Analiză reală

Seminar 1 26. 02. 2024

A brief recap of Definition and Properties

1. Sets $\mathbb{N}=\{1,2,3,\dots\}$, X - set, $\mathcal{P}(X)=\{A \mid A\subseteq X\}$ is the family of all subsets of X.

$A\subseteq X$, $C_A = X\setminus A$ - complement of A (wrt X)

$C(C_A)=A$

De Morgan's Laws: Let $(A_i)_{i\in I}\subseteq \mathcal{P}(X)$

$$C\left(\bigcup_{i\in I} A_i\right)=\bigcap_{i\in I} C A_i \ ; \ C\left(\bigcap_{i\in I} A_i\right)=\bigcup_{i\in I} C A_i$$

$X_1,\dots,X_m$ sets, $X_1\times\dots\times X_m=\{(x_1,\dots,x_m) \mid x_i\in X_i,\ i=\overline{1,m}\}$ is the Cartesian product of sets $X_1,\dots,X_m$

2. Countability: A set A is called countable if there exists a bijective function $f:\mathbb{N}\to A$

A is countable $\Leftrightarrow$ A is not finite and its element can be enumerated: $A=\{a_1,a_2,\dots,a_n,\dots\}$

A is called at most countable if A is finite or countable

Examples

1. $\mathbb{N},\mathbb{Z},\mathbb{Q}$ are countable
2. $X_1,\dots,X_m$ are countable $\Rightarrow X_1\times\dots\times X_m$ is countable

If $X=\{x_i \mid i\in\mathbb{N}\}$, $Y=\{y_j \mid j\in\mathbb{N}\}$. The elements of $X\times Y$ can be enumerated by using the following pattern:

$(x_1,y_1)\to(x_1,y_2)\ (x_1,y_3)\dots$

$(x_2,y_1)\ (x_2,y_2)\ (x_2,y_3)\dots$

$(x_3,y_1)\ (x_3,y_2)\ (x_3,y_3)\dots$

3. A countable union of countable sets is countable
4. $\mathbb{R}$, $\mathbb{R}\setminus\mathbb{Q}$ are not countable

3. The Euclidian space $\mathbb{R}^m$: $m\in\mathbb{N}$,

$\mathbb{R}^m=\underbrace{\mathbb{R}\times\dots\times\mathbb{R}}_{m \text{ times}}=\{(x_1,\dots,x_m) \mid x_i\in\mathbb{R},\ i=\overline{1,m}\}$

Together with the following two operations:

- addition: $\mathbb{R}^m\times\mathbb{R}^m\ni(x,y)\mapsto x+y\in\mathbb{R}^m$
- scalar multiplication: $\mathbb{R}\times\mathbb{R}^m\ni(\alpha,x)\mapsto\alpha x\in\mathbb{R}^m$

$\mathbb{R}^m$ is a real vector space

The function $\|\cdot\|:\mathbb{R}^m\to[0,\infty)$, $\|x\|=\sqrt{x_1^2+\dots+x_m^2}$ is the Euclidian norm in $\mathbb{R}^m$, $\forall x=(x_1,\dots,x_m)\in\mathbb{R}^m$

The function $d:\mathbb{R}^m\times\mathbb{R}^m\to[0,\infty)$, $d(x,y)=\|x-y\|=$ $=\sqrt{(x_1-y_1)^2+\dots+(x_m-y_m)^2}$ $\forall x=(x_1,\dots,x_m)$

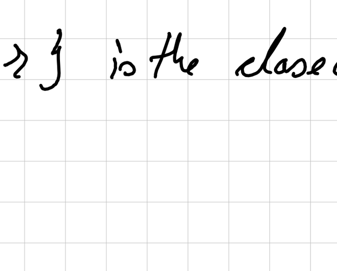
$y=(y_1,\dots,y_m)\in\mathbb{R}^m$ is the Euclidian distance (metric) in $\mathbb{R}^m$.

Indeed, we have:

(i) $d(x,y)=0\Leftrightarrow x=y$

(ii) $\forall x,y\in\mathbb{R}^m$, $d(x,y)=d(y,x)$

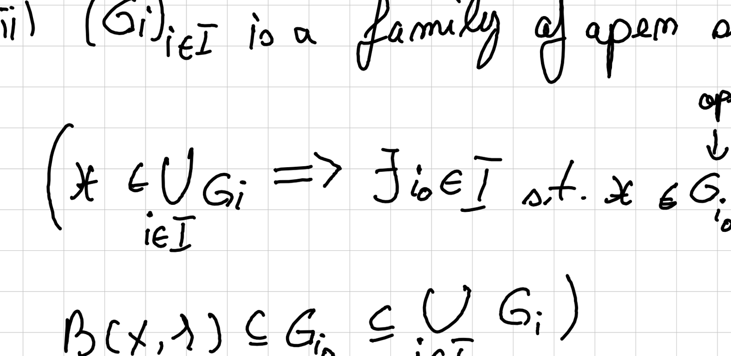
(iii) $\forall x,y,z\in\mathbb{R}^m$, $d(x,z)\le d(x,y)+d(y,z)$



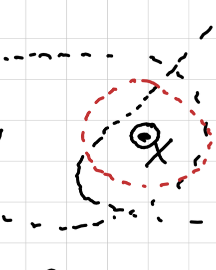
So, $(\mathbb{R}^m,d)$ is a metric space.

Let $x\in\mathbb{R}^m$, $r>0$. $B(x,r)=\{y\in\mathbb{R}^m \mid d(x,y)=\|x-y\|<r\}$ is the open ball of center x and radius r.

$B[x,r]=\{y\in\mathbb{R}^m \mid d(x,y)=\|x-y\|\le r\}$ is the closed ball of center x and radius r.



Let $A\subseteq\mathbb{R}^m$. We say that A is open if $\forall x\in A$, $\exists r>0$ s.t. $B(x,r)\subseteq A$



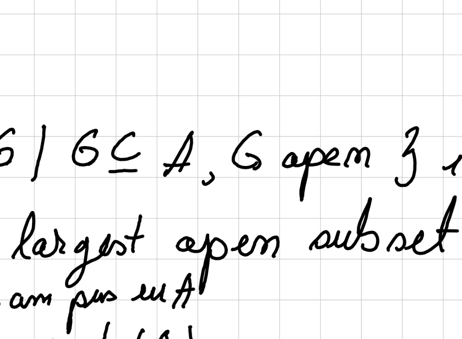
We have: (i) $\emptyset,\mathbb{R}^m$ are open sets in $\mathbb{R}^m$

(ii) $(G_i)_{i\in I}$ is a family of open sets $\Rightarrow\bigcup_{i\in I}G_i$ is open

$\Big(x\in\bigcup_{i\in I}G_i\Rightarrow\exists i_0\in I$ s.t. $x\in\overset{\text{open}}{G_{i_0}}\Rightarrow\exists r>0$ s.t.

$B(x,r)\subseteq G_{i_0}\subseteq\bigcup_{i\in I}G_i\Big)$

(iii) G_1,G_2 open $\Rightarrow G_1\cap G_2$ open



$\Big(x\in G_1\cap G_2\Rightarrow x\in G_1,\ x\in G_2\Rightarrow$

$\Rightarrow\forall i\in\{1,2\}$, $\exists r_i>0$ s.t. $B(x,r_i)\subseteq G_i\Rightarrow$

$\Rightarrow B(x,\underbrace{\min\{r_1,r_2\}}_{>0})\subseteq G_1\cap G_2\Big)$

So, $\mathbb{R}^m$ together with the family of all open subsets of $\mathbb{R}^m$ is a topological space.

Let $A\subseteq\mathbb{R}^m$ $A^\circ=\operatorname{int}(A)=\bigcup\{G \mid G\subseteq A,\ G\text{ open}\}$ is called the interior of A and is the largest open subset of A
↪ nu pusese minimc, am pus cu A

$\operatorname{int}(A)\subseteq A$ and A is open $\Leftrightarrow A=\operatorname{int}(A)$.

Let $A\subseteq\mathbb{R}^m$. We say that A is closed if $C_A=\mathbb{R}^m\setminus A$ is open

We have: (i) $\emptyset$ and $\mathbb{R}^m$ are closed in $\mathbb{R}^m$.

(ii) $(F_i)_{i\in I}$ is a family of closed sets $\Rightarrow\bigcap_{i\in I}F_i$ closed.

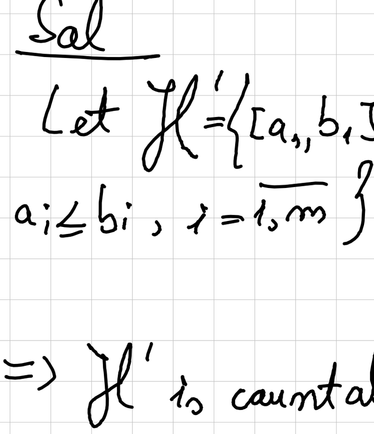
(iii) F_1,F_2 are closed $\Rightarrow F_1\cup F_2$ closed.

Let $A\subseteq\mathbb{R}^m$. $\bar{A}=\operatorname{cl}(A)=\bigcap\{F \mid A\subseteq F,\ F\text{ closed}\}$ is called the closure of A, and is the smallest closed set which contains A. $A\subseteq\bar{A}$ and A is closed $\Leftrightarrow A=\bar{A}$

Remark: The only sets in $\mathbb{R}^m$ that are open and closed are $\emptyset$ and $\mathbb{R}^m$. For any $x\in\mathbb{R}^m$ and $r>0$, $B(x,r)$ is open and $B[x,r]=\overline{B(x,r)}$ is closed.

$\mathcal{H}=\{[a_1,b_1]\times\dots\times[a_m,b_m] \mid a_i,b_i\in\mathbb{R},\ a_i\le b_i,\ i=\overline{1,m}\}$ is the family of all closed hyperrectangles (rectangles) in $\mathbb{R}^m$.

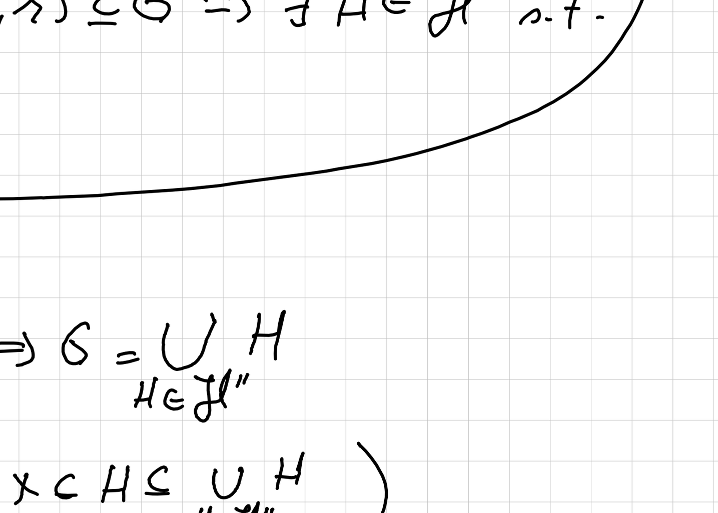
Let $H=[a_1,b_1]\times\dots\times[a_m,b_m]$



(i) H is closed

(ii) $\operatorname{int}H=(a_1,b_1)\times\dots\times(a_m,b_m)$ (with the convention $(\alpha,\alpha)=\emptyset$, $\alpha\in\mathbb{R}$)

Ex.1 Show that every non-empty open set $G\subseteq\mathbb{R}^m$ is countable union of rectangles.



Sol

Let $\mathcal{H}'=\{[a_1,b_1]\times\dots\times[a_m,b_m],\ a_i,b_i\in\mathbb{Q},$ $a_i\le b_i,\ i=\overline{1,m}\}$

$\Rightarrow\mathcal{H}'$ is countable, since $\mathbb{Q}^{2m}=\underbrace{\mathbb{Q}\times\dots\times\mathbb{Q}}_{2m\text{ times}}$ is countable

Let $x\in G\Rightarrow\exists r>0$ s.t. $B(x,r)\subseteq G\Rightarrow\exists H\in\mathcal{H}'$ s.t. $x\in H\subseteq B(x,r)$

$\mathcal{H}'':=\{H\in\mathcal{H}' \mid H\subseteq G\}\Rightarrow G=\bigcup_{H\in\mathcal{H}''}H$

$\Big($ I. $x\in G\Rightarrow\exists H\in\mathcal{H}'$ s.t. $x\in H\subseteq\bigcup_{H\in\mathcal{H}''}H$

II $x\in\bigcup_{H\in\mathcal{H}''}H\Rightarrow\exists H\in\mathcal{H}''$ s.t. $x\in H\subseteq G\Big)$

Next, we show $\mathcal{H}''$ is countable

$\mathcal{H}''\subseteq\mathcal{H}'$ - countable $\Rightarrow\mathcal{H}''$ is at most countable

If we assume that $\mathcal{H}''$ is finite $\Rightarrow G=\bigcup_{H\in\mathcal{H}''}H$ is closed

$\Rightarrow G=\mathbb{R}^m$, but $\mathbb{R}^m$ cannot be covered by a finite family of rectangles. So, $\mathcal{H}''$ is not finite $\Rightarrow\mathcal{H}''$ is countable.

Remark: It can be shown that any non-empty open subset of $\mathbb{R}^m$ is a countable union of rectangles with disjoint interiors.

4. Compact Sets: Let $A\subseteq\mathbb{R}^m$. A family $(A_i)_{i\in I}$ is called a covering of $A\subseteq\bigcup_{i\in I}A_i$. If $\forall i\in I$ A_i is open, $(A_i)_{i\in I}$ is an open covering of A.

A subcovering of the covering $(A_i)_{i\in I}$ is a subfamily $(A_j)_{j\in J}$, where $J\subseteq I$ and $(A_j)_{j\in J}$ is a covering of A.

If J is finite, then we say that $(A_j)_{j\in J}$ is finite subcovering of the covering $(A_i)_{i\in I}$.

We say that the set A is called

- bounded, if $\exists M\ge0$ s.t. $\forall x,y\in A$, $\|x-y\|\le M$
- compact, if any open covering of A has a finite subcovering.

Theorem 1: Let $A\subseteq\mathbb{R}^m$. The following statements are equivalent:

(i) A is compact.

(ii) A is bounded and closed.

(iii) A is sequentially compact, meaning that every sequence of points in A has a convergent subsequence with limit in A.

For example, closed balls and rectangles are compact.

5. Continuous Functions: Let $A\subseteq\mathbb{R}^m$ and $f:A\to\mathbb{R}$ a function

We say that f is cont. at $x\in A$ if: distance

$\forall\varepsilon>0$, $\exists\delta>0$ s.t. $\forall y\in A$ with $\|x-y\|<\delta$ we have $\underbrace{|f(x)-f(y)|}_{<\varepsilon}$

Theorem 2: Let $A\subseteq\mathbb{R}^m$ and $f:A\to\mathbb{R}$. The following are equivalent:

(i) A is continuous

(ii) $\forall G\subseteq\mathbb{R}$ open, $\underbrace{f^{-1}(G)}_{=\{x\in A \mid f(x)\in G\}}=A\cap U$, $U\subseteq\mathbb{R}^m$ open.

Theorem 3: $A\subseteq\mathbb{R}^m$ compact, $f:A\to\mathbb{R}$ cont, then $f(A)$ is compact.