Analiză reală

Seminar 12.

21.05.2025.

Ex. 1: Show that:

a) if $a>0$, $\alpha>0$, then the function $f:[a,\infty)\to\mathbb{R}$, $f(x)=x^{-\alpha}$ is Lebesgue integrable $\iff \alpha>1$

b) if $a>0$, $\alpha>0$, then the function $f:(0,a]\to\mathbb{R}$, $f(x)=x^{-\alpha}$ is Lebesgue integrable $\iff \alpha\in(0,1)$

c) there exists a Lebesgue integrable function f s.t. f^2 is not Leb. integrable

Sol: a) $f>0$ and continuous $\Rightarrow f$ is Lebesgue measurable

Let $f_n:[a,\infty)\to\mathbb{R}$, $f_n=f\underbrace{\chi_{[a,n]}}$, $n\in\mathbb{N}$, $n>a$

$\forall x\in[a,\infty)$, $\forall n_0\in\mathbb{N}$ s.t. $\forall n\geq n_0$, $x\in[a,n]\Rightarrow \chi_{[a,n]}(x)=1$

$\Rightarrow f_n(x)=f(x)$

$\Rightarrow \lim\limits_{n\to\infty} f_n=f$

$\forall n\in\mathbb{N}$, $n>a$, $[a,n]\subseteq[a,n+1]\Rightarrow \chi_{[a,n]}\leq\chi_{[a,n+1]}\overset{f>0}{\Longrightarrow}$

$\Rightarrow f_n\leq f_{n+1}$

$\Rightarrow (f_n)_{n\in\mathbb{N}}$ is non-decreasing. Hence, by MCT:

$$\int_{[a,\infty)} f\,d\lambda=\int_{[a,\infty)}\lim_{n\to\infty} f_n\,d\lambda=\lim_{n\to\infty}\int_{[a,\infty)} f_n\,d\lambda=\lim_{n\to\infty}\int_{[a,n]} f\,d\lambda$$

$$\int_a^n f(x)\,dx=\int_a^n x^{-\alpha}\,dx=\begin{cases}\left.\dfrac{x^{-\alpha+1}}{-\alpha+1}\right|_a^n, & \alpha\neq1\\[2mm] \left.\ln x\right|_a^n, & \alpha=1\end{cases}=$$

$$=\begin{cases}\dfrac{1}{-\alpha+1}\left(n^{-\alpha+1}-a^{-\alpha+1}\right), & \alpha\neq1\\ \ln n-\ln a, & \alpha=1\end{cases}$$

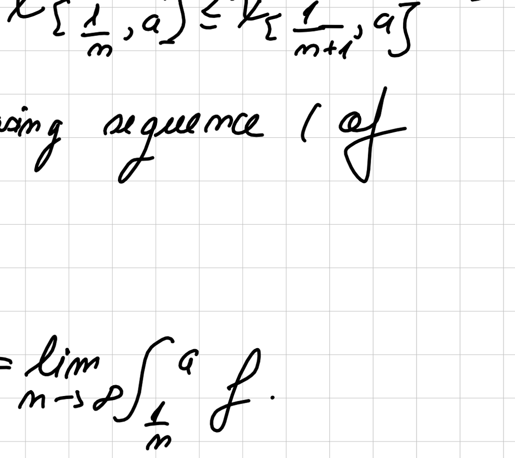
Case 1: $\alpha=1\Rightarrow \lim\limits_{n\to\infty}\int_a^n f=\infty$

Case 2: $\alpha\neq1\Rightarrow \lim\limits_{n\to\infty}\int_a^n f=\lim\limits_{n\to\infty}\dfrac{1}{-\alpha+1}\big(\overbrace{n^{-\alpha+1}}^{n^{-(\alpha-1)}\to0,\ \alpha>1;\ \to\infty,\ \alpha<1}-a^{-\alpha+1}\big)=$

$$=\begin{cases}\dfrac{a^{-\alpha+1}}{\alpha-1}, & \alpha>1\\ \infty, & \alpha\in(0,1)\end{cases}$$

$$\Rightarrow \int_{[a,\infty)} f\,d\lambda=\begin{cases}\infty, & \alpha\in(0,1]\\ \underbrace{\dfrac{a^{-\alpha+1}}{\alpha-1}}_{<\infty}, & \alpha>1\end{cases}\Rightarrow f \text{ is Lebesgue integrable} \iff \alpha>1$$

b) $f>0$ and continuous $\Rightarrow f$ is Lebesgue measurable



Let $f_n:(0,a]\to\mathbb{R}$, $f_n=f\chi_{[\frac1n,a]}$, $n\in\mathbb{N}$ s.t. $\frac1n<a$ ($\iff n>\frac1a$)

$\lim\limits_{n\to\infty} f_n=f$ ($\forall x\in(0,a]$, $\exists n_0\in\mathbb{N}$ s.t. $\forall n\geq n_0$ $x\in[\frac1n,a]\Rightarrow f_n(x)=f(x)$)

$\forall n\in\mathbb{N}$, $n>\frac1a$, $[\frac1n,a]\subseteq[\frac{1}{n+1},a]\Rightarrow \chi_{[\frac1n,a]}\leq\chi_{[\frac{1}{n+1},a]}\overset{f>0}{\Longrightarrow}$

$\Rightarrow f_{n+1}\geq f_n$. So, $(f_n)_{n\in\mathbb{N}}$ is a non-decreasing sequence (of non-negative measurable functions)

By MCT, $\int_{(0,a]} f\,d\lambda=\int_{(0,a]}\lim f_n\,d\lambda=\lim\limits_{n\to\infty}\int_{\frac1n}^a f$.

$$\int_{\frac1n}^a f(x)\,dx=\int_{\frac1n}^a x^{-\alpha}\,dx=\begin{cases}\left.\dfrac{x^{-\alpha+1}}{-\alpha+1}\right|_{\frac1n}^a=\dfrac{1}{-\alpha+1}\left(a^{-\alpha+1}-\dfrac{1}{n^{-\alpha+1}}\right), & \alpha\neq1\\ \left.\ln x\right|_{\frac1n}^a=\ln a+\ln n, & \alpha=1\end{cases}$$

Case 1: $\alpha=1\Rightarrow \lim\limits_{n\to\infty}\int_{\frac1n}^a f(x)\,dx=\infty$

Case 2: $\alpha\neq1$. $\lim\limits_{n\to\infty}\int_{\frac1n}^a f(x)\,dx=\begin{cases}\dfrac{a^{-\alpha+1}}{1-\alpha}, & \alpha\in(0,1)\\ \infty, & \alpha>1\end{cases}$

$$\Rightarrow \int_{(0,a]} f\,d\lambda=\begin{cases}\infty, & \alpha\geq1\\ \dfrac{a^{-\alpha+1}}{1-\alpha}, & \alpha\in(0,1)\end{cases}\Rightarrow f \text{ is Leb. integrable}$$

$\iff \alpha\in(0,1)$

c) We are looking for $\alpha\in(0,1)$ s.t. $2\alpha\notin(0,1)\iff\alpha\notin(0,\frac12)$

So, let $\alpha=\frac12$ and by using b) $f:(0,a]\to\mathbb{R}$, $f(x)=\frac{1}{\sqrt{x}}$ is Lebesgue integrable

$f^2:(0,a]\to\mathbb{R}$, $f^2(x)=\frac1x$ is not Lebesgue integrable.

Remark: The product of the two Leb. integrable functions is not necessary a Leb. integrable function.

$f:[a,b]\to\mathbb{R}$ is Riemann integrable $\iff f$ is continuous λ-a.e. (continuă peste tot în afară de o mulțime)

Ex. 2: Let us consider the functions $f,g,h:[0,\frac{\pi}{2}]\to\mathbb{R}$, $f(x)=\sin^2x$, $g(x)=\sin x$

$h(x)=\begin{cases}f(x), & x\in[0,\frac\pi2]\setminus\mathbb{Q}\\ g(x), & x\in[0,\frac\pi2]\cap\mathbb{Q}\end{cases}$. Study the Riemann and Lebesgue integrability of h.

Sol: Let $x\in[0,\frac\pi2]\Rightarrow\exists(x_n)_{n\in\mathbb{N}}\subseteq[0,\frac\pi2]\setminus\mathbb{Q}$ and $\exists(y_n)_{n\in\mathbb{N}}\in[0,\frac\pi2]\cap\mathbb{Q}$ s.t. $x_n\to x$, $y_n\to x$, as $n\to\infty$.

$h(x_n)=f(x_n)=\sin^2x_n\to\sin^2x$, as $n\to\infty$

$h(y_n)=g(y_n)=\sin y_n\to\sin x$, as $n\to\infty$.

If h is continuous at $x\Rightarrow\sin^2x=\sin x\Rightarrow\sin x\in\{0,1\}$

$\Rightarrow x\in\{0,\frac\pi2\}$

$\Rightarrow f$ is not continuous on $(0,\frac\pi2)$, $(0,\frac\pi2)\in\mathcal{L}(\mathbb{R})$, $\lambda((0,\frac\pi2))=\frac\pi2\neq0$

$\Rightarrow f$ is not continuous λ-a.e. $\Rightarrow f$ is not Rieman integrable.

$h=\underset{\text{Leb. measure}}{f}\cdot\underbrace{\chi_{[0,\frac\pi2]\setminus\mathbb{Q}}}_{=?}+g\,\chi_{[0,\frac\pi2]\cap\mathbb{Q}}$ is Leb. measurable

$\lambda([0,\frac\pi2]\cap\mathbb{Q})=0\Rightarrow h=f$ λ-a.e. $\Rightarrow\int_{[0,\frac\pi2]}h\,d\lambda=\int_{[0,\frac\pi2]}f\,d\lambda$

f is continuous $\Rightarrow f$ is Riemann integrable $[0,\frac\pi2]\Rightarrow$

$\Rightarrow f$ is Leb. integrable $\Rightarrow h$ is Leb. integrable

$$\int_{[0,\frac\pi2]}h\,d\lambda=\int_0^{\frac\pi2}\sin^2x\,dx=\int_0^{\pi/2}\frac{1-\cos2x}{2}\,dx=\frac\pi4-\int_0^{\frac\pi2}\frac{\cos2x}{2}\,dx.$$

$$=\frac\pi4-\underbrace{\left.\frac{\sin2x}{4}\right|_0^{\frac\pi2}}_{=0}=\frac\pi4$$

$\cos2x=\cos^2x-\sin^2x$
$\quad\ =\cos^2x+\sin^2x$
$1-\cos2x=2\sin^2x$

Thm. 1: (Differentiation of the integral depending on a parameter)

Let $(X,\mathcal{A},\mu)$ be a measurable space and $g:X\to[0,\infty]$ be an integrable function. Assume that $I\subseteq\mathbb{R}$ is an open interval and $f:I\times X\to\mathbb{R}$ is a function s.t.:

(i) $\forall t\in I$, the function $x\in X\mapsto f(t,x)$ is integrable

(ii) $\forall t\in I$, $\forall x\in X$, $\exists\frac{\partial f}{\partial t}(t,x)$ and $\left|\frac{\partial f}{\partial t}(t,x)\right|\leq g(x)$

Then the function $\varphi:I\to\mathbb{R}$, $\varphi(t)=\int f(t,x)\,d\mu(x)$ is differentiable and $\forall t_0\in I$, $\varphi'(t_0)=\int\frac{\partial f}{\partial t}(t_0,x)\,d\mu(x)$.

Ex. 3: (Laplace transform) Let $h:[0,\infty)\to\mathbb{R}$ be a Lebesgue integrable function. We define the function $\mathcal{L}\{h\}:(0,\infty)\to\mathbb{R}$, $\mathcal{L}\{h\}(t)=\int_{[0,\infty)}e^{-tx}h(x)\,d\lambda(x)$ and let us assume that $\int_{[0,\infty)}x|h(x)|\,d\lambda(x)<\infty$. Show that $\mathcal{L}\{h\}$ is differentiable.

Proof: Let $f:(0,\infty)\times[0,\infty)\to\mathbb{R}$, $f(t,x)=e^{-tx}h(x)$.

$\forall t>0$, $f(t,\cdot):[0,\infty)\to\mathbb{R}$ is Lebesgue measurable

$|f(t,x)|=\underbrace{e^{-tx}}_{\leq1}|h(x)|\leq|h(x)|\Rightarrow$

$\Rightarrow|f(t,\cdot)|$ is Leb. integrable. $\Rightarrow f(t,\cdot)$ is Lebesgue integrable.

$\forall t>0$, $\forall x\geq0$, $\frac{\partial f}{\partial t}(t,x)=-x\,h(x)\,e^{-tx}\Rightarrow$

$\Rightarrow\left|\frac{\partial f}{\partial t}(t,x)\right|=x\underbrace{e^{-tx}}_{\leq1}|h(x)|\leq x\,|h(x)|$

$x\in[0,\infty)\mapsto x|h(x)|$ is Leb. integrable

In view of Thm. 1, we obtain that $\mathcal{L}\{h\}$ is diff. and

$$\frac{d}{dt}\mathcal{L}\{h\}(t)=\int_{[0,\infty)}\frac{\partial f}{\partial t}(t,x)\,d\lambda(x)=$$

$$=-\int_{[0,\infty)}e^{-tx}\cdot x\,h(x)\,d\lambda(x)=-\mathcal{L}\{x\,h(x)\}(t).$$

Remark: In the classical definition of Laplace transform, $t\in\mathbb{C}$ s.t. the integral exists.