# Seminar 12

## Dynamical systems generated by autonomous scalar diff. eq.

$x' = f(x)$ $\qquad x = x(t)$. $\qquad$, $f \in C^1$

The flow = the saturated solution of the IVP:

$$\begin{cases} x' = f(x) \\ x(0) = \eta, \quad \eta \in \mathbb{R}. \end{cases} \Rightarrow x(t, \eta)$$

$x(\cdot, \eta) : I_\eta \to \mathbb{R}$, $I_\eta$ – maximal.

$I_\eta = (\alpha_\eta, \beta_\eta)$, $0 \in I_\eta$.

$W = \{ I_\eta \times \mathbb{R} \mid \eta \in \mathbb{R} \}$.

if $I_\eta = \mathbb{R} \to W = \mathbb{R}^2$

$\varphi : W \to \mathbb{R}$

$\varphi(t, \eta) = x(t, \eta)$

Properties of the flow:

1. $\varphi(0, \eta) = \eta$
2. $\varphi(t+s, \eta) = \varphi(t, \varphi(s, \eta))$
3. $\varphi$ is cont.

__Orbits:__

$$\gamma^+(\eta) = \bigcup_{t\in[0,\beta_\eta)} \varphi(t,\eta)$$ — the positive orbit of $\eta$

$$\gamma^-(\eta) = \bigcup_{t\in(\alpha_\eta,0]} \varphi(t,\eta)$$ — the negative orbit of $\eta$.

$$\gamma(\eta) = \gamma^+(\eta)\cup\gamma^-(\eta).$$ the orbit of $\eta$.

Phase portrait (the phase line): the collection of all orbits with the describing direction.

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1) Let's consider the eq. $x' = x+1$.

a) find the generated flow

b) find the orbits for $\eta=-1$, $\eta=0$, $\eta=-2$.

c) find the phase portrait.

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a) $\begin{cases} x' = x+1 \\ x(0) = \eta . \end{cases}$

$\frac{dx}{dt} = x+1 \to \int \frac{dx}{x+1} = \int dt$

$\Rightarrow \ln|x+1| = t + \ln c.$

$x+1 = c \cdot e^t$

$\boxed{x(t) = c \cdot e^t - 1, \ c \in \mathbb{R}}$ the gen. sol. of the diff. eq.

$x(0) = \eta \Rightarrow c - 1 = \eta \Rightarrow c = \eta + 1.$

$\Rightarrow \quad x(t,\eta) = (\eta+1) \cdot e^t - 1.$

$x(\cdot,\eta) : I_\eta \to \mathbb{R} \quad , I_\eta$ maximal.

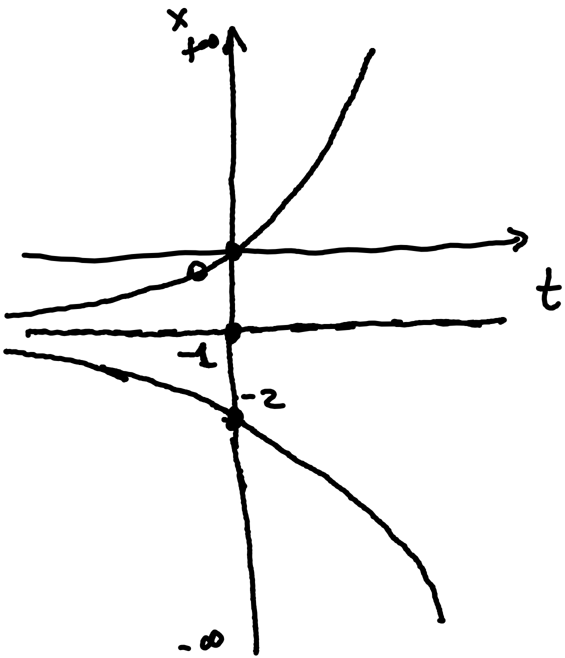
$\Rightarrow I_\eta = \mathbb{R} \ , \ \forall \eta \in \mathbb{R}.$

$\left.\begin{array}{l} \varphi(t,\eta) = x(t,\eta) = (\eta+1)e^t - 1 \\ \varphi : \mathbb{R}\times\mathbb{R} \to \mathbb{R} \end{array}\right\}$ the flow.

b) $\gamma(-1) = ?$ $\gamma(\eta) = \bigcup_{t \in I_\eta} \varphi(t, \eta)$

$\varphi(t, -1) = -1$

$$\gamma(-1) = \bigcup_{t \in \mathbb{R}} \varphi(t, -1) = \bigcup_{t \in \mathbb{R}} (-1) = \{-1\}.$$




$\gamma(0) = ?$

$\varphi(t, 0) = e^t - 1$

$$\gamma^+(0) = \bigcup_{t \in [0, +\infty)} \varphi(t, 0) = [0, +\infty) \longrightarrow$$

$$\gamma^-(0) = \bigcup_{t \in (-\infty, 0]} \varphi(t, 0) = (-1, 0] \longrightarrow$$

$$\gamma(0) = \gamma^+(0) \cup \gamma^-(0) = (-1, +\infty) \longrightarrow$$

$\gamma(-2) = ?$

$$\varphi(t,-2) = -e^{t} - 1$$

$$\gamma^{+}(-2) = \bigcup_{t \in [0,+\infty)} \varphi(t,-2) = (-\infty, -2]$$

$$\gamma^{-}(-2) = \bigcup_{t \in (-\infty,0]} \varphi(t,-2) = [-2,-1)$$

$$\gamma(-2) = \gamma^{+}(-2) \cup \gamma^{-}(-2) = (-\infty, -1)$$

c) we have 3 cases:

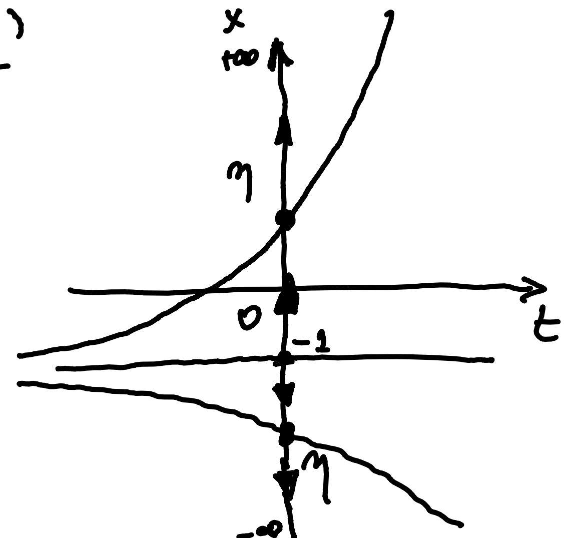
1. $\eta = -1 \Rightarrow \varphi(t,-1) \equiv -1.$

$$\gamma(-1) = \{-1\}$$

2. $\eta > -1$, $\varphi(t,\eta) = \underbrace{(\eta+1)}_{>0} e^{t} - 1$

$$\gamma^{+}(\eta) = \bigcup_{t \in [0,+\infty)} \varphi(t,\eta) = [\eta, +\infty)$$

$$\gamma^{-}(\eta) = \bigcup_{t \in (-\infty,0]} \varphi(t,\eta) = (-1, \eta]$$




$$\gamma(\eta) = \gamma^{+}(\eta) \cup \gamma^{-}(\eta) = (-1, +\infty)$$

3. $\eta < -1 \Rightarrow \varphi(t,\eta) = \underbrace{(\eta+1)}_{<0} e^t - 1$

$$\gamma^+(\eta) = \bigcup_{t\in[0,+\infty)} \varphi(t,\eta) = (-\infty, \eta] \quad \longleftarrow$$

$$\gamma^-(\eta) = \bigcup_{t\in(-\infty,0]} \varphi(t,\eta) = [\eta, -1) \quad \longleftarrow$$

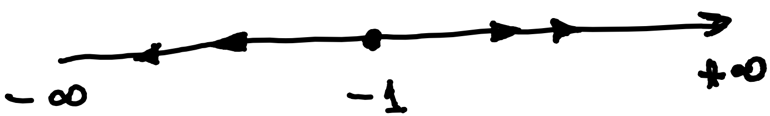
$$\gamma(\eta) = \gamma^+(\eta) \cup \gamma^-(\eta) = (-\infty, -1) \quad \longleftarrow$$

$f(x) = x+1$

$f(x) = 0 \Rightarrow x+1 = 0$

$x = -1$

| $x$ | $-\infty$ | | $-1$ | | $+\infty$ |
|---|---|---|---|---|---|
| $f(x)$ | $-$ | $-$ | $0$ | $+$ $+$ | $+$ |
| | $\longleftarrow$ | | | $\longrightarrow$ | |



phase portrait.

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$x' = f(x)$. $\quad f(x) = 0 \rightarrow x_1, x_2, \ldots, x_n$ real roots.

$x_1 < x_2 < \ldots < x_n$

| $x$ | $-\infty$ | | $x_1$ | | $x_2$ | | $x_3$ | $\ldots$ | $x_n$ |
|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | | $-$ | $0$ | $+$ | $0$ | $+$ | $0$ | $\ldots$ | $0$ |
| | | $\longleftarrow$ | | $\longrightarrow$ | | $\longleftarrow$ | | | |

2) Find the flow generated by the given diff. eq. and the corresponding phase portrait using the sign table of $f$.

a) $x' = 2x+1$

b) $x' = x^2$

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a) $\begin{cases} x' = 2x+1 \\ x(0) = \eta \end{cases}, \eta \in \mathbb{R}.$ $\qquad x(t) \equiv -\frac{1}{2}$

$\frac{dx}{dt} = 2x+1 \Rightarrow \frac{dx}{2x+1} = dt \,\Big|\cdot 2$

$\Rightarrow \int \frac{2 \cdot dx}{2x+1} = \int 2\,dt \Rightarrow \ln(2x+1) = 2t + \ln c.$

$2x+1 = c \cdot e^{2t}$

$x(t) = \frac{c \cdot e^{2t} - 1}{2}, c \in \mathbb{R}.$

$x(0) = \eta \Rightarrow \frac{c-1}{2} = \eta \Rightarrow c - 1 = 2\eta$

$c = 2\eta + 1$

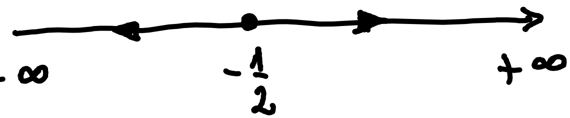
$\Rightarrow \boxed{x(t,\eta) = \frac{(2\eta+1)e^{2t} - 1}{2}} \quad I_\eta = \mathbb{R}$

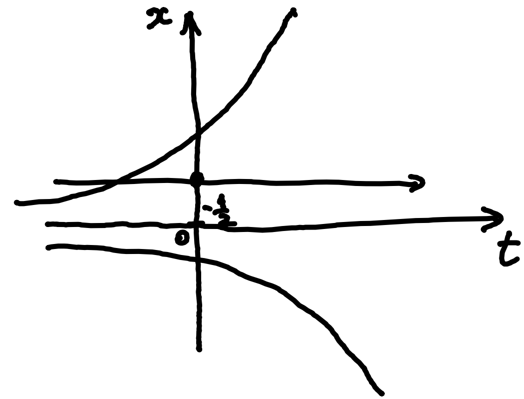
$\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \; \varphi(t,\eta) = x(t,\eta)$

$f(x) = 2x + 1$

$f(x) = 0 \Rightarrow 2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$

| $x$ | | $-\frac{1}{2}$ | |
|---|---|---|---|
| $f(x)$ | $- \; -$ | $0$ | $+ \; + \; +$ |
| | $\longleftarrow$ | | $\longrightarrow$ |




b) $x' = x^2$

$$\begin{cases} x' = x^2 \\ x(0) = \eta \end{cases}$$

$x(t) \equiv 0$ is singular solution.

$$\frac{dx}{dt} = x^2 \Rightarrow \frac{dx}{x^2} = dt \quad \Big| \cdot (-1)$$

$$\int -\frac{dx}{x^2} = \int -dt \Rightarrow \frac{1}{x} = -t + c \Rightarrow x(t) = \frac{1}{-t + c}, \; c \in \mathbb{R}$$

gen. sol.

$x(0)=\eta \Rightarrow \frac{1}{c}=\eta \Rightarrow c=\frac{1}{\eta}, \eta \neq 0.$

$$\Rightarrow \begin{cases} x(t,\eta)=\dfrac{1}{-t+\frac{1}{\eta}} & , \eta\neq 0 \\ x(t,\eta)\equiv 0 & , \eta=0 \end{cases}$$

$$\boxed{x(t,\eta)=\frac{\eta}{1-\eta t} \quad , \forall \eta\in\mathbb{R}}$$

$I_\eta = ?$ $\quad \eta=0 \Rightarrow x(t,0)\equiv 0 \Rightarrow \underline{I_0=\mathbb{R}}.$

$\eta\neq 0 \Rightarrow (-\infty,\frac{1}{\eta})$ or $(\frac{1}{\eta},+\infty)$

$0\in I_\eta$

$$I_\eta=\begin{cases} (-\infty,\frac{1}{\eta}) & ,\eta>0 \\ (\frac{1}{\eta},+\infty) & ,\eta<0 \\ \mathbb{R} & ,\eta=0 \end{cases}$$

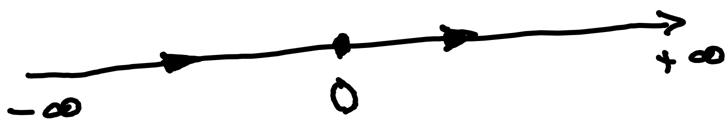
$\varphi: W\to\mathbb{R}$

$W=\{\underline{\underline{I_\eta}}\times\mathbb{R} \mid \eta\in\mathbb{R}\}$

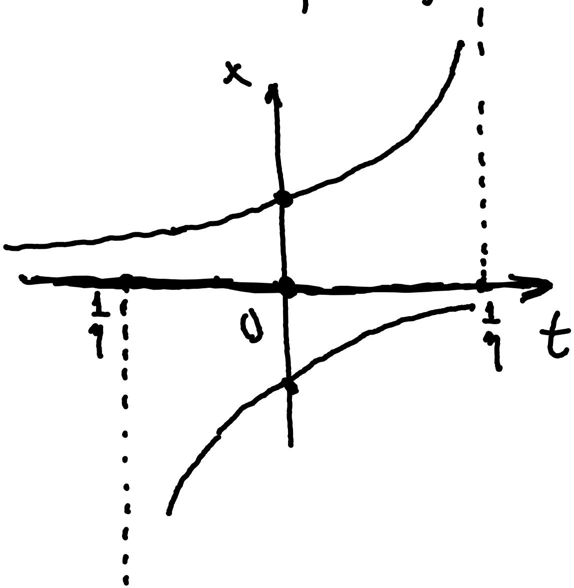
$\varphi(t,\eta)=x(t,\eta)=\dfrac{\eta}{1-\eta t}$

$f(x) = x^2$ $\qquad$ $f(x) = 0 \Rightarrow x = 0$

| $x$ | $-\infty$ | | $0$ | | $+\infty$ |
|---|---|---|---|---|---|
| $f(x)$ | | $+$ | $0$ | $+$ | |
| | | $\rightarrow$ | | $\rightarrow$ | |




phase portrait



$x' = f(x)$

sol. $x(t) \equiv x^*$ — equilibrium sol.

$x^* \in \mathbb{R}$ — the equilibrium point.

the equilibrium points are real solutions of the eq.

$$\boxed{f(x) = 0}$$

$x^*$

locally asymptotically stable

$x^*$

unstable eq. point

**Theorem.** (The Stability Theorem in the first approx.)

$f \in C^1$, $x^*$ is an eq. point

a) If $f'(x^*) < 0 \Rightarrow x^*$ is locally as. stable

b) If $f'(x^*) > 0 \Rightarrow x^*$ is unstable.

3) Find the equilibrium points and study their stability for the eqs:

a) $x' = -2x$

b) $x' = 2+x$

c) $x' = x(1-x)$

d) $x' = \sin x$

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a) $x' = -2x$

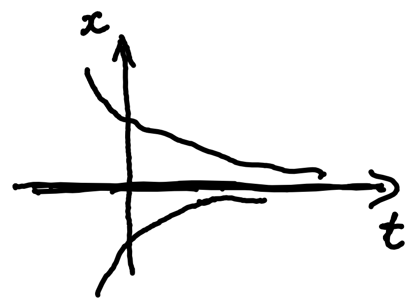
$f(x) = -2x$ $\qquad$ $f(x)=0 \Rightarrow -2x=0 \Rightarrow x^*=0$ eq. point

$f'(x) = -2 \Rightarrow f'(0) = -2 < 0 \Rightarrow x^*=0$ is asympt. stable

| $x$ | $-\infty$ | | $0$ | | $+\infty$ |
|---|---|---|---|---|---|
| $f(x)$ | | $+\ +$ | $0$ | $-\ -\ -$ | |
| | | $\rightarrow$ | | $\leftarrow$ | |

$-\infty$ $\qquad$ $0$

asympt. stable



b) $x' = 2 + x$

$f(x) = 2 + x \implies f(x) = 0$

$2 + x = 0$

$x^* = -2$ eq. point.

$f'(x) = 1 > 0 \implies x^* = -2$ unstable.

| $x$ | | $-2$ | |
|---|---|---|---|
| $f(x)$ | $- - -$ | $0$ | $+ + +$ |
| | $\longleftarrow$ | | $\longrightarrow$ |


unstable

c) $x' = x(1-x)$

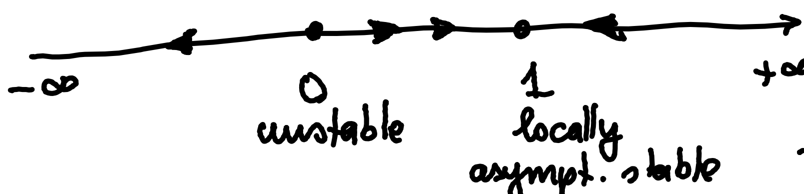
$f(x) = x(1-x)$ $\quad f(x) = 0 \Rightarrow x(1-x) = 0$ $\begin{cases} x_1^* = 0 \\ x_2^* = 1 \end{cases}$ eq. points

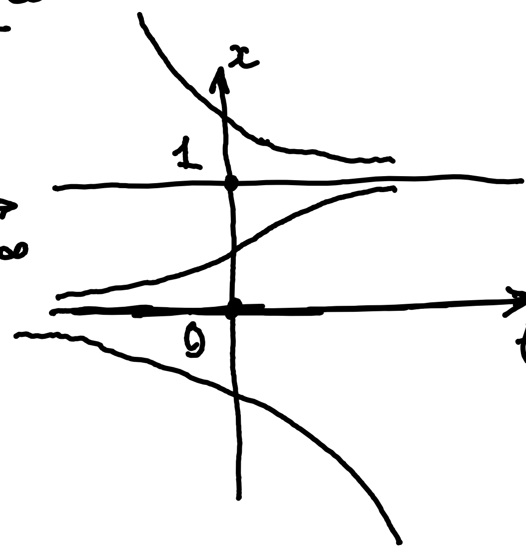
$f(x) = x - x^2 \rightarrow f'(x) = 1 - 2x$

$x_1^* = 0$ : $\quad f'(0) = 1 > 0 \Rightarrow x_1^* = 0$ is unstable

$x_2^* = 1$ : $\quad f'(1) = 1 - 2 \cdot 1 = -1 < 0 \Rightarrow x_2^* = 1$ locally asympt. stable.

| $x$ | $-\infty$ | | 0 | | 1 | | $+\infty$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | | $- -$ | 0 | $+ +$ | 0 | $- - -$ | |
| | | $\leftarrow$ | | $\rightarrow$ | | $\leftarrow$ | |



d) $x' = \sin x$.

$f(x) = \sin x$ $\qquad$ $f(x) = 0$

$\sin x = 0 \Rightarrow x_k^* = k\pi$, $k \in \mathbb{Z}$ eq. points.

$f'(x) = \cos x$

$$f'(x_k^*) = \cos(x_k^*) = \cos(k\pi) = \begin{cases} -1, & k \text{ odd.} \\ 1 & k \text{ even.} \end{cases}$$

$x_k^*$ $\begin{cases} \text{locally asympt. stable for } k \text{ odd} \\ \text{unstable for } k \text{ even} \end{cases}$

| $x$ | $-2\pi$ | | $-\pi$ | | $0$ | | $\pi$ | | $2\pi$ |
|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | 0 | + | 0 | − | 0 | + | 0 | − | 0 |
| | | $\rightarrow$ | | $\leftarrow$ | | $\rightarrow$ | | $\leftarrow$ | |