Seminar 10

Linear systems of differential equations

(1) $\boxed{Y' = AY + B}$ $\quad A \in C([a,b], \mathcal{M}_n(\mathbb{R}))$
$B \in C([a,b], \mathbb{R}^n)$

(2) $\boxed{Y' = AY}$ the homogeneous system

$Y' = AY + B$ the nonhomogeneous system.

S_0 the sol. set of (2)

S_0 is a linear subspace of linear space $\mathcal{C}^1([a,b], \mathbb{R}^n)$ with $\dim S_0 = n$.

$\{Y^1, \dots, Y^n\}$ basis in S_0 (the fundam. system of solutions)

$$S_0 = \{ c_1 Y^1 + \dots + c_n Y^n \mid c_1, \dots, c_n \in \mathbb{R} \}$$

$U(x) = (Y^1 \dots Y^n)$ – the fundam. matrix of solution

$$S_0 = \left\{ U(x) \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \;\middle|\; c_1, \dots, c_n \in \mathbb{R} \right\}$$

The Wronskian criterion

$\{\underline{Y}^1, \dots, \underline{Y}^n\}$ is a fundam. system of solutions $\Leftrightarrow$

$\Leftrightarrow$ $\underline{Y}^1, \dots, \underline{Y}^n$ are sol. of. the system (2) and

$\exists x_0 \in [a,b]$ such that $W(x_0; \underline{Y}^1, \dots, \underline{Y}^n) \neq 0$.

where $W(x; \underline{Y}^1, \dots, \underline{Y}^n) = \begin{vmatrix} y_1^1(x) & \dots & y_1^n(x) \\ \vdots & & \vdots \\ y_n^1(x) & \dots & y_n^n(x) \end{vmatrix}$ the Wronskian.

The nonhomogeneous case

$\underline{Y}' = A\underline{Y} + B$ the general sol.

$$\underline{Y} = \underline{Y}^0 + \underline{Y}^P$$

where $\underline{Y}^0$ – is the gen. sol. of (2)

$\underline{Y}^P$ – is a partic. sol. of (1) which can be found using the variation of the constants method.

if $U(x)$ is a fundam. matrix of sol.

$$\Rightarrow \underline{Y}^P(x) = U(x) \begin{pmatrix} \varphi_1(x) \\ \vdots \\ \varphi_n(x) \end{pmatrix}$$

1) Let's consider the syst.

$$\begin{cases} y_1' = y_1 \cos^2 x - (1 - \sin x \cdot \cos x) \cdot y_2 \\ y_2' = (1 + \sin x \cdot \cos x) \cdot y_1 + \sin^2 x \cdot y_2 \end{cases}$$

a) Prove that $Y^1 = \begin{pmatrix} e^x \cdot \cos x \\ e^x \sin x \end{pmatrix}$, $Y^2 = \begin{pmatrix} -\sin x \\ \cos x \end{pmatrix}$ generate a fundam. system. of solutions

b) Find the sol. of the system which satisfies

$$\begin{cases} y_1(0) = 1 \\ y_2(0) = 0 \end{cases}$$

$Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ $\qquad Y' = A \cdot Y$ where

$$A = \begin{pmatrix} \cos^2 x & -(1 - \sin x \cdot \cos x) \\ 1 + \sin x \cos x & \sin^2 x \end{pmatrix}$$

$\{Y^1, Y^2\}$ is a fundam system of sol $\Leftrightarrow$ Y^1, Y^2 are sol. and $W(x; Y^1, Y^2) \neq 0$

$\underline{Y}^1$ is a sol. of the system. $\Longleftrightarrow$ $y_1(x) = e^x \cos x$, $y_2(x) = e^x \sin x$ satisfy the system equations.

$$y_1' \stackrel{?}{=} y_1 \cos^2 x - (1 - \sin x . \cos x) . y_2$$

$$e^x \cos x - e^x . \sin x \stackrel{?}{=} e^x . \cos x . \cos^2 x - (1 - \sin x . \cos x) . e^x \sin x$$

$$= e^x \cos^3 x - e^x \sin x + e^x \underbrace{\sin^2 x}_{1-\cos^2 x} . \cos x$$

$$e^x \cos x - e^x \sin x = \cancel{e^x \cos^3 x} - e^x \sin x + e^x \cos x - \cancel{e^x \cos^3 x} \quad (T)$$

$$y_2' \stackrel{?}{=} (1 + \sin x \cos x) . y_1 + \sin^2 x . y_2$$

$$e^x \sin x + e^x . \cos x \stackrel{?}{=} (1 + \sin x \cos x) . e^x \cos x + \sin^2 x . e^x \sin x$$

$$\text{— " —} \stackrel{?}{=} e^x \cos x + e^x \sin x . \underbrace{\cos^2 x}_{1-\sin^2 x} + e^x \sin^3 x$$

$$\text{— " —} \stackrel{!}{=} e^x \cos x + e^x \sin x \cancel{- e^x \sin^3 x} \cancel{+ e^x \sin^3 x} \quad (T)$$

$\Rightarrow$ $\underline{Y}^1$ is a sol. of the system.

Y^2 is a sol. of the system $\Leftrightarrow$ $\begin{array}{l} y_1(x) = -\sin x \\ y_2(x) = \cos x \end{array}$ satisfy the syst. eqs.

$$y_1' \overset{?}{=} y_2 \cos^2 x - (1 - \sin x \cdot \cos x) \cdot y_2$$

$$-\cos x \overset{?}{=} -\sin x \cdot \cos^2 x - (1 - \sin x \cdot \cos x) \cdot \cos x$$

$$-\cos x = -\cancel{\sin x \cdot \cos^2 x} - \cos x + \cancel{\sin x \cdot \cos^2 x} \quad (T)$$

$$y_2' \overset{?}{=} (1 + \cos x \cdot \sin x) \cdot y_1 + \sin^2 x \cdot y_2$$

$$-\sin x \overset{?}{=} (1 + \cos x \cdot \sin x) \cdot (-\sin x) + \sin^2 x \cdot \cos x$$

$$-\sin x = -\sin x - \cancel{\cos x \sin^2 x} + \cancel{\sin^2 x \cdot \cos x} \quad (T).$$

$\Rightarrow$ Y^2 is a sol. of the syst.

$$W(x; Y^1, Y^2) = \begin{vmatrix} e^x \cos x & -\sin x \\ e^x \sin x & \cos x \end{vmatrix} = e^x \cos^2 x + e^x \sin^2 x =$$

$$= e^x (\cos^2 x + \sin^2 x) = \underbrace{e^x}_{>0} \neq 0$$

$\Rightarrow$ $\{Y^1, Y^2\}$ are linearly indep. $\Rightarrow$ $\{Y^1, Y^2\}$ is f.s.s.

b) $\{Y^1, Y^2\}$ is a fundam. syst. of sol.

$$U(x) = (Y^1\ Y^2) = \begin{pmatrix} e^x\cos x & -\sin x \\ e^x\sin x & \cos x \end{pmatrix}$$ is a fundam matrix of. sol.

$\Rightarrow$ the gen. sol. of the system:

$$Y = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = U(x)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} e^x\cos x & -\sin x \\ e^x\sin x & \cos x \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \begin{pmatrix} c_1e^x\cos x - c_2\sin x \\ c_1e^x\sin x + c_2\cos x \end{pmatrix}$$

$$\begin{cases} y_1(x) = c_1e^x\cos x - c_2\sin x \\ y_2(x) = c_1e^x\sin x + c_2\cos x \end{cases}, c_1, c_2 \in \mathbb{R}.$$

$$\begin{cases} y_1(0) = 1 \\ y_2(0) = 0 \end{cases} \Rightarrow \begin{cases} c_1 = 1 \\ c_2 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} y_1(x) = e^x\cos x \\ y_2(x) = e^x\sin x \end{cases}$$

2) Find the IVP solution:

$$\begin{cases} y' = -e^x y - z \cdot \cos x \\ z' = -y - (1+x^4) \cdot z \\ y(0) = 0 \\ z(0) = 0 \end{cases}$$

$$\underline{Y} = \begin{pmatrix} y \\ z \end{pmatrix}$$

$$\underline{Y}' = A \cdot \underline{Y}, \quad A = \begin{pmatrix} -e^x & -\cos x \\ -1 & -(1+x^4) \end{pmatrix}$$

$\underline{Y} \equiv 0 \Rightarrow \begin{cases} y(x) \equiv 0 \\ z(x) \equiv 0 \end{cases}$ is a solution of the given (IVP).

$\Rightarrow$ $\underline{Y} \equiv 0$ is the only one sol. of the IVP.

$\exists!$ Th

(any IVP attached to a linear system has an unique sol.)

3) Let's consider the system:

$$\begin{cases} y_1' = y_2 \\ y_2' = y_1 + 2 - x^2 \end{cases}$$

a) Prove that $\underline{Y}^1 = \begin{pmatrix} e^x \\ e^x \end{pmatrix}$, $\underline{Y}^2 = \begin{pmatrix} e^{-x} \\ -e^{-x} \end{pmatrix}$ generate a f.s.s.

b) Find the general sol. of the system.

$\underline{Y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Rightarrow \underline{Y}' = A \cdot \underline{Y} + B$ where:

$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 2 - x^2 \end{pmatrix}$ a nonhomogeneous system.

the gen. sol.: $\underline{Y} = \underline{Y}^0 + \underline{Y}^P$

$\underline{Y}^0$ – the gen. sol. of the homog. syst.

$\underline{Y}^P$ – a partic. sol. of the nonhomog. syst.

a) $Y^1 = \begin{pmatrix} e^x \\ e^x \end{pmatrix}$ is a sol. of the homog. syst $Y' = AY$

$\begin{cases} y_1(x) = e^x \\ y_2(x) = e^x \end{cases}$ verify the eqs of the homog. syst:

$$\boxed{\begin{cases} y_1' = y_2 \\ y_2' = y_1 \end{cases}}$$

Y^1: $e^x = e^x$

$Y^2 = \begin{pmatrix} e^{-x} \\ -e^{-x} \end{pmatrix}$ is a sol. of the homog. syst:

$\begin{cases} y_1(x) = e^{-x} \\ y_2(x) = -e^{-x} \end{cases}$ $-e^{-x} = -e^{-x}$

$\Rightarrow$ Y^1, Y^2 are sol. of the homogeneous system

$\{Y^1, Y^2\}$ l.i.

$$W(x; Y^1, Y^2) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -1 - 1 = -2 \neq 0$$

$\Rightarrow$ $\{Y^1, Y^2\}$ is a f.s.s.

$$U(x) = (Y^1 Y^2) = \begin{pmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{pmatrix}$$ the fundam. matrix of. sol.

$$\Rightarrow \boxed{Y^0 = U(x) \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \ c_1, c_2 \in \mathbb{R}}$$

$Y^p = ?$ a partic. sol. of $Y' = AY + B$

$$Y^p(x) = U(x) \cdot \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} = \begin{pmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{pmatrix} \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} =$$

$$Y^p(x) = \begin{pmatrix} y_p^1(x) \\ y_p^2(x) \end{pmatrix} = \begin{pmatrix} e^x \cdot \varphi_1(x) + e^{-x} \cdot \varphi_2(x) \\ e^x \varphi_1(x) - e^{-x} \varphi_2(x) \end{pmatrix}$$

$$\begin{cases} (y_p^1)'(x) = (y_p^2)(x) \\ (y_p^2)'(x) = (y_p^1)(x) + 2 - x^2 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \cancel{e^x \cdot \varphi_1} + e^x \cdot \varphi_1' - \cancel{e^{-x} \cdot \varphi_2} + e^{-x} \cdot \varphi_2' = \cancel{e^x \cdot \varphi_1} - \cancel{e^{-x} \cdot \varphi_2} \\ \cancel{e^x \cdot \varphi_1} + e^x \cdot \varphi_1' + \cancel{e^{-x} \cdot \varphi_2} - e^{-x} \cdot \varphi_2' = \cancel{e^x \cdot \varphi_1} + \cancel{e^{-x} \cdot \varphi_2} + 2 - x^2 \end{cases}$$

$$\Rightarrow \begin{cases} e^x \cdot \varphi_1' + e^{-x} \cdot \varphi_2' = 0 \\ e^x \cdot \varphi_1' - e^{-x} \varphi_2' = 2 - x^2 \end{cases} \quad (+)$$

$$2e^x \cdot \varphi_1' \;/\; = 2 - x^2 \Rightarrow \boxed{\varphi_1'(x) = (2-x^2) \cdot \frac{1}{2} \cdot e^{-x}}$$

$$e^{-x} \cdot \varphi_2' = -e^x \cdot \varphi_1'$$

$$e^{-x} \cdot \varphi_2' = -\cancel{e^x} \cdot (2-x^2) \cdot \frac{1}{2} \cdot \cancel{e^{-x}} \Rightarrow \boxed{\varphi_2'(x) = -\frac{1}{2}(2-x^2) \cdot e^x}$$

$$\varphi_1'(x) = e^{-x} - \frac{1}{2} x^2 e^{-x} \Rightarrow$$

$$\Rightarrow \varphi_1(x) = \int \left(e^{-x} - \frac{1}{2} x^2 \cdot e^{-x}\right) dx = -e^{-x} - \frac{1}{2} \int x^2 \cdot e^{-x} dx =$$

$$= -e^{-x} + \frac{1}{2} \int x^2 \cdot \underbrace{(-e^{-x})}_{(e^{-x})'} dx = -e^{-x} + \frac{1}{2} x^2 \cdot e^{-x} - \frac{1}{\cancel{2}} \int \cancel{2} x \cdot e^{-x} dx =$$

$$= -e^{-x} + \frac{1}{2} x^2 e^{-x} + \int x \cdot \underbrace{(-e^{-x})}_{(e^{-x})'} dx = -e^{-x} + \frac{1}{2} x^2 e^{-x} + x \cdot e^{-x} - \int e^{-x} dx$$

$$= -\cancel{e^{-x}} + \frac{1}{2} x^2 e^{-x} + x e^{-x} + \cancel{e^{-x}}$$

$$\boxed{\varphi_1(x) = \left(\frac{x^2}{2} + x\right) e^{-x}}$$

$$\varphi_2'(x) = -e^x + \frac{x^2}{2} e^x$$

$$\Rightarrow \varphi_2(x) = \int \left(-e^x + \frac{x^2}{2} e^x\right) dx = -e^x + \int \frac{x^2}{2} e^x dx =$$

$$= -e^x + \frac{x^2}{2} e^x - \int \frac{1}{\cancel{2}} \cdot \cancel{2}x \cdot e^x \, dx = -e^x + \frac{x^2}{2} e^x - \int x e^x dx =$$

$$= -e^x + \frac{x^2}{2} e^x - x e^x + \int e^x dx = \cancel{-e^x} + \frac{x^2}{2} e^x - x e^x \cancel{+ e^x}$$

$$\boxed{\varphi_2(x) = \left(\frac{x^2}{2} - x\right) e^x}$$

$$y^p(x) = \begin{pmatrix} y_p^1(x) \\ y_p^2(x) \end{pmatrix} = \begin{pmatrix} e^x \cdot \varphi_1(x) + e^{-x} \cdot \varphi_2(x) \\ e^x \cdot \varphi_1(x) - e^{-x} \cdot \varphi_2(x) \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{x^2}{2} \cancel{+x} + \frac{x^2}{2} \cancel{-x} \\ \cancel{\frac{x^2}{2}} + x - \cancel{\frac{x^2}{2}} + x \end{pmatrix} = \begin{pmatrix} x^2 \\ 2x \end{pmatrix} \Rightarrow \begin{cases} y_p^1(x) = x^2 \\ y_p^2(x) = 2x \end{cases}$$

the gen. sol. of the nonhomog. syst. :

$$\underline{Y} = \underline{Y}^0 + \underline{Y}^P$$

$$\underline{Y} = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \begin{pmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} x^2 \\ 2x \end{pmatrix}$$

$$\Rightarrow \begin{cases} y_1(x) = c_1 e^x + c_2 e^{-x} + x^2 \\ y_2(x) = c_1 e^x - c_2 e^{-x} + 2x \end{cases}, \quad \underline{c_1, c_2 \in \mathbb{R}}.$$