Lecture 10

Linear systems with constant coefficients

(1) $\underline{Y}' = A \cdot \underline{Y} + B$

$$\underline{Y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \quad A = (a_{ij})_{1 \le i,j \le n} \quad B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$A \in \mathcal{M}_n(\mathbb{R}) \qquad B \in C(I, \mathbb{R}^n)$$

the general solution of (1)

$$\underline{Y} = \underline{Y}^0 + \underline{Y}^p$$

where $\underline{Y}^0$ is the gen. sol. of the homogeneous syst.

$$\underline{Y}' = A \cdot \underline{Y}$$

$\underline{Y}^p$ is a partic. sol. of the nonhomogeneous syst. $\underline{Y}' = A\underline{Y} + B$, which can be found by variation of the constants method.

The homogeneous case

(2) $Y' = A \cdot Y$ $\qquad Y^0 = U(x) \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$, $c_1, \dots, c_n \in \mathbb{R}$

$U(x)$ is the fundam. matrix of sol.

$U = (Y^1 \dots Y^n)$

I The Exponential Matrix method

$y' = a \cdot y \Rightarrow$ the gen. sol. $y(x) = c.e^{ax}$

e^{ax} – is a fundam. sol.

$$\boxed{(e^{ax})' = a.\,e^{ax}}$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$$

$A \in \mathcal{M}_n(\mathbb{R})$

$$e^A = I + \frac{A}{1!} + \frac{A^2}{2!} + \dots + \frac{A^n}{n!} + \dots$$

$\mathbb{R} \mapsto \mathcal{M}_n(\mathbb{R})$

$x \mapsto e^{xA}$ the exponential matrix function

$$e^{xA} = I + \frac{x.A}{1!} + \frac{x^2.A^2}{2!} + \dots + \frac{x^n.A^n}{n!} + \dots$$

$$e^{0.A} = I \;, \quad \left(e^{xA}\right)' = A.e^{xA}$$

$U(x) = e^{xA}$, $\qquad U(x) = (Y^1 \dots Y^n)$

$W(x; Y^1, \dots, Y^n) = \det U(x)$

$\det U(0) = \det I = 1 \neq 0 \Rightarrow U(x) = e^{xA}$ is a fundam matrix of sol.

$$\Rightarrow Y^0(x) = e^{xA} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} , \; \underline{c_1, \dots, c_n \in \mathbb{R}}$$

$e^{xA} = ?$

II The reduction method to a n order linear diff. eq. with constant coeff.

$$(2)\ \begin{cases} y_1' = a_{11} y_1 + \dots + a_{1n} \cdot y_n \\ \vdots \\ y_n' = a_{n1} y_1 + \dots + a_{nn} \cdot y_n \end{cases}$$

$$Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \text{ a sol. of } (2) \Rightarrow Y \in C^\infty$$

we choose one eq. of (2).

$$y_1' = a_{11} y_1 + \dots + a_{1n} y_n$$

we derivate this eq with respect to x.

$$y_1'' = a_{11} y_1' + \dots + a_{1n} \cdot y_n' \underset{\uparrow}{=} a_{11}(a_{11} y_1 + \dots + a_{1n} y_n) + \dots +$$

we replace with the relations from the system (2)

$$+ a_{1n}(a_{n1} y_1 + \dots + a_{nn} y_n)$$

$\Rightarrow \dots \Rightarrow$

$$y_1'' = a_{11}^1 y_1 + \dots + a_{1n}^1 y_n$$

we derivate again with respect to x

$$y_1''' = a_{11}^1 y_1' + \dots + a_{1n}^1 \cdot y_n' \underset{\uparrow}{=} \dots = \dots$$

we replace
with rel. from (2)

$$y_1''' = a_{11}^2 y_1 + \dots + a_{1n}^2 y_n$$

we continue with this procedure until we get $y_1^{(n)}$

....

$$y_1^{(i)} = a_{11}^{i-1} \cdot y_1 + \dots + a_{1n}^{i-1} y_n \,, \quad i = \overline{2,n}$$

$$\begin{cases} a_{12} y_2 + \dots + a_{1n} y_n = y_1' - a_{11} y_1 \\ a_{12}^1 y_2 + \dots + a_{1n}^1 y_n = y_1'' - a_{11}^1 y_1 \\ \dots\dots\dots\dots \\ a_{12}^{n-1} y_2 + \dots + a_{1n}^{n-1} y_n = y_1^{(n)} - a_{11}^{n-1} \cdot y_1 \end{cases}$$

n equation system.

we solve the system.

$$\begin{cases} a_{12} y_2 + \dots + a_{1n} y_n = y_1' - a_{11} y_1 \\ a_{12}^1 y_2 + \dots + a_{1n}^1 y_n = y_1'' - a_{11}^1 \cdot y_1 \\ \vdots \\ a_{12}^{n-2} \cdot y_2 + \dots + a_{1n}^{n-2} \cdot y_n = y_1^{(n-1)} - a_{11}^{n-2} \cdot y_1 \end{cases}$$

$n-1$ eq. syst.

we solve this system with respect to

$y_2, y_3, \dots, y_n$

$$\Rightarrow \dots \Rightarrow (3) \; \boxed{y_k = \alpha_{k1} y_1 + \alpha_{k2} y_1' + \dots + \alpha_{kn} \cdot y_1^{(n-1)}}, \; k = \overline{2, n}$$

replaceing these relations in the last eq.

$$y_1^{(n)} = a_{11}^{n-1} \cdot y_1 + a_{12}^{n-1} \cdot y_2 + a_{13}^{n-1} \cdot y_3 + \dots + a_{1n}^{n-1} \cdot y_n$$

$$\Rightarrow \dots \Rightarrow \boxed{y_1^{(n)} + b_1 \cdot y_1^{(n-1)} + \dots + b_n y_1 = 0}$$

a homog. linear diff. eq with const. coeff.

$\Rightarrow \lambda^n + b_1 \lambda^{n-1} + \dots + b_n = 0$ the charact. eq.

$\Rightarrow \varphi_1, \varphi_2, \dots, \varphi_n$ the fundam syst. of sol.

$\Rightarrow y_1(x) = c_1 \varphi_1(x) + \dots + c_n \varphi_n(x)$, $c_1, \dots, c_n \in \mathbb{R}$

replacing $y_1(x), y_1'(x), \dots, y_1^{(n-1)}(x)$ in (3) $\Rightarrow$

$\Rightarrow y_2(x), \dots, y_n(x)$.

III The characteristic equation method

$$Y' = A \cdot Y$$

$$Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} \alpha_1 e^{\lambda x} \\ \alpha_2 e^{\lambda x} \\ \vdots \\ \alpha_n e^{\lambda x} \end{pmatrix}, \quad Y \neq 0 \Leftrightarrow \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \neq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$Y' - AY = 0$$

$$\Rightarrow \begin{pmatrix} \alpha_1 \lambda \cdot e^{\lambda x} \\ \vdots \\ \alpha_n \lambda \cdot e^{\lambda x} \end{pmatrix} - A \cdot \begin{pmatrix} \alpha_1 e^{\lambda x} \\ \vdots \\ \alpha_n \cdot e^{\lambda x} \end{pmatrix} = 0$$

$$\begin{pmatrix} \alpha_1 \lambda \\ \vdots \\ \alpha_n \lambda \end{pmatrix} \cdot e^{\lambda x} - A \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} e^{\lambda x} = 0 \quad \Big| : e^{\lambda x}$$

$$\begin{pmatrix} \alpha_1 \lambda \\ \vdots \\ \alpha_n \lambda \end{pmatrix} - A \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = 0$$

$$\lambda \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} - A \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = 0$$

$$(4) \quad \boxed{(\lambda I - A) \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = 0}$$

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \neq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow (5) \quad \boxed{\det(\lambda I - A) = 0}$$

the characteristic equation

the solutions of the charact. eq. (5) are the eigenvalues of the matrix A.

the characteristic eq.(5) is a n-degree polinomial eq.

a) The case of simple real eigenvalues

$\lambda_1, \dots, \lambda_n$ eigenvalues of A, $\lambda_i \neq \lambda_j$, $i \neq j$.

for each λ_j, $j = \overline{1,n}$, we construct a nonzero sol. of the system (4)

$$\begin{pmatrix} \alpha_1^j \\ \vdots \\ \alpha_n^j \end{pmatrix} \neq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \quad j = \overline{1,n}$$

$$\Rightarrow Y^j = \begin{pmatrix} \alpha_1^j e^{\lambda_j x} \\ \vdots \\ \alpha_n^j e^{\lambda_j x} \end{pmatrix}, \quad j = \overline{1,n}$$

$\Rightarrow U(x) = (Y^1 \; Y^2 \; \dots \; Y^n)$ a fundam. matrix of solutions.

b) The case of complex eigenvalue

$\boxed{\lambda = \alpha + i\beta}$ is an eigenvalue of A

$\Rightarrow \bar{\lambda} = \alpha - i\beta$ is an eigenvalue of .

$Z(x) = Z_1(x) + i\cdot Z_2(x)$ $\qquad Z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$

$Z(x)$ is a sol. of the system (2) $\Leftrightarrow$

$\Leftrightarrow$ $Z_1(x)$, $Z_2(x)$ are sol. of the system (2)

$$Z(x) = \begin{pmatrix} \alpha_1 e^{\lambda x} \\ \vdots \\ \alpha_n e^{\lambda x} \end{pmatrix}, \alpha_1, \dots, \alpha_n \in \mathbb{C} \qquad \boxed{e^{a+ib} = e^{a}(\cos b + i \sin b)}$$

$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ is a non-zero sol. of the system (4)

$$\Rightarrow Z(x) = \begin{pmatrix} (a_1 + i b_1) \cdot e^{(\alpha + i\beta)x} \\ \vdots \\ (a_n + i b_n) \cdot e^{(\alpha + i\beta)x} \end{pmatrix} = \begin{pmatrix} (a_1 + i b_1) e^{\alpha x} (\cos \beta x + i \sin \beta x) \\ \vdots \\ (a_n + i b_n) e^{\alpha x} (\cos \beta x + i \sin \beta x) \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} e^{\alpha x}(a_1 \cos\beta x - b_1 \sin\beta x) \\ \vdots \\ e^{\alpha x}(a_n \cos\beta x - b_n \sin\beta x) \end{pmatrix}}_{Y^1} + i \cdot \underbrace{\begin{pmatrix} e^{\alpha x}(a_1 \sin\beta x + b_1 \cos\beta x) \\ \vdots \\ e^{\alpha x}(a_n \sin\beta x + b_n \cos\beta x) \end{pmatrix}}_{Y^2}$$

$\operatorname{Re} Z(x) = Y^1$, $\operatorname{Im} Z(x) = Y^2$ are sol. of (2)

c) The case of multiple eigenvalues

λ is a multiple real eigenvalue with multiplicity order $\mu > 1$

$$\Rightarrow \quad Y^1(x) = e^{\lambda x} \cdot u_1$$

$$Y^2(x) = e^{\lambda x}\left(\frac{x}{1!} u_1 + u_2\right)$$

$$\vdots$$

$$Y^\mu(x) = e^{\lambda x}\left(\frac{x^{\mu-1}}{(\mu-1)!} \cdot u_1 + \dots + \frac{x}{1!} u_{\mu-1} + u_\mu\right)$$

where u_1 is a non-zero sol. of (4) $\quad Au_1 = \lambda u_1$

and $u_2, \dots, u_\mu$ are non-zero sol. of the systems.

$$\begin{cases} Au_2 = \lambda u_2 + u_1 \\ Au_3 = \lambda u_3 + u_2 \\ \vdots \\ A \cdot u_\mu = \lambda u_\mu + u_{\mu-1} \end{cases}$$