Lecture 11

The dynamical systems generated by scalar autonomous diff. eq

$x = x(t)$ $\qquad x' = f(t, x)$ a nonautonomous diff. eq.

(1) $\boxed{x' = f(x)}$ an autonomous diff. eq

scalar $\longrightarrow$ x is one dimensional

Theorem. If $f \in C^1(\mathbb{R})$ then the ivp

$$\begin{cases} x' = f(x) & (1) \\ x(0) = \eta, \quad \eta \in \mathbb{R} & (2) \end{cases}$$

has a unique maximal solution for every $\eta \in \mathbb{R}$.

maximal solution = a solution defined on the largest possible interval.

let's denote by $x(t, \eta)$ the maximal sol. of (1)+(2).

$$x(\cdot, \eta) : I_\eta \to \mathbb{R}$$

$x(\cdot, \eta)$ is maximal $\Rightarrow$ I_η is maximal.

$$I_\eta = (\alpha_\eta, \beta_\eta)$$

$0 \in I_\eta \Rightarrow \alpha_\eta < 0 < \beta_\eta$

$W = \{ I_\eta \times \mathbb{R} \mid \eta \in \mathbb{R} \}$.

$\varphi : W \to \mathbb{R}$

$\varphi(t, \eta) = x(t, \eta)$

Remark If. $I_\eta = \mathbb{R}$, $\forall \eta \in \mathbb{R} \Rightarrow W = \mathbb{R}^2$

the function φ is called the flow generated by the eq. (1)

the map $\eta \mapsto \varphi(t, \eta) \Rightarrow$ the dynamical system generated by (1).

Properties.

1) $\varphi(0, \eta) = \eta$, $\forall \eta \in \mathbb{R}$.
2) $\varphi(t+s, \eta) = \varphi(t, \varphi(s, \eta))$, $\forall t, s \in I_\eta$, $\forall \eta \in \mathbb{R}$.
3) φ is continuous

<u>Def</u>. $\gamma^+(\eta) = \bigcup_{t\in[0,\beta_\eta)} \varphi(t,\eta)$ the positive orbit of η

$\gamma^-(\eta) = \bigcup_{t\in(\alpha_\eta,0]} \varphi(t,\eta)$ the negative orbit of η

$\gamma(\eta) = \gamma^+(\eta) \cup \gamma^-(\eta) = \bigcup_{t\in(\alpha_\eta,\beta_\eta)} \varphi(t,\eta)$

The collection of all orbits together with the direction of the flow is called the phase portrait of the diff. eq. (1).

<u>Examples</u>

1) $x' = -x \qquad f(x) = -x$

flow: $\begin{cases} x' = -x \\ x(0) = \eta, \ \eta\in\mathbb{R} \end{cases}$

$\frac{dx}{dt} = -x \Rightarrow \int \frac{dx}{x} = \int -dt \rightarrow$

$\ln x = -t + \ln c$

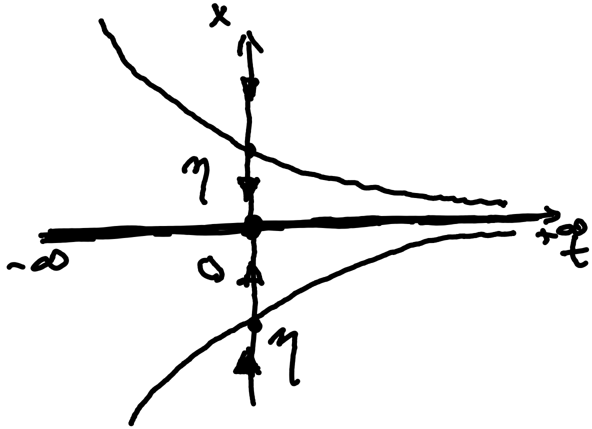
$x(t) = c e^{-t}, \ c\in\mathbb{R}$

$x(0) = \eta \rightarrow c = \eta \rightarrow x(t,\eta) = \eta \cdot e^{-t}$

$I_\eta = \mathbb{R}, \ \forall \eta\in\mathbb{R}$

$\varphi: \mathbb{R}^2 \to \mathbb{R}$

$\varphi(t,\eta) = x(t,\eta) = \eta \cdot e^{-t}$ the flow.



1. $\underline{\eta = 0} \Rightarrow \varphi(t,0) = 0, \ \forall t$

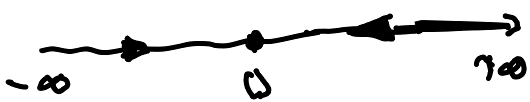
$$\gamma(0) = \bigcup_{t\in\mathbb{R}} \varphi(t,0) = \{0\}$$

2. $\underline{\eta > 0}$

$$\gamma^+(\eta) = \bigcup_{t\in[0,+\infty)} \varphi(t,\eta) = \underleftarrow{(0,\eta]}$$

$$\gamma^-(\eta) = \bigcup_{t\in(-\infty,0]} \varphi(t,\eta) = \underleftarrow{[\eta,+\infty)}$$

$$\gamma(\eta) = \gamma^+(\eta) \cup \gamma^-(\eta) = \underleftarrow{(0,+\infty)}$$



phase portrait

3. $\underline{\eta < 0}$

$$\gamma^+(\eta) = \bigcup_{t\in[0,+\infty)} \varphi(t,\eta) = \underrightarrow{[\eta,0)}$$

$$\gamma^-(\eta) = \bigcup_{t\in(-\infty,0]} \varphi(t,\eta) = \underrightarrow{(-\infty,\eta]}$$

$$\gamma(\eta) = \underrightarrow{(-\infty,0)}$$

2) $x' = x$

flow: $\begin{cases} x' = x \\ x(0) = \eta \end{cases}$

$\frac{dx}{dt} = x \Rightarrow \int \frac{dx}{x} = \int dt \Rightarrow \ln x = t + \ln c$

$\Rightarrow x(t) = c \cdot e^t, \ c \in \mathbb{R}$

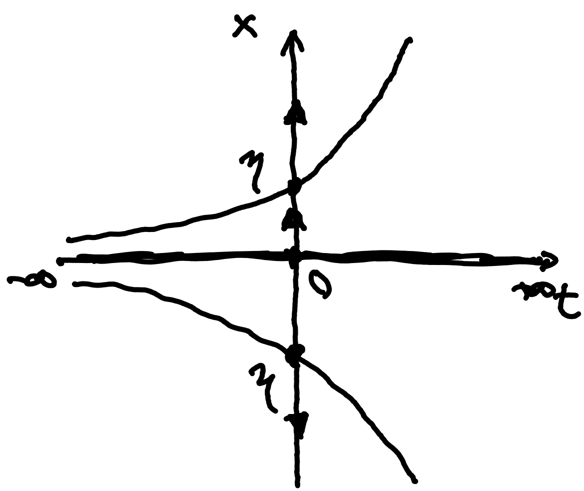
$x(0) = \eta \Rightarrow c = \eta$

$x(t, \eta) = \eta \cdot e^t$

maximal interval $I_\eta = \mathbb{R}, \ \forall \eta$

$\varphi : \mathbb{R}^2 \to \mathbb{R}$

$\varphi(t, \eta) = x(t, \eta) = \eta \cdot e^t$ the flow.



1. $\eta = 0 \Rightarrow \varphi(t, 0) = 0, \ \forall t$

   $\gamma(0) = \{0\}$.

2. $\eta > 0$

   $\gamma^+(\eta) = \bigcup_{t \in [0, +\infty)} \varphi(t, \eta) = [\eta, +\infty)$

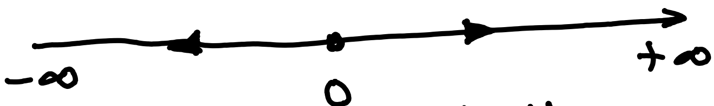
   $\gamma^-(\eta) = \bigcup_{t \in (-\infty, 0]} \varphi(t, \eta) = (0, \eta]$

   $\gamma(\eta) = (0, +\infty)$

3. $\underline{\eta < 0}$: $\gamma^+(\eta) = \bigcup_{t \in [0,+\infty)} \varphi(t,\eta) = (-\infty, \eta]$

$\gamma^-(\eta) = \bigcup_{t \in (-\infty,0]} \varphi(t,\eta) = [\eta, 0)$

$\gamma(\eta) = (-\infty, 0)$

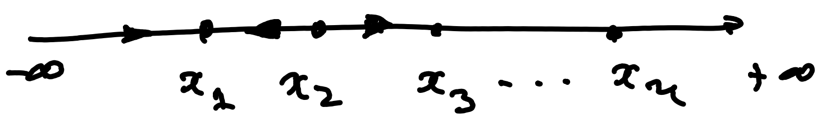


phase portrait.

In general $x' = f(x)$

$f(x) = 0 \Rightarrow x_1, x_2, \dots, x_n$ real roots

x	$-\infty$		x_1		x_2		x_3	.....	x_n	$+\infty$
$f(x)$		$+$	0	$-$	0	$+$	0	.....	0	
		$\rightarrow$		$\leftarrow$						



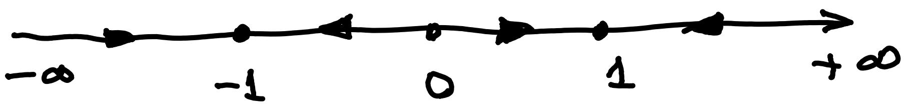
phase portrait

3) $x' = x - x^3$

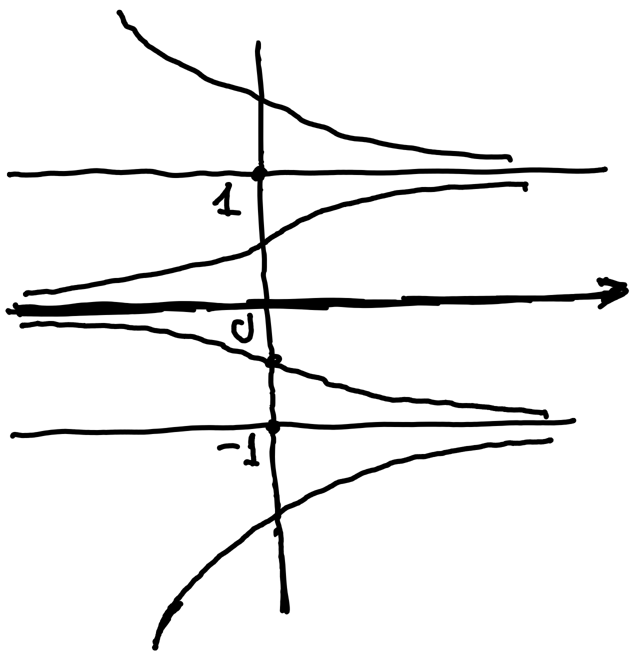
$f(x) = x - x^3$ $\quad$ $f(x) = 0 \rightarrow x - x^3 = 0 \rightarrow x(1 - x^2) = 0$

$x_1 = -1$, $x_2 = 0$, $x_3 = 1$

x		-1		0		1	
$f(x)$	$+$	0	$-$	0	$+\ +$	0	$-$
	$\longrightarrow$		$\longleftarrow$				



phase portrait



Def. The constant solutions $x(t) \equiv x^*$ of the eq. (1) are called equilibrium solutions (stationary)

The value $\underline{x^* \in \mathbb{R}}$ is called the equilibrium point

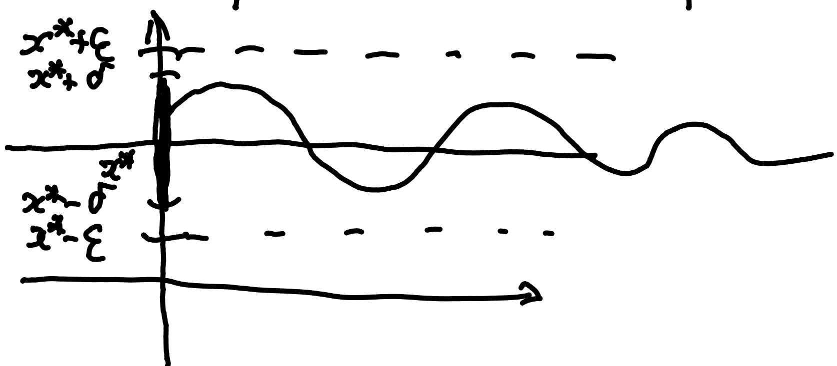
in our examples

1) $x' = -x \Rightarrow x^* = 0$ is an eq. point
2) $x' = x \Rightarrow x^* = 0$ — " — " —
3) $x' = x - x^3 \Rightarrow x_1^* = -1,\ x_2^* = 0,\ x_3^* = 1$ eq. points.

$$\left.\begin{array}{l} x' = f(x) \\ x(t) \equiv x^* \end{array}\right\} \Rightarrow x^* \text{ is a sol. of the eq. } \boxed{f(x) = 0}$$

Def. An equilibrium point $x^* \in \mathbb{R}$ of the eq. (1) is

a) locally stable $\Leftrightarrow$ $\forall \varepsilon > 0$ $\exists\, \delta = \delta(\varepsilon)$ such that for every η for which $|\eta - x^*| < \delta \Rightarrow |\varphi(t,\eta) - x^*| < \varepsilon$, $\forall t \geq 0$



b) locally asimptotically stable $\Leftrightarrow$ if it is locally stable and $|\varphi(t,\eta) - x^*| \xrightarrow[t \to +\infty]{} 0$

c) unstable $\Leftrightarrow$ it is not locally stable

Examples

1) $x' = -x$ $\qquad x^* = 0$ is the eq. point.

$$\varphi(t,\eta) = \eta \cdot e^{-t}, \quad \varphi : \mathbb{R}^2 \to \mathbb{R}.$$

$$|\varphi(t,\eta) - x^*| = |\eta \cdot e^{-t}| = |\eta| \cdot \underbrace{e^{-t}}_{\leq 1,\ t \geq 0} \leq |\eta| = |\eta - x^*|$$

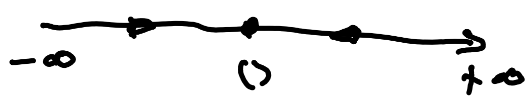
for $\varepsilon > 0$ $\exists\, \delta = \varepsilon$ such that if $|\eta - x^*| < \delta = \varepsilon$

we have $|\varphi(t,\eta) - x^*| \leq |\eta - x^*| < \delta = \varepsilon$

$\Rightarrow$ $x^* = 0$ is locally stable.

$$|\varphi(t,\eta) - x^*| \leq |\eta| \cdot e^{-t} \xrightarrow[t \to \infty]{} 0$$

$\Rightarrow$ $x^* = 0$ is ~~locally~~ asimpt. stable.



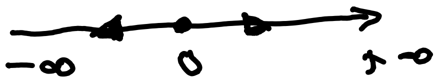
$x^* = 0$ is globally asimpt. stable

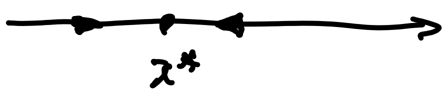
2) $x' = -x$ $\qquad$ $x^* = 0$ is the eq. point

$$\varphi(t,\eta) = \eta e^{t}$$

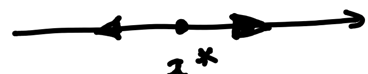
$$|\varphi(t,\eta) - x^*| = |\eta \cdot e^{t}| = |\eta| \cdot e^{t} \xrightarrow[t \to +\infty]{} +\infty$$

$\Rightarrow x^* = 0$ is unstable.





x^* locally as. stable

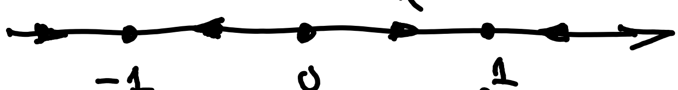


unstable.

3) $x' = x - x^3$

$x_1^* = -1$, $x_2^* = 0$, $x_3^* = 1$

unstable



locally as. stable $\qquad$ locally as. stable

<u>Theorem</u> (The Stability criterion in the first approx.)

$x^* \in \mathbb{R}$ eq. point of (1), $f \in C^1$

a) if $f'(x^*) < 0 \Rightarrow x^*$ is locally as. stable

b) if $f'(x^*) > 0 \Rightarrow x^*$ is unstable.

$x' = x - x^3$

$f(x) = x - x^3$

$f'(x) = 1 - 3x^2$

$x^*_{1,3} = \mp 1 \qquad f'(\pm 1) = -2 < 0 \Rightarrow x^*_{1,3} = \mp 1$ are locally as. stable

$x^*_2 = 0 \qquad f'(0) = 1 > 0 \Rightarrow x^*_2 = 0$ is unstable