Seminar 11

Linear systems with constant coefficients

$$Y' = A \cdot Y + B \quad , \; A \in \mathcal{M}_n(\mathbb{R}) \; , \; B \in C(J, \mathbb{R}^n)$$

the general sol. $Y = Y^0 + Y^p$

Y^0 – is the gen. sol. of the homog. syst. $Y' = AY$

Y^p – is a particular sol. of the nonhomog. syst. $Y' = AY + B$, which can be found using the variation of the constants method.

U – fundamental matrix of solutions.

$U = (Y^1 \dots Y^n)$ $\quad Y^1, \dots, Y^n$ sol. linearly indep. of the syst. $Y' = A \cdot Y$.

$$Y^0 = U \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} , \; \underline{c_1, \dots, c_n \in \mathbb{R}}$$

The homogeneous case

$$Y' = A \cdot Y.$$

I The reduction method to a n-order linear diff. eq. with const. coeff.

1) $\begin{cases} y_1' = y_1 - 5y_2 \\ y_2' = 2 \cdot y_1 - y_2 \end{cases}$

$y_1 = y_1(x)$
$y_2 = y_2(x)$

$$y_1' = y_1 - 5y_2$$

we derivate with respect to x.

$$y_1'' = y_1' - 5y_2'$$

$$y_1'' = y_1 - 5y_2 - 5 \cdot (2y_1 - y_2)$$

$$y_1'' = y_1 - \cancel{5y_2} - 10y_1 + \cancel{5y_2}$$

$$y_1'' = -9y_1$$

$$\boxed{y_1'' + 9y_1 = 0}$$

the second order linear homog. eq. with const. coeff.

$r^2+9=0$ the charact. eq.

$r^2=-9 \Rightarrow r_{1,2}=\pm 3i$

$\alpha=0,\ \beta=3$

$y_{11}(x)=e^{\alpha x}\cdot\cos\beta x=\cos 3x$

$y_{12}(x)=\sin 3x$ f.s.s.

$\Rightarrow y_1(x)=c_1 y_{11}(x)+c_2 y_{12}(x)$

$$\boxed{y_1(x)=c_1\cos 3x+c_2\cdot\sin 3x,\ c_1,c_2\in\mathbb{R}}$$

$y_1'=y_1-5y_2 \Rightarrow 5y_2=y_1-y_1'$

$\Rightarrow y_2=\frac{1}{5}(y_1-y_1')=\frac{1}{5}(c_1\cos 3x+c_2\sin 3x+c_1 3.\sin 3x-3c_2\cos 3x)$

$$\Rightarrow \boxed{y_2(x)=\frac{1}{5}\left[c_1\cdot(\cos 3x+3\sin 3x)+c_2(\sin 3x-3\cos 3x)\right]}$$

$$\begin{cases} y_1(x)=c_1\cos 3x+c_2\sin 3x \\ y_2(x)=\frac{c_1}{5}(\cos 3x+3\sin 3x)+\frac{c_2}{5}(\sin 3x-3\cos 3x) \end{cases} c_1,c_2\in\mathbb{R}.$$

$$U(x) = \begin{pmatrix} \cos 3x & \sin 3x \\ \frac{1}{5}(\cos 3x + 3\sin 3x) & \frac{1}{5}(\sin 3x - 3\cos 3x) \end{pmatrix}$$

a fundam. matrix of sol. for the system.

$$Y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = U(x) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad c_1, c_2 \in \mathbb{R}.$$

2) $$\begin{cases} y_1' = -4y_1 + 2y_2 + 5y_3 \\ y_2' = 6y_1 - y_2 - 6y_3 \\ y_3' = -8y_1 + 3y_2 + 9y_3 \end{cases}$$

$$y_1' = -4y_1 + 2y_2 + 5y_3$$

we derivate with respect to x:

$$y_1'' = -4y_1' + 2y_2' + 5y_3' = -4(-4y_1 + 2y_2 + 5y_3) +$$
$$+ 2(6y_1 - y_2 - 6y_3) + 5(-8y_1 + 3y_2 + 9y_3)$$

$$y_1'' = \underline{16y_1} - 8y_2 - 20y_3 + \underline{12y_1} - 2y_2 - 12y_3 - \underline{40y_1} + 15y_2 + 45y_3$$

$$\boxed{y_1'' = -12y_1 + 5y_2 + 13y_3}$$

$$\begin{array}{r} 45- \\ 32 \\ \hline 13 \end{array}$$

we derivate with respect to x.

$$\Rightarrow y_1''' = -12y_1' + 5y_2' + 13 \cdot y_3' =$$

$$= 48y_1 - 24y_2 - 60y_3$$

$$30y_1 - 5y_2 - 30y_3$$

$$-104y_1 + 39y_2 + 117y_3$$

$$\boxed{y_1''' = -26y_1 + 10y_2 + 27y_3}$$

$$\left\{\begin{array}{l} y_1' = -4y_1 + 2y_2 + 5y_3 \\ y_1'' = -12y_1 + 5y_2 + 13y_3 \\ y_1''' = -26y_1 + 10y_2 + 27y_3 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} 2y_2 + 5y_3 = y_1' + 4y_1 \mid (-5) \\ 5y_2 + 13y_3 = y_1'' + 12y_1 \mid \cdot 2 \end{array}\right.$$

$$\Downarrow$$

$$\left\{\begin{array}{l} -10y_2 - 25y_3 = -5y_1' - 20y_1 \\ 10y_2 + 26y_3 = 2y_1'' + 24y_1 \end{array}\right.$$

$$/ \qquad y_3 = 2y_1'' - 5y_1' + 4y_1$$

$$\boxed{y_3 = 2y_1'' - 5y_1' + 4y_1} \quad \Big| \cdot 27$$

$$2y_2 + 5y_3 = y_1' + 4y_1 \quad \Rightarrow \quad y_2 = \frac{1}{2}\left(y_1' + 4y_1 - 5y_3\right) =$$

$$= \frac{1}{2}\left(y_1' + 4y_1 - 10y_1'' + 25y_1' - 20y_1\right)$$

$$= \frac{1}{2}\left(-10y_1'' + 26y_1' - 16y_1\right)$$

$$\Rightarrow \boxed{y_2 = -5y_1'' + 13y_1' - 8y_1} \quad \Big| \cdot 10$$

$$\Rightarrow \quad y_1''' = -26y_1 + 10y_2 + 27y_3$$

$$y_1''' = -26y_1 - 50y_1'' + 130y_1' - 80y_1$$
$$+ 54y_1'' - 135y_1' + 108y_1$$

$$y_1''' = 4y_1'' - 5y_1' + 2y_1 \quad \Rightarrow$$

$$\Rightarrow \quad \boxed{y_1''' - 4y_1'' + 5y_1' - 2y_1 = 0}$$

$$r^3 - 4r^2 + 5r - 2 = 0 \quad \text{charact. eq.}$$

$$r^3 - 2r^2 - 2r^2 + 4r + r - 2 = 0$$

$$r^2(r-2) - 2r(r-2) + r - 2 = 0$$

$$(r-2)(r^2 - 2r + 1) = 0$$

$$(r-2)(r-1)^2 = 0 \Rightarrow \quad r_1 = 2 \rightarrow e^{2x}$$

$$r_2 = r_3 = 1 \rightarrow e^x, \; x \cdot e^x$$

$$\Rightarrow \boxed{y_1(x) = c_1 e^{2x} + c_2 e^x + c_3 x e^x, \; c_1, c_2, c_3 \in \mathbb{R}}$$

$$y_2 = -5y_1'' + 13y_1' - 8y_1 = \dots$$

(homework)

$$y_3 = 2y_1'' - 5y_1' + 4y_1 = \dots$$

II The method of characteristic equation

$Y' = AY$, $A \in \mathcal{M}_n(\mathbb{R})$ $\qquad Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

we look for sol. of the form

$$Y = \begin{pmatrix} \alpha_1 e^{\lambda x} \\ \vdots \\ \alpha_n e^{\lambda x} \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \neq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\Rightarrow$ λ is a sol. of the eq.

$$\boxed{\det(\lambda I_n - A) = 0}$$ the charact. eq.

and $\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ is a nonzero sol. of the system:

$$(\lambda I_n - A) \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = 0$$

1) $\begin{cases} y_1' = y_1 - 5y_2 \\ y_2' = 2y_1 - y_2 \end{cases}$ $A = \begin{pmatrix} 1 & -5 \\ 2 & -1 \end{pmatrix}$

$\det(\lambda I_2 - A) = 0$

$\begin{vmatrix} \lambda - 1 & 5 \\ -2 & \lambda + 1 \end{vmatrix} = 0 \Rightarrow (\lambda - 1)(\lambda + 1) + 10 = 0$

$\lambda^2 - 1 + 10 = 0$

$\lambda^2 + 9 = 0$

$\lambda_{1,2} = \pm 3i$

$(\lambda I_2 - A)\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = 0$

we find a nonzero sol. in $\mathbb{C}$

$\lambda = 3i$ $\begin{pmatrix} 3i-1 & 5 \\ -2 & 3i+1 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{cases} (-1+3i)\alpha_1 + 5\alpha_2 = 0 \\ -2\alpha_1 + (1+3i)\alpha_2 = 0 \end{cases}$

$-2\alpha_1 + (1+3i)\alpha_2 = 0 \Rightarrow \alpha_2 = \frac{+2}{1+3i}\alpha_1$

we choose $\alpha_1 = 1 \Rightarrow \alpha_2 = \frac{\overset{1-3i}{\cancel{2}}}{1+3i} = \frac{2-6i}{10}$

$= \frac{1}{5} - \frac{3}{5}i$

$$\boxed{\alpha_2 = \frac{1}{5} - \frac{3}{5}i} \qquad \boxed{e^{\alpha x + i\beta x} = e^{\alpha x}(\cos\beta x + i\sin\beta x)}$$

$$\Rightarrow Y = \begin{pmatrix} \alpha_1 e^{\lambda x} \\ \alpha_2 e^{\lambda x} \end{pmatrix} = \begin{pmatrix} e^{3ix} \\ \left(\frac{1}{5} - \frac{3}{5}i\right) e^{3ix} \end{pmatrix} =$$

$$= \begin{pmatrix} \cos 3x + i\sin 3x \\ \left(\frac{1}{5} - \frac{3}{5}i\right)(\cos 3x + i\sin 3x) \end{pmatrix} =$$

$$= \underbrace{\begin{pmatrix} \cos 3x \\ \frac{1}{5}\cos 3x + \frac{3}{5}\sin 3x \end{pmatrix}}_{Y^1} + i \cdot \underbrace{\begin{pmatrix} \sin 3x \\ -\frac{3}{5}\cos 3x + \frac{1}{5}\sin 3x \end{pmatrix}}_{Y^2}$$

$$U(x) = \begin{pmatrix} \cos 3x & \sin 3x \\ \frac{1}{5}\cos 3x + \frac{3}{5}\sin 3x & -\frac{3}{5}\cos 3x + \frac{1}{5}\sin 3x \end{pmatrix}$$

fundam. matrix of sol.

the gen. sol:

$$Y = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = U(x) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\Rightarrow \begin{cases} y_1(x) = c_1 \cos 3x + c_2 \sin 3x \\ y_2(x) = c_1 \left(\frac{1}{5} \cos 3x + \frac{3}{5} \sin 3x \right) + c_2 \left(-\frac{3}{5} \cos 3x + \frac{1}{5} \sin 3x \right) \end{cases}$$

$c_1, c_2 \in \mathbb{R}$.

2) $$\begin{cases} y_1' = y_2 + y_3 \\ y_2' = 3y_1 + y_3 \\ y_3' = 3y_1 + y_2 \end{cases} \qquad A = \begin{pmatrix} 0 & 1 & 1 \\ 3 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix}$$

$$\det(\lambda I_3 - A) = 0$$

$$\begin{vmatrix} \lambda & -1 & -1 \\ -3 & \lambda & -1 \\ -3 & -1 & \lambda \end{vmatrix} = 0 \Rightarrow \dots \Rightarrow \lambda^3 - 7\lambda - 6 = 0$$

$$(\lambda + 1)(\lambda + 2)(\lambda - 3) = 0$$

$\lambda_1 = -1 \;,\; \lambda_2 = -2 \;,\; \lambda_3 = 3$

$$Y = \begin{pmatrix} \alpha_1 e^{\lambda x} \\ \alpha_2 e^{\lambda x} \\ \alpha_3 e^{\lambda x} \end{pmatrix} \qquad \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$(\lambda I_3 - A)\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\underline{\lambda_1 = -1}$: $\quad (\lambda I_3 - A)\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} -1 & -1 & -1 \\ -3 & -1 & -1 \\ -3 & -1 & -1 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -\alpha_1 - \alpha_2 - \alpha_3 = 0 \\ -3\alpha_1 - \alpha_2 - \alpha_3 = 0 \\ -3\alpha_1 - \alpha_2 - \alpha_3 = 0 \end{cases} \qquad \begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = 0 \\ +3\alpha_1 + \alpha_2 + \alpha_3 = 0 \end{cases}$$

$$\boxed{\alpha_3 = 1} \Rightarrow \begin{cases} \alpha_1 + \alpha_2 = -1 \;|\cdot(-1) \\ +3\alpha_1 + \alpha_2 = -1 \end{cases}$$

$$\begin{cases} -\alpha_1 - \alpha_2 = 1 \\ 3\alpha_1 + \alpha_2 = -1 \end{cases}$$
$$\overline{2\alpha_1 \;/\; = 0} \Rightarrow \boxed{\alpha_1 = 0} \Rightarrow \boxed{\alpha_2 = -1} \Rightarrow$$

$$Y^1 = \begin{pmatrix} 0 \\ -e^{-x} \\ e^{-x} \end{pmatrix}$$

$\underline{\lambda_2 = -2}$: $(\lambda I_3 - A)\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 0$

$$\begin{pmatrix} -2 & -1 & -1 \\ -3 & -2 & -1 \\ -3 & -1 & -2 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -2\alpha_1 - \alpha_2 - \alpha_3 = 0 \\ -3\alpha_1 - 2\alpha_2 - \alpha_3 = 0 \\ -3\alpha_1 - \alpha_2 - 2\alpha_3 = 0 \end{cases}$$

$\boxed{\alpha_1 = 1} \Rightarrow \begin{cases} -\alpha_2 - \alpha_3 = 2 \\ -2\alpha_2 - \alpha_3 = 3 \quad | \cdot (-1) \end{cases}$

$$\begin{cases} -\alpha_2 - \alpha_3 = 2 \\ 2\alpha_2 + \alpha_3 = -3 \end{cases}$$

$\alpha_2 \quad = -1$

$\boxed{\alpha_2 = -1} \Rightarrow \alpha_3 = -\alpha_2 - 2 = \underline{-3}$

$$\Rightarrow Y^2 = \begin{pmatrix} e^{-2x} \\ -e^{-2x} \\ -3e^{-2x} \end{pmatrix}$$

$\underline{\lambda_3 = 3}$: $(\lambda I_3 - A)\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix} = 0$

$$\begin{pmatrix} 3 & -1 & -1 \\ -3 & 3 & -1 \\ -3 & -1 & 3 \end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix} = 0$$

$$\begin{cases} 3\alpha_1 - \alpha_2 - \alpha_3 = 0 \\ -3\alpha_1 + 3\alpha_2 - \alpha_3 = 0 \\ -3\alpha_1 - \alpha_2 + 3\alpha_3 = 0 \end{cases}$$

$$\begin{cases} -\alpha_2 - \alpha_3 = -3\alpha_1 \\ 3\alpha_2 - \alpha_3 = 3\alpha_1 \end{cases}$$

for $\boxed{\alpha_1 = 1}$ $\Rightarrow$

$$\Rightarrow \begin{cases} -\alpha_2 - \alpha_3 = -3 \mid (-1) \\ 3\alpha_2 - \alpha_3 = 3 \end{cases} \Rightarrow \begin{cases} \alpha_2 + \alpha_3 = 3 \\ 3\alpha_2 - \alpha_3 = 3 \end{cases}$$

$$4\alpha_2 = 6$$

$$\boxed{\alpha_2 = \frac{6}{4} = \frac{3}{2}}$$

$$\alpha_2 = \frac{3}{2} \Rightarrow \boxed{\alpha_3 = 3 - \alpha_2 = 3 - \frac{3}{2} = \frac{3}{2}}$$

$$\Rightarrow Y^3 = \begin{pmatrix} e^{3x} \\ \frac{3}{2}e^{3x} \\ \frac{3}{2}e^{3x} \end{pmatrix}$$

.....

$$U = (\underline{Y}^1\ \underline{Y}^2\ \underline{Y}^3) = \begin{pmatrix} 0 & e^{-2x} & e^{3x} \\ -e^{-x} & -e^{-2x} & \frac{3}{2}e^{3x} \\ e^{-x} & -3e^{-2x} & \frac{3}{2}e^{3x} \end{pmatrix}$$

$$\underline{Y} = \begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{pmatrix} = U \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$\Rightarrow \begin{cases} y_1(x) = c_2 e^{-2x} + c_3 e^{3x} \\ y_2(x) = -c_1 e^{-x} - c_2 e^{-2x} + \frac{3}{2} e^{3x} \\ y_3(x) = c_1 e^{-x} - 3c_2 e^{-2x} + \frac{3}{2} e^{3x} \end{cases}, \quad \underline{c_1, c_2, c_3 \in \mathbb{R}}.$$

the general solution.