Seminar 13

Dynamical systems generated by planar systems

$$(1)\begin{cases} x' = f_1(x,y) \\ y' = f_2(x,y) \end{cases}$$

Flow = the saturated solution of IVP:

$$(2)\begin{cases} x' = f_1(x,y) \\ y' = f_2(x,y) \\ x(0) = \eta_1 \\ y(0) = \eta_2 \end{cases} \qquad (\eta_1,\eta_2)\in\mathbb{R}^2$$

Theorem: if. $f=(f_1,f_2)\in C^1$ then the IVP(2) has an unique saturated solution for every $\eta=(\eta_1,\eta_2)\in\mathbb{R}^2$

$x(t,\eta_1,\eta_2)\ ,\ y(t,\eta_1,\eta_2) : I_\eta \to \mathbb{R}$

sat. sol. of (2) $\Longrightarrow$ $\quad I_\eta$ – maximal.

$$\left.\begin{array}{l} I_\eta = (\alpha_\eta, \beta_\eta) \\ 0 \in I_\eta \end{array}\right\} \Rightarrow \alpha_\eta < 0 < \beta_\eta$$

$$W = \{ I_\eta \times \mathbb{R}^2 \mid \eta \in \mathbb{R}^2 \}.$$

$$\varphi : W \to \mathbb{R}^2$$

$$\varphi(t, \eta_1, \eta_2) = \big(x(t, \eta_1, \eta_2), y(t, \eta_1, \eta_2) \big)$$

if $I_\eta = \mathbb{R}, \forall \eta \in \mathbb{R}^2 \Rightarrow W = \mathbb{R} \times \mathbb{R}^2 = \mathbb{R}^3$

φ — the flow generated by (1).

<u>Properties:</u>

1. $\varphi(0, \eta) = \varphi(0, \eta_1, \eta_2) = (\eta_1, \eta_2)$
2. $\varphi(t+s, \eta) = \varphi(t, \varphi(s, \eta))$
3. φ is continuous.

Orbits: $\gamma^+(\eta_1,\eta_2) = \bigcup_{t\in[0,\beta_\eta)} \varphi(t,\eta)$ positive orbit of $\eta=(\eta_1,\eta_2)$

$\gamma^-(\eta_1,\eta_2) = \bigcup_{t\in(\alpha_\eta,0]} \varphi(t,\eta)$ negative orbit

$$\gamma(\eta) = \gamma(\eta_1,\eta_2) = \gamma^+(\eta) \cup \gamma^-(\eta)$$

the orbit of $\eta=(\eta_1,\eta_2)$

Phase portrait = collection of all orbits with their describing sense.

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1) Let's consider the system:

$$\begin{cases} x' = -x \\ y' = -2y \end{cases}$$

a) find the flow generated

b) find the orbits of $(0,0)$, $(-1,0)$, $(0,1)$, $(1,1)$

c) find the phase portrait.

a) $\begin{cases} x' = -x \\ y' = -2y \\ x(0) = \eta_1 \\ y(0) = \eta_2 \end{cases}$

$x' = -x$

$x(t) = c_1 \cdot e^{-t}$

$x(0) = \eta_1$

$\Downarrow$

$c_1 = \eta_1$

$y' = -2y$

$y(t) = c_2 \cdot e^{-2t}$

$y(0) = \eta_2$

$\Downarrow$

$c_2 = \eta_2$

$$\Rightarrow \begin{cases} x(t, \eta_1, \eta_2) = \eta_1 e^{-t} \\ y(t, \eta_1, \eta_2) = \eta_2 e^{-2t} \end{cases} \qquad I_\eta = \mathbb{R}, \ \forall \eta = (\eta_1, \eta_2) \in \mathbb{R}^2$$

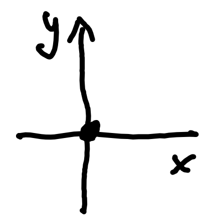
$$\Rightarrow \varphi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$$

$$\varphi(t, \eta_1, \eta_2) = \big(x(t, \eta_1, \eta_2), y(t, \eta_1, \eta_2) \big) =$$

$$= \left(\eta_1 e^{-t}, \eta_2 e^{-2t} \right).$$

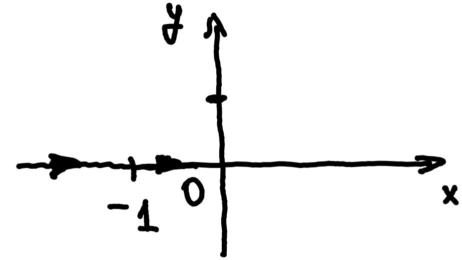
b) $\gamma(0,0) = ?$

$$\gamma(0,0) = \bigcup_{t \in \mathbb{R}} \varphi(t, 0, 0) = \bigcup_{t \in \mathbb{R}} (0,0) = \{(0,0)\}$$

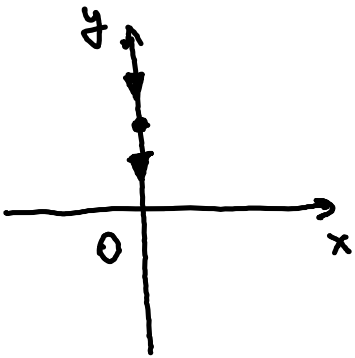


$\underline{\gamma(-1,0)}$: $\quad \gamma(-1,0) = \bigcup_{t\in\mathbb{R}} \varphi(t,-1,0) = \bigcup_{t\in\mathbb{R}} (-e^{-t}, 0) =$

$= \{ (x,0) \mid x<0 \}$



$\underline{\gamma(0,1)}$: $\quad \gamma(0,1) = \bigcup_{t\in\mathbb{R}} \varphi(t,0,1) =$



$= \bigcup_{t\in\mathbb{R}} (0, e^{-2t}) =$

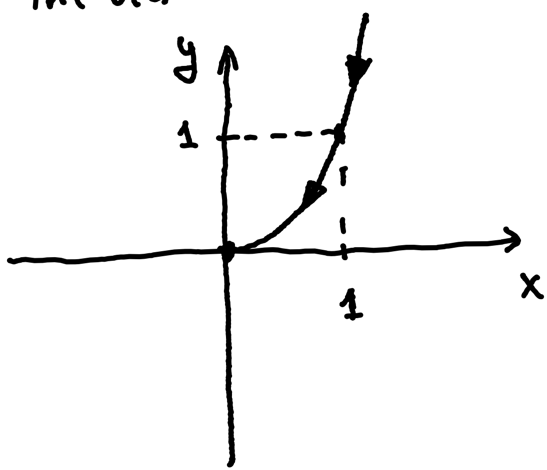
$= \{ (0,y) \mid y>0 \}.$

$\underline{\gamma(1,1)}$: $\quad \gamma(1,1) = \bigcup_{t\in\mathbb{R}} \varphi(t,1,1) = \bigcup_{t\in\mathbb{R}} (\underbrace{e^{-t}}_{x}, \underbrace{e^{-2t}}_{y})$

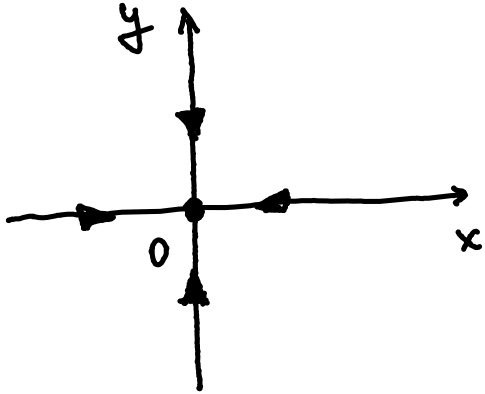
$M \in \gamma(1,1) \quad \begin{cases} x_M = e^{-t} \\ y_M = e^{-2t} \end{cases}, t\in\mathbb{R}$

$y_M = e^{-2t} = (e^{-t})^2 = x_M^2$

the orbit $\gamma(1,1)$ has the equation $y = x^2$
with $x>0$, $y>0$



c) phase portrait.



1. $\eta_1 = \eta_2 = 0$: $\gamma(0,0) = \{(0,0)\}$

2. $\eta_1 = 0$, $\eta_2 \neq 0$

$$\gamma(0, \eta_2) = \bigcup_{t \in \mathbb{R}} \varphi(t, 0, \eta_2) =$$

$$= \bigcup_{t \in \mathbb{R}} (0, \eta_2 e^{-2t}) =$$

$$= \left\{ (0, y) \,\middle|\, \begin{array}{l} y > 0 \text{ if } \eta_2 > 0 \\ y < 0 \text{ if } \eta_2 < 0 \end{array} \right\}.$$

3. $\eta_1 \neq 0$, $\eta_2 = 0$:

$$\gamma(\eta_1, 0) = \bigcup_{t \in \mathbb{R}} \varphi(t, \eta_1, 0) = \bigcup_{t \in \mathbb{R}} (\eta_1 e^{-t}, 0) =$$

$$= \left\{ (x, 0) \,\middle|\, \begin{array}{l} x > 0 \text{ if } \eta_1 > 0 \\ x < 0 \text{ if } \eta_1 < 0 \end{array} \right\}$$

4. $\eta_1 \neq 0$, $\eta_2 \neq 0$

$$\gamma(\eta_1, \eta_2) = \bigcup_{t \in \mathbb{R}} \varphi(t, \eta_1, \eta_2) = \bigcup_{t \in \mathbb{R}} (\eta_1 e^{-t}, \eta_2 e^{-2t})$$

$\gamma(\eta_1,\eta_2)$ is a curve given by the parametric eqs.

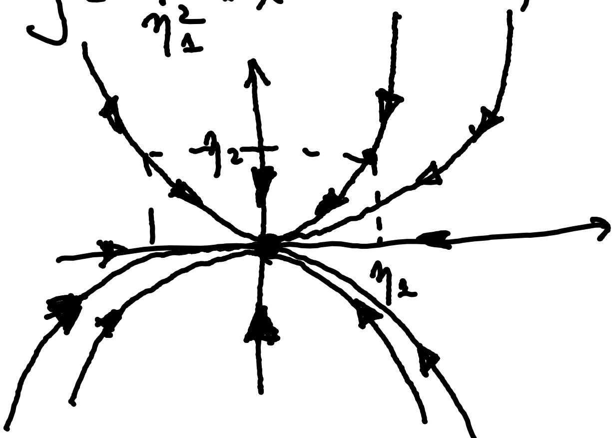
$$\begin{cases} x=\eta_1 e^{-t} \\ y=\eta_2 e^{-2t} \end{cases}, \quad t\in\mathbb{R}$$

$$x=\eta_1 e^{-t} \Rightarrow e^{-t}=\frac{x}{\eta_1}$$

$$y=\eta_2 e^{-2t} \Rightarrow e^{-2t}=\frac{y}{\eta_2} \Rightarrow \frac{y}{\eta_2}=\left(e^{-t}\right)^2=\left(\frac{x}{\eta_1}\right)^2$$

$\Rightarrow$ $\gamma(\eta_1,\eta_2)$ is given by the equation

$$y=\frac{\eta_2}{\eta_1^2}\cdot x^2$$ — a parabola



the phase portrait.

$$\begin{cases} x' = f_1(x,y) \\ y' = f_2(x,y) \end{cases}$$

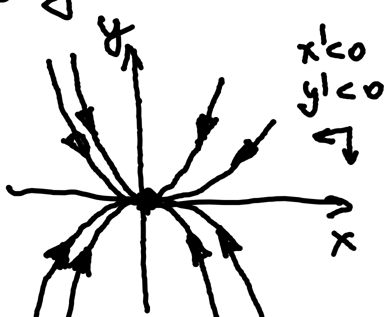
$$\left.\begin{aligned} \frac{dx}{dt} &= f_1(x,y) \\ \frac{dy}{dt} &= f_2(x,y) \end{aligned}\right\} \Rightarrow \boxed{\underset{x'(y)}{\frac{dx}{dy}} = \frac{f_1(x,y)}{f_2(x,y)}}$$

$$\boxed{\underset{y'(x)}{\frac{dy}{dx}} = \frac{f_2(x,y)}{f_1(x,y)}}$$ the differential eq. of the orbits.

$$\begin{cases} x' = -x \\ y' = -2y \end{cases} \Rightarrow \frac{dx}{dy} = \frac{-x}{-2y} \Rightarrow \frac{dx}{dy} = \frac{x}{2y} \Rightarrow$$

$$\Rightarrow \int \frac{dy}{y} = \int \frac{2\,dx}{x} \Rightarrow \ln y = 2\ln x + \ln c$$

$$\boxed{y = cx^2, \ c \in \mathbb{R}}$$ parabolas.



2) Find the phase portrait of the following systems using the diff. eq. of the orbits.

a) $\begin{cases} x' = x \\ y' = -2y \end{cases}$

b) $\begin{cases} x' = y \\ y' = -a^2 \cdot x \end{cases}$, $a \in \mathbb{R}^*$

c) $\begin{cases} x' = x \\ y' = x + 2y \end{cases}$

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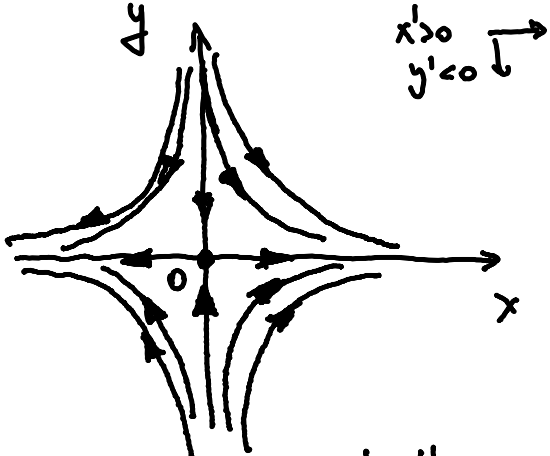
a) $\begin{cases} x' = x \\ y' = -2y \end{cases}$ $\quad \dfrac{dx}{dy} = \dfrac{x}{-2y}$ the diff. eq. of the orbits.

$$\int \frac{dy}{y} = \int -\frac{2\,dx}{x}$$

$$\ln y = -2 \ln x + \ln c$$

$$\boxed{y = c \cdot x^{-2}, \ c \in \mathbb{R}}$$



$$y = \frac{-c}{x^2}$$

phase portrait

b) $\begin{cases} x' = y \\ y' = -a^2 x \end{cases}$

$$\frac{dx}{dy} = \frac{y}{-a^2 x}$$

$$y\, dy = -a^2 x \cdot dx \quad / \cdot 2$$

$$\int 2y\, dy = \int -2a^2 x \cdot dx$$

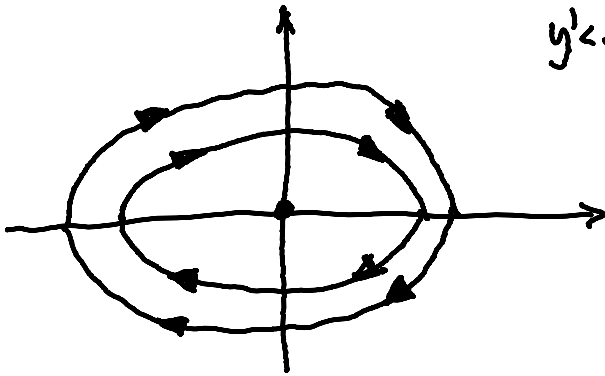
$$y^2 = -a^2 x^2 + c$$

$$\boxed{a^2 x^2 + y^2 = -c\ ,\ c \in \mathbb{R}}$$

$$\Rightarrow \left(\frac{x}{\frac{\sqrt{-c}}{a}}\right)^2 + \left(\frac{y}{\sqrt{-c}}\right)^2 = 1 \quad \text{ellipses.}$$

$x' > 0$ $\longrightarrow$

$y' < 0$ $\downarrow$



c) $\begin{cases} x' = x \\ y' = x + 2y \end{cases}$

$$\frac{dx}{dy} = \frac{x}{x+2y}$$

$$\frac{dy}{dx} = \frac{x+2y}{x}$$

$$\underbrace{\frac{dy}{dx}}_{y'(x)} = 1 + \frac{2}{x} \cdot y$$

$y' = 1 + \frac{2}{x} \cdot y \quad \Rightarrow \boxed{y' - \frac{2}{x} y = 1}$ the diff. eq. of. the orbits.

$y' - \frac{2}{x} y = 0$

$y' = \frac{2}{x} y$

$\frac{dy}{dx} = \frac{2}{x} \cdot y \quad \Rightarrow \int \frac{dy}{y} = \int \frac{2}{x} \cdot dx$

$\ln y = 2 \ln x + \ln c$

$y_0(x) = c \cdot x^2, c \in \mathbb{R}.$

$y_p(x) = c(x) . x^2$

$y_p' - \frac{2}{x} \cdot y_p = 1$

$c'(x) . x^2 + \cancel{c(x) . 2x} - \cancel{\frac{2}{x} \cdot c(x)} . x^2 = 1$

$c'(x) . x^2 = 1 \quad \Rightarrow \quad c'(x) = \frac{1}{x^2} \Rightarrow c(x) = \int \frac{1}{x^2} dx = -\frac{1}{x}$

$\Rightarrow y_p(x) = c(x) . x^2 = -\frac{1}{x} \cdot x^2 = -x.$

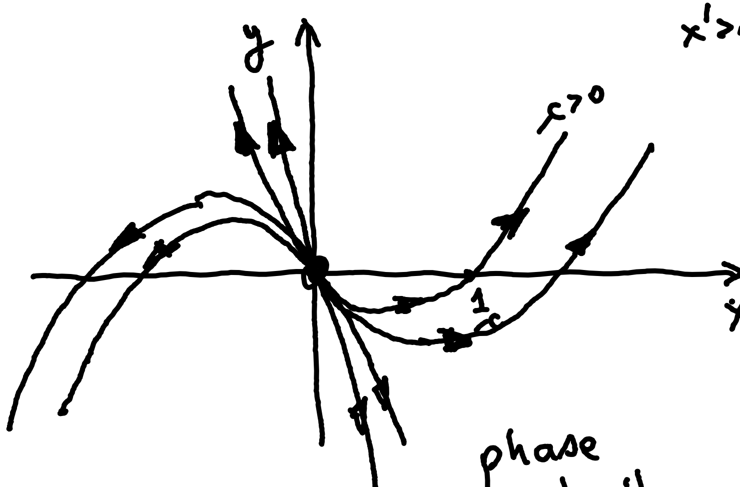
$y = y_0 + y_p$

$y(x) = c \cdot x^2 - x \ , \ c \in \mathbb{R}$.

$y = cx^2 - x$

$y = x(cx - 1)$

$x = 0 \ , \ x = \frac{1}{c}$.



phase portrait .

$x' > 0 \longrightarrow \begin{cases} x' = x \\ y' = x + 2y \end{cases}$