Seminar 4

1. Four electronic devices have the property that, for every $i \in \{1,2,3,4\}$, the probability that any i fixed devices are all functional is $\dfrac{1}{4^i}$. Using the inclusion-exclusion principle, compute the probability of the event A:"none of the devices is functional".

A: We will compute $P(\bar{A})$, where $\bar{A}$ is the event that at least one device is functional. For $i \in \{1,2,3,4\}$, let A_i be the event that the ith device is functional. Then $\bar{A} = A_1 \cup A_2 \cup A_3 \cup A_4$. By the inclusion-exclusion principle, we have:

$$\begin{aligned} P(\bar{A}) = {} & P(A_1) + P(A_2) + P(A_3) + P(A_4) \\ & - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_1 \cap A_4) - P(A_2 \cap A_3) - P(A_2 \cap A_4) - P(A_3 \cap A_4) \\ & + P(A_2 \cap A_3 \cap A_4) + P(A_1 \cap A_3 \cap A_4) + P(A_1 \cap A_2 \cap A_4) + P(A_1 \cap A_2 \cap A_3) \\ & - P(A_1 \cap A_2 \cap A_3 \cap A_4) \end{aligned}$$

and thus $P(A) = 1 - P(\bar{A}) = 1 - \left(4 \cdot \frac{1}{4} - 6 \cdot \frac{1}{4^2} + 4 \cdot \frac{1}{4^3} - \frac{1}{4^4}\right) = \frac{6 \cdot 16 - 4 \cdot 4 + 1}{4^4} = \frac{81}{256} \approx 0.316\,.$

2. Four antivirus programs are tested by scanning independently an infected file. They detect the virus with corresponding probabilities: $\dfrac{3}{4}, \dfrac{1}{4}, \dfrac{2}{4}, \dfrac{1}{4}$. Compute the probabilities of the following events:
A:"All programs detect the virus."
B:"Exactly one program detects the virus."
C:"Exactly three programs detect the virus."
D:"At most one program detects the virus."
E:"At least one program detects the virus."

A: Let V_n:"The nth program detects the virus.", $k = \overline{1,4}$.

$$P(A) = P(V_1 \cap V_2 \cap V_3 \cap V_4) = P(V_1) \cdot P(V_2) \cdot P(V_3) \cdot P(V_4) = \frac{3}{128} \approx 0.023.$$

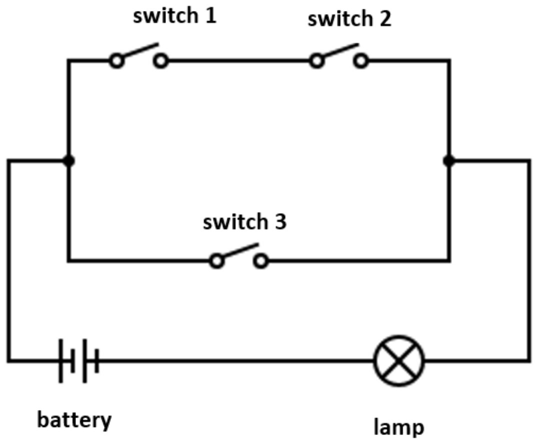
$$\begin{aligned} P(B) &= P(V_1 \cap \overline{V_2} \cap \overline{V_3} \cap \overline{V_4}) + P(\overline{V_1} \cap V_2 \cap \overline{V_3} \cap \overline{V_4}) + P(\overline{V_1} \cap \overline{V_2} \cap V_3 \cap \overline{V_4}) + P(\overline{V_1} \cap \overline{V_2} \cap \overline{V_3} \cap V_4) \\ &= \frac{3 \cdot 3 \cdot 2 \cdot 3 + 1 \cdot 1 \cdot 2 \cdot 3 + 1 \cdot 3 \cdot 2 \cdot 3 + 1 \cdot 3 \cdot 2 \cdot 1}{256} = \frac{84}{256} = \frac{21}{64} \approx 0.382. \end{aligned}$$

$$\begin{aligned} P(C) &= P(\overline{V_1} \cap V_2 \cap V_3 \cap V_4) + P(V_1 \cap \overline{V_2} \cap V_3 \cap V_4) + P(V_1 \cap V_2 \cap \overline{V_3} \cap V_4) + P(V_1 \cap V_2 \cap V_3 \cap \overline{V_4}) \\ &= \frac{1 \cdot 1 \cdot 2 \cdot 1 + 3 \cdot 3 \cdot 2 \cdot 1 + 3 \cdot 1 \cdot 2 \cdot 1 + 3 \cdot 1 \cdot 2 \cdot 3}{256} = \frac{44}{256} = \frac{11}{64} \approx 0.171. \end{aligned}$$

$$P(D) = P(B) + P(\overline{V_1} \cap \overline{V_2} \cap \overline{V_3} \cap \overline{V_4}) = \frac{84}{256} + \frac{1 \cdot 3 \cdot 2 \cdot 3}{256} = \frac{102}{256} = \frac{51}{128} \approx 0.398.$$

$$P(E) = 1 - P(\overline{V_1} \cap \overline{V_2} \cap \overline{V_3} \cap \overline{V_4}) = 1 - \frac{18}{256} = \frac{238}{256} = \frac{119}{128} \approx 0.929.$$

3. In the diagram below the three switches are either ON or OFF, independently, with probability $\frac{1}{2}$ for each state. Compute the probability that the circuit operates.



A: Let S_i:"Switch i is ON", $i = \overline{1,3}$. Using the independence of the switches, we compute

$$\begin{aligned} P(\text{"circuit operates"}) &= P((S_1 \cap S_2) \cup S_3) = P(S_1 \cap S_2) + P(S_3) - P(S_1 \cap S_2 \cap S_3) \\ &= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{2+4-1}{8} = \frac{5}{8}. \end{aligned}$$

4. The owner of three shops decides to give a bonus to the salary of a randomly chosen employee. The first shop has 50 employees and 50% of them are men, the second shop has 75 employees and 60% of them are men and the third shop has 100 employees and 70% are men.
a) Find the probability that the lucky employee works in the third shop, given that the lucky employee is a woman.
b) Find the probability that the lucky employee is a woman, given that the lucky employee works in the third shop.

A: Let S_3:"The lucky employee works in the third shop." and W:"The lucky employee is a woman."
a) $P(S_3|W) = \frac{P(S_3 \cap W)}{P(W)} = \frac{\frac{30}{225}}{\frac{25+30+30}{225}} = \frac{30}{85} = \frac{6}{17}$. b) $P(W|S_3) = \frac{P(W \cap S_3)}{P(S_3)} = \frac{\frac{30}{225}}{\frac{100}{225}} = \frac{30}{100} = \frac{3}{10}$.

5. Three dice are rolled. Let N_k be number that showed on the kth die, $k \in \{1,2,3\}$. Find:
a) $P(N_1 = 1, N_2 = 2, N_3 = 3)$.
b) $P(N_1 = N_2 = N_3)$.
c) $P(N_1 + N_2 + N_3 \geq 5)$.
d) $P(N_1 + N_2 + N_3 \geq 5 | N_1 < N_2 < N_3)$.
e) $P(N_1 < N_2 < N_3 | N_1 < N_2)$.
f) $P(N_1 > N_2 < N_3 | N_1 = N_3)$.
g) $P(N_1 = N_2, N_2 > 2 | N_3 > 2)$.

A: a) $P(N_1 = 1, N_2 = 2, N_3 = 3) = P(N_1 = 1)P(N_2 = 2)P(N_3 = 3) = \frac{1}{6^3} = \frac{1}{216}$.

b) $P(N_1 = N_2 = N_3) = \sum_{i=1}^{6} P(N_1 = N_2 = N_3 = i) = \frac{6}{6^3} = \frac{1}{36}$.
c) $P(N_1 + N_2 + N_3 \geq 5) = 1 - P(N_1 + N_2 + N_3 \in \{3, 4\}) = 1 - \frac{4}{6^3} = \frac{212}{216} = \frac{53}{54}$.

d) $P(N_1 + N_2 + N_3 \geq 5 | N_1 < N_2 < N_3) = 1$, because $N_1 < N_2 < N_3 \implies N_1 \geq 1, N_2 \geq 2, N_3 \geq 3 \implies N_1 + N_2 + N_3 \geq 6$.

e) $P(N_1 < N_2 < N_3 | N_1 < N_2) = \frac{P(N_1 < N_2 < N_3)}{P(N_1 < N_2)} = \frac{\frac{C_6^3}{6^3}}{\frac{C_6^2}{6^2}} = \frac{C_6^3}{C_6^2 \cdot 6} = \frac{20}{90} = \frac{2}{9}$.

f) $P(N_1 > N_2 < N_3 | N_1 = N_3) = \frac{P(N_1=N_3>N_2)}{P(N_1=N_3)} = \frac{\frac{C_6^2}{6^3}}{\frac{6}{6^2}} = \frac{C_6^2}{6^2} = \frac{15}{36} = \frac{5}{12}$.

g) $P(N_1 = N_2, N_2 > 2 | N_3 > 2) = \frac{\sum_{i=3}^{6}\sum_{j=3}^{6} P(N_1=i)P(N_2=i)P(N_3=j)}{P(N_3>2)} = \sum_{i=3}^{6} P(N_1 = i)P(N_2 = i) = \frac{4}{36} = \frac{1}{9}$, here we used that $\{N_1 = i\}$, $\{N_2 = i\}$ and $\{N_3 = j\}$ are independent events, for all $i, j \in \{3, 4, 5, 6\}$.

6. John has in his pocket 2 red dice and 3 blue dice. He takes randomly a die. If the chosen die is red, he rolls it 3 times. On the other hand, if the chosen die is blue, he rolls it 2 times. Compute the probability of the event E:"the sum of the numbers that show up after the rolls is 10."

A: Let R:"the chosen die is red", B:"the chosen die is blue". Note that $\{R, B\}$ forms a partition. By the formula of total probability, $P(E) = P(E|R)P(R) + P(E|B)P(B) = \frac{27}{6^3} \cdot \frac{2}{5} + \frac{3}{6^2} \cdot \frac{3}{5} = \frac{1}{10}$.

7. Mary studies Probability Theory. She arrives late at the seminar with probability 0.2, if the day is rainy, and with probability 0.1, if the day is sunny. According to the weather forecast, the next day, when Mary has the seminar, is rainy with probability 0.8. Compute the probabilities of the events:

A: "Mary arrives on time at the next seminar."

B: "The next day is rainy, given that Mary arrives on time at the seminar ."

A: Let L: "Mary is late at the next seminar" and S:"the next day is sunny". Note that $\{S, \bar{S}\}$ forms a partition. By the formula of total probability, $P(\bar{L}) = P(\bar{L}|S)P(S) + P(\bar{L}|\bar{S})P(\bar{S}) = 0.9 \cdot 0.2 + 0.8 \cdot 0.8 = 0.82$. By the Bayes formula, $P(\bar{S}|\bar{L}) = \frac{P(\bar{L}|\bar{S})P(\bar{S})}{P(\bar{L})} = \frac{0.8 \cdot 0.8}{0.82} = \frac{32}{41}$.

8. A die is rolled. Let N be the number that is obtained. Next, the die is rolled N times. What is the probability that $N = 3$, given that $N \geq 2$ and the numbers obtained after the N rolls are pairwise **a)** distinct? **b)** equal?

A: Let D: "$N \geq 2$ and the numbers obtained after the N rolls are pairwise distinct" and E: "$N \geq 2$ and the numbers obtained after the N rolls are all equal". Note that $\{\{N = i\} : i \in \{1, \ldots, 6\}\}$ forms a partition. The Bayes formula implies:

a) $P(N = 3|D) = \frac{P(D|N=3)P(N=3)}{\sum_{i=1}^{6} P(D|N=i)P(N=i)} = \frac{\frac{A_6^3}{6^4}}{\sum_{i=2}^{6} \frac{A_6^i}{6^{i+1}}}$; b) $P(N = 3|E) = \frac{P(E|N=3)P(N=3)}{\sum_{i=1}^{6} P(E|N=i)P(N=i)} = \frac{\frac{1}{6^3}}{\sum_{i=2}^{6} \frac{1}{6^i}}$.

9. A pair of dice - one white die and one red die - is rolled two times. Compute the probability that the two pairs of numbers, obtained after the two rolls, are equal. (Example of favorable case: the white die shows number 2 and the red die shows number 4, both after the first roll and the second roll; example of unfavorable case: first roll "2 on white die, 4 on red die", second roll "4 on white die, 2 on red die".)

A: Let (W_k, R_k) be the pair of numbers obtained by the white die, respectively, the red die after the kth roll, $k \in \{1, 2\}$. The desired probability is

$$p = \sum_{i=1}^{6}\sum_{j=1}^{6} P\left(\{W_1 = i\} \cap \{R_1 = j\} \cap \{W_2 = i\} \cap \{R_2 = j\}\right) = \sum_{i=1}^{6}\sum_{j=1}^{6} \frac{1}{6^4} = \frac{6^2}{6^4} = \frac{1}{36},$$

where we use the independence of the events $\{W_1 = i\}, \{R_1 = j\}, \{W_2 = i\}, \{R_2 = j\}, \forall\, i, j \in \{1, \ldots, 6\}$.
Alternatively, by the formula of total probability,

$$p = \sum_{i,j=1}^{6} P\left(\{W_2 = i\} \cap \{R_2 = j\} \middle| \{W_1 = i\} \cap \{R_1 = j\}\right) P\left(\{W_1 = i\} \cap \{R_1 = j\}\right) = \sum_{i,j=1}^{6} \frac{1}{6^2} \cdot \frac{1}{6^2} = \frac{1}{36}.$$

10. A computer center has three printers A, B, and C, which print at different speeds. Programs are routed to the first available printer. The probability that a program is routed to printers A, B, and C are 0.5, 0.3, and 0.2, respectively. Occasionally a printer will jam and destroy a printout. The probability that printers A, B, and C will jam are 0.02, 0.06 and 0.1, respectively. Your program is destroyed when a printer jams. What is the probability that printer A is involved? Printer B is involved? Printer C is involved?

A: Let $A_i,\ i=\overline{1,3}$ denote the events that the program was routed to printers A, B and C, respectively, and let E denote the event that the program was destroyed. Then $\{A_1, A_2, A_3\}$ form a partition and we have

$$P\left(A_1\right)=0.5,\ P\left(A_2\right)=0.3,\ P\left(A_3\right)=0.2,$$

and

$$P\left(E|A_1\right)=0.02,\ P\left(E|A_2\right)=0.06,\ P\left(E|A_3\right)=0.1.$$

By the formula of total probability we have

$$P(E)=P\left(E|A_1\right)P(A_1)+\left(E|A_2\right)P(A_2)+\left(E|A_3\right)P(A_3)=0.5\cdot 0.02+0.3\cdot 0.06+0.2\cdot 0.1=0.048.$$

By Bayes' formula, we get

$$P\left(A_1|E\right)=\frac{0.5\cdot 0.02}{0.048}\approx 0.2083;\,P\left(A_2|E\right)=\frac{0.3\cdot 0.06}{0.048}=0.375;\,P\left(A_3|E\right)=\frac{0.2\cdot 0.1}{0.048}\approx 0.4166.$$

Optional:

11. Let $(S,\mathcal{K},P)$ is be a probability space and $A,B\in\mathcal{K}$.

(a) Prove that

$$P(A)+P(B)-1\leq P(A\cap B)\leq\min\{P(A),P(B)\}$$
$$\max\{P(A),P(B)\}\leq P(A\cup B)\leq P(A)+P(B).$$

(b) It is known that $P(A)=\frac{2}{5}, P(B)=\frac{9}{10}$. Indicate four concrete values $a,b,c,d\in(0,1]$ such that

$$a\leq P(A\cap B)\leq b<c\leq P(A\cup B)\leq d.$$

Explain why your choice is correct!

(c) It is known that $P(A)=\frac{2}{5}, P(B)=\frac{9}{10}, P(A|B)=\frac{10}{27}$.

(c1) Compute the probabilities: $P(A\cap B), P(A\cup B), P(B\cap\bar{A}), P(B|A)$.

(c2) Are A and B independent events?

A: (a) Using the properties of P we write

- $0\leq P(A\cup B)\leq 1\Rightarrow P(A)+P(B)-P(A\cap B)\leq 1\Rightarrow P(A)+P(B)-1\leq P(A\cap B)$;
- $P(A\cap B)\leq P(A)\leq P(A\cup B)$ and $P(A\cap B)\leq P(B)\leq P(A\cup B)$

$\Rightarrow P(A\cap B)\leq\min\{P(A),P(B)\}\leq\max\{P(A),P(B)\}\leq P(A\cup B)$;

- $P(A\cup B)=P(A)+P(B)-P(A\cap B)\leq P(A)+P(B),$ since $-P(A\cap B)\leq 0.$

(b) We use the above inequalities

$$P(A)+P(B)-1\leq P(A\cap B)\leq\min\{P(A),P(B)\}\leq\max\{P(A),P(B)\}\leq P(A\cup B)\leq P(A)+P(B)$$

$$\Rightarrow\frac{2}{5}+\frac{9}{10}-1\leq P(A\cap B)\leq\frac{2}{5}<\frac{9}{10}\leq P(A\cup B)\leq\frac{2}{5}+\frac{9}{10}$$
$$\Rightarrow\frac{3}{10}\leq P(A\cap B)\leq\frac{2}{5}<\frac{9}{10}\leq P(A\cup B)\leq\frac{13}{10}$$

$$\Rightarrow a = \frac{3}{10}, b = \frac{2}{5}, c = \frac{9}{10}, d = 1.$$

(c1) From the definition of conditional probability

$$\Rightarrow P(A|B) = \frac{P(A \cap B)}{P(B)} \Rightarrow P(A \cap B) = P(A|B) \cdot P(B) = \frac{10}{27} \cdot \frac{9}{10} = \frac{1}{3};$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{2}{5} + \frac{9}{10} - \frac{1}{3} = \frac{29}{30};$$

$$P(B \cap \bar{A}) = P(B \setminus A) = P(B) - P(A \cap B) = \frac{9}{10} - \frac{1}{3} = \frac{17}{30};$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{3}}{\frac{2}{5}} = \frac{5}{6}.$$

(c2) Are A and B are not independent events, since

$$P(A \cap B) = \frac{1}{3} \neq P(A) \cdot P(B) = \frac{2}{5} \cdot \frac{9}{10} = \frac{9}{25}.$$

12. a) Let $(S, \mathcal{K}, P)$ be a probability space and $B \in \mathcal{K}$ such that $P(B) > 0$. Prove that $(B, \mathcal{K}_B, P(\cdot|B))$ is a probability space, where $\mathcal{K}_B := \{B \cap A : A \in \mathcal{K}\}$ and $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}, A \in \mathcal{K}_B$.
b) Give examples, by considering a random experiment and its corresponding probability space $(S, \mathcal{K}, P)$, for the probability space $(B, \mathcal{K}_B, P(\cdot|B))$ from **a)**.
A: a) Since $(S, \mathcal{K})$ is a measurable space, we deduce that $(B, \mathcal{K}_B)$ is a measurable space (note that $\mathcal{K}_B$ has the properties of a σ-field):
(i) $\mathcal{K}_B = B \cap \mathcal{K} \neq \emptyset$, because $B \neq \emptyset$ and $\mathcal{K} \neq \emptyset$;
(ii) if $A \in \mathcal{K}_B$, then $\bar{A} \in \mathcal{K}_B$, because there is $C \in \mathcal{K}$ such that $A = B \cap C$ and thus $\bar{A} = B \setminus A = B \setminus (B \cap C) = B \setminus C = B \cap \bar{C} \in B \cap \mathcal{K}$;
(iii) if $A_n \in \mathcal{K}_B$, $n \in \mathbb{N}^*$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{K}_B$, because there are $C_n \in \mathcal{K}$, $n \in \mathbb{N}^*$, such that $A_n = B \cap C_n$, so $\bigcup\limits_{n=1}^{\infty} A_n = \bigcup\limits_{n=1}^{\infty} (B \cap C_n) = B \cap \left(\bigcup\limits_{n=1}^{\infty} C_n \right) \in \mathcal{K}_B$.
Since P is a probability, we deduce that $P(\cdot|B)$ is a probability:
(1) $P(B|B) = \frac{P(B)}{P(B)} = 1$;
(2) $P(A|B) = \frac{P(A)}{P(B)} \geq 0$, for $A \in \mathcal{K}_B$;
(3) if $(A_n)_{n \geq 1}$ is a sequence of pairwise disjoint events from $\mathcal{K}_B$, then

$$P(\bigcup_{n=1}^{\infty} A_n | B) = \frac{P(\bigcup\limits_{n=1}^{\infty} A_n)}{P(B)} = \frac{\sum\limits_{n=1}^{\infty} P(A_n)}{P(B)} = \sum_{n=1}^{\infty} P(A_n|B).$$

b) Consider the experiment of rolling a die. Then we can choose $S = \{1, 2, 3, 4, 5, 6\}, \mathcal{K} = \mathcal{P}(S), P(A) = \frac{\#A}{6}, A \in \mathcal{K}$. Next, let $B = \{2, 4, 6\}$ (i.e., an even number shows up on the die). Then $\mathcal{K}_B = B \cap \mathcal{K} = \mathcal{P}(\{2, 4, 6\})$ and $P(A|B) = \frac{P(A)}{P(B)} = \frac{\#A}{3}, A \in \mathcal{K}_B$.