Laboratory 3: Systems of differential equations

Exercise 1 *Find the general solution of the following systems of differential equations:*

$$(a)\quad \begin{cases} x'(t) = & x(t) + 4y(t) \\ y'(t) = & x(t) + y(t) \end{cases}$$

$$(b)\quad \begin{cases} x'(t) = & 2x(t) - y(t) \\ y'(t) = & x(t) + 2y(t) \end{cases}$$

$$(c)\quad \begin{cases} x'(t) = & x(t) - y(t) + z(t) \\ y'(t) = & x(t) + y(t) - z(t) \\ z'(t) = & -y(t) + 2z(t) \end{cases}$$

$$(d)\quad \begin{cases} x'(t) = & 3x(t) - y(t) + z(t) \\ y'(t) = & 2x(t) + z(t) \\ z'(t) = & x(t) - y(t) + 2z(t) \end{cases}$$

$$(e)\quad \begin{cases} x'(t) = & 5x(t) + 3y(t) + 1 \\ y'(t) = & -6x(t) - 4y(t) + e^t \end{cases}$$

$$(f)\quad \begin{cases} x'(t) = & x(t) + 3y(t) + \cos(t) \\ y'(t) = & x(t) - y(t) + 2t \end{cases}$$

$$(g)\quad \begin{cases} x'(t) = & x(t) - 2y(t) - 2z(t) + e^{-t} \\ y'(t) = & -2x(t) + y(t) + 2z(t) \\ z'(t) = & 2x(t) - y(t) - 3z(t) + e^t \end{cases}$$

$$(h)\quad \begin{cases} x'(t) = & -x(t) + 3y(t) - 4z(t) + 25t \\ y'(t) = & -2x(t) - 6z(t) + 12e^t \\ z'(t) = & -2x(t) - 6y(t) + 6z(t) + 12 \end{cases}$$

Exercise 2 *Find the solutions of the following initial value problems and represent their graphs:*

$$(a)\quad \begin{cases} x'(t) = & x(t) + 4y(t) \\ y'(t) = & x(t) + y(t) \end{cases} \qquad x(0) = 1,\ y(0) = 2$$

$$(b)\quad \begin{cases} x'(t) = & x(t) - y(t) + t - 1 \\ y'(t) = & -2x(t) + 4y(t) + e^t \end{cases} \qquad x(0) = 0,\ y(0) = 1$$

$$(c)\quad \begin{cases} x'(t) = & x(t) + 2y(t) + e^{-t} \\ y'(t) = & -2x(t) + y(t) + 1 \end{cases} \qquad x(0) = 0,\ y(0) = 1$$

$$(d)\quad \begin{cases} x'(t) = & -x(t) + 3y(t) + 3z(t) + 27t^2 \\ y'(t) = & 2x(t) - 2y(t) - 5z(t) + 3t \\ z'(t) = & -2x(t) + 3y(t) + 6z(t) + 3 \end{cases} \qquad x(0) = 50,\ y(0) = -30,\ z(0) = 26$$

Exercise 3 *Let's consider the system*

$$\begin{cases} x'(t) = & x(t) + y(t) \\ y'(t) = & -2x(t) + 4y(t) \end{cases}$$

(a) Find the system solution which satisfies the following initial conditions:

$$\begin{cases} x(0) = & 3 \\ y(0) = & 0 \end{cases}, \begin{cases} x(0) = & 0 \\ y(0) = & 3 \end{cases}, \begin{cases} x(0) = & -3 \\ y(0) = & 0 \end{cases}, \begin{cases} x(0) = & 0 \\ y(0) = & -3 \end{cases}$$

(b) For each solution from the point (a) calculate $\lim_{t\to+\infty} x(t),\ \lim_{t\to+\infty} y(t)$;

(c) Represent the phase portrait which contains the solution orbits from the point (a).

Exercise 4 *Let's consider the system:*

$$\begin{cases} x'(t) = & y(t) \\ y'(t) = & -x(t) - 2y(t) \end{cases}$$

(a) Find the general solution;

(b) Calculate $\lim_{t\to+\infty} x(t),\ \lim_{t\to+\infty} y(t)$;

(c) Represent the phase portrait.

Exercise 5 *Represent the phase portrait and specify (without solving the system) for which systems the following property* $\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} y(t) = 0$ *hold:*

(a) $\begin{cases} x'(t) = & 2x(t) + y(t) \\ y'(t) = & x(t) + 2y(t) \end{cases}$

(b) $\begin{cases} x'(t) = & -x(t) - y(t) \\ y'(t) = & x(t) - y(t) \end{cases}$

(c) $\begin{cases} x'(t) = & y(t) \\ y'(t) = & -x(t) \end{cases}$

(d) $\begin{cases} x'(t) = & -2x(t) \\ y'(t) = & -4x(t) - 2y(t) \end{cases}$

(e) $\begin{cases} x'(t) = & x(t) - 4y(t) \\ y'(t) = & 5x(t) - 3y(t) \end{cases}$

(f) $\begin{cases} x'(t) = & 3x(t) - y(t) \\ y'(t) = & y(t) \end{cases}$