Seminar 1 Analiză Complexă

(1) a) Să se determine rădăcinile de ordinul $n\in\mathbb{N}^*$ dintr-un număr complex nenul. Interpretare geometrică.

b) Să se calculeze: $\sqrt{\frac{1+i\sqrt{3}}{2}}$, $\sqrt[4]{i}$, $\sqrt[3]{\frac{1+i}{1-i}}$, $\sqrt{\sqrt{3}+i}$.

c) Să se calculeze $\arg i$, $\operatorname{Arg} i$; $\arg\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)$, $\operatorname{Arg}\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)$; $\arg\left(\frac{1+i}{1-i}\right)^3$, $\operatorname{Arg}\left(\frac{1+i}{1-i}\right)^3$.

Soluție. Fie $w\in\mathbb{C}^*$. Căutăm $z\in\mathbb{C}^*$ astfel ca $z^n=w$.

$w\in\mathbb{C}^* \Rightarrow w=r(\cos\theta+i\sin\theta)$, unde $r=|w|$, $\theta\in\operatorname{Arg} w$.

Fie $z=\rho(\cos\varphi+i\sin\varphi)$, $\rho=|z|$, $\varphi\in\operatorname{Arg} z$.

Atunci $z^n=\rho^n(\cos\varphi+i\sin\varphi)^n \underset{\text{f. Moivre}}{=} \rho^n(\cos n\varphi+i\sin n\varphi)$

Deci $z^n=w \Leftrightarrow$

$$\rho^n(\cos(n\varphi)+i\sin(n\varphi))=r(\cos\theta+i\sin\theta)$$

$\Leftrightarrow$ $\rho^n=r$ și $n\varphi=\theta \pmod{2\pi}$ $\Leftrightarrow$

$\Leftrightarrow$ $\rho=\sqrt[n]{r}$ și $n\varphi\in\{\theta+2k\pi : k\in\mathbb{Z}\}$

$\Leftrightarrow$ $\rho=\sqrt[n]{r}$ și $\varphi\in\left\{\frac{\theta+2k\pi}{n} : k\in\mathbb{Z}\right\}$.

Fie $\varphi_k=\frac{\theta+2k\pi}{n}$, $k\in\mathbb{Z}$ $\Rightarrow$ soluțiile distincte sunt:

(*) $z_k = \sqrt[n]{r}(\cos\varphi_k + i\sin\varphi_k)$, $k=\overline{0,n-1}$, unde $\varphi_k = \frac{\theta+2k\pi}{n}$

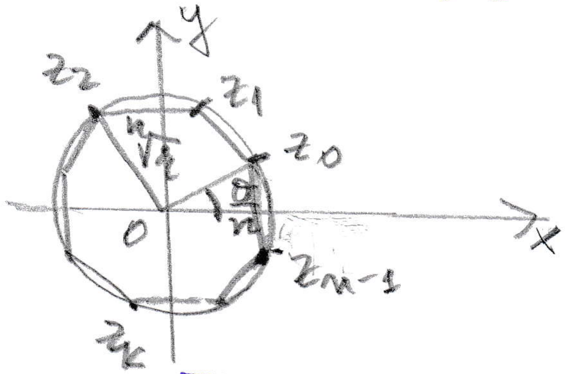
$\uparrow$ soluțiile distincte

Interpretare geometrică

Deoarece $z_k = \sqrt[n]{r}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right)$,

$\Rightarrow |z_k| = \sqrt[n]{r}$, $k=\overline{0,n-1}$ $\Rightarrow z_k \in \partial U(0, \sqrt[n]{r})$, $k=\overline{0,n-1}$

$\Rightarrow z_0, z_1, \ldots, z_{n-1}$ sunt vârfurile (afixele vârfurilor) unui poligon regulat cu n laturi înscris în cercul de centru O și rază $\sqrt[n]{r}$.



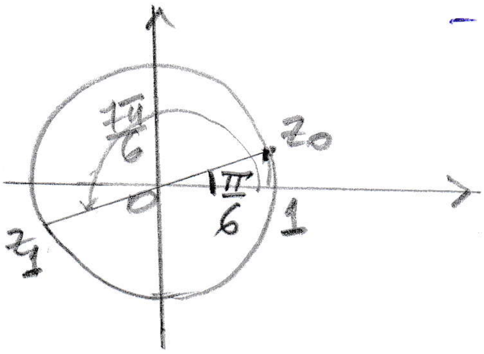
b) $\sqrt{\frac{1+i\sqrt{3}}{2}} = \sqrt{\frac{1}{2}+i\frac{\sqrt{3}}{2}}$. Fie $n=2$ și $w = \frac{1}{2}+i\frac{\sqrt{3}}{2} \in \mathbb{C}^* \Rightarrow$

$\Rightarrow |w| = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1$; dacă $\theta = \arg w \in (-\pi, \pi]$

$\Rightarrow \cos\theta = \frac{1}{2}$, $\sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \boxed{\theta = \frac{\pi}{3}} \Rightarrow$

$\Rightarrow w = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} \xRightarrow{(*)} z_k = \cos\left(\frac{\frac{\pi}{3}+2k\pi}{2}\right) + i\sin\left(\frac{\frac{\pi}{3}+2k\pi}{2}\right)$, $k=\overline{0,1}$.

$z_0 = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$, $z_1 = \cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}$.



Deci: $\sqrt{\frac{1+i\sqrt{3}}{2}} \in \left\{ \cos\left(\frac{\frac{\pi}{3}+2k\pi}{2}\right) + i\sin\left(\frac{\frac{\pi}{3}+2k\pi}{2}\right) : k=\overline{0,1} \right\}$.

- $\sqrt[4]{i} = ?$ $\quad n=4,\ w=i \Rightarrow w = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$\Rightarrow$ ec. $z^4 = i$ are soluțiile $z_k = \cos\left(\frac{\frac{\pi}{2}+2k\pi}{4}\right) + i\sin\left(\frac{\frac{\pi}{2}+2k\pi}{4}\right)$, $k=\overline{0,3} \Rightarrow |z_k| = 1$, $k=\overline{0,3} \Rightarrow z_k \in \partial U(0,1)$, $k=\overline{0,3}$.

- $\sqrt[3]{\frac{1+i}{1-i}} = ?$ $\quad \frac{1+i}{1-i} = \frac{(1+i)^2}{2} = \frac{1-1+2i}{2} = i$

$\Rightarrow \sqrt[3]{i} \in \left\{ \cos\left(\frac{\frac{\pi}{2}+2k\pi}{3}\right) + i\sin\left(\frac{\frac{\pi}{2}+2k\pi}{3}\right) ; k=\overline{0,2} \right\}$

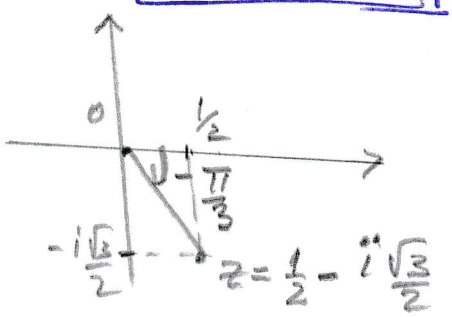
c) $\arg i = ?$; $\operatorname{Arg} i = ?$ $\quad i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \Rightarrow$

$\Rightarrow \arg i = \frac{\pi}{2} \Rightarrow \operatorname{Arg} i = \left\{ \frac{\pi}{2} + 2k\pi : k \in \mathbb{Z} \right\}$.

$\arg\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = ?$ $\quad z = \frac{1}{2} - i\frac{\sqrt{3}}{2} \Rightarrow |z| = 1$.

Fie $\theta_0 = \arg z \in (-\pi, \pi] \Rightarrow \cos\theta_0 = \frac{1}{2}$, $\sin\theta_0 = -\frac{\sqrt{3}}{2}$

$\Rightarrow \boxed{\theta_0 = -\frac{\pi}{3}} \Rightarrow \operatorname{Arg} z = \left\{ -\frac{\pi}{3} + 2k\pi : k \in \mathbb{Z} \right\}$.



$$\arg\left(\frac{1+i}{1-i}\right)^3 \underset{\substack{\uparrow \\ \frac{1+i}{1-i}=i}}{=} \arg i^3 = \arg(-i) = -\frac{\pi}{2} \Rightarrow$$

$$\Rightarrow \mathrm{Arg}\left(\frac{1+i}{1-i}\right)^3 = \left\{-\frac{\pi}{2} + 2k\pi : k\in\mathbb{Z}\right\}.$$

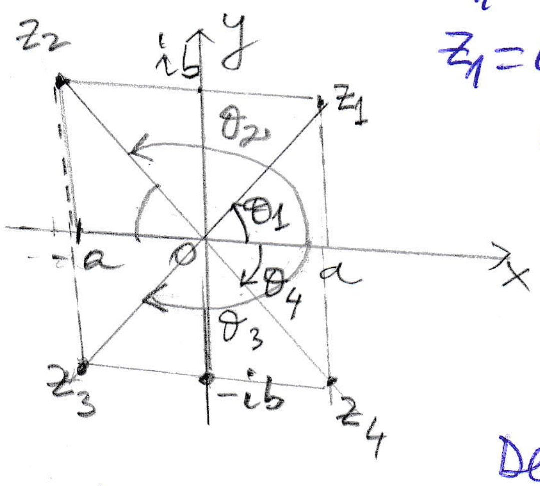
<u>sau</u>

$$\mathrm{Arg}\left(\frac{1+i}{1-i}\right)^3 = 3\,\mathrm{Arg}\left(\frac{1+i}{1-i}\right) = 3\,\mathrm{Arg}\, i = \left\{\frac{3\pi}{2} + 2n\pi : n\in\mathbb{Z}\right\}.$$

(2) Fie $a, b>0$, $z_1=a+ib$, $z_2=-a+ib$, $z_3=-a-ib$, $z_4=a-ib$. Să se determine $\arg z_k$ și $\mathrm{Arg}\, z_k$, $k=\overline{1,4}$.

<u>Soluție</u>. Fie $\theta_1 = \arg z_1$

$$\left.\begin{matrix} \theta_1=\arg z_1 \\ z_1=a+ib,\ a,b>0 \end{matrix}\right\} \Rightarrow \theta_1 \in \left(0, \frac{\pi}{2}\right).$$



Avem că $\begin{cases} \cos\theta_1 = \dfrac{a}{\sqrt{a^2+b^2}} \\ \sin\theta_1 = \dfrac{b}{\sqrt{a^2+b^2}} \end{cases} \Rightarrow$

$$\mathrm{tg}\,\theta_1 = \frac{b}{a} \Rightarrow \boxed{\theta_1 = \mathrm{arctg}\frac{b}{a}}$$

Deci $\arg z_1 = \mathrm{arctg}\dfrac{b}{a} \Rightarrow$

$$\mathrm{Arg}\, z_1 = \left\{\mathrm{arctg}\frac{b}{a} + 2k\pi : k\in\mathbb{Z}\right\}.$$

Fie $\theta_2 = \arg z_2 \Rightarrow \underline{\theta_2 = \pi - \theta_1} = \pi - \mathrm{arctg}\dfrac{b}{a} \in \left(\dfrac{\pi}{2}, \pi\right)$.

$\Rightarrow \mathrm{Arg}\, z_2 = \{\theta_2 + 2k\pi : k\in\mathbb{Z}\}$.

Fie $\theta_3 = \arg z_3 \Rightarrow \underline{\theta_3 = -\theta_2} = -(\pi - \theta_1) = \theta_1 - \pi$

$\Rightarrow \arg z_3 = \mathrm{arctg}\dfrac{b}{a} - \pi \Rightarrow \mathrm{Arg}\, z_3 = \{\theta_3 + 2k\pi : k\in\mathbb{Z}\}$.

Fie $\theta_4 = \arg z_4 \Rightarrow \underline{\theta_4 = -\theta_1} \Rightarrow \underline{\arg z_4 = -\arg z_1}$

$\Rightarrow \mathrm{Arg}\, z_4 = \{\theta_4 + 2k\pi : k\in\mathbb{Z}\}$.

3) Fie $z_1, z_2 \in \mathbb{C}$. Să se arate că au loc relațiile:

a) $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm 2\mathrm{Re}(z_1 \cdot \overline{z_2}) = |z_1|^2 + |z_2|^2 \pm 2\mathrm{Re}(\overline{z_1} z_2)$

b) $||z_1| - |z_2|| \le |z_1 + z_2| \le |z_1| + |z_2|$.

Să se studieze egalitatea în fiecare inegalitate.

__Soluție:__ a) $|z_1 \pm z_2|^2 \underset{\uparrow}{=} (z_1 \pm z_2) \cdot \overline{(z_1 \pm z_2)} =$

$|z|^2 = z \cdot \overline{z}, \forall z \in \mathbb{C}$

$= (z_1 \pm z_2)(\overline{z_1} \pm \overline{z_2}) = z_1 \cdot \overline{z_1} \pm z_1 \cdot \overline{z_2} \pm z_2 \cdot \overline{z_1} + z_2 \cdot \overline{z_2}$

$= |z_1|^2 + |z_2|^2 \pm (z_1 \cdot \overline{z_2} + \underbrace{z_2 \cdot \overline{z_1}}_{= \overline{z_1 \cdot \overline{z_2}}}) = |z_1|^2 + |z_2|^2 +$

$\pm \underbrace{(z_1 \cdot \overline{z_2} + \overline{z_1 \cdot \overline{z_2}})}_{= 2\mathrm{Re}(z_1 \cdot \overline{z_2})} = |z_1|^2 + |z_2|^2 \pm 2\mathrm{Re}(z_1 \cdot \overline{z_2})$ $\quad (z + \overline{z} = 2\mathrm{Re}\, z, \forall z \in \mathbb{C})$

$\underset{\uparrow}{=} |z_1|^2 + |z_2|^2 \pm 2\mathrm{Re}(\overline{z_1} \cdot z_2)$

$\mathrm{Re}\, z = \mathrm{Re}\, \overline{z}, \forall z \in \mathbb{C} \Rightarrow \mathrm{Re}(z_1 \overline{z_2}) = \mathrm{Re}(\overline{z_1} \cdot z_2)$.

b) $|z_1 + z_2| \le |z_1| + |z_2| \Leftrightarrow |z_1 + z_2|^2 \le (|z_1| + |z_2|)^2 \Leftrightarrow$

$\Leftrightarrow |z_1|^2 + |z_2|^2 + 2\mathrm{Re}(z_1 \cdot \overline{z_2}) \le |z_1|^2 + |z_2|^2 + 2|z_1| \cdot |z_2|$

$\Leftrightarrow \mathrm{Re}(z_1 \cdot \overline{z_2}) \le |z_1| \cdot |z_2| = |z_1| \cdot |\overline{z_2}| = |z_1 \overline{z_2}| \Leftrightarrow$

$\Leftrightarrow \mathrm{Re}(z_1 \cdot \overline{z_2}) \underset{\uparrow}{\le} |z_1 \cdot \overline{z_2}|$ – evident.

$\mathrm{Re}\, z \le |z|, \forall z \in \mathbb{C}$

Cazul egalității: $|z_1+z_2| = |z_1| + |z_2| \Leftrightarrow$

$\Leftrightarrow \operatorname{Re}(z_1\cdot\overline{z_2}) = |z_1\cdot\overline{z_2}| \Leftrightarrow \operatorname{Im}(z_1\cdot\overline{z_2}) = 0$ și $\operatorname{Re}(z_1\cdot\overline{z_2}) \geq 0 \Leftrightarrow$

$\Leftrightarrow z_1\cdot\overline{z_2} \geq 0$. Fie $a = \underbrace{z_1\cdot\overline{z_2}}_{\geq 0} \Rightarrow a z_2 = z_1\cdot|z_2|^2 \,/ : |z_2|^2$

$\Rightarrow z_1 = \underbrace{\frac{a}{|z_2|^2}}_{\text{not } b \geq 0}\cdot z_2 = b z_2 \Rightarrow \boxed{z_1 = b z_2 \text{ cu } b \geq 0.}$

Inegalitatea $||z_1|-|z_2|| \leq |z_1+z_2|$ se tratează în mod analog.

4 a) Fie $z\in\mathbb{C}$ cu $|z|=1$. Să se arate că are loc relația

$$2 \leq |1+z| + |1+z^2| + |1+z^3| \leq 6.$$

b) Dacă $z\in\mathbb{C}^*$ cu $\left|z+\frac{1}{z}\right| \geq 2$, atunci $\left|z^3+\frac{1}{z^3}\right| \geq 2$.

c) Dacă $z, a\in\mathbb{C}$ astfel ca $|z|=1$ și $1-\overline{a}z \neq 0$, atunci $\left|\frac{z-a}{1-\overline{a}z}\right| = 1$.

d) Dacă $z, a\in\mathbb{C}$ cu $|z|<1$, $|a|<1$, atunci $\left|\frac{z-a}{1-\overline{a}z}\right| < 1$.

e) Dacă $z, a\in\mathbb{C}$ cu $\operatorname{Im} z>0$, $\operatorname{Im} a>0$, atunci $\left|\frac{z-a}{z-\overline{a}}\right| < 1$.

Soluție: a) $|1+z| + |1+z^2| + |1+z^3| \leq 1 + \underbrace{|z|}_{=1} + 1 + \underbrace{|z|}_{=1}^2 + 1 + \underbrace{|z|}_{=1}^3 = 6.$

Dar $2=1+1=(1+z)+(1-z)=|(1+z)+(1-z)|\leq|1+z|+|1-z|$

$=|1+z|+|(1+z^3)-(z^3+z)|\leq|1+z|+|1+z^3|+|z^3+z|$

$=|1+z|+|1+z^3|+\underbrace{|z|}_{=1}\cdot|1+z^2|=|1+z|+|1+z^2|+|1+z^3|.$

b) $\left|z^3+\frac{1}{z^3}\right|=\left|\left(z+\frac{1}{z}\right)^3-3\left(z+\frac{1}{z}\right)\right|=\underbrace{\left|z+\frac{1}{z}\right|}_{\geq 2}\cdot\left|\left(z+\frac{1}{z}\right)^2-3\right|$

$\geq 2\cdot\left|\underbrace{\left|z+\frac{1}{z}\right|^2}_{\geq 4}-3\right|\geq 2|4-3|=2.$

c) Deoarece $|z|=1 \Rightarrow z\cdot\bar{z}\underset{\substack{\uparrow\\ |z|^2=z\cdot\bar{z},\ \forall z\in\mathbb{C}}}{=}1 \Rightarrow$

$\Rightarrow\left|\frac{z-a}{1-\bar{a}z}\right|=\left|\frac{z-a}{z\cdot\bar{z}-\bar{a}z}\right|=\underbrace{\frac{1}{|z|}}_{=1}\cdot\left|\frac{z-a}{\bar{z}-\bar{a}}\right|=\left|\frac{z-a}{\overline{(z-a)}}\right|=1.$

d) Deoarece $|z|<1$ şi $|a|<1 \Rightarrow |\bar{a}\cdot z|=|a|\cdot|z|<1$

$\Rightarrow 1-\bar{a}\cdot z\neq 0.$

Avem că $\left|\frac{z-a}{1-\bar{a}z}\right|<1 \Leftrightarrow |z-a|<|1-\bar{a}z| \Leftrightarrow$

Prob. 3, a)

$\Leftrightarrow |z-a|^2<|1-\bar{a}z|^2 \Leftrightarrow |z|^2+|a|^2-2\mathrm{Re}(z\bar{a})<$

$<1+|\bar{a}\cdot z|^2-2\mathrm{Re}(\bar{a}\cdot z) \Leftrightarrow |z|^2+|a|^2<1+|a|^2\cdot|z|^2$

$\Leftrightarrow |z|^2(1-|a|^2)+|a|^2-1<0 \Leftrightarrow \underbrace{(1-|a|^2)}_{>0}\underbrace{(|z|^2-1)}_{<0}<0$ OK.

e) Fie $z,a\in\mathbb{C}$ cu $\mathrm{Im}\,z>0$ şi $\mathrm{Im}\,a>0$.

Atunci $z\neq\bar{a}$ pt. că $\mathrm{Im}\,z>0$ şi $\mathrm{Im}\,\bar{a}=-\mathrm{Im}\,a<0$.

Avem că $\left|\frac{z-a}{z-\bar{a}}\right|<1 \Leftrightarrow |z-a|<|z-\bar{a}| \Leftrightarrow$

$\Leftrightarrow |z-a|^2<|z-\bar{a}|^2 \Leftrightarrow |z|^2+|a|^2-2\mathrm{Re}(z\cdot\bar{a})<$

$<|z|^2+|a|^2-2\mathrm{Re}(z\cdot a) \Leftrightarrow \mathrm{Re}(z\cdot\bar{a})>\mathrm{Re}(z\cdot a)$

$\Leftrightarrow Re[z(\underbrace{a-\bar{a}}_{=2i\,Im\,a})]<0 \Leftrightarrow Re[z\cdot 2i\cdot Im\,a]<0 \Leftrightarrow$

$\Leftrightarrow Im\,a\cdot Re(iz)<0.$ $\binom{**}{**}$

Dacă $z=x+iy \Rightarrow iz=-y+ix \Rightarrow Re(iz)=-y$

$\Rightarrow \boxed{Re(iz)=-Im\,z}$.

Deci $Im\,a\cdot Re(iz)<0 \underset{\binom{**}{**}}{\Leftrightarrow} \underbrace{Im\,a}_{>0}\cdot\underbrace{Im\,z}_{>0}>0.$ OK.

5 Din calculul expresiei $\frac{1+i\sqrt{3}}{1+i}$, să se determine valorile lui $\cos\frac{\pi}{12}$ și $\sin\frac{\pi}{12}$.

Soluție. Fie $z=\frac{1+i\sqrt{3}}{1+i}$, $z_1=1+i\sqrt{3}$, $z_2=1+i$

$\Rightarrow z=\frac{z_1}{z_2}=\frac{z_1\cdot\bar{z}_2}{|z_2|^2}$.

Dar $|z_1|=\sqrt{1+3}=2$; dacă $\theta_1:=\arg z_1 \Rightarrow$

$\Rightarrow \cos\theta_1=\frac{1}{2}, \sin\theta_1=\frac{\sqrt{3}}{2} \Rightarrow \boxed{\theta_1=\frac{\pi}{3}}$.

$|z_2|=\sqrt{2}$; dacă $\theta_2=\arg z_2 \Rightarrow \boxed{\theta_2=\frac{\pi}{4}}$.

$\Rightarrow z_1=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$ și $z_2=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$.

Deci: $z=\frac{z_1}{z_2}=\frac{2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)}{\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)}$

$=\sqrt{2}\left(\cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)\right)$

$=\sqrt{2}\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$.

$$\Rightarrow z=\sqrt{2}\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right). \quad (***)$$

Pe de altă parte,

$$z=\frac{z_1\cdot\overline{z_2}}{|z_2|^2}=\frac{(1+i\sqrt{3})(1-i)}{2}=\frac{1-i+i\sqrt{3}+\sqrt{3}}{2}=$$

$$=\frac{1+\sqrt{3}}{2}+i\,\frac{\sqrt{3}-1}{2}.$$

$$\Rightarrow z=\frac{1+\sqrt{3}}{2}+i\,\frac{\sqrt{3}-1}{2}.$$

Așadar, din $(***)$ și relația precedentă obținem că

$$\frac{1+\sqrt{3}}{2}+i\,\frac{\sqrt{3}-1}{2}=\sqrt{2}\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right) \Leftrightarrow$$

$$\Leftrightarrow \frac{1+\sqrt{3}}{2}=\sqrt{2}\cos\frac{\pi}{12} \text{ și } \frac{\sqrt{3}-1}{2}=\sqrt{2}\sin\frac{\pi}{12} \Leftrightarrow$$

$$\Leftrightarrow \cos\frac{\pi}{12}=\frac{1+\sqrt{3}}{2\sqrt{2}} \text{ și } \sin\frac{\pi}{12}=\frac{\sqrt{3}-1}{2\sqrt{2}}.$$

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Temă 1.

1. Să se calculeze: $(1+i\sqrt{3})^3$, $(1+i)^{100}$.
2. Să se determine toate valorile lui $\sqrt{\sqrt{3}+i}$ și $\sqrt[4]{1-i}$.
3. Să se determine toate numerele complexe z care satisfac condiția:
$$|z|-\bar{z}=\frac{1}{2}+i.$$
4. Să se arate că dacă $z_1, z_2, z_3 \in \mathbb{C}$ satisfac relațiile:
$$z_1+z_2+z_3=0 \text{ și } z_1^2+z_2^2+z_3^2=0,$$
atunci $|z_1|=|z_2|=|z_3|$.
5. Să se arate că dacă $z_1, z_2, z_3 \in \mathbb{C}$ satisfac relația:
$$z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_1z_3,$$
atunci $|z_1-z_2|=|z_2-z_3|=|z_1-z_3|$.
6. Fie $z_1, z_2 \in \mathbb{C}$. Să se arate că are loc egalitatea:
$$|z_1+z_2|^2+|z_1-z_2|^2=2(|z_1|^2+|z_2|^2).$$