Mathematical Logic
Lecture 3
23.10.2023

Chapter 3. Sets

"Naively", a set is a collection of uniquely determined abstract objects.

This theory was introduced by Georg Cantor (~1870)

Later, contradictions have been found in this theory (i.e. Cantor's set theory is not consistent)

Around 1920 $\longrightarrow$ axiomatic set theory has been developed.

- The concepts of
  - set
  - element

  } variables of the language
  - belongs $\in$ – binary predicate symbol

  are primary, i.e. they are not defined.

A set can be given by:
- enumerating the elements e.g. $A = \{a, b, c\}$
- a property e.g. $A = \{x \mid P(x)\}$

  actually P is a predicate

  e.g. $\{x \mid x \neq x\} = \emptyset$ the empty set (it is unique)

  $\neg(x \overset{\Pi}{=} x)$

- two sets are equal if they have the same elements:

  $A = B \Longleftrightarrow \forall x\, (x \in A \longleftrightarrow x \in B)$.

- A is a subset of B

  $A \subseteq B \Longleftrightarrow \forall x\, (x \in A \longrightarrow x \in B)$

- the <u>power set</u> of A is the set of all subsets of A:

$$\mathcal{P}(A) \overset{\text{def}}{=} \{X \mid X \subseteq A\}.$$

<u>rem</u> $X \subseteq A \iff X \in \mathcal{P}(A)$.

<u>Example</u> $\mathcal{P}(\emptyset) = \{\emptyset\}$.

$$\mathcal{P}(\mathcal{P}(\emptyset)) = \mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$$

$$\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset))) = \mathcal{P}(\{\emptyset, \{\emptyset\}\}) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$$

<u>Rem.</u> we will consider that all our sets are subsets of a "big" set U (universe)

Operations with sets

$\vee$ (connective)

1) <u>union</u> $A \cup B \overset{\text{def}}{=} \{x \in U \mid x \in A \text{ or } x \in B\}$

2) <u>intersection</u> $A \cap B \overset{\text{def}}{=} \{x \mid x \in A \text{ and } x \in B\}$

3) <u>difference</u> $A \setminus B \overset{\text{def}}{=} \{x \mid x \in A \text{ and } x \notin B\}$

$\not\in B$: $\neg(x \in B)$

particular: $\complement A = U \setminus A = \{x \mid x \notin A\}$

complement of A

hence $A \setminus B = A \cap \complement B$

4) <u>symmetric difference</u>

$\begin{cases} \Delta - \text{Delta} \\ \delta - \text{delta} \end{cases}$

$$A \,\Delta\, B = (A \setminus B) \cup (B \setminus A) = (A \cap \complement B) \cup (B \cap \complement A)$$

$$= \{x \mid (x \in A \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \notin A)\}$$

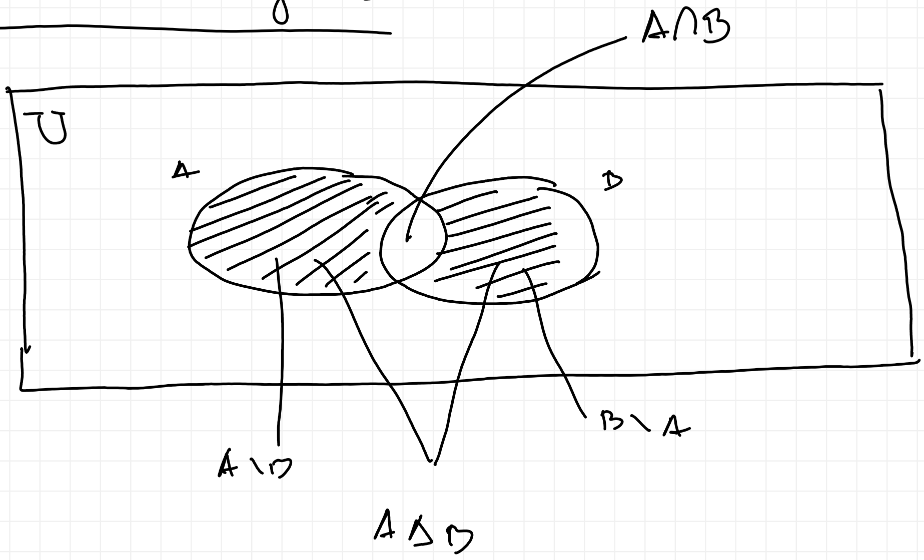
$$= \{x \mid x \in A \text{ xor } x \in B\}$$

$$= (A \cup B) \setminus (A \cap B) = (A \cup B) \cap \complement(A \cap B)$$

truth table

p	q	p xor q
0	0	0
0	1	1
1	0	1
1	1	0

Euler-Venn diagrams



Ordered pair of elements

Consider the element $a, b \in U$.

recall : $\cdot \{a, b\} = \{b, a\}$

$\cdot$ if $a = b$ $\quad \{a, b\} = \{a\}$

We want to define the ordered pair (a, b) such that the foll. property is satisfied:

$$(a, b) = (c, d) \iff a = c \text{ and } b = d$$

Def (Kuratowski, 1921).

$$(a, b) \overset{\text{def}}{=} \{\{a\}, \{a, b\}\}$$

Variant: $\quad (a, b) = \{a, \{a, b\}\}$

Def by recursion we define ordered n-tuples:

$$(a, b, c) \overset{\text{def}}{=} ((a, b), c)$$

$$(a_1, a_2, \dots, a_n) \overset{def}{=} ((a_1, \dots, a_{n-1}), a_n)$$

5). <u>Cartesian product</u> (René Descartes ~1650
Renatus Cartesius)

$$A \times B \overset{def}{=} \{(a,b) \mid a \in A \text{ and } b \in B\}$$

$$A_1 \times \dots \times A_n \overset{def}{=} \{(a_1, \dots, a_n) \mid \forall i \in \{1, \dots, n\} \; a_i \in A_i\}$$

Russell's paradox

(Bertrand Russell).

We may consider "the set of all sets" $\mathcal{S} \in \mathcal{S}$??

Let $R = \{X \mid X \text{ set}, X \notin X\}$. — this is a set according to Cantor

<u>Question</u>: does $R \in R$?

<u>Case 1</u> Assume $R \in R$. Then R does not satisfy the condition in the def. of R, hence $R \notin R$, contrad.

<u>Case 2</u> Assume $R \notin R$. Then R satisfies the condition in the definition of R, hence $R \in R$, contrad.

This leads to axiomatic set theory

* $X \in X$ is not allowed

$\boxed{\text{Homework ex 19–26}}$

Chapter 4. Relations (correspondences)

Def A <u>relation</u> is a triple $\rho = (A, B, R)$ where $R \subseteq A \times B$ and A, B are sets.

- A is called the <u>domain of ρ</u> : dom ρ
- B is called the <u>codomain</u> of ρ : codom ρ
- $R \subseteq A \times B$ is the <u>graph of</u> ρ

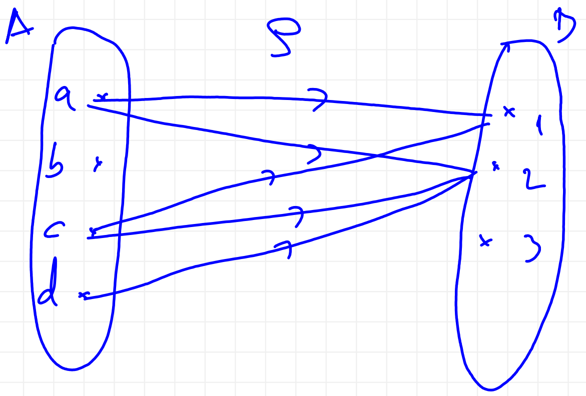
Rem a) equality of relations $\rho = (A, B, R)$, $\sigma = (C, D, S)$.

$$\rho = \sigma \iff \begin{cases} A = C & \text{(have the same domain)} \\ B = D & \text{(have the same codomain)} \\ R = S & \text{(have the same graph)} \end{cases}$$

b). We write: $(a, b) \in R \overset{\text{not.}}{\iff} a \rho b$ e.g. $a < b$, $a \mid b$, $d_1 \perp d_2$

$a \rho b \iff a \sigma b$

c). We may represent relations by using <u>oriented graphs</u> (<u>quiver</u>)



$R = \{(a,1), (a,2), (c,1), (c,2), (d,2)\}$

$R^{-1} = \{(1,a), (2,a), (1,c), (2,c), (2,d)\}$
(we invert the arrows)

Examples 1) the <u>universal relation</u> $\rho = (A, B, A \times B)$.
i.e. $\forall a \in A \ \forall b \in B \ a \rho b$

the <u>empty relation</u> $(A, B, \emptyset)$

2). The identity (equality) relation on the set A.

$$1_A = (A, A, \Delta_A) \quad \text{where}$$

$$\Delta_A = \{(a,a) \mid a \in A\}.$$

i.e. $a \, 1_A \, b \overset{def}{\iff} (a,b) \in \Delta_A$

$\iff a = b.$

Operations with relations

1) union Let $\rho = (A, B, R)$, $\rho' = (A, B, R')$

$$\rho \cup \rho' \overset{def}{=} (A, B, R \cup R').$$

i.e. $\forall (a,b) \in A \times B$: $a \, \rho \cup \rho' \, b \iff a \rho b$ or $a \rho' b$.

2) intersection $\rho \cap \rho' \overset{def}{=} (A, B, R \cap R')$

i.e. $\forall (a,b) \in A \times B$: $a \, \rho \cap \rho' \, b \overset{def}{\iff} a \rho b$ and $a \rho' b$

3) the inverse of a relation Let $\rho = (A, B, R)$.

Then $\rho^{-1} \overset{def}{=} (B, A, R^{-1})$, where $R^{-1} \subseteq B \times A$

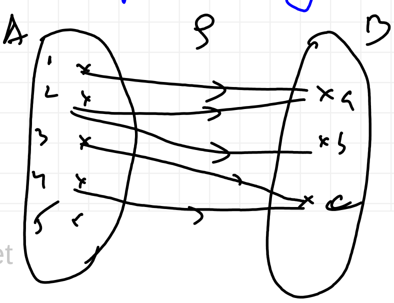
$R^{-1} \overset{def}{=} \{(b,a) \mid (a,b) \in R\}$

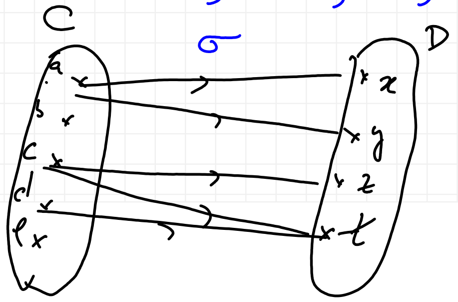
i.e. $\forall (a,b) \in A \times B$: $a \rho b \iff b \rho^{-1} a$

(ρ^{-1} is not the inverse w.r.t. an operation)

4) The composition of relations Let $\rho = (A, B, R)$, $\sigma = (C, D, S)$

Example





$S \subseteq C \times D$

$$\underset{2^{nd}}{\sigma} \circ \underset{1^{st}}{\rho} = (\underset{\text{dom}\,\rho}{A}, \underset{\text{codom}\,\sigma}{D}, S\circ R)$$

where $S\circ R \subseteq A\times D$

In our example: $S\circ R = \{(1,x),(1,y),(2,x),(2,y),(3,z),(3,t),(4,z),(4,t)\}$

In general:

$$S\circ R \overset{def}{=} \{(a,d)\in A\times D \mid (\exists x)\ x\in \underset{\text{codom}\,\rho}{B}\cap \underset{\text{dom}\,\sigma}{C} \text{ s.t. } (a,x)\in R \text{ and } (x,d)\in S\}$$

equivalently.

For $(a,d)\in A\times D$ we have:

$$a\ \sigma\circ\rho\ d \overset{def}{\Longleftrightarrow} (\exists x)\ x\in B\cap C \text{ and } a\rho x \text{ and } x\sigma d$$

<u>Theorem</u> (properties of the composition).

1). the identity relation is <u>neutral element</u> w.r.t "$\circ$".

i.e. $\rho\circ 1_A = 1_B\circ\rho = \rho$, where $\rho=(A,B,R)$

2). "$\circ$" is <u>associative</u>: if $\rho=(A,B,R)$, $\sigma=(C,D,S)$, $\tau=(E,F,T)$

then $\tau\circ(\sigma\circ\rho) = (\tau\circ\sigma)\circ\rho$

<u>Proof</u> 1). $\rho\circ 1_A = (A, B, R\circ\Delta_A)$.

$1_B\circ\rho = (A, B, \Delta_B\circ R)$.

so we have the same domain and the same codomain

Let $(a,b)\in A\times B$. We have:

- $a\ \rho\circ 1_A\ b \Longleftrightarrow (\exists x)\ x\in A\cap A$ and $a 1_A x$ and $x\rho b \Longleftrightarrow a\rho b$
- $a\ 1_B\circ\rho\ b \Longleftrightarrow (\exists x)\ x\in B\cap B$ and $a\rho x$ and $x 1_B b \Longleftrightarrow a\rho b$.

b). $\sigma\circ\rho = (A, D, S\circ R)$, $\tau\circ(\sigma\circ\rho) = (A, F, T\circ(S\circ R))$

$\tau\circ\sigma = (C, F, T\circ S)$; $(\tau\circ\sigma)\circ\rho = (A, F, (T\circ S)\circ R)$

So both relations have the same domain and the same codomain.

Let $(a,f) \in A\times F$. We have:

$\underline{a\ (\tau\circ\sigma)\circ\rho\ f} \overset{\text{def. }\circ}{\Longleftrightarrow} (\exists x)\ x\in B\cap C$ and $a\rho x$ and $x\ \underline{\tau\circ\sigma}\ f \overset{\text{def. }\circ}{\Longleftrightarrow}$

$\Longleftrightarrow (\exists x)\ x\in B\cap C$ and $a\rho x$ and $(\exists y)\ y\in D\cap E$ and $x\sigma y$ and $y\tau f$

$\Longleftrightarrow (\exists x)(\exists y)\ x\in B\cap C$ and $a\rho x$ and $y\in D\cap E$ and $x\sigma y$ and $y\tau f$

$\Longleftrightarrow (\exists y)(\exists x)\ y\in D\cap E$ and $x\in B\cap C$ and $a\rho x$ and $x\sigma y$ and $y\tau f$

$\Longleftrightarrow (\exists y)\ y\in B\cap E$ and $\underline{(\exists x)\ x\in B\cap C \text{ and } a\rho x \text{ and } x\sigma y}$ and $y\tau f$

$\overset{\text{def. }\circ}{\Longleftrightarrow} (\exists y)\ y\in B\cap E$ and $a\ \sigma\circ\rho\ y$ and $y\tau f$

$\overset{\text{def. }\circ}{\Longleftrightarrow}\ \underline{a\ \tau\circ(\sigma\circ\rho)\ f}$

We have used the following tautologies:

- $A\wedge(B\wedge C) \Longleftrightarrow (A\wedge B)\wedge B$ (assoc of $\wedge$)
- $A\wedge B \Longleftrightarrow B\wedge A$ (comm. of $\wedge$)
- $\exists x\exists y\ A \Longleftrightarrow \exists y\exists x\ A$
- $\exists x(A\wedge C(x)) \Longleftrightarrow A\wedge\exists x\ C(x)$ (A does not depend on the var x)

Manual 2.3.3 (1) and (8) pg. 17

$\boxed{\text{Homework: ex. 27 – 30}}$