Seminar 9

1. Are the following sets subrings of the field $\mathbb{C}$:
(i) $A = \{b\mathrm{i} \mid b \in \mathbb{R}\}$;
(ii) $B = \{a + b\sqrt[3]{2} + c\sqrt[3]{4} \mid a, b, c \in \mathbb{Q}\}$;
(iii) $C = \{z \in \mathbb{C} \mid |z| \leq 1\}$?

2. Show that the set $\mathbb{Z}[\mathrm{i}] = \{a + b\mathrm{i} \mid a, b \in \mathbb{Z}\}$ is a subring of the field $\mathbb{C}$, called *the ring of Gauss integers*. Determine its invertible elements.

3. Are the following sets subrings of the ring $M_2(\mathbb{R})$:
(i) $\mathcal{A} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b, c \in \mathbb{R} \right\}$;
(ii) $\mathcal{B} = \left\{ \begin{pmatrix} a & a \\ 0 & b \end{pmatrix} \middle| a, b \in \mathbb{R} \right\}$;
(iii) $\mathcal{C} = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \middle| a, b \in \mathbb{R} \right\}$?

4. Are the following sets subrings of the ring $\mathbb{R}[X]$:
(i) $A = \{f \in \mathbb{R}[X] \mid \text{the free term of } f \text{ is } 0\}$;
(ii) $B = \{f \in \mathbb{R}[X] \mid \text{the free term of } f \text{ is } 1\}$;
(iii) $C = \{f \in \mathbb{R}[X] \mid \text{the coefficient of the term of degree 1 of } f \text{ is } 0\}$?

5. Give examples of:
(i) subring without identity of a ring with identity.
(ii) subring with identity of a ring with identity, which have different identities.
(iii) non-commutative finite ring.

6. Show that the set $\mathbb{Q}(\sqrt{2}) = \{a+b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ is a subfield of the field $\mathbb{R}$. Generalization.

7. Is the set $A = \{a + b\sqrt[3]{2} \mid a, b \in \mathbb{Q}\}$ a subring of the field $\mathbb{R}$?

8. Let $m, n \in \mathbb{N}$. Show that $n\mathbb{Z}$ is a subring of the ring $m\mathbb{Z} \Leftrightarrow m|n$.

9. Let $(R, +, \cdot)$ be a ring. Show that:

$$Z(R) = \{a \in R \mid a \cdot r = r \cdot a \,, \forall r \in R\}$$

is a subring of R, called the *center of* R. When does the equality $Z(R) = R$ hold?

10. Show that:

$$Z(M_2(\mathbb{R}), +, \cdot) = \{a \cdot I_2 \mid a \in \mathbb{R}\},$$

where I_2 is the identity matrix. Generalization for $M_n(\mathbb{R})$ with $n \in \mathbb{N}$, $n \geq 2$.