COURSE 12

Examples of other multi-step methods

1. THE BISECTION METHOD

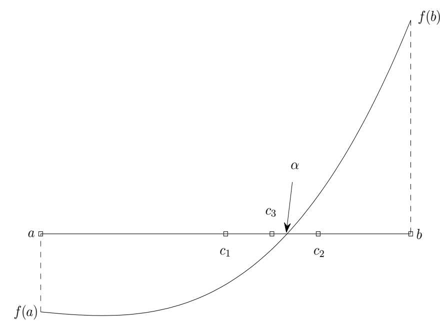
Let f be a given function, continuous on an interval $[a, b]$, such that

$$f(a)f(b) < 0. \tag{1}$$

By Bolzano Theorem, it follows that there exists at least one zero α of f in (a, b).

The bisection method is based on halving the interval $[a, b]$ to determine a smaller and smaller interval within α must lie.

First we give the midpoint of $[a, b]$, $c = (a + b)/2$ and then compute the product $f(c)f(b)$. If the product is negative, then the root is in the interval $[c, b]$ and we take $a_1 = c$, $b_1 = b$. If the product is positive, then the root is in the interval $[a, c]$ and we take $a_1 = a$, $b_1 = c$. Thus, a new interval containing α is obtained.



Bisection method

The algorithm:

Suppose $f(a)f(b) \leq 0$. Let $a_0 = a$ and $b_0 = b$.

for $n = 0, 1, ...,$ITMAX

$c \leftarrow \frac{a_n + b_n}{2}$

if $f(a_n)f(c) \leq 0$, set $a_{n+1} = a_n, b_{n+1} = c$

else, set $a_{n+1} = c, b_{n+1} = b_n$

The process of halving the new interval continues until the root is located as accurately as desired, namely

$$\frac{|a_n - b_n|}{|a_n|} < \varepsilon,$$

where a_n and b_n are the endpoints of the n-th interval $[a_n, b_n]$ and ε is a specified precision. The approximation of the solution will be $\frac{a_n+b_n}{2}$.

Some other stopping criterions: $|a_n - b_n| < \varepsilon$ or $|f(a_n)| < \varepsilon$.

Example 1 *The function* $f(x) = x^3 - x^2 - 1$ *has one zero in* $[1, 2]$. *Use the bisection algorithm to approximate the zero of* f *with precision* 10^{-4}.

Sol. *Since* $f(1) = -1 < 0$ *and* $f(2) = 3 > 0$, *then (1) is satisfied. Starting with* $a_0 = 1$ *and* $b_0 = 2$, *we compute*

$$c_0 = \frac{a_0 + b_0}{2} = \frac{1+2}{2} = 1.5 \text{ and } f(c_0) = 0.125.$$

Since $f(1.5)f(2) > 0$, *the function changes sign on* $[a_0, c_0] = [1, 1.5]$.

To continue, we set $a_1 = a_0$ *and* $b_1 = c_0$; *so*

$$c_1 = \frac{a_1 + b_1}{2} = \frac{1+1.5}{2} = 1.25 \text{ and } f(c_1) = -0.609375$$

Again, $f(1.25)f(1.5) < 0$ *so the function changes sign on* $[c_1, b_1] = [1.25,\ 1.5]$. *Next we set* $a_2 = c_1$ *and* $b_2 = b_1$. *Continuing in this manner we obtain a sequence* $(c_i)_{i>0}$ *which converges to* 1.465454, *the solution of the equation.*

2. THE METHOD OF FALSE POSITION

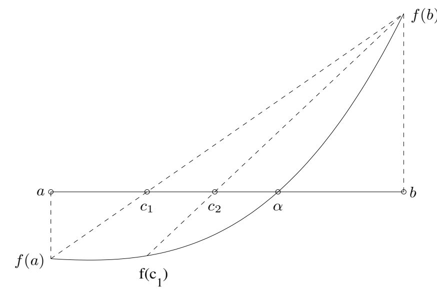
This method is also known as *regula falsi*, is similar to the Bisection method but has the advantage of being slightly faster than the latter. The function have to be continuous on $[a, b]$ with

$$f(a)f(b) < 0.$$

The point c is selected as point of intersection of the Ox-axis, and the straight line joining the points $(a,\ f(a))$ and $(b, f(b))$. From the equation of the secant line, it follows that

$$c = b - f(b)\frac{b-a}{f(b)-f(a)} = \frac{af(b)-bf(a)}{f(b)-f(a)} \qquad (2)$$

Compute $f(c)$ and repeat the procedure between the values at which the function changes sign, that is, if $f(a)f(c) < 0$ set $b = c$, otherwise set $a = c$. At each step we get a new interval that contains a root of f and the generated sequence of points will eventually converge to the root.



Method of false position.

The algorithm:

Given a function f continuous on $[a_0, b_0]$, with $f(a_0)f(b_0) < 0$,

input: a_0, b_0

for $n = 0, 1, ..., ITMAX$

$c \leftarrow \frac{f(b_n)a_n - f(a_n)b_n}{f(b_n) - f(a_n)}$

if $f(a_n)f(c) < 0$, set $a_{n+1} = a_n, b_{n+1} = c$ **else** set $a_{n+1} = c, b_{n+1} = b_n$.

Stopping criterions: $|f(a_n)| \leq \varepsilon$ or $|a_n - a_{n-1}| \leq \varepsilon$, where ε is a specified tolerance value.

Remark 2 *The bisection and the false position methods converge at a very low speed compared to the secant method.*

Example 3 *The function* $f(x) = x^3 - x^2 - 1$ *has one zero in* $[1, 2]$*. Use the method of false position to approximate the zero of* f *with precision* 10^{-4}*.*

Sol. *A root lies in the interval* $[1, 2]$ *since* $f(1) = -1$ *and* $f(2) = 3$. *Starting with* $a_0 = 1$ *and* $b_0 = 2$, *we get using (2)*

$$c_0 = 2 - \frac{3(2-1)}{3-(-1)} = 1.25 \text{ and } f(c_0) = -0.609375.$$

Here, $f(c_0)$ *has the same sign as* $f(a_0)$ *and so the root must lie on the interval* $[c_0, b_0] = [1.25, 2]$. *Next we set* $a_1 = c_0$ *and* $b_1 = b_0$ *to get the next approximation*

$$c_1 = 2 - \frac{3-(2-1.25)}{3-(-0.609375)} = 1.37662337 \text{ and } f(c_1) = -0.2862640.$$

Now $f(x)$ *change sign on* $[c_1, b_1] = [1.37662337, 2]$. *Thus we set* $a_2 = c_1$ *and* $b_2 = b_{1.}$ *Continuing in this manner the iterations lead to the approximation* 1.465558.

Example 4 *Compare the false position method, the secant method and Newton's method for solving the equation* $x = cos x$, *having as starting points* $x_0 = 0.5$ *și* $x_1 = \pi/4$, *respectively* $x_0 = \pi/4$.

n	(a) x_n False position	(b) x_n Secant	(c) x_n Newton
0	0.5	0.5	0.5
1	0.785398163397	0.785398163397	0.785398163397
2	0.736384138837	0.736384138837	0.739536133515
3	0.739058139214	0.739058139214	0.739085178106
4	0.739084863815	0.739085149337	0.739085133215
5	0.739085130527	0.739085133215	
6	0.739085133188		
7	0.739085133215		

The extra condition from the false position method usually requires more computation than the secant method, and the simplifications from the secant method come with more iterations than in the case of Newton's method.

Example 5 *Consider the equation* $x^2 - x - 3 = 0$*. Give the next two iterations for approximating the solution of this equation using:*

a) Newton's method starting with $x_0 = 0$.

b) secant, false position and bisection methods starting with $x_0 = 0$ *and* $x_1 = 4$.

Hermite inverse interpolation

Let $f:\Omega\to\mathbb{R}$, $\Omega\subset\mathbb{R}$. Consider the equation

$$f\left(x\right)=0,\quad x\in\Omega. \tag{3}$$

Assume that α is a solution of equation $f(x)=0$ and $V\left(\alpha\right)$ is a neighborhood of α. If $y_k=f(x_k)$, where $x_k\in V(\alpha)$, $k=0,...,m$, are approximations of α, $r_k\in\mathbb{N}$, then if there exist $g^{(j)}(y_k)=(f^{-1})^{(j)}(y_k), j=0,...,r_k$, one considers the Hermite type interpolation problem.

Theorem 6 *Let α be a solution of equation $f(x)=0$, $V(\alpha)$ a neighborhood of α and $x_0,x_1...,x_m\in V(\alpha)$. For $n=r_0+...+r_m+m$, where r_k represents the multiplicity order of the nodes x_k, $k=0,...,m$, if $f\in C^{n+1}(V(\alpha))$ and $f'(x)\neq 0$ for $x\in V(\alpha)$, we have the following Hermite approximation method for α :*

$$\alpha\approx F_n^H(x_0,...,x_m)= \tag{4}$$

$$=\left(H_n g\right)(0)=\sum_{k=0}^{m}\sum_{j=0}^{r_k}\sum_{\nu=0}^{r_k-j}\frac{(-1)^{j+\nu}}{j!\nu!}f_k^{j+\nu}v_k(0)\left(\frac{1}{v_k(y)}\right)_{y=f_k}^{(\nu)}g^{(j)}(f_k),$$

where $f_k = f(x_k), k = 0, ..., m, g = f^{-1}$, *and*

$v_k(y) = (y - f_0)^{r_0+1} ... (y - f_{k-1})^{r_{k-1}+1}(y - f_{k+1})^{r_{k+1}+1} ... (y - f_m)^{r_m+1}.$

For $g = f^{-1}$ the corresponding Hermite polynomial is

$$(H_n g)(y) = \sum_{k=0}^{m} \sum_{j=0}^{r_k} h_{kj}(y) g^{(j)}(y_k),$$

and it satisfies the conditions:

$$(H_n g)^{(j)}(y_k) = g^{(j)}(y_k), \quad j = 0, ..., r_k; \quad k = 0, ..., m,$$

where h_{kj} are the fundamental interpolating polynomials, i.e.,

$$h_{kj}^{(p)}(y_\nu) = 0,\ k \neq \nu,\ p = 0, ..., r_\nu$$
$$h_{kj}^{(p)}(y_k) = \delta_{pj},\ p = 0, ..., r_k$$

and the corresponding interpolation formula is

$$g = H_n g + R_n g,$$

where $R_n g$ is the remainder term.

Taking into account that

$$\alpha = g(0) \approx (H_n g)(0),$$

defines a new approximation to α, we have that

$$F_n^H(x_0, ..., x_m) = (H_n g)(0)$$

is an approximation method for α.

Regarding the order of Hermite-type inverse interpolation method F_n^H, we have two results, first for the case of equal information (the same multiplicity order for all the nodes x_k, $k = 0, ..., m$) and then for different multiplicities.

Theorem 7 *(Equal information) The order $ord(F_n^H)$ is the unique positive root of the equation:*

$$t^{m+1} - (r+1)\sum_{j=0}^{m} t^j = 0,$$

where r *is the multiplicity order of the points* x_k, $\forall k = 0, ..., m$.

Theorem 8 *(Unequal information) The order of* F_n^H *is the unique positive and real root of the equation:*

$$t^{m+1} - (r_m + 1)t^m - (r_{m-1} + 1)t^{m-1} - .... - (r_1 + 1)t - (r_0 + 1) = 0,$$

where $r_0, ..., r_m$ *are real numbers, permutation of the multiplicity orders of the nodes* x_k, $k = 0, ..., m$ *satisfying the conditions:*

$$r_0 + r_1 + ... + r_m > 1 \tag{5}$$

and

$$r_m \geq r_{m-1} \geq ... \geq r_1 \geq r_0. \tag{6}$$

Remark 9 *The order of the Taylor-type inverse interpolation method, can be expressed as the solution of equation*

$$t - (r_0 + 1) = 0,$$

where r_0 *is the multiplicity order of the node* x_0.

Particular cases.

1) For $x_0,\ x_1 \in V(\alpha)$ with $r_0 = 0,\ r_1 = 1$, we have the following approximation method:

$$F_2^H(x_0,x_1) = x_1 - \left[\frac{f(x_1)}{f(x_0)-f(x_1)}\right]^2 (x_1 - x_0) - \frac{f(x_1)}{f(x_0)-f(x_1)}\frac{f(x_0)}{f\,'(x_1)}.$$

The *order* of this method is the solution of the equation:

$$t^2 - r_1 t - r_0 = 0,$$

so,

$$t^2 - 2t - 1 = 0,$$

and $p = 1 + \sqrt{2}$.

2) For nodes $x_0,\ x_1,\ x_2 \in V(\alpha)$ with $r_0 = r_1 = 0;\ r_2 = 1$, the method is:

$$F_4^H(x_0,x_1,x_2) =$$

$$= \frac{f(x_2)^2}{f(x_1)-f(x_0)}\left[\frac{x_0 f(x_1)}{[f(x_0)-f(x_2)]^2} - \frac{x_1 f(x_0)}{[f(x_1)-f(x_2)]^2}\right]$$

$$+ \frac{f(x_0)f(x_1)}{[f(x_2)-f(x_0)][f(x_2)-f(x_1)]}\left[1 + \frac{f(x_2)}{[f(x_2)-f(x_0)][f(x_2)-f(x_1)]}\right]\left[x_2 - \frac{f(x_2)}{f'(x_2)}\right].$$

The *order* of this method is the solution of the equation:

$$t^3 - 2t^2 - t - 1 = 0,$$

so $p = 2.548$.

3) For x_0, $x_1 \in V(\alpha)$ with $r_0 = r_1 = 1$, (double nodes), the approximation method is

$$\begin{aligned} F_3^H(x_0, x_1) =& x_1 - \tfrac{3f(x_0)f(x_1)^2 - f(x_1)^3}{[f(x_0)-f(x_1)]^3}(x_0 - x_1) \\ &+ \tfrac{f(x_0)f(x_1)}{[f(x_0)-f(x_1)]^2}\left[\tfrac{f(x_1)}{f\,'(x_0)} - \tfrac{f(x_0)}{f\,'(x_1)}\right]. \end{aligned}$$

The *order* of this method is the solution of the equation:

$$t^{m+1} - (r+1)\sum_{j=0}^{m} t^j = 0,$$

so, $t^2 - 2t - 2 = 0$, and $p = 1 + \sqrt{3}$.

Birkhoff inverse interpolation

Assume that α is a solution of equation $f(x) = 0$ and $V(\alpha)$ is a neighborhood of α. If $y_k = f(x_k)$, where $x_k \in V(\alpha)$, $k = 0, ..., m$, are approximations of α, $r_k \in N$ and $I_k \subset \{0, ..., r_k\}$, then if there exist $g^{(j)}(y_k) = (f^{-1})^{(j)}(y_k), j \in I_k$, one considers the Birkhoff type interpolation problem.

The Birkhoff polynomial

$$(B_n g)(y) = \sum_{k=0}^{m} \sum_{j \in I_k} b_{kj}(y) g^{(j)}(y_k),$$

satisfies the conditions:

$$(B_n g)^{(j)}(y_k) = g^{(j)}(y_k), \;\; j \in I_k, \; k = 0, ..., m,$$

where b_{kj} are the fundamental interpolating polynomials, i.e.,

$$b_{kj}^{(p)}(y_\nu) = 0, \; k \neq \nu, \; p \in I_\nu$$
$$b_{kj}^{(p)}(y_k) = \delta_{pj}, \; p \in I_k$$

and the corresponding interpolation formula is

$$g = B_n g + R_n g,$$

where $R_n g$ is the remainder term.

Taking into account that

$$\alpha = g(0) \approx (B_n g)(0),$$

defines a new approximation to α we have that

$$F_n^B(x_0, ..., x_m) = (B_n g)(0)$$

is an approximation method for α.

Particular case.

1) Let $x_0, x_1 \in V(\alpha)$, $I_0 = \{0\}$, $I_1 = \{1\}$ and $y_0 = f(x_0), y_1 = f(y_1)$. Find the corresponding $F-$method for approximating the solution of equation $f(x) = 0$.

Sol. Let $x_0, x_1 \in V(x^*)$, $I_0 = \{0\}$, $I_1 = \{1\}$, so $n = 1$ și $y_0 = f(x_0)$, $y_1 = f(y_1)$.

Taking

$$F_1^B(x_0, x_1) = (B_1 g)(0),$$

we obtain the method defined by

$$F_1^B(x_0, x_1) = x_0 - \frac{f(x_0)}{f'(x_1)}.$$