Lecture 1 February 24, 2022

$3 \times 45 = 135$ → $14^{50} - 16^{00}$, $16^{10} - 17^{15}$

Chapter 1. TOPOLOGY IN $\mathbb{R}^n$

1. The Euclidean Space $\mathbb{R}^n$

1.1. Definition (the vector space $\mathbb{R}^n$) Let $n \in \mathbb{N} := \{1, 2, 3, \dots\}$ and let

$$\mathbb{R}^n := \{ (x_1, \dots, x_n) \mid x_j \in \mathbb{R} \quad \forall\, j = \overline{1, n} \}.$$

On $\mathbb{R}^n$ we consider

- an internal composition law

$$+ : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \qquad \forall (x, y) \in \mathbb{R}^n \times \mathbb{R}^n \longmapsto x + y \in \mathbb{R}^n$$

- an external composition law with operators in $\mathbb{R}$

$$\cdot : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \qquad \forall (\alpha, x) \in \mathbb{R} \times \mathbb{R}^n \longmapsto \alpha \cdot x \in \mathbb{R}^n$$

defined as follows: given $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$

we set $\quad x + y := (x_1 + y_1, \dots, x_n + y_n) \qquad \alpha \cdot x := (\alpha x_1, \dots, \alpha x_n).$

$(\mathbb{R}^n, +, \cdot)$ is a vector space over $\mathbb{R}$

- the origin $\quad 0_n = (0, \dots, 0)$
- the symmetric (inverse) element of $x = (x_1, \dots, x_n)$ is $-x = (-x_1, \dots, -x_n)$
- we write αx instead of $\alpha \cdot x$.

Elements of $\mathbb{R}^n$ $\nearrow$ n-tuples, $\rightarrow$ vectors, $\searrow$ points

$$(x_1, x_2, x_3) \longleftrightarrow x = x_1 e_1 + x_2 e_2 + x_3 e_3 = x_1 \vec{i} + x_2 \vec{j} + x_3 \vec{k}$$

$$\updownarrow$$

$$P(x_1, x_2, x_3)$$

__1.2. Definition__ (the canonical basis). In $\mathbb{R}^n$ we consider the vectors

$$e_1 := (1, 0, 0, \dots, 0)$$
$$e_2 := (0, 1, 0, \dots, 0)$$
$$\vdots$$
$$e_n := (0, 0, 0, \dots, 1)$$

$\Rightarrow$ $\{e_1, e_2, \dots, e_n\}$ is an algebraic basis in the vector space $\mathbb{R}^n$, called the canonical (standard) basis in $\mathbb{R}^n$.

If $x = (x_1, \dots, x_n) \in \mathbb{R}^n \Rightarrow x = x_1 e_1 + \dots + x_n e_n$.

In $\mathbb{R}^2$ $\quad e_1 = (1, 0) = \vec{i} \qquad x = (x_1, x_2) = x_1 \vec{i} + x_2 \vec{j}$

$e_2 = (0, 1) = \vec{j}$

In $\mathbb{R}^3$ $\quad e_1 = (1, 0, 0) = \vec{i} \qquad e_2 = (0, 1, 0) = \vec{j} \qquad e_3 = (0, 0, 1) = \vec{k}$

<u>1.3. Definition</u> (the inner / dot product) Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$

$y = (y_1, \dots, y_n) \in \mathbb{R}^n$

We define

$$\langle x, y \rangle = x \cdot y := x_1 y_1 + \dots + x_n y_n$$

↳ the inner or dot product of the vectors x and y.

<u>Properties</u>.

1° $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle \quad \forall x, y, z \in \mathbb{R}^n$

2° $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle \quad \forall \alpha \in \mathbb{R}, \ \forall x, y \in \mathbb{R}^n$

3° $\langle x, y \rangle = \langle y, x \rangle \quad \forall x, y \in \mathbb{R}^n$

4° $\langle x, x \rangle > 0 \quad \forall x \in \mathbb{R}^n, \ x \neq 0_n$

Let X be a real vector space. A function $\langle \cdot , \cdot \rangle : X \times X \to \mathbb{R}$ is said to be an inner product on X if it satisfies the following axioms:

(IP_1) $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

(IP_2) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$

(IP_3) $\langle x, y \rangle = \langle y, x \rangle$

$\forall x, y, z \in X \quad \forall \alpha \in \mathbb{R}$

(IP_4) $\langle x, x \rangle > 0 \quad \forall x \in X, \ x \neq 0_X$

The ordered pair $(X, \langle \cdot , \cdot \rangle)$ is called inner product space

By $1^\circ \div 4^\circ \Rightarrow (\mathbb{R}^n, \langle \cdot , \cdot \rangle)$ is an inner product space = the Euclidean space $\mathbb{R}^n$

Other examples a) $C[a,b] = \{ f : [a,b] \to \mathbb{R} \mid f \text{ is continuous on } [a,b] \}$

$\hookrightarrow$ real vector space

$$\langle f, g \rangle := \int_a^b f(x) g(x) \, dx \qquad \forall f, g \in C[a,b]$$

$\hookrightarrow$ an inner product on $C[a,b]$

b) $R[a,b] = \{ f : [a,b] \to \mathbb{R} \mid f \text{ is Riemann integrable over } [a,b] \}$

↳ real vector space

$$\langle f, g \rangle = \int_a^b f(x) g(x)\, dx \qquad \forall f, g \in R[a,b]$$

↳ is <u>NOT</u> an inner product on $R[a,b]$

$\langle f, f \rangle \not> 0 \quad \forall f \in R[a,b], \ f \neq 0$

$$f(x) = \begin{cases} 1 & x = b \\ 0 & x \in [a,b) \end{cases} \qquad f \neq 0, \text{ but } \langle f, f \rangle = 0$$

<u>1.4 Definition</u> (the Euclidean norm) Let $\|\cdot\| : \mathbb{R}^n \to [0, \infty)$, defined by

$$\|x\| := \sqrt{\langle x, x \rangle} = \sqrt{x_1^2 + \cdots + x_n^2} \qquad \forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n$$

↳ the Euclidean norm (or length) of the vector x

Properties of the Euclidean norm

1° $\|x\| = 0 \iff x = 0_n$

2° $\|\alpha x\| = |\alpha| \cdot \|x\| \quad \forall \alpha \in \mathbb{R}, \ \forall x \in \mathbb{R}^n$

3° $\|x+y\| \leq \|x\| + \|y\| \quad \forall x, y \in \mathbb{R}^n$ (the triangle inequality)

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Cauchy-Schwarz inequality : $\forall x, y \in \mathbb{R}^n : |\langle x, y\rangle| \leq \sqrt{\langle x, x\rangle} \cdot \sqrt{\langle y, y\rangle} = \|x\| \cdot \|y\|$ (*)

$x = (x_1, \dots, x_n) \in \mathbb{R}^n \qquad y = (y_1, \dots, y_n) \in \mathbb{R}^n$

(*) $\iff |x_1 y_1 + \dots + x_n y_n| \leq \sqrt{x_1^2 + \dots + x_n^2} \sqrt{y_1^2 + \dots + y_n^2}$

$\iff (x_1 y_1 + \dots + x_n y_n)^2 \leq (x_1^2 + \dots + x_n^2)(y_1^2 + \dots + y_n^2)$

Let X be a real vector space. A function $\|\cdot\| : X \to [0, \infty)$ is called a norm on X if it satisfies the following axioms :

(N_1) $\|x\| = 0 \iff x = 0_X$

(N_2) $\|\alpha x\| = |\alpha| \cdot \|x\|$

(N_3) $\|x+y\| \leq \|x\| + \|y\|$

$\forall x, y \in X \quad \forall \alpha \in \mathbb{R}$

The ordered pair $(X, \|\cdot\|)$ = a normed space

$(\mathbb{R}^n, \|\cdot\|)$ is a normed space

$\hookrightarrow$ Euclidean norm

<u>Other examples</u> a) $\|f\| := \sqrt{\langle f, f\rangle} = \sqrt{\int_a^b f^2(x)\,dx} \quad \forall f \in C[a,b]$

$\hookrightarrow$ is a norm on $C[a,b]$

b) $\|\cdot\|_1 : \mathbb{R}^n \to [0,\infty)$ $\quad \|x\|_1 := |x_1| + \cdots + |x_n| \quad \forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n$

$\hookrightarrow$ is a norm on $\mathbb{R}^n$, called the Minkowski norm

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c) $\|\cdot\|_\infty : \mathbb{R}^n \to [0,\infty)$ $\quad \|x\|_\infty := \max\{|x_1|, \ldots, |x_n|\} \quad \forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n$

$\hookrightarrow$ is a norm on $\mathbb{R}^n$, called the Tchebyshev norm

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If $x, y \in \mathbb{R}^n$ are non-zero vectors $\Rightarrow$ $|\langle x, y\rangle| \le \|x\| \cdot \|y\| \iff \left|\dfrac{\langle x, y\rangle}{\|x\| \cdot \|y\|}\right| \le 1$

$$\iff \quad -1 \le \frac{\langle x, y\rangle}{\|x\| \cdot \|y\|} \le 1$$

The angle between x and y is defined by $\cos(\widehat{x, y}) = \dfrac{\langle x, y\rangle}{\|x\| \cdot \|y\|}$

1.5. Definition (the Euclidean distance) Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$

$y = (y_1, \dots, y_n) \in \mathbb{R}^n$

We define $d(x,y) := \|x-y\| = \sqrt{(x_1-y_1)^2 + \dots + (x_n-y_n)^2}$

↳ the Euclidean distance between x and y

$$d_1(x,y) := \|x-y\|_1 = |x_1-y_1| + \dots + |x_n-y_n|$$

↳ the Minkowski (taxi cab) distance between x and y

$$d_\infty(x,y) := \|x-y\|_\infty = \max\{|x_1-y_1|, \dots, |x_n-y_n|\}$$

↳ the Tchebyshev distance between x and

Remark $\|x\|_p := \left(|x_1|^p + \dots + |x_n|^p\right)^{1/p}$ $\quad p \geq 1$

↳ the p-norm on $\mathbb{R}^n$

2. The Topological Structure of $\mathbb{R}^n$

2.1. Definition (balls) Let $a \in \mathbb{R}^n$ and let $r > 0$. We define

$$B(a,r) := \{ x \in \mathbb{R}^n \mid d(x,a) < r \} = \{ x \in \mathbb{R}^n \mid \|x - a\| < r \}$$

↳ the open ball about a with radius r

$$\overline{B}(a,r) := \{ x \in \mathbb{R}^n \mid d(x,a) \le r \} = \{ x \in \mathbb{R}^n \mid \|x - a\| \le r \}$$

↳ closed ball about a with radius r

Example Let $\overline{B}_2^n := \{ x \in \mathbb{R}^n \mid d(x, 0_n) \le 1 \} = \{ (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \dots + x_n^2 \le 1 \}$

↳ closed unit ball in $\mathbb{R}^n$ wrto the Euclidean distance

$\overline{B}_1^n := \{ x \in \mathbb{R}^n \mid d_1(x, 0_n) \le 1 \} = \{ (x_1, \dots, x_n) \in \mathbb{R}^n \mid |x_1| + \dots + |x_n| \le 1 \}$

↳ closed unit ball in $\mathbb{R}^n$ wrto the Minkowski distance

$\overline{B}_\infty^n := \{ x \in \mathbb{R}^n \mid d_\infty(x, 0_n) \le 1 \} = \{ (x_1, \dots, x_n) \in \mathbb{R}^n \mid \max\{|x_1|, \dots, |x_n|\} \le 1 \}$

↳ closed unit ball in $\mathbb{R}^n$ wrto the Tchebyshev distance

$n=1$

$$\overline{B}_2^1 = \{x_1 \in \mathbb{R} \mid x_1^2 \leq 1\} = [-1,1]$$
$$\overline{B}_1^1 = \{x_1 \in \mathbb{R} \mid |x_1| \leq 1\} = [-1,1]$$
$$\overline{B}_\infty^1 = \{x_1 \in \mathbb{R} \mid \max |x_1| \leq 1\} = [-1,1]$$

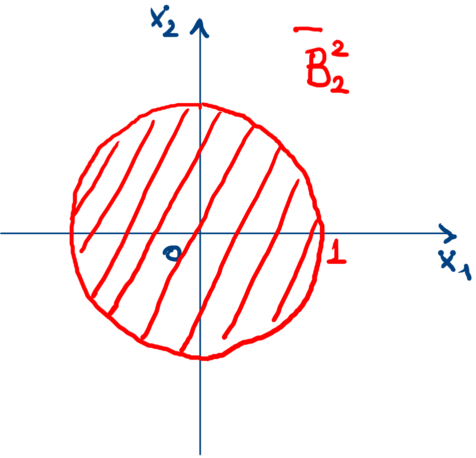
$\overline{\text{I}}$ $x_1 \geq 0$, $x_2 \geq 0$

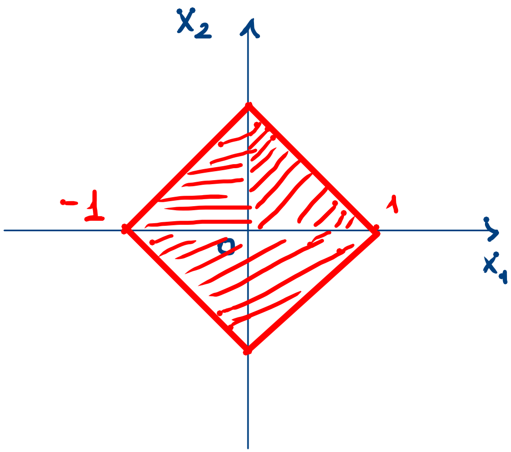
$x_1 + x_2 \leq 1$

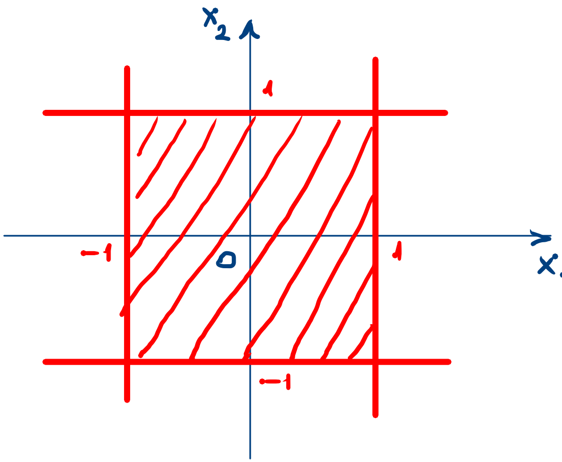
$\dfrac{x_1}{1} + \dfrac{x_2}{1} = 1$

$n=2$

$$\overline{B}_2^2 = \{(x_1,x_2) \mid x_1^2 + x_2^2 \leq 1\}$$
$$\overline{B}_1^2 = \{(x_1,x_2) \mid |x_1| + |x_2| \leq 1\}$$
$$\overline{B}_\infty^2 = \{(x_1,x_2) \mid \max\{|x_1|, |x_2|\} \leq 1\} \Leftrightarrow \begin{cases} |x_1| \leq 1 \\ |x_2| \leq 1 \end{cases} \Leftrightarrow \begin{cases} -1 \leq x_1 \leq 1 \\ -1 \leq x_2 \leq 1 \end{cases}$$







2.2. Definition (neighbourhoods) Let $a \in \mathbb{R}^n$. A set $V \subseteq \mathbb{R}^n$ is called a neighbourhood of a if $\exists r > 0$ s.t. $B(a, r) \subseteq V$.

$\mathcal{V}(a)$ = the family consisting of all neighbourhoods of a

2.3. Definition. Let $A \subseteq \mathbb{R}^n$ and let $a \in \mathbb{R}^n$. One says that a is:

- an interior point of A if $A \in \mathcal{V}(a)$, i.e., $\exists r > 0$ s.t. $B(a, r) \subseteq A$;
- an adherent / closure point of A if $\forall\, V \in \mathcal{V}(a)$: $V \cap A \neq \emptyset$;
- a boundary point of A if $\forall\, V \in \mathcal{V}(a)$: $V \cap A \neq \emptyset$ and $V \cap (\mathbb{R}^n \setminus A) \neq \emptyset$;
- a limit / accumulation point of A if $\forall\, V \in \mathcal{V}(a)$: $V \cap A \setminus \{a\} \neq \emptyset$
- an isolated point of A if $\exists\, V \in \mathcal{V}(a)$ s.t. $V \cap A = \{a\}$.

The set consisting of all interior points of A will be denoted by $\operatorname{int} A$ (the interior of A).

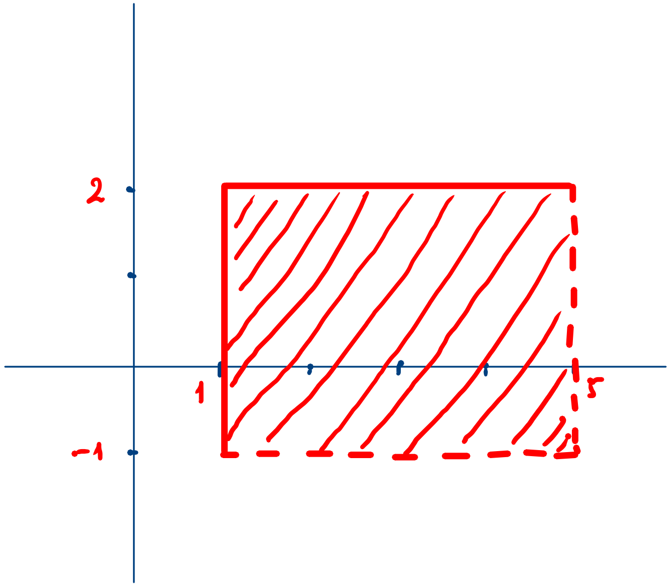
——— " ——— adherent ——— " ——— $\operatorname{cl} A$ (the closure of A).

——— " ——— boundary ——— " ——— $\operatorname{bd} A$ (the boundary of A)

——— " ——— limit ——— " ——— A'

Examples. In $\mathbb{R}^2$ we consider the following sets:

a) $A = [1,5) \times (-1,2] = \{ (x,y) \mid x \in [1,5) \text{ and } y \in (-1,2] \}$



$\operatorname{int} A = (1,5) \times (-1,2)$

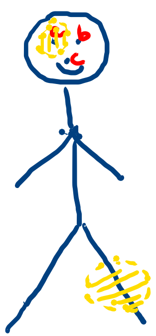
$\operatorname{cl} A = [1,5] \times [-1,2]$

$\operatorname{bd} A = \{1,5\} \times [-1,2] \cup [1,5] \times \{-1,2\}$

$A' = [1,5] \times [-1,2]$

b) $A = \mathbb{Q} \times \mathbb{Q}$ $\qquad \operatorname{int} A = \emptyset \qquad \operatorname{cl} A = A' = \operatorname{bd} A = \mathbb{R}^2$

c) $A =$



$\operatorname{int} A = \emptyset$

$\operatorname{cl} A = A$

$\operatorname{bd} A = A$

$A' = A \setminus \{a, b, c\}$