Seminar 2

1. Let " $*$ " be the operation on $\mathbb{R}$ defined by:

$$x * y = xy - 5x - 5y + 30.$$

Is $(\mathbb{R}, *)$ a group? What about $(\mathbb{R} \setminus \{5\}, *)$?

2. Let $n \in \mathbb{N}$, $n \geq 2$. Show that the set

$$GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) \neq 0\}$$

is a stable subset of the monoid $(M_n(\mathbb{R}), \cdot)$ and $(GL_n(\mathbb{R}), \cdot)$ is a group.

3. Let $n \in \mathbb{N}^*$. Show that the set

$$U_n = \{z \in \mathbb{C} \mid z^n = 1\}$$

is a stable subset of the group $(\mathbb{C}^*, \cdot)$, $(U_n, \cdot)$ is an abelian group, and determine the elements of U_n.

4. Let $n \in \mathbb{N}$ and $\mathbb{Z}_n = \{\widehat{x} \mid x \in \mathbb{Z}\}$, where $\widehat{x} = x + n\mathbb{Z} = \{x + nk \mid k \in \mathbb{Z}\}$. Let "+" be the operation on $\mathbb{Z}_n$ defined by:

$$\widehat{x} + \widehat{y} = \widehat{x + y}, \quad \forall\, \widehat{x}, \widehat{y} \in \mathbb{Z}_n.$$

Show that $(\mathbb{Z}_n, +)$ is an abelian group and determine its cardinal (discussion on n).

5. Let $M \neq \emptyset$ be a set and

$$S_M = \{f : M \to M \mid f \text{ bijective}\}.$$

(i) Show that $(S_M, \circ)$ is a group.

(ii) If $|M| = n \in \mathbb{N}^*$, then we denote S_M by S_n. Determine the operation table for the group $(S_3, \circ)$.

6. Determine the operation table for the dihedral group $(D_3, \cdot)$ of rotations and symmetries of an equilateral triangle.

7. Determine the operation table for the dihedral group $(D_4, \cdot)$ of rotations and symmetries of a square.

8. Let $(G, \cdot)$ and $(G', \cdot)$ be groups with identity elements e and e' respectively. Let " $\cdot$ " be the operation on $G \times G'$ defined by:

$$(g_1, g_1') \cdot (g_2, g_2') = (g_1 \cdot g_2, g_1' \cdot g_2'), \quad \forall (g_1, g_1'), (g_2, g_2') \in G \times G'.$$

Show that $(G \times G', \cdot)$ is a group, called the *direct product* of the groups G and G'.

9. Determine the group of invertible elements of the monoids $(\mathbb{N}, +)$, $(\mathbb{N}, \cdot)$, $(\mathbb{Z}, \cdot)$, $(\mathbb{Q}, \cdot)$, $(\mathbb{R}, \cdot)$, $(\mathbb{C}, \cdot)$, $(M_n(\mathbb{R}), \cdot)$ $(n \in \mathbb{N},\ n \geq 2)$ and $(M^M, \circ)$, where $M \neq \emptyset$ is a set and M^M denotes the set of all functions $f : M \to M$.

10. Let $(G, \cdot)$ be a group. Show that:

(i) G is abelian $\iff \forall x, y \in G,\ (xy)^2 = x^2y^2$.

(ii) $\forall x \in G,\ x^2 = 1 \implies G$ is abelian.