# LU decomposition. Methods for solving nonlinear equations (1.25p)

**A**

1. Consider the system $A\mathbf{x} = b$, where

$$A = \begin{pmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 1 & 3 & 4 & 4 \\ 0 & 2 & 1 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 4 \\ 15 \\ 3 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

   Solve the system using the LU decomposition of the matrix A.

2. Approximate $\sqrt[3]{2}$ using 2 iterations of Newton's method and the starting point $x_0 = 1$. (Hint: You may use the equation $x^3 = 2$).

3. Compute the next two iterates of the secant, false position and bisection methods to solve the equation $x^3 = 2x + 2$, using $x_0 = 1$, $x_1 = 2$.

4. Show that the function $g(x) = \pi + \frac{1}{2}\sin\frac{x}{2}$ has a unique fixed point in the interval $[0, 2\pi]$. How many iterations are necessary to find a fixed point that is accurate to within $10^{-2}$, starting with $x_0 = 0$? Compute $x_1$ and $x_2$.

Facultative:

- Check that the function $f(x) = x^3 + 2x^2 - 1$ has at least one root in $[0, 2]$. Compute the first 3 iterations for the bisection method.

- Show that the formula for the secant method is algebraically equivalent to

$$x_{n+1} = \frac{f(x_n)x_{n-1} - f(x_{n-1})x_n}{f(x_n) - f(x_{n-1})}$$

  and then compute $x_2$ to approximate the solution of the equation $x = x^2 - e^{-x}$, starting with $x_0 = -1,\ x_1 = 1$.

- Consider Newton's method for finding the positive square root of $A > 0$. Assuming that $x_0 > 0,\ x_0 \neq \sqrt{A}$, show that the sequence of iterates can be written as

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{A}{x_n}\right).$$

- Suppose that $a$ is a zero of multiplicity $m$ of $f$, where $f^{(m)}$ is continuous on an open interval that contains $a$. Show that the following fixed-point method has $g'(a) = 0$:

$$g(x) = x - m\frac{f(x)}{f'(x)}.$$

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**B**

1. Consider the system $A\mathbf{x} = b$, where

$$A = \begin{pmatrix} 2 & 0 & 4 & 1 \\ 0 & 2 & 4 & 1 \\ 2 & 4 & 3 & 0 \\ 1 & 2 & 0 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} -2 \\ -4 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

   Solve the system using the LU decomposition of the matrix A.

2. Approximate $\sqrt[3]{3}$ using 2 iterations of Newton's method and the starting point $x_0 = 1$. (Hint: You may use the equation $x^3 = 3$).

3. Compute the next two iterates of the secant, false position and bisection methods to solve the equation $x^3 = 2x - 2$, using $x_0 = -2$, $x_1 = 0$.

4. Show that the function $g(x) = \pi + \frac{1}{2}\sin\frac{x}{2}$ has a unique fixed point in the interval $[0, 2\pi]$. How many iterations are necessary to find a fixed point that is accurate to within $10^{-2}$, starting with $x_0 = 0$? Compute $x_1$ and $x_2$.

Facultative:

- Check that the function $f(x) = x^3 + 2x^2 - 1$ has at least one root in $[0, 2]$. Compute the first 3 iterations for the bisection method.
- Show that the formula for the secant method is algebraically equivalent to

$$x_{n+1} = \frac{f(x_n)x_{n-1} - f(x_{n-1})x_n}{f(x_n) - f(x_{n-1})}$$

  and then compute $x_2$ to approximate the solution of the equation $x = x^2 - e^{-x}$, starting with $x_0 = -1,\ x_1 = 1$.
- Consider Newton's method for finding the positive square root of $A > 0$. Assuming that $x_0 > 0,\ x_0 \neq \sqrt{A}$, show that the sequence of iterates can be written as

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{A}{x_n}\right).$$

- Consider Newton's method for finding the positive square root of $A > 0$. Assuming that $x_0 > 0,\ x_0 \neq \sqrt{A}$, show that the sequence of iterates can be written as

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{A}{x_n}\right).$$

- Suppose that $a$ is a zero of multiplicity $m$ of $f$, where $f^{(m)}$ is continuous on an open interval that contains $a$. Show that the following fixed-point method has $g'(a) = 0$:

$$g(x) = x - m\frac{f(x)}{f'(x)}.$$