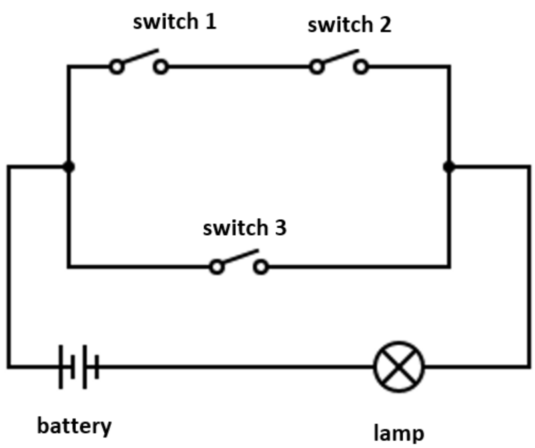
Seminars 4 and 5 - 2025

1. Four electronic devices have the property that, for every $i \in \{1, 2, 3, 4\}$, the probability that any i fixed devices are all functional is $\frac{1}{4^i}$. Using the inclusion-exclusion principle, compute the probability of the event
A:"none of the devices is functional".

2. Four antivirus programs are tested by scanning independently an infected file. They detect the virus with corresponding probabilities: $\frac{3}{4}, \frac{1}{4}, \frac{2}{4}, \frac{1}{4}$. Compute the probabilities of the following events:
A:"All programs detect the virus."
B:"Exactly one program detects the virus."
C:"Exactly three programs detect the virus."
D:"At most one program detects the virus."
E:"At least one program detects the virus."

3. In the diagram below the three switches are either ON or OFF, independently, with probability $\frac{1}{2}$ for each state. Compute the probability that the circuit operates.



4. The owner of three shops decides to give a bonus to the salary of a randomly chosen employee. The first shop has 50 employees and 50% of them are men, the second shop has 75 employees and 60% of them are men and the third shop has 100 employees and 70% are men.
a) Find the probability that the lucky employee works in the third shop, given that the lucky employee is a woman.
b) Find the probability that the lucky employee is a woman, given that the lucky employee works in the third shop.

5. (Coursework, for one "*" (see Teams)) Three dice are rolled. Let N_k be number that showed on the kth die, $k \in \{1, 2, 3\}$. Find:
a) $P(N_1 = 1, N_2 = 2, N_3 = 3)$.
b) $P(N_1 = N_2 = N_3)$.
c) $P(N_1 + N_2 + N_3 \geq 5)$.
d) $P(N_1 + N_2 + N_3 \geq 5 | N_1 < N_2 < N_3)$.
e) $P(N_1 < N_2 < N_3 | N_1 < N_2)$.
f) $P(N_1 > N_2 < N_3 | N_1 = N_3)$.
g) $P(N_1 = N_2, N_2 > 2 | N_3 > 2)$.

6. John has in his pocket 2 red dice and 3 blue dice. He takes randomly a die. If the chosen die is red, he rolls it 3 times. On the other hand, if the chosen die is blue, he rolls it 2 times. Compute the probability of the event E:"the sum of the numbers that show up after the rolls is 10."

7. Mary studies Probability Theory. She arrives late at the seminar with probability 0.2, if the day is rainy, and with probability 0.1, if the day is sunny. According to the weather forecast, the next day, when Mary has the

seminar, is rainy with probability 0.8. Compute the probabilities of the events:
A: "Mary arrives on time at the next seminar."
B: "The next day is rainy, given that Mary arrives on time at the seminar ."

8. A die is rolled. Let N be the number that is obtained. Next, the die is rolled N times. What is the probability that $N = 3$, given that $N \geq 2$ and the numbers obtained after the N rolls are pairwise **a)** distinct? **b)** equal?

9. A pair of dice - one white die and one red die - is rolled two times. Compute the probability that the two pairs of numbers, obtained after the two rolls, are equal. (Example of favorable case: the white die shows number 2 and the red die shows number 4, both after the first roll and the second roll; example of unfavorable case: first roll "2 on white die, 4 on red die", second roll "4 on white die, 2 on red die".)

10. A computer center has three printers A, B, and C, which print at different speeds. Programs are routed to the first available printer. The probability that a program is routed to printers A, B, and C are 0.5, 0.3, and 0.2, respectively. Occasionally a printer will jam and destroy a printout. The probability that printers A, B, and C will jam are 0.02, 0.06 and 0.1, respectively. Your program is destroyed when a printer jams. What is the probability that printer A is involved? Printer B is involved? Printer C is involved?

Optional:

11. Let $(S, \mathcal{K}, P)$ is be a probability space and $A, B \in \mathcal{K}$.
(a) Prove that

$$P(A) + P(B) - 1 \leq P(A \cap B) \leq \min\{P(A), P(B)\}$$

$$\max\{P(A), P(B)\} \leq P(A \cup B) \leq P(A) + P(B).$$

(b) It is known that $P(A) = \frac{2}{5}, P(B) = \frac{9}{10}$. Indicate four concrete values $a, b, c, d \in (0, 1]$ such that

$$a \leq P(A \cap B) \leq b < c \leq P(A \cup B) \leq d.$$

Explain why your choice is correct!
(c) It is known that $P(A) = \frac{2}{5}, P(B) = \frac{9}{10}, P(A|B) = \frac{10}{27}$.
(c1) Compute the probabilities: $P(A \cap B), P(A \cup B), P(B \cap \bar{A}), P(B|A)$.
(c2) Are A and B independent events?

12. a) Let $(S, \mathcal{K}, P)$ be a probability space and $B \in \mathcal{K}$ such that $P(B) > 0$. Prove that $(B, \mathcal{K}_B, P(\cdot|B))$ is a probability space, where $\mathcal{K}_B := \{B \cap A : A \in \mathcal{K}\}$ and $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}, A \in \mathcal{K}_B$.
b) Give examples, by considering a random experiment and its corresponding probability space $(S, \mathcal{K}, P)$, for the probability space $(B, \mathcal{K}_B, P(\cdot|B))$ from **a)**.