Seminar 7

15.04.2025

1) Let $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = x+y+z$ and let $C = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1,\ 2x+y+2z = 1\}$. Det $\min f(C)$ and $\max f(C)$
(You can plug this in GeoGebra)

Solution: C is bounded and closed $\Rightarrow$ C is compact
f is continuous

$\overset{\text{Weierstrass}}{\Longrightarrow}$ f is bounded and reaches its bounds

$\Rightarrow \exists\, (a,b,c) \in C$ s.t. $f(a,b,c) = \min f(C)$

$\Rightarrow \exists\, (a',b',c') \in C$ s.t. $f(a',b',c') = \max f(C)$

↳ constrained extrema $\Rightarrow$ NO FERMAT TH.

$\Downarrow$

LAGRANGE MULT. RULE

By the Lagrange Mult. rule $\Rightarrow \exists\, \lambda_0, \mu_0 \in \mathbb{R}$ and $\exists\, \lambda_0', \mu_0' \in \mathbb{R}$ s.t. (a,b,c,λ_0,μ_0) and $(a',b',c',\lambda_0',\mu_0')$ are critical points of the Lagrange function

It remains to construct the Lagrange function and to find its critical points

Let $F_1(x,y,z) = x^2+y^2+z^2-1$

$F_2(x,y,z) = 2x+y+2z-1$

$$L(x,y,z,\lambda,\mu) = f(x,y,z) + \lambda \cdot F_1(x,y,z) + \mu \cdot F_2(x,y,z)$$
$$= x+y+z + \lambda(x^2+y^2+z^2-1) + \mu(2x+y+2z-1)$$

The critical points of L are solutions of the system

$$\begin{cases} L'_x(x,y,z,\lambda,\mu) = 1+2\lambda x + 2\mu = 0 \\ L'_y(x,y,z,\lambda,\mu) = 1+2\lambda y + \mu = 0 \\ L'_z(\ \ldots\ ) = 1+2\lambda z + 2\mu = 0 \end{cases} \overset{\ominus}{\Longrightarrow} 2\lambda(x-z) = 0$$

c1) $\lambda = 0$

$\Rightarrow \mu = -\frac{1}{2}$ (1st eq.)

$$\begin{cases} L'_\lambda(\ldots) = x^2+y^2+z^2-1 = 0 \\ L'_\mu(\ldots) = 2x+y+2z-1 = 0 \end{cases}$$

$\Rightarrow \mu = -1$ (2^{nd} eq) $\lightning$

c2) $x = z \Rightarrow \begin{cases} 2x^2+y^2-1=0 \\ 4x+y-1=0 \Rightarrow y = 1-4x \end{cases}$

$2x^2+(1-4x)^2-1=0 \Leftrightarrow 2x^2+\cancel{1}-8x+16x^2\cancel{-1}=0$

$\Leftrightarrow 18x^2-8x=0 \Leftrightarrow 2x(9x-4)=0$

- $x=0$
  $y=1$
  $z=0$
  $\mu = -\frac{1}{2}$ (1^{st} eq.)
  $\lambda = \frac{-1}{4}$ (2^{nd} eq.)

- $x = \frac{4}{9}$
  $y = -\frac{7}{9}$
  $z = \frac{4}{9}$
  $\mu = \ldots$
  $\lambda = \ldots$

The critical points of L are $(0,1,0,-\frac{1}{4},-\frac{1}{2})$ and $(\frac{4}{9},-\frac{7}{9},\frac{4}{9},\ldots)$

$f(0,1,0) = 1 = \max f(C)$
$f(\frac{4}{9},-\frac{7}{9},\frac{4}{9}) = \frac{1}{9} = \min f(C)$

2) Let $f:\mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = x^2+y^2+z^2-2x+2\sqrt{2}zy+2z$ and let $B = \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2 \le 1\}$. Det $\min f(B)$, $\max f(B)$.

Sol: $\left.\begin{array}{l} B \text{ is bounded \& closed} \Rightarrow B\text{ - compact} \\ f \text{ is cont.} \end{array}\right\} \overset{\text{Weir.}}{\Longrightarrow}$

$\Rightarrow \exists (a,b,c), (a',b',c') \in B$ s.t. $f(a,b,c) = \min f(B)$ and $f(a',b',c') = \max f(B)$

If $(a,b,c) \in \text{int } B$ OR $(a',b',c') \in \text{int } B \overset{\text{FERMAT}}{\Longrightarrow} \nabla f(a,b,c) = (0,0,0)$ OR $\nabla f(a',b',c') = (0,0,0)$

$$\nabla f(x,y,z) = (0,0,0) \iff \begin{cases} \frac{\partial f}{\partial x}(x,y,z) = 2x-2 = 0 \\ \frac{\partial f}{\partial y}(x,y,z) = 2y+2\sqrt{2} = 0 \\ \frac{\partial f}{\partial z}(x,y,z) = 2z+2 = 0 \end{cases}$$

$$\iff (x,y,z) = (1, -\sqrt{2}, -1)$$

$$1^2 + 2 + 1^2 = 4 < 1 \Rightarrow (1, -\sqrt{2}, -1) \notin \text{int } B$$

$\Rightarrow (a,b,c), (a',b',c') \in \text{bd } B$

$$\text{bd } B = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\} \Rightarrow (a,b,c), (a',b',c')$$

are CONSTRAINED EXTREMA OF F W.R.T. bd B

By the L.M.R. $\Rightarrow \exists \lambda_0, \lambda_0' \in \mathbb{R}$ s.t. (a,b,c,λ_0) and (a',b',c',λ_0') are critical points for the Lagrange function

Let $F(x,y,z) = x^2+y^2+z^2-1$

$$L(x,y,z,\lambda) = f(x,y,z) + \lambda F(x,y,z)$$
$$= x^2+y^2+z^2-2x+2\sqrt{2}y+2z+\lambda(x^2+y^2+z^2-1)$$

$$\begin{cases} L'_x(x,y,z,\lambda) = 2x-2+2\lambda x = 0 \Rightarrow x(1+\lambda) = 1 \Rightarrow x = \frac{1}{1+\lambda} \\ L'_y(x,y,z,\lambda) = 2y+2\sqrt{2}+2\lambda y = 0 \Rightarrow y(1+\lambda) = -\sqrt{2} \Rightarrow y = \frac{-\sqrt{2}}{1+\lambda} \\ L'_z(x,y,z,\lambda) = 2z+2+2\lambda z = 0 \Rightarrow z(1+\lambda) = -1 \Rightarrow z = \frac{-1}{1+\lambda} \\ L'_\lambda(x,y,z,\lambda) = x^2+y^2+z^2-1 = 0 \end{cases}$$

$$\Downarrow$$

$$\frac{1}{(1+\lambda)^2} + \frac{2}{(1+\lambda)^2} + \frac{1}{(1+\lambda)^2} = 1 \iff \frac{4}{(1+\lambda)^2} = 1 \iff \underbrace{1+\lambda = \pm 2}$$

• $\lambda = 1$	• $\lambda = -3$
$x = \frac{1}{2}$	$x = -\frac{1}{2}$
$y = -\frac{\sqrt{2}}{2}$	$y = \frac{\sqrt{2}}{2}$
$z = \frac{-1}{2}$	$z = \frac{1}{2}$

"Un sfert de pită cu 2 sferturi de pită și iară un sfert îi pita întreagă"

$$f\left(\frac{1}{2}, -\frac{\sqrt{2}}{2}, -\frac{1}{2}\right) = \cancel{\frac{1}{4}} + \cancel{\frac{2}{4}} + \cancel{\frac{1}{4}} - \cancel{1} - 2 - 1 = -3 \to \min f(B)$$

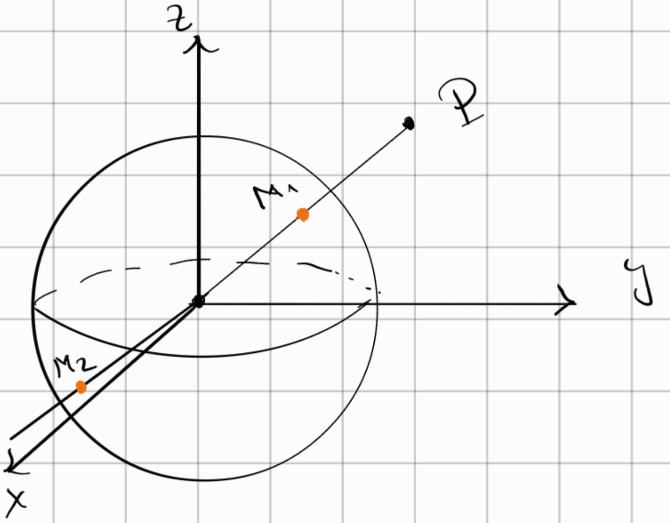
$$f\left(-\frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{1}{2}\right) = 5 \to \max f(B)$$

Sol 2: (geometric sol.)

$$f(x,y,z) = (x^2-2x+1) + (y^2+2\sqrt{2}y+2) + (z^2+2z+1) - 4$$
$$= (x-1)^2 + (y+\sqrt{2})^2 + (z+1)^2 - 4$$
$$= MP^2 - 4 \quad \text{where } M(x,y,z)$$
$$P(1,-\sqrt{2},1)$$

$f(x,y,z)$ reaches its minimum when MP reaches its minimum

$f(x,y,z)$ reaches its maximum when MP reaches its maximum



$$OP: \begin{cases} x = t \\ y = -\sqrt{2}t \\ z = -t \end{cases}$$

$$t^2 + 2t^2 + t^2 - 1 = 0$$
$$4t^2 = 1 \Rightarrow \boxed{t = \pm\frac{1}{2}}$$

$$M_1\left(\frac{1}{2}, -\frac{\sqrt{2}}{2}, -\frac{1}{2}\right) \qquad M_2\left(-\frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{1}{2}\right)$$

3) Find the critical points of the fct. below and specify their nature

a) $f(x,y,z) = x^3 + y^3 + z^3 + 12xy - 3z$

Sol:

I. We det. the critical points of f. They are sol. of the system

$$\begin{cases} \frac{\partial f}{\partial x}(x,y,z) = 3x^2 + 12y = 0 \\ \frac{\partial f}{\partial y}(x,y,z) = 3y^2 + 12x = 0 \\ \frac{\partial f}{\partial z}(x,y,z) = 3z^2 - 3 = 0 \Rightarrow \boxed{z = \pm 1} \end{cases}$$

$$\begin{cases} x^2 = -4y \Rightarrow y = -\frac{x^2}{4} \\ y^2 = -4x \Rightarrow \frac{x^4}{16} + 4x = 0 \end{cases}$$

$$\Updownarrow$$

$$x^4 + 64x = 0$$
$$x(x^3 + 64) = 0$$

- $x = 0 \Rightarrow y = 0$
- $x^3 = -64 \Rightarrow x = -4$

$\Rightarrow y = -4$

$\Rightarrow$ the critical points of f are:

$(0,0,-1)$, $(0,0,1)$, $(-4,-4,-1)$, $(-4,-4,1)$

II We compute the Hessian Matrix of f

$f''_{xx}(x,y,z) = 6x$

$f''_{yy}(x,y,z) = 6y$

$f''_{zz}(x,y,z) = 6z$

$f''_{xy}(x,y,z) = f''_{yx}(x,y,z)$

$= 12$

$f''_{xz}(x,y,z) = f''_{zx}(x,y,z) = 0$

$f''_{yz}(x,y,z) = f''_{zy} = 0$

$$H(f)(x,y,z) = \begin{pmatrix} 6x & 12 & 0 \\ 12 & 6y & 0 \\ 0 & 0 & 6z \end{pmatrix}$$

The next step must be performed for every critical point

III.1. $H(f)(0,0,-1) = \begin{pmatrix} 0 & 12 & 0 \\ 12 & 0 & 0 \\ 0 & 0 & -6 \end{pmatrix}$

$\Delta_1 = 0$

$\Delta_2 = \begin{vmatrix} 0 & 12 \\ 12 & 0 \end{vmatrix} = -144$

$\Delta_3 = \begin{vmatrix} \cancel{0} & \cancel{12} & \cancel{0} \\ 12 & 0 & 0 \\ 0 & 0 & -6 \end{vmatrix} = -12 \cdot \begin{vmatrix} 12 & 0 \\ 0 & -6 \end{vmatrix} = -12 \cdot 12 \cdot (-6) = 144 \cdot 6 = 864$

- If all determinants are POSITIVE (>0) then, the critical point is a LOCAL MINIMUM

- If the signs of these determinants ALTERNATE STARTED WITH $\ominus$ $\Rightarrow$ critical point is a LOCAL MAXIMUM

- If all are $\neq 0$ but we are neither in the first case nor in the second case, then the critical point is a SADDLE POINT

- If at least ONE is ZERO $\Rightarrow$ the nature of the critical point must be established by other methods

$$d^2f(0,0,-1)(h_1,h_2,h_3) = f''_{xx}(0,0,-1)\cdot h_1^2 + f''_{yy}(0,0,-1)\cdot h_2^2$$
$$+ f''_{zz}(0,0,-1)\cdot h_3^2 + 2\cdot f''_{xy}(0,0,-1)\cdot h_1\cdot h_2 + 2\cdot f''_{xz}(0,0,-1)\cdot h_1\cdot h_3$$
$$+ 2 f''_{yz}(0,0,-1)\cdot h_2\cdot h_3$$

$$d^2f(0,0,-1)(h_1,h_2,h_3) = -6h_3^2 + 24 h_1\cdot h_2$$

$$\left.\begin{array}{l} d^2f(0,0,-1)(1,1,0) = 24 > 0 \\ d^2f(0,0,-1)(0,0,1) = -6 < 0 \end{array}\right\} \Rightarrow d^2f(0,0,-1) \text{ is an indefinite quadratic form}$$

$\Downarrow$

$(0,0,-1)$ saddle point

Nature of $(-4,-4,-1)$

$$H(f)(-4,-4,-1) = \begin{pmatrix} -24 & 12 & 0 \\ 12 & -24 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

$\Delta_1 = -24$

$$\Delta_2 = \begin{vmatrix} -24 & 12 \\ 12 & -24 \end{vmatrix} = 24^2 - 12^2 > 0$$

$$\Delta_3 = \begin{vmatrix} -24 & 12 & 0 \\ 12 & -24 & 0 \\ 0 & 0 & -6 \end{vmatrix} = (-6)\cdot \Delta_2 < 0$$

2^{nd} bullet $\Rightarrow$ LOCAL MAX

Nature of $(-4,-4,1) \Rightarrow$ SADDLE POINT

Rem: Even if, f has only 1 local extremum, which is a maximum, it is NOT a global extremum

ex: $f(x,0,0) = x^3 \xrightarrow{x\to\infty} \infty$

b) $f(x,y) = x^4 + y^4 - 4(x-y)^2$

$$\begin{cases} f'_x(x,y) = 4x^3 - 8(x-y) = 0 \\ f'_y(x,y) = 4y^3 + 8(x-y) = 0 \end{cases} \Leftrightarrow \begin{cases} x^3 = 2x-2y \Rightarrow x^3 = 4x \\ y^3 = -2x+2y \end{cases}$$

$x(x^2-4)=0$

$\oplus$

$x^3 + y^3 = 0 \Rightarrow y^3 = (-x)^3$

$\Downarrow$

$\boxed{y=-x}$

C.P. of f are: $(0,0)$, $(2,-2)$, $(-2,2)$

$$\begin{cases} f''_{xx}(x,y) = 12x^2 - 8 \\ f''_{yy}(x,y) = 12y^2 - 8 \end{cases} \quad \Bigg| \quad f''_{xy}(x,y) = f''_{yx}(x,y) = 8$$

$$H(f)(x,y) = \begin{pmatrix} 12x^2-8 & 8 \\ 8 & 12y^2-8 \end{pmatrix}$$

$$H(f)(0,0) = \begin{pmatrix} -8 & 8 \\ 8 & -8 \end{pmatrix}$$

$\Delta_1 = -8$

$\Delta_2 = \begin{vmatrix} -8 & 8 \\ 8 & -8 \end{vmatrix} = 0$

$$d^2f(0,0)(h_1,h_2) = f''_{xx}(0,0)h_1^2 + f''_{yy}(0,0)h_2^2 + 2f''_{xy}(0,0)\cdot h_1 \cdot h_2$$

$$= -8h_1^2 - 8h_2^2 + 16h_1\cdot h_2 \quad = -8(h_1^2 - 2h_1h_2 + h_1^2)$$

$= -8(h_1-h_2)^2 \leq 0$

$d^2 f(0,0)$ is a NEGATIVE SEMI DEFINITE QUADRATIC FORM

but it is NOT a NEGATIVE DEFINITE QUADRATIC FORM

$f(x,x) = 2x^4 > 0 = f(0,0) \quad \forall x \in \mathbb{R} \setminus \{0\}$

$\Rightarrow (0,0)$ CAN'T be a local maximum

$f(x,-x) = 2x^4 - 16x^2 = 2x^2(2x^2-8) < 0 = f(0,0) \quad \forall x \in [-2\sqrt{2}, 2\sqrt{2}] \setminus \{0\}$

$\Rightarrow (0,0)$ can't be a local minimum

$\Rightarrow (0,0)$ is a saddle point

$$H(f)(2,-2) = \begin{pmatrix} 40 & 8 \\ 8 & 40 \end{pmatrix} = H(f)(-2,2)$$

$\left.\begin{array}{l} \Delta_1 = 40 \\ \Delta_2 > 0 \end{array}\right\} \Rightarrow (2,-2)$ LOCAL MINIMUM

$(-2,2)$ —— " ——