# Seminar 2 - 2025

## Maximum Likelihood Estimator and Method of Moments

Maximum Likelihood Estimation (MLE) provides a structured approach for estimating parameters in a probability model from a given data sample. Suppose we have a sample $x_1, \ldots, x_n$ drawn from a probability model specified by a mass or density function $f_X(x;\theta)$, dependent on parameter(s) $\theta$ within the parameter space $\Theta$. The maximum likelihood estimate, or m.l.e., is obtained as follows:
**Step 1:** Define the likelihood function, $L(\theta)$, where

$$L(\theta) = \prod_{i=1}^{n} f_X(x_i;\theta)$$

which is the product of $n$ mass or density terms (each term being $f_X$ evaluated at $x_i$) viewed as a function of $\theta$.
**Step 2:** Take the natural log of the likelihood function to get $\ln L(\theta)$, and isolate terms involving $\theta$.
**Step 3:** Identify the value $\widehat{\theta} \in \Theta$ that maximizes $\ln L(\theta)$. For a single parameter, solve for $\widehat{\theta}$ by setting

$$\frac{\partial(\ln L(\theta))}{\partial\theta} = 0$$

within $\Theta$. For vector-valued $\theta$, say $\theta = (\theta_1, \ldots, \theta_k)$, find $\widehat{\theta} = (\widehat{\theta}_1, \ldots, \widehat{\theta}_k)$ by solving the $k$ equations

$$\frac{\partial(\ln L(\theta))}{\partial\theta_j} = 0 \quad \text{for } j = 1, \ldots, k$$

within $\Theta$. Note that if $\Theta$ is bounded, the maximum likelihood estimate might lie on its boundary.
**Step 4:** Verify that $\widehat{\theta}$ indeed corresponds to a maximum of the log-likelihood by examining the second derivative with respect to $\theta$. In the single-parameter case, if the second derivative of $\ln L(\theta)$ is negative at $\theta = \widehat{\theta}$, then $\widehat{\theta}$ is confirmed as the m.l.e. of $\theta$. (Other techniques may also be used to confirm a maximum at $\widehat{\theta}$.)

**Problem 1** Suppose a sample $x_1, \ldots, x_n$ is modeled by a Poisson distribution with parameter $\lambda$, where

$$f_X(x;\theta) \equiv f_X(x;\lambda) = \frac{\lambda^x}{x!}e^{-\lambda} \quad \text{for } x = 0, 1, 2, \ldots$$

and $\lambda > 0$. Estimate $\lambda$ via maximum likelihood method.

**Quick example:** Given the following observed frequencies for domestic accidents, we have $n = 647$ data points:

| Number of accidents | Frequency |
|---|---|
| 0 | 447 |
| 1 | 132 |
| 2 | 42 |
| 3 | 21 |
| 4 | 3 |
| 5 | 2 |

Assuming a Poisson model, estimate $\lambda$ using ML method. Then compare with the estimator obtained using the method of moments.

**Problem 2**

Estimate the unknown parameter $\theta$ from a sample

$$3, 3, 3, 3, 3, 7, 7, 7$$

drawn from a discrete distribution with the probability mass function

$$\begin{cases} \mathbb{P}(X = 3) = \theta \\ \mathbb{P}(X = 7) = 1 - \theta \end{cases}$$

Compute two estimators of $\theta$:

1. **the method of moments estimator**;
2. **the maximum likelihood estimator**.

**Problem 3**

The number of times a computer code is executed until it runs without errors has a Geometric distribution with unknown parameter $p$. For 5 independent computer projects, a student records the following numbers of runs:

$$3 \quad 7 \quad 5 \quad 3 \quad 2$$

Estimate $p$:

1. **by the method of moments**;
2. **by the method of maximum likelihood**.

**Problem 4** A sample of 3 observations $X_1 = 0.4, X_2 = 0.7, X_3 = 0.9$ is collected from a continuous distribution with density

$$f(x) = \begin{cases} \theta x^{\theta - 1} & \textbf{if } 0 < x < 1 \\ 0 & \textbf{otherwise} \end{cases}$$

Estimate $\theta$ using MM and ML.

**Homework:**
**Problem 5** A sample $(X_1, \ldots, X_{10})$ is drawn from a distribution with a probability density function

$$f(x) = \frac{1}{2}\left(\frac{1}{\theta}e^{-x/\theta} + \frac{1}{10}e^{-x/10}\right), \quad 0 < x < \infty$$

The sum of all 10 observations equals 150. Estimate $\theta$ by the method of moments.