Mathematical Logic
Lecture 12
08.01.2024

Finite, infinite, and countable sets

Recall: the set of natural numbers:

$$\mathbb{N} = \{0,1,2,\dots\}, \text{ where } 0 = \emptyset,\ 1 = \emptyset^+ = \{\emptyset\},\ 2 = 1^+$$

where the successor of the set X is $X^+ = X \cup \{X\}$

Not. 1) If $n \in \mathbb{N}$, then we denote by the same n the cardinal number of n:

$|n| = n$ so nat. numbers are also regarded as cardinal numbers.

(recall: $|A| = \{B \text{ set} \mid \underset{\text{equipotent}}{A \sim B}\}$)

2). $|\mathbb{N}| \overset{\text{not}}{=}: \aleph_0$ aleph zero

3) $|\mathbb{R}| \overset{\text{not}}{=}: c$ – the power of the continuum.

Def Let A be a set.

1) We say that A is finite if A is equipotent to a nat. number (i.e. $\exists n \in \mathbb{N}$ s.t. $A \sim n$)

2). A is infinite if A is not finite (i.e. $\forall n \in \mathbb{N}$ $A \nsim n$.)

3) A is countable if A is equipotent to a subset of $\mathbb{N}$.

Rem A is countable means that either A is finite or A is infinite countable i.e. $A \sim \mathbb{N}$.

Theorem (characterization of infinite sets). Let A be a set. The foll. statements are equivalent:

(i) A is infinite

(ii) A is equipotent to some proper subset of A

(i.e. $\exists\, B \subsetneq A$ s.t. $A \sim B$)

(iii) $\exists\, f : \mathbb{N} \to A$ injective function.

<u>Rem</u> cond (iii) says that $\aleph_0$ is the smallest transfinite cardinal number.

<u>Theorem</u> (characterization of finite sets)

Let A be a set. The foll. statements are equivalent:

(i) A is finite

(ii) $\forall\, B \subsetneq A \quad A \not\sim B$.

(iii) $\forall\, f : \mathbb{N} \to A$ is not injective

<u>Example</u> Consider $2\mathbb{N} = \{2k \mid k \in \mathbb{N}\} \subsetneq \mathbb{N}$

Let $f : \mathbb{N} \to 2\mathbb{N}$, $f(k) = 2k$. Then f is bijective.

so $|\mathbb{N}| = |2\mathbb{N}|$ Similarly $|\mathbb{N}| \sim |\mathbb{N}^*|$

<u>Theorem</u> (Cantor). 1). $\mathbb{R}$ is not countable (i.e. $\aleph_0 < c$)

2). More precisely: $c = 2^{\aleph_0}$

<u>Proof</u> 1) we use the decimal representation of real numbers, and "Cantor's diagonal argument"

By an (ex) $\mathbb{R} \sim (a,b) \sim [a,b] \sim [a,b) \sim (a,b]$

So we will prove that the interval $[0,1)$ is not countable

If $a \in [0,1)$, then we may write:

$$a = 0.a_1 a_2 a_3 \cdots \quad (10)$$

$$= \sum_{k=1}^{\infty} \frac{a_k}{10^k}, \quad \text{where } a_k \in \{0,\dots,9\}.$$

Recall also that this representation is unique if we exclude period 9 :

$$0.(9) = 0.999\ldots = 1.000\ldots = 1.(0)$$

So some rational number has two representations:

$$0.a_1a_2\ldots a_m(9) = 0.a_1a_2\ldots a_{m-1}(a_m+1)(0)$$

$a_m \neq 9$ $\quad$ (0)

$$= \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_m+1}{10^m} \in \mathbb{Q}.$$

Assume by contradiction that $\exists\ f : \mathbb{N}^* \to [0,1)$ bijective.

Let:

$f(1) = 0.a_{11}a_{12}a_{13}a_{14}a_{15}\ldots$

$f(2) = 0.a_{21}a_{22}a_{23}a_{24}a_{25}\ldots$ $\quad$ (without period 9)

$f(3) = 0.a_{31}a_{32}a_{33}a_{34}a_{35}\ldots$

$f(4) = 0.a_{41}a_{42}a_{43}a_{44}a_{45}\ldots$

$f(5) = 0.a_{51}a_{52}a_{53}a_{54}a_{55}\ldots$

$\vdots$

Let $a = 0.a_1a_2a_3a_4a_5\ldots$, where we choose

$a_1 \neq a_{11}, 0, 9$

$a_2 \neq a_{22}, 0, 9$

$\ldots$

$a_n \neq a_{nn}, 0, 9.$

We have $a \in [0,1)$, and by the uniqueness of representation,

we have that $a \neq f(k)\ \forall k \in \mathbb{N}^*$.

But this contradicts the surjectivity of f.

2) We know that $2^{\aleph_0} = |\mathrm{Hom}(\mathbb{N}^*, \{0,1\})|$

Recall that a function $f : \mathbb{N}^* \to \{0,1\}$ is just a sequence $(a_k)_{k \in \mathbb{N}^*}$, $a_k \in \{0,1\}$

$$\overset{\shortparallel}{(a_1, a_2, \dots)}$$

Again we can get $\mathbb{R} \sim [0,1)$, i.e. $|[0,1)| = c$.

We use the representation of numbers from $[0,1)$ in base 2.

recall: $a = 0.a_1 a_2 a_3 \dots_{(2)}$

$$= \sum_{k=1}^{\infty} \frac{a_k}{2^k}, \text{ where } a_k \in \{0,1\}$$

We have to avoid period 1 because

$$0.(1) = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1 = \lim_{n \to \infty} \frac{1}{1 - \frac{1}{2^n}}$$

So $0.a_1 a_2 \dots a_k 0(1) = 0.a_1 a_2 \dots a_k 1(0) \in \mathbb{Q}$

$$= \frac{a_1}{2} + \frac{a_2}{2^2} + \dots + \frac{a_k}{2^k} + \frac{1}{2^{k+1}}$$

So some rational numbers have exactly two representations, and if we avoid period 1 the repres. is unique.

Let $f : [0,1) \longrightarrow \mathrm{Hom}(\mathbb{N}^*, \{0,1\}) = \{0,1\}^{\mathbb{N}^*}$

where if $a = 0.a_1 a_2 a_3 \dots_{(2)}$

then we define $f(a) = (a_k)_{k \in \mathbb{N}^*} \in \{0,1\}^{\mathbb{N}^*}$.

By the uniqueness of repr. of a, we get that f is a well defined function and f is injective.

It follows that $c = |[0,1)| = |\mathrm{Im}\, f|$

Let $A \subseteq \{0,1\}^{\mathbb{N}}$ be the set of sequences having period 1. Then we have the disjoint union:

$$\{0,1\}^{\mathbb{N}} = \mathrm{Inf} \cup A \text{, hence}$$

$$2^{\aleph_0} = |\{0,1\}^{\mathbb{N}}| = c + |A|$$

But sequences with period 1 represent rational numbers, and by (ex) we have $|A| = \aleph_0$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \| \quad |\mathbb{Q}|$

We get $2^{\aleph_0} = c + \aleph_0$

$$\stackrel{ex}{=} c \, .$$

Rem $c \stackrel{?}{=} \aleph_1$, the continuum hypothesis is indep from other axioms of set theory.

(i.e. between $\aleph_0$ and c there are no other cardinals)

Combinatorics

We will compute the number of elements of certain finite sets.

Let $k, n \in \mathbb{N}$. Consider the totally ordered finite sets

$$A = \{a_1 < a_2 < \dots < a_k\}$$

$$B = \{b_1 < b_2 < \dots < b_n\}.$$

1). Arrangements with repetition

Def A k-arrangement with repetition of n elements is a sequence of length k of elements of B

Example: Let $k=2$, $n=4$. We write down all the 2-arrangements with rep. of 4 elem:

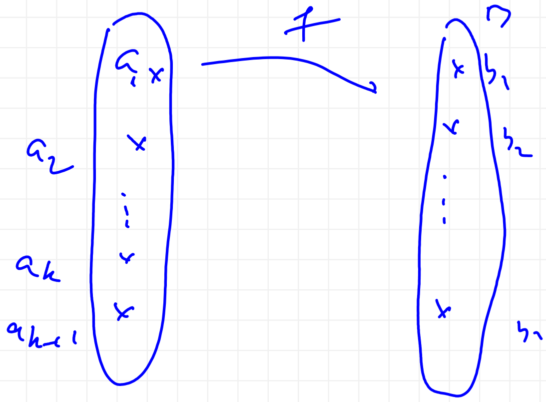
b_1b_1	b_2b_1	b_3b_1	b_4b_1
b_1b_2	b_2b_2	b_3b_2	b_4b_2
b_1b_3	b_2b_3	b_3b_3	b_4b_3
b_1b_4	b_2b_4	b_3b_1	b_4b_4

$\bar{A}_4^2 = 16$

Not. $\bar{A}_n^k$ = the number of k-arrangements with rep. of n elem.

Problem: Calculate $\bar{A}_n^k = ?$

Observe that a k-arrangement with rep of n elem is identified with a function $f: A \to B$.



Let $k=1$.
$|B^A| = n$
Let $k=2$
$|B^A| = n^2$

induction on k: $P(k)$: $\bar{A}_n^k = n^k$

Assume $P(k)$ is true. Then for any choice of $f(a_1), \dots, f(a_k)$ we have n choices for $f(a_{k+1})$.

We get $\bar{A}_n^{k+1} = \bar{A}_n^k \cdot n \overset{P(k)}{=} n^{k+1}$.

Conclusion. $\boxed{\bar{A}_n^k = |\mathrm{Hom}(A,B)| = n^k}$

e). Arrangements

Def: a k-arrangement of n elements is a sequence of length n of elements of B, such that every element occurs at most once.

Expl $n = 4$, $k = 2$ we write down all the 2-arrangements of 4 elements:

$$\begin{array}{cccc} b_1 b_2 & b_2 b_1 & b_3 b_1 & b_4 b_1 \\ b_1 b_3 & b_2 b_3 & b_3 b_2 & b_4 b_2 \\ b_1 b_4 & b_2 b_4 & b_3 b_4 & b_4 b_3 \end{array} \qquad A_4^2 = 12$$

Not $A_n^k :=$ the number of k-arrangements of n elements.

Problem: Calculate $A_n^k = ?$

Observe that a k-arrangement of n elements can be identified with an injective function $f : A \to B$

hence $A_n^k = |\mathrm{Hom}_{inj}(A, B)|$

We proceed by induction on k.

- for $k = 1$, any funct $f : A \to B$ is injective
- for $k = 2$, for any values of $f(e_1)$ we have $n-1$ possibilities for $f(e_2)$
- from k to $k+1$: for any values of $f(e_1), \dots, f(e_k)$ we have $n-k$ possibilities for $f(e_k)$

we get $A_n^{k+1} = A_n^k \cdot (n-k)$.

Hence $\boxed{A_n^k = n(n-1)(n-2)\cdots(n-k+1)}$

Rem 1) if $k > n$, then $A_n^k = 0$

2) If $k = 0$, i.e. $A = \emptyset$, then $\mathrm{Hom}(\emptyset, B) = \{\emptyset\}$

hence $\bar{A}_n^0 = A_n^0 = 1$. in part $0^0 = 1$ in this context!!

3) Permutations

Def A <u>perm. of n elmts</u> is a sequence of length n of elmts of B s.t. each elem occurs exactly once.

Rm a perm. of n elmts is identified with a bijective function $f: A \to B$, where $k=n$.

Not P_n = no. of perm. of n elmts.

We have: $\boxed{P_n = A_n^n = n!}$

Rm $P_0 = 1$, so $0! = 1$ by convention.

$P_0 = |\mathrm{Hom}_{bij}(\emptyset, \emptyset)|$

4) Combinations

Def A <u>k-combination of n elements</u> is a strictly increasing sequence of length k of elem of B.

Ex $n = 5$, $k = 3$. We write down all the 3-combinations of 5 elements:

$b_1 b_2 b_3$	$b_1 b_3 b_4$	$b_2 b_3 b_4$	$b_3 b_4 b_5$.
$b_1 b_2 b_4$	$b_1 b_3 b_5$	$b_2 b_3 b_5$	
$b_1 b_2 b_5$	$b_1 b_4 b_5$	$b_2 b_4 b_5$	

Not $\binom{n}{k} = C_n^k$ = no. of k-combinations of n elmts

Problem: calculate $\binom{n}{k} = ?$

Observe that from any k-comb. of n elements we get P_k k-arrangem. of n elements.

then $A_n^k = C_n^k \cdot P_k$.

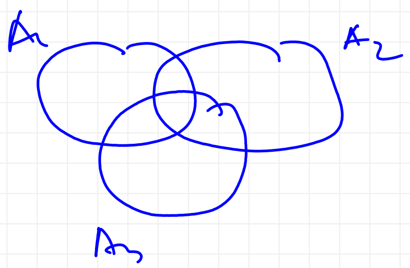
Then
$$\binom{n}{k} = \frac{A_n^k}{P_k} = \frac{n(n-1)\dots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$$
(for $k \le n$)

(for $k > n$ $\binom{n}{k} = 0$)

Rem: any k-combination of n elements is determined uniquely by a subset with k-elements of B.

hence $\binom{n}{k}$ = number of subsets with k-elements of a set with n elements.

5). The inclusion exclusion principle



- $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$
- $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$

Theorem: let $A_1, \dots, A_n$ be finite sets.

Then
$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i_1 < i_2 < i_3 \le n} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| - \dots + (-1)^{k+1} \sum_{1 \le i_1 < \dots < i_k \le n} |A_{i_1} \cap \dots \cap A_{i_k}| + \dots + (-1)^{n-1} \left|\bigcap_{i=1}^{n} A_i\right|$$

Proof HW

Pag. 8.4.3 / 55.

$\boxed{\text{Homework: ex. 95 – 106.}}$