Lecture 12

Dynamical systems generated by planar autonomous systems

$x(t), y(t)$ unknown functions

$$(1)\quad \begin{cases} x'(t) = f_1(x(t), y(t)) \\ y'(t) = f_2(x(t), y(t)) \end{cases} \qquad f = (f_1, f_2)$$

Theorem If $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ then the IVP

$$(2)\quad \begin{cases} x' = f_1(x, y) \\ y' = f_2(x, y) \\ x(0) = \eta_1 \\ y(0) = \eta_2 \end{cases}$$

has a unique saturated solution for every $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$.

We denote by $(x(t, \eta_1, \eta_2), y(t, \eta_1, \eta_2))$ the unique sol of (2).

$x(\cdot, \eta_1, \eta_2), y(\cdot, \eta_1, \eta_2) : I_\eta \to \mathbb{R}$

I_η is the maximal interval

$I_\eta = (\alpha_\eta, \beta_\eta)$, $0 \in I_\eta \Rightarrow \alpha_\eta < 0 < \beta_\eta$

$W = \{ I_\eta \times \mathbb{R}^2 \mid \eta = (\eta_1, \eta_2) \in \mathbb{R}^2 \}$

The flow generated by (1)

$\varphi : W \to \mathbb{R}^2$

$\varphi(t, \eta) = \varphi(t, \eta_1, \eta_2) = \left(x(t, \eta_1, \eta_2), y(t, \eta_1, \eta_2) \right)$

The map $\forall \eta = (\eta_1, \eta_2)$

$\eta \longmapsto \varphi(\cdot, \eta_1, \eta_2)$ is called the dynamical system generated by (1).

<u>Remark</u> If. $I_\eta = \mathbb{R}$ $\forall \eta \in \mathbb{R}^2 \Rightarrow$

$\Rightarrow W = \mathbb{R} \times \mathbb{R}^2 = \mathbb{R}^3$

Properties of the flow

1. $\varphi(0, \eta_1, \eta_2) = (\eta_1, \eta_2)$, $\forall \eta = (\eta_1, \eta_2) \in \mathbb{R}^2$
2. $\varphi(t+s, \eta_1, \eta_2) = \varphi(t, \varphi(s, \eta_1, \eta_2))$, $\forall t, s \in I_\eta$, $\eta = (\eta_1, \eta_2)$
3. φ is continuous.

Definition

$\gamma^+(\eta) = \gamma^+(\eta_1, \eta_2) = \bigcup_{t \in [0, \beta_\eta)} \varphi(t, \eta)$ the positive orbit of η

$\gamma^-(\eta) = \gamma^-(\eta_1, \eta_2) = \bigcup_{t \in (\alpha_\eta, 0]} \varphi(t, \eta)$ the negative orbit of η

$\gamma(\eta) = \gamma^+(\eta) \cup \gamma^-(\eta)$ the orbit of η

Phase portrait: is the collection of all orbits together with the developing direction.

Example

$$\begin{cases} x' = y \\ y' = -x \end{cases}$$

flow: $\begin{cases} x' = y \\ y' = -x \\ x(0) = \eta_1 \\ y(0) = \eta_2 \end{cases}$ $\quad \eta_1, \eta_2 \in \mathbb{R}.$

$$\left.\begin{aligned} x' = y \Rightarrow x'' &= y' \\ y' &= -x \end{aligned}\right\} \Rightarrow x'' = -x \Rightarrow \boxed{x'' + x = 0}$$

$\Rightarrow r^2 + 1 = 0$ char. eq.

$$r_{1,2} = \pm i \quad \begin{aligned} &\nearrow x_1(t) = e^{\alpha t} \cos \beta t = \cos t \\ &\searrow x_2(t) = e^{\alpha t} \cos \beta t = \sin t \end{aligned}$$

$\alpha = 0, \ \beta = 1$

$$\Rightarrow \boxed{x(t) = c_1 \cos t + c_2 \sin t, \ c_1, c_2 \in \mathbb{R}}$$

$$y = x' \Rightarrow \boxed{y(t) = -c_1 \sin t + c_2 \cos t, \ c_1, c_2 \in \mathbb{R}}$$

the gen. sol. of the syst:

$$\begin{cases} x(t) = c_1 \cos t + c_2 \sin t \\ y(t) = -c_1 \sin t + c_2 \cos t \end{cases}, \ c_1, c_2 \in \mathbb{R}.$$

$x(0)=\eta_1 \Rightarrow c_1=\eta_1$

$y(0)=\eta_2 \Rightarrow c_2=\eta_2$

$$\Rightarrow \begin{cases} x(t,\eta_1,\eta_2) = \eta_1\cos t + \eta_2\sin t \\ y(t,\eta_1,\eta_2) = -\eta_1\sin t + \eta_2\cos t \end{cases}$$

$x(\cdot,\eta_1,\eta_2), y(\cdot,\eta_1,\eta_2) : I_\eta \to \mathbb{R}$

$I_\eta = \mathbb{R}$, $\forall \eta = (\eta_1,\eta_2) \in \mathbb{R}^2$ the maximal interval.

$\varphi : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}^2$

$\varphi(t,\eta_1,\eta_2) = (x(t,\eta_1,\eta_2), y(t,\eta_1,\eta_2)) =$

$= (\eta_1\cos t + \eta_2\sin t, -\eta_1\sin t + \eta_2\cos t)$.

φ — is the flow generated by the syst.

Orbits.

1. $(\eta_1,\eta_2)=(0,0) \implies \varphi(t,0,0)=(0,0)$

$$\gamma(0,0)=\bigcup_{t\in\mathbb{R}}\varphi(t,0,0)=\{(0,0)\}.$$

2. $(\eta_1,\eta_2)\neq(0,0)$

$$\gamma(\eta_1,\eta_2)=\bigcup_{t\in\mathbb{R}}\varphi(t,\eta_1,\eta_2)=$$

$$=\bigcup_{t\in\mathbb{R}}(\underbrace{\eta_1\cos t+\eta_2\sin t}_{x},\underbrace{-\eta_1\sin t+\eta_2\cos t}_{y})$$

$$\begin{cases} x=\eta_1\cos t+\eta_2\sin t \\ y=-\eta_1\sin t+\eta_2\cos t \end{cases},\ t\in\mathbb{R}$$

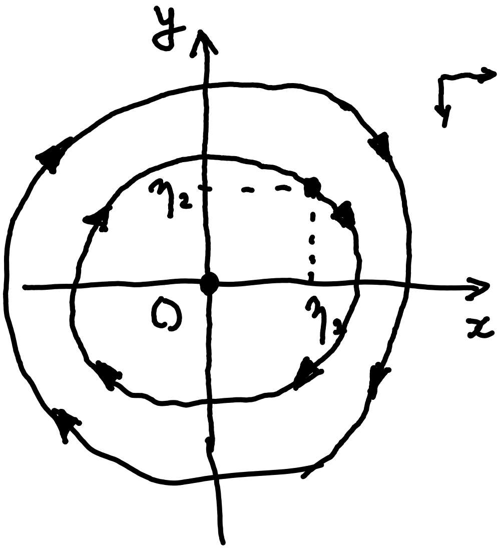
$$x^2+y^2=(\eta_1\cos t+\eta_2\sin t)^2+(-\eta_1\sin t+\eta_2\cos t)^2=$$

$$=\eta_1^2\cos^2 t+2\cancel{\eta_1\eta_2\cos t}\sin t+\eta_2^2\sin^2 t+$$

$$\eta_1^2\sin^2 t-2\eta_1\cancel{\eta_2\sin t}\cos t+\eta_2^2\cos^2 t=$$

$$=\eta_1^2+\eta_2^2$$

$x^2+y^2=\eta_1^2+\eta_2^2$ the orbit is a circle centered in $(0,0)$ with the radius $\sqrt{\eta_1^2+\eta_2^2}$



phase portrait

$$\begin{cases} x' = y \\ y' = -x \end{cases}$$

In general

$$\begin{cases} x' = f_1(x,y) \\ y' = f_2(x,y) \end{cases} \Rightarrow \begin{cases} \dfrac{dx}{dt} = f_1(x,y) \\ \dfrac{dy}{dt} = f_2(x,y) \end{cases} \Rightarrow \boxed{\dfrac{dx}{dy} = \dfrac{f_1(x,y)}{f_2(x,y)}}$$

$x = x(y) \Rightarrow \dfrac{dx}{dy} = x'(y)$

the diff. eq of the orbits.

or. $$\boxed{\dfrac{dy}{dx} = \dfrac{f_2(x,y)}{f_1(x,y)}}$$

$\uparrow$

$y'(x)$

$$\begin{cases} x' = y \\ y' = -x \end{cases} \qquad \begin{aligned} f_1(x,y) &= y \\ f_2(x,y) &= -x \end{aligned}$$

$\dfrac{dx}{dy} = \dfrac{y}{-x}$ the diff. eq. of the orbits

$y\,dy = -x\,dx \;\big|\cdot 2 \Rightarrow \int 2y\,dy = \int -2x\,dx \Rightarrow$

$\Rightarrow y^2 = -x^2 + c \Rightarrow \boxed{x^2 + y^2 = c}$

Definition

A constant solution of the system (1) is called an equilibrium solution.

$$\begin{cases} x(t) = x^* \\ y(t) = y^* \end{cases}$$

$(x^*, y^*) \in \mathbb{R}^2$ is called the equilibrium point

$$\begin{cases} x' = f_1(x,y) \\ y' = f_2(x,y) \end{cases} \Rightarrow (x^*, y^*) \text{ is a solution of the system } \begin{cases} f_1(x,y) = 0 \\ f_2(x,y) = 0 \end{cases}$$

Definition. An equilibrium point $X^*(x^*, y^*)$ of the syst. (1) is called:

a) locally stable if $\forall \varepsilon > 0$ $\exists \delta = \delta(\varepsilon) > 0$ such that if

$$\| \eta - X^* \|_{\mathbb{R}^2} < \delta \Rightarrow \| \varphi(t, \eta) - X^* \|_{\mathbb{R}^2} \leq \varepsilon, \ \forall t > 0$$

b) locally asimptotically stable if it is locally stable

and $\| \varphi(t,\eta) - X^* \| \xrightarrow[t \to +\infty]{} 0$

c) unstable if it is not locally stable.

Linear case

(3) $\begin{cases} x' = a_{11}x + a_{12}y \\ y' = a_{21}x + a_{22}y \end{cases}$, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$X^*(0,0)$ is an equilibrium point of (3).

the characteristic eq.

(4) $\det(\lambda I_2 - A) = 0$

Theorem (The Stability Theorem in the linear case)

Let's consider the linear system (3).

a) If $Re\lambda < 0$, $\forall \lambda$ eigenvalue of $A \Rightarrow X^*(0,0)$ is asimptotically stable.

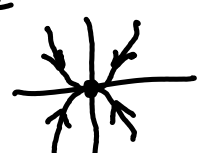
b) If $Re\lambda \leq 0$, $\forall \lambda$ eigenvalue of A, but $Re\lambda = 0$ holds for simple eigenvalue $\Rightarrow X^*(0,0)$ is locally stable

c) If $\exists \lambda$ eigenvalue with $Re\lambda > 0$ or $Re\lambda = 0$ and λ is not a simple eigenvalue $\Rightarrow X^*(0,0)$ is unstable.

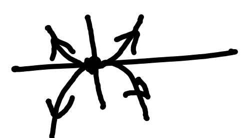
The classification of $(0,0)$:

I $\lambda_1, \lambda_2 \in \mathbb{R}$

- if $\lambda_1 \cdot \lambda_2 > 0 \Rightarrow X^*(0,0)$ is a node type

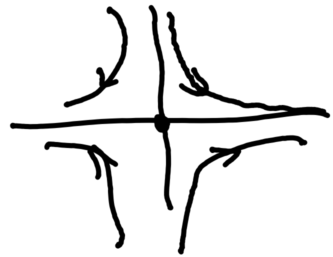


$\lambda_1 < 0$
$\lambda_2 < 0$
sink node



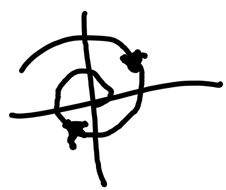
$\lambda_1 > 0$
$\lambda_2 > 0$
source node

• $\lambda_1 \cdot \lambda_2 < 0 \Rightarrow (0,0)$ is a saddle point (always unstable)

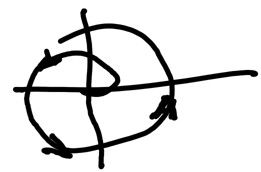


II $\lambda_{1,2} = \alpha \pm i\beta \in \mathbb{C}$

• $\alpha \neq 0 \Rightarrow (0,0)$ is a focus type

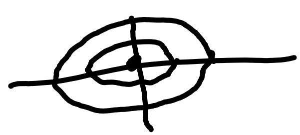


$\alpha < 0$
stable focus



$\alpha > 0$
unstable focus

• $\alpha = 0 \Rightarrow (0,0)$ is center type



(always locally stable)

Nonlinear case

$$\begin{cases} x' = f_1(x,y) \\ y' = f_2(x,y) \end{cases}$$

$X^*(x^*, y^*)$ eq. point

$f = (f_1, f_2)$

$$J_f(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}$$ the jacobian of $f = (f_1, f_2)$

$$\boxed{\underline{Y}' = J_f(x^*, y^*) \cdot \underline{Y}}$$ the linearized system.

Theorem (Stab. Th. in the first approx.)

a) If $\operatorname{Re}\lambda < 0$, $\forall \lambda$ eigenvalue of $J_f(x^*, y^*)$ $\Rightarrow$

$\Rightarrow X^*(x^*, y^*)$ is locally asimpt. stable

b) If $\exists \lambda$ with $\operatorname{Re}\lambda > 0$, λ eigenvalue of $J_f(x^*, y^*)$

$\Rightarrow X^*(x^*, y^*)$ is unstable.