Seminar 1

First order solvable differential equations

$$\boxed{y' = f(x,y)}$$

1) Antiderivative problem

$y' = f(x)$ $\quad f \in C(I)$ given

$$\boxed{y(x) = \int_{x_0}^{x} f(s)\,ds + c,\ c \in \mathbb{R}}$$ the general solution

2) Separable differential equations

General form: $\boxed{y'(x) = f(x).g(y)}$

$y = y(x) \Rightarrow dy = y'(x).dx \Rightarrow \boxed{y'(x) = \frac{dy}{dx}}$

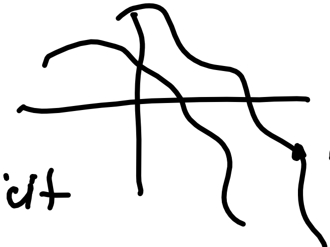
$$\frac{dy}{dx} = f(x).g(y) \Rightarrow \frac{dy}{g(y)} = f(x).dx$$

$$\underbrace{\int \frac{dy}{g(y)}}_{G(y)} = \underbrace{\int f(x)\,dx}_{F(x)} + c \Rightarrow$$

$\rightarrow \boxed{G(y) = F(x) + c, \ c \in \mathbb{R}}$ the general sol in the implicit form.

$G^{-1} \Rightarrow \boxed{y(x) = G^{-1}(F(x) + c), \ c \in \mathbb{R}}$

the general solution in explicit form.



Remark. If there exists $y_0 \in \mathbb{R}$ such that $g(y_0) = 0$ then $y(x) \equiv y_0$ is a solution of the separable eq, called singular solution.

- $g(y) = 0 \Rightarrow y(x) \equiv y_0$
- $y' = \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = f(x) \cdot g(y)$

Exercise 1 : Solve the following diff. eq.

a) $y' = 2x(1+y^2)$

b) $(x^2-1)y' + 2xy^2 = 0$

c) $xy' = y^3 + y$

d) $y' = k \cdot \frac{y}{x}$, $k \in \mathbb{R}^*$.

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a) $y' = \underbrace{2x}_{f(x)} \underbrace{(1+y^2)}_{g(y)}$

$g(y) = 0$

$1+y^2 = 0 \Rightarrow y^2 = -1$

$y_{1,2} = \pm i \notin \mathbb{R}$.

$\Rightarrow$ the eq has not sing. sol.

$y' = \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = 2x(1+y^2)$

$$\int \frac{dy}{1+y^2} = \int 2x\,dx$$

$$\boxed{\operatorname{arctg} y = x^2 + c,\ c \in \mathbb{R}}$$ general sol. in implicit form.

$$\Rightarrow \boxed{y(x) = \operatorname{tg}(x^2 + c),\ c \in \mathbb{R}}$$ general sol in explicit form.

b) $(x^2-1)y' + 2xy^2 = 0$

$(x^2-1)y' = -2xy^2 \Rightarrow \boxed{y' = -\frac{2xy^2}{x^2-1}}$

$y' = \underbrace{-\frac{2x}{x^2-1}}_{f(x)} \cdot \underbrace{y^2}_{g(y)}$ separable eq.

$g(y)=0 \Rightarrow y^2=0 \Rightarrow y=0 \Rightarrow$ $\boxed{y(x)\equiv 0 \text{ is a singular solution}}$

$y' = \frac{dy}{dx}$ $\qquad \frac{dy}{dx} = -\frac{2x}{x^2-1}\cdot y^2$

$$-\frac{dy}{y^2} = \frac{2x}{x^2-1}\,dx$$

$\Rightarrow -\int \frac{dy}{y^2} = \underline{\int \frac{2x}{x^2-1}\,dx} \Rightarrow \boxed{\frac{1}{y} = \ln|x^2-1| + c,\ c\in\mathbb{R}}$

$$\Downarrow$$

$\int \frac{u'(x)}{u(x)}dx = \ln|u(x)|$

$u(x) = x^2-1$

$\boxed{y(x) = \frac{1}{\ln|x^2-1|+c},\ c\in\mathbb{R}}$

the gen. sol.

c) $xy' = y^3 + y \implies y' = \frac{y^3+y}{x}$

$$y' = \underbrace{\frac{1}{x}}_{f(x)} \cdot \underbrace{(y^3+y)}_{g(y)}$$

$g(y) = 0 \implies y^3 + y = 0$

$y(y^2+1) = 0$

$y_1 = 0 \implies \boxed{y(x) \equiv 0 \quad \text{is a sing. sol}}$

$y_{2,3} = \pm i \notin \mathbb{R}$

$$\frac{dy}{dx} = \frac{1}{x} \cdot (y^3 + y) \implies \frac{dy}{y^3+y} = \frac{1}{x}\, dx$$

$$\int \frac{dy}{y^3+y} = \int \frac{1}{x} dx + C.$$

$$\int \frac{dy}{y(y^2+1)}$$

$$\frac{1}{y(y^2+1)} = \frac{A}{y} + \frac{By+C}{y^2+1} \implies 1 = A(y^2+1) + y(By+C)$$

$y = 0 \implies \boxed{1 = A}$

$y = i \implies 1 = i(B.i + C)$

$1 = -B + C.i \implies \begin{matrix} 1 = -B \implies B = -1 \\ C = 0 \end{matrix}$

$$\frac{1}{y(y^2+1)} = \frac{1}{y} - \frac{1}{y^2+1} \Rightarrow \int \frac{dy}{y(y^2+1)} = \int \left(\frac{1}{y} - \frac{1}{y^2+1}\right) dy =$$

$$= \int \frac{1}{y}\, dy - \int \frac{dy}{y^2+1} = \underline{\ln|y| - \operatorname{arctg} y}$$

$$\Rightarrow \underline{\int \frac{dy}{y(y^2+1)}} = \int \frac{1}{x}\, dx + c.$$

$$\boxed{\ln|y| - \operatorname{arctg} y = \ln|x| + c, \; c \in \mathbb{R}}$$ the gen. sol. in implicit form.

d) $y' = k \cdot \frac{y}{x}, \; k \in \mathbb{R}^*$

$y(x) \equiv 0$ singular solution

$$\ln x = \frac{1}{k} \ln y + c.$$

$$y = e^{k \ln x + c}$$

$$y(x) = e^{k \cdot \ln x + c} = e^{\ln x^k + c} = e^{\ln x^k} \cdot e^c$$

$$y(x) = x^k \cdot e^c, \; c \in \mathbb{R} \qquad e^c = c_1 \qquad \boxed{y(x) = c_1 \cdot x^k}$$

$$\frac{dy}{dx} = k \cdot \frac{y}{x} \implies \int \frac{dy}{y} = \int k \cdot \frac{dx}{x}$$

$$\ln|y| = k \cdot \ln|x| + c$$

$$|y| = e^{k \cdot \ln|x| + c}.$$

$$y = \pm e^{k \ln|x| + c} \longrightarrow y = \underbrace{\pm e^{c}}_{c_1} \cdot e^{\ln|x|^{k}}$$

$$\boxed{y(x) = c_1 \cdot x^{k}, \quad c_1 \in \mathbb{R}}$$

Exercise 2: Solve the following IVP:

a) $\begin{cases} (1+e^{x}) \cdot y \cdot y' - e^{x} = 0 \\ y(0) = 1 \end{cases}$

b) $\begin{cases} y' \cdot \sin x - y \ln y = 0 \\ y\left(\frac{\pi}{2}\right) = 1 \end{cases}$

c) $\begin{cases} y' = k \cdot \frac{y}{x}, & k \in \mathbb{R}^{*} \\ y(0) = 1 \end{cases}$

a) $(1+e^x)\,y\cdot y' - e^x = 0 \Rightarrow y' = \underbrace{\frac{e^x}{1+e^x}}_{f(x)} \cdot \underbrace{\frac{1}{y}}_{g(y)}$

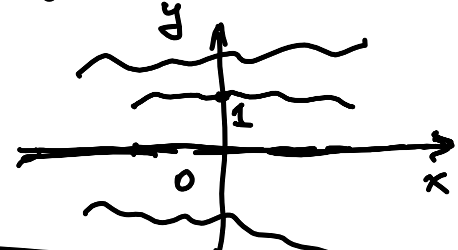
$g(y) = 0 \Rightarrow \frac{1}{y} = 0 \Rightarrow$ no real sol. $\Rightarrow$ no sing. sol.

$$\frac{dy}{dx} = \frac{e^x}{1+e^x}\cdot\frac{1}{y} \Rightarrow y\cdot dy = \frac{e^x}{1+e^x}\cdot dx \;\Big/\cdot 2.$$

$$\int 2y\,dy = \int \frac{2e^x}{1+e^x}\,dx$$

$$\Rightarrow y^2 = 2\cdot\ln(1+e^x) + c.$$

$$\boxed{y(x) = \pm\sqrt{2\ln(1+e^x)+c}\,,\; c\in\mathbb{R}}$$ the gen. sol.



$y(0) = 1$

$$y(x) = \sqrt{2\ln(1+e^x)+c}$$

$$y(0) = 1 \Rightarrow \sqrt{2\ln(2)+c} = 1$$

$$2\ln(2)+c = 1$$

$$\boxed{c = 1 - 2\ln(2)}$$

$\Rightarrow$ the IVP solution:

$$y(x) = \sqrt{2\ln(1+e^x)+1-2\ln 2}$$

3. Reducible diff. eq. to the separable diff. eq

General form: $\boxed{y'(x) = f(ax+b\cdot y+c)+d}$, $a, b, c, d \in \mathbb{R}$
$b \neq 0$

subst: $z(x) = ax + b\cdot y(x) + c \Rightarrow$

$$\Rightarrow \boxed{y(x) = \frac{z(x) - ax - c}{b}} \Rightarrow y'(x) = \frac{1}{b}\cdot z'(x) - \frac{a}{b}$$

$$\frac{1}{b}\cdot z' - \frac{a}{b} = f(z) + d \quad \Big|\cdot b.$$

$$z' - a = b\,(f(z)+d)$$

$$\boxed{z' = a + b\cdot(f(z)+d)} \quad \text{sep. eq.}$$

$$z' = f_1(x)\cdot f_2(z)$$

$$f_1(x) \equiv 1, \quad f_2(z) = a + b\,(f(z)+d)$$

Exercise 3: Solve the following diff. eqs.:

a) $y' = (y-x)^2 + 1$.

b) $y' = (8x+2y+1)^2$

c) $y' = \sin(x-y)$

d) $y' - 1 = e^{x+2y}$

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a) $y' = (y-x)^2 + 1$

subst $z = y - x \Rightarrow \boxed{y = z + x}$

$z(x) = y(x) - x$ $\quad y' = z' + 1$.

$\Rightarrow z' + 1 = z^2 + 1 \Rightarrow \boxed{z' = z^2}$.

$z(x) \equiv 0$ sing. sol.

$\frac{dz}{dx} = z^2 \Rightarrow \int \frac{dz}{z^2} = \int dx \Rightarrow -\frac{1}{z} = x + c$

$\Rightarrow \boxed{z(x) = -\frac{1}{x+c}, c \in \mathbb{R}}$ (**)

$z(x) \equiv 0 \Rightarrow \boxed{y(x) = x \text{ sing. sol.}}$

$z(x) = \ldots$(**) $\Rightarrow$

$\Rightarrow \boxed{y(x) = x - \frac{1}{x+c}, c \in \mathbb{R}}$

4. Homogeneous diff. eq. (in the Euler sens)

General form: $\boxed{y' = f(x,y)}$

where f is homogeneous of 0 degree.

($f(x,y)$ is homogeneous of k. degree $\Leftrightarrow$
$\Leftrightarrow \quad f(tx,ty) = t^k \cdot f(x,y)$)

f is homogeneous of 0 degree $\Leftrightarrow$ $\boxed{f(tx,ty) = f(x,y)}$

$$y' = f(x,y) \longrightarrow \boxed{y' = F\left(\frac{y}{x}\right)}.$$

subst $z = \frac{y}{x} \quad \Rightarrow \boxed{y = x \cdot z}$

$$y' = z + x \cdot z'$$

$$z + xz' = F(z) \quad \Rightarrow \boxed{z' = \frac{1}{x}\left(F(z) - z\right)} \text{ sep. diff. eq.}$$

Exercise 4 : Solve the following eqs :

a) $2x^2 y' = x^2 + y^2$

b) $y' = -\frac{x+y}{y}$

c) $y' = e^{\frac{y}{x}} + \frac{y}{x}$

d) $xy' = \sqrt{x^2 - y^2} + y$

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a) $y' = \frac{x^2+y^2}{2x^2} \Rightarrow \boxed{y' = \frac{1}{2} + \frac{1}{2}\left(\frac{y}{x}\right)^2}$

subst $z = \frac{y}{x} \Rightarrow y = xz \Rightarrow y' = z + xz'$

$\Rightarrow z + xz' = \frac{1}{2} + \frac{1}{2} z^2$

$xz' = \frac{1}{2} + \frac{1}{2} z^2 - z$

$z' = \frac{1}{x}\left(\frac{1}{2} + \frac{1}{2} z^2 - \overset{2}{\cancel{z}}\right) \Rightarrow z' = \frac{1}{2 \cdot x}\left(1 + \underline{\underline{z^2 - 2z}}\right)$

$$\Rightarrow \boxed{z' = \underbrace{\frac{1}{2\cdot x}}_{f(x)} \underbrace{(z-1)^2}_{g(z)}}$$

$g(z)=0 \Rightarrow z=1$

$\boxed{z(x)\equiv 1}$ is a sing. sol.

$$\frac{dz}{dx} = \frac{1}{2\cdot x}\cdot (z-1)^2 \Rightarrow \int \frac{dz}{(z-1)^2} = \int \frac{1}{2\cdot x}\cdot dx$$

$$\Rightarrow -\frac{1}{z-1} = \frac{1}{2}\ln|x| + c$$

$$\Rightarrow z-1 = -\frac{1}{\frac{1}{2}\ln|x|+c}$$

$$\boxed{z(x) = 1 - \frac{1}{\frac{1}{2}\ln|x|+c}, \quad c\in\mathbb{R}}$$

$z(x)\equiv 1 \Rightarrow y(x)=x$ sing. sol.

$z(x)=\dots$

$$\Rightarrow \boxed{y(x) = x - \frac{x}{\frac{1}{2}\ln|x|+c}, \quad c\in\mathbb{R}}$$ gen. sol.