Seminar 2 Analiză complexă

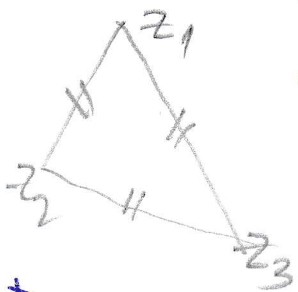
Aplicații ale numerelor complexe în geometrie

① Fie $z_1, z_2, z_3 \in \mathbb{C}^*$ trei puncte distincte astfel încât

$$|z_1| = |z_2| = |z_3| \text{ și } z_1 + z_2 + z_3 = 0.$$

Să se arate că z_1, z_2, z_3 sunt afixele vârfurilor unui triunghi echilateral.

Soluție: z_1, z_2, z_3 sunt afixele vârfurilor unui Δ echilateral $\Longleftrightarrow$



$$|z_1 - z_2| = |z_1 - z_3| = |z_2 - z_3|. \; (*)$$

Notăm $r = |z_j|$, $j = \overline{1,3}$.

Deoarece $\left.\begin{array}{l} z_1, z_2, z_3 \in \mathbb{C}^* \\ |z_j| = r, \; j = \overline{1,3} \end{array}\right\} \Longrightarrow z_j = r(\cos\theta_j + i \sin\theta_j)$,

unde $\theta_j \in \operatorname{Arg} z_j$, $j = \overline{1,3}$.

Atunci $(*) \Longleftrightarrow |z_1 - z_2|^2 = |z_1 - z_3|^2 = |z_2 - z_3|^2$.

Dar $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\operatorname{Re}(z_1 \cdot \overline{z_2}) =$

$= r^2 + r^2 - 2\operatorname{Re}\big(r(\cos\theta_1 + i\sin\theta_1) \cdot r(\cos\theta_2 - i\sin\theta_2)\big)$

$= 2r^2 - 2r^2 \operatorname{Re}\big(\cos\theta_1 \cdot \cos\theta_2 + \sin\theta_1 \sin\theta_2$

$+ i(\sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2)\big)$

$= 2r^2 - 2r^2 \big(\underbrace{\cos\theta_1 \cdot \cos\theta_2 + \sin\theta_1 \cdot \sin\theta_2}_{= \cos(\theta_1 - \theta_2)}\big)$

$= 2r^2(1-\cos(\theta_1-\theta_2))$.

Deci: $|z_1-z_2|^2 = 2r^2(1-\cos(\theta_1-\theta_2))$.

Analog, $|z_1-z_3|^2 = 2r^2(1-\cos(\theta_1-\theta_3))$

și $|z_2-z_3|^2 = 2r^2(1-\cos(\theta_2-\theta_3))$.

Așadar, $|z_1-z_2|^2 = |z_1-z_3|^2 = |z_2-z_3|^2 \iff$

$$\boxed{\cos(\theta_1-\theta_2) = \cos(\theta_1-\theta_3) = \cos(\theta_2-\theta_3)} \quad (**)$$

Dar $z_1+z_2+z_3=0 \iff r\left(\sum_{j=1}^{3}\cos\theta_j + i\sum_{j=1}^{3}\sin\theta_j\right)=0$

$\iff \sum_{j=1}^{3}\cos\theta_j = 0$ și $\sum_{j=1}^{3}\sin\theta_j = 0$.

$$\Rightarrow \begin{cases} \cos\theta_1+\cos\theta_2 = -\cos\theta_3 \\ \sin\theta_1+\sin\theta_2 = -\sin\theta_3 \end{cases} \Rightarrow \begin{cases} (\cos\theta_1+\cos\theta_2)^2 = \cos^2\theta_3 \\ (\sin\theta_1+\sin\theta_2)^2 = \sin^2\theta_3 \end{cases}$$

adunare

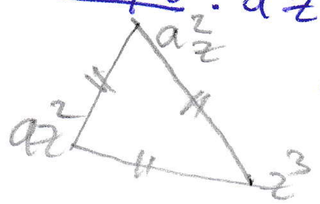
$$\Rightarrow 1+1+2(\underbrace{\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2}_{=\cos(\theta_1-\theta_2)}) = 1 \Rightarrow$$

$$\Rightarrow \boxed{\cos(\theta_1-\theta_2) = -\frac{1}{2}}.$$

Analog, $\cos(\theta_1-\theta_3) = \cos(\theta_2-\theta_3) = -\frac{1}{2}$ $\Big\} \Rightarrow (**)$ - OK

2 Fie $a \in \mathbb{C}^*$. Să se determine toate punctele $z \in \mathbb{C}^*$ cu proprietatea că a^2z, az^2 și z^3 sunt afixele vârfurilor unui triunghi echilateral.

Soluție: a^2z, az^2 și z^3 sunt afixele vf. unui Δ echilateral



$\iff |a^2z-az^2| = |az^2-z^3| = |a^2z-z^3| \iff$

$|a|\cdot|z|\cdot|a-z| = |z|^2\cdot|a-z| = |z|\cdot|a-z|\cdot|a+z| \;\Big|:|z|$

$\Leftrightarrow |a|\cdot|a-z| = |z|\cdot|a-z| = |a-z|\cdot|a+z| \;\binom{*}{*}$

Dacă $|a-z|=0 \Leftrightarrow z=a \Rightarrow az^2=a^2z=z^3=a^3$.

Presupunem că $z\neq a$. Atunci $\binom{*}{*} \Leftrightarrow |a|=|z|=|a+z|$.

Fie $A:=\{z\in\mathbb{C}: |z|=|z+a|=|a|\}$

$B:=\{z\in\mathbb{C}: |z|=|a|\}$

$C:=\{z\in\mathbb{C}: |z+a|=|a|\}$.

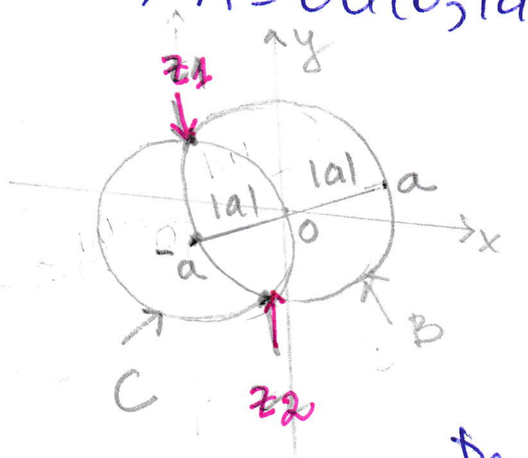
$\Rightarrow \boxed{A=B\cap C}$.

Dar $z\in B \Leftrightarrow z\in\partial U(0,|a|) \Rightarrow \boxed{B=\partial U(0,|a|)}$

$z\in C \Leftrightarrow |z-(-a)|=|a| \Leftrightarrow z\in\partial U(-a,|a|)$

$\Rightarrow \boxed{C=\partial U(-a,|a|)}$

$\Rightarrow A=\partial U(0,|a|)\cap\partial U(-a,|a|)=\{z_1,z_2\}$.



<u>Soluția a doua</u> (analitică)

$z\in A \Leftrightarrow |z|=|a|=|z+a| \Leftrightarrow$

$|z|^2=|a|^2=|z+a|^2$.

Dar $|z+a|^2=|z|^2+|a|^2+2\mathrm{Re}(z\cdot\bar a)$

<u>Deci</u>: $z\in A \Leftrightarrow |z|^2=|a|^2$ și

$\cancel{|z|^2}+|a|^2+2\mathrm{Re}(z\cdot\bar a)=\cancel{|z|^2}$

$\Rightarrow |z|^2=|a|^2$ și $2\mathrm{Re}(z\cdot\bar a)=-|a|^2 \;\Big|:|a|^2\neq 0$

$\Rightarrow |z|^2=|a|^2$ și $\mathrm{Re}\left(\frac{z\cdot\bar a}{|a|^2}\right)=-\frac{1}{2} \Rightarrow$

$\Rightarrow |z|^2 = |a|^2$ și $Re\left(\frac{z}{a}\right) = -\frac{1}{2}$.

Deci, am obținut că $Re\left(\frac{z}{a}\right) = -\frac{1}{2}$ și $\left|\frac{z}{a}\right| = 1 \Rightarrow$

$$\Rightarrow \left(Im\,\frac{z}{a}\right)^2 = \left|\frac{z}{a}\right|^2 - \left(Re\,\frac{z}{a}\right)^2 = 1 - \frac{1}{4} = \frac{3}{4} \Rightarrow Im\,\frac{z}{a} = \pm\frac{\sqrt{3}}{2}.$$

$$\Rightarrow \frac{z}{a} = Re\left(\frac{z}{a}\right) + i\,Im\left(\frac{z}{a}\right) = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} \Rightarrow$$

$$\Rightarrow \boxed{z = a\left(-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right)} \text{ (2 soluții).}$$

3 Fie $A, C \in \mathbb{R}$ și $B \in \mathbb{C}$ astfel încât $|B|^2 > AC$. Să se arate că expresia

(1) $A|z|^2 + \bar{B}z + B\bar{z} + C = 0$

reprezintă ecuația unui cerc în sens larg (cerc sau dreaptă). În cazul cercului, să se determine centrul și raza sa.

<u>Soluție</u>: Cazul I. $\boxed{A = 0} \Rightarrow (1) \Leftrightarrow \bar{B}z + B\bar{z} + C = 0$

$\Leftrightarrow 2Re(\bar{B}z) + C = 0$. (1')

Fie $B = b_1 + ib_2$ și $z = x + iy \Rightarrow Re(\bar{B}z) = Re[(b_1 - ib_2)(x+iy)]$

$= b_1x + b_2y$.

Deci (1') $\Leftrightarrow 2(b_1x + b_2y) + C = 0 \Leftrightarrow$

$b_1x + b_2y + \frac{C}{2} = 0 \Rightarrow$ ecuația unei drepte.

Cazul II: $\boxed{A\neq 0} \Rightarrow (1) \Leftrightarrow |z|^2+\frac{\overline{B}}{A}z+\frac{B}{A}\overline{z}+\frac{C}{A}=0 \quad (2).$

Dar $A\in\mathbb{R} \Leftrightarrow A=\overline{A} \Rightarrow \overline{\left(\frac{B}{A}\right)}=\frac{\overline{B}}{A}.$

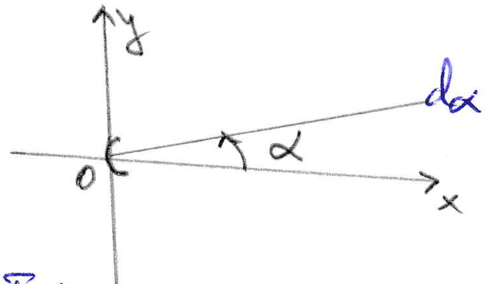
Deci $(2) \Leftrightarrow |z|^2+2\operatorname{Re}\left(\frac{\overline{B}}{A}z\right)+\frac{C}{A}=0 \Leftrightarrow$

$\Leftrightarrow |z|^2+2\operatorname{Re}\left(\frac{\overline{B}}{A}z\right)+\left|\frac{B}{A}\right|^2+\frac{C}{A}-\left|\frac{B}{A}\right|^2=0 \Leftrightarrow$

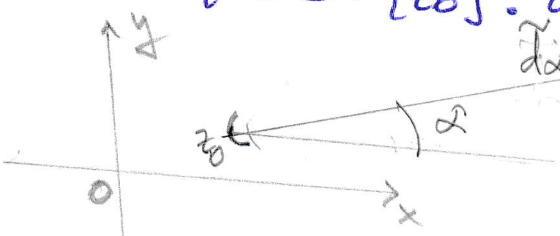
$\Leftrightarrow \left|z+\frac{B}{A}\right|^2+\frac{C}{A}-\left|\frac{B}{A}\right|^2=0 \Leftrightarrow \left|z+\frac{B}{A}\right|^2=\overbrace{\frac{|B|^2-AC}{A^2}}^{>0}$

$\Leftrightarrow \left|z+\frac{B}{A}\right|=\sqrt{\frac{|B|^2-AC}{A^2}} \Rightarrow$ cercul cu centrul în $-\frac{B}{A}$ și raza $\sqrt{\frac{|B|^2-AC}{A^2}}$.

Obs: Fie $\alpha\in\left[0,\frac{\pi}{2}\right] \Rightarrow \{z\in\mathbb{C}^*: \arg z=\alpha\}$ - semidreaptă deschisă care pornește din origine (nu conține originea) și determină $\measuredangle\alpha$ cu sensul pozitiv al axei reale. OK



Fie $z_0\in\mathbb{C} \Rightarrow \{z\in\mathbb{C}\setminus\{z_0\}: \arg(z-z_0)=\alpha\}=?$



$z-z_0\in d_\alpha \Rightarrow$

$\Rightarrow z\in z_0+d_\alpha=\tilde{d}_\alpha.$

④ Să se reprezinte în planul complex următoarele mulțimi de puncte:

a) $\left\{z\in\mathbb{C}\setminus\{-i\}:\ \frac{\pi}{4}<\arg(z+i)<\frac{\pi}{2}\right\}$

b) $\left\{z\in\mathbb{C}:\ \operatorname{Re}[z(1-i)]<\sqrt{2}\right\}$

c) $\left\{z\in\mathbb{C}\setminus\{\pm i\}:\ 0<\arg\left(\frac{i-z}{i+z}\right)<\frac{\pi}{2}\right\}$

d) $\left\{z\in\mathbb{C}:\ |z-i|+|z+i|<4\right\}$

e) $\left\{z\in\mathbb{C}^*:\ \operatorname{Re}\left(\frac{1}{z}\right)<\frac{1}{2}\right\}$.

Soluţii

a) $A:=\left\{z\in\mathbb{C}\setminus\{-i\}:\ \frac{\pi}{4}<\arg(z+i)<\frac{\pi}{2}\right\}$.

$B:=\left\{z\in\mathbb{C}\setminus\{-i\}:\ \arg(z+i)=\frac{\pi}{2}\right\}$

$C=\left\{z\in\mathbb{C}\setminus\{-i\}:\ \arg(z+i)=\frac{\pi}{4}\right\}$

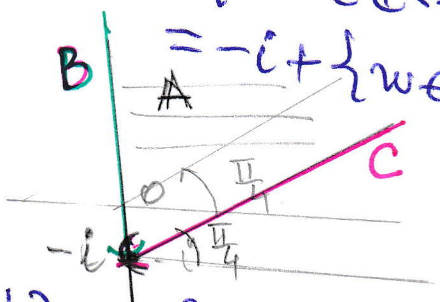
$$\Rightarrow B=\left\{z\in\mathbb{C}\setminus\{-i\}:\ \arg(\underbrace{z+i}_{=w})=\frac{\pi}{2}\right\}= \quad \Rightarrow z=w-i$$

$$=\left\{w-i\in\mathbb{C}\setminus\{-i\}:\ \arg w=\frac{\pi}{2}\right\}$$

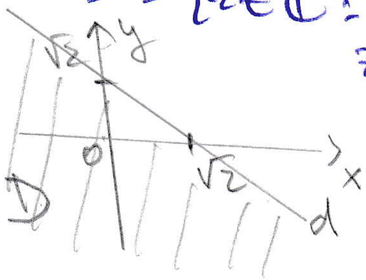
$$=-i+\left\{w\in\mathbb{C}^*:\ \arg w=\frac{\pi}{2}\right\}$$

$$C=\left\{w-i\in\mathbb{C}\setminus\{-i\}:\ \arg w=\frac{\pi}{4}\right\}$$

$$=-i+\left\{w\in\mathbb{C}^*:\ \arg w=\frac{\pi}{4}\right\}.$$



b) $D:=\left\{z\in\mathbb{C}:\ \operatorname{Re}[z(1-i)]<\sqrt{2}\right\}=\left\{(x,y)\in\mathbb{R}^2:\ x+y<\sqrt{2}\right\}$



$z=x+iy\Rightarrow \operatorname{Re}[z(1-i)]=\operatorname{Re}[(x+iy)(1-i)]=$

$=x+y$

c) $E := \left\{ z \in \mathbb{C} \setminus \{\pm i\} : 0 < \arg\left(\frac{i-z}{i+z}\right) < \frac{\pi}{2} \right\}$.

$\Rightarrow E = \left\{ z \in \mathbb{C} \setminus \{\pm i\} : \mathrm{Re}\left(\frac{i-z}{i+z}\right) > 0, \mathrm{Im}\left(\frac{i-z}{i+z}\right) > 0 \right\}$.

Dar $\frac{i-z}{i+z} = \frac{(i-z)(-i+\bar{z})}{|i+z|^2} = \frac{1+i\bar{z}+iz-|z|^2}{|i+z|^2} =$

$= \frac{1-|z|^2+i(z+\bar{z})}{|i+z|^2} = \frac{1-|z|^2+2i\cdot \mathrm{Re}\,z}{|i+z|^2}$

$= \frac{1-|z|^2}{|i+z|^2} + \frac{2\mathrm{Re}\,z}{|i+z|^2} \cdot i$

$\Rightarrow \mathrm{Re}\left(\frac{i-z}{i+z}\right) = \frac{1-|z|^2}{|i+z|^2}$ şi $\mathrm{Im}\left(\frac{i-z}{i+z}\right) = \frac{2\mathrm{Re}\,z}{|i+z|^2}$.

Deci: $E = \left\{ z \in \mathbb{C} \setminus \{\pm i\} : \frac{1-|z|^2}{|i+z|^2} > 0, \frac{2\mathrm{Re}\,z}{|i+z|^2} > 0 \right\}$

$= \left\{ z \in \mathbb{C} \setminus \{\pm i\} : 1-|z|^2 > 0, \mathrm{Re}\,z > 0 \right\}$

$= \left\{ z \in \mathbb{C} \setminus \{\pm i\} : |z| < 1, \mathrm{Re}\,z > 0 \right\}$

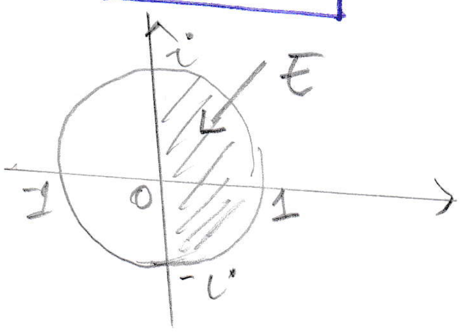
$= \left\{ z \in \mathbb{C} : |z| < 1, \mathrm{Re}\,z > 0 \right\}$.

$\Rightarrow E = \left\{ z \in \mathbb{C} : |z| < 1, \mathrm{Re}\,z > 0 \right\}$.

Fie $F = \{ z \in \mathbb{C} : |z| < 1 \} \Rightarrow \boxed{F = U(0,1)}$

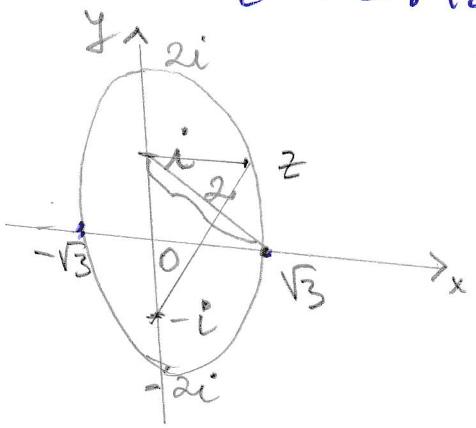
$G = \{ z \in \mathbb{C} : \mathrm{Re}\,z > 0 \}$ - semiplanul drept.

$\Rightarrow \boxed{E = F \cap G}$.



d) $A := \{z\in\mathbb{C} : |z-i|+|z+i|<4\}$

$\Rightarrow \partial A = \{z\in\mathbb{C} : |z-i|+|z+i|=4\}$ – elipsa



Deci A = interiorul elipsei obținute

$z=x\in\mathbb{R}$ cu $|x-i|+|x+i|=4$

$\Rightarrow \sqrt{x^2+1}+\sqrt{x^2+1}=4 \Leftrightarrow 2\sqrt{x^2+1}=4$

$\Leftrightarrow \sqrt{x^2+1}=2 \Leftrightarrow x^2=3 \Leftrightarrow x=\pm\sqrt{3}$

Dacă $z=iy$, $y\in\mathbb{R}$ cu $|y|>1 \Rightarrow$

$\Rightarrow |iy-i|+|iy+i|=4 \Leftrightarrow$

$\Leftrightarrow \left.\begin{array}{l}|y-1|+|y+1|=4\\ |y|>1\end{array}\right\} \Rightarrow y=\pm 2$

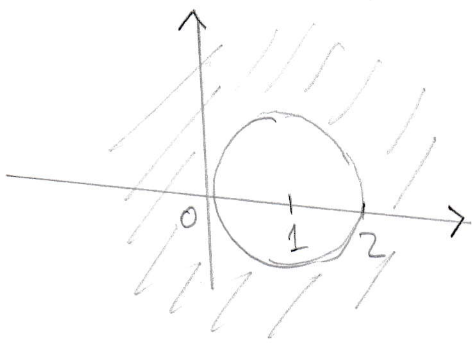
Sol 2: $z=x+iy \Rightarrow A=\{(x,y)\in\mathbb{R}^2 : \sqrt{x^2+(y-1)^2}+\sqrt{x^2+(y+1)^2}<4\}$

$\Downarrow$ ......

$|z-i|=|x+i(y-1)|=\sqrt{x^2+(y-1)^2}$

$|z+i|=\sqrt{x^2+(y+1)^2}$

e) $\{z\in\mathbb{C}^* : Re\left(\frac{1}{z}\right)<\frac{1}{2}\}$.



Fie $B=\{z\in\mathbb{C}^* : Re\left(\frac{1}{z}\right)<\frac{1}{2}\}$.

$z\in B \Leftrightarrow z\in\mathbb{C}^*$ și $Re\left(\frac{1}{z}\right)<\frac{1}{2}$.

Dar $Re\left(\frac{1}{z}\right)=Re\left(\frac{\bar{z}}{|z|^2}\right)=\frac{Re\,\bar{z}}{|z|^2}=\frac{Re\,z}{|z|^2}$.

Deci: $Re\left(\frac{1}{z}\right)<\frac{1}{2} \Leftrightarrow \frac{Re\,z}{|z|^2}<\frac{1}{2} \Leftrightarrow 2Re\,z<|z|^2$

$\Leftrightarrow \underbrace{|z|^2-2Re\,z+1}_{=|z-1|^2}-1>0 \Leftrightarrow |z-1|^2-1>0 \Leftrightarrow |z-1|>1$.

$\Rightarrow B=\{z\in\mathbb{C}^* : |z-1|>1\}=\{z\in\mathbb{C} : |z-1|>1\}$ – exteriorul discului cu centrul în $z_0=1$ și raza 1.