

Chapter 2. First order logic (predicate logic)

In addition to what we did in Propositional Logic, we will find other expressions from the natural language

- we introduce quantifiers \exists \forall
 existential (there is) universal (for all)

Remark 1) "First order" means that we only quantify elements of sets: $\forall x$, $\exists x$, $x \in M$

"second order" means that we also quantify sets
 $\exists M$, $\forall M$, M set

2) "Naively", a predicate is a kind of "open sentence"
 e.g., $x + y = 1$, in that if we replace the variables with elements of a set, we get a proposition (i.e. true or false)

First order language

Def A first order language consists of the following data:

- a) symbols:
 - 1) parentheses: $(,)$
 - 2) connectives: $\neg, \vee, \wedge, \rightarrow, \leftrightarrow$
 - 3) quantifiers: \exists, \forall
 - 4) symbol of equality: $=$
 - 5) constants: $a, b, c, \dots, a_1, a_2, \dots$
 - 6) variables: $x, y, z, \dots, x_1, x_2, \dots$

We are also given a natural number $n \in \mathbb{N}$, called the arity (nullary, unary, binary, ..., n-ary, ...)

7) functions: $f, g, h, \dots, f_1, h, \dots$

8) predicates: $P, Q, R, \dots, P_1, Q_2, \dots$

→ shows the number of variables

b) terms (expressions) - these are defined recursively (by induction)

1) constants are terms
2) variables are terms } atomic terms

3) If f is a symbol of an n -ary function, and t_1, \dots, t_n are terms, then $f(t_1, \dots, t_n)$ is also a term.

c) formulas: defined recursively

atomic formulas

1) If P is a symbol of an n -ary predicate, and t_1, \dots, t_n are terms, then $P(t_1, \dots, t_n)$ is a formula.

2) If t_1, t_2 are terms, then $t_1 = t_2$ is a formula.

3) If A and B are formulas, then

$(\neg A), (A \vee B), (A \wedge B), (A \rightarrow B), (A \leftrightarrow B)$ are also formulas.

4) If $A(x)$ is a formula which depends on the variable x , then $(\exists x A(x)), (\forall x A(x))$ are also formulas.

Remarks 1) In the formula $\exists x A(x, y)$, x is called a bound variable and y is called a free variable.

2) If a formula like $\forall x \exists y A(x, y)$ has no free variables, then this formula is called a closed formula.

A closed formula may be regarded as a proposition. (it will have a truth value) e.g. $\forall y \exists x (x + y = 1)$ closed formula

The structure of a first order language. Interpretation

We will give values to expressions, and truth values to formulas.

Def A structure of a first order language consists of the following data:

- 1) a set M
- 2) to every constant symbol a we associate a fixed element $\bar{a} \in M$
- 3) to every n -ary function symbol f we associate a $\tilde{f} : M^n \longrightarrow M$
" $M \times \dots \times M = \{ (x_1, \dots, x_n) \mid x_i \in M, i=1, \dots, n \}$
" n times
- 4) to any n -ary predicate symbol P we associate a subset $\tilde{P} \subseteq M^n$

5) To the equality symbol $=$ we associate the equality relation on M , i.e. the subset

$$\Delta(M) = \{ (x, x) \mid x \in M \} \subseteq M^2$$

" $M \times M$

Def An interpretation of a first order language.

(w.r.t. a given structure) is a function

$$\mathcal{I} : \bigcup \longrightarrow \{ \text{true}, \text{false} \}$$

" the set of variables

Def the value of a term (w.r.t. a given structure and interpretation)

is defined recursively:

- 1) the value of a constant a is the elem $\bar{a} \in M$
- 2) The value of a variable x is the elem $\bar{x} = \mathcal{I}(x) \in M$
- 3) if f is an n -ary function symbol, and t_1, \dots, t_n are terms, then the value of the expression $f(t_1, \dots, t_n)$ is $\tilde{f}(\bar{t}_1, \dots, \bar{t}_n) \in M$

Def the truth value of a formula (w.r.t. a given structure and interpretation) is defined recursively:

- 1) the truth value of the formula $P(t_1, \dots, t_n)$ is 1 if and only if $(\tilde{t}_1, \dots, \tilde{t}_n) \in \tilde{P}$, and is 0 otherwise.
- 2) the truth value of the formula $t_1 = t_2$ is 1 if and only if $\tilde{t}_1 = \tilde{t}_2$ in M , and is 0 otherwise.
- 3) the truth values of the formulas $(\neg A)$, $(A \vee B)$, $(A \wedge B)$, $(A \rightarrow B)$, $(A \leftrightarrow B)$ are defined as in the propositional logic.
- 4) - The truth value of the formula $\exists x A$ is 1 if and only if there is an element $x \in M$ such that $A(x)$ is true.
 - The truth value of the formula $\forall x A$ is 1 if and only if for every $x \in M$, $A(x)$ is true.

Example consider the formula $x + y = 1$

here: x, y are variables
 1 constant

$+$ is a symbol of binary function; alternate notation

$$x + y = +(x, y)$$

$x, y, 1$ are terms.

structure: let $M = \mathbb{R}$, $\bar{1} = 1 \in \mathbb{R}$

$$\tilde{+} = + : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} ; \quad U = \{x, y\}$$

interpretation: $D : U \rightarrow M$, $D(x) = 2$, $D(y) = 3$

so in this case $\tilde{x} = 2$, $\tilde{y} = 3$; the value of the term $x + y$ is $5 \in \mathbb{R}$

so the truth value of our formula is 0 (false)

- consider the closed formulas:

- $\forall y \exists x (x+y=1)$ is true ($x=1-y$)
- $\exists x \forall y (x+y=1)$ is false

Def a) A formula is called a tautology if it takes the truth value 1 for any structure M and any interpretation \mathcal{D} .

b) a formula is called a contradiction if it takes the truth value 0 for any structure M and any interpretation \mathcal{D} .

c) a formula is satisfiable if it is not a contradiction

Examples $\left\{ \begin{array}{l} \neg(\forall x A(x)) \Leftrightarrow \exists x \neg A(x) \\ \neg(\exists x A(x)) \Leftrightarrow \forall x \neg A(x) \end{array} \right.$ (ie. the formula $\neg(\forall x A(x)) \Leftrightarrow \exists x \neg A(x)$ is a tautology)

De Morgan

$$\bullet \exists x \forall y A(x,y) \Rightarrow \forall y \exists x A(x,y)$$

Remark the previous example shows that the converse implication is not true in general.

$$\text{ie. } \forall y \exists x A(x,y) \longrightarrow \exists x \forall y A(x,y)$$

can be 1 can be 0

List of fundamental tautologies: 2.3.2. p.17.

Homework: \rightarrow ex. 19.