Analiză reală

Seminar 13 28.05.2025

Let $(X, \mathcal{A}, \mu)$ be a measure space and $1 \le p < \infty$. We define $\mathcal{L}^p(X, \mathcal{A}, \mu) = \{ f : X \to \mathbb{R} \mid f$ - $\mathcal{A}$-measurable and $|f|^p$ is integrable$\}$ and for $p = \infty$:

$\mathcal{L}^\infty(X, \mathcal{A}, \mu) = \{ f : X \to \mathbb{R} \mid f$ - $\mathcal{A}$-measurable and $\exists M \ge 0$ s.t. $|f| \le M$ $\mu$-a.e.$\}$.

For simplicity, we use the notation $\mathcal{L}^p(X, \mathcal{A}, \mu) = \mathcal{L}^p(X) = \mathcal{L}^p$.

For $f, g \in \mathcal{L}^p$, we define the equivalence relation:

$f \sim g \iff f = g$ $\mu$-a.e.

The quotient set $L^p(X, \mathcal{A}, \mu) = \mathcal{L}^p(X, \mathcal{A}, \mu)/_\sim = \{[f] \mid f \in \mathcal{L}^p\}$, where $[f] = \{ g \in \mathcal{L}^p \mid g \sim f \}$.

For simplicity, we use the notation $L^p(X, \mathcal{A}, \mu) = L^p(X) = L^p$ and we identify an equivalence class $[f]$ with a representative $f \in L^p$ (suffering for $[f]$).

For each set $L^p$ [$p \in [1, \infty]$], we associate a norm $\|\cdot\| : L^p \to [0, \infty)$,

$$\begin{cases} 1 \le p < \infty : \|f\|_p = \|[f]\|_p = \left(\int |f|^p \, d\mu\right)^{\frac{1}{p}} \\ p = \infty : \|f\|_\infty = \|[f]\|_\infty = \inf\{ M \mid |f| \le M \ \mu\text{-a.e.}\} \end{cases}$$

Ex.1: Let $\mu$ be the counting measure on $\mathcal{P}(\mathbb{N})$. Determine $L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu) = \ell^p(\mathbb{N})$ and $\|\cdot\|_p$, where $p \in [1, \infty]$. Show that if $1 \le p < q \le \infty$, we have $\ell^p(\mathbb{N}) \subsetneq \ell^q(\mathbb{N})$ and $\|f\|_q \le \|f\|_p$, $\forall f \in \ell^p(\mathbb{N})$.

Sol: All functions $f : \mathbb{N} \to \mathbb{R}$ are $\mathcal{P}(\mathbb{N})$-measurable.

For $p \in [1, \infty)$, $L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu) = \{ f : \mathbb{N} \to \mathbb{R} \mid |f|^p$ integrable$\}$

$= \{ f : \mathbb{N} \to \mathbb{R} \mid \sum_{n=1}^\infty |f(n)|^p < \infty \}$, since (by seminar 9)

$\int |f|^p \, d\mu = \sum_{n=1}^\infty |f|^p(n) = \sum_{n=1}^\infty |f(n)|^p$

Let $f, g \in L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ s.t. $f \sim g \Rightarrow f = g$ $\mu$-a.e. $\Rightarrow$

$\Rightarrow \exists A \in \mathcal{P}(\mathbb{N})$ with $\mu(A) = 0$ s.t. $\forall n \in \mathbb{N} \setminus A$, $f(n) = g(n)$.

$\mu(A) = 0 \Rightarrow A = \emptyset \Rightarrow f(n) = g(n), \forall n \in \mathbb{N}$ ($\mu$ is the counting measure.)

$\Rightarrow f = g$.

Any function $f : \mathbb{N} \to \mathbb{R}$ is a sequence of real numbers. $(x_n = f(n))$, so we have:

$L^p(\mathbb{N}) = \{ (x_n)_{n \in \mathbb{N}} \subseteq \mathbb{R} \mid \sum_{n=1}^\infty |x_n|^p < \infty \} = \ell^p(\mathbb{N})$

$\|(x_n)_{n \in \mathbb{N}}\|_p = \left( \sum_{n=1}^\infty |x_n|^p \right)^{\frac{1}{p}}$

For $p = \infty$, $\mathcal{L}^\infty(\mathbb{N}) = \{ f : \mathbb{N} \to \mathbb{R} \mid \exists M \ge 0$ s.t. $|f| \le M$ $\mu$-a.e.$\}$ ($\Leftrightarrow |f| \le M$)

$L^\infty(\mathbb{N}) = \{ (x_n)_{n \in \mathbb{N}} \subseteq \mathbb{R} \mid (x_n)_{n \in \mathbb{N}}$ is bounded$\} =: \ell^\infty(\mathbb{N})$

$\|(x_n)_{n \in \mathbb{N}}\|_\infty = \inf\{ M \mid |x_n| \le M \} = \sup_{n \in \mathbb{N}} |x_n|$

Let $p \in [1, \infty)$ and $(x_n)_{n \in \mathbb{N}} \in \ell^p(\mathbb{N})$. For the first case, we consider $q = \infty$.

Let $m \in \mathbb{N}$, $|x_m| = (|x_m|^p)^{\frac{1}{p}} \le \left( \sum_{n=1}^\infty |x_n|^p \right)^{\frac{1}{p}} = \|(x_n)_{n \in \mathbb{N}}\|_p < \infty$

$\Rightarrow \sup_{m \in \mathbb{N}} |x_m| \le \|(x_n)_{n \in \mathbb{N}}\|_p < \infty \Rightarrow (x_n)_{n \in \mathbb{N}} \in \ell^\infty(\mathbb{N})$ and

$\|(x_n)_{n \in \mathbb{N}}\|_\infty \le \|(x_n)_{n \in \mathbb{N}}\|_p$.

We look for a sequence $(x_n)_{n \in \mathbb{N}} \in \ell^\infty(\mathbb{N}) \setminus \ell^p(\mathbb{N})$.

Let $x_n = \frac{1}{n^r}$, $n \in \mathbb{N}$, $r \in (0, \frac{1}{p}]$

$\boxed{\sum_{n=1}^\infty |x_n|^p = \sum_{n=1}^\infty \frac{1}{n^{rp}}. \quad rp \le 1 \Rightarrow r \le \frac{1}{p}}$

$(x_n)_{n \in \mathbb{N}}$ is bounded $\Rightarrow (x_n)_{n \in \mathbb{N}} \in \ell^\infty(\mathbb{N})$

and $\sum_{n=1}^\infty |x_n|^p = \sum_{n=1}^\infty \frac{1}{n^{pr}} = \infty$ $(pr \le 1) \Rightarrow (x_n)_{n \in \mathbb{N}} \notin \ell^p(\mathbb{N})$.

Let $q \in [1, \infty)$ s.t. $p < q$. For any $m \in \mathbb{N}$.

$|x_m|^q = |x_m|^{q-p} \cdot |x_m|^p \le \|(x_n)_{n \in \mathbb{N}}\|_p^{q-p} \cdot |x_m|^p$

$\Rightarrow \sum_{m=1}^\infty |x_m|^q \le \|(x_n)_{n \in \mathbb{N}}\|_p^{q-p} \cdot \underbrace{\sum_{m=1}^\infty |x_m|^p}_{= \|(x_n)_{n \in \mathbb{N}}\|_p^p} = \|(x_n)_{n \in \mathbb{N}}\|_p^q$

$\Rightarrow (x_n)_{n \in \mathbb{N}} \in \ell^q(\mathbb{N})$ and $\|(x_n)_{n \in \mathbb{N}}\|_q = \left( \sum_{m=1}^\infty |x_m|^q \right)^{\frac{1}{q}} \le \left( \|(x_n)_{n \in \mathbb{N}}\|_p^q \right)^{\frac{1}{q}} =$

$= \|(x_n)_{n \in \mathbb{N}}\|_p$

$$\boxed{\begin{array}{l} \sum \frac{1}{n^{rq}} < \infty \quad rq > 1 \Rightarrow r > \frac{1}{q} \\ \sum \frac{1}{n^{rp}} = \infty \\ rp \le 1 \\ r \le \frac{1}{p} \end{array}}$$

Let $x_n = \frac{1}{n^r}$, $n \in \mathbb{N}$ and $r \in (\frac{1}{q}, \frac{1}{p}]$

$\sum_{n=1}^\infty |x_n|^p = \sum_{n=1}^\infty \frac{1}{n^{rp}} = \infty \Rightarrow (x_n)_{n \in \mathbb{N}} \notin \ell^p(\mathbb{N})$

$\sum_{n=1}^\infty |x_n|^q = \sum_{n=1}^\infty \frac{1}{n^{rq}} < \infty \Rightarrow (x_n)_{n \in \mathbb{N}} \in \ell^q(\mathbb{N})$

$\Rightarrow \forall 1 \le p < q \le \infty$, $\ell^p(\mathbb{N}) \subsetneq \ell^q(\mathbb{N})$.

Hölder's inequality: Let $(X, \mathcal{A}, \mu)$ be a measure space, $p, q \in [1, \infty)$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$, $f \in \mathcal{L}^p$, $g \in \mathcal{L}^q \Rightarrow f g \in \mathcal{L}^1$, $\|fg\|_1 \le \|f\|_p \cdot \|g\|_q$

probabil de folosit la examen

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EX.2: a) Compute $\int_0^\infty e^{-x} \sin x \, dx$ and $\int_{[0,\infty)} e^{-x} \sin x \, d\lambda(x)$

b) Show that $\int_{[0,\infty)} e^{-x} \sqrt{3 + 2\sin x} \, d\lambda(x) \le 2$.

Sol: a) Let $f : [0, \infty) \to \mathbb{R}$, $f(x) = e^{-x} \sin x$ is a continuous function $\Rightarrow$ $f$ is Lebesgue measurable.

$\forall t \in (0, \infty)$, $\int_0^t f(x) \, dx = \int_0^t e^{-x} \sin x \, dx = -\int_0^t (e^{-x})' \sin x \, dx =$

$= -e^{-x} \sin x \Big|_0^t + \int_0^t e^{-x} \cos x \, dx = -e^{-t} \sin t - \int_0^t (e^{-x})' \cos x \, dx =$

$= -e^{-t} \sin t - e^{-x} \cos x \Big|_0^t - \int_0^t e^{-x} \sin x \, dx = -e^{-t}(\sin t + \cos t) + 1 - \int_0^t f(x) \, dx$

$\Rightarrow \int_0^t f(x) \, dx = -\frac{1}{2} e^{-t}(\sin t + \cos t) + \frac{1}{2} \xrightarrow{t \to \infty} \frac{1}{2}$

$\Rightarrow f$ is Riemann integrable in the improper sense and $\int_0^\infty f(x) \, dx = \frac{1}{2}$.

$\forall x \in [0, \infty)$, $|f(x)| \le e^{-x}$ and $x \in [0, \infty) \mapsto e^{-x}$ is Riemann integrable

$\Rightarrow |f|$ is Riemann integrable

$\Rightarrow f$ is absolutely Riemann integrable in the improper sense

$\Rightarrow f$ is Lebesgue integrable and $\int_{[0,\infty)} f \, d\lambda = \int_0^\infty f = \frac{1}{2}$.

b) Let $f$ and $g : [0, \infty) \to \mathbb{R}$, $f(x) = e^{-\frac{x}{2}}$, $g(x) = e^{-\frac{x}{2}} \sqrt{3 + 2\sin x}$

$\Rightarrow f^2(x) = e^{-x}$, which is Lebesgue integrable $\Rightarrow f \in \mathcal{L}^2$

$0 \le g^2(x) = e^{-x}(3 + 2\underbrace{\sin x}_{\le 1}) \le 5e^{-x}$

$x \in [0, \infty) \longmapsto 5e^{-x}$ is Lebesgue integrable $\Big\} \Rightarrow$

$$\boxed{\begin{array}{l} \int_{[0,\infty)} |f|^p \, d\lambda < \infty \\ \int_{[0,\infty)} |g|^q \, d\lambda < \infty \\ \frac{1}{p} + \frac{1}{q} = 1 \\ p = 2 \\ q = 2 \end{array}}$$

$\Rightarrow g^2$ is Lebesgue integrable $\Rightarrow g \in \mathcal{L}^2$

$\Rightarrow f \cdot g \in \mathcal{L}^1$ and $\|f \cdot g\|_1 \le \|f\|_2 \cdot \|g\|_2$

$\boxed{\|f\|_2 = \left(\int_{[0,\infty)} |f|^2 \, d\mu\right)^{\frac{1}{2}}}$

$\Rightarrow \int_{[0,\infty)} e^{-x} \sqrt{3 + 2\sin x} \, dx \le \Big( \underbrace{\int_{[0,\infty)} e^{-x} \, d\lambda(x)}_{= \int_0^\infty e^{-x} dx = 1} \Big)^{\frac{1}{2}} \cdot$

$\cdot \Big( \underbrace{\int_{[0,\infty)} e^{-x}(3 + 2\sin x) \, d\lambda(x)}_{= 3 \cdot 1 + \cancel{2} \cdot \frac{1}{\cancel{2}} = 4} \Big)^{\frac{1}{2}} =$

$= 1 \cdot \sqrt{4} = 2$

$\Rightarrow \int_{[0,\infty)} e^{-x} \sqrt{3 + 2\sin x} \, d\lambda(x) \le 2$.

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Ex.4: Let $(X, \mathcal{A}, \mu)$ be a probability space and $f : X \to \mathbb{R}$ be an integrable function s.t. $\int f \, d\mu = 0$ and $\int f^2 \, d\mu = 1$. Show that, $\forall t > 0$, $\mu(\{f > t\}) \le \frac{1}{1 + t^2}$

Sol:

Let $t > 0$ and $g : X \to \mathbb{R}$, $g = f - t \Rightarrow g$ and $g^2 = f^2 - 2tf + t^2$ are measurable.

$\int t \, d\mu = t \cdot \underbrace{\mu(X)}_{=1} = t < \infty$, $\int t^2 \, d\mu = t^2 \mu(X) = t^2 < \infty \Rightarrow$

$\Rightarrow g$ and $g^2$ are integrable, with $\int g \, d\mu = \int f \, d\mu - \int t \, d\mu = -t$ and

$\int g^2 \, d\mu = \underbrace{\int f^2 \, d\mu}_{=1} - 2t \underbrace{\int f \, d\mu}_{=0} + \int t^2 \, d\mu = 1 + t^2$

On the other hand $\{f > t\} = \{f - t > 0\} = \{g > 0\}$

$-t = \int g \, d\mu = \underbrace{\int g^+ \, d\mu}_{\ge 0} - \int g^- \, d\mu \ge -\int_{\{g \le 0\}} |g| \, d\mu$.

$g^- = |g| \chi_{\{g \le 0\}}$

$\int_{\{g \le 0\}} |g| \, d\mu \le t < \infty \Rightarrow \int |g| \chi_{\{g \le 0\}} \, d\mu < \infty$

$\Rightarrow |g| \chi_{\{g \le 0\}} \in \mathcal{L}^1$.

$|g| \in \mathcal{L}^2$, $\chi_{\{g \le 0\}} \in \mathcal{L}^2 \underset{\text{Hölder}}{\Rightarrow} t \le \left( \int |g|^2 \, d\mu \right)^{\frac{1}{2}} \cdot \left( \int \chi_{\{g \le 0\}} \, d\mu \right)^{\frac{1}{2}}$

$\Rightarrow t^2 \le (1 + t^2) \mu(\{g \le 0\}) \Rightarrow \mu(\{g \le 0\}) \ge \frac{t^2}{1 + t^2}$

$\mu(\{g \le 0\}) + \mu(\{g > 0\}) = 1$ → measure of the union

$\Rightarrow \mu(\{g > 0\}) = 1 - \mu(\{g \le 0\}) \le \frac{1}{1 + t^2}$