<u>Lecture 4</u> March 17, 2022

**6.6. Theorem** (K. Weierstrass) If $A \subseteq \mathbb{R}^n$ is a compact set and $f : A \to \mathbb{R}$ is continuous on $A$, then $f$ is bounded and it attains its bounds, i.e.,

$$\exists\, a \in A \text{ s.t. } f(a) = \inf f(A)$$

$$\exists\, b \in A \text{ s.t. } f(b) = \sup f(A).$$

**Proof.**

$$\left.\begin{array}{l} A - \text{compact} \\ f - \text{continuous on } A \end{array}\right\} \overset{\boxed{T6.5}}{\Longrightarrow} f(A) \text{ is compact in } \mathbb{R} \overset{\substack{\text{Characterization} \\ \text{of compact sets}}}{\Longrightarrow}$$

$$\Rightarrow f(A) \text{ is } \left[\begin{array}{l} \text{bounded} \Rightarrow f \text{ is bounded} \\ \text{closed} \Rightarrow \inf f(A) \in f(A) \qquad \sup f(A) \in f(A) \end{array}\right.$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\exists\, a \in A : f(a) = \inf f(A) \qquad \exists\, b \in A : f(b) = \sup f(A).$$

Chapter 2. <u>DIFFERENTIAL CALCULUS IN $\mathbb{R}^m$</u>

## 1. The normed space of linear mappings

**1.1. Definition.** A function $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ is called a <u>linear mapping</u> if

$$\forall\, \alpha, \beta \in \mathbb{R}, \quad \forall\, x, y \in \mathbb{R}^n : \quad \varphi(\alpha x + \beta y) = \alpha \varphi(x) + \beta \varphi(y).$$

$$L(\mathbb{R}^n, \mathbb{R}^m) := \{ \varphi : \mathbb{R}^n \to \mathbb{R}^m \mid \varphi \text{ is a linear mapping} \}$$

If $\varphi \in L(\mathbb{R}^n, \mathbb{R}^m) \implies \varphi(0_n) = 0_m$

$$\varphi(-x) = -\varphi(x) \qquad \forall\, x \in \mathbb{R}^n$$

$$\forall\, \alpha_1, \dots, \alpha_k \in \mathbb{R}, \quad \forall\, x_1, \dots, x_k \in \mathbb{R}^n :$$

$$\varphi(\alpha_1 x_1 + \dots + \alpha_k x_k) = \alpha_1 \varphi(x_1) + \dots + \alpha_k \varphi(x_k).$$

**1.2. Theorem.** Let $\varphi : \mathbb{R}^n \to \mathbb{R}^m$. Then

$$\varphi \in L(\mathbb{R}^n, \mathbb{R}^m) \iff \exists\, v_1, \dots, v_n \in \mathbb{R}^m \text{ s.t. } \forall\, x = (x_1, \dots, x_n) \in \mathbb{R}^n :$$

$$\varphi(x) = x_1 v_1 + \dots + x_n v_n$$

<u>Proof</u> $\boxed{\Leftarrow}$ Obvious $\qquad \boxed{\Rightarrow}$ Just take $v_1 = \varphi(e_1), \dots, v_n = \varphi(e_n)$

$\{e_1, \dots, e_n\}$ canonical basis of $\mathbb{R}^n$

**1.3. Corollary.** Let $\varphi : \mathbb{R}^n \to \mathbb{R}$. Then

$$\varphi \in L(\mathbb{R}^n, \mathbb{R}) \iff \exists\, v \in \mathbb{R}^n \text{ s.t. } \forall\, x \in \mathbb{R}^n : \ \varphi(x) = \langle x, v \rangle .$$

**1.4. Definition.** Let $\varphi \in L(\mathbb{R}^n, \mathbb{R}^m)$, and let $\{e_1, \dots, e_n\}$ be the canonical basis in $\mathbb{R}^n$. We define

$$v_1 := \varphi(e_1) \in \mathbb{R}^m \implies v_1 = (v_{11}, v_{12}, \dots, v_{1m})$$

$$v_2 := \varphi(e_2) \in \mathbb{R}^m \implies v_2 = (v_{21}, v_{22}, \dots, v_{2m})$$

$$\vdots$$

$$v_n := \varphi(e_n) \in \mathbb{R}^m \implies v_n = (v_{n1}, v_{n2}, \dots, v_{nm})$$

We define

$$[\varphi] = \begin{pmatrix} v_{11} & v_{21} & \dots & v_{n1} \\ v_{12} & v_{22} & \dots & v_{n2} \\ \vdots & \vdots & & \vdots \\ \underbrace{v_{1m}}_{\varphi(e_1)} & \underbrace{v_{2m}}_{\varphi(e_2)} & & \underbrace{v_{nm}}_{\varphi(e_n)} \end{pmatrix} \in \mathcal{M}_{m\times n}(\mathbb{R})$$

the matrix of the linear mapping $\varphi$

$\forall\ x = (x_1, \dots, x_n) \in \mathbb{R}^n$ :

$\varphi = (\varphi_1, \dots, \varphi_m) : \mathbb{R}^n \to \mathbb{R}^m$

$$\begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \\ \vdots \\ \varphi_m(x) \end{pmatrix} = [\varphi] \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad (*)$$

Convention: if a vector $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ appears in a matrix relation, then it will be identified with the column matrix $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$$(*) \iff \underbrace{\varphi(x)}_{\mathcal{M}_{m\times 1}(\mathbb{R})} = \underbrace{[\varphi]}_{\mathcal{M}_{m\times n}(\mathbb{R})} \underbrace{x}_{\mathcal{M}_{n\times 1}(\mathbb{R})}$$

$$\langle x, y \rangle = x_1 y_1 + \dots + x_n y_n = (x_1\ x_2\ \dots\ x_n) \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$$\langle x, y \rangle = x^t \cdot y = y^t \cdot x$$

1.5. Theorem. If $\varphi, \psi \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $\alpha, \beta \in \mathbb{R} \Rightarrow$

$\Rightarrow\ \alpha\varphi + \beta\psi \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $[\alpha\varphi + \beta\psi] = \alpha[\varphi] + \beta[\psi]$

Remark $L(\mathbb{R}^n, \mathbb{R}^m)$ is a vector subspace of the vector space of all functions from $\mathbb{R}^n$ to $\mathbb{R}^m$

The mapping $\forall \varphi \in L(\mathbb{R}^n, \mathbb{R}^m) \longmapsto [\varphi] \in \mathcal{M}_{m\times n}(\mathbb{R})$

is an isomorphism between the vector spaces $L(\mathbb{R}^n, \mathbb{R}^m)$ and $\mathcal{M}_{m\times n}(\mathbb{R})$

**1.6. Theorem.** If $\varphi \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $\psi \in L(\mathbb{R}^m, \mathbb{R}^p)$, then

$$\psi \circ \varphi \in L(\mathbb{R}^n, \mathbb{R}^p) \quad \text{and} \quad [\psi \circ \varphi] = [\psi] \cdot [\varphi]$$

**1.7. Theorem.** Let $\varphi = (\varphi_1, \dots, \varphi_m) : \mathbb{R}^n \to \mathbb{R}^m$. Then

$$\varphi \in L(\mathbb{R}^n, \mathbb{R}^m) \iff \varphi_i \in L(\mathbb{R}^n, \mathbb{R}) \quad \forall\, i = \overline{1, m}$$

**1.8. Theorem.** Given $\varphi \in L(\mathbb{R}^n, \mathbb{R}^n)$, the following assertions are equivalent:

1° $\varphi$ is bijective

2° $\varphi$ is injective $\qquad n = \dim_{\mathbb{R}} \operatorname{Ker}\varphi + \dim_{\mathbb{R}} \operatorname{Im}\varphi$

3° $\varphi$ is surjective

4° $\det[\varphi] \neq 0$

**1.9. Theorem.** If $\varphi \in L(\mathbb{R}^n, \mathbb{R}^n)$ is bijective $\Rightarrow$ $\varphi^{-1} \in L(\mathbb{R}^n, \mathbb{R}^n)$ and $[\varphi^{-1}] = [\varphi]^{-1}$.

**1.10. Theorem.** Every linear mapping $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz function.

**Proof.** A function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is called a Lipschitz function if $\exists\, \alpha \geq 0$ s.t.
(Lipschitz constant)

$$\forall\, x, y \in A : \qquad \| f(x) - f(y) \| \leq \alpha \cdot \| x - y \|$$

Ex $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \arctan x$ is a Lipschitz function

$x, y \in \mathbb{R} \underset{\substack{\uparrow \\ \text{Lagrange MVT}}}{\Longrightarrow} \exists\, c$ between $x$ and $y$ s.t. $f(x) - f(y) = (x-y) \cdot f'(c)$

$$|f(x) - f(y)| = |x - y| \cdot |f'(c)|$$

$$0 < f'(c) = \frac{1}{1+c^2} \leq 1$$

$$\Rightarrow \quad |f(x) - f(y)| \leq |x - y|$$

Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n \Rightarrow x = x_1 e_1 + \dots + x_n e_n \Rightarrow$

$$\varphi(x) = x_1 \varphi(e_1) + \dots + x_n \varphi(e_n)$$

$$\| \varphi(x) \| = \| x_1 \varphi(e_1) + \dots + x_n \varphi(e_n) \| \leq$$

$$\leq \| x_1 \varphi(e_1) \| + \dots + \| x_n \varphi(e_n) \|$$

$$= \underbrace{|x_1|}_{\leq \|x\|} \cdot \| \varphi(e_1) \| + \dots + \underbrace{|x_n|}_{\leq \|x\|} \cdot \| \varphi(e_n) \|$$

$$\leq \|x\| \Big( \underbrace{\| \varphi(e_1) \| + \dots + \| \varphi(e_n) \|}_{:= \alpha} \Big)$$

$\Rightarrow \quad \forall x \in \mathbb{R}^n : \quad \|\varphi(x)\| \leq \alpha \|x\|$

$\Rightarrow \quad \forall x, y \in \mathbb{R}^n : \quad \|\varphi(x) - \varphi(y)\| = \|\varphi(x-y)\| \leq \alpha \|x-y\|$

$\Rightarrow \quad \varphi$ is a Lipschitz function

**1.11. Definition** (norm of a linear mapping). Let $\varphi \in L(\mathbb{R}^n, \mathbb{R}^m)$ and let

$$S^{n-1} := \{ x \in \mathbb{R}^n \mid \|x\| = 1 \} = \{ (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \dots + x_n^2 = 1 \}$$

↳ the unit sphere in $\mathbb{R}^n$

It can be proved that $S^{n-1}$ is bounded and closed, hence it is compact $\xRightarrow[\text{Weierstrass Thm}]{\varphi \text{ continuous}}$

$\Rightarrow$ we can define

$$\|\varphi\| := \max_{x \in S^{n-1}} \|\varphi(x)\|$$

↳ the norm of the linear mapping $\varphi$

**1.12. Theorem** The function $\|\cdot\| : L(\mathbb{R}^n, \mathbb{R}^m) \to [0, \infty)$, defined by

$$\|\varphi\| := \max_{x \in S^{n-1}} \|\varphi(x)\| \qquad \varphi \in L(\mathbb{R}^n, \mathbb{R}^m)$$

is a norm on the real vector space $L(\mathbb{R}^n, \mathbb{R}^m)$ (in the sense of the axiomatic definition form Lecture 1)

**1.13. Theorem.** Given $\varphi \in L(\mathbb{R}^n, \mathbb{R}^m)$, $\psi \in L(\mathbb{R}^m, \mathbb{R}^p)$, the following assertions are true:

1° $\forall x \in \mathbb{R}^n$ : $\|\varphi(x)\| \leq \|\varphi\| \cdot \|x\|$.

2° $\|\psi \circ \varphi\| \leq \|\psi\| \cdot \|\varphi\|$.

**Remark.** $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$, $a \in A \cap A'$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

## 2. Derivatives of vector functions of one real variable

**2.1. Definition.** Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}^m$, $a \in A \cap A'$. The derivative of $f$ at $a$ is defined by

$$f'(a) := \lim_{x \to a} \frac{1}{x - a} [f(x) - f(a)]$$

provided that the limit in the right side exists in $\mathbb{R}^m$.

$$f'(a) \in \mathbb{R}^m$$

**2.2. Theorem.** Let $A \subseteq \mathbb{R}$, $f = (f_1, \dots, f_m) : A \to \mathbb{R}^m$, $a \in A \cap A'$. Then:

1° If $f$ has a derivative at $a$ $\Rightarrow$ $f_1, \dots, f_m$ have all a derivative at $a$, and

$$f'(a) = \left( f_1'(a), \dots, f_m'(a) \right). \qquad (*)$$

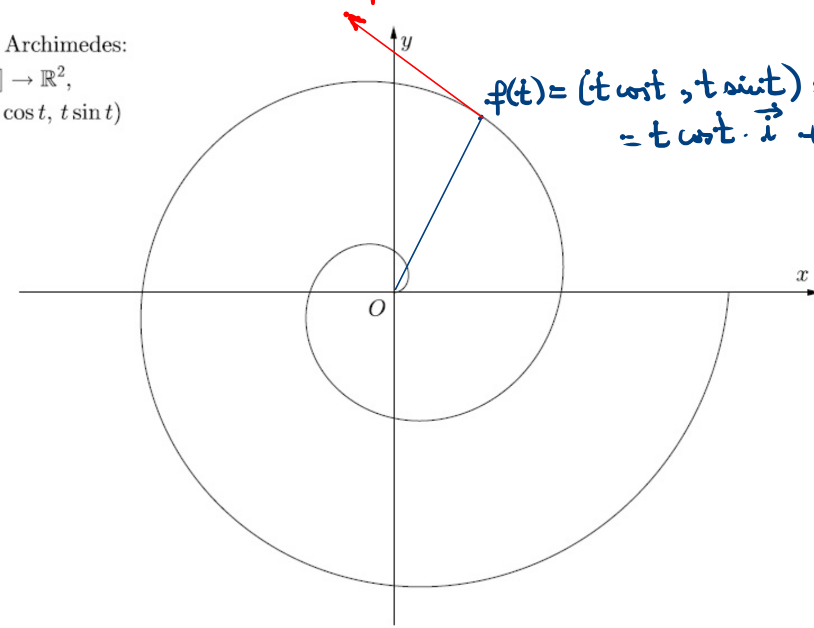
2° If $f_1, \dots, f_m$ have all a derivative at $a$ $\Rightarrow$ $f$ has a derivative at $a$, ~~too~~, and $(*)$ holds.

**Example.** Spiral of Archimedes:

$f : [0, 4\pi] \to \mathbb{R}^2$,

$f(t) = (t\cos t,\ t\sin t)$

$f'(t)$




$f(t) = (t\cos t, t\sin t) =$

$= t\cos t \cdot \vec{i} + t\sin t \cdot \vec{j}$

$f'(t) = (\cos t - t\sin t,\ \sin t + t\cos t)$

The Lagrange MVT cannot be extended to vector functions.

Counterexample $f:[0,2\pi]\to\mathbb{R}^2$ $f(t)=(\cos t, \sin t)$

Suppose $\exists\, c\in(0,2\pi)$ s.t.
$$\left.\begin{aligned} &f(2\pi)-f(0)=2\pi f'(c)\\ &f(2\pi)=(1,0)=f(0)\\ &f'(c)=(-\sin c, \cos c)\end{aligned}\right\}\Rightarrow (0,0)=2\pi(-\sin c, \cos c)$$
$$\Downarrow$$
$$\sin c = 0 = \cos c \quad \Rightarrow\Leftarrow$$

<u>2.3. Theorem</u> (The MVT for vector functions of one real variable). Let $f:[a,b]\to\mathbb{R}^m$ s.t. $f$ is continuous on $[a,b]$ and $f$ has a derivative at each point of $(a,b)$. Then

$$\exists\, c\in(a,b) \text{ s.t. } \|f(b)-f(a)\|\le(b-a)\|f'(c)\|$$

<u>Proof</u>. If $f(a)=f(b)$ $\Rightarrow$ $c$ can be chosen arbitrarily in $(a,b)$

Suppose that $f(a)\neq f(b)$. We consider the <u>unit</u> vector

$$v:=\frac{1}{\|f(b)-f(a)\|}\left[f(b)-f(a)\right]\in\mathbb{R}^m \qquad \|v\|=1$$

Let $v=(v_1,\dots,v_m)$, $f=(f_1,\dots,f_m)$, and let $g:[a,b]\to\mathbb{R}$

$$g(x):=\langle v, f(x)\rangle = v_1f_1(x)+\dots+v_mf_m(x)$$

$f$ is continuous on $[a,b]$ $\Rightarrow$ $f_i : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ $\forall\, i = \overline{1,m}$

$\Rightarrow$ $g$ is continuous on $[a,b]$

$f$ has a derivative at each point of $(a,b)$ $\overset{\boxed{T2.2}}{\Longrightarrow}$ $f_i$ is differentiable on $(a,b)$ $\forall\, i=\overline{1,m}$

$\Rightarrow$ $g$ is differentiable on $(a,b)$

By applying the Lagrange MVT to $g$ $\Rightarrow$ $\exists\, c \in (a,b)$ s.t.

$$g(b) - g(a) = (b-a)\, g'(c) \qquad (1)$$

But
$$g(b) - g(a) = \langle v, f(b) \rangle - \langle v, f(a) \rangle = \langle v, f(b) - f(a) \rangle =$$

$$= \left\langle \frac{1}{\|f(b)-f(a)\|} \cdot [f(b)-f(a)],\ f(b)-f(a) \right\rangle$$

$$= \frac{1}{\|f(b)-f(a)\|} \cdot \langle f(b)-f(a),\ f(b)-f(a) \rangle$$

$$= \frac{1}{\cancel{\|f(b)-f(a)\|}} \cdot \|f(b)-f(a)\|^{\cancel{2}}$$

$$= \|f(b)-f(a)\| \qquad (2)$$

$$g'(x) = v_1 f_1'(x) + \cdots + v_m f_m'(x) = \langle v, f'(x) \rangle$$

$$g'(c) = \langle v, f'(c) \rangle \qquad (3)$$

By (1), (2), (3) $\Longrightarrow$ $\|f(b)-f(a)\| = (b-a)\cdot\langle v, f'(c)\rangle \leq (b-a)\cdot \underbrace{\|v\|}_{=1}\cdot\|f'(c)\|$

Cauchy-Schwarz inequality

$|\langle x, y\rangle| \leq \|x\|\cdot\|y\|$

$= (b-a)\cdot\|f'(c)\|$