# Seminar 2

06.03.2025

It was proved in the previous seminar that

(1) $cl\,A \subseteq \mathbb{R}^n \setminus (int(\mathbb{R}^n \setminus A))$

Now, we claim that:

(2) $cl\,A \overset{?}{\supseteq} \mathbb{R}^n \setminus (int(\mathbb{R}^n \setminus A))$

Let $a \in \mathbb{R}^n \setminus (int(\mathbb{R}^n \setminus A)) \Rightarrow a \notin int(\mathbb{R}^n \setminus A)$

Assume by the contrary that $a \notin cl\,A$

$a \in cl\,A \iff \forall V \in \mathcal{V}(a) : V \cap A \neq \emptyset$ (def.)

$a \notin cl\,A \iff \exists V \in \mathcal{V}(a) : V \cap A = \emptyset \Rightarrow V \subseteq \mathbb{R}^n \setminus A$

$\Rightarrow \mathbb{R}^n \setminus A \in \mathcal{V}(a) \Rightarrow a \in int(\mathbb{R}^n \setminus A)$ $\lightning$

The obtained contradiction shows that $a \in cl\,A$, hence (2) holds

By (1),(2) $\Rightarrow cl\,A = \mathbb{R}^n \setminus (int(\mathbb{R}^n \setminus A))$

b) $cl\,A = (int\,A) \cup (bd\,A)$

We know that $int\,A \subseteq cl\,A$ ALWAYS

From the def of a boundary point, it follows that: $bd\,A = (cl\,A) \cap (\mathbb{R}^n \setminus A)$

$\Rightarrow bd\,A \subseteq cl\,A$

$\Rightarrow \underline{(int\,A) \cup (bd\,A) \subseteq cl\,A}$, (1)

We claim that $cl\,A \overset{?}{\subseteq} (int\,A) \cup (bd\,A)$ holds

Let $a \in cl\,A$

Assume by the contrary that $a \notin (int\,A) \cup (bd\,A)$

$\Rightarrow a \notin int\,A$ and $a \notin bd\,A$

$\exists V \in \mathcal{V}(a) : V \cap (\mathbb{R}^n \setminus A) = \emptyset$ or $V \cap A = \emptyset$

If $V \cap A = \emptyset \Rightarrow a \notin \mathrm{cl}\,A$ ↯
If $V \cap (\mathbb{R}^n \setminus A) = \emptyset \Rightarrow V \subseteq A \Rightarrow A \in \mathcal{V}(a) \Rightarrow a \in \mathrm{int}\,A$ ↯

The obtained contradictions show that $a \in (\mathrm{int}\,A) \cup (\mathrm{bd}\,A)$, hence $\mathrm{cl}\,A \subseteq (\mathrm{int}\,A) \cup (\mathrm{bd}\,A)$
By (1),(2) $\Rightarrow$ b) holds

c) $\mathrm{cl}\,A = A \cup A'$
We know that $A \subseteq \mathrm{cl}\,A$
From the def. of an adhearent/limit point $\}$
$\Rightarrow A' \subseteq \mathrm{cl}\,A$

$\Rightarrow \boxed{A \cup A' \subseteq \mathrm{cl}\,A}$ (1)

We prove now that $\mathrm{cl}\,A \subseteq A \cup A'$ (2)
Let $a \in \mathrm{cl}\,A$
Assume by the contrary $a \notin A \cup A' \Rightarrow a \notin A$ and $a \notin A'$
$\Downarrow$
$\exists V \in \mathcal{V}(a)$ s.t.
$V \cap A \setminus \{a\} = \emptyset$

$\left.\begin{array}{l} \Rightarrow V \cap A \subseteq \{a\} \\ \text{But } a \notin A \end{array}\right\} \Rightarrow V \cap A = \emptyset$ ↯

The obtained contradiction shows that $a \in A \cup A'$
$\Rightarrow \mathrm{cl}\,A \subseteq A \cup A'$
By (1),(2) $\Rightarrow$ c) holds

Vine la examen!

1.] Prove that every open (H) ball in $\mathbb{R}^n$ is an open set and that every closed ball in $\mathbb{R}^n$ is a closed set.

Solution: Let $a \in \mathbb{R}^n$ and let $r \in (0;\infty)$

(?) $\bar{B}(a,r)$ is closed $\Leftrightarrow \mathbb{R}^n \setminus \bar{B}(a,r)$ is open

$\Leftrightarrow \forall x \in \mathbb{R}^n \setminus \bar{B}(a,r)$, $\exists r'>0$ s.t. $B(x,r') \subseteq \mathbb{R}^n \setminus \bar{B}(a,r)$

nu trebuie scris. TRIVIAL

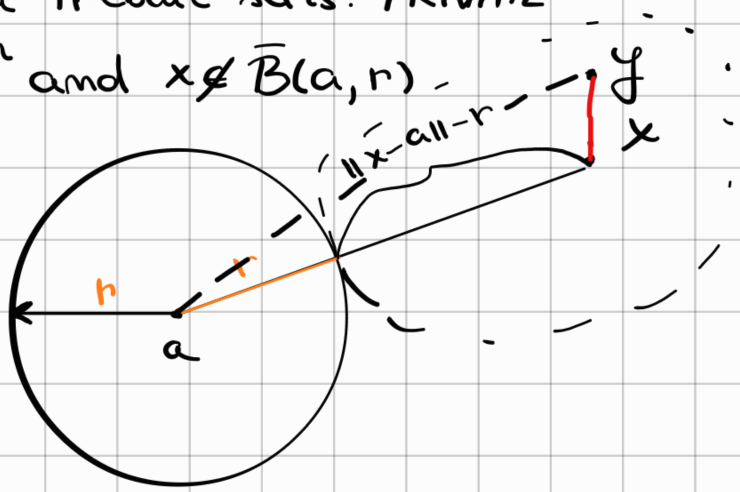
Let $x \in \mathbb{R}^n \setminus \bar{B}(a,r) \Rightarrow x \in \mathbb{R}^n$ and $x \notin \bar{B}(a,r)$

$\Rightarrow \|x-a\| > r$

Let $r' = \|x-a\| - r > 0$

We claim that

$B(x,r') \subseteq \mathbb{R}^n \setminus \bar{B}(a,r)$




Let $y \in B(x,r')$. We have $\|y-a\| \geq \|x-a\| - \underbrace{\|x-y\|}_{<r'} > \|x-a\| - \underbrace{r'}_{= r}$

$\Rightarrow \|y-a\| > r \Rightarrow y \notin \bar{B}(a,r) \Rightarrow y \in \mathbb{R}^n \setminus \bar{B}(a,r)$

2.] Given two sets $A_1, A_2 \subseteq \mathbb{R}^n$, prove that $cl(A_1 \cup A_2) = (cl\, A_1) \cup (cl\, A_2)$. Is it true that $cl(\bigcup_{i\in I} A_i) = \bigcup_{i\in I}(cl\, A_i)$ $\forall (A_i)_{i\in I}$ family of subsets of $\mathbb{R}^n$?

Solution: From the def. of an adhearant point

$\Rightarrow$ $A \subseteq B \Rightarrow cl\, A \subseteq cl\, B$

$$\left.\begin{array}{l} A_1 \subseteq A_1 \cup A_2 \Rightarrow cl\, A_1 \subseteq cl(A_1 \cup A_2) \\ A_2 \subseteq A_1 \cup A_2 \Rightarrow cl_2\, A \subseteq cl(A_1 \cup A_2) \end{array}\right\} \Rightarrow$$

$\Rightarrow (cl\, A_1) \cup (cl\, A_2) \subseteq cl(A_1 \cup A_2)$

We claim that $cl(A_1 \cup A_2) \overset{?}{\subseteq} (cl\, A_1) \cup (cl\, A_2)$

Let $x \in cl(A_1 \cup A_2) \Rightarrow \forall V \in \mathcal{V}(x): V \cap (A_1 \cup A_2) \neq \emptyset$

$\Rightarrow (V \cap A_1) \cup (V \cap A_2) \neq \emptyset \quad \forall V \in \mathcal{V}(x)$

$\Rightarrow V \cap A_1 \neq 0 \quad \text{or} \quad V \cap A_2 \neq \emptyset \quad \forall V \in \mathcal{V}(x)$

$\cancel{\Rightarrow} [\forall V \in \mathcal{V}(x) \quad V \cap A_1 \neq \emptyset] \text{ OR } [\forall V \in \mathcal{V}(x) \; V \cap A_2 \neq \emptyset]$

$\Rightarrow x \in \mathrm{cl}\, A_1 \quad \text{or} \quad x \in \mathrm{cl}\, A_2$

$\Rightarrow x \in (\mathrm{cl}\, A_1) \cup (\mathrm{cl}\, A_2)$

$\forall m \in \mathbb{Z}$, $m$ is even or $m$ is odd $\rightarrow$ TRUE

$[\forall m \in \mathbb{Z}$, $m$ is even$]$ or $[\forall m \in \mathbb{Z}$, $m$ is odd$]$ $\rightarrow$ FALSE

Correct solution: Let $x \in \mathrm{cl}\,(A_1 \cup A_2)$

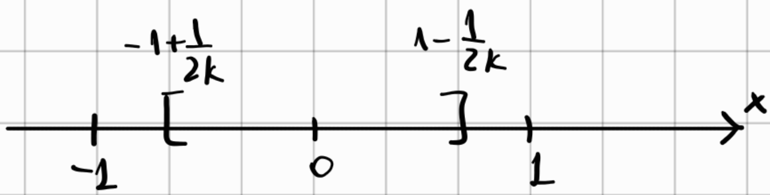
Assume by the contrary that $x \notin (\mathrm{cl}\, A_1) \cup (\mathrm{cl}\, A_2)$

$\Rightarrow x \notin \mathrm{cl}\, A_1$ and $x \notin \mathrm{cl}\, A_2$

$\Rightarrow \exists V_1 \in \mathcal{V}(x) : V_1 \cap A_1 = \emptyset$ and $\exists V_2 \in \mathcal{V}(x) : V_2 \cap A_2 = \emptyset$

Let $V = V_1 \cap V_2 \Rightarrow V \in \mathcal{V}(x)$

$V \cap (A_1 \cup A_2) = \underbrace{(V \cap A_1)}_{=\emptyset} \cup \underbrace{(V \cap A_2)}_{=\emptyset} = \emptyset$ in contradiction with $x \in \mathrm{cl}\,(A_1 \cup A_2)$




$A_k = [-1 + \frac{1}{2k} ; 1 - \frac{1}{2k}] = \mathrm{cl}\, A_k$

$\bigcup_{k \in \mathbb{N}} A_k = \bigcup_{k \in \mathbb{N}} \mathrm{cl}\, A_k = (-1, 1)$

$\neq$ ↯

$\mathrm{cl} \bigcup_{k \in \mathbb{N}} A_k = \mathrm{cl}\,(-1; 1) = [-1; 1]$

3] Given two sets $A, B \subseteq \mathbb{R}^n$, prove that

a) If $A \cup B = \mathbb{R}^n$, then $(\mathrm{cl}\, A) \cup (\mathrm{int}\, B) = \mathbb{R}^n$

b) If $A \cap B = \emptyset$, then $(\mathrm{cl}\, A) \cap (\mathrm{int}\, B) = \emptyset$

<u>Solution:</u> a) Obviously, $(\mathrm{cl}\, A) \cup (\mathrm{int}\, B) \subseteq \mathbb{R}^n$ (1)

We claim that $\mathbb{R}^n \overset{?}{\subseteq} (\mathrm{cl}\, A) \cup (\mathrm{int}\, B)$ (2)

Let $a \in \mathbb{R}^n$. Assume by the contrary that $a \notin (\mathrm{cl}\, A) \cup (\mathrm{int}\, B)$

$\Rightarrow a \notin \mathrm{cl}\, A$ and $a \notin \mathrm{int}\, B$

$$\Downarrow$$

$$\left.\begin{array}{l} \exists V \in \mathcal{V}(a) : V \cap A = \emptyset \Rightarrow V \subseteq \mathbb{R}^n \setminus A \\ A \cup B = \mathbb{R}^n \Rightarrow \mathbb{R}^n \setminus A \subseteq B \end{array}\right\} \Rightarrow V \subseteq B \Rightarrow B \in \mathcal{V}(a)$$

$$\Downarrow$$

$$a \in \mathrm{int}\, B \quad ⚡$$

b) Assume that $(\mathrm{cl}\, A) \cap (\mathrm{int}\, B) \neq \emptyset \Rightarrow \exists\, x \in (\mathrm{cl}\, A) \cap (\mathrm{int}\, B)$

$\Rightarrow x \in \mathrm{cl}\, A$ and $x \in \mathrm{int}\, B$

$$\underbrace{\forall V \in \mathcal{V}(x) : V \cap A \neq \emptyset \qquad B \in \mathcal{V}(x)}$$

$$\Downarrow$$

$A \cap B \neq \emptyset$ ⚡ with hypothesis $(A \cap B = \emptyset)$

4) Let $(x_k)$ be a conv. seq. in $\mathbb{R}^2$ and let $x = \lim_{k\to\infty} x_k$.

Prove that the set $A = \{x\} \cup \{x_1, x_2, x_3, \ldots\}$ is compact.

(?) A - compact $\Leftrightarrow$ A - bounded & closed

(?) A - bounded

$(x_k) \to x \Leftrightarrow \|x_k - x\| \to 0 \quad \Rightarrow \exists\, r > 0$ s.t. $\|x_k - x\| < r \ \forall k \in \mathbb{N}$

$$\Rightarrow \|x_k\| = \|x_k - x + x\| \leq \underbrace{\|x_k - x\|}_{<r} + \|x\| < r + \|x\|$$

$$\left.\begin{array}{ll} \Rightarrow \|x_k\| < r + \|x\| & \forall k \in \mathbb{N} \\ \|x\| < r + \|x_k\| & \forall k \in \mathbb{N} \end{array}\right\} \Rightarrow A \subseteq \bar{B}(0_2, r + \|x\|)$$

$$\Downarrow$$

A is bounded

(?) A - closed $\Leftrightarrow \mathbb{R}^2 \setminus A$ is open

$\Leftrightarrow \forall a \in \mathbb{R}^2 \setminus A, \exists r > 0$ s.t. $B(a, r) \subseteq \mathbb{R}^2 \setminus A$

Let $a \in \mathbb{R}^2 \setminus A \Rightarrow a \neq x \Rightarrow \|x - a\| > 0$

Let us denote by : $\varepsilon = \frac{\|x - a\|}{2}$

Since $(x_k) \to x \Rightarrow \exists k_0 \in \mathbb{N}$ s.t. $\forall k > k_0, \|x_k - x\| < \varepsilon$

$\Leftrightarrow \forall k > k_0, x_k \in B(x_k, \varepsilon)$

Let $r = \min\{\varepsilon, \|x_1 - a\|, \|x_2 - a\|, \dots, \|x_{k_0} - a\|\} > 0$

Then $B(a, r) \cap A = \emptyset \Rightarrow \underline{B(a, r) \subseteq \mathbb{R}^2 \setminus A}$