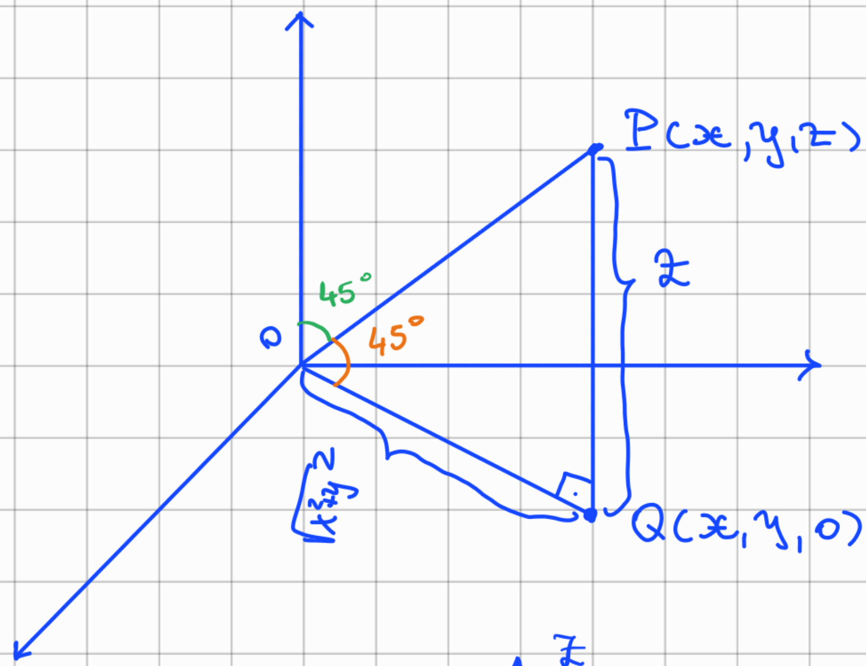
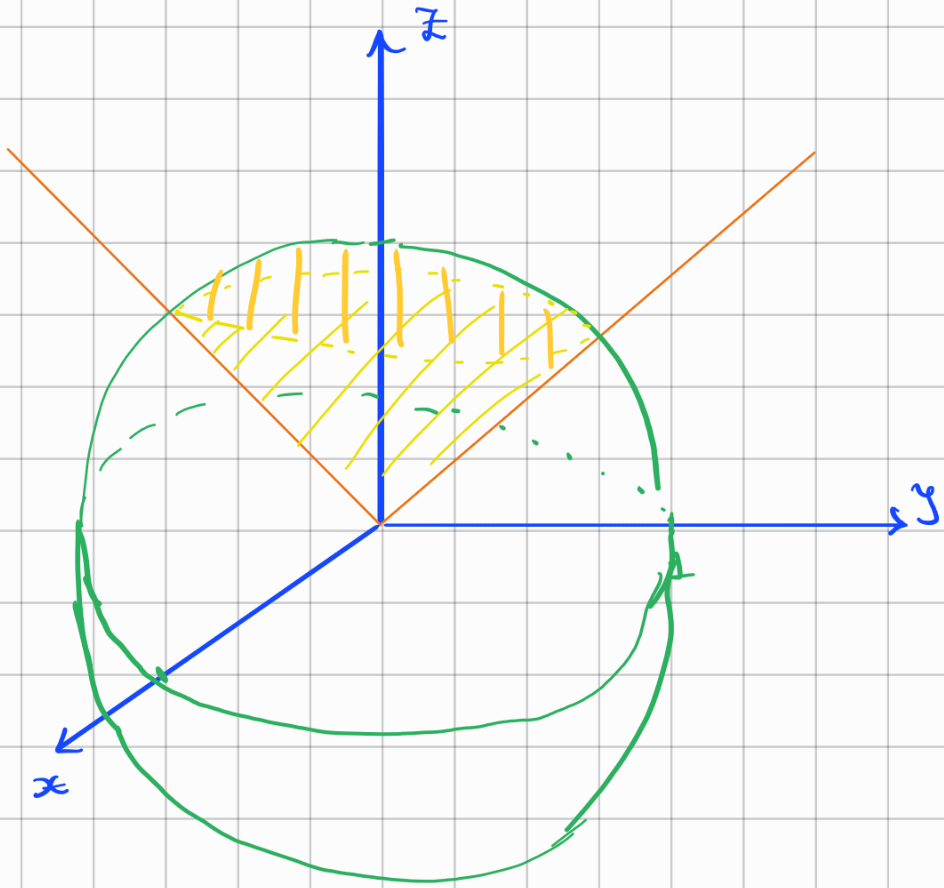
Seminar 10

15.05.2025

1 Find the center of mass of the homogeneus solid bounded by the cone $z=\sqrt{x^2+y^2}$ and by the sphere $x^2+y^2+z^2=1$.

Sol: Let $P(x,y,z)$ s.t. $z=\sqrt{x^2+y^2}$





Since the z-axis is a symmetry axis $\Rightarrow G \in Oz$

$\Rightarrow x_G = y_G = 0$

$$z_G = \frac{\iiint_A z \, dx\,dy\,dz}{vol(A)}$$

$$vol(A) = \iiint_A dx\,dy\,dz = \int_{\rho=0}^{\rho=1} \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\frac{\pi}{4}} \rho^2 \sin\varphi \, d\rho\,d\theta\,d\varphi$$

$x = \rho \cdot \sin\varphi \cos\theta$

$y = \rho \cdot \sin\varphi \cdot \sin\theta$

$z = \rho \cdot \cos\varphi$

$\rho \in [0; 1]$

$\varphi \in [0; \frac{\pi}{4}]$

$\theta \in [0; 2\pi]$

$$= \int_0^1 \rho^2 \, d\rho \cdot \int_0^{2\pi} d\theta \cdot \int_0^{\frac{\pi}{4}} \sin\varphi \, d\varphi$$

$$= \frac{1}{3} \cdot 2\pi \cdot (-\cos\varphi)\Big|_0^{\frac{\pi}{4}}$$

$$= \frac{1}{3} \cdot 2\pi \left(1 - \frac{\sqrt{2}}{2}\right) = \frac{2\pi}{3}\left(1 - \frac{\sqrt{2}}{2}\right)$$

$$\iiint_A z \, dx\,dy\,dz = \int_{\rho=0}^{\rho=1} \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\frac{\pi}{4}} \rho \cdot \cos\varphi \cdot \rho^2 \sin\varphi \, d\rho\,d\theta\,d\varphi$$

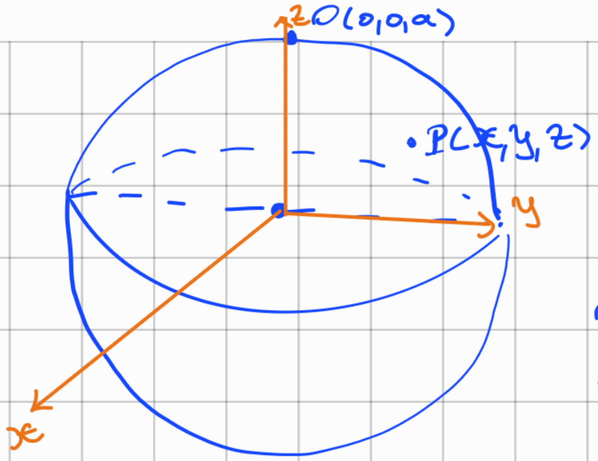
$$= \int_0^1 \rho^3 \, d\rho \cdot \int_0^{2\pi} d\theta \cdot \int_0^{\frac{\pi}{4}} \sin\varphi \cdot \cos\varphi \, d\varphi$$

$$= \frac{1}{4} \cdot 2\pi \cdot \left(\frac{\sin^2\varphi}{2} \Big|_0^{\frac{\pi}{4}} \right) = \frac{\pi}{2} \cdot \frac{\left(\frac{\sqrt{2}}{2}\right)^2}{2} = \frac{\pi}{2} \cdot \frac{\frac{2}{4}}{2} = \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{\pi}{8}$$

$$z = \frac{\cancel{\pi}}{8} \cdot \frac{3}{2\cancel{\pi}\left(\frac{2}{2-\sqrt{2}}\right)} = \frac{3}{16 - 8\sqrt{2}}$$

$$\Rightarrow G = \left(0, 0, \frac{3}{16 - 8\sqrt{2}}\right)$$

$\boxed{2}$ Find the mass of a ball of radius a knowing that its density is directly proportional with the dist. from a fixed point O, lying on the boundry of the ball.



$$\rho(P) = k \cdot OP \qquad [k] = [g/cm^4]$$

We choose a cartesian system with the origin at the center of the ball and s.t. O belongs to Oz

$$\rho(x,y,z) = k \cdot \sqrt{x^2+y^2+(z-a)^2}$$

$$m = \iiint_A \rho(x,y,z)\, dx\,dy\,dz = k \cdot \iiint_A \sqrt{x^2+y^2+(z-a)^2}\, dx\,dy\,dz$$

$x = \rho \cdot \sin\varphi \cdot \cos\theta$

$y = \rho \cdot \sin\varphi \cdot \sin\theta$

$z = \rho \cdot \cos\varphi$

$\rho \in [0;a]$

$\varphi \in [0;\pi]$

$\theta \in [0;2\pi]$

$$= k \cdot \int_{\rho=0}^{\rho=a} \int_{\varphi=0}^{\varphi=\pi} \int_{\theta=0}^{\theta=2\pi} \sqrt{\rho^2 \cdot \sin^2\varphi + (\rho \cdot \cos\varphi - a)^2} \cdot \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$= k \cdot \underbrace{\int_{\theta=0}^{\theta\, 2\pi} d\theta}_{2\pi} \cdot \int_{\rho=0}^{\rho=a} \int_{\varphi=0}^{\varphi=2\pi} \rho^2 \cdot \sin\varphi \cdot \sqrt{\rho^2 - 2a\rho\cos\varphi + a^2}\, d\rho\, d\varphi$$

$$= 2\pi \cdot k \cdot \int_0^a \left(\int_0^{2\pi} \rho^2 \cdot \sin\varphi \cdot \sqrt{\rho^2 - 2a\rho\cos\varphi + a^2}\, d\varphi \right) d\rho$$

$$t = \sqrt{\rho^2 - 2a\rho\cos\varphi + a^2}$$

$$t^2 = \rho^2 - 2a\rho\cos\varphi + a^2 \Rightarrow \cancel{2}t\, dt = \cancel{2}a\rho\sin\varphi\, d\varphi$$

$$m = 2\pi k \int_{\rho=0}^{\rho=a} \left(\int_{t=|\rho-a|=a-\rho}^{t=a+\rho} \frac{\rho}{a} \cdot t \cdot t\, dt \right) d\rho$$

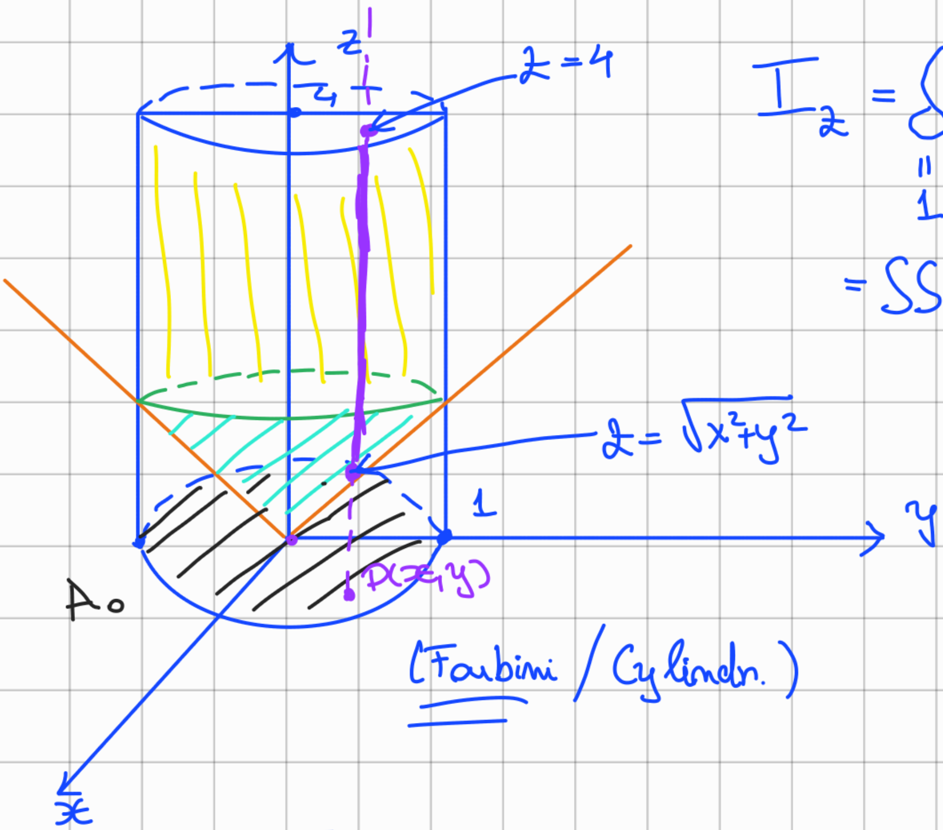
$$= \frac{2k\pi}{a} \int_{\rho=0}^{\rho=a} \left(\rho \cdot \frac{t^3}{3} \Big|_{t=a-\rho}^{t=a+\rho} \right) d\rho$$

$$= \frac{2k\pi}{3a} \int_0^a \rho \cdot (\cancel{a^3} + \rho^3 + 3a^2\rho + \cancel{3a\rho^2} - \cancel{a^3} + \rho^3 + 3a^2\rho - \cancel{3a\rho^2})\, d\rho$$

comm. fact 2

$$= \frac{4k\pi}{3a} \int_0^a \rho^4 + 3a^2\rho^2\, d\rho = \ldots = \frac{8\pi}{5} \cdot k \cdot a^4$$

3 Compute the moment of inertia w.r.t. Oz of a homogeneous solid of density 1, bounded by the planes $z=0$ and $z=4$, lying inside the cone $z=\sqrt{x^2+y^2}$ and inside the cylinder $x^2+y^2=1$.



$$I_z = \underset{\overset{\parallel}{1}}{\delta} \cdot \iiint_A (x^2+y^2)\,dx\,dy\,dz$$

$$= \iiint_A (x^2+y^2)\,dx\,dy\,dz$$

(Fubini / Cylindr.)

$$\begin{cases} z=\sqrt{x^2+y^2} \\ x^2+y^2=1 \end{cases} \Rightarrow \begin{cases} x^2+y^2=1 \\ z=1 \end{cases}$$

$$\iiint_A (x^2+y^2)\,dx\,dy\,dz = \iint_{A_0}\left(\int_{z=\sqrt{x^2+y^2}}^{z=4}(x^2+y^2)\,dz\right)dx\,dy$$

$$= \iint_{A_0}(x^2+y^2)\cdot\left(4-\sqrt{x^2+y^2}\right)dx\,dy$$

$x=\rho\cdot\cos\theta$

$y=\rho\cdot\sin\theta$

$\rho\in[0;1]$

$\theta\in[0;2\pi]$

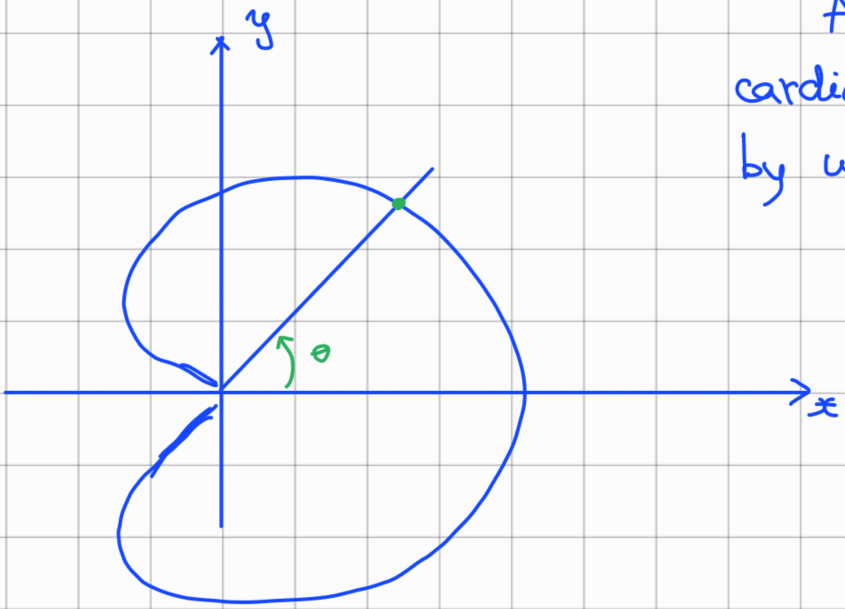
$$I_z = \int_{\rho=0}^{\rho=1}\int_{\theta=0}^{\theta=2\pi}\rho^2\cdot(4-\rho)\,\rho\,d\rho\,d\theta$$

$$= \int_0^1 \rho^3(4-\rho)\,d\rho\cdot\int_0^{2\pi}d\theta$$

$$=\left(4\int_0^1\rho^3-\int_0^1\rho^4\,d\rho\right)\cdot 2\pi = \left(4\cdot\frac{1}{4}-\frac{1}{5}\right)2\pi$$

$$=\frac{4}{5}\cdot 2\pi = \frac{8\pi}{5}$$

$\boxed{5}$ Cardioid is the plane curve whose cartesian implicit eq. is $x^2+y^2=2a\cdot(x+\sqrt{x^2+y^2})$, $a>0$. Calculate its arclength.



A parametrization of the cardioid can be obtained by using polar coord.

$$\left.\begin{array}{l} x=\rho\cos\theta \\ y=\rho\sin\theta \end{array}\right\} \Rightarrow \rho^2=2a(\rho\cos\theta+\rho)$$

$$\underline{\left| \rho=2a(\cos\theta+1) \right|}$$ the polar eq. of the cardioid

$$\gamma:\begin{cases} x=2a(1+\cos\theta)\cdot\cos\theta \\ y=2a(1+\cos\theta)\cdot\sin\theta \end{cases}, \theta\in[0;2\pi]$$

$$l(\gamma)=\int_0^{2\pi}\|\gamma'(\theta)\|\,d\theta$$

$$\gamma(\theta)=\left(2a(\cos\theta+\cos^2\theta),\ 2a(\sin\theta+\sin\theta\cos\theta)\right)$$

$$\gamma'(\theta)=\left(2a(-\sin\theta-\underbrace{2\cos\theta\cdot\sin\theta}_{\sin 2\theta}),\ 2a(\cos\theta+\underbrace{\cos^2\theta-\sin^2\theta}_{\cos 2\theta})\right)$$

$$=\left(-2a(\sin\theta+\sin 2\theta),\ 2a(\cos\theta+\cos 2\theta)\right)$$

$$\|\gamma'(\theta)\|^2=4a^2(\sin^2\theta+2\sin\theta\cdot\sin 2\theta+\sin^2 2\theta)+$$
$$4a^2(\cos^2\theta+2\cos\theta\cdot\cos 2\theta+\cos^2 2\theta)$$

$$=4a^2(2+2(\underbrace{\sin\theta\sin 2\theta+\cos\theta\cdot\cos 2\theta}_{\cos(2\theta-\theta)}))$$

$$= 8a^2(\underbrace{1+\cos\theta}_{2\cos^2(\frac{\theta}{2})}) = 16a^2\cdot\cos^2(\tfrac{\theta}{2})$$

$$\Rightarrow \|\gamma'(\theta)\| = 4a\left|\cos(\tfrac{\theta}{2})\right|$$

$$\ell(\gamma) = \int_0^{2\pi} 4a\left|\cos(\tfrac{\theta}{2})\right| d\theta = 4a\left(\int_0^{\pi}\cos\tfrac{\theta}{2}\,d\theta - \int_{\pi}^{2\pi}\cos\tfrac{\theta}{2}\,d\theta\right)$$

$$= 4a\left(2\cdot\sin\tfrac{\theta}{2}\Big|_0^{\pi} - 2\sin\tfrac{\theta}{2}\Big|_{\pi}^{2\pi}\right) = \underline{\underline{16a}}$$

$\boxed{6}$ Compute the arc length of the parametrized path
$\gamma:[0;2\pi]\to\mathbb{R}^3$, $\gamma(t) = (\cos^3 t, \sin^3 t, \cos 2t)$

<u>Sol</u>: $\ell(\gamma) = \int_0^{2\pi} \|\gamma'(t)\|\,dt$

$$\gamma'(t) = (-3\cos^2 t\cdot\sin t,\ 3\sin^2 t\cdot\cos t,\ -2\cdot\sin 2t)$$

$$\|\gamma'(t)\|^2 = 9\cos^4 t\cdot\sin^2 t + 9\sin^4 t\cdot\cos^2 t + 4\sin^2 2t$$

$$= 9\cos^2 t\cdot\sin^2 t(\cos^2 t + \sin^2 t) + 4\sin^2 2t$$

$$= 9\cos^2 t\cdot\sin^2 t + 4\cdot 4\cdot\sin^2 t\cdot\cos^2 t$$

$$= 25\cos^2 t\cdot\sin^2 t$$

$$\|\gamma'(t)\| = 5|\cos t\cdot\sin t|$$

$$\ell(\gamma(t)) = 5\int_0^{2\pi} |\sin t\cdot\cos t|\,dt$$

The function $f(t) = |\sin t\cdot\cos t|$ is periodic with $T = \frac{\pi}{2}$

$$\ell(\gamma) = 5\cdot 4\cdot\int_0^{\frac{\pi}{2}} \sin t\cdot\cos t\,dt = 20\cdot\frac{\sin^2 t}{2}\Big|_0^{\frac{\pi}{2}} = \underline{\underline{10}}$$

$\boxed{7}$ Compute $I = \int_\gamma y^2\,ds$, where $\gamma:[0;2\pi]\to\mathbb{R}^2$,
$\gamma(t) = (a(t-\sin t), a(1-\cos t))$, $a>0$

$$I = \int_0^{2\pi} a^2(1-2\cos t+\cos^2 t)\cdot\|\gamma'(t)\|\,dt$$

$$= \int_0^{2\pi} a^2 \overbrace{(1-2\cos t+\cos^2 t)}^{(1-\cos t)^2=(2\cdot\sin^2\frac{t}{2})^2}\cdot 2a\sin\frac{t}{2}\,dt$$

$$= 2a^3\int_0^{2\pi} 4\sin^5\frac{t}{2}\,dt = 8a^3\int_0^{2\pi}\sin^5\frac{t}{2}\,dt$$

$$\frac{t}{2} = u \Rightarrow 8a^3\int_0^{\pi} \sin^5 u\cdot 2\,du = 16a^3\int_0^{\pi} \sin^4 u\cdot\sin u\,du$$

$$= 16a^3\int_0^{\pi} (1-\cos^2 u)^2\cdot \sin u\,du$$

$$\cos u = v \Rightarrow -16a^3\cdot\int_1^{-1}(1-v^2)^2\cdot dv = 16a^3\int_{-1}^{1} (1-v^2)^2\,dv$$

$$= 16a^3\int_{-1}^{1}(1-2v^2+v^4)\,dv = \dots$$