Seminar 5

Problems attached to differential equations

Initial value problem (Cauchy problem) (IVP)

in general IVP for an n-order diff. eq.:

$$\begin{cases} y^{(n)} = f(x, y, y', \dots, y^{(n-1)}) \\ y(x_0) = y_0 \\ y'(x_0) = y_1 \\ \vdots \\ y^{(n-1)}(x_0) = y_{n-1} \end{cases}$$

- find gen. sol. of the diff. eq.
- using the initial conditions we find the values of the integration constants
- replace the constants in gen. sol. expression $\Rightarrow$ IVP sol.

1) Find the following ivp solutions:

a) $\begin{cases} (1+e^x)\cdot y\cdot y' - e^x = 0 \\ y(0) = 1 \end{cases}$

d) $\begin{cases} xy' + y = e^x \\ y(a) = b \end{cases}$, $a, b \in \mathbb{R}$, $a \neq 0$. linear eq.

b) $\begin{cases} y'\cdot \sin x - y\cdot \ln y = 0 \\ y\left(\frac{\pi}{2}\right) = 1 \end{cases}$

e) $\begin{cases} y' + \frac{y}{x} = \frac{1}{x^2 y^2} \\ y(1) = 0 \end{cases}$ Bernoulli eq

c) $\begin{cases} x^2 y' = y^2 - xy \\ y(-1) = -1 \end{cases}$

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a) $y' = \frac{e^x}{(1+e^x)\cdot y} \Rightarrow \boxed{y' = \underbrace{\frac{e^x}{1+e^x}}_{f(x)} \cdot \underbrace{\frac{1}{y}}_{g(y)}}$ separable diff. eq.

$\frac{dy}{dx} = \frac{e^x}{(1+e^x)}\cdot\frac{1}{y}$

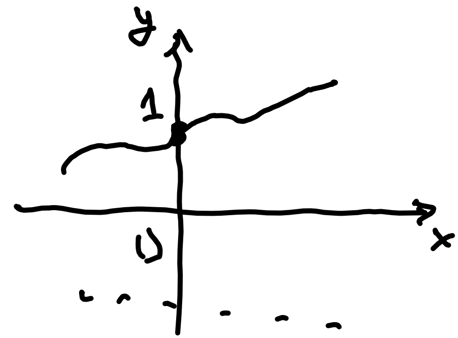
$g(y) = 0 \Rightarrow \frac{1}{y} = 0$ has no sol.

$y\cdot dy = \frac{e^x}{1+e^x}\cdot dx \ \Big|\cdot 2$

$\Rightarrow$ diff. has has no singular sol.

$$\int 2y\cdot dy = \int \frac{2e^x}{1+e^x}\,dx \Rightarrow y^2 = 2\cdot\ln(e^x+1)+c.$$

$$\Rightarrow \boxed{y(x) = \pm\sqrt{2\ln(e^x+1)+c}\ ,\ c\in\mathbb{R}}$$ the gen. solution



$$\boxed{y(0)=1} \Rightarrow$$

$$y(x)=\sqrt{2\ln(e^x+1)+c}$$

$$\Rightarrow \sqrt{2\ln 2 + c} = 1$$

$$2\ln 2 + c = 1$$

$$\boxed{c = 1-2\ln 2}$$

$\Rightarrow$ ivp. sol. $\boxed{y(x)=\sqrt{2\ln(e^x+1)+1-2\ln 2}}$

b) $\begin{cases} y' \cdot \sin x - y \ln y = 0 \\ y\left(\frac{\pi}{2}\right) = 1 \end{cases}$

$$y' \cdot \sin x - y \ln y = 0 \Rightarrow \boxed{y' = \underbrace{\frac{1}{\sin x}}_{f(x)} \cdot \underbrace{y \ln y}_{g(y)}}$$ separable diff. eq.

$g(y) = 0 \Rightarrow y \ln y = 0$

$y = 0$ not convenient

$\ln y = 0 \Rightarrow y = 1$

$\boxed{y(x) \equiv 1}$ is singular sol.

$$\frac{dy}{dx} = \frac{1}{\sin x} \cdot y \ln y$$

$$\boxed{\int \frac{dy}{y \ln y} = \int \frac{1}{\sin x} dx}$$

$$\int \frac{1}{y} \cdot \frac{1}{\ln y} dy = \int \frac{u'(y)}{u(y)} dy = \ln u(y) = \boxed{\ln(\ln y)}$$

$u(y) = \ln y \Rightarrow u'(y) = \frac{1}{y}$

$$\int \frac{1}{\sin x} dx = \int \frac{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}}{2 \cdot \sin \frac{x}{2} \cdot \cos \frac{x}{2}} dx = \frac{1}{2}\left(\int \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} dx + \int \frac{\cos \frac{x}{2}}{\sin \frac{x}{2}} dx \right)$$

$$= \frac{1}{2}\left(\int tg \frac{x}{2} dx + \int ctg \frac{x}{2} dx \right) =$$

$$= \frac{1}{2}\left(-2 \cdot \ln\left(\cos \frac{x}{2}\right) + 2 \ln\left(\sin \frac{x}{2}\right) \right)$$

$$= \ln \sin \frac{x}{2} - \ln \cos \frac{x}{2} = \ln \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} = \ln\left(tg \frac{x}{2}\right)$$

$$\Rightarrow \ln(\ln y) = \ln\left(tg \frac{x}{2}\right) + \ln c$$

$$\ln y = c \cdot tg \frac{x}{2} \Rightarrow \boxed{y(x) = e^{c \cdot tg \frac{x}{2}}, c \in \mathbb{R}}$$

the gen. solution

Remark: In this case the singular sol. is included in the gen. sol.

$$y\left(\frac{\pi}{2}\right) = 1 \Rightarrow e^{c \cdot tg \frac{\pi}{4}} = 1 \Rightarrow c \cdot 1 = 0 \Rightarrow c = 0$$

$\Rightarrow y(x) = 1$ is the ivp. sol.

c) $\begin{cases} x^2 y' = y^2 - xy \\ y(-1) = -1 \end{cases}$ $\Rightarrow$ $y' = \dfrac{y^2 - xy}{x^2}$

$$\boxed{y' = \left(\frac{y}{x}\right)^2 - \left(\frac{y}{x}\right)}$$ homogeneous diff. eq in the Euler sens.

subst. $z = \dfrac{y}{x}$ $\Rightarrow$ $y = x \cdot z$ $\quad y(x) = x \cdot z(x)$

$$\boxed{y' = z + x \cdot z'}$$

$\cancel{z} + x \cdot z' = z^2 - z$

$x z' = z^2 - 2z$

$$\boxed{z' = \frac{z(z-2)}{x}}$$

$z(x) \equiv 0$
$z(x) \equiv 2$ singular sol.

$$\frac{dz}{dx} = \frac{z(z-2)}{x}$$

$$\int \frac{dz}{z(z-2)} = \int \frac{dx}{x}$$

$$\frac{1}{z \cdot (z-2)} = \frac{A}{z} + \frac{B}{z-2}$$

$1 = A(z-2) + B \cdot z$

$z = 2 \Rightarrow 1 = B \cdot 2 \Rightarrow B = \frac{1}{2}$

$z = 0 \Rightarrow 1 = A \cdot (-2) \Rightarrow A = -\frac{1}{2}$

$$\int -\frac{1}{2} \cdot \frac{1}{z} dz + \frac{1}{2} \int \frac{1}{z-2} dz = \int \frac{dx}{x}$$

$$-\frac{1}{2}\ln z+\frac{1}{2}\ln(z-2)=\ln x+\frac{1}{2}\ln c \,\Big|\cdot 2.$$

$$-\ln z+\ln(z-2)=2\ln x+\ln c.$$

$$\ln\frac{z-2}{z}=\ln c\cdot x^2 \Rightarrow \frac{z-2}{z}=c\cdot x^2$$

$$\Rightarrow z-2=c\cdot x^2\cdot z$$

$$z(1-cx^2)=2$$

$$\boxed{z(x)=\frac{2}{1-cx^2},\ c\in\mathbb{R}}$$

$$y=x\cdot z \Rightarrow \boxed{y(x)=\frac{2\cdot x}{1-cx^2}},\ c\in\mathbb{R}$$

$$z(x)\equiv 0 \Rightarrow \boxed{y(x)\equiv 0}, \text{ sing. sol.}$$

$$z(x)=2 \Rightarrow y(x)=2x \text{ sing. sol.}$$

$$y(-1)=-1 \qquad y(x)\equiv 0 \text{ is not a sol. of IVP.}$$

$y(-1)=-1 \Rightarrow \dfrac{-2}{1-c}=-1 \Rightarrow -1+c=-2$

$c=1-2=-1$

$\Rightarrow \boxed{y(x)=\dfrac{2x}{1+x^2}}$ is the sol. of. IVP.

2) Find the solutions for:

a) $\begin{cases} y''=xe^{-x} \\ y(0)=0 \\ y'(0)=0 \end{cases}$

c) $\begin{cases} x^2yy''-(y-xy')^2=0 \\ y(1)=\frac{1}{e} \\ y'(1)=\frac{2}{e} \end{cases}$

b) $\begin{cases} y'''-\frac{2}{x}y''=x^3 \\ y(1)=1 \\ y'(1)=\frac{11}{10} \\ y''(1)=\frac{1}{2} \end{cases}$

b) $\begin{cases} y''' - \frac{2}{x} y'' = x^3 \\ y(1) = 1 \\ y'(1) = \frac{11}{10} \\ y''(1) = \frac{1}{2} \end{cases}$

$y''' - \frac{2}{x} y'' = x^3$

subst $y'' = z \Rightarrow y''' = z'$

$\Rightarrow \boxed{z' - \frac{2}{x} z = x^3}$ linear eq.

$\boxed{z' - \frac{2}{x} z = 0} \Rightarrow$

$\Rightarrow z' = \frac{2}{x} z \Rightarrow \frac{dz}{dx} = \frac{2}{x} z \Rightarrow \int \frac{dz}{z} = \int \frac{2}{x} dx \Rightarrow$

$\Rightarrow \ln z = 2 \cdot \ln x + \ln c \Rightarrow \boxed{z_0(x) = c \cdot x^2, c \in \mathbb{R}}$

$z_p(x) = c(x) \cdot x^2$

$z_p' - \frac{2}{x} z_p = x^3$

$c'(x) \cdot x^2 + \cancel{c(x) \cdot 2x} - \frac{2}{\cancel{x}} \cdot \cancel{c(x) \cdot x^2} = x^3 \Rightarrow c'(x) \cdot x^2 = x^3$

$\Rightarrow c'(x) = x \Rightarrow c(x) = \frac{x^2}{2}$

$\Rightarrow z_p(x) = c(x) \cdot x^2 \Rightarrow z_p(x) = \frac{x^4}{2}$

$\Rightarrow z = z_0 + z_p \Rightarrow \boxed{z(x) = c \cdot x^2 + \frac{x^4}{2}, c \in \mathbb{R}}$

$$\boxed{y'' = z} \Rightarrow y'' = c_1 x^2 + \frac{x^4}{2} \;\vdots$$

$$y' = \int \left(c_1 x^2 + \frac{x^4}{2} \right) dx + c_2$$

$$y' = c_1 \frac{x^3}{3} + \frac{x^5}{10} + c_2 \;\vdots$$

$$y = \int \left(c_1 \frac{x^3}{3} + \frac{x^5}{10} + c_2 \right) dx + c_3$$

$$\boxed{y(x) = c_1 \cdot \frac{x^4}{12} + \frac{x^6}{60} + c_2 x + c_3 \, , \; c_1, c_2, c_3 \in \mathbb{R}} \;\vdots$$

$$\begin{aligned}
y(1) = 1 \quad &\Rightarrow \quad c_1 \cdot \frac{1}{12} + \frac{1}{60} + c_2 + c_3 = 1 \\
y'(1) = \frac{11}{10} \quad &\Rightarrow \quad c_1 \cdot \frac{1}{3} + \frac{1}{10} + c_2 = \frac{11}{10} \\
y''(1) = \frac{1}{2} \quad &\Rightarrow \quad c_1 + \frac{1}{2} = \frac{1}{2} \quad \Rightarrow \boxed{c_1 = 0}
\end{aligned}$$

$$\Rightarrow \frac{1}{10} + c_2 = \frac{11}{10} \Rightarrow \boxed{c_2 = 1}$$

$$\Rightarrow \frac{1}{60} + \cancel{1} + c_3 = \cancel{1} \Rightarrow \boxed{c_3 = -\frac{1}{60}}$$

$\Rightarrow$ ivp. sol. $\boxed{y(x) = \frac{x^6}{60} + x - \frac{1}{60}}$

c) $x^2 y y'' - (y - x y')^2 = 0 \mid : y^2$ ($y(x) \equiv 0$ is not a sol. of ivp)

$$x^2 \cdot \frac{y''}{y} - \left(1 - x \cdot \frac{y'}{y}\right)^2 = 0$$

$\Leftarrow$ $F\left(x, \frac{y'}{y}, \frac{y''}{y} \ldots\right) = 0$

subst $\left| z = \frac{y'}{y} \right.$

$$z' = \frac{y'' \cdot y - y' \cdot y'}{y^2}$$

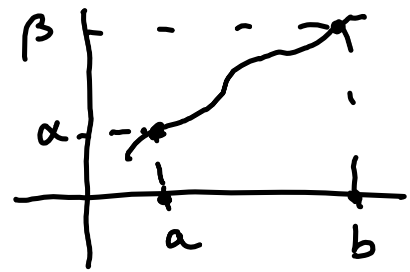
$$z' = \frac{y''}{y} - \left(\frac{y'}{y}\right)^2 =$$

$$\Rightarrow \frac{y''}{y} = z' + \left(\frac{y'}{y}\right)^2$$

$$\frac{y''}{y} = z' + z^2$$

Boundary value problem for second. diff. eq

$$\begin{cases} y'' = f(x, y, y') \\ y(a) = \alpha \\ y(b) = \beta \end{cases}$$



3) Find

a) $\begin{cases} y'' = 2 \\ y(0) = 1 \\ y(1) = 3 \end{cases}$

c) $\begin{cases} y'' = y' \ln(y') \\ y(0) = 0 \\ y(1) = 1 \end{cases}$

b) $\begin{cases} xy'' - y' = 1 \\ y(0) = -4 \\ y(2) = 2 \end{cases}$

Homework.