Analiză reală

Seminar 9 30.04.2025

Integration of measurable functions

Ex. 1: Let us consider the measure space $(\mathbb{R}, \mathcal{L}, \lambda)$. For $\bar{x} \in \mathbb{R}$ and $f:\mathbb{R}\to\mathbb{R}$, we define the func. $f_{\bar{x}}:\mathbb{R}\to\mathbb{R}$, $f_{\bar{x}}(x)=f(x-\bar{x})$

a) If f is Lebesgue measurable, show that $f_{\bar{x}}$ is Leb. meas.

Sol: $\forall c\in\mathbb{R}$, $\{f_{\bar{x}} < c\} = \underset{\in\mathbb{R}}{\bar{x}} + \overbrace{\{f<c\}}^{\in\mathcal{L}} \in \mathcal{L}$

$x \in \{f_{\bar{x}}<c\} \iff f_{\bar{x}}(x)<c \iff f(x-\bar{x})<c \iff x-\bar{x} \in \{f<c\}$

$\iff x \in \bar{x}+\{f<c\}$

$\Rightarrow f_{\bar{x}}$ is Leb. measurable.

b) Let $\bar{x}\in\mathbb{R}$ and $s:\mathbb{R}\to[0,\infty)$ be a simple function. Show that $s_{\bar{x}}$ is simple and $\int s_{\bar{x}}\, d\lambda = \int s\, d\lambda$

Sol: s is Lebesgue measurable $\overset{a)}{\Rightarrow}$ $s_{\bar{x}}$ is Leb. measurable.

$s_{\bar{x}}(\mathbb{R}) = s(\mathbb{R})$ - finite $\Rightarrow s_{\bar{x}}(\mathbb{R})$ is finite.

$\Rightarrow s_{\bar{x}}$ is simple

$s=\sum_{i=1}^{m} \alpha_i \chi_{A_i}$ - the standard representation of s.

$s_{\bar{x}}(x) = s(x-\bar{x}) = \sum_{i=1}^{m} \alpha_i \cdot \chi_{A_i}(x-\bar{x})$, $\forall x\in\mathbb{R}$

$\chi_{A_i}(x-\bar{x}) = \begin{cases} 1, & x-\bar{x}\in A_i \iff x\in\bar{x}+A_i \\ 0, & x-\bar{x}\notin A_i \iff x\notin\bar{x}+A_i \end{cases} \Rightarrow \chi_{A_i}(x-\bar{x}) = \chi_{\bar{x}+A_i}(x)$

$\Rightarrow s_{\bar{x}}(x) = \sum_{i=1}^{m}\alpha_i \chi_{\bar{x}+A_i}(x) \Rightarrow \int s_{\bar{x}}\, d\lambda = \sum_{i=1}^{m}\alpha_i \int \chi_{\bar{x}+A_i}\, d\lambda =$

$= \sum_{i=1}^{m}\alpha_i \cdot \underbrace{\lambda(\bar{x}+A_i)}_{=\lambda(A_i)\text{ (invariance to translations)}} = \sum_{i=1}^{m}\alpha_i\, \lambda(A_i) = \int s\, d\lambda$

c) Let $\bar{x}\in\mathbb{R}$ and $f:\mathbb{R}\to[0,\infty]$ be a Leb. meas. function. Show that $\int f_{\bar{x}}\, d\lambda = \int f\, d\lambda$

$(X,\mathcal{A},\mu)$ - meas. space

Step 1: $s=\sum_{i=1}^{m}\alpha_i A_i$ the standard representation of the simple function

Sol:

$s: X\to[0,\infty)$

$\int s\, d\mu = \sum_{i=1}^{m}\alpha_i\mu(A_i)\in[0,\infty]$. If $\int s\, d\mu<\infty$, we say that s is integrable.

Step 2: $f:X\to[0,\infty]$ be a measurable function

$S_f=\{s:X\to[0,\infty) \mid s \text{ simple}, s\le f\}$

$\int f\, d\mu = \sup_{s\in S_f}\int s\, d\mu \in [0,\infty)$. If $\int f\, d\mu<\infty$, we say that f is integrable.

Step 3: $f:X\to\overline{\mathbb{R}}$ be a measurable function.

$f^+=\max\{f,0\}$, $f^-=\max\{-f,0\}$, $f=f^+-f^-$

If at least one of the integrals $\int f^+ d\mu$ and $\int f^- d\mu$ is finite $\Rightarrow$

$\Rightarrow \int f\, d\mu = \int f^+ d\mu - \int f^- d\mu \in \overline{\mathbb{R}}$

If both int. $\int f^+ d\mu$ and $\int f^- d\mu$ are finite, we say that f is integrable, $\int f^+ d\mu\in\mathbb{R}$.

$S_f=\{s:\mathbb{R}\to[0,\infty) \mid s \text{ simple}, s\le f\}$

$S_{f_{\bar{x}}}=\{t:\mathbb{R}\to[0,\infty) \mid t \text{ simple}, t\le f_{\bar{x}}\}$

$s\in S_f \Rightarrow s_{\bar{x}}\in S_{f_{\bar{x}}}$ and $\int s_{\bar{x}}\, d\lambda = \int s\, d\lambda \Rightarrow$

$\Rightarrow \int f\, d\lambda \le \int f_{\bar{x}}\, d\lambda$

$t\in S_{f_{\bar{x}}} \Rightarrow t_{-\bar{x}}\in S_f$ and $\int t\, d\lambda \overset{b)}{=} \int t_{-\bar{x}}\, d\lambda \Rightarrow \int f_{\bar{x}}\, d\lambda \le \int f\, d\lambda$

$\Rightarrow \int f_{\bar{x}}\, d\lambda = \int f\, d\lambda$

d) For $f:\mathbb{R}\to\mathbb{R}$ integrable and $\bar{x}\in\mathbb{R}$. Show that $f_{\bar{x}}$ is integrable and $\int f_{\bar{x}}\, d\lambda = \int f\, d\lambda$.

Sol: $(f_{\bar{x}})^+(x) = \max\{f_{\bar{x}}(x),0\} = \max\{f(x-\bar{x}),0\} =$

$= f^+(x-\bar{x}) = (f^+)_{\bar{x}}(x) \Rightarrow \int (f_{\bar{x}})^+ d\lambda = \int (f^+)_{\bar{x}}\, d\lambda \Rightarrow$

$\overset{d)}{\Rightarrow} \int (f_{\bar{x}})^+ d\lambda = \int f^+ d\lambda \in [0,\infty)$

Similar, $\int (f_{\bar{x}})^- d\lambda = \int f^- d\lambda \in [0,\infty)$ $\Bigg\} \Rightarrow f_{\bar{x}}$ is int. and

$\int f_{\bar{x}}\, d\lambda = \int f^+ d\lambda - \int f^- d\lambda = \int f\, d\lambda$.

The Monotone Convergence Theorem (MCT): Let $(X,\mathcal{A},\mu)$ be a measure space and $(f_n)_{n\in\mathbb{N}}$ be a non-decreasing seq. of nonnegative measurable functions, $f_n:X\to[0,\infty]$

Then, $\int \lim_{n\to\infty} f_n\, d\mu = \lim_{n\to\infty}\int f_n\, d\mu$.

Remark 1: $(f_n)_{n\in\mathbb{N}}$ is a non-decreasing sequence:

$\forall n\in\mathbb{N}$, $f_n\le f_{n+1} \iff \forall n\in\mathbb{N}, \forall x\in X, f_n(x)\le f_{n+1}(x)$

Fatou's Lemma: Let $(X,\mathcal{A},\mu)$ be a measure space and $(f_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative measurable functions. $f_n:X\to[0,\infty]$. Then $\int \liminf_{n\to\infty} f_n\, d\mu \le \liminf_{n\to\infty}\int f_n\, d\mu$.

Ex. 1: Let us consider the space $(\mathbb{R},\mathcal{L},\lambda)$ and the functions $f_n:\mathbb{R}\to\mathbb{R}$, $n\in\mathbb{N}$, defined by:

a) $f_n=\chi_{[n,n+1)}$, b) $f_n = n\cdot\chi_{(0,\frac{1}{n}]}$, c) $f_n=\frac{1}{n}$, d) $f_n=-\frac{1}{n}$

Does the conclusion of MCT hold for $(f_n)_{n\in\mathbb{N}}$?
What can be said about the conclusion of Fatou's Lemma

Sol: We note that all the functions f_n are Lebesgue measurable.

a) $f_n=\chi_{[n,n+1)}$. $\chi_{[n-1,n)}(n)=0$

$\forall n\in\mathbb{N}\setminus\{1\}$, $f_{n-1}(n)=0$, $f_n(n)=1$, $f_{n+1}(n)=0$

$\Rightarrow (f_n)$ is not a monotone sequence.

$\forall x\in\mathbb{R}$, $\exists n_0\in\mathbb{N}$ s.t. $\forall n\ge n_0$, $x\notin[n,n+1) \Rightarrow f_n(x)=0$

$\Rightarrow \lim_{n\to\infty} f_n(x) = 0 \Rightarrow f_n\to 0 \Rightarrow \int \lim_{n\to\infty} f_n\, d\lambda = \int 0\cdot d\lambda = 0$

$\forall n\in\mathbb{N}$, $\int f_n\, d\lambda = \int \chi_{[n,n+1)}\, d\lambda = \lambda([n,n+1))=1$

$\Rightarrow \lim_{n\to\infty}\int f_n\, d\lambda = 1$.

So, the conclusion of MCT does not hold and inequality in Fatou's Lemma is strict.

b) $f_n = n\cdot\chi_{(0,\frac{1}{n}]}$, $\forall n\in\mathbb{N}\setminus\{1\}$,

$f_{n-1}(\frac{1}{n}) = (n-1)\cdot\underbrace{\chi_{(0,\frac{1}{n-1}]}(\tfrac{1}{n})}_{=1} = n-1$

$f_n(\frac{1}{n}) = n$, $f_{n+1}(\frac{1}{n})=0 \Rightarrow (f_n)$ is not monotone.

$\forall x\in\mathbb{R}$, $\exists n_0\in\mathbb{N}$ s.t. $\forall n\in\mathbb{N}$, $n\ge n_0$, $x\notin(0,\frac{1}{n}] \Rightarrow$

$\Rightarrow f_n(x)=0 \Rightarrow \lim_{n\to\infty} f_n = 0 \Rightarrow \int\lim_{n\to\infty} f_n\, d\lambda = 0$.

$\forall n\in\mathbb{N}$, $\int f_n\, d\lambda = n\cdot\lambda((0,\frac{1}{n}]) = n\cdot\frac{1}{n} = 1. \Rightarrow$

$\Rightarrow \lim_{n\to\infty}\int f_n\, d\lambda = 1$.

So, the conclusion of MCT does not hold and the inequality in Fatou's Lemma is strict.

c) $f_n=\frac{1}{n}$, $n\in\mathbb{N} \Rightarrow (f_n)$ is a decreasing sequence.

$\lim_{n\to\infty} f_n = 0 \Rightarrow \int \lim_{n\to\infty} f_n\, d\lambda = 0$

$\forall n\in\mathbb{N}$, $\int f_n\, d\lambda = \int \frac{1}{n}\, d\lambda = \frac{1}{n}\cdot\underbrace{\lambda(\mathbb{R})}_{=\infty} = \infty \Rightarrow$

$\Rightarrow \lim_{n\to\infty}\int f_n\, d\lambda = \infty$.

So, the conclusion of MCT does not hold and the inequality in Fatou's Lemma is strict.

d) $f_n=-\frac{1}{n}$, $n\in\mathbb{N}$ is a non-decreasing sequence, but $f_n<0$, $\forall n\in\mathbb{N}$.

$\lim_{n\to\infty} f_n = 0 \Rightarrow \int\lim_{n\to\infty} f_n\, d\lambda = 0$

$\forall n\in\mathbb{N}$, $f_n^+=0 \Rightarrow \int f_n^+\, d\lambda = 0$, $f_n^- = \frac{1}{n} \Rightarrow \int f_n^-\, d\lambda = \infty$

$\Rightarrow \int f_n\, d\lambda = \int f_n^+ d\lambda - \int f_n^- d\lambda = 0-\infty = -\infty \Rightarrow$

$\Rightarrow \lim_{n\to\infty}\int f_n\, d\lambda = -\infty$

So, the conclusion of MCT and Fatou's Lemma does not hold.

Ex. 3: Let $(X,\mathcal{A},\mu)$ be a measure space, $(f_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative measurable functions, $f_n:X\to[0,\infty]$. Suppose that $f_n\to f$, $f_n\le f$, $\forall n\in\mathbb{N}$

Show that $\lim_{n\to\infty}\int f_n\, d\mu = \int f\, d\mu$.

Sol: $\forall n\in\mathbb{N}$, $f_n\le f \Rightarrow \int f_n\, d\mu \le \int f\, d\mu$.

$\Rightarrow \limsup_{n\to\infty}\int f_n\, d\mu \le \int f\, d\mu$ (1)

In view of Fatou's Lemma, $\int f\, d\mu = \int \liminf_{n\to\infty} f_n\, d\mu \le$

$\le \liminf_{n\to\infty}\int f_n\, d\mu \le \limsup_{n\to\infty}\int f_n\, d\mu$. (2)

(1), (2) $\Rightarrow \lim_{n\to\infty}\int f_n\, d\mu = \int f\, d\mu$.