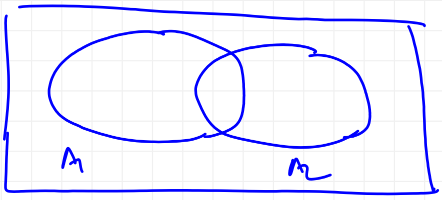
Mathematical Logic
Lecture 13
15.01.2024.

5. Inclusion-exclusion principle

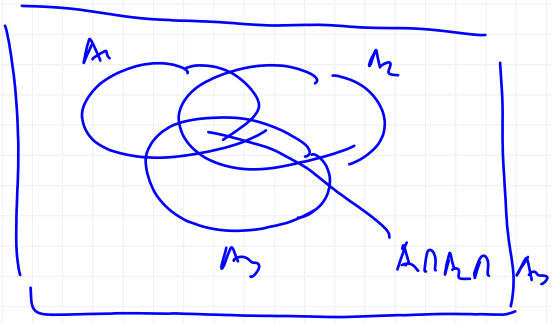
We want to calculate the number of elements of a union of finite sets. $A_1 \cup \dots \cup A_n$.

Particular cases $n = 2$.



$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

$n = 3$



$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

Theorem Let $n \in \mathbb{N}^*$, $A_1, \dots, A_n$ be finite sets. Then:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \underbrace{\sum_{i=1}^{n} |A_i|}_{C_n^1 = n \text{ terms}} - \underbrace{\sum_{1 \le i < j \le n} |A_i \cap A_j|}_{C_n^2 \text{ terms}} + \underbrace{\sum_{1 \le i_1 < i_2 < i_3 \le n} |A_{i_1} \cap A_{i_2} \cap A_{i_3}|}_{C_n^3 \text{ terms}} -$$

$$+ \dots + (-1)^{m+1} \underbrace{\sum_{1 \le i_1 < \dots < i_m \le n} |A_{i_1} \cap \dots \cap A_{i_m}|}_{C_n^m \text{ terms}} + \dots + (-1)^{n+1} \underbrace{\left|\bigcap_{i=1}^{n} A_i\right|}_{C_n^n = 1 \text{ term}}$$

Proof induction on n. (Homework !).

ex. 106 Let $|A| = k$, $|B| = n$. Find the number of surjective functions $f : A \to B$.

Rem If $f : A \to B$ is surjective, then f has a section (right inverse). $s : B \to A$ st $s \circ f = 1_B$

But f is a left inverse of f so s is injective

$\Rightarrow |B| \le |A|$

So if $k < n$ then $\nexists$ surj functions

We assume $k \ge n$, denote $s(k, n) \overset{\text{not}}{=} |Hom_{surj}(A, B)|$

We know $|Hom(A,B)| = n^k = |B|^{|A|}$

We will compute $|Hom_{non\,surj}(A,B)|$ by applying the incl.-excl. principle.

Let $B = \{b_1, \dots, b_n\}$

$f : A \to B$ is not surjective $\iff Im f \neq B$

$\iff \exists\, i \in \{1, \dots, n\}$ s.t. $b_i \notin Im f$

Denote $A_i := \{ f : A \to B \mid b_i \notin Im f \}$.

then f is not surj $\iff \exists\, i$ s.t. $f \in A_i$

$\iff f \in \bigcup_{i=1}^{n} A_i$

so $|Hom_{non\,surj}(A,B)| = |\bigcup_{i=1}^{n} A_i|$

- $|A_i| = |Hom(A, B \setminus \{b_i\})| = (n-1)^k$
- $|A_i \cap A_j| = |Hom(A, B \setminus \{b_i, b_j\})| = (n-2)^k$

...

- $|A_{i_1} \cap \dots \cap A_{i_m}| = |Hom(A, B \setminus \{b_{i_1}, \dots, b_{i_m}\})| = (n-m)^k$

...

- If $m = n-1$ then we get 1^k
- $\bigcap_{i=1}^{n} A_i = Hom(A, \emptyset) = \emptyset$

$$|Hom_{non\,surj}(A,B)| = |\bigcup_{i=1}^{n} A_i| = C_n^1 (n-1)^k - C_n^2 (n-2)^k + C_n^3 (n-3)^k - \dots + (-1)^{m+1} C_n^m (n-m)^k + \dots + (-1)^n C_n^{n-1} 1^k$$

hence

$$\boxed{\Delta(k,n) = n^k - |Hom_{non\,surj}(A,B)| = n^k - C_n^1 (n-1)^k + C_n^2 (n-2)^k - C_n^3 (n-3)^k + \dots + (-1)^m C_n^m (n-m)^k + \dots + (-1)^{n-1} C_n^{n-1} 1^k}$$

97 a) Prove that (1) $\aleph_0 + \aleph_0 = \aleph_0$ where $\aleph_0 = |\mathbb{N}|$

(2) $\aleph_0 \cdot \aleph_0 = \aleph_0$

<u>Solution</u> (1) We write: $\mathbb{N} = \underset{\text{even } n.}{2\mathbb{N}} \cup \underset{\text{odd } n.}{(2\mathbb{N}+1)}$

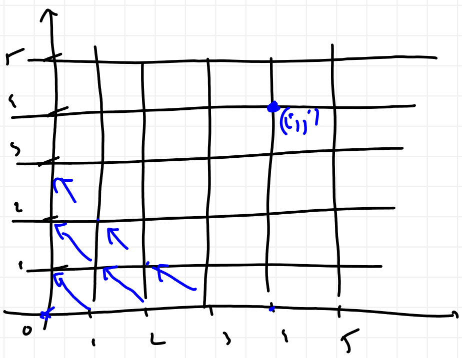
so $\aleph_0 = |\mathbb{N}| = |2\mathbb{N}| + |2\mathbb{N}+1| = \aleph_0 + \aleph_0$

the funct $f : \mathbb{N} \to 2\mathbb{N}$, $f(x) = 2x$ is bij

$g : \mathbb{N} \to 2\mathbb{N}+1$, $g(x) = 2x+1$ bij

(2) $\aleph_0 \cdot \aleph_0 = |\mathbb{N} \times \mathbb{N}|$, so we need a bijective function

<u>1st method</u> $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$



We define $f(0) = (0,0)$ $\quad f(6) = (0,3)$

$f(1) = (0,1)$ $\quad f(7) = (2,1)$

$f(2) = (1,0)$ $\quad f(8) = (1,2)$

$f(3) = (2,0)$ $\quad f(9) = (0,3)$

$f(4) = (1,1)$ $\quad f(10) = (4,0)$

$f(5) = (0,2)$ $\quad \vdots$

So $\forall (i,j) \in \mathbb{N}\times\mathbb{N}$ $\exists!$ $n \in \mathbb{N}$ st $f(n) = (i,j)$

<u>2nd method</u> We use the fact that any $\neq 0$ nat number can be written uniquely as a product between a power of 2 and an odd number.

i.e. $\forall n \in \mathbb{N}^*$ $\exists! (m,k) \in \mathbb{N}^* \times \mathbb{N}^*$ st $n = 2^{m-1}(2k-1)$

e.g. $18 = 2 \cdot 9$, $m = 2$, $k = 5$

$8 = 2^3 \cdot 1$, $m = 4$, $k = 1$

$56 = 8 \cdot 7$, $m = 4$, $k = 4$

So the funct $\left\{ \begin{array}{l} f : \mathbb{N}^* \times \mathbb{N}^* \to \mathbb{N}^* \\ f(m,k) = 2^{m-1} \cdot (2k-1) \end{array} \right.$ is bijective

then $|\mathbb{N}^* \times \mathbb{N}^*| = |\mathbb{N}^*|$ i.e. $\aleph_0 \cdot \aleph_0 = \aleph_0$.

96 Let A be an infinite set. Prove that:

a) $|A| + n = |A|$

b) $|A| + \aleph_0 = |A|$

Sol we have $n \leq \aleph_0$ for $n \in \mathbb{N}$.

hence $|A| \leq |A| + n \leq |A| + \aleph_0 \overset{?}{\underset{b)}{=}} |A|$

then we get a)

So it's enough to prove b)

We know that A is infinite $\Longleftrightarrow \exists f : \mathbb{N} \to A$ injective function

We write the disjoint union $A = \operatorname{Im} f \cup (A \setminus \operatorname{Im} f)$

we have $|\operatorname{Im} f| = |\mathbb{N}| = \aleph_0$

So $|A| = |\operatorname{Im} f| + |A \setminus \operatorname{Im} f|$

$= \aleph_0 + |A \setminus \operatorname{Im} f| = (\aleph_0 + \aleph_0) + |A \setminus \operatorname{Im} f|$

$\overset{\text{assoc}}{=} \aleph_0 + \underbrace{(\aleph_0 + |A \setminus \operatorname{Im} f|)}_{|A|} = \aleph_0 + |A|$

99 a) $c^2 = c^{\aleph_0} = c$

b) $c + c = c \cdot \aleph_0 = \underline{\aleph_0^{\aleph_0}} = c$

where $c := |\mathbb{R}| = 2^{\aleph_0}$ power of continuum

Sol a) $c^2 = (2^{\aleph_0})^2 = 2^{2 \cdot \aleph_0} = 2^{\aleph_0} = c$

$c^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0} = c$

b) $c \leq c + c \leq 2c \leq \aleph_0 \cdot c \leq c^2 = c$

hence $c + c = c \cdot \aleph_0 = c$

$c = 2^{\aleph_0} \leq \aleph_0^{\aleph_0} \leq c^{\aleph_0} \overset{a)}{=} c$ hence $c = \aleph_0^{\aleph_0}$

97 d) Prove that $\mathbb{Q}$ is countable (i.e. $|\mathbb{Q}| = |\mathbb{N}| = \aleph_0$)

Sol $\mathbb{N} \subseteq \mathbb{Q}$ hence $\aleph_0 \leq |\mathbb{Q}|$

we have to show that $|\mathbb{Q}| \leq \aleph_0$

We write $\mathbb{Q} = \mathbb{Q}^*_- \cup \{0\} \cup \mathbb{Q}^*_+$

Clearly, $|\mathbb{Q}^*_-| = |\mathbb{Q}^*_+|$, so it is enough to show that $|\mathbb{Q}^*_+| \leq \aleph_0$

$$\mathbb{Q}^*_+ = \left\{ \frac{a}{b} \mid a, b \in \mathbb{N}^* \right\} \qquad \frac{3}{2} = \frac{6}{4}$$

Let $f : \mathbb{N}^* \times \mathbb{N}^* \longrightarrow \mathbb{Q}^*_+$, $f(a,b) = \frac{a}{b}$

f is surjective (not inj)

hence $|\mathbb{Q}^*_+| \leq |\mathbb{N}^* \times \mathbb{N}^*| = \aleph_0 \cdot \aleph_0 = \aleph_0$

so we are done

98 a) Prove that any interval has the power of the continuum, i.e.

$$\mathbb{R} \sim (a,b) \sim [a,b) \sim [a,b] \sim (a,b]$$

equipotent

$$A \sim B \iff |A| = |B|$$

b) the set $\mathbb{R} \setminus \mathbb{Q}$ of irrational numbers is uncountable

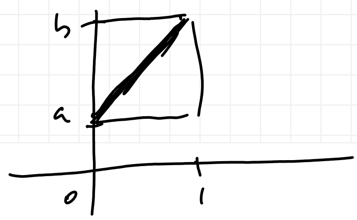
ie $\aleph_0 < |\mathbb{R} \setminus \mathbb{Q}|$

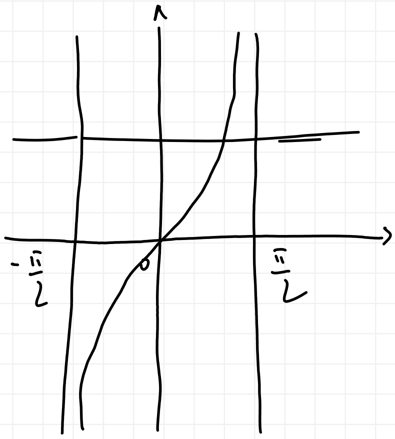
Sol a) We have seen $|(a,b)| = |[a,b)|$

$= |[a,b]| \ldots$

First, we prove that $[0,1] \sim [a,b]$

Let $\begin{cases} f : [0,1] \longrightarrow [a,b] \\ f(x) = (b-a)x + a \end{cases}$ so f is bijective





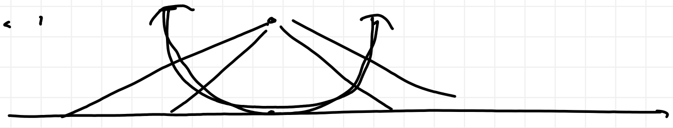
Let $\begin{cases} f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longrightarrow \mathbb{R} \\ f(x) = tg\, x \end{cases}$

so f is bijective

so $|\mathbb{R}| = \left|\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right|$

$= |(a,b)|$

other idea:



b) We assume by contradiction that $\mathbb{R} \setminus \mathbb{Q}$ is countable, i.e. $|\mathbb{R} \setminus \mathbb{Q}| = \aleph_0$

We write $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$
disjoint union

e.g. $\sqrt{2} \notin \mathbb{Q}$
$\pi, e \notin \mathbb{Q}$

$$|\mathbb{R}| = \underset{\aleph_0}{\underset{\parallel}{|\mathbb{Q}|}} + \underset{\aleph_0}{\underset{\parallel}{|\mathbb{R} \setminus \mathbb{Q}|}} \text{ (by assumption)}$$

We know that $\aleph_0 + \aleph_0 = \aleph_0$
and $|\mathbb{R}| = c > \aleph_0$

contradiction