Seminar 8

17.04.2025

Compute the following integrals

$\boxed{1}$ $I = \int_0^1 \int_1^2 \frac{1}{x+y}\, dx\,dy$

<u>Solution:</u> $I = \int_{x=0}^{x=1} \left(\int_{y=1}^{y=2} \frac{1}{x+y}\, dy \right) dx = \int_{y=1}^{y=2} \left(\int_{x=0}^{x=1} \frac{1}{x+y}\, dx \right) dy$

$\Rightarrow I = \int_{x=0}^{x=1} \left(\ln(x+y) \Big|_{y=1}^{y=2} \right) dx = \int_0^1 (\ln(x+2) - \ln(x+1))\, dx$

$= \int_0^1 (x+2)' \cdot \ln(x+2)\, dx - \int_0^1 (x+1)' \cdot \ln(x+1)\, dx$

$= (x+2)\ln(x+2)\Big|_0^1 - \cancel{\int_0^1 dx} - (x+1)\cdot\ln(x+1)\Big|_0^1 + \cancel{\int_0^1 dx}$

$= 3\ln 3 - 2\ln 2 - 2\ln 2 + \underbrace{1\ln 1}_{0} = \underline{3\ln 3 - 4\ln 2}$

$\Rightarrow \underline{I = \ln\frac{27}{16}}$

$\boxed{2}$ $I = \int_0^1 \int_0^1 \frac{x}{(1+x^2+y^2)^{\frac{3}{2}}}\, dx\,dy$

<u>Solution:</u> $I = \int_{y=0}^{y=1} \left(\int_{x=0}^{x=1} \frac{x}{(1+x^2+y^2)^{3/2}}\, dx \right) dy$

$t = 1 + x^2 + y^2 \Rightarrow dt = 2x\, dx$

$I = \int_{y=0}^{y=1} \left(\int_{t=1+y^2}^{t=2+y^2} \frac{\frac{1}{2}\, dt}{t^{3/2}} \right) dy$ $\qquad \frac{1}{t^{\frac{3}{2}}} = t^{-\frac{3}{2}}$

$= \int_{y=0}^{y=1} \left(\frac{1}{\cancel{2}} \cdot \frac{t^{-\frac{1}{2}}}{-\frac{1}{\cancel{2}}} \Bigg|_{t=1+y^2}^{t=2+y^2} \right) dy$

$$= \int_0^1 \left(-\frac{1}{\sqrt{y^2+2}} + \frac{1}{\sqrt{y^2+1}} \right) dy = \ln\left(y + \sqrt{y^2+1}\right)\Big|_0^1 - \ln\left(y + \sqrt{y^2+2}\right)\Big|_0^1$$

$$\boxed{I = \ln \frac{2+\sqrt{2}}{1+\sqrt{3}}}$$

$\boxed{3}$ $$I = \int_0^{\pi} \int_0^{\frac{\pi}{2}} \frac{x \sin x \sin y}{1+\cos^2 x} \, dx dy \qquad 1+\cos^2 x =$$

Solution: $$I = \int_{x=0}^{x=\pi} \left(\int_{y=0}^{y=\frac{\pi}{2}} \frac{x \sin x \sin y}{1+\cos^2 x} \, dy \right) dx$$

constant

$$= \int_{x=0}^{x=\pi} \left(\frac{x \sin x}{1+\cos^2 x} \underbrace{\int_{y=0}^{y=\frac{\pi}{2}} \sin y \, dy}_{= I_2} \right) dx$$

$$I = I_2 \cdot \underbrace{\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} \, dx}_{= I_1} = I_1 \cdot I_2$$

$$\int_a^b f(x)\, dx \qquad x = a+b-t$$

$$I_2 = -\cos y \Big|_0^{\frac{\pi}{2}} = 1$$

$$I_1 = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} \, dx = -\int_{\pi}^{0} \frac{(\pi - t)\sin(\pi - t)}{1+\cos^2(\pi - t)} \, dt = \int_0^{\pi} \frac{(\pi - t)\sin t}{1+\cos^2 t} \, dt$$

$$x = \pi - t$$

$$= \int_0^{\pi} \frac{\pi \sin t}{1+\cos^2 t} \, dt - \underbrace{\int_0^{\pi} \frac{t \sin t}{1+\cos^2 t} \, dt}_{I_1}$$

$$\Rightarrow 2I_1 = \pi \int_0^{\pi} \frac{\sin t}{1+\cos^2 t} \, dt = -\pi \int_0^{\pi} \frac{(\cos t)'}{1+\cos^2 t} \, dt$$

$$= -\pi \operatorname{arctg}(\cos t) \Big|_0^{\pi} = -\pi \left(-\frac{\pi}{4} - \frac{\pi}{4} \right) = -\pi \cdot \left(-\frac{\pi}{2} \right) = \frac{\pi^2}{2}$$

$$2I_1 = \frac{\pi^2}{2} \Rightarrow \boxed{I_1 = \frac{\pi^2}{4}}$$

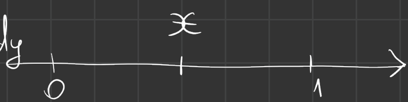
So, $\left.\begin{array}{l} I_1 = \frac{\pi^2}{4} \\ I_2 = 1 \end{array}\right\} = \_\_ = \boxed{I_1 \cdot I_2 = \frac{\pi^2}{4}}$

Rem: $$\boxed{\int_a^b \int_c^d f(x)\cdot g(y)\, dx\, dy = \left(\int_a^b f(x)\, dx\right)\left(\int_c^d g(y)\, dy\right)}$$

$\boxed{4}$ $I = \int_0^1 \int_0^1 \max\{x, y\}\, dx\, dy \rightarrow$ symmetric w.r.t. x and y

Solutions: $$I = \int_{x=0}^{x=1} \underbrace{\left(\int_{y=0}^{y=1} \max\{x, y\}\, dy \right)}_{= F_1(x)} dx = \int_0^1 F_1(x)\, dx$$

$$F_1(x) = \int_{y=0}^{y=x} \max\{x, y\}\, dy + \int_{y=x}^{y=1} \max\{x, y\}\, dy$$



$$= \int_{y=0}^{y=x} x\, dy + \int_{y=x}^{y=1} y\, dy$$

$$= xy\Big|_{y=0}^{y=x} + \frac{y^2}{2}\Big|_{y=x}^{y=1} = x^2 + \frac{1}{2} - \frac{x^2}{2} = \boxed{\frac{x^2+1}{2}}$$

$$I = \int_0^1 \frac{x^2+1}{2}\, dx = \frac{1}{2}\int_0^1 x^2\, dx + \int_0^1 dx = \frac{1}{2}\left(\frac{x^3}{3}\Big|_0^1 + x\Big|_0^1\right)$$

$$= \frac{1}{2}\cdot\frac{4}{3} = \frac{2}{3} \Rightarrow \boxed{I = \frac{2}{3}}$$

$\boxed{5}$ $I = \int_0^a \int_0^b \int_0^c (x+y+z)\, dx\, dy\, dz$, $\quad a, b, c > 0$

Solutions

$$I = \int_{x=0}^{x=a} \left(\int_{y=0}^{y=b} \left(\int_{z=0}^{z=c} (x+y+z)\, dz \right) dy \right) dx$$

$$= \int_{x=0}^{x=a} \left(\int_{y=0}^{y=b} \left(xz + yz + \frac{z^2}{2}\right)\Big|_{z=0}^{z=c} \right) dy \Big) dx$$

$$= \int_{x=0}^{x=a} \left(\int_{y=0}^{y=b} \left(cx + cy + \frac{c^2}{2}\right) dy \right) dx$$

$$= \int_{x=0}^{x=a} \left(cxy + c\frac{y^2}{2} + \frac{c^2}{2} y \Big|_{y=0}^{y=b} \right) dx$$

$$= \int_{x=0}^{x=a} \left(cbx + c\frac{b^2}{2} + \frac{c^2}{2} b \right) dx$$

$$= bc\frac{x^2}{2} + \frac{b^2c}{2}x + \frac{c^2b}{2}x \Big|_0^a = \frac{bca^2}{2} + \frac{b^2ca}{2} + \frac{c^2ba}{2}$$

$$\boxed{I = \frac{abc(a+b+c)}{2}}$$

$\boxed{6}$ $I = \int_1^2 \int_1^2 \int_1^2 \frac{1}{(x+y+z)^3} \, dx\,dy\,dz$

Solution $I = \int_{x=1}^{x=2} \left(\int_{y=1}^{y=2} \left(\int_{z=1}^{z=2} \frac{1}{(x+y+z)^3} dz \right) dy \right) dx$

$$= \int_{x=1}^{x=2} \left(\int_{y=1}^{y=2} \frac{(x+y+z)^{-2}}{-2} \Big|_{z=1}^{z=2} dy \right) dx$$

$$\frac{(x+y+z)^{-2}}{-2} = \frac{-1}{2(x+y+z)^2}$$

$$I = \frac{1}{2} \int_{x=1}^{x=2} \left(\int_{y=1}^{y=2} \left(\frac{-1}{(x+y+2)^2} + \frac{1}{(x+y+1)^2} \right) dy \right) dx$$

$$-\frac{1}{2}\int_{x=1}^{x=2}\left(\frac{1}{x+y+2}-\frac{1}{x+y+1}\Bigg|_{y=1}^{y=2}\right)dx$$

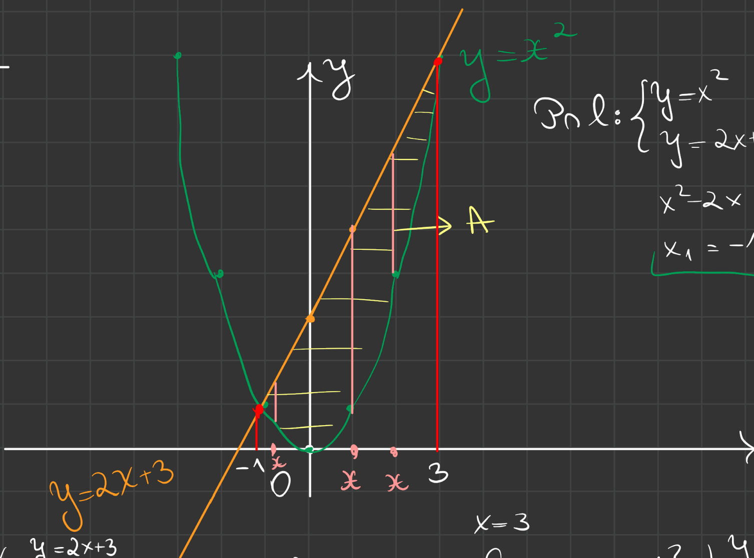
$$=\frac{1}{2}\int_{1}^{2}\left(\frac{1}{x+4}-\frac{1}{x+3}-\frac{1}{x+3}+\frac{1}{x+2}\right)dx$$

$$=\frac{1}{2}\left(\ln(x+4)-2\ln(x+3)+\ln(x+2)\right)\Bigg|_{1}^{2}$$

$$\Rightarrow \boxed{I=\frac{1}{2}\ln\frac{128}{125}}$$

$\boxed{7}$ $I=\iint_A (x+y)\,dx\,dy$ where A is the set bounded by the parabola $y=x^2$ and by the line $y=2x+3$

Solution



$$P\cap l:\begin{cases} y=x^2\\ y=2x+3\end{cases}$$

$$x^2-2x-3=0$$

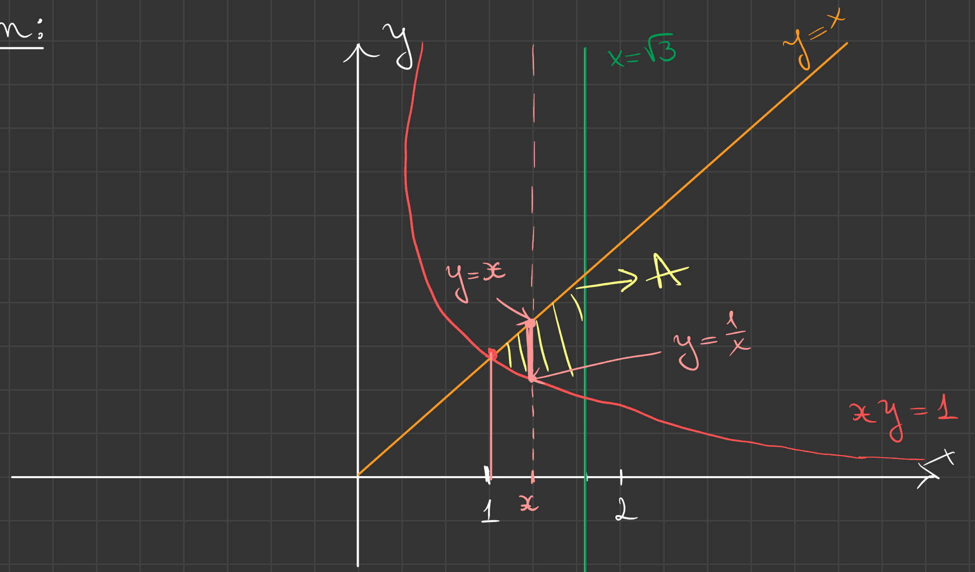
$$x_1=-1\;;\;x_2=3$$

$$I=\int_{x=-1}^{x=3}\left(\int_{y=x^2}^{y=2x+3}(x+y)\,dy\right)dx=\int_{x=-1}^{x=3}\left(xy+\frac{y^2}{2}\Bigg|_{y=x^2}^{y=2x+3}\right)dx$$

$$=\int_{x=-1}^{x=3}\left(x(2x+3-x^2)+\frac{(2x+3)^2-x^4}{2}\right)dx=\ldots$$

8 $I = \iint_A \frac{x}{y^2+1}\,dx\,dy$ where A is the set b e lines $x=\sqrt{3}$, $y=x$ and by the hyperbola $xy=1$ $\left(y=\frac{1}{x}\right)$

Solution:



$$I = \int_{x=1}^{x=\sqrt{3}} \left(\int_{y=\frac{1}{x}}^{y=x} \frac{x}{y^2+1}\,dy \right) dx = \int_{x=1}^{x=\sqrt{3}} \left(x \cdot \operatorname{arctg} y \Big|_{y=\frac{1}{x}}^{y=x} \right) dx$$

$$\operatorname{arctg} x + \operatorname{arctg} \frac{1}{x} = \begin{cases} \frac{\pi}{2}, & x>0 \\ -\frac{\pi}{2}, & x<0 \end{cases}$$

$$= \int_1^{\sqrt{3}} x\left(\operatorname{arctg} x - \operatorname{arctg}\frac{1}{x}\right) dx$$

Int. by parts

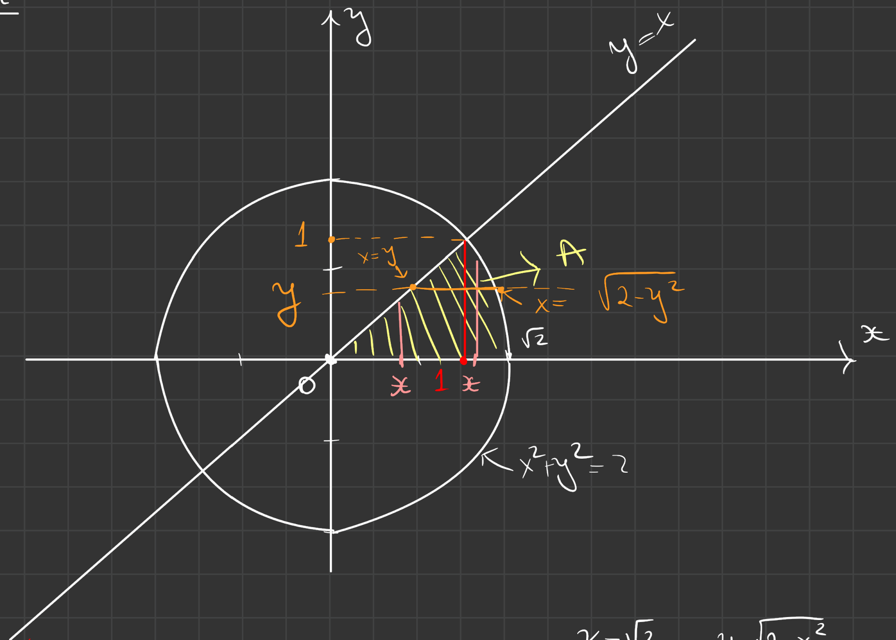
$$= \int_1^{\sqrt{3}} x\left(2\operatorname{arctg} x - \frac{\pi}{2}\right) dx = \int_1^{\sqrt{3}} 2x\operatorname{arctg} x\,dx - \frac{\pi}{2}\,\frac{x^2}{2}\Big|_1^{\sqrt{3}}$$

$$= \int_1^{\sqrt{3}} (x^2)' \cdot \operatorname{arctg} x\,dx - \frac{\pi}{2} = \ldots$$

$$\Rightarrow \boxed{I = 1 - \sqrt{3} + \frac{\pi}{3}}$$

$\boxed{9}$ $I = \iint_A \frac{x}{1+y^2}\,dx\,dy$ where A is $\{(x,y)\in\mathbb{R}^2 \mid x \geq y \geq 0,\ x^2+y^2 \leq 2\}$

Solution



$$I = \int_{x=0}^{x=\cancel{\sqrt{2}}\,1} \left(\int_{y=0}^{y=x} \frac{x}{y^2+1}\,dy \right) dx + \int_{x=1}^{x=\sqrt{2}} \left(\int_{y=0}^{y=\sqrt{2-x^2}} \frac{x}{y^2+1}\,dy \right) dx$$

$$I = \int_{y=0}^{y=1} \left(\int_{x=y}^{x=\sqrt{2-y^2}} \frac{x}{y^2+1}\,dx \right) dy$$

$$= \int_{y=0}^{y=1} \left(\frac{1}{y^2+1} \cdot \frac{x^2}{2} \Bigg|_{x=y}^{x=\sqrt{2-y^2}} \right) dy$$

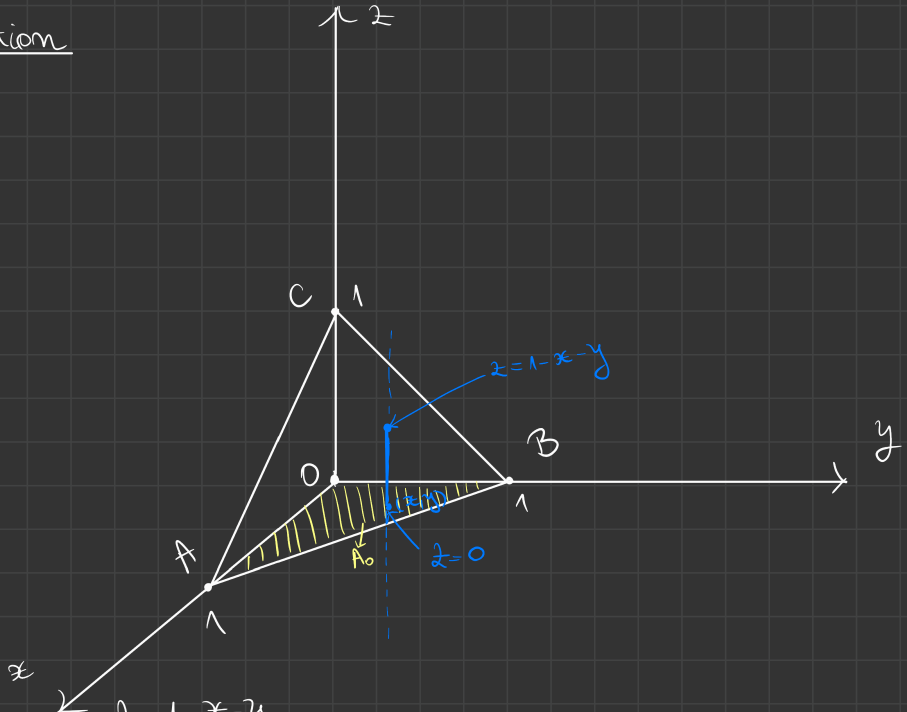
$$= \int_0^1 \left(\frac{1}{y^2+1} \cdot \frac{2-y^2-y^2}{2} \right) dy = \int_0^1 \frac{1-y^2}{1+y^2}\,dy$$

$$= \dots \Rightarrow \boxed{I = \frac{\pi}{2} - 1}$$

$\boxed{10}$ $I = \iiint_A \frac{1}{(x+y+z+1)^2}\, dx\, dy\, dz$

where A is bounded by the planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$

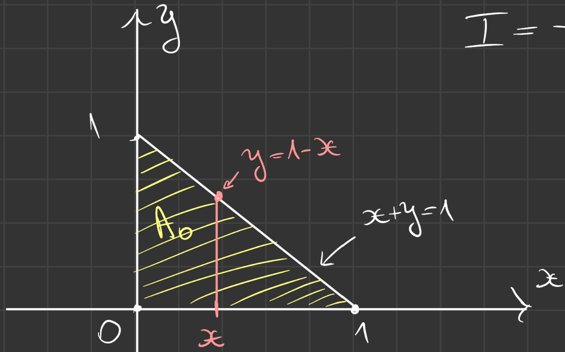
<u>Solution</u>



$$I = \iint_{A_0} \left(\int_{z=0}^{z=1-x-y} \frac{1}{(x+y+z+1)^2}\, dz \right) dx\, dy$$

$$= \iint_{A_0} \left(-\frac{1}{x+y+z+1} \Bigg|_{z=0}^{z=1-x-y} \right) dx\, dy$$

$$I = \iint_{A_0} \left(-\frac{1}{2} + \frac{1}{x+y+1} \right) dx\, dy$$



$$I = -\frac{1}{2}\underbrace{\iint_{A_0} dx\,dy}_{\text{area}(A_0)=\frac{1}{2}} + \iint_{A_0} \frac{1}{x+y+1}\,dx\,dy$$

$$I = -\frac{1}{4} + \int_{x=0}^{x=1}\left(\int_{y=0}^{y=1-x} \frac{1}{x+y+1}\,dy\right)dx$$

$$= -\frac{1}{4} + \int_{x=0}^{x=1} \ln(x+y+1)\Big|_{y=0}^{y=1-x}\,dx$$

$$= -\frac{1}{4} + \int_0^1 \ln 2 - \ln(x+1)\,dx = \ldots \Rightarrow \boxed{I = \frac{3}{4} - \ln 2}$$