Seminar 5

1. Determine the order of each element and all generators of the cyclic groups $(\mathbb{Z}_8, +)$ and $(U_6, \cdot)$.

2. Determine the order of each element of Klein's group $(K, \cdot)$, permutation group $(S_3, \circ)$, dihedral group $(D_4, \cdot)$ and quaternion group $(Q, \cdot)$. Are they cyclic groups?

3. (i) Consider the matrices $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ in the group $(GL_2(\mathbb{R}), \cdot)$. Determine $\operatorname{ord} A$, $\operatorname{ord} B$, $\operatorname{ord}(A \cdot B)$ and $\operatorname{ord}(B \cdot A)$.

(ii) Give an example of group in which there exist two elements of infinite order, whose product has finite order.

4. Let $(G, \cdot)$ be a group, and let $x, y \in G$ be such that $xy = yx$, $\operatorname{ord} x = m$ and $\operatorname{ord} y = n$ $(m, n \in \mathbb{N}^*)$. Then:

(i) $\operatorname{ord}(xy)$ is finite and divides $[m, n]$.

(ii) If $< x > \cap < y >= \{1\}$, then $\operatorname{ord}(xy) = [m, n]$.

(iii) If $(m, n) = 1$, then $\operatorname{ord}(xy) = m \cdot n$.

5. Let $(G, \cdot)$ be a group and $x, y \in G$. Show that:

$$\operatorname{ord}(xy) = \operatorname{ord}(yx).$$

6. Let $(G, \cdot)$ be an abelian group. Show that

$$t(G) = \{x \in G \mid \operatorname{ord} x \text{ is finite}\}$$

is a subgroup of G. Is the property still true if G is not abelian?

7. Let $(G, \cdot)$ and $(G', \cdot)$ be abelian groups. Show that if $G \simeq G'$, then $t(G) \simeq t(G')$.

Using this, show that the following groups are not isomorphic:

(i) $(\mathbb{Q}, +)$ and $(\mathbb{Q}^*, \cdot)$.

(ii) $(\mathbb{R}, +)$ and $(\mathbb{R}^*, \cdot)$.

8. Let $f : G \to G'$ be a group homomorphism and let $x \in G$ be an element of finite order. Prove that:

(i) $\operatorname{ord} f(x)$ is finite and $\operatorname{ord} f(x) | \operatorname{ord} x$.

(ii) If f is injective, then $\operatorname{ord} f(x) = \operatorname{ord} x$.

9. Using Exercise **8.**, show that the groups $(\mathbb{Z}_4, +)$ and $(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$ are not isomorphic, and determine all non-isomorphic groups with 4 elements.

10. Let $m, n \in \mathbb{N}$ with $m, n \geq 2$. Show that the group $(\mathbb{Z}_m \times \mathbb{Z}_n, +)$ is cyclic if and only if $(m, n) = 1$.