Mathematical logic
Lecture 9
04.12.2023

Chapter 6. Lattices and Boole algebras

Lattices as algebraic structures

Def 1. An ordered set $(A, \leq)$ is called a lattice if for any $x, y \in A$ $\exists \inf\{x,y\}$, $\exists \sup\{x,y\}$

Def 2 The algebraic structure $(A, \vee, \wedge)$ is called a lattice if the following axioms are satisfied:

(1) commutativity: $\forall x, y \in A$ $\quad x \vee y = y \vee x$
$\quad x \wedge y = y \wedge x$

(2) associativity $\forall x, y, z \in A$ $\quad (x \vee y) \vee z = x \vee (y \vee z)$
$\quad (x \wedge y) \wedge z = x \wedge (y \wedge z)$

(3) absorbtion: $\forall x, y \in A$ $\quad x \vee (x \wedge y) = x$
$\quad x \wedge (x \vee y) = x$.

We'll show that the two notions are equivalent

Theorem 1 1). Let $(A, \leq)$ be a lattice (as an ordered set).
We define on A the following operations:

$$\forall x, y \in A: \quad x \vee y \overset{def}{=} \sup\{x, y\}$$
$$x \wedge y \overset{def}{=} \inf\{x, y\}.$$

Then $(A, \vee, \wedge)$ is a lattice (as an alg. structure)

2). Let $(A, \vee, \wedge)$ be a lattice (as an alg. structure).
We define on A the relation „$\leq$": $\forall x, y \in A$ $\quad x \leq y \overset{def}{\Longleftrightarrow} x \vee y = y$.
Then „$\leq$" is an order relation on A, and moreover, $(A, \leq)$ is a lattice, where $\sup\{x, y\} = x \vee y$ and $\inf\{x, y\} = x \wedge y$.

Proof 1). Homework! see Th. 6.1.2 / 39.

2) First, we prove that $\forall x, y \in A$.

$$x \vee y = y \iff x \wedge y = x$$ (so we may also use the oper. $\wedge$ to define "$\leq$")

"$\Rightarrow$" $x \vee y = y \xRightarrow{x\wedge\cdot} x \wedge (x \vee y) = x \wedge y \xRightarrow{abs} x = x \wedge y$

"$\Leftarrow$" $x \wedge y = x \xRightarrow{\cdot \vee y} (x \wedge y) \vee y = x \vee y \xRightarrow{abs} y = x \vee y$

We check that "$\leq$" is an order relation:

(R). $\forall a \in A$: $a \overset{?}{\leq} a \overset{def}{\iff} a \vee a \overset{?}{=} a$

$\iff a \wedge a = a.$

(so we need to prove that both operations are idempotent)

Indeed, by absorbtion, we have,

$\cdot$ $a = a \vee (a \wedge a) \xRightarrow{a\wedge\cdot} a \wedge a = a \wedge (a \vee \underbrace{(a \wedge a)}_{b})$

$\xRightarrow{abs} a \wedge a = a$

$\cdot$ similarly with $a = a \wedge (a \vee a) \xRightarrow{a\vee\cdot} \underline{a \vee a} = a \vee (a \wedge \underline{(a \vee a)}) = \underline{a}$

(T) $\forall a, b, c \in A$ if $a \leq b$, $b \leq c \Rightarrow$

$\Rightarrow a \vee b = b$, $b \vee c = c \Rightarrow \underline{a \vee c} = a \vee (b \vee c) \overset{ass.}{=}$

$= \underline{(a \vee b)} \vee c = b \vee c = \underline{c}$, hence $a \leq c$

(A) Let $a, b \in A$ s.t. $a \leq b$ and $b \leq a$.

Then: $a \vee b = b$ and $b \vee a = a \xRightarrow{comm} a = b$

Next, we prove that $\sup\{x, y\} = x \vee y$.

$\cdot$ we check that $x \vee y$ is an upper bound for $\{x, y\}$.

$x \vee (x \vee y) \overset{ass}{=} (x \vee x) \vee y \xRightarrow{idemp} x \vee y. \Rightarrow x \leq x \vee y$

$$y \vee (x \vee y) = x \vee (y \vee y) = x \vee y \Rightarrow y \leq x \vee y.$$

Let z be another upper bound for $\{x, y\}$

$$z \vee (x \vee y) = (z \vee x) \vee y = z \vee y = z$$

hence $x \vee y \leq z$

hence $x \vee y$ is the least upper bound.

similarly: $\inf \{x, y\} = x \wedge y$ (Homework!!)

<u>Examples</u>. 1). Let $(A, \leq)$ be a totally ordered set. so

$$\inf \{x, y\} = \min \{x, y\} \ ; \ \sup \{x, y\} = \max \{x, y\}$$

2). Let M be a set and consider $(\mathcal{P}(M), \subseteq)$

Let $X, Y \subseteq M$. Then: $\inf \{X, Y\} = X \cap Y$

$\sup \{X, Y\} = X \cup Y$.

3). Let $(\mathbb{N}, |)$. Let $x, y \in \mathbb{N}$.

$\inf \{x, y\} = (x, y)$ g.c.d.

$\sup \{x, y\} = [x, y]$ l.c.m.

<u>Def 3</u> A function $f : (A, \vee, \wedge) \to (A', \vee', \wedge')$ is called a <u>morphism of lattices</u> if $\forall x, y \in A$:

$$f(x \vee y) = f(x) \vee' f(y) \text{ and } f(x \wedge y) = f(x) \wedge' f(y)$$

<u>Rem</u> (see the exercises!)

$$f \text{ morph of lattices} \;\substack{\Longrightarrow \\ \not\Longleftarrow}\; f \text{ is monotony.}$$

Boole algebras and Boole rings

Defn. A lattice $(A, \vee, \wedge)$ is called a Boole lattice (Boole algebra) if the following additional axioms hold:

(1) distributivity: $\forall x, y, z \in A$: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
(one is enough to state! see (?))

(2). $\exists \min A =: 0$; $\exists \max A =: 1$
(i.e.: $0 \wedge x = 0$, $0 \vee x = x$; $1 \wedge x = x$, $1 \vee x = 1$.)

(3). $\forall x \in A$ has a complement x' satisfying:
$$x \vee x' = 1 \quad ; \quad x \wedge x' = 0$$

Proposition 1 Let $(A, \vee, \wedge, 0, 1, ')$ be a Boole algebra. Then:

1). $\forall x \in A$, the complement of x is unique

2). $0' = 1$, $1' = 0$, $(x')' = x$

3). $\forall x, y \in A$ we have:
$$(x \vee y)' = x' \wedge y'$$
$$(x \wedge y)' = x' \vee y'$$
De Morgan

Proof 1). We assume that x', $\bar{x}$ are both complements of $x \in A$.
We have:

- $x' = x' \wedge 1 = x' \wedge (x \vee \bar{x}) \overset{distr}{=} (x' \wedge x) \vee (x' \wedge \bar{x}) =$
$= 0 \vee (x' \wedge \bar{x}) = x' \wedge \bar{x}$

- $\bar{x} = \bar{x} \wedge 1 =$ . . . Homework $= x' \wedge \bar{x}$

hence $x' = \bar{x}$

2) • $0 \vee 1 = 1$, $0 \wedge 1 = 0 \Rightarrow 0' = 1$, $1' = 0$

• $x \vee x' = 1$, $x \wedge x' = 0 \Rightarrow (x')' = x$

3) we calculate

• $(x \vee y) \wedge (x' \wedge y') \overset{dist}{=} (\underbrace{x \wedge x'}_{0} \wedge y') \vee (\underbrace{y \wedge x' \wedge y'}_{0})$

$= (0 \wedge y') \vee (0 \wedge x') = 0 \vee 0 = 0$

• $(x \vee y) \vee (x' \wedge y') \overset{dist}{=} (\underbrace{x \vee y \vee x'}) \wedge (x \vee \underbrace{y \vee y'}) =$

$= (1 \vee y) \wedge (x \vee 1) = 1 \wedge 1 = 1$

thus $(x \vee y)' = x' \wedge y'$

Similarly, we prove the other equality! HW!

<u>Def 5.</u> Any $(B, +, \cdot)$ (assoc. with 1) is called a <u>Boole ring</u> if $\forall x \in B$ we have $x^2 = x$

(every elem. is <u>idempotent</u>)

$\left\{ \begin{array}{l} (B,+) \text{ ab. group} \\ (B,\cdot) \text{ monoid} \\ \cdot \text{ is distrib. w.r.t. } + . \end{array} \right.$

<u>Prop</u> Let $(B, +, \cdot, 0, 1)$ be a Boole ring. Then:

1). $1 + 1 = 0$ (hence $x + x = 0 \quad \forall x \in B$)

(we say that B has <u>characteristic 2</u>)

2) B is commutative, i.e. $xy = yx \quad \forall x, y \in B$.

<u>Proof</u> 1). $\cancel{1+1} = (1+1)^2 = \cancel{1+1}+1+1$; but $(B,+)$ is a group, hence $0 = 1+1$

2). $\cancel{x+y} = (x+y)^2 = \underline{x^2} + xy + yx + \underline{y^2} = \cancel{x+y} + xy + yx$

$\Rightarrow \quad 0 = xy + yx \quad \Rightarrow \quad yx = -xy \quad = xy$
because $1 = -1$

Theorem 2 (Stone).

1) Let $(A, \vee, \wedge, 0, 1, ')$ be a Boole algebra. We define on A the following operations:

- $x + y \overset{def}{=} (x \wedge y') \vee (x' \wedge y)$
- $xy \overset{def}{=} x \wedge y$

Then: $(A, +, \cdot, 0, 1)$ is a Boole ring.

2) Let $(B, +, \cdot, 0, 1)$ be a Boole ring.
We define on B the foll. operations:

- $x \vee y \overset{def}{=} x + y + xy$
- $x \wedge y \overset{def}{=} xy$
- $x' \overset{def}{=} 1 + x$

Then $(B, \vee, \wedge, 0, 1, ')$ is a Boole algebra.

3). The constructions from 1) and 2) are inverse of each other:

- $\underbrace{(A, \vee, \wedge, 0, 1, ')}_{\text{Boole algebra}} \xmapsto{1)} \underbrace{(A, +, \cdot, 0, 1)}_{\text{Boole ring}} \xmapsto{2)} \underbrace{(A, \cup, \cap, 0, 1, ^-)}_{\text{Boole algebra}}$

Then: $\cup = \vee$, $\cap = \wedge$, $^- = '$

- $\underbrace{(B, +, \cdot, 0, 1)}_{\text{Boole ring}} \xmapsto{2)} \underbrace{(B, \vee, \wedge, 0, 1, ')}_{\text{Boole algebra}} \xmapsto{1)} \underbrace{(B, \oplus, \odot, 0, 1)}_{\text{Boole ring}}$

Then: $\oplus = +$, $\odot = \cdot$

Proof see Thm. 6.2.6 / 41. HW!

Examples 1) Let $\mathbb{Z}_2 = \{\hat{0}, \hat{1}\}$ be the ring of residues modulo 2.

$$\hat{0} = 2\mathbb{Z} \text{ (even)}, \qquad \hat{1} = 2\mathbb{Z}+1 \text{ (odd)}$$

So $(\mathbb{Z}_2, +, \cdot)$ is a Boole ring: $\hat{0}^2 = \hat{0}$, $\hat{1}^2 = \hat{1}$

We determine the associated Boole algebra.

$$\begin{cases} x \wedge y = xy \\ x \vee y = x + y + xy \end{cases} \qquad x' = 1 + x$$

$\vee$	0	1
0	0	1
1	1	1

$\wedge$	0	1
0	0	0
1	0	1

x	x'
0	1
1	0

$0 \vee 0 = 0 + 0 + 0 \cdot 0 = 0$

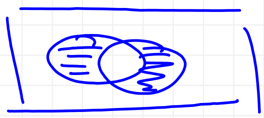
$0 \vee 1 = 0 + 1 + 0 \cdot 1 = 1$

$1 \vee 1 = 1 + 1 + 1 \cdot 1 = 1$

2). Consider the structure $(\mathcal{P}(M), \cup, \cap, \underset{\hat{0}}{\emptyset}, \underset{\hat{1}}{M}, \complement)$.

From set theory we know that this is a Boole algebra. We determine the corresponding Boole ring.

Let $X, Y \in \mathcal{P}(M)$. Then,



$$X \cdot Y = X \cap Y$$

$$X + Y = (X \cap \complement Y) \cup (Y \cap \complement X)$$
$$= (X \setminus Y) \cup (Y \setminus X)$$
$$= X \,\Delta\, Y \quad \text{(symmetric difference)}$$

We get the Boole ring $(\mathcal{P}(M), \Delta, \cap, \emptyset, M)$.

3) The Lindenbaum – Tarski algebra

Let $X = \{x_1, \dots, x_n\}$ be a set of atomic formulas in propositional logic.

Let $\mathcal{F}$ be the set of formulas containing these atoms.

On $\mathcal{F}$ we consider the equivalence relation "$\Longleftrightarrow$":

$A \Longleftrightarrow B$ if and only if the formula: $(A \Longleftrightarrow B)$ is a tautology.

We consider the quotient set $\mathcal{F}/_{\Longleftrightarrow} = \{ \hat{A} \mid A \in \mathcal{F} \}$

where $\hat{A} = \{ B \in \mathcal{F} \mid A \Longleftrightarrow B \}$

On $\mathcal{F}/_{\Longleftrightarrow}$ we define the following operations:

$\underset{\substack{\uparrow \\ \text{operation}}}{\hat{A} \vee \hat{B}} \overset{\text{def}}{=} \underset{\substack{\uparrow \\ \text{connective}}}{\widehat{A \vee B}}$ $\qquad \hat{0} =$ the set of all contradictions

$\hat{A} \wedge \hat{B} \overset{\text{def}}{\Longleftrightarrow} \widehat{A \wedge B}$ $\qquad \hat{1} =$ the set of all tautologies.

$\hat{A}' \overset{\text{def}}{=} \widehat{\neg A}$

From propositional logic it follows that $(\mathcal{F}/_{\Longleftrightarrow}, \vee, \wedge, \hat{0}, \hat{1}, ')$ is a Boole algebra.

Rem Examples 2) and 3) give the possibility to use algebraic methods in set theory and propositional logic.

Homework Fill in the proofs of the above theorems.
ex 76 — 83.