0 \in \mathbb{R}.}$$

## The Abstract Data Dependence Th.

$(X, \|\cdot\|)$ a Banach space, $A, B : X \to X$ such that

(i) $A$ is a $L_A$-contraction $(\{y^*\} = F_A)$

(ii) $\exists\, z^* \in F_B$

(iii) $\| A(y) - B(y) \| \le \eta$, $\forall y \in X$

Then: $\quad \| y^* - z^* \| \le \dfrac{\eta}{1 - L_A}.$

$(1)+(2) \Longleftrightarrow (3) \quad y(x) = y^0 + \int_{x_0}^{x} f(s, y(s))\,ds.$

$(4)+(5) \Longleftrightarrow (6) \quad z(x) = z^0 + \int_{x_0}^{x} g(s, z(s))\,ds.$

$A, B : C[x_0 - a, x_0 + a] \to C[x_0 - a, x_0 + a]$

$$A(y)(x) = y^0 + \int_{x_0}^{x} f(s, y(s))\,ds$$

$$B(y)(x) = z^0 + \int_{x_0}^{x} g(s, y(s))\,ds$$

$(1)+(2)$ satisfy the cond from Th. 1 $\Rightarrow F_A = \{y^*\}$

$(4)+(5)$ suppose that has at least one sol. $z^* \Rightarrow z^* \in F_B$

$$|A(y)(x) - B(y)(x)| = \left| y^0 + \int_{x_0}^{x} f(s, y(s))\,ds - z^0 - \int_{x_0}^{x} g(s, y(s))\,ds \right|$$

$$\leq |y^0 - z^0| + \left| \int_{x_0}^{x} (f(s, y(s)) - g(s, y(s)))\,ds \right| \leq$$

$$\leq \underbrace{|y^0 - z^0|}_{\leq \eta_1} + \left| \int_{x_0}^{x} \underbrace{|f(s, y(s)) - g(s, y(s))|}_{\leq \eta_2}\,ds \right| \leq$$

$$\leq \eta_1 + \eta_2 \left| \int_{x_0}^{x} ds \right| = \eta_1 + \eta_2 \underbrace{|x - x_0|}_{\leq a} \leq \eta_1 + \eta_2 \cdot a .$$

$$\Rightarrow \quad |A(y)(x) - B(y)(x)| \leq \eta_1 + \eta_2 \cdot a \quad \Big| \cdot e^{-\tilde{c}|x - x_0|}$$

$$\Rightarrow \quad |A(y)(x) - B(y)(x)| \, e^{-\tilde{c}|x - x_0|} \leq ( \quad ) \cdot \underbrace{e^{\cdots}}_{\leq 1} \leq \eta_1 + \eta_2 a$$

$$\overset{\max}{\Rightarrow} \quad \| A(y) - B(y) \|_B \leq \underbrace{\eta_1 + \eta_2 a}_{\eta} .$$

$$\Rightarrow \| y^* - z^* \|_B \leq \frac{\eta}{1 - L_A}$$

$$\left.\begin{array}{l} \eta_1 \to 0 \\ \eta_2 \to 0 \end{array}\right\} \Rightarrow \eta \to 0 \Rightarrow \| y^* - z^* \| \to 0$$

**Theorem 2** (The Continuous Data Dependence Theorem)

Let us consider the problems (1)+(2) and (4)+(5).

Suppose that:

(i) $f, g \in C([x_0-a, x_0+a]\times\mathbb{R}, \mathbb{R})$

(ii) $f$ is lipschitz with respect to the second variable on $\mathbb{R}$

(iii) $|f(x,u)-g(x,u)| \le \eta_2$, $\forall x \in [x_0-a, x_0+a]$, $\forall u \in \mathbb{R}$

(iv) the probl (4)+(5) has at least one sol. $z^*$

Then: $\|y^*-z^*\| \le \dfrac{\eta}{1-L_A}$, where $L_A = \dfrac{L_f}{L_f+1}$,

$y^*$ is the unique sol. of (1)+(2), $\eta = |y^0 - z^0| + \eta_2 \cdot a$