Seminars 5 and 6 - 2025

Exercise 1:
Let $X_1, \dots, X_n$ be an i.i.d. sample from the exponential distribution with density

$$f(x;\lambda) = \lambda e^{-\lambda x}, \qquad x > 0,\ \lambda > 0.$$

Compute the Fisher information $I_n(\lambda)$.

Exercise 2:
Let $X_1, \dots, X_n$ be i.i.d. $N(\mu, \sigma^2)$ where σ^2 is known. Compute $I_n(\mu)$ and verify $I_n(\mu) = nI_1(\mu)$.

Exercise 3:
Using Exercise 1 result, give a lower bound for the variance of any unbiased estimator $\widehat{\lambda}$ of λ based on the sample of size n.

Exercise 4:
Let $X \sim \text{Bernoulli}(\theta)$, $0 < \theta < 1$. Compute $I_1(\theta)$ and $I_n(\theta)$.

Exercise 5:
For $X_1, \dots, X_n$ i.i.d. $N(\mu, \sigma^2)$ with known σ^2 and sample variance $\bar{X}$:

(a) Compute $\text{Var}(\overline{X})$.

(b) Compare it with the Cramér–Rao lower bound and conclude whether $\overline{X}$ is efficient.

Exercise 6:
Let $X_1, \dots, X_n$ be i.i.d. Poisson(λ). Show that $S = \sum_{i=1}^n X_i$ is sufficient for λ using the factorization theorem.

Exercise 7:
Let $X_1, \dots, X_n$ i.i.d. Poisson(λ). Consider the unbiased estimator $\widehat{\lambda}_0 = X_1$. (This is unbiased because $\mathbb{E}[X_1] = \lambda$.) Let $S = \sum_{i=1}^n X_i$ be sufficient. Use Rao–Blackwell by setting

$$\widehat{\lambda}^* = \mathbb{E}[X_1 \mid S]$$

and compute $\widehat{\lambda}^*$. Compare variances $\text{Var}(\widehat{\lambda}_0)$ and $\text{Var}(\widehat{\lambda}^*)$.

Exercise 8:
Let $X_1, \dots, X_n$ i.i.d. Bernoulli(p). Consider the crude unbiased estimator $\widehat{p}_0 = X_1$. Let $S = \sum_{i=1}^n X_i$, which is sufficient. Define $\widehat{p}^* = \mathbb{E}[X_1 \mid S]$.

(a) Compute $\widehat{p}^*$ explicitly as a function of S.

(b) Show $\widehat{p}^* = \dfrac{S}{n}$ and compare variances.

Exercise 9:
Let $X_1, \dots, X_n$ be i.i.d. Gamma(α, β) with pdf

$$f(x;\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \qquad x > 0,$$

where $\alpha > 0$ is known and $\beta > 0$ is unknown.

(a) Compute the Fisher information $I_n(\beta)$.
(b) Show that $S = \sum_{i=1}^n X_i$ is sufficient for β.
(c) Rao–Blackwellize $\widehat{\beta}_0 = X_1/\alpha$ and compare variances with the original estimator.

Exercise 10:
Let $X_1, \ldots, X_n \sim \text{Uniform}(0, \theta)$, $\theta > 0$. Consider the estimator $\widehat{\theta}_0 = X_1$.
(a) Show that $T = \max(X_1, \ldots, X_n)$ is sufficient for θ.
(b) Rao–Blackwellize $\widehat{\theta}_0$ using T and compare variances.