Lecture 13

Applications of dynamical system theory

1) Harvesting renewable resources

Suppose that a pop. growth is described by the logistical model

$$\begin{cases} x' = rx\left(1-\frac{x}{K}\right) \\ x(0)=x_0 \end{cases}$$

r – unrestricted growth rate
$r>0$
K – environmental support constant
$K>0$

Problem: for fixed parameters r and K determine the effect of harvesting.

a) Constant harvesting rate

$$\begin{cases} x' = rx\left(1-\frac{x}{K}\right) - h \\ x(0)=x_0 \end{cases}$$

h – the constant rate of harvesting.
$h>0$

$$x' = f(x) \quad \text{where} \quad f(x) = rx\left(1 - \frac{x}{K}\right) - h$$
$$= -\frac{r}{K}x^2 + rx - h$$

Equilibrium points

$$f(x) = 0 \Rightarrow -\frac{r}{K}x^2 + rx - h = 0$$

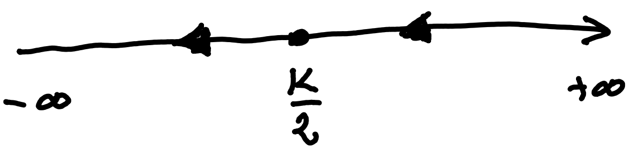
$$\Delta = r^2 - 4 \cdot \frac{r}{K} \cdot h = \underbrace{r}_{>0} \cdot \left(r - \frac{4h}{K}\right)$$

I $\Delta = 0 \Leftrightarrow r - \frac{4h}{K} = 0 \Leftrightarrow \boxed{h = \frac{rK}{4}}$

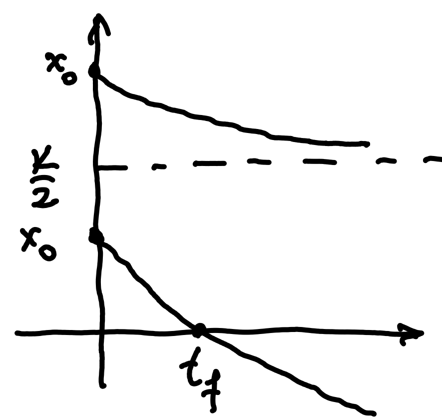
$$x_1 = x_2 = \frac{-r}{2 \cdot \left(-\frac{r}{K}\right)} = \frac{K}{2} \Rightarrow x^* = \frac{K}{2} > 0 \text{ eq. point.}$$

x		$\frac{K}{2}$	
$f(x)$	$- - -$	0	$- - - -$

unstable eq. point



phase portrait



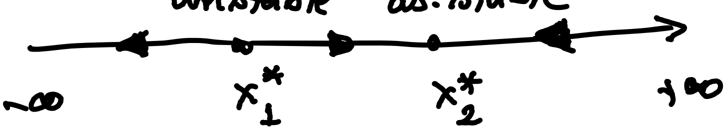
- if $x_0 > \frac{K}{2}$ then $x(t) \xrightarrow[t\to+\infty]{} \frac{K}{2}$
- if $x_0 \in \left(0, \frac{K}{2}\right)$ then $\exists\, t_f > 0$ such that $x(t_f) = 0$

II $\Delta > 0 \iff r - \frac{4h}{K} > 0 \iff \boxed{h < \frac{rK}{4}.}$

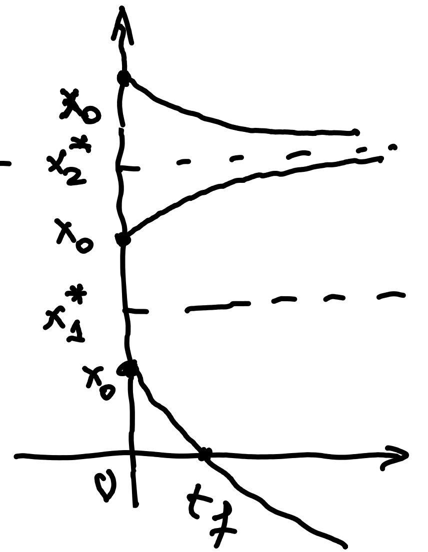
$$x^*_{1,2} = \frac{-r \pm \sqrt{r^2 - \frac{4rh}{K}}}{-2\cdot\frac{r}{K}} = \frac{K}{2r}\left(r \mp \sqrt{r^2 - \frac{4rh}{K}}\right)$$

$\Rightarrow x^*_{1,2} > 0 \qquad 0 < x^*_1 < x^*_2$

x		x^*_1		x^*_2	
$f(x)$	$- - -$	0	$+ \ +$	0	$- - -$
	$\leftarrow$		$\rightarrow$		$\leftarrow$
		unstable		as. stable	



phase portrait

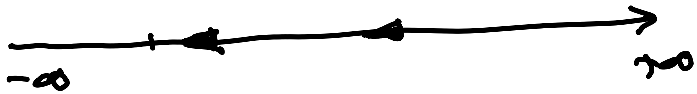


- if $x_0 \in (0, x_1^*) \Rightarrow \exists\, t_f > 0$ s.t $x(t_f) = 0$
- if $x_0 \in (x_1^*, +\infty) \Rightarrow x(t) \xrightarrow[t \to +\infty]{} x_2^*$

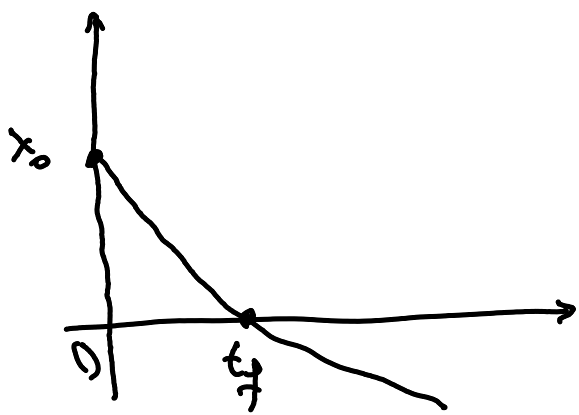
III $\Delta < 0 \iff h > \frac{rK}{4}$

$f(x) = 0$ has no real solutions $\Rightarrow$ the diff. eq. has no equilibrium point

x	$-\infty$ $\qquad\qquad\qquad$ $+\infty$
$f(x)$	$- \; - \; - \; - \; - \; - \; - \; -$



$\forall x_0 > 0 \Rightarrow \exists\, t_f > 0$ s.t $x(t_f) = 0$



Conclusion

- if the harvesting rate h satisfies

$$0 < h \leq \frac{rK}{4}$$

then there exists a treshold value x_T such that if the initial size of the pop. x_0 is less than x_T then the pop. is exterminated in finite time

if the initial size of the pop. x_0 is above x_T then the pop. will tend to an equilibrium sol.

$$\left(\begin{array}{ll} \underline{\mathrm{I}} & x_T = \frac{K}{2} \;,\; x^* = \frac{K}{2} \\ \underline{\mathrm{II}} & x_T = x_1^* \;,\; x^* = x_2^* \end{array} \right)$$

- if the harvesting rate h satisfies

$$h > \frac{rK}{4}$$

then the pop. is exterminated in finite time regardless of its initial size x_0

(the excessive harvesting).

b) <u>The proportional rate harvesting</u>

- the pop. is harvested with a rate proportional to the size of the pop.

$$\begin{cases} x' = rx\left(1-\frac{x}{K}\right) - E.x \\ x(0) = x_0 \end{cases}$$

$r, K, E > 0$

E – the effort made to harvest

<u>Problem</u>: Study the effect of harvesting on the growth of the pop.

$x' = f(x)$ where $f(x) = rx\left(1-\frac{x}{K}\right) - Ex =$

$= x\left[r\left(1-\frac{x}{K}\right) - E\right]$

<u>Equilibrium points</u>

$f(x) = 0 \Rightarrow x\left[r\left(1-\frac{x}{K}\right) - E\right] = 0$

$\Rightarrow \boxed{x_1^* = 0}$, $r\left(1-\frac{x}{K}\right) - E = 0$

eq. point

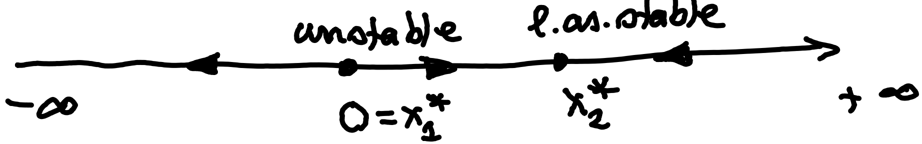
$r - \frac{rx}{K} - E = 0 \Rightarrow \frac{rx}{K} = r - E$

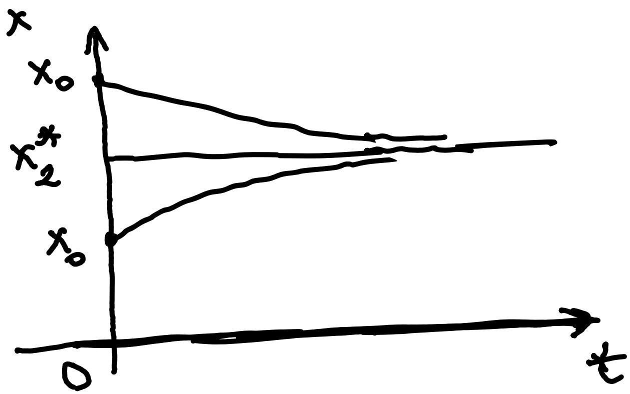
$\boxed{x_2^* = \frac{K}{r}(r-E)}$ eq. point.

$x_2^* \geq 0 \iff r \geq E$

I $\quad E < r \implies x_1^* = 0 \,,\; x_2^* = \frac{K}{r}(r - E) > 0$

x	$-\infty$	$0 = x_1^*$		x_2^*	
$f(x)$	$- \; -$	0	$+ \; +$	0	$- \; - \; -$
	$\longleftarrow$		$\longrightarrow$		$\longleftarrow$



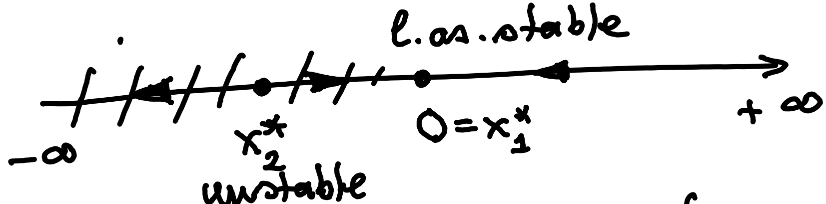


if $x_0 > 0 \implies$

$\implies x(t) \xrightarrow[t \to +\infty]{} x_2^*$

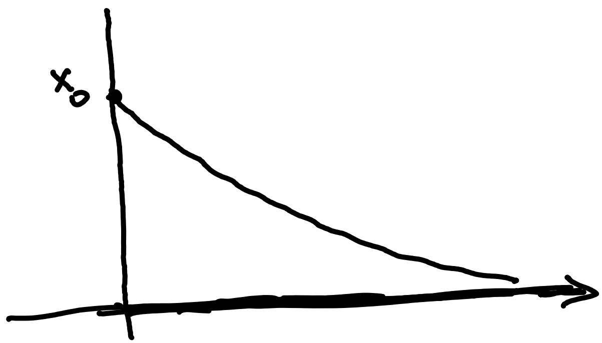
II $E \geq r \implies x_1^* = 0 \ , \ x_2^* \leq 0$

x		$x_2^* < 0$		$x_1^* = 0$	
$f(x)$	$- \ -$	0	$+ \ +$	0	$- \ -$
	$\leftarrow$		$\rightarrow$		$\leftarrow$



$\forall x_0 > 0 \implies x(t) \xrightarrow[t \to +\infty]{} 0$

the pop. disappears in time



2) <u>The prey-predator model</u> (Lotka-Volterra model)

x – the prey pop.

y – the predator pop.

x_0, y_0 – the initial size of the populations.

$x(0) = x_0$, $y(0) = y_0$

- if the predator pop. is not present in the area, we suppose that prey pop. has unlimited food resources
  $\Rightarrow$ the prey pop. will develop according to the Malthus model with positive growth rate

$$x' = a \cdot x \quad , \quad a > 0$$

- the predator pop. has no other food resources than the pop. x, so if the prey pop. is not present in the area then the predators have no food so will develop according to the Malthus model but with a negative growth rate

$$y' = -c \cdot y \quad , \quad c > 0$$

- the predator pop. harvest prey pop with a proportional rate to the total number of possible interactions between them

$x \cdot y$ – the total number of possible interactions

$$x' = ax - b \cdot x \cdot y \quad , a, b > 0$$

- the decreasing of the predator pop. is limited by the harvesting with a rate prop. to the total number of possible interactions

$$y' = -cy + d \cdot xy \quad , c, d > 0$$

$$\Rightarrow \begin{cases} x' = ax - bxy \\ y' = -cy + dxy \end{cases} , a, b, c, d > 0$$

the Lotka-Volterra model.

Equilibrium points

$$\begin{cases} x' = f_1(x,y) \\ y' = f_2(x,y) \end{cases} \qquad \begin{aligned} f_1(x,y) &= ax - bxy \\ f_2(x,y) &= -cy + dxy \end{aligned}$$

$$\begin{cases} f_1(x,y) = 0 \\ f_2(x,y) = 0 \end{cases} \Rightarrow \begin{cases} x(a - by) = 0 \\ y(-c + dx) = 0 \end{cases} \Rightarrow$$

$$\Rightarrow \quad x = 0 \quad \text{or} \quad a - by = 0$$

$$x = 0 \Downarrow y \cdot (-c) = 0 \Downarrow y = 0$$

$X_1^*(0,0)$ eq. point

$$a - by = 0 \Downarrow y = \frac{a}{b} \Downarrow \underbrace{\frac{a}{b}}_{\neq 0}(-c + dx) = 0 \Downarrow -c + dx = 0 \Rightarrow x = \frac{c}{d}$$

$X_2^*\left(\underbrace{\frac{c}{d}}_{>0}, \underbrace{\frac{a}{b}}_{>0}\right)$ eq. point

Stability of the eq. points

$$J_f(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} a-by & -bx \\ dy & -c+dx \end{pmatrix}$$

$f=(f_1,f_2)$

$\underline{X_1^*(0,0)}$: $\quad J_f(0,0) = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix}$

$$\det(\lambda I_2 - J_f(0,0)) = 0 \Rightarrow \begin{vmatrix} \lambda - a & 0 \\ 0 & \lambda + c \end{vmatrix} = 0$$

$$\Rightarrow (\lambda - a)(\lambda + c) = 0 \Rightarrow \begin{array}{l} \lambda_1 = a > 0 \\ \lambda_2 = -c < 0 \end{array}$$

$\lambda_1 > 0 \Rightarrow X_1^*(0,0)$ is unstable

$\underline{x_2^*\left(\frac{c}{d}, \frac{a}{b}\right)}$: $J_f\left(\frac{c}{d}, \frac{a}{b}\right) = \begin{pmatrix} 0 & -\frac{bc}{d} \\ \frac{d \cdot a}{b} & 0 \end{pmatrix}$

$$\det\left(\lambda I_2 - J_f(x_2^*)\right) = 0$$

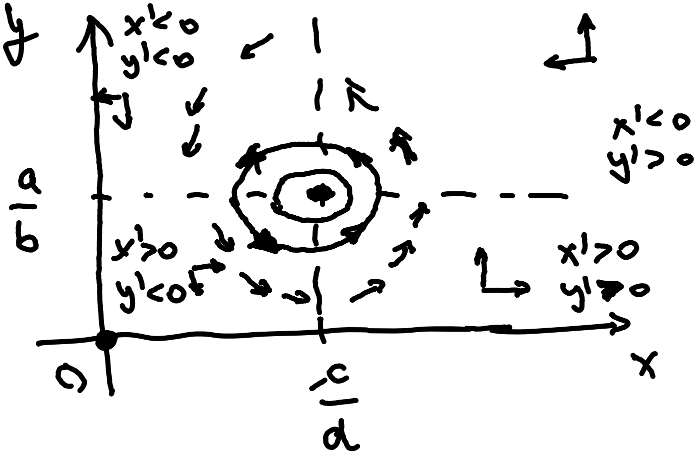
$$\begin{vmatrix} \lambda & \frac{bc}{d} \\ -\frac{da}{b} & \lambda \end{vmatrix} = 0 \quad \Rightarrow \lambda^2 + ac = 0$$

$$\lambda^2 = \underbrace{-ac}_{<0}$$

$$\Rightarrow \lambda_{1,2} = \pm i\sqrt{ac}$$

$\left.\begin{array}{l} \operatorname{Re}\lambda_{1,2} = 0 \\ \text{the system is nonlinear} \end{array}\right\} \Rightarrow$ we cannot apply the Stab. Th. in the first approx.

The sketch of the phase portrait



$$\begin{cases} x' = x(a - by) \\ y' = y(-c + dx) \end{cases}$$

$$x' = rx\left(1 - \frac{x}{k}\right) - \alpha \cdot xy$$

$$y' = -cy + \beta xy - \gamma \cdot y^2$$

it can be proved that the orbits are closed curves

$\Rightarrow X_e^*$ is locally stable.

The orbits eq.
$$\frac{dx}{dy} = \frac{x(a-by)}{y(-c+dx)}$$

$$\Rightarrow \ldots \Rightarrow \ldots .$$