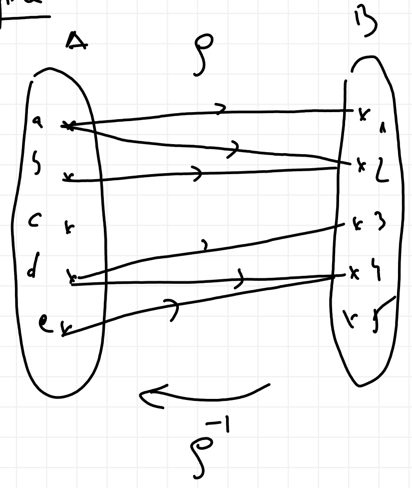
Mathematical Logic
Lecture 4
30.10.2023

The section of a relation w.r.t. a subset
(the image of a subset under a relation)

Example



$\rho(\{a\}) = \{1,2\}$

$\rho(\{a,c,d\}) = \{1,2,3,4\}$

$\rho(\{c\}) = \emptyset$

$\rho^{-1}(\{2\}) = \{a,b\}$

$\rho^{-1}(B) = \{a,b,d,e\}$

$\rho^{-1}(\{5\}) = \emptyset$

Def Let $\rho = (A, B, R)$ be a relation, and let $X \subseteq A$.
The section of ρ w.r.t. X is:

$$\rho(X) = \{ b \in B \mid \exists x \;\; x \in X \text{ and } x \rho b \} \subseteq B$$

that is, for $b \in B$, we have

$$b \in \rho(X) \overset{def}{\Longleftrightarrow} \exists x \;\; x \in X \text{ and } x \rho b$$

(or formally $\exists x \in X$ s.t. $x \rho b$)

Particular case: if $X = \{x\}$, we denote $\rho\langle x\rangle := \rho(\{x\}) = \{b \in B \mid x\rho b\}$

Proposition (the behaviour of the section w.r.t. composition).
Let $\rho = (A, B, R)$, $\sigma = (C, D, S)$, and let $X \subseteq A$.
Then:
$$\boxed{(\sigma \circ \rho)(X) = \sigma\big(C \cap \rho(X)\big)}$$

In particular, if $\rho(X) \subseteq C$, then $(\sigma \circ \rho)(X) = \sigma(\rho(X))$

Proof. Both sets are subsets of D. Let $d \in D$. We have:

$$d \in (\sigma\circ\rho)(X) \overset{\text{by def}}{\iff} \exists x \;\; x\in X \text{ and } x(\sigma\circ\rho)d$$

$$\overset{\text{def of } \circ}{\iff} \exists x \;\; x\in X \text{ and } \exists y \;\; y\in B\cap C \text{ and } x\rho y \text{ and } y\sigma d$$

$$\overset{(*)}{\iff} \exists y \;\; y\in B\cap C \text{ and } \exists x \;\; x\in X \text{ and } x\rho y \text{ and } y\sigma d$$

$$\underset{\text{sect}}{\overset{\text{def of}}{\iff}} \exists y \;\; y\in B\cap C \text{ and } y\in\rho(X) \text{ and } y\sigma d$$

$$\underset{\cap}{\overset{\text{def}}{\iff}} \exists y \;\; y\in B\cap \underbrace{C\cap\rho(X)} \text{ and } y\sigma d$$

$$\underset{\text{sect}}{\overset{\text{def}}{\iff}} d\in\sigma(C\cap\rho(X))$$

(*) We have used the foll. tautologies:

$$(A\wedge B)\wedge C \iff A\wedge(B\wedge C) \qquad (\text{assoc})$$

$$A\wedge B \iff B\wedge A \qquad (\text{comm})$$

$$\exists x\exists y\; A(x,y) \iff \exists y\exists x\; A(x,y) \qquad (2.3.5\ (1)\ \text{p.}17)$$

$$\exists x\,(A\wedge C(x)) \iff A\wedge\exists x\, C(x) \qquad (2.3.5\ (2)\quad)$$

Functions (as particular cases of relations)

Def. Let $f=(A,B,F)$ be a relation, where $F\subseteq A\times B$.

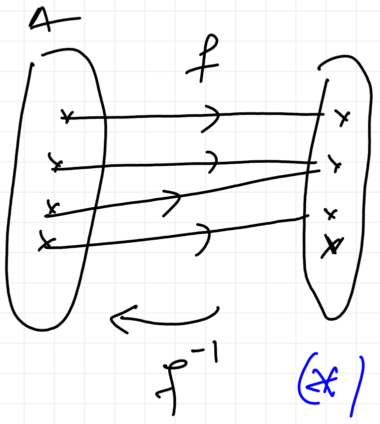
We say that f is a function if $\forall x\in A$ the section $f\langle x\rangle = \{b\in B \mid x f b\}$ contains exactly one element, i.e. $|f\langle x\rangle| = 1$.

We denote: $f\langle x\rangle = \{f(x)\}$ Euler's notation

$$f: A\to B, \quad \begin{cases} A \xrightarrow{f} B \\ x \underset{\text{maps to}}{\longmapsto} f(x) \end{cases}$$

__Example__. In the previous example, ρ is not a function because $|\rho\langle a\rangle| = 2$, and also because $|\rho\langle c\rangle| = \emptyset$.

i.e. $|\rho\langle c\rangle| = 0$



f is a function

but the inverse relation f^{-1} is __not__ a function.

Remarks and examples

1) the equality relation $1_A = (A, A, \Delta_A)$ is a function because $1_A\langle a\rangle = \{a\}$ $\forall a \in A$.

We call $1_A : A \longrightarrow A$, $1_A(a) = a$

the __identity function of A__

2) • Assume that $A = \emptyset$. Then the relation $(\emptyset, B, \emptyset)$ is a function

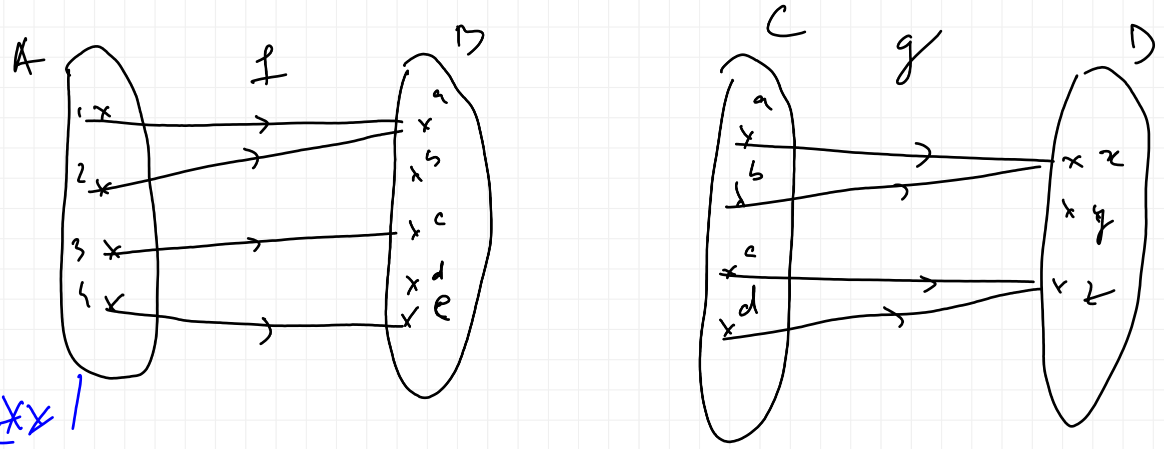
• Assume that $A \neq \emptyset$ and $B = \emptyset$. Then the relation $(A, \emptyset, \emptyset)$ is __not__ a function.

3). __Equality of functions__: the functions $f : A \to B$ and $g : C \to D$ are equal $(f = g)$ iff

$$\underbrace{f = (A, B, F)} = \underbrace{g = (C, D, G)} \Longleftrightarrow \begin{cases} A = C & \text{(same domain)} \\ B = D & \text{(same codomain)} \\ F = G \Leftrightarrow f(a) = g(a) \ \forall a \in A \end{cases}$$

$$F = \{(a, f(a)) \mid a \in A\}$$

4) Composition of functions



(**)

In this example, the composed relation

$$g\circ f = (A, D, G\circ F) \text{ is } \underline{\text{not}} \text{ a function}$$

because $(g\circ f)\langle 4\rangle = \emptyset$

It is clear, that, in general, $g\circ f$ is a function

$$\Longleftarrow \quad f(A) \subseteq C$$

5) Properties of the composition of function

(a) the identity function is a neutral element

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[rr, bend right, "f\circ 1_A"'] & A \arrow[r, "f"] \arrow[rr, bend right, "1_B\circ f"'] & B \arrow[r, "1_B"] & B \end{tikzcd}$$

$$f\circ 1_A = 1_B\circ f = f$$

(b) the comp. of function is associative:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] \arrow[rrr, bend right=60, "h\circ(g\circ f)"'] & B \arrow[r, "g"] \arrow[rr, bend right, "h\circ g"'] & C \arrow[r, "h"] & D \end{tikzcd}$$

$(h\circ g)\circ f$

We have $h\circ(g\circ f) = (h\circ g)\circ f$

Proof: $\forall a\in A$ we have

$$(h\circ(g\circ f))(a) = h((g\circ f)(a)) = h(g(f(a)))$$

$$= (h\circ g)(f(a)) = ((h\circ g)\circ f)(a)$$

c). Let $f: A \to B$ be a function. We may consider the inverse relation $f^{-1} = (B, A, \bar{F}^{-1})$.

The f^{-1} is <u>not</u> a function in general! (see ex. (*))

Image and inverse image (pre image)

(particular case of the notion of a relation)

Let $f: A \to B$ be a function.

- If $X \subseteq A$, then $f(X) \overset{def}{=} \{ f(x) \mid x \in X \}$

(the <u>image of X under f</u>)

i.e. if $b \in B$, then

$b \in f(X) \overset{def}{\iff} \exists x \in X$ s.t. $b = f(x)$

$(\exists x \;\; x \in X$ and $b = f(x))$

In example (**) $f(\{1,2,3\}) = \{a, c\}$

particular case. $\operatorname{Im} f = f(A) = \{ f(a) \mid a \in A \}$.

the image of f

In ex. (**) $\operatorname{Im} f = \{a, c, e\}$

- If $Y \subseteq B$, then $f^{-1}(Y) \overset{def}{=} \{ a \in A \mid f(a) \in Y \}$.

In ex. (**). $f^{-1}(a) = \{1, 2\}$

$f^{-1}(\{b, c\}) = \{3\}$

$f^{-1}(b) = \emptyset$

$f^{-1}(B) = A$ always!

Commutative diagrams

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C \arrow[ur, "h"'] & \end{tikzcd}$$

this diagram is comm $\Longleftrightarrow$ $f = h \circ g$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "k"] \\ C \arrow[r, "h"'] & D \end{tikzcd}$$

is comm $\Longleftrightarrow$ $k \circ f = h \circ g$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C \arrow[r, "h"'] & D \arrow[u, "k"'] \end{tikzcd}$$

is comm $\Longleftrightarrow$ $k \circ h \circ g = f$

Families of elements and sets

example: consider the sequence (pair)

string = coordi

$(a_n)_{n \in \mathbb{N}}$, $a_n = (-1)^n$

$1, -1, 1, -1, \dots$

the set of elements of the sequence is $\{1, -1\}$

Def. A family of elements of a set A indexed by the index set I is a function $f : I \to A$.

Not $f(i) = a_i$, $f = (a_i)_{i \in I}$.

Def A family of sets (subsets of U) indexed by I is a function $f : I \longrightarrow \mathcal{P}(U)$

Not $f(i) = A_i$, $f = (A_i)_{i \in I}$.

Operations with families of sets:

- $\bigcup_{i \in I} A_i \overset{\text{def}}{=} \{ x \in U \mid \exists i \ i \in I \text{ and } x \in A_i \}$
- $\bigcap_{i \in I} A_i \overset{\text{def}}{=} \{ x \in U \mid \forall i \ i \in I \text{ and } x \in A_i \}$

Particular case: assume $I = \emptyset$.

$\bigcup_{i \in \emptyset} A_i = \emptyset$ $\qquad$ $\bigcap_{i \in \emptyset} A_i = U$

Injective, surjective and bijective functions

one-to-one $\qquad$ onto.

$\boxed{p \to q \Longleftrightarrow \neg q \to \neg p}$
contraposition

Def 1 A function $f : A \to B$ is injective if

$\forall x_1, x_2 \in A \quad x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2)$

(or equivalently) $\forall x_1, x_2 \in A \quad f(x_1) = f(x_2) \Longrightarrow x_1 = x_2$

Rem f is not inj $\Longleftrightarrow \exists x_1, x_2 \in A$ s.t. $x_1 \neq x_2$ and $f(x_1) = f(x_2)$

$$\boxed{\begin{aligned} \neg(p \to q) &\Longleftrightarrow \neg(\neg p \vee q) \\ &\Longleftrightarrow p \wedge \neg q \end{aligned}}$$

Def 2 A function $f: A \to B$ is <u>surjective</u> if

$\forall y \in B \ \exists x \in A$ s.t. $y = f(x)$

or equivalently, $\operatorname{Im} f = B$.

$\operatorname{Im} f = f(A) = \{ f(x) \mid x \in A \}$

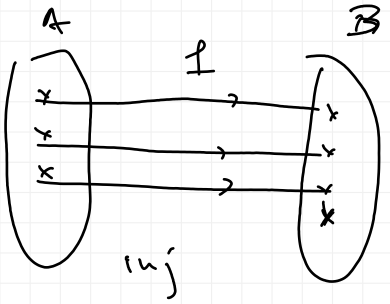
Rm f is <u>not</u> surjective $\iff \exists y \in B \ \forall x \in A \ \ y \neq f(x)$

Def 3 A function $f: A \to B$ is <u>bijective</u> if f is injective and surjective,

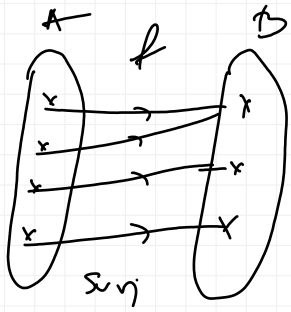
i.e. $\forall y \in B \ \exists! \, x \in A$ s.t. $y = f(x)$

(unique)

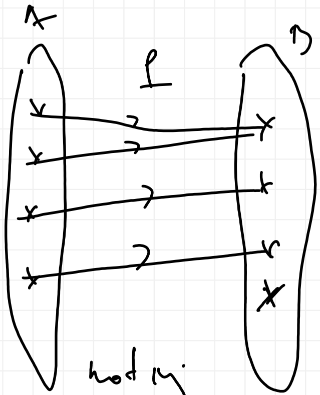
Example



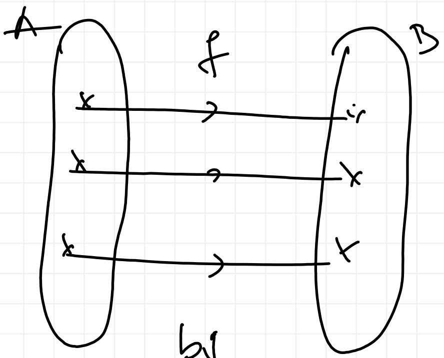
inj
not surj



surj
not inj



not inj
not surj



bij

Georg Cantor

$\boxed{\text{Homework: ex. 31 – 41}}$