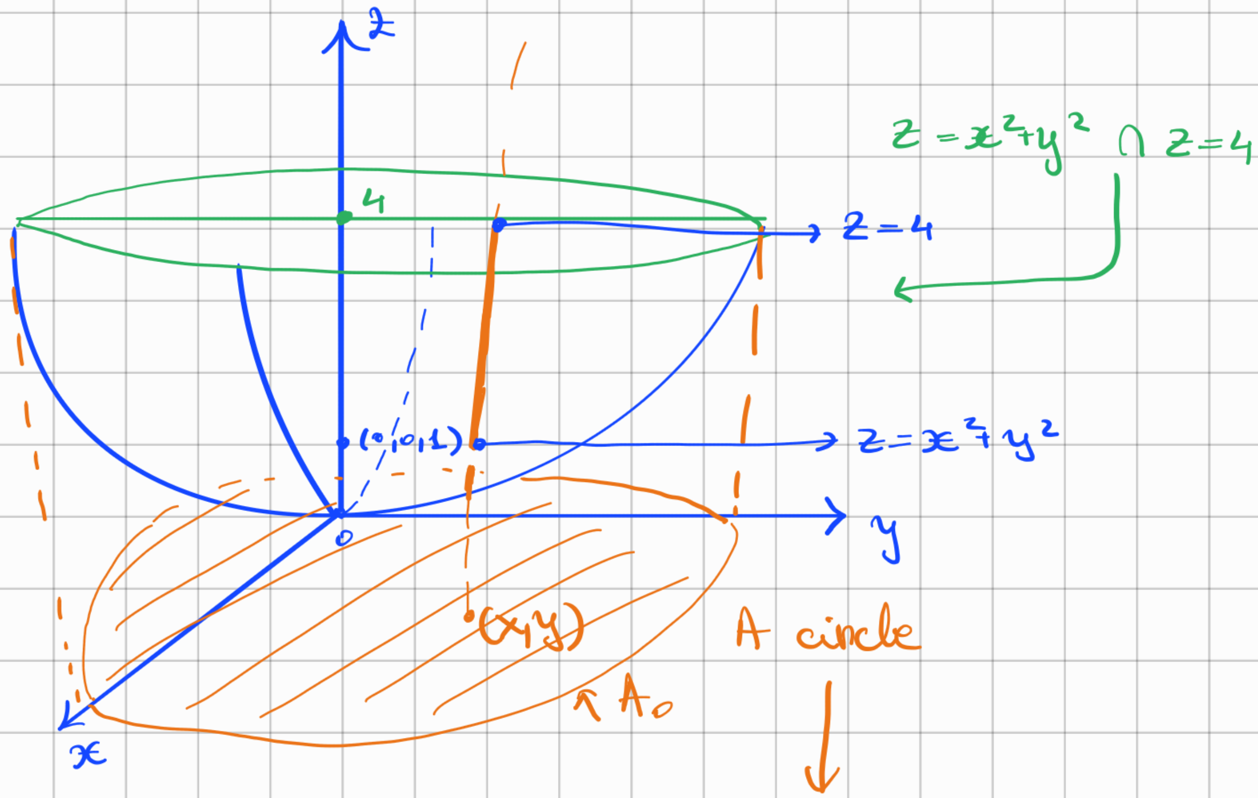
Seminar 9

08.05.2025

$\boxed{1}$ Compute $I = \iiint_A (x^2+y^2)\,dx\,dy\,dz$;

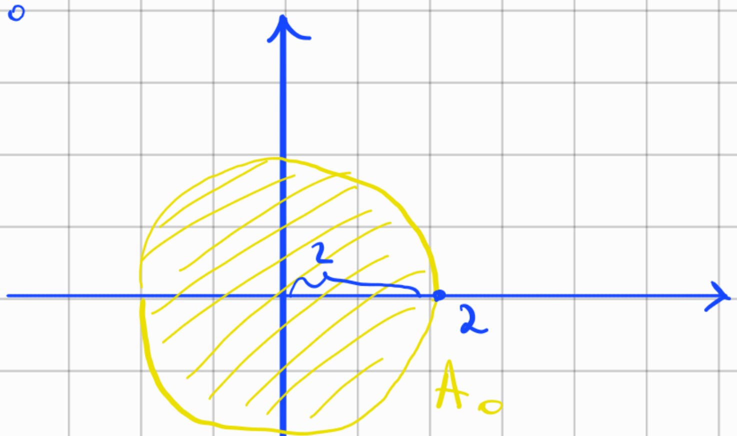
$A = \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2 \le z \le 4\}$



We project the set onto OXY plane

$$I = \iint_{A_0}\left(\int_{z=x^2+y^2}^{z=4}(x^2+y^2)\,dz\right)dx\,dy$$

$$= \iint_{A_0}(x^2+y^2)(4-x^2-y^2)\,dx\,dy$$



$x = \rho\cos\theta$ $\qquad \rho\in[0;2)$ $\qquad x^2+y^2=\rho^2$

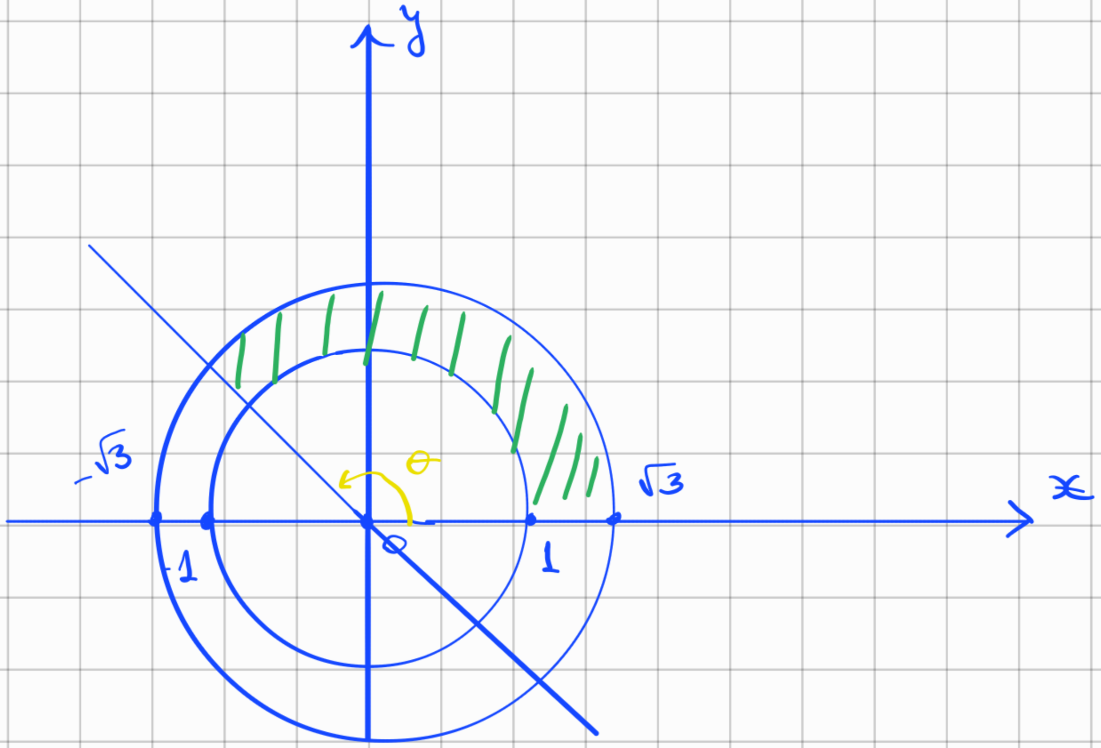
$y = \rho\sin\theta$ $\qquad \theta\in[0;2\pi]$

$$I = \int_{\rho=0}^{\rho=2}\left(\int_{\theta=0}^{\theta=2\pi} \rho^2(4-\rho^2)\right)\cdot \rho \, d\rho \, d\theta$$

$$= \int_0^2 (4\cdot\rho^3 - \rho^5)\,d\rho \cdot \int_0^{2\pi} d\theta$$

$$= 2\pi\cdot\left(4\cdot \frac{\rho^4}{4}\Big|_0^2 - \frac{\rho^6}{6}\Big|_0^2\right) = 2\pi\cdot\left(16 - \frac{32}{3}\right)$$

$\boxed{2}$ Compute $I = \iint_A x\cdot\sqrt{3-x^2-y^2}\,dx\,dy$ if $A = \{(x,y)\in\mathbb{R}^2 \mid 1\le x^2+y^2\le 3,\ x+y\ge 0,\ y\ge 0\}$



$x = \rho\cos\theta \qquad \rho\in[1;\sqrt{3}]$

$y = \rho\sin\theta \qquad \theta\in[0;\frac{3\pi}{4}]$

$$I = \int_{\rho=1}^{\rho=\sqrt{3}}\int_{\theta=0}^{\theta=\frac{3\pi}{4}} \rho\cos\theta\cdot\sqrt{3-\rho^2}\cdot\rho\,d\rho\,d\theta$$

$$= \int_1^{\sqrt{3}} \rho^2\cdot\sqrt{3-\rho^2}\,d\rho \cdot \underbrace{\int_0^{\frac{3\pi}{4}}\cos\theta\,d\theta}_{\sin\theta\big|_0^{\frac{3\pi}{4}} = \frac{\sqrt{2}}{2}}$$

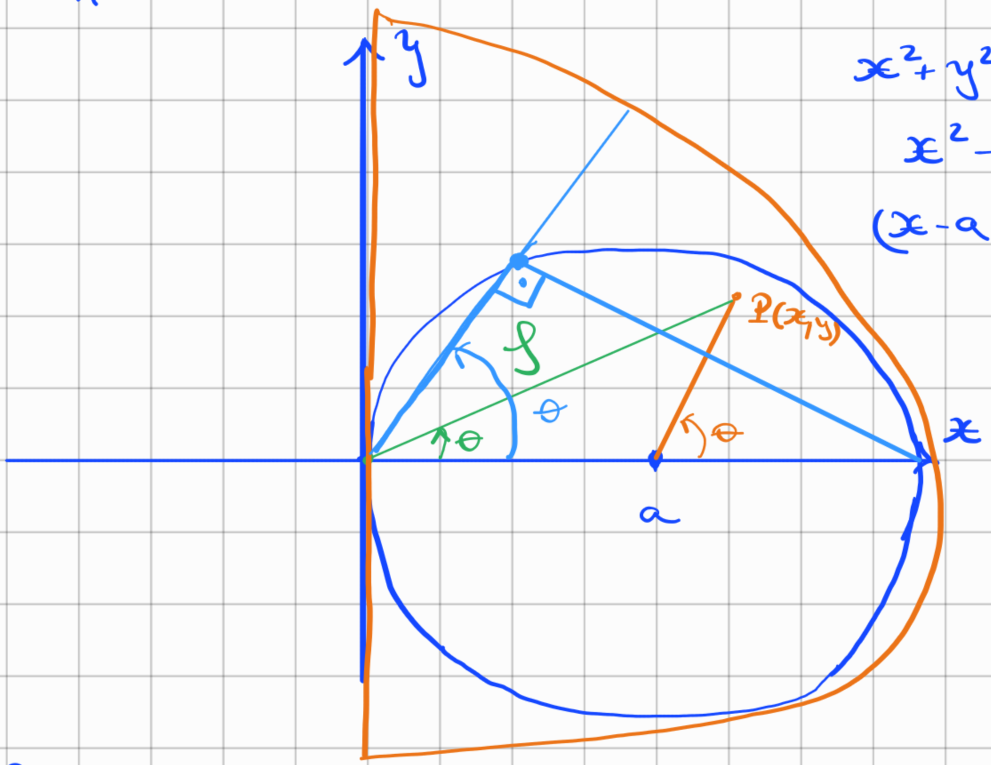
$\rho = \sqrt{3} \cdot \sin t$ $\qquad \boxed{\alpha = \arcsin \frac{1}{\sqrt{3}}}$

$$= \frac{\sqrt{2}}{2} \cdot \int_{\alpha}^{\frac{\pi}{2}} 3 \cdot \sin^2 t \; \sqrt{\underbrace{3 - 3\sin^2 t}_{3(1-\sin^2 t)}} \; \sqrt{3} \cdot \cos t \, dt$$

$$= \frac{1}{4} \cdot \frac{9\sqrt{2}}{2} \cdot \int_{\alpha}^{\frac{\pi}{2}} 4 \sin^2 t \cdot \cos^2 t \, dt = \frac{9\sqrt{2}}{8} \int_{\alpha}^{\frac{\pi}{2}} \underbrace{\sin^2 2t}_{\frac{1-\cos 4t}{2}} \, dt$$

$$= \frac{9\sqrt{2}}{16} \cdot \left(t - \frac{\sin 4t}{4} \Big|_{\alpha}^{\frac{\pi}{2}} \right) = \underline{\frac{9\sqrt{2}}{16} \cdot \left(\frac{\pi}{2} - \alpha + \frac{\sin 4\alpha}{4} \right)}$$

$\boxed{3}$ $I = \iint_A \sqrt{x^2+y^2} \, dx dy$; $A = \{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 2ax,$ $a > 0 \}$



$x^2 + y^2 - 2ax \le 0$

$x^2 - 2ax + a^2 + y^2 \le a^2$

$(x-a)^2 + y^2 \le a^2$

M1

$x - a = \rho \cos\theta$ $\qquad \rho \in [0; a]$

$y = \rho \sin\theta$ $\qquad \theta \in [0; 2\pi]$

$$I = \int_{\rho=0}^{\rho=a} \int_{\theta=0}^{\theta=2\pi} \sqrt{(a+\rho\cos\theta)^2 + \rho^2 \sin^2\theta} \cdot \rho \, d\rho \, d\theta$$

$$= \int_{\rho=0}^{\rho=a} \int_{\theta=0}^{\theta=2\pi} \rho \cdot \sqrt{a^2 + 2a\rho\cos\theta + \rho^2} \, d\rho \, d\theta$$

X

$\textcircled{M2}$ → with green

$x = \rho\cos\theta$
$y = \rho\sin\theta$

we integr. over a bigger set (with orange)

~~$\rho \in [0; 2a]$~~
$\theta \in [-\frac{\pi}{2}; \frac{\pi}{2}]$
$\rho \in [0; 2a\cos\theta]$

$$x^2+y^2 \le 2ax \Leftrightarrow \rho^{\cancel{2}} \le 2a\cdot\cancel{\rho}\cdot\cos\theta \Leftrightarrow \rho \le 2a\cos\theta$$

AND $\rho \ge 0$ (it's a distance)

$$I = \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \left(\int_{\rho=0}^{\rho=2a\cos\theta} \rho^2\, d\rho \right) d\theta = \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \frac{\rho^3}{3}\Big|_0^{2a\cos\theta} d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{8a^3\cos^3\theta}{3}\, d\theta = \frac{8a^3}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-\sin^2\theta)\cos\theta\, d\theta$$

$\sin\theta = t$

$$= \frac{8a^3}{3}\int_{-1}^{1}(1-t^2)\,dt = \dots$$

$\boxed{4}$ $I = \iiint_A \ln(1+\sqrt{x^2+y^2+z^2})\,dx\,dy\,dz$ where

$A: x^2+y^2+z^2 \le 1$

$x = \rho\sin\varphi\cos\theta$ $\qquad \rho \in [0; 1]$
$y = \rho\sin\varphi\sin\theta$ $\qquad \varphi \in [0; \pi]$
$z = \rho\cos\varphi$ $\qquad \theta \in [0; 2\pi]$

$$I = \int_{\rho=0}^{\rho=1}\int_{\varphi=0}^{\varphi=\pi}\int_{\theta=0}^{\theta=2\pi} \ln(1+\rho)\,\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \underbrace{\int_0^1 \rho^2\cdot\ln(1+\rho)\,d\rho}_{I_1} \cdot \underbrace{\int_0^{\pi}\sin\varphi\, d\varphi}_{2} \cdot \underbrace{\int_0^{2\pi} d\theta}_{2\pi}$$

$$\boxed{I = 4\pi\cdot I_1}$$

$$I_1 = \int_0^1 \underset{\downarrow \left(\frac{y^3}{3}\right)'}{y^2} \cdot \ln(1+y)\,dy = \frac{y^3}{3}\cdot \ln(1+y)\Big|_0^1 - \int_0^1 \frac{y^3}{3}\cdot\frac{1}{1+y}\,dy$$

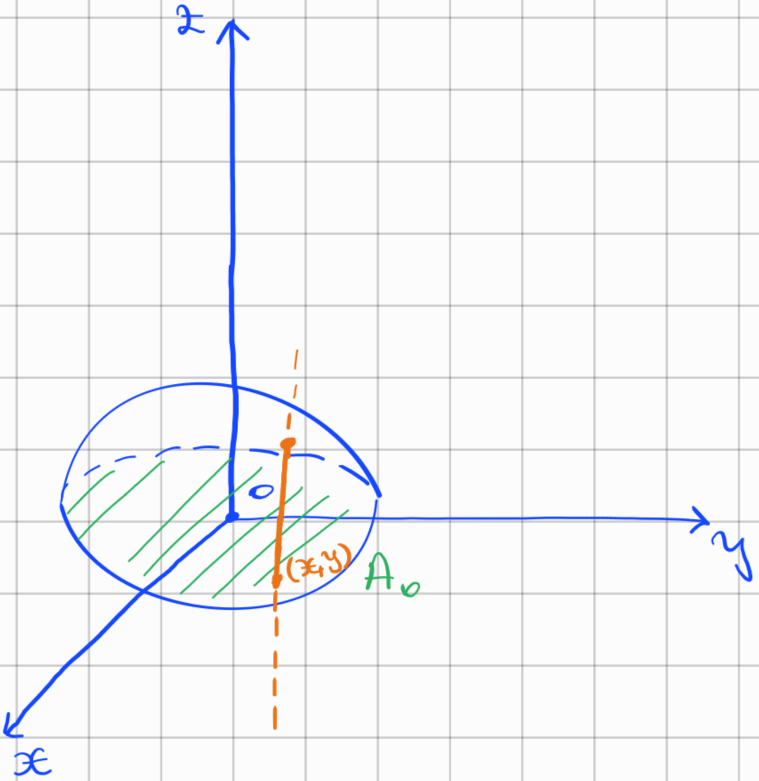
$$= \frac{1}{3}\cdot \ln 2 - \frac{1}{3}\int_0^1 \frac{y^3}{1+y}\,dy$$

$$= \frac{1}{3}\cdot \ln 2 - \frac{1}{3}\int_0^1 \frac{1+y^3-1}{1+y}\,dy = \frac{1}{3}\left(\ln 2 - \int_0^1 \left(\frac{\cancel{(1+y)}(y^2-y+1)}{\cancel{1+y}} - \frac{1}{1+y}\right)dy\right)$$

$$= \frac{1}{3}\left(\ln 2 - \frac{y^3}{3}\Big|_0^1 - \frac{y^2}{2}\Big|_0^1 + y\Big|_0^1 + \ln(1+y)\Big|_0^1\right)$$

$$= \frac{1}{3}\left(\ln 2 - {}^{2/}\frac{1}{3} - {}^{3/}\frac{1}{2} + {}^{6/}1 + \ln 2\right) = \frac{1}{3}\left(2\ln 2 + \frac{1}{3}\right) = \frac{2\ln 2}{3} + \frac{1}{9}$$

$\boxed{5}$ $I = \iiint_A \frac{z}{(x^2+y^2+z^2)^2}\,dx\,dy\,dz$, $A: x^2+y^2+z^2 \leq 1$, $z \geq 0$



$$I = \iint_{A_0}\left(\int_{z=0}^{z=\sqrt{1-x^2-y^2}} \frac{z}{(x^2+y^2+1)^2}\,dz\right)dx\,dy$$

$$= \iint_{A_0} \frac{1}{(x^2+y^2+1)^2}\cdot \frac{z^2}{2}\Big|_0^{\sqrt{1-x^2-y^2}} = \frac{1}{2}\iint_A \frac{1-x^2-y^2}{(x^2+y^2+1)^2}\,dx\,dy$$

$x = \rho\cos\theta \qquad \rho = [0;1]$

$y = \rho\sin\theta \qquad \theta = [0;2\pi]$

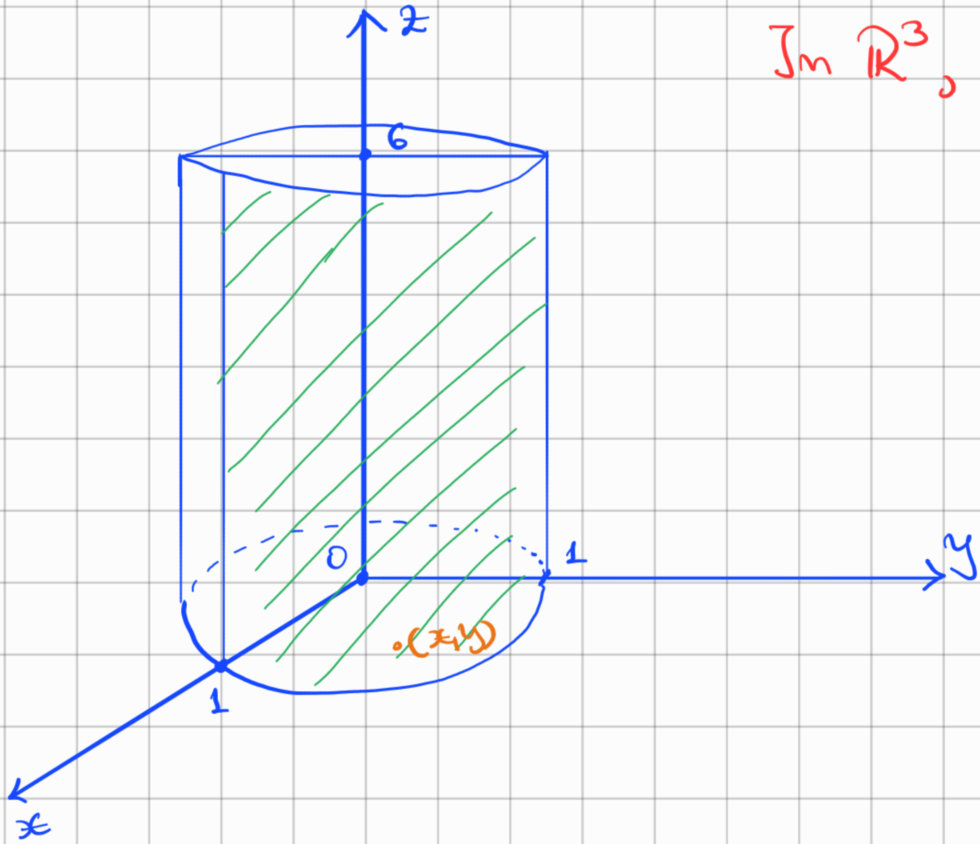
$$I = \frac{1}{2}\int_{\rho=0}^{\rho=1}\int_{\theta=0}^{\theta=2\pi} \frac{1-\rho^2}{(\rho^2+1)^2}\,\rho\, d\rho\, d\theta = \frac{1}{2}\int_0^1 \frac{1-\rho^2}{(\rho^2+1)^2}\cdot\rho\, d\rho \cdot \underbrace{\int_0^{2\pi} d\theta}_{2\pi}$$

$$\rho^2+1 = t \Rightarrow dt = 2\rho\, d\rho$$

$$I = \pi \cdot \int_1^2 \frac{1-t+1}{t^2}\cdot\frac{1}{2}\, dt = \ldots$$

$\boxed{6}$ $I = \iiint_A xyz\sqrt{x^2+4y^2}\, dx\,dy\,dz$, $A = \{(x,y,z)\in\mathbb{R}^3 \mid x \geq 0, y \geq 0, x^2+y^2 \leq 1, 0 \leq z \leq 6\}$

In $\mathbb{R}^3$, $x^2+y^2=1 \Rightarrow$ CYLINDER



$x = \rho\cos\theta \qquad \rho\in[0;1]$

$y = \rho\sin\theta \qquad \theta\in[0;\frac{\pi}{2}]$

$z = z \qquad z\in[0;6]$

$$I = \int_{\rho=0}^{\rho=1}\int_{\theta=0}^{\theta=\frac{\pi}{2}}\int_{z=0}^{z=6} \rho\cos\theta\cdot\rho\sin\theta\cdot z\sqrt{\rho^2\cos^2\theta+4\rho^2\sin^2\theta}\;\rho\, d\rho\, d\theta\, dz$$

$$= \int_{\rho=0}^{\rho=1} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{z=0}^{z=6} \rho^4 \cdot z \cdot \sin\theta \cos\theta \cdot \sqrt{\cos^2\theta + 4\sin^2\theta} \; d\rho \, d\theta \, dz$$

$$= \underbrace{\int_{\rho=0}^{\rho=1} \rho^4 \, d\rho}_{\frac{\rho^5}{5}\Big|_0^1 = \frac{1}{5}} \cdot \int_{z=0}^{z=6} z \, dz \cdot \int_{\theta=0}^{\theta=\frac{\pi}{2}} \sin\theta \cdot \cos\theta \cdot \sqrt{1+3\sin^2\theta} \; d\theta$$

$t = \sqrt{1+3\sin^2\theta} \iff t^2 = 1+3\sin^2\theta$

$2t\,dt = 6\sin\theta\cos\theta\,d\theta$

$\sin\theta\cos\theta\,d\theta = \frac{1}{3}t\,dt$

$$= \frac{1}{5} \cdot 18 \cdot \int_1^2 t \cdot \frac{1}{3} t \, dt = \frac{6}{5} \cdot \frac{t^3}{3}\Big|_1^2 = \frac{6}{5} \cdot \left(\frac{8}{3} - \frac{1}{3}\right)$$

$$= \frac{6}{5} \cdot \frac{7}{3} = \frac{42}{15}$$

$\boxed{7}$ $I = \iiint_A \frac{1}{\sqrt{x^2+y^2+(3-z)^2}} \, dx\,dy\,dz$, $A: x^2+y^2 \leq 1$, $0 \leq z \leq 2$

$x = \rho\cos\theta$ $\quad \rho \in [0;1]$

$y = \rho\sin\theta$ $\quad \theta \in [0;2\pi]$

$z = z$ $\quad z \in [0;2]$

$$I = \int_{\rho=0}^{\rho=1} \int_{\theta=0}^{\theta=2\pi} \int_{z=0}^{z=2} \frac{1}{\sqrt{\rho^2+(3-z)^2}} \, \rho \, d\rho \, d\theta \, dz$$

$$= \int_{\rho=0}^{\rho=1} \int_{z=0}^{z=2} \frac{\rho}{\sqrt{\rho^2+(3-z)^2}} \, d\rho \, dz \cdot \underbrace{\int_{\theta=0}^{\theta=2\pi} d\theta}_{2\pi}$$

$$= 2\pi \cdot \int_{z=0}^{z=2} \left(\int_{\rho=0}^{\rho=1} \frac{\rho}{\sqrt{\rho^2+(3-z)^2}} \, d\rho \right) dz$$

$$= 2\pi \int_{z=0}^{z=2} \sqrt{\rho^2+(3-z)^2} \Big|_{\rho=0}^{\rho=1} dz = 2\pi \left(\int_0^2 \sqrt{1+(3-z)^2} \, dz - \int_0^2 (3-z) \, dz \right)$$

$= \ldots$