Seminar 7

The Cauchy problem. The existence and uniqueness theorems

$$\begin{cases} (1) \quad y' = f(x,y) \\ (2) \quad y(x_0) = y^0 \end{cases} \qquad f: [a,b]\times\mathbb{R} \to \mathbb{R}$$

$x_0 \in [a,b]$, $y^0 \in \mathbb{R}$.

$(1)+(2) \Leftrightarrow (3)$

$$(3) \quad y(x) = y^0 + \int_{x_0}^{x} f(s, y(s))\,ds$$ the Volterra integral eq.

Theorem 1. (The $\exists!$ Th. in the space)

Let us consider the problem $(1)+(2)$. We suppose that:

(i) $f \in C([a,b]\times\mathbb{R}, \mathbb{R})$.

(ii) f is lipschitz with respect to the second variable, i.e.

$\exists L_f > 0$ such that $|f(x,u) - f(x,v)| \le L_f \cdot |u-v|$, $\forall u, v \in \mathbb{R}$ ....

Then the problem $(1)+(2)$ has an unique solution $y^* \in C([a,b], \mathbb{R})$.

Remark.

a) Let be a function $g \in C^1(\mathbb{R})$.
If there exists $M>0$ such that $|g'(x)| \leq M, \forall x \in \mathbb{R}$
then g is lipschitz with $L_g = M$.

b) Let $f \in C^1([a,b] \times \mathbb{R}, \mathbb{R})$. If there exists $M>0$ such that

$$\left| \frac{\partial f}{\partial y}(x,y) \right| \leq M, \ \forall (x,y) \in [a,b] \times \mathbb{R} \text{ then}$$

f is lipschitz with respect to the second variable y,
with $L_f = M$

Theorem 2 (The $\exists!$ Th. in the ball).

Let us consider the problem (1)+(2)

$f : D_f \to \mathbb{R}$, $D_f \subseteq \mathbb{R}^2$ domain (connex and open set)

$\overline{D} = [x_0 - a, x_0 + a] \times [y^0 - b, y^0 + b]$, $a, b > 0$.

Suppose that:

(i) $f \in C(\overline{D}, \mathbb{R})$

(ii) f is lipschitz with respect to the second variable on $\overline{D}$.

$\Leftrightarrow \exists L_f > 0$ s.t $|f(x,u) - f(x,v)| \leq L_f \cdot |u - v|$, $\forall x \in [x_0 - a, x_0 + a]$
$\forall u, v \in [y^0 - b, y^0 + b]$
$\Downarrow$
$(x,u), (x,v) \in \overline{D}$.

Then the problem (1)+(2) has a unique solution $y^* \in C([x_0 - h, x_0 + h], [y^0 - b, y^0 + b])$ where

$h = \min\left\{a, \frac{b}{M_f}\right\}$, $M_f = \max\limits_{\overline{D}} |f(x,u)|$.

Exercise 1

Give an existence and uniqueness result for the following Cauchy problems:

a) $\begin{cases} y' = \sin(xy) + 2y \\ y(0) = 0 \end{cases}$

b) $\begin{cases} xy' = y - 1 \\ y(1) = 0 \end{cases}$

c) $\begin{cases} y' = 3x + 4y^3 \\ y(0) = 0 \end{cases}$

d) $\begin{cases} y' = e^{-x} + y^2 \\ y(0) = 1 \end{cases}$

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a) $\begin{cases} y' = \sin(xy) + 2y \\ y(0) = 0 \end{cases}$

$x_0 = 0,\ y^0 = 0$

$f(x,y) = \sin(xy) + 2y \qquad f : \mathbb{R}^2 \to \mathbb{R}$

$I = [-a, a],\ \boxed{a > 0}$

f is cont on $[-a, a] \times \mathbb{R}$

$$\left|\frac{\partial f}{\partial y}(x,y)\right| = \left|\cos(xy)\cdot x + 2\right| \leq \left|\cos(xy)\cdot x\right| + 2 =$$

$$= \underbrace{|x|}_{} \cdot \underbrace{|\cos(xy)|}_{\leq 1} + 2 \leq \underline{a+2}$$

$x \in [-a,a] \Leftrightarrow |x| \leq a$

$\Rightarrow \frac{\partial f}{\partial y}$ is bounded on $[-a,a]\times\mathbb{R} \Rightarrow f$ is lipschitz with respect to variable y on $[-a,a]\times\mathbb{R}$ $\left(L_f = a+2\right)$

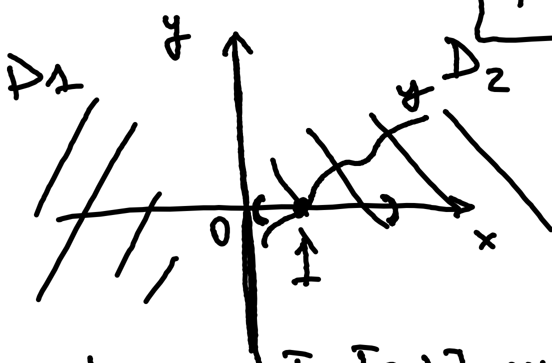
$\underset{\text{Th.1}}{\Rightarrow} \exists!\, y^* \in C([-a,a],\mathbb{R})$.

<u>Theorem</u> For all <u>$a>0$</u> the Cauchy problem (a) has a unique solution $y^* \in C([-a,a],\mathbb{R})$.

b) $\begin{cases} xy' = y-1 \\ y(1)=0 \end{cases} \iff \begin{cases} y' = \dfrac{y-1}{x} \\ y(1) = 0 \end{cases}$ $\frac{1}{x}\cdot(y-1)$ ....

$x_0 = 1,\ y^0 = 0$

$f(x,y) = \dfrac{y-1}{x}$ $f: \mathbb{R}^* \times \mathbb{R} \to \mathbb{R}$.



$f: (\underbrace{(-\infty,0)\times\mathbb{R}}_{D_1}) \cup (\underbrace{(0,+\infty)\times\mathbb{R}}_{D_2}) \to \mathbb{R}$

$(x_0, y^0) \in D_2$

we take $I = [a,b]$ such that $x_0 = 1 \in [a,b]$

$\iff \boxed{0 < a \le 1}$ and $\boxed{b > 1}$

$f: [a,b]\times\mathbb{R} \to \mathbb{R}$ $f \in C([a,b]\times\mathbb{R}, \mathbb{R})$.

$\left|\dfrac{\partial f}{\partial y}(x,y)\right| = \left|\dfrac{1}{x}\right|$

$\left.\begin{array}{l} x \in [a,b] \\ 0 < a \le x \le b \end{array}\right\} \Rightarrow \left|\dfrac{\partial f}{\partial y}(x,y)\right| = \dfrac{1}{x} \le \dfrac{1}{a}$

$\Rightarrow \dfrac{1}{b} \le x \le \dfrac{1}{a}$

$\Rightarrow \frac{\partial f}{\partial y}$ is bounded on $[a,b]\times\mathbb{R}$

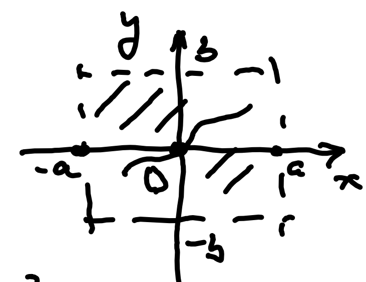
$\Rightarrow f$ is lipschitz with respect to the variable y on $[a,b]\times\mathbb{R}$

$\left(L_f = \frac{1}{a}\right)$

$\underset{Th1}{\Rightarrow} \exists!\ y^* \in C([a,b],\mathbb{R})$

<u>Theorem</u>. If $0<a\leq 1$ and $b>1$ then the Cauchy problem (b) has a unique solution $y^*\in C([a,b],\mathbb{R})$.

c) $\begin{cases} y' = 3x + 4y^3 \\ y(0)=0 \end{cases}$ $f:\mathbb{R}^2\to\mathbb{R}$



$x_0=0,\ y^0=0$

$f(x,y) = 3x+4y^3 \Rightarrow f$ is continuous on $\mathbb{R}^2$

$I\subseteq\mathbb{R}$ s.t $x_0=0\in I$.

$a>0 \Rightarrow 0\in[-a,a]=I$ f is cont on $[-a,a]\times\mathbb{R}$.

$\left|\frac{\partial f}{\partial y}(x,y)\right| = |12y^2| = 12 \cdot y^2 \xrightarrow[|y|\to+\infty]{} +\infty \implies \frac{\partial f}{\partial y}$ is not bounded on $[-a,a]\times\mathbb{R}$.

$(x,y)\in[-a,a]\times\mathbb{R}$; $y\in\mathbb{R}$

we cannot apply Th. 1.

We consider $\overline{D} = [-a,a]\times[-b,b]$, $a,b>0$

$f\in C(\overline{D},\mathbb{R})$.

f is lipschitz with respect to variable y on $\overline{D}$?

$$\left|\frac{\partial f}{\partial y}(x,y)\right| = 12\cdot y^2 \leq 12\cdot b^2$$

$(x,y)\in\overline{D}=[-a,a]\times[-b,b] \implies y\in[-b,b]$

$-b\leq y\leq b \iff |y|\leq b$

$y^2\leq b^2$

$\implies \frac{\partial f}{\partial y}$ is bounded on $\overline{D} \implies$

$\implies f$ is lipschitz with respect to variable y on $\overline{D}$

$(L_f = 12\cdot b^2)$

Th. 2 $\Rightarrow \exists!\, y^* \in C([-h,h],[-b,b])$

where $h = \min\{a, \frac{b}{M_f}\}$

$$M_f = \max_{\overline{D}} |f(x,y)|$$

$$|f(x,y)| = |3x + 4y^3| \le 3\cdot|x| + 4\cdot|y|^3 \le 3\cdot a + 4\cdot b^3$$

$x \in [-a,a]$, $y \in [-b,b]$

$$\Rightarrow h = \min\left\{a, \frac{b}{3a+4b^3}\right\}.$$

__Theorem__: If $a>0, b>0$ then the IVP $(*)$ has a unique solution $y^* \in C([-h,h],[-b,b])$ where

$$h = \min\left\{a, \frac{b}{3a+4b^3}\right\}.$$

For example if we take $a=1$, $b=1$ then:

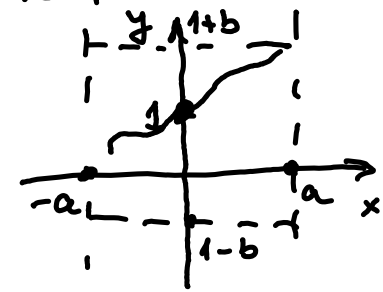
$$h = \min\left\{1, \frac{1}{7}\right\} = \frac{1}{7} \Rightarrow \exists!\, y^* \in C\left(\left[-\tfrac{1}{7}, \tfrac{1}{7}\right], [-1,1]\right)$$

d) $\begin{cases} y' = e^{-x} + y^2 \\ y(0) = 1 \end{cases}$

$\underline{x_0 = 0}$, $y^0 = 1$

$f(x,y) = e^{-x} + y^2$ $\quad f: \mathbb{R}^2 \to \mathbb{R}$.

f is continuous on $\mathbb{R}^2$.



$I \subseteq \mathbb{R}$ such that $x_0 = 0 \in I$

we can take $I = [-a,a]$, $a > 0$

$f: [-a,a] \times \mathbb{R} \to \mathbb{R}$ is cont.

$\left|\frac{\partial f}{\partial y}(x,y)\right| = |2y| = 2\cdot|y| \xrightarrow[|y| \to +\infty]{} +\infty \Rightarrow$

$\Rightarrow \frac{\partial f}{\partial y}$ is not bounded on $[-a,a] \times \mathbb{R} \Rightarrow f$ is not lipschitz with respect to variable y on $[-a,a] \times \mathbb{R} \Rightarrow$ we cannot apply Th. 1.

We take $\overline{D} = [-a,a] \times [1-b, 1+b]$, $a,b > 0$

$f \in C(\overline{D}, \mathbb{R})$ $\qquad y \in [1-b, 1+b] \Leftrightarrow |y-1| \leq b$.

$$\left|\frac{\partial f}{\partial y}(x,y)\right| = 2\cdot|y| = 2\cdot|y-1+1| \leq 2\cdot|y-1| + 2 \leq 2b+2$$

$\Rightarrow f$ is lipschitz with respect to y on $\overline{D}$

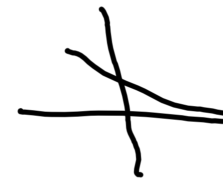
$$\underset{\text{Th.2}}{\Longrightarrow} \exists!\, y^* \in C([-h,h],[1-b,1+b]) \text{ solution}$$

where $h = \min\left\{a, \dfrac{b}{M_f}\right\}$, $\underline{M_f} = \max\limits_{D} |f(x,y)|$

$$|f(x,y)| = |e^{-x} + y^2| \le |e^{-x}| + |y^2| = e^{-x} + y^2$$

$$-a \le x \le a \Rightarrow e^{-(-a)} \ge e^{-x} \ge e^{-a} \Rightarrow$$

$$e^{-a} \le e^{-x} \le \underline{\underline{e^{a}}}$$



$$y \in [1-b, 1+b] \Longleftrightarrow |y-1| \le b$$

$$|y| = |y-1+1| \le |y-1| + 1 \le b+1$$

$$\Rightarrow |y|^2 \le (b+1)^2$$

$$|f(x,y)| \le e^{a} + (b+1)^2 \Rightarrow h = \min\left\{a, \frac{b}{e^{a} + (b+1)^2}\right\}.$$

<u>Theorem</u>: If $a>0$, $b>0$ then the ivp (d) has a unique solution $y^* \in C([-h,h],[1-b,1+b])$ where

$$h = \min\left\{a, \frac{b}{e^{a}+(b+1)^2}\right\}.$$

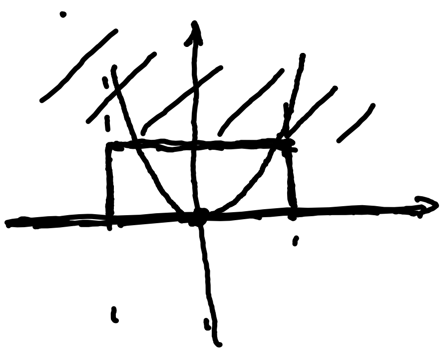
Exercise 2

Study the solution existence for the IVP:

$$\begin{cases} y' = \sqrt{y} \\ y(0) = 0 \end{cases}$$

$x_0 = 0$, $y^0 = 0$

$f(x,y) = \sqrt{y}$ $\quad f : \mathbb{R} \times [0,+\infty) \to \mathbb{R}$



we take $\overline{D} = [-a,a] \times [0,b]$ $\quad a, b > 0$

$f \in C(\overline{D}, \mathbb{R})$

$$\left| \frac{\partial f}{\partial y}(x,y) \right| = \left| \frac{1}{2\sqrt{y}} \right| = \frac{1}{2\sqrt{y}} \xrightarrow[y \to 0]{} +\infty$$

$\Rightarrow$ f is not lipschitz with respect to variable y $\Rightarrow$ we cannot apply Th. 2.

the IVP has two solutions.

$y_1(x) \equiv 0$

$y_2(x) = \dfrac{x^2}{4}$

so the IVP has no unique sol.