DATA STRUCTURES

Binary Trees Traversals. Binary Search Trees

Lect. PhD. Diana-Lucia Miholca

2023 - 2024

Babeş - Bolyai University
Faculty of Mathematics and Computer Science

In the previous lecture:

- Trees
  - Terminology
- Binary trees (I)
  - Terminology
  - Properties
  - Possible representations
  - Traversals (I)
    - Level order
    - Preorder

In the today's lecture:

- Binary Trees
  - Traversals
    - Inorder
    - Postorder
- Binary Search Trees

Tree traversals

Traversing a tree means visiting all of its nodes.

For a binary tree there are 4 possible traversals:

- Level order (breadth first) - the same as in case of a (non-binary) tree ✓ (discussed in the previous lecture)
- Preorder ✓ (discussed in the previous lecture)
- Inorder
- Postorder

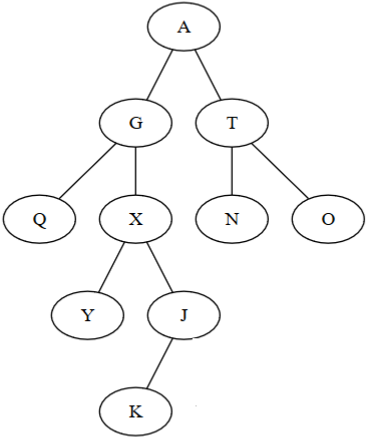
Inorder traversal

In case of *inorder* traversal:

- Traverse the left subtree - if exists
- Visit the *root* of the tree
- Traverse the right subtree - if exists

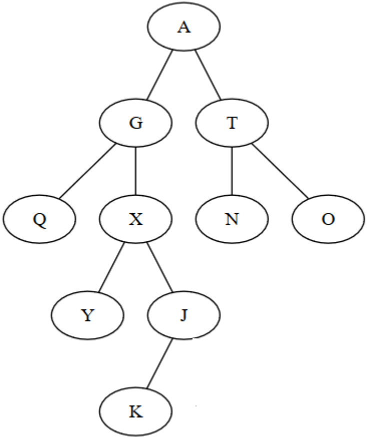
When traversing the subtrees (left or right) the same inorder traversal is applied.

Inorder traversal example



- Inorder traversal:

Inorder traversal example



- Inorder traversal: Q, G, Y, X, K, J, A, N, T, O

Binary tree - linked representation with dynamic allocation

Linked representation with dynamic allocation:

- There is one node for every element of the tree
- The structure representing a node contains:
  - the information
  - a pointer to the left child
  - a pointer to the right child
  - optionally, a pointer to the parent
- NIL denotes the absence of a node
  - $\Rightarrow$ the root of an empty tree is NIL

Binary tree representation

- In the following, we are going to use the dynamically allocated linked representation for a binary tree:

Representation of a node in a Binary Tree:

<u>BTNode:</u>

info: TElem

left: ↑ BTNode

right: ↑ BTNode

Representation of a Binary Tree:

<u>BinaryTree:</u>

root: ↑ BTNode

Inorder traversal - recursive implementation

- The simplest implementation for inorder traversal is with a recursive algorithm.

Recursive inorder traversal:

subalgorithm inorder_recursive(node) **is:**

//pre: node is a ↑ BTNode

Inorder traversal - recursive implementation

- The simplest implementation for inorder traversal is with a recursive algorithm.

Recursive inorder traversal:

subalgorithm inorder_recursive(node) **is:**
//pre: node is a $\uparrow$ *BTNode*
    if node $\neq$ NIL **then**

Inorder traversal - recursive implementation

- The simplest implementation for inorder traversal is with a recursive algorithm.

Recursive inorder traversal:

subalgorithm inorder_recursive(node) **is:**
//pre: node is a $\uparrow$ BTNode
    if node $\neq$ NIL **then**
        inorder_recursive([node].left)

Inorder traversal - recursive implementation

- The simplest implementation for inorder traversal is with a recursive algorithm.

Recursive inorder traversal:

subalgorithm inorder_recursive(node) **is:**
//pre: node is a $\uparrow$ BTNode
    if node $\neq$ NIL **then**
        inorder_recursive([node].left)
        @visit node

Inorder traversal - recursive implementation

- The simplest implementation for inorder traversal is with a recursive algorithm.

Recursive inorder traversal:

subalgorithm inorder_recursive(node) **is:**
//pre: node is a $\uparrow$ *BTNode*
    if node $\neq$ NIL **then**
        inorder_recursive([node].left)
        @visit node
        inorder_recursive([node].right)
    end-if
end-subalgorithm

We need a wrapper subalgorithm to perform the first call to *inorder_recursive* with the root of the tree as parameter.

Inorder traversal - recursive implementation

- The simplest implementation for inorder traversal is with a recursive algorithm.

Recursive inorder traversal:

subalgorithm inorder_recursive(node) **is:**
//pre: node is a $\uparrow$ *BTNode*
    if node $\neq$ NIL **then**
        inorder_recursive([node].left)
        @visit node
        inorder_recursive([node].right)
    end-if
end-subalgorithm

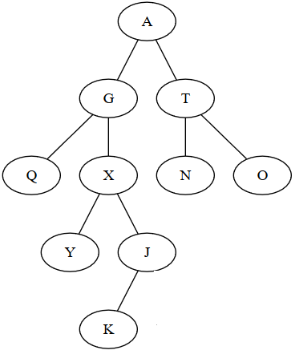
We need a wrapper subalgorithm to perform the first call to *inorder_recursive* with the root of the tree as parameter.

$\Theta(n)$ for a tree with n nodes.

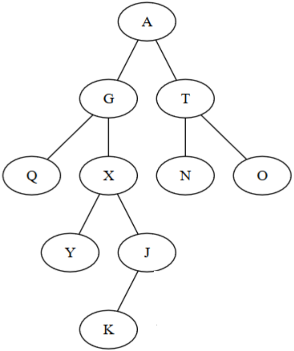
Inorder traversal - non-recursive implementation

- We can implement the inorder traversal algorithm without recursion, using an auxiliary stack to store the nodes.
  - We start with an empty stack and an auxiliary current (pointer to) node (*currentNode*) is set to the root
  - While *currentNode* is not NIL, push it to the stack and set it to its left child
  - While stack not empty:
    - Pop a node and visit it
    - Set *currentNode* to the right child of the popped node
    - While *currentNode* is not NIL, push it to the stack and set it to its left child

Inorder traversal - non-recursive implementation example



Inorder traversal - non-recursive implementation example



- currentNode: A (Stack:)
- currentNode: NIL (Stack: A G Q)
- Visit Q, currentNode NIL (Stack: A G)
- Visit G, currentNode X (Stack: A)
- currentNode: NIL (Stack: A X Y)
- Visit Y, currentNode NIL (Stack: A X)
- Visit X, currentNode J (Stack: A)
- currentNode: NIL (Stack: A J K)
- Visit K, currentNode NIL (Stack: A J)
- Visit J, currentNode NIL (Stack: A)
- Visit A, currentNode T (Stack:)
- currentNode: NIL (Stack: T N)
- ...

Inorder traversal - non-recursive implementation

Iterative inorder traversal:

subalgorithm inorder(tree) **is:**

//pre: tree is a BinaryTree

Inorder traversal - non-recursive implementation

Iterative inorder traversal:

subalgorithm inorder(tree) **is:**
//pre: tree is a BinaryTree
    init(s) *//s:Stack is an auxiliary stack*

Inorder traversal - non-recursive implementation

Iterative inorder traversal:

subalgorithm inorder(tree) **is:**
//pre: tree is a BinaryTree
    init(s) *//s:Stack is an auxiliary stack*
    currentNode $\leftarrow$ tree.root

Inorder traversal - non-recursive implementation

Iterative inorder traversal:

subalgorithm inorder(tree) **is:**
//pre: tree is a BinaryTree
    init(s) *//s:Stack is an auxiliary stack*
    currentNode $\leftarrow$ tree.root
    while currentNode $\neq$ NIL **execute**
        push(s, currentNode)
        currentNode $\leftarrow$ [currentNode].left
    end-while

Inorder traversal - non-recursive implementation

Iterative inorder traversal:

subalgorithm inorder(tree) **is:**
//pre: tree is a BinaryTree
    init(s) *//s:Stack is an auxiliary stack*
    currentNode $\leftarrow$ tree.root
    while currentNode $\neq$ NIL **execute**
        push(s, currentNode)
        currentNode $\leftarrow$ [currentNode].left
    end-while
    while not isEmpty(s) **execute**

Inorder traversal - non-recursive implementation

Iterative inorder traversal:

subalgorithm inorder(tree) **is:**
//pre: tree is a BinaryTree
    init(s) *//s:Stack is an auxiliary stack*
    currentNode $\leftarrow$ tree.root
    while currentNode $\neq$ NIL **execute**
        push(s, currentNode)
        currentNode $\leftarrow$ [currentNode].left
    end-while
    while not isEmpty(s) **execute**
        currentNode $\leftarrow$ pop(s)
        @visit currentNode

Inorder traversal - non-recursive implementation

Iterative inorder traversal:

subalgorithm inorder(tree) **is:**
//pre: tree is a BinaryTree
    init(s) *//s:Stack is an auxiliary stack*
    currentNode $\leftarrow$ tree.root
    while currentNode $\neq$ NIL **execute**
        push(s, currentNode)
        currentNode $\leftarrow$ [currentNode].left
    end-while
    while not isEmpty(s) **execute**
        currentNode $\leftarrow$ pop(s)
        @visit currentNode
        currentNode $\leftarrow$ [currentNode].right

Inorder traversal - non-recursive implementation

Iterative inorder traversal:

```
subalgorithm inorder(tree) is:
//pre: tree is a BinaryTree
    init(s) //s:Stack is an auxiliary stack
    currentNode ← tree.root
    while currentNode ≠ NIL execute
        push(s, currentNode)
        currentNode ← [currentNode].left
    end-while
    while not isEmpty(s) execute
        currentNode ← pop(s)
        @visit currentNode
        currentNode ← [currentNode].right
        while currentNode ≠ NIL execute
            push(s, currentNode)
            currentNode ← [currentNode].left
        end-while
    end-while
end-subalgorithm
```

Inorder traversal - non-recursive implementation - complexity

Time complexity: $\Theta(n)$

Extra space complexity: $O(n)$

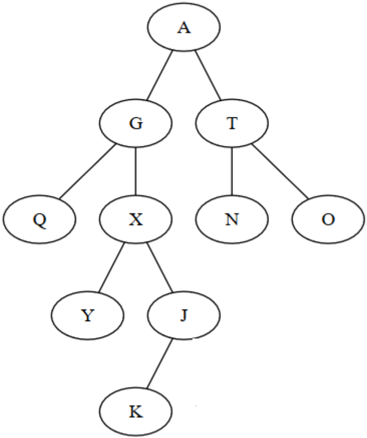
Postorder traversal

In case of *postorder* traversal:

- Traverse the left subtree - if exists
- Traverse the right subtree - if exists
- Visit the *root* of the tree

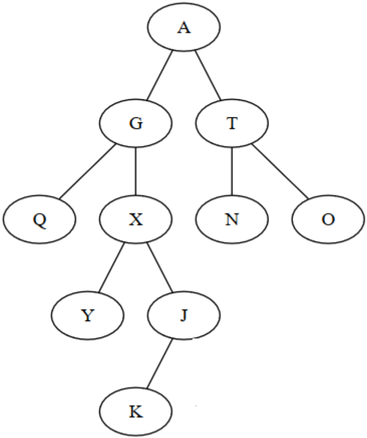
When traversing the subtrees (left or right) the same postorder traversal is applied.

Postorder traversal example



- Postorder traversal:

Postorder traversal example



- Postorder traversal: Q, Y, K, J, X, G, N, O, T, A

Postorder traversal - recursive implementation

- The simplest implementation for postorder traversal is with a recursive algorithm.

Recursive postorder traversal:

subalgorithm postorder_recursive(node) **is:**
//pre: node is a $\uparrow$ *BTNode*
    if node $\neq$ NIL **then**
        postorder_recursive([node].left)
        postorder_recursive([node].right)
        @visit node
    end-if
end-subalgorithm

We need again a wrapper subalgorithm to perform the first call to *postorder_recursive* with the root of the tree as parameter.

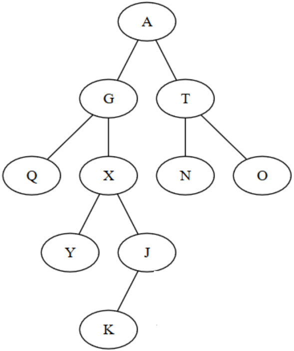
The traversal takes $\Theta(n)$ time for a tree with n nodes.

Postorder traversal - non-recursive traversal

- We start with an empty stack and a current node (*currentNode*) set to the root of the tree
- While *currentNode* is not NIL, push to the stack its right child, the *currentNode* and then set *currentNode* to its left child.
- While the stack is not empty:
  - Pop a node from the stack (*currentNode*)
  - If it has a right child, the stack is not empty and contains the right child on top of it, then pop the right child, push *currentNode* and set *currentNode* to its right child.
  - Otherwise, visit *currentNode* and set it to NIL
  - While *currentNode* is not NIL, push to the stack its right child, the *currentNode* and then set *currentNode* to its left child.

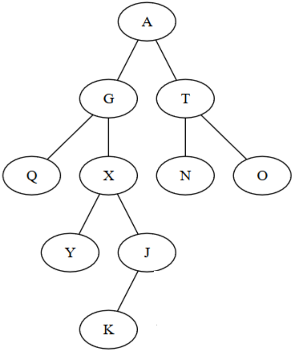
Postorder traversal - non-recursive implementation example

- currentNode: A (Stack:)



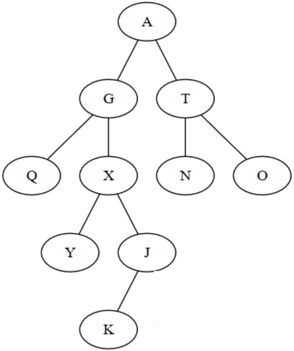
Postorder traversal - non-recursive implementation example

- currentNode: A (Stack:)
- currentNode: NIL (Stack: T A X G Q)

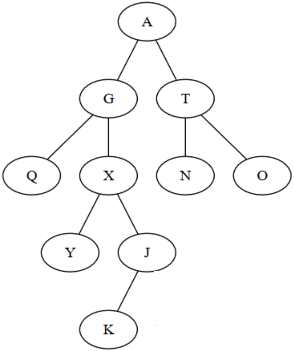


Postorder traversal - non-recursive implementation example

- currentNode: A (Stack:)
- currentNode: NIL (Stack: T A X G Q)
- Visit Q, currentNode NIL (Stack: T A X G)

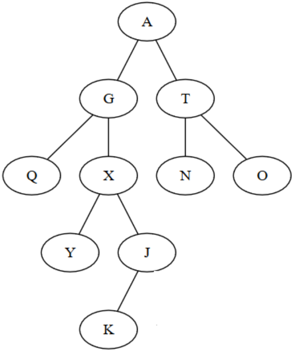


Postorder traversal - non-recursive implementation example



- currentNode: A (Stack:)
- currentNode: NIL (Stack: T A X G Q)
- Visit Q, currentNode NIL (Stack: T A X G)
- currentNode: X (Stack: T A G)

Postorder traversal - non-recursive implementation example



- currentNode: A (Stack:)
- currentNode: NIL (Stack: T A X G Q)
- Visit Q, currentNode NIL (Stack: T A X G)
- currentNode: X (Stack: T A G)
- currentNode: NIL (Stack: T A G J X Y)
- Visit Y, currentNode: NIL (Stack: T A G J X)
- currentNode: J (Stack: T A G X)
- currentNode: NIL (Stack: T A G X J K)
- Visit K, currentNode: NIL (Stack: T A G X J)
- Visit J, currentNode: NIL (Stack: T A G X)
- Visit X, currentNode: NIL (Stack: T A G)
- Visit G, currentNode: NIL (Stack: T A)
- currentNode: T (Stack: A)
- currentNode: NIL (Stack: A O T N)

...

Postorder traversal - non-recursive implementation

Iterative postorder traversal:

```
subalgorithm postorder(tree) is:
//pre: tree is a BinaryTree
    init(s) //s: Stack is an auxiliary stack
    currentNode ← tree.root
    while currentNode ≠ NIL execute
        if [currentNode].right ≠ NIL then
            push(s, [currentNode].right)
        end-if
        push(s, currentNode)
        currentNode ← [currentNode].left
    end-while
    while not isEmpty(s) execute
        currentNode ← pop(s)
        if [currentNode].right ≠ NIL and (not isEmpty(s)) and [currentNode].right =
top(s) then
            pop(s)
            push(s, currentNode)
            currentNode ← [currentNode].right
//continued on the next slide
```

Postorder traversal - non-recursive implementation

Iterative postorder traversal:

```
    else
        @visit currentNode
        currentNode ← NIL
    end-if
    while currentNode ≠ NIL execute
        if [currentNode].right ≠ NIL then
            push(s, [currentNode].right)
        end-if
        push(s, currentNode)
        currentNode ← [currentNode].left
    end-while
end-while end-subalgorithm
```

Time complexity: $\Theta(n)$

Extra space complexity: $O(n)$

Binary tree iterator

The interface of the binary tree contains the *iterator* operation, which should return an iterator.

- This operation receives a parameter that specifies what type of traversal we want to perform with the iterator (*preorder*, *inorder*, *postorder*, *level order*).

The traversal algorithms discussed so far traverse all the elements of the binary tree at once, but an iterator has to do an element-by-element traversal.

For defining an iterator, we have to divide the code into the functions of an iterator: *init*, *getCurrent*, *next*, *valid*.

Inorder binary tree iterator

Representation of an inorder iterator:

InorderIterator:

bt: BinaryTree

s: Stack

currentNode: ↑BTNode

Inorder binary tree iterator - init

The constructor of an inorder iterator over a binary tree:

subalgorithm init (it, bt) **is:**
//pre: it - is an InorderIterator, bt is a BinaryTree
  it.bt $\leftarrow$ bt
  init(it.s)
  node $\leftarrow$ bt.root
  while node $\neq$ NIL **execute**
    push(it.s, node)
    node $\leftarrow$ [node].left
  end-while
  if not isEmpty(it.s) **then**
    it.currentNode $\leftarrow$ top(it.s)
  else
    it.currentNode $\leftarrow$ NIL
  end-if
end-subalgorithm

Inorder binary tree iterator - getCurrent

The function for getting the current element of an inorder iterator over a binary tree:

function getCurrent(it) **is:**
  if not valid(it) **then**
    @throw an exception
  end-if
  getCurrent ← [it.currentNode].info
end-function

Inorder binary tree iterator - valid

The function for checking the validity of an inorder iterator over a binary tree:

```
function valid(it) is:
    if it.currentNode = NIL then
        valid ← false
    else
        valid ← true
    end-if
end-function
```

Inorder binary tree iterator - next

The operation for advancing to the next element of an inorder iterator over a binary tree:

subalgorithm next(it) **is:**
    node ← pop(it.s)
    if [node].right ≠ NIL **then**
        node ← [node].right
        while node ≠ NIL **execute**
            push(it.s, node)
            node ← [node].left
        end-while
    end-if
    if not isEmpty(it.s) **then**
        it.currentNode ← top(it.s)
    else
        it.currentNode ← NIL
    end-if
end-subalgorithm

Preorder, Inorder, Postorder

How to remember the difference between traversals?

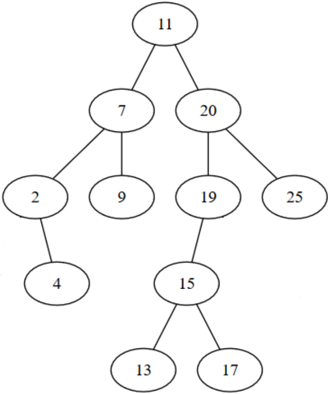
- Left subtree is always traversed before the right subtree.
- The visiting of the root node is what changes:
  - **PRE**order - visit the root **before** the left and right
  - **IN**order - visit the root **between** the left and right
  - **POST**order - visit the root **after** the left and right

Binary search trees

D A **binary search tree** is a binary tree that satisfies the following property:

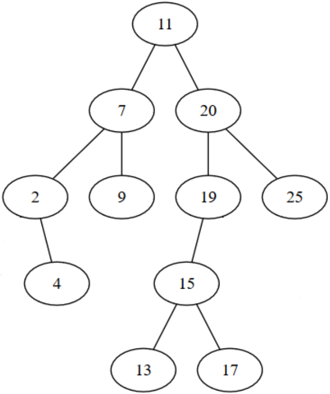
- Let x be a node in a binary search tree.
  - For every node y from the left subtree of x, the information from y is less than or equal to the information from x
  - For every node y from the right subtree of x, the information from y is greater than or equal to the information from x

Binary Search Tree Example



Inorder:

Binary Search Tree Example



- Inorder: 2 4 7 9 11 13 15 17 19 20 25
- An inorder traversal of a binary search tree will visit the elements in increasing order.

Binary Search Tree - terminology

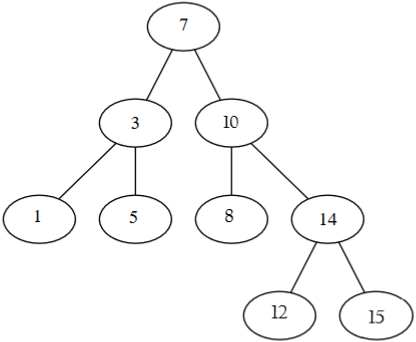
The terminology discussed for binary trees is valid for binary search trees as well:

We can have a binary search trees that are:

- full
- complete
- almost complete
- degenerated
- balanced

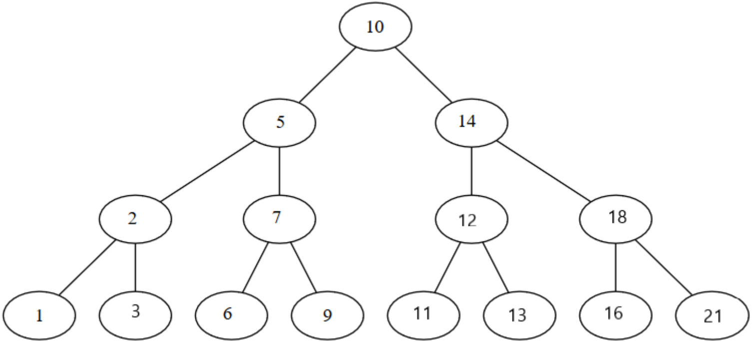
Binary tree - Terminology

A binary search tree is called **full** if every internal node has exactly two children.



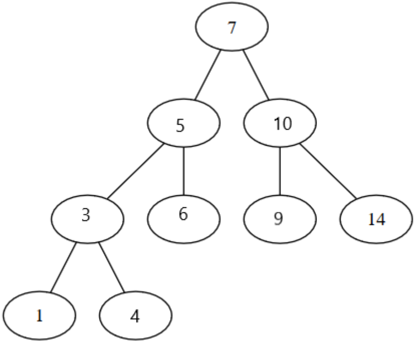
Binary tree - Terminology

A binary search tree is called **complete** if every level of the tree is completely filled.



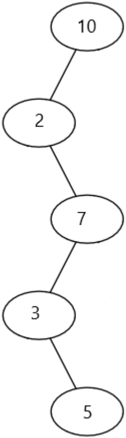
Binary tree - Terminology

A binary search tree is called **almost complete** if every level of the tree is completely filled, except possibly the bottom level, which is filled from left to right.



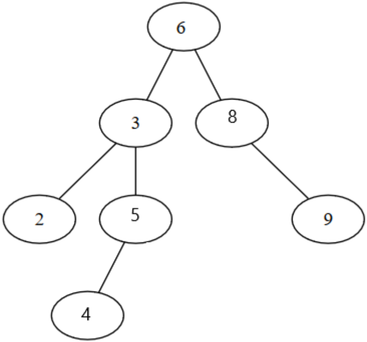
Binary tree - Terminology

A binary search tree is called **degenerated** if every internal node has exactly one child (it is actually a chain of nodes).



Binary tree - Terminology

A binary search tree is called **balanced** if the difference between the height of the left and right subtrees is at most 1 for every node from the tree.



Binary Search Tree

Binary search trees inherit the numerical properties of binary trees:

- The number of nodes in a complete binary search tree of height N is

Binary Search Tree

Binary search trees inherit the numerical properties of binary trees:

- The number of nodes in a complete binary search tree of height N is $2^{N+1} - 1$ (it is $1 + 2 + 4 + 8 + ... + 2^N$)
- The maximum number of nodes in a binary search tree of height N is

Binary Search Tree

Binary search trees inherit the numerical properties of binary trees:

- The number of nodes in a complete binary search tree of height N is $2^{N+1} - 1$ (it is $1 + 2 + 4 + 8 + ... + 2^N$)
- The maximum number of nodes in a binary search tree of height N is $2^{N+1} - 1$ - if the tree is complete.
- The minimum number of nodes in a binary search tree of height N is

Binary Search Tree

Binary search trees inherit the numerical properties of binary trees:

- The number of nodes in a complete binary search tree of height N is $2^{N+1} - 1$ (it is $1 + 2 + 4 + 8 + ... + 2^N$)
- The maximum number of nodes in a binary search tree of height N is $2^{N+1} - 1$ - if the tree is complete.
- The minimum number of nodes in a binary search tree of height N is $N + 1$ - if the tree is degenerated.
- A binary search tree with N nodes has a height between $[log_2 N]$ and $N - 1$.

Binary Search Tree

Binary search trees can be used as representation for **sorted** containers

- Sorted Set
- Sorted Map
- etc.

Basic operations:

- searching for an element
- inserting an element
- removing an element

Binary Search Tree - other operations

Other operations:

- get the minimum element
- get the maximum element
- find the successor of an element
- find the predecessor of an element

Binary Search Tree - Representation

- A linked representation with dynamic allocation for binary search trees:

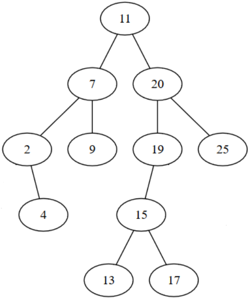
Representation of a node in a Binary Search Tree:

BSTNode:
  info: TComp
  left: ↑ BSTNode
  right: ↑ BSTNode

Representation of a Binary Search Tree:

Binary Search Tree:
  root: ↑ BSTNode

Binary Search Tree - search operation



How can we search for 15? What about the 14?

Binary Search Tree - search operation

How can we **search** for an element in a binary search tree?

The idea of searching for an element *elem*:

- We start at the root of the tree.
- For each node *node* encountered:
  - If $node = NIL \Rightarrow$ unsuccessful search
  - If $[node].info = elem \Rightarrow$ successful search
  - If $[node].info > elem \Rightarrow$ search recursively the left subtree
  - If $[node].info < elem \Rightarrow$ search recursively the right subtree

BST - search operation - recursive implementation

Recursively searching in a Binary Search Tree:

function search_rec (node, elem) **is:**

//pre: node is a BSTNode and elem is the TElem we are searching for

BST - search operation - recursive implementation

Recursively searching in a Binary Search Tree:

function search_rec (node, elem) **is:**
//pre: node is a BSTNode and elem is the TElem we are searching for
  if node = NIL **then**
    search_rec ← false
  else

BST - search operation - recursive implementation

Recursively searching in a Binary Search Tree:

function search_rec (node, elem) **is:**
//pre: node is a BSTNode and elem is the TElem we are searching for
    if node = NIL **then**
        search_rec ← false
    else
        if [node].info = elem **then**
            search_rec ← true

BST - search operation - recursive implementation

Recursively searching in a Binary Search Tree:

function search_rec (node, elem) **is:**
//pre: node is a BSTNode and elem is the TElem we are searching for
  if node = NIL **then**
    search_rec ← false
  else
    if [node].info = elem **then**
      search_rec ← true
    else if [node].info < elem **then**
      search_rec ← search_rec([node].right, elem)
    else
      search_rec ← search_rec([node].left, elem)
  end-if
end-function

BST - search operation - recursive implementation - complexity

The time complexity of the *search* operation is

BST - search operation - recursive implementation - complexity

The time complexity of the *search* operation is $O(h)$, where h is the height of the tree.

The maximum height of a tree with n nodes is

BST - search operation - recursive implementation - complexity

The time complexity of the *search* operation is $O(h)$, where h is the height of the tree.

The maximum height of a tree with n nodes is $n - 1$ (if the tree is degenerated).

Therefore, the time complexity of the *search* operation can also be expressed as $O(n)$.

.

BST - search operation - recursive implementation

We need a wrapper to call the recursive function with the root of the tree:

Recursive searching function - initial call

function search (tree, e) **is:**
//pre: tree is a Binary Search Tree, e is the elem we are looking for
  search ← search_rec(tree.root, e)
end-function

BST - search operation - non-recursive implementation

- The iterative implementation of the *search* operation:

Iteratively searching in a Binary Search Tree:

function search (tree, elem) **is:**
//pre: tree is a Binary Search Tree and elem is the TElem we are searching for

BST - search operation - non-recursive implementation

- The iterative implementation of the *search* operation:

Iteratively searching in a Binary Search Tree:

function search (tree, elem) **is:**
//pre: tree is a Binary Search Tree and elem is the TElem we are searching for
    currentNode $\leftarrow$ tree.root
    found $\leftarrow$ false

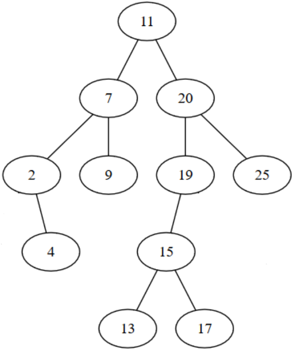
BST - search operation - non-recursive implementation

- The iterative implementation of the *search* operation:

Iteratively searching in a Binary Search Tree:

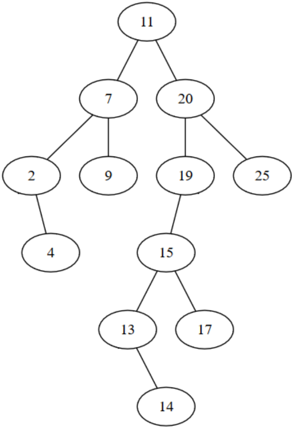
function search (tree, elem) **is:**
//pre: tree is a Binary Search Tree and elem is the TElem we are searching for
    currentNode $\leftarrow$ tree.root
    found $\leftarrow$ false
    while currentNode $\neq$ NIL and not found **execute**
        if [currentNode].info = elem **then**
            found $\leftarrow$ true
        else if [currentNode].info $<$ elem **then**
            currentNode $\leftarrow$ [currentNode].right
        else
            currentNode $\leftarrow$ [currentNode].left
        end-if
    end-while
    search $\leftarrow$ found
end-function

BST - insert operation



Where can we insert element 14?

BST - insert operation



BST - insert operation - recursive implementation

- An function that creates a new node with a given element:

Creating a node with a given element:

function initNode(e) **is:**
//pre: e is a TComp
//post: initNode: $\uparrow$ BSTNode $\leftarrow$ a node with e as information
    allocate(newNode)
    [newNode].info $\leftarrow$ e
    [newNode].left $\leftarrow$ NIL
    [newNode].right $\leftarrow$ NIL
    initNode $\leftarrow$ newNode
end-function

BST - insert operation - recursive implementation

Recursively inserting in a Binary Search Tree::

function insert_rec(node, e) **is:**
//pre: node is a BSTNode, e is TComp
//post: a node containing e was added in the tree starting from node
    if node = NIL **then**
        node ← initNode(e)
    else if [node].info ≥ e **then**
        [node].left ← insert_rec([node].left, e)
    else
        [node].right ← insert_rec([node].right, e)
    end-if
    insert_rec ← node
end-function

BST - insert operation - recursive implementation

The time complexity of the *insert* operation is

BST - insert operation - recursive implementation

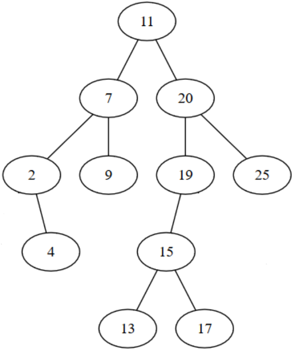
The time complexity of the *insert* operation is $O(h)$ (or $O(n)$)

We need a wrapper function to call *insert_rec* with the root of the tree:

Recursive insertion function - initial call:

function insert (tree, e) **is:**
//pre: tree is a Binary Search Tree, e is the elem to be inserted
  tree.root $\leftarrow$ insert_rec(tree.root, e)
end-function

BST - Finding the minimum element



 How can we find the minimum element of the binary search tree?

BST - Finding the minimum element

Finding the minimum in a Binary Search Tree:

function minimum(tree) **is:**

//pre: tree is a Binary Search Tree

//post: minimum = the minimum value from the tree

BST - Finding the minimum element

Finding the minimum in a Binary Search Tree:

function minimum(tree) **is:**
//pre: tree is a Binary Search Tree
//post: minimum = the minimum value from the tree
  currentNode $\leftarrow$ tree.root
  if currentNode = NIL **then**
    @empty tree, no minimum
  else
    while [currentNode].left $\neq$ NIL **execute**
      currentNode $\leftarrow$ [currentNode].left
    end-while
    minimum $\leftarrow$ [currentNode].info
  end-if
end-function

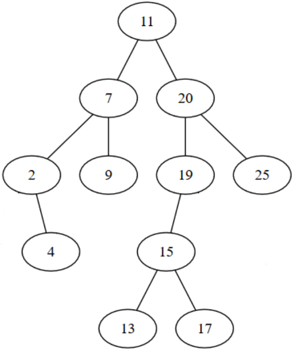
BST - Finding the minimum element

- The time complexity of the *minimum* operation is

BST - Finding the minimum element

- The time complexity of the *minimum* operation is $O(h)$ (or $O(n)$)
- We can adapt the *minimum* function for finding the minimum of a subtree
  - The parameter would be a pointer to the root of the subtree
- Maximum can be found symmetrically
  - It is in the rightmost node of the BST

Finding the parent of a node



How can we find the parent of the node?

Finding the parent of a node

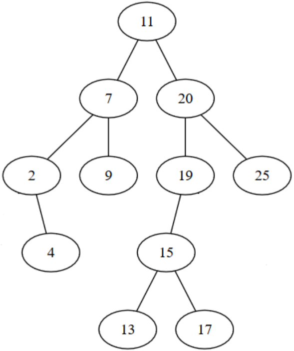
Finding the parent of a node in a Binary Search Tree:

function parent(tree, node) **is:**
//pre: tree is a Binary Search Tree, node is a pointer to a BSTNode, node $\neq$ NIL
//post: returns the parent of node, or NIL if node is the root

```
    c ← tree.root
    if c = node then //node is the root
        parent ← NIL
    else
        while c ≠ NIL and [c].left ≠ node and [c].right ≠ node execute
            if [c].info ≥ [node].info then
                c ← [c].left
            else
                c ← [c].right
            end-if
        end-while
        parent ← c
    end-if
end-function
```

Complexity: $O(h)$ (or $O(n)$)

BST - Finding the successor



 How can we find the successor of a node?

 How can we find the successor of 11? What about the successor of 17?

BST - Finding the successor of a node

How can we find the **successor** of a node x in a binary search tree?

The idea is the following:

- If x has a non-empty right subtree:
  - Its successor is just the leftmost node in x's right subtree
- If x does not have a right subtree:
  - The successor of x is the lowest ancestor of the node whose left child is also an ancestor of x
  - We go up the tree from x until we encounter a node that is the left child of its parent. The parent of that node is the succesor.

BST - Finding the successor of a node

Finding the successor of a node in a Binary Search Tree:

function successor(tree, x) **is:**
//pre: tree is a Binary Search Tree, x is a pointer to a BSTNode, x $\neq$ NIL
//post: returns the node with the next value after the value from x
//or NIL if x is the maximum
  if [x].right $\neq$ NIL **then**
    c $\leftarrow$ [x].right
    while [c].left $\neq$ NIL **execute**
      c $\leftarrow$ [c].left
    end-while
    successor $\leftarrow$ c
  else
    p $\leftarrow$ parent(tree, x)
    while p $\neq$ NIL **and** [p].left $\neq$ x **execute**
      x $\leftarrow$ p
      p $\leftarrow$ parent(tree, p)
    end-while
    successor $\leftarrow$ p
  end-if
end-function

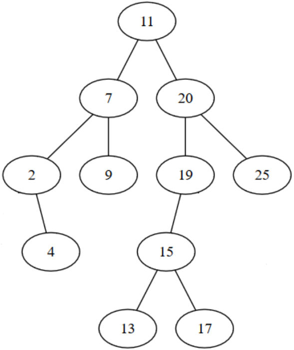
BST - Finding the successor of a node

Time complexity of *successor* depends on *parent* function:

- If *parent* runs in $\Theta(1) \Rightarrow$ complexity of successor is $O(h)$ (or $O(n)$)
- If *parent* runs in $O(n) \Rightarrow$ complexity of successor is $O(h^2)$ (or $O(n^2)$)

Similar to *successor*, we can define a *predecessor* function as well.

BST - Remove a node



How can we remove the value 25? What about removing 2? What about removing 11?

BST - Remove a node

When we want to remove a value (a node *x* containing the given value) from a BST we have 3 cases:

- *x* has no children
  - Set the corresponding child of the parent to NIL
- *x* has one descendant
  - Set the corresponding child of the parent to the child of *x*
- *x* has two children
  - Find its predecessor *p*, let it take the position of *x* in the tree and delete the node containing *p*

    OR
  - Find its successor *s*, let it replace *x* in the tree and remove the node currently containing it from the tree.

BST - Removing an element

Recursively removing an element from a BST:

```
function remove_rec(p, e) is:
//pre: p: ↑ BSTNode, e: TComp
//post: e is removed from the BST rooted by p; return the root of the new (sub)tree
    if p = NIL then
        remove_rec ← p
    else
        if e < [p].info then
            [p].left ← remove_rec([p].left,e)
            remove_rec ← p
        else if e > [p].info then
            [p].right ← remove_rec([p].right,e)
            remove_rec ← p
        else
            if [p].left ≠ NIL and [p].right ≠ NIL then
                temp ← minimum([p].right)
                [p].info ← [temp].info
                [p].right ← remove_rec([p].right, [p].info)
                remove_rec ← p
//continued on the next slide
```

BST - Removing an element

Recursively removing an element from a BST:

```
            else if [p].left = NIL then
                remove_rec← [p].right
            else
                remove_rec ← [p].left
            end-if
        end-if
    end-if
end-function
```

The time complexity of the *remove* operation is $O(h)$ (or $O(n)$)

BST - Removing an element

We need a wrapper function to call *remove_rec* with the root of the tree:

Recursively deletion an from a BST - - initial call:

function remove (tree, e) **is:**
//pre: tree is a Binary Search Tree, e is the elem to be removed
//post: tree' is a Binary Search Tree obtained by removing e from tree
    tree.root ← remove_rec(tree.root, e)
end-function

Other operations

- Traversals
  - The same as for binary trees

Containers represented using BSTs

BSTs are used for representing the following containers:

- ADT Set
  - set in C++ STL, TreeSet in Guava (Google Core Libraries for Java) (implemented using balanced BST)
- ADT Map (Sorted Map)
  - map in C++ STL, TreeMap in Guava (for Java)
- ADT MultiMap (Sorted MultiMap)
  - multimap in C++ STL, TreeMultimap in Guava (for Java)
- ADT Bag
  - TreeMultiset in Guava (for Java)

Binary Search Trees- Applications

Real-world applications of binary search tree data structure:

Unix Kernel

- Managing Virtual Memory Areas (VMAs)

Compilers

- For implementing Syntax Tress

Routing tables

- A routing table is used to link routers in a network. It is usually implemented with a variation of a binary search tree.

Data compression

- Huffman coding

Bibliography

- David M. Mount, *Lecture notes for the course Data Structures* (CMSC 420), at the Dept. of Computer Science, University of Maryland, College Park, 2001

- Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein, *Introduction to Algorithms*, Third Edition, The MIT Press, 2009

- Narasimha Karumanchi, *Data Structures and Algorithms Made Easy: Data Structures and Algorithmic Puzzles*, Fifth Edition, 2016

- Clifford A. Shaffer, *A Practical Introduction to Data Structures and Algorithm Analysis*, Third Edition, 2010

Thank you