Lecture 12

12 January 2024 10:07

Series of Functions

Recall:

$\mathcal{F}(A,\mathbb{R}) = \{ f \mid f: A \to \mathbb{R} \}$ → the set of all real-valued functions whose domain is A $(\subseteq \mathbb{R})$

$h: \mathbb{N} \to \mathcal{F}(A,\mathbb{R})$

$\forall m \in \mathbb{N}$ $h(m) := f_m$ $\quad f_m: A \to \mathbb{R}$ is a sequence of functions

$(f_m) = (f_m)_{m \geq 1} = (f_m)_{m \in \mathbb{N}}$

- $\mathcal{C} = \{ x \in A : \lim\limits_{m \to \infty} f_m(x) \in \mathbb{R} \}$ is the convergence set of a sequence of functions
- $f: \mathcal{C} \to \mathbb{R}$ $\quad \forall x \in \mathcal{C}$ $f(x) = \lim\limits_{m \to \infty} f_m(x)$ is called the POINTWISE LIMIT FUNCTION of the sequence of functions (f_m)

$\boxed{f_m \xrightarrow{\mathcal{C}} f}$

$\Leftrightarrow \forall \varepsilon > 0, \forall x \in \mathcal{C}$ $\exists m_\varepsilon \in \mathbb{N}$ s.t. $\forall m \geq m_\varepsilon$ $|f_m(x) - f(x)| < \varepsilon$

- $f \underset{\mathcal{C}}{\rightrightarrows} f$ converges uniformly to f

if $\forall \varepsilon > 0$ $\exists m_\varepsilon \in \mathbb{N}$ s.t. $\forall m \geq m_\varepsilon$, $\forall x \in \mathcal{C}$ $|f_m(x) - f(x)| < \varepsilon$

$\rightrightarrows \Rightarrow \rightarrow$

$\not\Leftarrow$

Def:

Each ordered pair $((f_m),(s_m))$ of two sequences of functions

$(f_m), (s_m) \subseteq \mathcal{F}(A,\mathbb{R})$ with the property that:

- $\boxed{\forall x \in \mathcal{C}}$ $s_1(x) = f_1(x)$
- $\boxed{\forall x \in \mathcal{C}}$ $s_2(x) = f_1(x) + f_2(x)$

⋮

- $\boxed{\forall x \in \mathcal{C}}$ $s_n(x) = f_1(x) + f_2(x) + \ldots + f_n(x)$

is a $\boxed{\text{SERIES of FUNCTIONS.}}$

Def:

- $\mathcal{C} = \{ x \in A : \text{the series } \sum f_m(x), \text{ of real numbers, is convergent} \}$

Def:

- $\mathcal{C} = \{x \in A :$ the series $\sum f_n(x)$, of real numbers, is convergent$\}$

$$= \{x \in A : \sum_{n=1}^{\infty} f_n(x) \in \mathbb{R}\}$$

- If $\mathcal{C} \neq \emptyset$ we define $s : \mathcal{C} \to \mathbb{R}$ $\forall x \in \mathcal{C}$ $s(x) = \sum_{n=1}^{\infty} f_n(x)$.

$$= \lim_{n\to\infty} s_n(x).$$

which is the POINTWISE SUM FUNCTION of the series of function $\sum f_n$

Notation: $\sum f_n \xrightarrow[\mathcal{C}]{} s$

- If $\boxed{s_n \rightrightarrows s}$ then we say that the series of functions $\sum f_n$ converges uniformly to s.

Notation: $\sum f_n \underset{\mathcal{C}}{\rightrightarrows} s$

Remark: A particular example of series of functions is the case of a power series

$$\sum_{n\geq 0} a_n \cdot x^n \text{ wher } \boxed{(a_n) \subseteq \mathbb{R}} = \sum_{n\geq 0} f_n(x)$$

$$\boxed{\forall n \in \mathbb{N}\cup\{0\} \quad f_n(x) = a_n \cdot x^n}$$

- Recall that $(-R, R) \subseteq \mathcal{C} \subseteq [-R, R]$

$$R = \frac{1}{\lambda} \quad \text{and} \quad \lambda = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \text{ or } \sqrt[n]{|a_n|}$$

Uniform convergence criteria for series of functions.

$\boxed{T_1}$ (Cauchy)

$(f_n) \subseteq \mathcal{F}(A, \mathbb{R})$, $\mathcal{C} \subseteq A$ $\Big|$ $\sum f_n \underset{\mathcal{C}}{\rightrightarrows} s \iff \forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\varepsilon$, $\forall p \in \mathbb{N}$, $\forall x \in \mathcal{C}$: $|f_{n+1}(x) + \dots + f_{n+p}(x)| < \varepsilon$

$\boxed{T_2}$ (Weirstrass)

$(f_n) \subseteq \mathcal{F}(A,\mathbb{R})$
$\mathcal{C} \subseteq A$

$(a_n) \subseteq \mathbb{R}$

If the series of real numbers

$$\sum a_n \text{ is C}$$

~~and~~ $\exists n' \in \mathbb{N}$ s.t. $\forall n \geq n'$ $|f_n(x)| \leq a_n$
$\forall x \in \mathcal{C}$

$$\sum f_n \overset{\rightarrow}{\underset{\mathcal{C}}{\rightarrow}}$$

(we only get the nature, not the sum)

Remark: For both sequences and series of functions properties such as:

- continuity
- Riemann Integrability
- differentiability

are inherited from the functions generating either (f_n) or $\sum f_n$

through $\boxed{\rightrightarrows}$ (to f or s)

Example Consider the following series of functions

$$\sum_{n\geq 1} \frac{(-1)^{n+1}}{n} \cdot x^n.$$

$\boxed{\text{determine its sum.}}$

Solution: $\forall n \in \mathbb{N}$ $f_n(x) = \frac{(-1)^{n+1}}{n} \cdot x^n$, $\forall x \in \mathbb{R}$ $f_n : \mathbb{R} \to \mathbb{R}$

$$\Rightarrow \boxed{\sum f_n} \subseteq \mathcal{F}(\mathbb{R},\mathbb{R})$$

$\boxed{\mathcal{C} = ?}$ It. is a power series $a_n = \frac{(-1)^{n+1}}{n}$

$$\lambda = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{n}{n+1} = 1 \Rightarrow \boxed{R = 1}$$

$$\Rightarrow \boxed{(-1,1) \subseteq \mathcal{C} \subseteq [-1,1]}$$

- $x = -1 \Rightarrow \sum_{n\geq 1} \frac{(-1)^{n+1} \cdot (-1)^n}{n} = \sum_{n\geq 1} \frac{-1}{n} = -\sum_{n\geq 1} \frac{1}{n}$ D.

$\Rightarrow -1 \notin \mathcal{C}$

$\boxed{x=1} \Rightarrow \sum_{m\geq 1} \frac{(-1)^{m+1}}{m} \cdot 1^m = -1 \sum_{m\geq 1} \frac{(-1)^m}{m}$ c. (Leibniz)

$\Rightarrow 1 \in \mathcal{C}$

Cauchy-Hadamard & Abel-Dirichlet for power series $\Rightarrow$ Hence $\boxed{\mathcal{C} = (-1, 1]}$

$\exists\, s : (-1, 1] \to \mathbb{R}$ $\forall x \in (-1, 1]$ $s(x) = \sum_{m=1}^{\infty} f_m(x)$

It can be shown that $\boxed{\sum f_m \rightrightarrows s}$ $\boxed{\text{on } (-1, 1]}$ Weierstrass $\boxed{a_n = f_n(x) \quad \forall m \in \mathbb{N}}$

$$\Rightarrow \boxed{s'(x) = \sum_{m\geq 1} f_m'(x)}$$

$\boxed{\forall x \in \mathcal{C}}$

$$s'(x) = \sum_{m=1}^{\infty} \left(\frac{(-1)^{m+1}}{m} \cdot x^m \right)' = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{\cancel{m}} \cdot \cancel{m} \cdot x^{m-1} =$$

$$= \sum_{m=1}^{\infty} (-1)^{m+1} \cdot x^{m-1} = \boxed{\sum_{m=1}^{\infty} (-x)^{m-1}}$$

$s : (-1, 1]$

$\boxed{f \text{ is dif. } (-1, 1)}$

For $\boxed{x \in \mathcal{C} \text{ randomly chose}}$

$$s'(x) = \sum_{m=1}^{\infty} (-x)^{m-1} = \frac{1}{1-(-x)} \qquad \boxed{\forall x \in (-1,1)}$$

$\boxed{-x \text{ is a constant}}$

$$= \frac{1}{1+x} \quad \forall x \in (-1, 1)$$

$\forall m \in \mathbb{N}$ f_m was c. with f_m' c.

s is c. with s' c.

$$\Rightarrow \boxed{\int s'(x)\,dx = \int \frac{1}{1+x}\,dx} \Rightarrow \boxed{\begin{aligned} s(x) &= \int \frac{1}{1+x}\,dx \\ &= \ln(1+x) + \mathcal{C} \end{aligned}}$$

$(*)$

$$s(x) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \cdot x^m \qquad \forall x \in (-1, 1]$$

$\boxed{\text{I}}$

$s(0) = 0$ (the sequence of the partial sums is constant 0)

$$\left.\begin{array}{l}\Rightarrow s(x) = \ln(1+x) + c \\ s(0) = 0\end{array}\right\} \Rightarrow \ln(1+0) + c = 0 \Rightarrow \boxed{c = 0}$$

Hence $\boxed{s(x) = \ln(1+x) \quad \forall x \in (-1,1)}$.

$$\left.\begin{array}{l}f_n \text{ is c. on } (-1,1] \\ s \text{ is c.}\end{array}\right\} \Rightarrow \lim_{x\to 1} s(x) = \boxed{s(1)}$$

$$\Rightarrow s(1) = \lim_{x\to 1} \ln(1+x) = \ln 2$$

$$\Rightarrow \boxed{s : (-1,1] \to \mathbb{R}, \quad s(x) = \ln(1+x)}$$

the sum function of the series of functions generated by the power series

$$\sum_{n\ge 1} \frac{(-1)^{n+1}}{n} x^n$$

$$\Leftrightarrow \boxed{\forall x \in (-1,1] \quad \ln(1+x) = \sum_{n\ge 1} \frac{(-1)^{n+1}}{n} x^n} \quad **$$

$$\boxed{\sum_{n=1}^{p} \frac{(-1)^{n+1}}{n} x^n = (T_{p;0} f)(x), \quad f(x) = \ln(1+x)}$$

the Taylor series expansion of $\ln(1+x)$

Therefore $\sum_{n\ge 1} \frac{(-1)^{n+1}}{n} \cdot 1^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln(1+1)$

$$\boxed{\Rightarrow \ln 2 = \lim_{n\to\infty}\left(-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \dots + \frac{(-1)^{n+1}}{n}\right)}$$

Examples for the study of the uniform convergence for series of functions with the help of the Weierstrass criterion:

with the help of the Weierstrass criterion:

$$\left.\begin{array}{l} |f_n(x)| \le a_n \\ \sum a_n \text{ is C} \end{array}\right\} \Rightarrow \boxed{\sum f_n(x) \text{ is UC (AC) on } \mathscr{C}}$$

a) $\forall n \in \mathbb{N}$ $f_n : \mathbb{R}^* \to \mathbb{R}$ $f_n(x) = \dfrac{1}{n!\, x^n}$

$$\sum_{n \ge 1} \frac{1}{n!\, x^n} \qquad |f_n| \le a_n$$

$$\boxed{a_n := \frac{1}{n!\, |x|^n}} \quad ? \ \sum a_n \text{ is C.}$$

is a $\boxed{\text{SPT}}$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{1}{(n+1)!\, |x|^{n+1}} \cdot \frac{\cancel{n!}\, |x|^n}{1} = \underset{=a_n}{}$$

$$= \lim_{n\to\infty} \frac{1}{|x|} \cdot \frac{1}{n+1} = 0 \Rightarrow \sum |a_n| \text{ is } \boxed{C}$$

regardless of the values of x

$$\Rightarrow \boxed{\sum f_n \text{ is UC}}$$

b) $\displaystyle\sum_{n \ge 1} \left(\frac{1}{x^2 + n^2} \right)$

$$\left.\begin{array}{l} |f_n(x)| = \dfrac{1}{x^2 + n^2} \le \dfrac{1}{n^2} \\ \sum \dfrac{1}{n^2} \text{ is C.} \end{array}\right\} \Rightarrow \boxed{\sum f_n \text{ UC}}$$

$\mathscr{C} = $ on $\boxed{\mathbb{R}}$

c) $\displaystyle\sum_{n \ge 1} \left(\frac{1}{x^2 + 2^n} \right)$

$$\left.\begin{array}{l} |f_n(x)| = \dfrac{1}{x^2 + 2^n} \le \dfrac{1}{2^n} \\ \sum \dfrac{1}{2^n} \text{ C.} \end{array}\right\} \Rightarrow \boxed{\sum f_n \text{ is UC}}$$

$\mathscr{C} = \boxed{\text{on } \mathbb{R}}$

The binomial series

... generalization of Newton's binom

The binomial series

– a generalization of Newton's binom

$$\textcircled{1+}\sum_{m\geq 1} \frac{k\cdot(k-1)\cdot\ldots\cdot(k-m+1)}{m!}x^n, \quad \forall m\in\mathbb{N}$$

$\boxed{k\in\mathbb{R}}$ is a fixed value

$a_0 = 1$

$a_1 \ldots$

$$\forall m\in\mathbb{N} \quad \boxed{a_m = \frac{k\cdot(k-1)\cdot\ldots\cdot(k-m+1)}{m!}}$$

$(a_m)\subseteq\mathbb{R}$

$\Rightarrow$ it is a POWER SERIES $\sum a_m\cdot x^m$

- $\lambda = \lim\limits_{m\to\infty} \dfrac{|a_{m+1}|}{|a_m|} =$

$$\lim_{m\to\infty}\left|\frac{\cancel{k\cdot(k-1)\cdot\ldots\cdot(k-m+1)}\cdot(k-m)}{(m+1)!}\right|\cdot\left|\frac{m!}{\cancel{k(k-1)\cdot\ldots\cdot}(k-m)}\right| =$$

$$= \lim_{m\to\infty}\frac{|k-m|}{(m+1)} = 1 \quad \Rightarrow R = \frac{1}{\lambda}$$

$$\Rightarrow (-1,1)\subseteq \mathcal{C}\subseteq[-1,1]$$

In order to fully determine $\mathcal{C}$ - particular cases for k have to be consider

H. cannot be dealt with easily.

We know for sure that $(-1,1)\subseteq\mathcal{C}$

$$\boxed{\sum f_m \underset{(-1,1)}{\rightrightarrows} s}$$

- $\forall m\in\mathbb{N}$ f_m is c. / dif / int. $\Rightarrow$ s is c. / d. / int on $(-1,1)$

$$\exists s:(-1,1)\to\mathbb{R} \quad s(x) = \sum_{m=1}^{\infty} f_m(x) + 1$$

$$= 1 + \sum^{\infty} \frac{k(k-1)\cdot\ldots\cdot(k-m+1)}{} x^m$$

$\boxed{a_0 = 1}$

$$= 1 + \sum_{m=1}^{\infty} \frac{k(k-1)\cdots(k-m+1)}{m!} x^m$$

$\boxed{a_0 = 1}$ for power series

$$= 1 + \frac{k}{1!}\cdot x + \frac{k(k-1)}{2!} x^2 + \cdots + \frac{k(k-1)\cdots(k-m+1)}{m!} x^m + \cdots + \cdots$$

- We differenciate it w.r.t. x

$$\cdot x \quad \Big| \quad s'(x) = k + \frac{k\cdot(k-1)}{2}\cdot 2x + \frac{k(k-1)\cdot(k-2)}{3!}\cdot 3x^2 + \cdots + \frac{k(k-1)\cdots(k-m+1)}{m!}\cdot m x^{m-1} + \cdots$$

$$= k + \frac{k\cdot(k-1)}{1!} x + \frac{k\cdot(k-1)(k-2)}{2!} x^2 + \cdots + k\frac{(k-1)\cdots(k-m+1)}{(m-1)!} x^{m-1}$$

$$\Rightarrow \quad x s'(x) = \frac{k}{1!} x + \frac{k(k-1)}{1!} x^2 + \cdots + \frac{k(k-1)(k-m+1)}{(m-1)!} x^m + \cdots$$

$$\frac{k(k-1)\cdots(k-m)}{(m-2)!} x^{m-1}$$

$$s'(x) + x s'(x) = k + \frac{k}{1} x + \frac{k(k-1)}{1} x + \frac{k(k-1)(k-2)}{2!} x^2 + \frac{k(k-1)}{1!} x^2 + \cdots$$

$$+ \cdots + \frac{k(k-1)\cdots(k-m+1)}{(m-1)!} x^{m-1} +$$

$$+ \frac{k(k-1)\cdots(k-m)}{(m-2)!} x^{m-1}$$

$$= k + k\cancel{x}\left(1 + \frac{k-1}{1}\right)x + \frac{k(k-1)}{1!}\cdot\left(\frac{k-2}{2} + 1\right)x^2 +$$

$$+ \cdots + \frac{k\cdot(k-1)\cdots(k-m)}{(m-1)!}\cdot\left(\frac{k-m+1}{m} + 1\right)x^m$$

$$= k + k\cdot x\left(\frac{1+k-1}{1}\right) + \frac{k(k-1)}{1!}\left(\frac{k-2+2}{2}\right)x^2 +$$

$$\cdots$$

$$= k + k\cdot x\cdot\frac{k}{1} + \frac{k(k-1)\cdot k\cdot x^2}{2} + \frac{k(k-1)(k-2)}{2!}\cdot k\cdot\frac{x^3}{3}$$

$$k(k-1)(k-2)\cdots(k-m+2)\,\frac{k}{}\cdot\frac{x^{m-1}}{} +$$

$$+ \dots + \frac{k(k-1)(k-2)\cdot\ldots\cdot(k-m+2)\,k}{(m-2)!} \cdot \frac{x^{m-1}}{m-1} +$$

$$+ \frac{k(k-1)\cdot\ldots(k-2)\ldots(k-m+1)}{(m-1)!}\,k \cdot \frac{x^m}{m} + \dots$$

$$= k \cdot s(x) \qquad \forall x \in (-1,1)$$

$$\Rightarrow \boxed{(1+x)\,s'(x) = k \cdot s(x)} \qquad \forall x \in (-1,1) \qquad ***$$

$$\boxed{!} \quad \left(\frac{s(x)}{(1+x)^k}\right)' = \frac{s'(x)\cdot(1+x)^k - s(x)\cdot k(1+x)^{k-1}}{(1+x)^{2k}} =$$

$$= \frac{s'(x)(1+x) - k\cdot s(x)}{(1+x)^{k+1}}$$

$$*** \Rightarrow \forall x \in (-1,1) \quad (1+x)s'(x) - k s(x) = 0 \Rightarrow \left(\frac{s(x)}{(1+x)^k}\right)' = 0$$

$$\Rightarrow \exists c \in \mathbb{R} \text{ s.t. } \frac{s(x)}{(1+x)^k} = c \iff \boxed{s(x) = c\cdot(1+x)^k} \quad \forall x \in (-1,1)$$

$$\boxed{?} \quad \left.\begin{aligned} s(0) &= c\cdot 1 = c \\ s(0) &= 1 \end{aligned}\right\} \Rightarrow \boxed{c=1}$$

$$\Rightarrow s(x) = (1+x)^k \qquad \forall x \in (-1,1)$$

$$\forall x \in (-1,1) \quad \boxed{(1+x)^k = \lim_{m\to\infty}\left(1 + \frac{k}{1}x + \frac{k(k-1)}{2}x^2 + \dots + \frac{k(k-1)\ldots(k-m+1)}{m!}x^m\right)}$$

$$k \in \mathbb{R}$$

$$\boxed{k=-1} \quad \frac{1}{1+x} = (1+x)^{-1} = \lim_{m\to\infty}\left(1 - x + x^2 + \dots + (-1)^m x^m\right)$$

$\boxed{k=-1}$ $\quad \boxed{\dfrac{1}{1+x} = (1+x)^{-1} = \lim\limits_{n\to\infty}\left(1 - x + x^2 + \dots + (-1)^n x^n\right)}$

$\forall\, x \in \underbrace{(-1,1)}_{\text{symmetric}}$

$\boxed{x=-t}$ $\quad \dfrac{1}{1-t} = \lim\limits_{n\to\infty}\left(1 + t + t^2 + \dots + t^n\right) \quad \forall x \in (-1,1)$