Seminar 9

Linear equations with constant coefficients.

General form:

(1) $y^{(n)} + a_1 \cdot y^{(n-1)} + \dots + a_{n-1} \cdot y' + a_n \cdot y = f(x)$

$a_1, a_2, \dots, a_n \in \mathbb{R}$, $f \in C(I)$

$\hookrightarrow$ nonhomogenous equation.

(2) $y^{(n)} + a_1 \cdot y^{(n-1)} + \dots + a_{n-1} \cdot y' + a_n \cdot y = 0.$

$\hookrightarrow$ homogenous equation.

Let $L[y] = y^{(n)} + a_1 \cdot y^{(n-1)} + \dots + a_n y$

$L : C^n(I) \to C(I)$ is a linear operator

We have that:

$(1) \Leftrightarrow L[y] = f$

$(2) \Leftrightarrow L[y] = 0.$

The general solution of (1): $y = y_0 + y_p$,

where y_0 - is the gen. sol of (2)

y_p - is a particular sol of (1).

A) Case $L[y]=0$

→ the characteristic equation:

(3) $$r^n + a_1 \cdot r^{n-1} + \dots + a_{n-1} \cdot r + a_n = 0 \rightarrow n \text{ roots}$$

- if r is a real root of (3) with the multiplicity μ, then:

$$y_1(x) = e^{rx}$$
$$y_2(x) = x e^{rx}$$
$$\dots$$
$$y_\mu(x) = x^{\mu-1} \cdot e^{rx}$$

- if $r = \alpha + i\beta \in \mathbb{C}$ - complex roots of (3) with the multiplicity μ, then:

$$y_1(x) = e^{\alpha x} \cdot \cos\beta x \quad ; \quad y_2(x) = e^{\alpha x} \cdot \sin\beta x$$
$$y_3(x) = x \cdot e^{\alpha x} \cdot \cos\beta x \quad ; \quad y_4(x) = x e^{\alpha x} \cdot \sin\beta x$$
$$\dots \qquad \dots$$
$$y_{2\mu-1}(x) = x^{\mu-1} e^{\alpha x} \cos\beta x \quad ; \quad y_{2\mu}(x) = x^{\mu-1} e^{\alpha x} \cdot \sin\beta x$$

$\Rightarrow \{y_1, y_2, \dots, y_n\}$ the fundamental system of solutions

$$\boxed{y_0(x) = c_1 \cdot y_1 + c_2 \cdot y_2 + \dots + c_n \cdot y_n \ , \ c_1, \dots c_n \in \mathbb{R}}$$

(B) Case $L[y] = f$

We look for $y_p = ?$

y_p can be found using the variation of constants method (see S8), $\forall f(x)$

$\{y_1, \dots, y_n\}$ f.s.s.

$\Rightarrow y_p = c_1(x) y_1 + \dots + c_n(x) y_n$

...

y_p can be found with different formulas (only in special cases for $f(x)$):

Special cases of $f(x)$ - (we can avoid the application of the var const meth)

I. $f(x) = P_m(x)$

a) $a_n \neq 0 \Rightarrow \boxed{y_p(x) = Q_m(x)}$

b) $a_n = a_{n-1} = \dots = a_{n-p+1} = 0$, $a_{n-p} \neq 0$

$\Rightarrow \boxed{y_p(x) = x^p \cdot Q_m(x)}$

II $f(x) = e^{rx} \cdot P_m(x)$

a) if r is NOT a root of (3)

$\Rightarrow \boxed{y_p(x) = e^{rx} \cdot Q_m(x)}$

b) if r is a root of (3) with multipl μ

$\Rightarrow \boxed{y_p(x) = x^{\mu} \cdot e^{rx} \cdot Q_m(x)}$

III $f(x) = e^{\alpha x} \cdot P_m(x) \cdot \cos\beta x$ or $f(x) = e^{\alpha x} \cdot P_m(x) \cdot \sin\beta x$

a) if $\alpha + i\beta$ is NOT a root of (3)

$\Rightarrow \boxed{y_p(x) = e^{\alpha x} \cdot \left(Q_m^1(x) \cdot \cos\beta x + Q_m^2(x) \cdot \sin\beta x\right)}$

b) if $\alpha + i\beta$ is a root of (3) with multiplicity μ

$\Rightarrow \boxed{y_p(x) = x^{\mu} \cdot e^{\alpha x} \cdot \left(Q_m^1(x) \cos\beta x + Q_m^2(x) \cdot \sin\beta x\right)}$

ex 1: $y''' - y = 0$.

- we write the characteristic eq:

$$r^3 - r^0 = 0 \iff r^3 - 1 = 0.$$

$$\iff (r-1)(r^2+r+1) = 0.$$

$r_1 = 1$

$\Delta = -3$

$$r_{2,3} = \frac{-1 \pm i\sqrt{3}}{2}$$

$\rightarrow r_1 = 1 \xRightarrow{\text{A real}} y_1(x) = e^{1\cdot x} = e^x$ ✓

$\rightarrow r_{2,3} = \underbrace{\boxed{-\frac{1}{2}}}_{\alpha = -\frac{1}{2}} \pm i \underbrace{\boxed{\frac{\sqrt{3}}{2}}}_{\beta = \frac{\sqrt{3}}{2}}$ (case A complex)

$$\Rightarrow y_2(x) = e^{\alpha x}\cos\beta x = e^{-\frac{1}{2}x}\cos\left(\frac{\sqrt{3}}{2}x\right) \checkmark$$

$$\Rightarrow y_3(x) = e^{\alpha x}\sin\beta x = e^{-\frac{1}{2}x}\cdot\sin\left(\frac{\sqrt{3}}{2}x\right) \checkmark$$

Then the general solution: $y = c_1 \cdot y_1 + c_2 \cdot y_2 + c_3 \cdot y_3$

$$\Rightarrow \boxed{y(x) = c_1 \cdot e^x + c_2 \cdot e^{-\frac{1}{2}x}\cos\left(\frac{\sqrt{3}}{2}x\right) + c_3 \cdot e^{-\frac{1}{2}x}\cdot\sin\left(\frac{\sqrt{3}}{2}x\right)}$$

$c_1, c_2, c_3 \in \mathbb{R}$

Ex2: $y''' - y'' = 0.$

- the charact eq: $r^3 - r^2 = 0.$

$\Leftrightarrow r^2(r-1) = 0.$

$r_1 = r_2 = 0$ $\qquad$ $r_3 = 1$

$\rightarrow r_1 = r_2 = 0$ $\rightarrow$ $y_1(x) = e^{r_1 x} = e^{0 \cdot x} = e^0 = 1$

$y_2(x) = x \cdot e^{0x} = x \cdot 1 = x$

$\rightarrow r_3 = 1 \longrightarrow y_3(x) = e^{r_3 \cdot x} = e^{1 \cdot x} = e^x$

$\Rightarrow$ The general solution:

$$y = c_1 \cdot y_1 + c_2 \cdot y_2 + c_3 \cdot y_3$$

$$y = c_1 \cdot 1 + c_2 \cdot x + c_3 \cdot e^x$$

$$\Rightarrow \boxed{y(x) = c_1 + c_2 x + c_3 e^x, \quad c_1, c_2, c_3 \in \mathbb{R}}$$

ex 3: $y'' + y = e^x$ $\to$ nonhomog eq.

$\Rightarrow$ sol : $y = y_0 + y_p$,

where $\to y_0$ = gen sol of $y'' + y = 0$.

y_p = a particular solution of $y'' + y = e^x$

st 1: $y'' + y = 0$.

• the charact eq : $r^2 + 1 = 0$. $\Rightarrow r_{1,2} = \pm i = \underbrace{0}_{\alpha=0} \pm \underbrace{1}_{\beta=1} \cdot i$

$$\Rightarrow \begin{aligned} y_1(x) &= e^{\alpha x} \cdot \cos\beta x = e^{0x} \cdot \cos(1\cdot x) = \cos x \\ y_2(x) &= e^{\alpha x} \cdot \sin\beta x = e^{0x} \cdot \sin(1\cdot x) = \sin x \end{aligned}$$

Follows that $\boxed{y_0 = c_1 \cdot y_1 + c_2 \cdot y_2 = c_1 \cdot \cos x + c_2 \cdot \sin x}$

$c_1, c_2 \in \mathbb{R}$

st 2: $y_p = ?$ We have $f(x) = e^x$ = case B II

$f(x) = e^x = \underbrace{1}_{\text{const} = \text{polinomial of degree } 0} \cdot e^{1\cdot x} = P_0(x) \cdot e^{1\cdot x}$

$\to r = 1 \neq$ root of charact eq.

$$\Longrightarrow y_p = Q_0(x) \cdot e^x = \underbrace{a}_{\text{constant}} \cdot e^x .$$

$$\left.\begin{aligned} y_p &= a\cdot e^x \quad \text{- sol of} \quad y_p'' + y_p = e^x \\ y_p' &= a e^x \\ y_p'' &= a e^x \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow a e^x + a e^x = e^x$$

$$2ae^x = e^x \quad \Big|: e^x$$

$$2a = 1 \quad \Rightarrow a = \frac{1}{2} \quad \Rightarrow \boxed{y_p = \frac{1}{2} e^x}$$

→ <u>st 3</u>: $y = y_0 + y_p$

$$\boxed{y(x) = c_1 \cdot \cos x + c_2 \cdot \sin x + \frac{1}{2} e^x, \quad c_1, c_2 \in \mathbb{R}}$$

the general solution of the initial equation.

ex4: $y''' - y'' = x + 1$ $\longrightarrow$ nonhomog eq.

$\longrightarrow$ st 1: $y''' - y'' = 0$.

- the charact eq: $r^3 - r^2 = 0$.

$$r^2(r-1) = 0.$$

$r_1 = r_2 = 0$; $r_3 = 1$

$y_1(x) = e^{0\cdot x} = 1$; $y_2(x) = x \cdot e^{0x} = x$; $y_3(x) = e^{1\cdot x} = e^x$

$$\Rightarrow \boxed{y_0(x) = c_1 + c_2 x + c_3 e^x \quad , c_1, c_2, c_3 \in \mathbb{R}.}$$

$\longrightarrow$ st 2: $y_p = ?$ here $f(x) = x + 1 = P_1(x)$

(case B. I) $m = 1$

R: here y, y' does not appear in the diff eq.

$\Rightarrow y_p = x^2 \cdot Q_1(x) = x^2 \cdot (ax + b) = ax^3 + bx^2$

But y_p is sol of $y_p''' - y_p'' = x + 1$ $\xrightarrow{\text{need}}$ y_p'', y_p'''

$$y_p = ax^3 + bx^2$$

$$y_p' = 3ax^2 + 2bx$$

$$y_p'' = 6ax + 2b$$

$$y_p''' = 6a$$

We replace y_p''', y_p'' in the nonhomog eq:

$\Rightarrow 6a - (6ax + 2b) = x + 1$

$-6ax + 6a - 2b = x + 1$, $\forall x$

$$\begin{cases} -6a = 1 \\ 6a - 2b = 1 \end{cases} \Rightarrow a = -1/6 \Rightarrow b = -1$$

$$\Rightarrow \boxed{y_p(x) = -\frac{1}{6}x^3 - x^2 = -\frac{x^2}{6}(x+6)}$$

st3: $y = y_0 + y_p$

$$\Rightarrow \boxed{y(x) = c_1 + c_2 x + c_3 e^x - \frac{x^2}{6}\cdot(x+6)} \quad , c_1, c_2, c_3 \in \mathbb{R}$$

ex 5: $y'' + y = 4x \cdot e^{-x} + 2\cdot \cos x.$

→ st1: $y'' + y = 0.$

- the charact eq : $r^2 + 1 = 0 \Rightarrow r_{1,2} = \pm i = 0 \pm 1\cdot i$

$$\left.\begin{array}{l} \Rightarrow y_1(x) = e^{0x}\cos(1\cdot x) = \cos x \\ y_2(x) = \sin x \end{array}\right\} \Rightarrow \boxed{y_0(x) = c_1\cdot \cos x + c_2 \cdot \sin x} \quad c_1, c_2 \in \mathbb{R}.$$

→ st2: we look for $y_p = ?$

Here : $f(x) = \underbrace{4x\cdot e^{-x}}_{f_1} + \underbrace{2\cdot\cos x}_{f_2} = f_1 + f_2$

(case B. II) (case B. III)

The superposition principle : in special cases,

For $L[y] = f_1 + f_2$, f_1, f_2

then we look for $\begin{cases} y_{p1} \text{ a partic. sol of } L[y] = f_1 \\ y_{p2} \text{ a partic sol of } L[y] = f_2 \end{cases}$

$\Rightarrow \boxed{y_p = y_{p1} + y_{p2}}$ is a particular sol. of $L[y] = f_1 + f_2$

R: $L[y_p] = L[y_{p1} + y_{p2}] = L[y_{p1}] + L[y_{p2}] = f_1 + f_2$

$\rightarrow f_1 = 4x \cdot e^{-x}$

- we look for y_{p1} sol of $y''_{p1} + y_{p1} = 4xe^{-x}$ $(*)$

- here $f_1 = 4x \cdot e^{-x} = P_1(x) \cdot e^{-1 \cdot x} \Rightarrow m = 1$

$r = -1$

is not a root of the charact eq

$$\Rightarrow y_{p1} = Q_1(x) \cdot e^{rx} = (ax+b) \cdot e^{-x}$$

$$y'_{p1} = -e^{-x}(ax+b) + e^{-x} \cdot a$$

$$= e^{-x}(-ax + a - b)$$

$$y''_{p1} = -e^{-x}(-ax + a - b) + e^{-x}(-a)$$

$$= e^{-x}(ax - 2a + b)$$

- replace y_{p1}, y''_{p1} in eq $(*)$:

$$e^{-x}(ax - 2a + b) + e^{-x}(ax + b) = 4x \cdot e^{-x} \quad | : e^{-x}$$

$$2ax - 2a + 2b = 4x \Rightarrow \begin{cases} 2a = 4 & \Rightarrow a = 2 \\ 2b - 2a = 0 & \Rightarrow b = 2 \end{cases}$$

$$\Rightarrow \boxed{y_{p1} = e^{-x}(2x+2) = 2e^{-x}(x+1)}$$

$$f_2(x) = 2\cos x = \underbrace{2}_{P_0 = 2} \cdot e^{0 \cdot x} \cdot \cos(1x)$$

$P_0 = 2 \rightarrow m = 0$; $\alpha = 0$; $\beta = 1$

? $\alpha + i\beta = 0 + i \cdot 1$ = is a root of charact eq.

YES $\rightarrow$ multiplicity $\mu = 1$

$$\Rightarrow y_{p2}(x) = x \cdot \left(\underbrace{Q_0^1(x)}_{\text{const}} \cdot e^{0 \cdot x} \cdot \cos(1 \cdot x) + \underbrace{Q_0^2(x)}_{\text{const}} \cdot e^{0 \cdot x} \cdot \sin x \right)$$

$$y_{p2}(x) = x \cdot (a \cdot \cos x + b \cdot \sin x)$$

y_{p2} is a solution of the eq: $y_{p2}'' + y_{p2} = 2 \cdot \cos x$

- we need y_{p2}'' in order to replace in equation.

$$y_{p2} = x(a\cos x + b\sin x)$$

$$y_{p2}' = a\cos x + b\sin x + x(-a\sin x + b\cos x)$$

$$y_{p2}'' = -a\sin x + b\cos x + (-a\sin x + b\cos x) + x(-a\cos x - b\sin x)$$

$$-2a\sin x+2b\cos x \underline{-x(a\cos x\cancel{+b\sin x})}+$$
$$+x(a\cos x\cancel{+b\sin x}) = 2\cos x$$

$$\Rightarrow -2a\cdot\sin x + 2b\cdot\cos x = 2\cdot\cos x$$

$$\Rightarrow \begin{cases} -2a=0 \\ 2b=2 \end{cases} \Rightarrow \begin{cases} a=0 \\ b=1 \end{cases}$$

$$\Longrightarrow \boxed{y_{p_2} = \sin x}$$

Thus: $y_p = y_{p_1} + y_{p_2}$

$$\boxed{y_p = 2e^{-x}(x+1) + \sin x}$$

→ $\underline{st3}$: $y = y_0 + y_p$

$$\boxed{y(x) = c_1\cdot\cos x + c_2\cdot\sin x + 2e^{-x}(x+1)+\sin x}$$

$\underline{c_1, c_2 \in \mathbb{R}}$.

ex 6: $y'' - y = \dfrac{2e^x}{e^x - 1}$

→ st 1 : $y'' - y = 0$

· the char eq : $r^2 - 1 = 0$.

$r_1 = 1 \Rightarrow y_1 = e^{1x} = e^x$; $r_2 = -1 \Rightarrow y_2 = e^{-1 \cdot x} = e^{-x}$

$$\Rightarrow \boxed{y_0 = C_1 \cdot e^x + C_2 \cdot e^{-x} \quad , C_1, C_2 \in \mathbb{R}}$$

→ st 2 : $y_p = ?$ $\qquad f(x) = \dfrac{2e^x}{e^x - 1}$

Here $f(x)$ does not belong to one of the above special cases. We have to apply the variation of the constants method.

We look for a particular solution of the form:

$$y_p(x) = c_1(x) \cdot e^x + c_2(x) \cdot e^{-x}$$

y_p - sol of $y_p'' - y_p = \dfrac{2e^x}{e^x - 1}$

$$y_p' = \underline{c_1'(x)\cdot e^{x}} + c_1(x)\cdot e^{x} + \underline{c_2'(x)\cdot e^{-x}} - c_2(x)e^{-x}$$

- we impose the cond: $c_1'(x)\cdot y_1 + c_2'(x) y_2 = 0.$

$\rightarrow$ here $\underline{c_1'(x)\cdot e^{x} + c_2'(x)\cdot e^{-x} = 0}$

$$\Rightarrow y_p' = c_1(x)\cdot e^{x} - c_2(x)\cdot e^{-x}$$

$$y_p'' = c_1'(x)\cdot e^{x} + c_1(x) e^{x} - c_2'(x)\cdot e^{-x} + c_2(x) e^{-x}$$

- We replace in equation:

$$c_1'(x)\cdot e^{x} + \underline{c_1(x)\, e^{x}} - c_2'(x)\cdot e^{-x} + c_2(x)\cdot e^{-x} - c_1(x) e^{x} - c_2(x)\cdot e^{-x} = \frac{2e^{x}}{e^{x}-1}$$

$$\Rightarrow c_1'(x) e^{x} - c_2'(x)\cdot e^{-x} = \frac{2e^{x}}{e^{x}-1}$$

Thus, we have the system:

$$\begin{cases} c_1'(x) e^{x} + c_2'(x) e^{-x} = 0 \\ c_1'(x) e^{x} - c_2'(x) e^{-x} = \dfrac{2e^{x}}{e^{x}-1} \end{cases} \quad +$$

$$2c_1'(x) e^{x} = \frac{2e^{x}}{e^{x}-1} \quad \Big| : 2e^{x}$$

$$\Rightarrow \boxed{C_1'(x) = \frac{1}{e^x - 1}}$$

Follows : $C_2'(x) \cdot e^{-x} = - C_1'(x) \cdot e^x -$

$$\boxed{C_2'(x) = - \frac{e^{2x}}{e^x - 1}}$$

$$\Rightarrow C_1(x) = \int \frac{dx}{e^x - 1} = \int \frac{1 - e^x + e^x}{e^x - 1} dx$$

$$= \int \left(-1 + \frac{e^x}{e^x - 1} \right) dx$$

$$= -x + \ln |e^x - 1|$$

$$\Rightarrow C_2(x) = \int - \frac{e^x \cdot e^x}{e^x - 1} dx = \int - \frac{(e^x - 1 + 1) e^x}{e^x - 1} dx =$$

$$= \int - \frac{(e^x - 1) e^x}{e^x - 1} dx - \int \frac{e^x}{e^x - 1} dx$$

$$= - \int e^x dx - \int \frac{e^x}{e^x - 1} dx$$

$$= - e^x - \ln |e^x - 1|$$

$$\Rightarrow \boxed{y_p(x) = \left(-x + \ln|e^x - 1|\right)e^x - 1 - e^{-x} \cdot \ln|e^x - 1|}$$

→ st 3 : $y = y_0 + y_p$

...