

COURSE 12

The matrix of a linear map

First, we define the matrix of a vector in a basis of a vector space. For certain reasons, it is presented as a column-matrix, but it must be said that this is rather a convention than a constraint. But if one changes the convention, the form of the next notions and results must be properly changed.

Let K be a field.

Definition 1. Let V be a K -vector space, $v \in V$ and $B = (v_1, \dots, v_n)$ a basis of V . If

$$v = k_1 v_1 + \cdots + k_n v_n \quad (k_1, \dots, k_n \in K)$$

is the unique representation of v as a linear combination of the vectors of B , then the **matrix of the vector** v in the basis B is

$$[v]_B = \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix}$$

Definition 2. Let $f : V \rightarrow V'$ be a K -linear map, let $B = (v_1, \dots, v_n)$ be a basis of V and let $B' = (v'_1, \dots, v'_m)$ be a basis of V' . Then we can uniquely write the vectors of $f(B)$ as linear combinations of the vectors of B' , i.e. there exist $a_{ij} \in K$ ($i \in \{1, \dots, m\}$, $j \in \{1, \dots, n\}$) uniquely determined such that

[illegible]

Then the **matrix of the K -linear map f** in the pair of bases (B, B') (or, simply, in the bases B and B') is the matrix whose columns consist of the coordinates of the vectors of $f(B)$ in the basis B' , that is,

$$[f]_{BB'} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

For $V = V'$ and $B = B'$, we denote $[f]_B = [f]_{BB}$ and we call it the **matrix of f** in the basis B .

Remarks 3. (1) We complete the matrix of a linear map by columns. This is also a part of the convention we mentioned at the beginning of this section.

(2) As we will see next, the matrix of a linear map depends on the map, on the considered bases, but also by the order of the elements in each basis.

Examples 4. a) For any n -dimensional K -vector space V and any basis B of V , we have

$$[1_V]_B = I_n.$$

b) Consider the \mathbb{R} -linear map $f : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ defined by

$$f(x, y, z, t) = (x + y + z, y + z + t, z + t + x), \quad \forall (x, y, z, t) \in \mathbb{R}^4.$$

Let $E = (e_1, e_2, e_3, e_4)$ and $E' = (e'_1, e'_2, e'_3)$ be the standard bases in \mathbb{R}^4 and \mathbb{R}^3 respectively. Since

$$\begin{cases} f(e_1) = f(1, 0, 0, 0) = (1, 0, 1) = e'_1 + e'_3 \\ f(e_2) = f(0, 1, 0, 0) = (1, 1, 0) = e'_1 + e'_2 \\ f(e_3) = f(0, 0, 1, 0) = (1, 1, 1) = e'_1 + e'_2 + e'_3 \\ f(e_4) = f(0, 0, 0, 1) = (0, 1, 1) = e'_2 + e'_3 \end{cases}$$

it follows that the matrix of f in the bases E and E' is

$$[f]_{EE'} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

c) Let $\mathbb{R}_n[X]$ be the \mathbb{R} - vector space of the polynomials with the degree at most n and real coefficients. The map

$$\varphi : \mathbb{R}_3[X] \rightarrow \mathbb{R}_2[X], \quad \varphi(a_0 + a_1X + a_2X^2 + a_3X^3) = a_1 + 2a_2X + 3a_3X^2$$

(which associates a polynomial f its formal derivative f') is a linear map. Let us write the matrix of φ in the pair of basis $B = (1, X, X^2, X^3)$, $B' = (1, X, X^2)$, and then in the pair of basis $B = (1, X, X^2, X^3)$, $B'' = (X^2, 1, X)$. We have

$$\begin{aligned} \varphi(1) &= 0 \cdot 1 + 0 \cdot X + 0 \cdot X^2 = 0 \cdot X^2 + 0 \cdot 1 + 0 \cdot X \\ \varphi(X) &= 1 \cdot 1 + 0 \cdot X + 0 \cdot X^2 = 0 \cdot X^2 + 1 \cdot 1 + 0 \cdot X \\ \varphi(X^2) &= 0 \cdot 1 + 2 \cdot X + 0 \cdot X^2 = 0 \cdot X^2 + 0 \cdot 1 + 2 \cdot X \\ \varphi(X^3) &= 0 \cdot 1 + 0 \cdot X + 3 \cdot X^2 = 3 \cdot X^2 + 0 \cdot 1 + 0 \cdot X \end{aligned}$$

thus,

$$[\varphi]_{BB'} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \quad \text{and} \quad [\varphi]_{BB''} = \begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}.$$

d) Let K be a field, $m, n \in \mathbb{N}^*$, $A \in M_{m,n}(K)$, E the standard basis of K^n and E' the standard basis of K^m . Then

$$f_A : K^n \rightarrow K^m, \quad f_A(x_1, \dots, x_n) = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

is a linear map and $[f_A]_{EE'} = A$.

Theorem 5. Let $f : V \rightarrow V'$ be a K -linear map, $B = (v_1, \dots, v_n)$ a basis of V , $B' = (v'_1, \dots, v'_m)$ a basis of V' and $v \in V$. Then

$$[f(v)]_{B'} = [f]_{BB'} \cdot [v]_B.$$

Proof. Let $[f]_{BB'} = (a_{ij}) \in M_{mn}(K)$. Let $v = \sum_{j=1}^n k_j v_j$ and $f(v) = \sum_{i=1}^m k'_i v'_i$ with $k_i, k'_i \in K$. On the other hand, using the definition of the matrix of f in the bases B and B' , we have

$$f(v) = f\left(\sum_{j=1}^n k_j v_j\right) = \sum_{j=1}^n k_j f(v_j) = \sum_{j=1}^n k_j \left(\sum_{i=1}^m a_{ij} v'_i\right) = \sum_{i=1}^m \left(\sum_{j=1}^n a_{ij} k_j\right) v'_i.$$

But there is only one way to write $f(v)$ as a linear combination of the vectors of the basis B' , hence we have

$$k'_i = \sum_{j=1}^n a_{ij} k_j, \forall i \in \{1, \dots, m\}.$$

Therefore, $[f(v)]_{B'} = [f]_{BB'} \cdot [v]_B$. □

For a K -linear map $f : V \rightarrow V'$ the dimension $\dim(\text{Im} f)$ is also called **the rank of f** . We denote it by $\text{rank}(f)$. The rank of a linear map and the rank of its matrix in a pair of bases are strongly connected.

Theorem 6. Let $f : V \rightarrow V'$ be a K -linear map. Then

$$\text{rank}(f) = \text{rank}[f]_{BB'},$$

where B and B' are arbitrary bases of V and V' respectively.

Proof. Let $B = (v_1, \dots, v_n)$ and $[f]_{BB'} = A$. We have

$$\begin{aligned} \text{rank}(f) &= \dim(\text{Im} f) = \dim f(V) = \dim f(\langle v_1, \dots, v_n \rangle) = \dim \langle f(v_1), \dots, f(v_n) \rangle = \\ &= \text{rank}({}^t A) = \text{rank}(A) = \text{rank}[f]_{BB'}. \end{aligned}$$

□

Remark 7. The matrices of a linear map in different pairs of bases have the same rank.

We continue this section by presenting one of the key results in Linear Algebra, connecting linear maps and matrices.

Theorem 8. Let V , V' and V'' be vector spaces over K with $\dim V = n$, $\dim V' = m$ and $\dim V'' = p$ and let B , B' and B'' be bases of V , V' and V'' respectively. If $f, g \in \text{Hom}_K(V, V')$, $h \in \text{Hom}_K(V', V'')$ and $k \in K$, then

$$[f + g]_{BB'} = [f]_{BB'} + [g]_{BB'}, \quad [kf]_{BB'} = k \cdot [f]_{BB'},$$

$$[h \circ f]_{BB''} = [h]_{B'B''} \cdot [f]_{BB'}.$$

Proof. If $[f]_{BB'} = (a_{ij}) \in M_{mn}(K)$, $[g]_{BB'} = (b_{ij}) \in M_{mn}(K)$ and $[h]_{B'B''} = (c_{ki}) \in M_{pm}(K)$ then

$$f(v_j) = \sum_{i=1}^m a_{ij} v'_i, \quad g(v_j) = \sum_{i=1}^m b_{ij} v'_i, \quad h(v'_i) = \sum_{k=1}^p c_{ki} v''_k$$

for any $j \in \{1, \dots, n\}$ and for any $i \in \{1, \dots, m\}$.

Then for any $k \in K$ and for any $j \in \{1, \dots, n\}$ we have

$$(f + g)(v_j) = f(v_j) + g(v_j) = \sum_{i=1}^m a_{ij} v'_i + \sum_{i=1}^m b_{ij} v'_i = \sum_{i=1}^m (a_{ij} + b_{ij}) v'_i,$$

$$(kf)(v_j) = kf(v_j) = k \cdot \left(\sum_{i=1}^m a_{ij} v'_i \right) = \sum_{i=1}^m (ka_{ij}) v'_i,$$

hence $[f + g]_{BB'} = [f]_{BB'} + [g]_{BB'}$ and $[kf]_{BB'} = k \cdot [f]_{BB'}$.

Finally, for any $j \in \{1, \dots, n\}$ we have

$$\begin{aligned} (h \circ f)(v_j) &= h(f(v_j)) = h \left(\sum_{i=1}^m a_{ij} v'_i \right) = \sum_{i=1}^m a_{ij} h(v'_i) = \sum_{i=1}^m a_{ij} \left(\sum_{k=1}^p c_{ki} v''_k \right) = \\ &= \sum_{k=1}^p \sum_{i=1}^m (c_{ki} a_{ij}) v''_k = \sum_{k=1}^p \left(\sum_{i=1}^m c_{ki} a_{ij} \right) v''_k, \end{aligned}$$

hence $[h \circ f]_{BB''} = [h]_{B'B''} \cdot [f]_{BB'}$. \square

Theorem 9. Let V and V' be vector spaces over K with $\dim V = n$ and $\dim V' = m$ and let B and B' be bases of V and V' respectively. Then the map $\varphi : \text{Hom}_K(V, V') \rightarrow M_{mn}(K)$ defined by

$$\varphi(f) = [f]_{BB'}, \quad \forall f \in \text{Hom}_K(V, V')$$

is an isomorphism of vector spaces.

Proof. Let us prove first that φ is bijective.

Let $f, g \in \text{Hom}_K(V, V')$ such that $\varphi(f) = \varphi(g)$. Then $[f]_{BB'} = [g]_{BB'} = (a_{ij})$ and

$$f(v_j) = a_{1j} v'_1 + a_{2j} v'_2 + \dots + a_{mj} v'_m = g(v_j), \quad \forall j \in \{1, \dots, n\}.$$

Then $f = g$ by the universal property of vector spaces. Thus, φ is injective.

Now let $A = (a_{ij}) \in M_{mn}(K)$, seen as a list of column-vectors (a^1, \dots, a^n) , where $a^j = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix}$.

Consider $B = (v_1, \dots, v_n)$ and $B' = (v'_1, \dots, v'_m)$ and consider the K -linear map $f : V \rightarrow V'$ defined on the basis B of V by

$$f(v_j) = a_{1j} v'_1 + \dots + a_{mj} v'_m, \quad \forall j \in \{1, \dots, n\}.$$

Then

$$\varphi(f) = [f]_{BB'} = (a_{ij}) = A.$$

Thus, φ is surjective.

The proof is completed by Theorem 8. \square

Remark 10. The extremely important isomorphism given in Theorem 9 allows us to work with matrices instead of linear maps, which is much simpler from a computational point of view.

As we previously saw, $(\text{End}_K(V), +, \circ)$ is a unitary ring.

Theorem 11. Let V be a vector space over K with $\dim V = n$ and let B be a basis of V . Then the map $\varphi : \text{End}_K(V) \rightarrow M_n(K)$ defined by

$$\varphi(f) = [f]_B, \quad \forall f \in \text{End}_K(V)$$

is an isomorphism of vector spaces and of rings.

Proof. Let $v \in V$ and let us write v in the two bases B and B' . Then

$$v = \sum_{i=1}^n k_i v_i \text{ and } v = \sum_{j=1}^n k'_j v'_j$$

for some $k_i, k'_j \in K$. Since $T_{BB'} = (t_{ij}) \in M_n(K)$, we have

$$v'_j = \sum_{i=1}^n t_{ij} v_i, \quad \forall j \in \{1, \dots, n\}.$$

It follows that

$$v = \sum_{j=1}^n k'_j \left(\sum_{i=1}^n t_{ij} v_i \right) = \sum_{i=1}^n \left(\sum_{j=1}^n t_{ij} k'_j \right) v_i.$$

By the uniqueness of writing of v as a linear combination of the vectors of B , it follows that

$$k_i = \sum_{j=1}^n t_{ij} k'_j,$$

hence $[v]_B = T_{BB'} \cdot [v]_{B'}$. □

Remark 17. Usually, we are interested in computing the coordinates of a vector v in the new basis B' , knowing the coordinates of the same vector v in the old basis B and the transition matrix from B to B' . Then by Theorem 16, we have

$$[v]_{B'} = T_{BB'}^{-1} \cdot [v]_B = T_{B'B} \cdot [v]_B.$$

Theorem 18. Let $f \in \text{Hom}_K(V, V')$, let B_1 and B_2 be bases of V and let B'_1 and B'_2 be bases of V' . Then

$$[f]_{B_2 B'_2} = T_{B'_1 B'_2}^{-1} \cdot [f]_{B_1 B'_1} \cdot T_{B_1 B_2}.$$

Proof. We have $T_{B_1 B_2} = [1_V]_{B_2 B_1}$ and $T_{B'_1 B'_2} = [1_{V'}]_{B'_2 B'_1}$ (see Remark 14 2)). Of course, we also have $T_{B'_1 B'_2}^{-1} = [1_{V'}]_{B'_1 B'_2}$ and by applying Theorem 8 to the equality $f = 1_{V'} \circ f \circ 1_V$ we get

$$[f]_{B_2 B'_2} = [1_{V'}]_{B'_1 B'_2} \cdot [f]_{B_1 B'_1} \cdot [1_V]_{B_2 B_1} = T_{B'_1 B'_2}^{-1} \cdot [f]_{B_1 B'_1} \cdot T_{B_1 B_2},$$

hence the expected conclusion. □

If we take $V' = V$, $B_1 = B'_1$ and $B_2 = B'_2$, we deduce:

Corollary 19. Let $f \in \text{End}_K(V)$ and let B_1 and B_2 be bases of V . Then

$$[f]_{B_2} = T_{B_1 B_2}^{-1} \cdot [f]_{B_1} \cdot T_{B_1 B_2}.$$