Seminar 6 - 2025

Theoretical part

The binomial probabilistic model

Repeated independent trials of an experiment such that there are only two possible outcomes for each trial - which we classify as either *success* or *failure* - and their probabilities remain the same throughout the trials are called **Bernoulli trials**. The binomial model describes the *number of successes* in a series of independent Bernoulli trials:

- *success* appears with probability p, *failure* with probability $1-p$;
- the experiment is repeated n times;
- the probability that success occurs k times in n trials for $k \in \mathbb{N}$, $k \in \{0, \dots, n\}$ is $\boldsymbol{C_n^k p^k (1-p)^{n-k}}$.

▶ $C_n^k p^k (1-p)^{n-k}$ represents the coefficient of x^k in the expansion $(px+1-p)^n$ for $k \in \{0, 1, \dots, n\}$.

▶ This model corresponds to the binomial distribution $Bino(n,p), n \in \mathbb{N}^*, p \in (0,1)$.

▶ **Example:** A die is rolled 10 times. The probability that the number 6 shows up 3 times is $C_{10}^3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^7$.

The multinomial probabilistic model

Consider $n \in \mathbb{N}^*$ independent trials such that each trial can have several possible mutually exclusive outcomes $O_1, \dots, O_j$ ($j \in \mathbb{N}^*$) with $P(O_i) = p_i \in (0,1)$, $i \in \{1, \dots, j\}$. Obviously, $p_1 + \dots + p_j = 1$. The probability that O_i occurs n_i times in n trials for $n_i \in \mathbb{N}$, $i \in \{1, \dots, j\}$ and $n_1 + \dots + n_j = n$ is $\boldsymbol{\dfrac{n!}{n_1! n_2! \dots n_j!} p_1^{n_1} p_2^{n_2} \dots p_j^{n_j}}$.

▶ $\dfrac{n!}{n_1! n_2! \dots n_j!} p_1^{n_1} p_2^{n_2} \dots p_j^{n_j}$ represents the coefficient of $x_1^{n_1} \cdot \ldots \cdot x_j^{n_j}$ in the expansion of $(p_1 x_1 + \ldots + p_j x_j)^n$.

▶ This model corresponds to the multinomial distribution $Multino(n, p_1, \dots, p_j)$, $n \in \mathbb{N}^*$, $p_1, \dots, p_j \in (0,1)$, $p_1 + \ldots + p_j = 1$.

▶ **Example:** Suppose that an urn contains 2 red marbles, 1 yellow marble and 3 blue marbles. 7 marbles are drawn randomly with replacement from the urn (each drawn marble is put back into the urn). The probability that there are drawn 3 red marbles, 2 yellow marbles and 2 blue marble is $\dfrac{7!}{3!2!2!}\left(\dfrac{2}{6}\right)^3 \left(\dfrac{1}{6}\right)^2 \left(\dfrac{3}{6}\right)^2$.

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1. Let S be the set of all positive integers less or equal than 50, with exactly 2 digits such that one is an even digit and the other is an odd digit. A number is randomly extracted from S. Let X be the sum of its digits. Write the probability distribution of X.

A: Let Y be the extracted number. We have:

- $X = 1$, if $Y \in \{10\}$.
- $X = 3$, if $Y \in \{12, 21, 30\}$.
- $X = 5$, if $Y \in \{14, 41, 23, 32, 50\}$.
- $X = 7$, if $Y \in \{16, 25, 34, 43\}$.
- $X = 9$, if $Y \in \{18, 27, 36, 45\}$.

- $X = 11$, if $Y \in \{29, 38, 47\}$.
- $X = 13$, if $Y \in \{49\}$.

So, $X \sim \left(\begin{array}{ccccccc} 1 & 3 & 5 & 7 & 9 & 11 & 13 \\ \frac{1}{21} & \frac{3}{21} & \frac{5}{21} & \frac{4}{21} & \frac{4}{21} & \frac{3}{21} & \frac{1}{21} \end{array} \right)$.

2. The probability that a chipset is defective equals 0.06. A circuit board has 12 such independent chipsets and it's functional if at least 11 chipsets are operating. 4 independent such circuit boards are installed in a computer unit. Compute the probabilities of the following events:
B:"A circuit board is functional."
C:"Exactly two circuit boards are functional in the computer unit."
D:"At least a circuit board is functional in the computer unit."
A: We use the binomial model: $p = P(B) = C_{12}^{11}(0.94)^{11}0.06 + (0.94)^{12}$; $P(C) = C_4^2 p^2(1-p)^2$; $P(D) = \sum_{k=1}^{4} C_4^k p^k (1-p)^{4-k} = 1 - (1-p)^4$.

3. Let (X, Y) be a discrete random vector with the joint probability distribution given by the following contingency table

X \ Y	-2	1	2
1	0.2	0.1	0.2
2	0.1	0.1	0.3

a) Find the probability distributions of X and Y.
b) Compute the probability that $|X - Y| = 1$, given that $Y > 0$.
c) Are the events $\{X = 2\}$ and $\{Y = 1\}$ independent?
d) Are the random variables X and Y independent?

A: a) $X \sim \begin{pmatrix} 1 & 2 \\ 0.5 & 0.5 \end{pmatrix}, Y \sim \begin{pmatrix} -2 & 1 & 2 \\ 0.3 & 0.2 & 0.5 \end{pmatrix}$.
b) $P(|X-Y| = 1 | Y > 0) = \frac{P(|X-Y|=1, Y>0)}{P(Y>0)} = \frac{P(X=1,Y=2)+P(X=2,Y=1)}{P(Y>0)} = \frac{0.2+0.1}{0.7} = \frac{3}{7}$.
c) $P(X = 2, Y = 1) = 0.1 = 0.5 \cdot 0.2 = P(X = 2) \cdot P(Y = 1) \implies$ the events $\{X = 2\}$ and $\{Y = 1\}$ are independent.
d) $P(X = 2, Y = 2) = 0.3 \neq 0.25 = 0.5 \cdot 0.5 = P(X = 2) \cdot P(Y = 2) \implies$ the random variables X and Y are not independent.

4. It was found that the probability to log on to a computer from a remote terminal is 0.7. Let X denote the number of attempts, which are independent, that must be made to gain access to the computer:
a) Write the probability distribution of X.
b) Write the cumulative distribution function of X.
c) Compute the probability that at most 4 attempts must be made to gain access to the computer.
d) Compute the probability that at least 3 attempts must be made to gain access to the computer.

A: a) $X \sim \begin{pmatrix} k \\ (0.7)(0.3)^{k-1} \end{pmatrix}_{k \in \{1,2,3,\ldots\}}$.
Note that, $X - 1$ has a geometric distribution with parameter $p = 0.7$.

b) The cumulative distribution function is $F : \mathbb{R} \to \mathbb{R}$,

$$F(x) = P(X \leq x) = \begin{cases} 0, & \text{if} \quad x < 1 \\ 0.7, & \text{if} \quad 1 \leq x < 2 \\ (0.7)[1 + (0.3)], & \text{if} \quad 2 \leq x < 3 \\ & \dots \\ (0.7)[1 + (0.3) + \cdots + (0.3)^{k-1}], & \text{if} \quad k \leq x < k+1 \\ & \dots \end{cases}.$$

In particular, using the formula for the sum of terms in geometric progression, we get

$$F(k) = P(X \leq k) = 1 - (0.3)^k, \text{ for } k \in \{1, 2, \dots\}.$$

c) $P(X \leq 4) = F_X(4) = 1 - (0.3)^4$.

d) $P(X \geq 3) = 1 - P(X \leq 2) = 1 - F_X(2) = (0.3)^2$.