Seminar 4

1. (i) Let $f : \mathbb{C}^* \to \mathbb{R}^*$ be defined by $f(z) = |z|$. Show that f is a group homomorphism between $(\mathbb{C}^*, \cdot)$ and $(\mathbb{R}^*, \cdot)$.

(ii) Let $n \in \mathbb{N}$ and $g : \mathbb{Z} \to \mathbb{Z}_n$ be defined by $g(x) = \widehat{x}$. Prove that g is a group homomorphism between $(\mathbb{Z}, +)$ and $(\mathbb{Z}_n, +)$.

2. (i) Let $n \in \mathbb{N}$, $n \geq 2$ and let $\alpha : GL_n(\mathbb{R}) \to \mathbb{R}^*$ be defined by $\alpha(A) = \det(A)$. Show that α is a group homomorphism between $(GL_n(\mathbb{R}), \cdot)$ and $(\mathbb{R}^*, \cdot)$.

(ii) Let $n \in \mathbb{N}$, $n \geq 2$ and let $\beta : M_n(\mathbb{R}) \to \mathbb{R}$ be defined by $\beta(A) = \det(A)$. Show that β is not a group homomorphism between $(M_n(\mathbb{R}), +)$ and $(\mathbb{R}, +)$.

3. Determine the kernel and the image of the group homomorphisms from Ex. **1.** and **2.**

4. Let $f : \mathbb{C}^* \to GL_2(\mathbb{R})$ be defined by $f(a + b\mathrm{i}) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Show that f is a group homomorphism between $(\mathbb{C}^*, \cdot)$ and $(GL_2(\mathbb{R}), \cdot)$.

5. Let $a, b \in \mathbb{N}$ and $f : \mathbb{C}^* \to \mathbb{R}^*$ be defined by $f(z) = a \cdot |z| + b$. Determine a, b such that f is a group homomorphism between $(\mathbb{C}^*, \cdot)$ and $(\mathbb{R}^*, \cdot)$.

6. Let $(G, \cdot)$ be a group and let $f : G \to G$ be defined by $f(x) = x^{-1}$. Show that $f \in \mathrm{End}(G) \iff G$ is abelian.

7. Show that the following groups are isomorphic: $(\mathbb{Z}_n, +)$ and $(U_n, \cdot)$ $(n \in \mathbb{N}^*)$.

8. Show that the following groups are isomorphic: Klein's group $(K, \cdot)$ and $(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$.

9. Show that the following groups are isomorphic: $(\mathbb{R}, +)$ and $(\mathbb{R}^*_+, \cdot)$.

10. Let $(G, \cdot)$ be a group with 3 elements. Determine $\mathrm{End}(G)$ and $\mathrm{Aut}(G)$.