Lecture 7

n-order linear differential equations

General form:

$$(1) \quad y^{(n)} + a_1(x)\cdot y^{(n-1)} + a_2(x)\cdot y^{(n-2)} + \dots + a_{n-1}(x)\cdot y' + a_n(x)\, y = f(x)$$

$a_1, \dots, a_n, f \in C(I)$

$f(x) \equiv 0 \Rightarrow$ homogeneous linear diff. eq.

$f(x) \not\equiv 0 \Rightarrow$ nonhomogeneous linear diff. eq.

Theorem 1. The IVP

$$\begin{cases} y^{(n)} + a_1 \cdot y^{(n-1)} + \dots + a_n y = f \\ y(x_0) = y_0 \\ y'(x_0) = y_1 \\ \vdots \\ y^{(n-1)}(x_0) = y_{n-1} \end{cases} \qquad x_0 \in I, \; y_0, \dots, y_{n-1} \in \mathbb{R}.$$

has a unique solution in $C^n(I)$.

$$L : C^n(I) \longrightarrow C(I)$$
$$y \longmapsto Ly$$
$$Ly(x) = y^{(n)}(x) + a_1(x) \cdot y^{(n-1)}(x) + \dots + a_n(x) \cdot y(x)$$

the operator L is a linear operator $\Longleftrightarrow$

$\Longleftrightarrow$ $u, v \in C^n(I)$, $\alpha, \beta \in \mathbb{R}$

$$L(\alpha u + \beta v) = \alpha \cdot Lu + \beta \cdot Lv$$

(1) $\Longleftrightarrow$ $Ly = f$.

the solution set S of eq. (1):

$$\boxed{S = \ker L + \{y_p\}}$$

where: $\ker L = \{ y \in C^n(I) \mid Ly = 0 \}$

y_p is a particular solution of the nonhom. eq. (1) $(Ly = f)$

The homogeneous case

$$Ly = 0$$

(2) $$y^{(n)} + a_1 \cdot y^{(n-1)} + \dots + a_{n-1} \cdot y' + a_n \cdot y = 0$$

$$a_1, \dots, a_n \in C(I)$$

S_0 - the sol. set of (2)

$S_0 = \ker L$

Theorem 2. S_0 is a linear subspace of the linear space $C^n(I)$ with $\dim S_0 = n$.

Proof. S_0 is a linear subspace of $C^n(I) \Leftrightarrow$

$\Leftrightarrow u, v \in S_0,\ \alpha, \beta \in \mathbb{R} \Rightarrow \alpha u + \beta v \in S_0$

$u \in S_0$ $\quad u^{(n)} + a_1 u^{(n-1)} + \dots + a_n \cdot u = 0 \quad | \cdot \alpha$

$v \in S_0$ $\quad v^{(n)} + a_1 v^{(n-1)} + \dots + a_n \cdot v = 0 \quad | \cdot \beta$ $\quad (+)$

$$(\alpha u + \beta v)^{(n)} + a_1 (\alpha u + \beta v)^{(n-1)} + \dots + a_n (\alpha u + \beta v) = 0 \Longrightarrow$$

$$\alpha u^{(n)} + \beta \cdot v^{(n)} = (\alpha \cdot u + \beta v)^{(n)}$$

$\Rightarrow \alpha u + \beta v \in S_0$

$\dim S_0 \overset{?}{=} n$ $\qquad \varphi : \mathbb{R}^n \to S_0$ $\qquad \alpha \in \mathbb{R}^n \Leftrightarrow$

$\qquad\qquad \alpha \mapsto y(\alpha)$ $\qquad \Leftrightarrow \alpha = (\alpha_1, \dots, \alpha_n)$

where $y(\alpha)$ is the solution of IVP

$$\begin{cases} Ly = 0 \\ y(x_0) = \alpha_1 \\ y'(x_0) = \alpha_2 \\ \vdots \\ y^{(n-1)}(x_0) = \alpha_n \end{cases} \underset{\text{Th.1}}{\Longrightarrow} \exists!\, y(\alpha) \in C^n(I) \text{ sol. of IVP} \Rightarrow$$

$\Rightarrow \varphi$ is a bijection

φ is isomorphism of linear spaces $\Leftrightarrow$

$$\Leftrightarrow \quad \varphi(\alpha+\beta) = \varphi(\alpha) + \varphi(\beta), \ \forall \alpha; \beta \in \mathbb{R}^n$$

$$\varphi(\lambda\alpha) = \lambda . \varphi(\alpha), \ \forall \alpha \in \mathbb{R}^n, \forall \lambda \in \mathbb{R}.$$

$$\varphi(\alpha+\beta) \overset{?}{=} \varphi(\alpha) + \varphi(\beta)$$

$$y(\alpha+\beta) \overset{?}{=} y(\alpha) + y(\beta)$$

$\alpha = (\alpha_1, \dots \alpha_n)$

$\beta = (\beta_1, \dots, \beta_n)$

$y(\alpha+\beta)$ is a sol of ivp

$$(3)\begin{cases} Ly = 0 \\ y(x_0) = \alpha_1 + \beta_1 \\ y'(x_0) = \alpha_2 + \beta_2 \\ \vdots \\ y^{(n-1)}(x_0) = \alpha_n + \beta_n \end{cases}$$

$y(\alpha)$ is the sol of ivp

$$\begin{cases} Ly = 0 \\ y(x_0) = \alpha_1 \\ \vdots \\ y^{(n-1)}(x_0) = \alpha_n \end{cases}$$

$y(\beta)$ is the sol of ivp

$$\begin{cases} Ly = 0 \\ y(x_0) = \beta_1 \\ y^{(n-1)}(x_0) = \beta_n \end{cases}$$

$y(\alpha), y(\beta) \in S_0 \Rightarrow$

$\Rightarrow y(\alpha) + y(\beta) \in S_0$

$(y(\alpha) + y(\beta))(x_0) = \alpha_1 + \beta_1$

$(y(\alpha) + y(\beta))'(x_0) = \alpha_2 + \beta_2$

$\vdots$

$(y(\alpha) + y(\beta))^{(n-1)}(x_0) = \alpha_n + \beta_n$

$\Rightarrow y(\alpha) + y(\beta)$ is a sol of ivp (3).

$\Rightarrow y(\alpha+\beta) = y(\alpha) + y(\beta)$

Th. 1

analogue $\varphi(\lambda\alpha) = \lambda \varphi(\alpha)$ $\quad \alpha = (\alpha_1, \dots, \alpha_n)$, $\lambda \in \mathbb{R}$.

$\varphi(\lambda\alpha) \to y(\lambda\alpha)$, $\quad \varphi(\alpha) \to y(\alpha)$

$$\begin{cases} Ly = 0 \\ y(x_0) = \lambda\alpha_1 \\ \vdots \\ y^{(n-1)}(x_0) = \lambda\alpha_n \end{cases} \qquad \begin{cases} Ly = 0 \\ y(x_0) = \alpha_1 \\ \vdots \\ y^{(n-1)}(x_0) = \alpha_n \end{cases}$$

S_0 is a linear subspace of $C^n(I)$

$\dim S_0 = n \iff \{y_1, \dots, y_n\} \subset S_0$ basis in $S_0 \iff$

$\iff \{y_1, \dots, y_n\} \subset S_0$ is linearly indep. system.

$\{y_1, \dots, y_n\}$ is called a fundamental system of sol.

Def. The system $\{y_1, \dots, y_n\}$ is

a) linearly dependent $\Leftrightarrow$ $\exists\, c_1, \dots, c_n \in \mathbb{R}$ with $(c_1, \dots, c_n) \neq (0, \dots, 0)$

such that $c_1 y_1 + \dots + c_n y_n = 0$.

b) linearly independent $\Leftrightarrow$

$\Leftrightarrow$ $c_1 y_1 + \dots + c_n y_n = 0 \Rightarrow c_1 = \dots = c_n = 0$

$$W(x; y_1, \dots, y_n) = \begin{vmatrix} y_1(x) & \dots & y_n(x) \\ y_1'(x) & & y_n'(x) \\ \vdots & & \vdots \\ y_1^{(n-1)}(x) & \dots & y_n^{(n-1)}(x) \end{vmatrix}$$

the wronskian (the determinant of Wronski)

Theorem 3

a) if $\{y_1, \dots, y_n\} \subset C^{(n-1)}(I)$ is linearly dependent $\Rightarrow$

$$\Rightarrow \quad W(x; y_1, \dots, y_n) \equiv 0 \text{ on } I.$$
$$\not\Leftarrow$$

b) if $\{y_1, \dots, y_n\} \subset S_0$ is linearly independent $\Rightarrow$

$$\Rightarrow \quad W(x; y_1, \dots, y_n) \neq 0, \quad \forall x \in I.$$

Proof. a) $\{y_1, \dots, y_n\}$ is linearly dependent $\Rightarrow$

$\Rightarrow \exists\ (c_1, \dots, c_n) \neq (0, \dots, 0)$ s.t $c_1 y_1 + \dots + c_n y_n = 0$

we suppose that $c_1 \neq 0 \Rightarrow \boxed{y_1 = \frac{1}{c_1}(-c_2 y_2 - \dots - c_n y_n)} \Rightarrow$

y_1 is a linear combination of $y_2, \dots, y_n$

$$\Rightarrow \quad y_1' = \frac{1}{c_1}\left(-c_2 y_2' - \dots - c_n y_n'\right)$$
$$\vdots$$
$$y_1^{(n-1)} = \frac{1}{c_1}\left(-c_2 y_2^{(n-1)} - \dots - c_n y_n^{(n-1)}\right) \Rightarrow$$

$\Rightarrow$ the first column of W is a linear combination of other $n-1$ columns of W

$\Rightarrow$ $W(x; y_2, \dots, y_n) = 0$, $\forall x \in I$.

b) We suppose that $\underline{\exists x_0 \in I}$ such that

$W(x_0; y_1, \dots, y_n) = 0$

We consider the system

$$(4) \begin{cases} c_1 y_1(x_0) + \dots + c_n y_n(x_0) = 0 \\ c_1 y_1'(x_0) + \dots + c_n y_n'(x_0) = 0 \\ \vdots \\ c_1 y_1^{(n-1)}(x_0) + \dots + c_n y_n^{(n-1)}(x_0) = 0 \end{cases}$$

with the unknowns $c_1, \dots, c_n$

the system (4) is an homogeneous linear system with the coefficients matrix:

$$A = \begin{pmatrix} y_1(x_0) & \dots & y_n(x_0) \\ \vdots & & \vdots \\ y_1^{(n-1)}(x_0) & \dots & y_n^{(n-1)}(x_0) \end{pmatrix}$$

$\det A = W(x_0; y_1, \dots, y_n) = 0 \Rightarrow$

$\Rightarrow$ the system (*) has at least one solution

$(\tilde{c}_1, \dots, \tilde{c}_n) \neq (0, \dots, 0)$

We construct the function

$$\left.\begin{aligned} &\tilde{y}(x) = \tilde{c}_1 y_1(x) + \dots + \tilde{c}_n y_n(x) \\ &y_1, \dots, y_n \in S_0 \end{aligned}\right\} \Rightarrow \tilde{y} \in S_0$$

$\tilde{y}$ is a solution of ivp:

$$\begin{cases} Ly = 0 \\ y(x_0) = 0 \\ y'(x_0) = 0 \\ \vdots \\ y^{(n-1)}(x_0) = 0 \end{cases}$$

also this ivp has as a solution the function $y(x) \equiv 0 \xRightarrow[\text{Th1}]{} \tilde{y} \equiv 0$

$$\left.\begin{aligned} &\Rightarrow \tilde{c}_1 y_1 + \dots + \tilde{c}_n y_n = 0, \quad \forall x \in I. \\ &(\tilde{c}_1, \dots, \tilde{c}_n) \neq (0, \dots, 0) \end{aligned}\right\} \Rightarrow$$

$\Rightarrow \{y_1, \dots, y_n\}$ is linearly dep $\Rightarrow$

$\Rightarrow$ contradiction

Conclusion

In S_0 we have the following possibilities for $y_1, \dots, y_n \in S_0$:

- $\{y_1, \dots, y_n\}$ is a linearly dependent syst. $\Rightarrow$

  $\Rightarrow W(x; y_1, \dots, y_n) = 0$, $\forall x \in I$.

- $\{y_1, \dots, y_n\}$ is a linearly independent syst $\Rightarrow$

  $\Rightarrow W(x; y_1, \dots, y_n) \neq 0$, $\forall x \in I$.

Theorem 4 (The Wronskian criterion)

The system $\{y_1, \dots, y_n\} \subset S_0$ is a fundamental system of solutions for (2) $\Leftrightarrow \exists\, x_0 \in I$ such that

$$W(x_0; y_1, \dots, y_n) \neq 0 .$$

If. $\{y_1, \dots y_n\}$ is a fundamental system of solution for (2) then for $\forall y \in S_0$ $\exists$ $c_1, \dots, c_n \in \mathbb{R}$ such that

$$y = c_1 y_1 + \dots + c_n y_n$$

$$\Rightarrow S_0 = \ker L = \{ c_1 y_1 + \dots + c_n y_n \mid c_1, \dots, c_n \in \mathbb{R} \}$$

or

$$\boxed{y_0 = c_1 y_1 + \dots + c_n y_n \ , c_1, \dots, c_n \in \mathbb{R}.}$$