Seminar 6

"Nu dăm cu tunul după muște"

03.04.2025

$\boxed{1.}$ Let $f: \mathbb{R}^3 \to \mathbb{R}^2$ be a differ. fct. on $\mathbb{R}^3$ and let $F: \mathbb{R}^2 \to \mathbb{R}^2$, $F(x,y) = f(\cos x + \sin y, \sin x + \cos y, e^{x-y})$

a) Prove that if f is of class C^1 on $\mathbb{R}^3$, then F is of class C^1 on $\mathbb{R}^2$.

b) If $J(f)(1,1,1) = \begin{pmatrix} 1 & 3 & 4 \\ 2 & -1 & 3 \end{pmatrix}$, determine $dF(\frac{\pi}{2}, \frac{\pi}{2})$

A function is said to be **of class C^1 on an open set** if all it's p.d. exist and are cont. on that set.

Solution: a) $F = f \circ g$ where $g(x,y) = (\underbrace{\cos x + \sin y}_{u(x,y)}, \underbrace{\sin x + \cos y}_{v(x,y)}, \underbrace{e^{x-y}}_{w(x,y)})$

$f = f(u,v,w)$ $\qquad g: \mathbb{R}^2 \to \mathbb{R}^3$

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial x}$$

$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial y}$$

$$\frac{\partial F}{\partial x}(x,y) = \frac{\partial f}{\partial u}(g(x,y)) \cdot \overbrace{\frac{\partial u}{\partial x}}^{-\sin x}(x,y) + \frac{\partial f}{\partial v}(g(x,y)) \cdot \overbrace{\frac{\partial v}{\partial x}}^{\cos x}(x,y) + \frac{\partial f}{\partial w}(g(x,y)) \cdot \underbrace{\frac{\partial w}{\partial x}(x,y)}_{e^{x-y}}$$

$$\frac{\partial F}{\partial x}(x,y) = -\sin x \cdot \frac{\partial f}{\partial u}(\underbrace{g(x,y)}_{(\cos x + \sin y,\ \sin x + \cos y,\ e^{x-y})}) + \cos x \cdot \frac{\partial f}{\partial v}(g(x,y)) + e^{x-y} \cdot \frac{\partial f}{\partial w}(g(x,y))$$

?

f is of class C^1 in $\mathbb{R}^3 \Rightarrow \frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, \frac{\partial f}{\partial w}$ are cont. on $\mathbb{R}^3$

$\Rightarrow \frac{\partial F}{\partial x}$ is cont. on $\mathbb{R}^2$

Analogously, it can be proved that $\frac{\partial F}{\partial y}$ is cont. on $\mathbb{R}^2$ $\Bigg\}$

$\Rightarrow F$ is of class C^1 on $\mathbb{R}^2$

b) $J(f)(1,1,1) = \begin{pmatrix} 1 & 3 & 4 \\ 2 & -1 & 3 \end{pmatrix}$

$dF(\frac{\pi}{2}; \frac{\pi}{2}) \in L(\mathbb{R}^2, \mathbb{R}^2)$

$[dF(\frac{\pi}{2}; \frac{\pi}{2})] = J(F)(\frac{\pi}{2}; \frac{\pi}{2}) = J(f \circ g)(\frac{\pi}{2}; \frac{\pi}{2})$

$= J(f)(g(\frac{\pi}{2}; \frac{\pi}{2})) \cdot J(g)(\frac{\pi}{2}; \frac{\pi}{2}) = J(f)(1,1,1) \cdot J(g)(\frac{\pi}{2}, \frac{\pi}{2})$

$$J(g)(x,y) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{pmatrix} = \begin{pmatrix} -\sin x & \cos y \\ \cos x & -\sin y \\ e^{x-y} & -e^{x-y} \end{pmatrix}$$

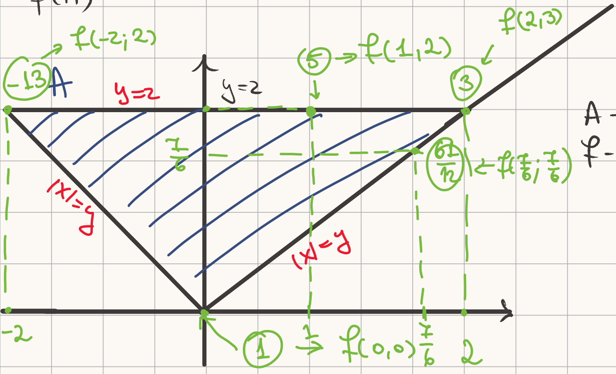
$$J(g)(\frac{\pi}{2}; \frac{\pi}{2}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & -1 \end{pmatrix}$$

$$[dF(\frac{\pi}{2}; \frac{\pi}{2})] = \begin{pmatrix} 1 & 3 & 4 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -1+4 & -3-4 \\ -2+3 & 1-3 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -7 \\ 1 & -2 \end{pmatrix}$$

$$dF(\frac{\pi}{2}; \frac{\pi}{2})(h_1, h_2) = \begin{pmatrix} 3 & -7 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 3h_1 - 7h_2 \\ h_1 - 2h_2 \end{pmatrix}$$

$\boxed{2}$ Let $A = \{(x,y) \in \mathbb{R}^2 \mid |x| \le y \le 2\}$ and let $f: A \to \mathbb{R}$, $f(x,y) = 1 + 4x + 3y - 2x^2 - y^2$. Determine $\min f(A)$, $\max f(A)$, $f(A)$



A - compact, f - cont $\Big\}$ T.W. $\Rightarrow$ f is cont. and reaches its bounds

Let $m := \min f(A)$

$M := \max f(A)$

Let $C = \{(x,y) \in \operatorname{int} A \mid \nabla f(x,y) = (0,0)\}$

$$\nabla f(x,y) = (0,0) \iff \begin{cases} \frac{\partial f}{\partial x}(x,y) = 4 - 4x = 0 \\ \frac{\partial f}{\partial y}(x,y) = 3 - 2y = 0 \end{cases} \iff \begin{cases} x = 1 \\ y = \frac{3}{2} \end{cases}$$

$\Rightarrow C = \{(1; \frac{3}{2})\}$

$$\left.\begin{array}{l} \text{Let } m_2 := \min f(C) \\ M_2 := \max f(C) \end{array}\right\} \Rightarrow m_2 = M_2 = f(1; \tfrac{3}{2}) = \frac{21}{4}$$

Let $m_1 := \min f(\operatorname{bd} A)$

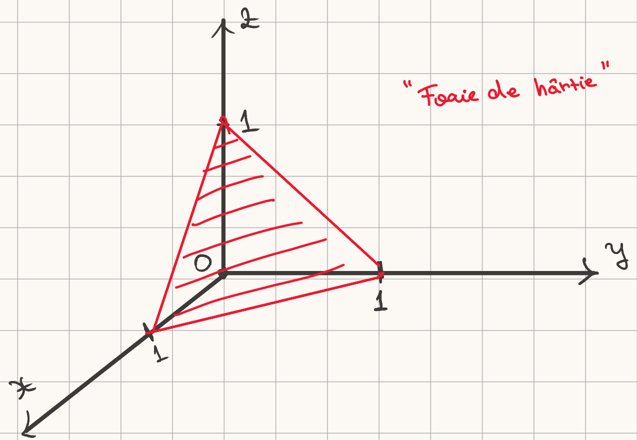
$M_1 := \max f(\operatorname{bd} A)$

$\operatorname{bd} A = \{(x,x) \mid x \in [0;2]\} \cup \{(x,-x) \mid x \in [-2;0]\}$
$\cup \{(x,2) \mid x \in [-2;2]\}$

$\boxed{3.}$ Let $x,y,z \in [0;\infty)$ s.t. $x+y+z=1$. Prove that

$$0 \le xy+yz+2x-2xyz \le \frac{7}{27}$$

<u>Solution:</u> Let $A = \{(x,y,z) \in [0;\infty)^3 \mid x+y+z=1\}$ and let $f: A \to \mathbb{R}$, $f(x,y,z) = xy+yz+2x-2xyz$



$$\left.\begin{array}{l} A\text{-compact} \\ f\text{-cont} \end{array}\right\} \xRightarrow{\text{T.W.}} f \text{ is bounded and reaches its bounds on } A$$

$m := \min f(A)$

$M := \max f(A)$

Unfortunately, $\operatorname{int} A = \emptyset \Rightarrow$ FERMAT TH. DOES NOT APPLY

$z = 1-x-y$

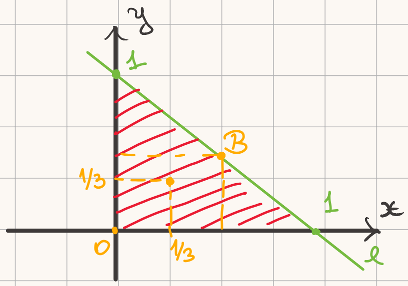
$f(x,y,1-x-y) = xy+y(1-x-y)+x(1-x-y)-2xy(1-x-y)$

$= \cancel{xy}+y-\cancel{xy}-y^2+x-x^2-xy-2xy+2x^2y+2xy^2$

$= \underbrace{2x^2y+2xy^2-x^2-y^2-3xy+x+y}_{g(x,y)}$

$1-x-y \ge 0$

$g: B \to \mathbb{R}$, wher $B = \{(x,y) \in [0;\infty)^2 \mid x+y \le 1\}$



$m = \min g(B)$

$M = \max g(B)$

Let $m_1 = \min g(\mathrm{bd}\,B)$

$M_1 = \max g(\mathrm{bd}\,B)$

$x+y=1$

$\boxed{y = 1-x}$

$\mathrm{bd}\,B = \{\underbrace{(x,0)}_{Ox} \mid x \in [0;1]\} \cup \{\underbrace{(0,x)}_{Oy} \mid x \in [0;1]\} \cup \{\underbrace{(x,1-x)}_{l} \mid x \in [0;1]\}$

(bc. of symmetry of f and g)

- $g(x,0) = g(0,x) = g(x,1-x) = -x^2+x = h(x)$ (put either of them in g(x))

x	0		$\frac{1}{2}$		1
$h(x)$	0	$\nearrow$	$\frac{1}{4}$	$\searrow$	0

$\Rightarrow m_1 = 0$

$\boxed{M_1 = \frac{1}{4}}$

Let $C = \{(x,y) \in \mathrm{int}\,B \mid \nabla g(x,y) = (0,0)\}$

Let $m_2 := \min f(C)$

$M_2 := \max f(C)$

$$\nabla g(x,y) = (0,0) \iff \begin{cases} \frac{\partial g}{\partial x}(x,y) = 4xy + 2y^2 - 2x - 3y + 1 = 0 \\ \frac{\partial g}{\partial y}(x,y) = 2x^2 + 4xy - 2y - 3x + 1 = 0 \end{cases}$$

$$-2y^2 + x - y - 2x^2 = 0 \iff (x-y) - 2(x-y)(x+y) = 0$$

$$\iff (x-y)(1-2(x+y)) = 0$$

$\textcircled{I}$ $x = y$

$$6x^2 - 5x + 1 = 0 \Rightarrow x_1 = \frac{5 + \sqrt{1}}{12} = \frac{1}{2} \; ; \; y_1 = \frac{1}{2}$$

But $\left(\frac{1}{2}; \frac{1}{2}\right) \notin$ int B

$$\Rightarrow x_2 = \frac{1}{3} \; ; \; y_2 = \frac{1}{3}$$

$\left(\frac{1}{3}; \frac{1}{3}\right) \in$ int B

$\textcircled{I}$

$$\left. \begin{array}{l} x + y = \frac{1}{2} \\ y = \frac{1}{2} - x \end{array} \right\} \Rightarrow \begin{array}{l} x_1 = 0 \; ; \; y_1 = \frac{1}{2} \\ x_2 = \frac{1}{2} \; ; \; y_2 = 0 \end{array}$$

$\notin$ int B

$$\Rightarrow C = \left\{ \left(\frac{1}{3}; \frac{1}{3}\right) \right\} \Rightarrow m_2 = M_2 = g\left(\frac{1}{3}; \frac{1}{3}\right) = \frac{7}{27}$$

$$m = \min\{\underset{0}{m_1}, \underset{\frac{7}{27}}{m_2}\} = \underline{\underline{0}}$$

$$M = \max\{\underset{\frac{1}{4}}{M_1}, \underset{\frac{7}{27}}{M_2}\} = \underline{\underline{\frac{7}{27}}}$$