Seminar 3

Solvable differential equations in implicit form

$F(x, y, y') = 0$

1) The Clairaut diff. eq.

the general form: $\boxed{y = x \cdot y' + \Psi(y')}$, $\Psi \in C^1$

- we derivate the eq. with respect to x

$$\cancel{y'} = \cancel{y'} + x \cdot y'' + \Psi'(y') \cdot y''$$

$$\boxed{y''(x + \Psi'(y')) = 0}$$

- we denote by $\boxed{p = y'}$

$$p' \cdot (x + \Psi'(p)) = 0$$

$\nearrow$ $p' = 0 \Rightarrow \boxed{p = c}$, $c \in \mathbb{R} \Rightarrow$

$\searrow$ $x + \Psi'(p) = 0 \Rightarrow$

$\Rightarrow \boxed{y(x) = x \cdot c + \Psi(c), \ c \in \mathbb{R}}$ the general solution

$x + \psi'(p) = 0 \Rightarrow \begin{cases} x = -\psi'(p) \\ y = \underbrace{x}_{-\psi'(p)} \cdot p + \psi(p) \end{cases}$ the singular solution in parametric form.

$y' = p$

<u>Exercise 1</u> Solve the following diff. eq.

a) $y = xy' - (y')^2$

b) $y = xy' + \sqrt{1 + y'^2}$

c) $y = xy' + \frac{1}{5}(y')^5$

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a) $y = xy' - (y')^2$ $\qquad\qquad \psi(y') = -(y')^2$

- we derivate the eq. with respect to x.

$\Rightarrow \cancel{y'} = \cancel{y'} + x \cdot y'' - 2y' \cdot y''$

$\Rightarrow y''(x - 2y') = 0$

- denote $y' = p \Rightarrow \boxed{p'(x - 2p) = 0} \Rightarrow$

$p' = 0 \Rightarrow p = c, c \in \mathbb{R}.$

$y' = p \quad , \; y = xy' - (y')^2 \Big\} \Rightarrow \boxed{y(x) = x \cdot c - c^2, c \in \mathbb{R}}$

the general sol.

$x - 2p = 0 \Rightarrow \begin{cases} x = 2p \\ y = xp - p^2 \end{cases}$

$\Rightarrow \begin{cases} x = 2p \\ y = 2p \cdot p - p^2 \end{cases} \Rightarrow \begin{cases} x = 2p \\ y = p^2 \end{cases}$ the singular sol. in parametric form.

$x = 2p \Rightarrow p = \frac{x}{2} \Rightarrow y(x) = \left(\frac{x}{2}\right)^2$

$\boxed{y(x) = \frac{x^2}{4}}$ the singular solution in explicit form.

b) $y = xy' + \sqrt{1+y'^2} \qquad \varphi(y') = \sqrt{1+y'^2}$

– we derivate

$\cancel{y'} = \cancel{y'} + xy'' + \frac{1}{\cancel{2} \cdot \sqrt{1+y'^2}} \cdot \cancel{2} \cdot y' \cdot y''$

$y'' \left(x + \frac{y'}{\sqrt{1+y'^2}} \right) = 0$

$y' = p \Rightarrow y'' = p'$

$$\Rightarrow \quad p'\left(x + \frac{p}{\sqrt{1+p^2}}\right) = 0$$

$$\left.\begin{aligned} &\Rightarrow p' = 0 \Rightarrow p = c,\ c \in \mathbb{R} \\ &\qquad y' = p = c \\ &\qquad \boxed{y = xy' + \sqrt{1+y'^2}} \end{aligned}\right\} \Rightarrow \boxed{y = x.c + \sqrt{1+c^2},\ c \in \mathbb{R}}$$

the general solution

$$x + \frac{p}{\sqrt{1+p^2}} = 0 \Rightarrow x = -\frac{p}{\sqrt{1+p^2}}$$

$$y = x.p + \sqrt{1+p^2} = -\frac{p^2}{\sqrt{1+p^2}} + \sqrt{1+p^2} =$$

$$= \frac{-p^2 + 1 + p^2}{\sqrt{1+p^2}} = \frac{1}{\sqrt{1+p^2}}$$

$$\Rightarrow \begin{cases} x = -\dfrac{p}{\sqrt{1+p^2}} \\ y = \dfrac{1}{\sqrt{1+p^2}} \end{cases}$$ the singular solution in parametric form.

$$x^2 + y^2 = \left(-\frac{p}{\sqrt{1+p^2}}\right)^2 + \left(\frac{1}{\sqrt{1+p^2}}\right)^2 = \frac{p^2}{1+p^2} + \frac{1}{1+p^2} = 1$$

$\Rightarrow \boxed{x^2+y^2=1}$, is the singular solution in implicit form.

$\Rightarrow y^2 = 1-x^2 \rightarrow \boxed{y(x) = \pm\sqrt{1-x^2}}$ the singular solution in explicit form.

2. The Lagrange differential equations

the general form: $\boxed{y = x.\varphi(y') + \psi(y')}$, $\varphi, \psi \in C^1$

- we derivate the eq. with respect to x.
- we denote $y'=p \Rightarrow \ldots \Rightarrow \begin{cases} x(p) = \ldots \\ y(p) = x(p).\varphi(p) + \psi(p) \end{cases}$

the general sol. in parametric form.

Exercise 2. Solve the diff. eqs:

a) $y = x(1+y') + (y')^2$

b) $y = \frac{3}{2}xy' + e^{y'}$

c) $y = x(y')^2 - \frac{1}{y'}$

a) $y = x(1+y') + (y')^2$ $\qquad \varphi(y') = 1+y'$, $\psi(y') = (y')^2$

- we derivate the eq. with respect to x.

$\Rightarrow \cancel{y'} = 1 + \cancel{y'} + x \cdot y'' + 2 \cdot y' \cdot y''$

$\Rightarrow 0 = 1 + y''(x + 2y')$

- we denote by $y' = p$

$\Rightarrow 0 = 1 + p'(x+2p)$

$\Rightarrow \boxed{p'(x+2p) = -1}$ $\qquad p = p(x) \qquad p' = p'(x) = \frac{dp}{dx}$

$\frac{dp}{dx}(x+2p) = -1 \quad \Big| \cdot \frac{dx}{dp}$

$x + 2p = -\underbrace{\frac{dx}{dp}}_{x'(p)}$ $\Rightarrow$ the unknown function $x = x(p)$

$x + 2p = x'$

$x(p) + 2p = -x'(p) \Rightarrow \boxed{x'(p) + x(p) = -2p}$ nonhomog. first order linear diff. eq.

$\boxed{x' + x = -2p}$. x is the unknown function with respect to the variable p.

$x = x(p)$

$x' + x = 0$

$x' = -x$

$\frac{dx}{dp} = -x \Rightarrow \int \frac{dx}{x} = \int -dp \Rightarrow \ln x = -p + \ln c$

$\boxed{x_0(p) = c \cdot e^{-p}, c \in \mathbb{R}}$

$x_{part}(p) = ?$

$x_{part}(p) = c(p) \cdot e^{-p}$

$x'_{part} + x_{part} = -2p$

$\Rightarrow c'(p) \cdot e^{-p} + \cancel{c(p)\, e^{-p} \cdot (-1)} + \cancel{c(p) \cdot e^{-p}} = -2p$

$(e^{p})'$

$\downarrow$

$\Rightarrow c'(p) \cdot e^{-p} = -2p \quad | \cdot e^{p}$

$c'(p) = -2p \cdot e^{p} \Rightarrow c(p) = -2 \int p \cdot e^{p} dp =$

$= -2p \cdot e^{p} + 2 \int e^{p} dp = -2p \cdot e^{p} + 2e^{p}$

$\Rightarrow x_{part}(p) = c(p) \cdot e^{-p} = (-2p + 2) \cdot e^{p} \cdot e^{-p} = -2p + 2$

$\Rightarrow x(p) = x_0(p) + x_{part}(p) \Rightarrow \boxed{x(p) = c \cdot e^{-p} - 2p + 2, c \in \mathbb{R}}$

$$\left.\begin{array}{l} y' = p \\ y = x\,(1+y') + (y')^2 \end{array}\right\} \Rightarrow y = x\,(1+p) + p^2$$

$$\Rightarrow \begin{cases} x(p) = c.e^{-p} - 2p + 2 \\ y(p) = \left(ce^{-p} - 2p + 2\right).(1+p) + p^2 \end{cases}, \; c \in \mathbb{R}.$$

the general solution in parametric form.

<u>Remark</u>. if there exist $\alpha \in \mathbb{R}$ such that $\varphi(\alpha) = \alpha$ then $y(x) = \alpha . x + \psi(\alpha)$ is a singular solution of the Lagrange equation.

$\varphi(\alpha) = 1 + \alpha$

$\varphi(\alpha) = \alpha$

$1 + \alpha = \alpha$

$1 = 0$ (False)

$\varphi(\alpha) = \alpha$ has not real sol. $\Rightarrow$ the diff. eq has not singular solution.

b) $\boxed{y = \frac{3}{2} x y' + e^{y'}}$

$\varphi(y') = \frac{3}{2} y'$ $\quad \varphi(p) = p \Rightarrow \frac{3}{2} p = p \Rightarrow \frac{1}{2} p = 0 \Rightarrow p = 0$

$y(x) = \frac{3}{2} x \cdot 0 + e^{0} = 1$ is a singular sol. of the eq.

derivate the eq.: $\quad y' = \frac{3}{2} y' + \frac{3}{2} x \cdot y'' + e^{y'} \cdot y''$

$0 = \frac{1}{2} y' + \frac{3}{2} x y'' + e^{y'} \cdot y''$

$$\boxed{0 = \frac{1}{2} y' + y'' \left(\frac{3}{2} x + e^{y'} \right)}$$

$y' = p \Rightarrow y'' = p' \Rightarrow 0 = \frac{1}{2} p + p' \left(\frac{3}{2} x + e^{p} \right)$ $\quad p' = \frac{dp}{dx}$

$\Downarrow$

$\frac{dx}{dp} = x'(p)$

$\Rightarrow p' \left(\frac{3}{2} x + e^{p} \right) = -\frac{1}{2} p$

$\frac{dp}{dx} \left(\frac{3}{2} x + e^{p} \right) = -\frac{1}{2} p \;\Big|\; \cdot \frac{dx}{dp}$

$\frac{3}{2} x + e^{p} = -\frac{1}{2} p \cdot x'(p) \Rightarrow \frac{1}{2} p x'(p) + \frac{3}{2} x(p) = -e^{p} \;\Big|\; \cdot \frac{2}{p}$

$\Rightarrow \boxed{x' + \frac{3}{p} x = -\frac{2e^{p}}{p}}$ nonhomogeneous linear diff. eq.

$x' + \frac{3}{p}x = 0$ homog. eq.

$x' = -\frac{3}{p}x \rightarrow \frac{dx}{dp} = -\frac{3}{p}x \rightarrow \int \frac{dx}{x} = \int -\frac{3}{p}\,dp$

$\Rightarrow \ln x = -3\ln p + \ln c \Rightarrow \boxed{x_0(p) = c \cdot p^{-3},\ c \in \mathbb{R}}$

$\boxed{x_{part}(p) = c(p) \cdot p^{-3}}$

$x'_{part} + \frac{3}{p}x_{part} = -\frac{2e^p}{p}$

$c'(p) \cdot p^{-3} + \cancel{c(p) \cdot (-3) \cdot p^{-4}} + \cancel{\frac{3}{p} \cdot c(p) \cdot p^{-3}} = -\frac{2e^p}{p}$

$c'(p)\, p^{-3} = -\frac{2e^p}{p} \quad | \cdot p^3 \Rightarrow c'(p) = -2e^p \cdot p^2$

$\Rightarrow c(p) = \int -2\underbrace{e^p}_{(e^p)'} \cdot p^2\,dp = -2e^p \cdot p^2 + 2\int e^p \cdot 2p\,dp =$

$= -2e^p p^2 + 4\int p \cdot e^p\,dp = -2e^p p^2 + 4p \cdot e^p - 4e^p$

$c(p) = e^p(-2p^2 + 4p - 4) \Rightarrow x_{part}(p) = e^p(-2p^2 + 4p - 4) \cdot p^{-3} =$

$= 2e^p\left(-\frac{1}{p} + \frac{2}{p^2} - \frac{2}{p^3}\right).$

$x(p) = x_0(p) + x_{part}(p)$

$$\boxed{x(p) = \frac{c}{p^3} + 2e^p\left(-\frac{1}{p} + \frac{2}{p^2} - \frac{2}{p^3}\right)}$$

$$y = \frac{3}{2}\underset{x(p)}{x}\underset{p}{y'} + e^{y'} \Rightarrow y(p) = \frac{3}{2}\left(\quad\quad\right)\cdot p + e^p$$

$$\Rightarrow \begin{cases} x(p) = \dfrac{c}{p^3} + 2e^p\left(-\dfrac{1}{p} + \dfrac{2}{p^2} - \dfrac{2}{p^3}\right) \\ y(p) = \dfrac{3}{2}\left(\dfrac{c}{p^3} + 2e^p\left(-\dfrac{1}{p} + \dfrac{2}{p^2} - \dfrac{2}{p^3}\right)\right)\cdot p + e^p \end{cases}$$

the general sol. in parametric form.

3) The exact differential equation

$$g(x,y) + h(x,y)\cdot y' = 0 \qquad y' = \frac{dy}{dx}$$

$$g(x,y) + h(x,y)\cdot \frac{dy}{dx} = 0 \quad \Big| \cdot dx$$

$$\boxed{g(x,y)\cdot dx + h(x,y)\cdot dy = 0}$$

$$u = u(x,y) \qquad du = \frac{\partial u}{\partial x}(x,y)\cdot dx + \frac{\partial u}{\partial y}(x,y)\cdot dy$$

Exact diff. eq: if $\exists\, u = u(x,y)$, $u \in C^1$, such that:

$$\begin{cases} g(x,y) = \dfrac{\partial u}{\partial x}(x,y) \\ h(x,y) = \dfrac{\partial u}{\partial y}(x,y) \end{cases}$$

The condition that a diff. eq. is an exact diff. eq.

iff: $\boxed{\dfrac{\partial g}{\partial y} = \dfrac{\partial h}{\partial x}}$

the function $\boxed{u(x,y) = \int_{x_0}^{x} g(s,y)\,ds + \int_{y_0}^{y} h(x_0,t)\,dt}$ ....

$g \cdot dx + h \cdot dy = 0$; $g \cdot dx + h\,dy = du$

$\Downarrow$

$du = 0$

$\Downarrow$

$\boxed{u(x,y) = c \ , c \in \mathbb{R}}$ the general sol. in implicit form.

Exercise 3 Solve the following diff. eqs.:

a) $8xy - 5y^2 + 2(2x^2 - 5y) \cdot y' = 0$.

b) $3x(x + 2x^2)\,dx + 2x(3x^2 + 2y^2)\,dy = 0$

c) $y(e^{xy} - 4x)\,dx + x(e^{xy} - 2x)\,dy = 0$

a) $\underbrace{(8xy-5y^2)}_{g(x,y)}dx+\underbrace{(4x^2-10xy)}_{h(x,y)}dy=0$

$$\frac{\partial g}{\partial y}(x,y)=8x-10y$$

$$\frac{\partial h}{\partial x}(x,y)=8x-10y$$

$\Rightarrow \frac{\partial g}{\partial y}=\frac{\partial h}{\partial x} \Rightarrow$ exact diff. eq.

the gen. sol. is $u(x,y)=c,\ c\in\mathbb{R}$.

where: $u(x,y)=\int_0^x g(s,y)\,ds+\int_0^y h(0,t)\,dt$

$x_0=0,\ y_0=0$

$$=\int_0^x (8sy-5y^2)\,ds+\int_0^y 0\,dt$$

$$=4y\cdot s^2\Big|_0^x-5y^2\cdot s\cdot\Big|_0^x=4x^2y-5xy^2$$

$\Rightarrow \boxed{4x^2y-5xy^2=c,\ c\in\mathbb{R}}$ the general solution in implicit form.

Proposition

If there exists $u \in C^1$, $u = u(x,y)$, such that:

$$\begin{cases} \dfrac{\partial u}{\partial x}(x,y) = g(x,y) \\ \dfrac{\partial u}{\partial y}(x,y) = h(x,y) \end{cases}$$

then $$u(x,y) = \int_{x_0}^{x} g(s,y)\,ds + \int_{y_0}^{y} h(x_0,t)\,dt$$

or $$u(x,y) = \int_{y_0}^{y} h(x,t)\,dt + \int_{x_0}^{x} g(s,y_0)\,ds.$$

Proof. $\dfrac{\partial u}{\partial x}(x,y) = g(x,y) \Rightarrow$

$$\Rightarrow u(x,y) = \int_{x_0}^{x} g(s,y)\,ds + C(y)$$

$$\left.\begin{aligned} &\Rightarrow \frac{\partial u}{\partial y}(x,y) = \int_{x_0}^{x} \frac{\partial g}{\partial y}(s,y)\,ds + C'(y) \\ &\frac{\partial u}{\partial y}(x,y) = h(x,y) \end{aligned}\right\} \Rightarrow$$

$$\int_{x_0}^{x} \frac{\partial g}{\partial y}(s,y)\,ds + c'(y) = h(x,y)$$

we make $x = x_0$

$$\Rightarrow \quad \underbrace{\int_{x_0}^{x_0} \frac{\partial g}{\partial y}(s,y)\,ds}_{=0} + c'(y) = h(x_0,y) \Rightarrow$$

$$\Rightarrow \quad c'(y) = h(x_0,y) \Rightarrow c(y) = \int_{y_0}^{y} h(x_0,t)\,dt + c$$

we take $c=0$ in order to get the simplest form of $u(x,y)$

$$\Rightarrow \boxed{u(x,y) = \int_{x_0}^{x} g(s,y)\,ds + \int_{y_0}^{y} h(x_0,t)\,dt}$$

analogue we can prove the second formula.

We start from $\frac{\partial u}{\partial y}(x,y) = h(x,y)$

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