Training problems for the final exam

1. Let $u, v : \mathbb{C} \to \mathbb{R}$ be given by

$$u(x+iy) = x^3 - 3xy^2, \quad (x,y) \in \mathbb{R}^2,$$

$$v(x+iy) = 3x^2y - y^3, \quad (x,y) \in \mathbb{R}^2,$$

and let $f : \mathbb{C} \to \mathbb{C}$ be given by $f = u + iv$. Prove that f is holomorphic on $\mathbb{C}$ and compute the derivative of f.

2. Let $u, v : \mathbb{C} \to \mathbb{R}$ be given by

$$u(x+iy) = ax^2 + e^{-y}\cos x, \quad (x,y) \in \mathbb{R}^2,$$

$$v(x+iy) = e^{-y}\sin x, \quad (x,y) \in \mathbb{R}^2,$$

where $a \in \mathbb{R}$, and let $f : \mathbb{C} \to \mathbb{C}$ be given by $f = u + iv$.
a) Find $a \in \mathbb{R}$ such that u is harmonic.
b) For the value found for a), prove that f is holomorphic and compute its derivative.

3. a) Find the Möbius transformation $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ such that $f(0) = \infty$, $f(i) = 1$, $f(\infty) = 2$.
b) Let $g : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be given by $g(z) = -\dfrac{1}{z}$, $z \in \mathbb{C}_\infty$, and let

$$D = \{z \in \mathbb{C} : \operatorname{Re} z > 0, \operatorname{Im} z > 0, |z| < 1\}.$$

Represent graphically the domain $g(D)$.

4. a) Find the Möbius transformation $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ such that

$$f(i) = \infty, \quad f(0) = 2i, \quad f(\infty) = 1.$$

b) Let $g : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be given by $g(z) = \dfrac{z+i}{z+1}$, $z \in \mathbb{C}_\infty$, and let

$$D = \{z \in \mathbb{C} : -1 < \operatorname{Re} z < 0, \operatorname{Im} z < 0\}.$$

Represent graphically the domain $g(D)$.

5. Let $\gamma : [0,1] \to \mathbb{C}$, $\gamma(t) = 2e^{2\pi i t}, t \in [0,1]$. Compute the complex integrals:
a) $\displaystyle\int_\gamma \frac{\sin z^2}{z^2} dz$; b) $\displaystyle\int_\gamma \frac{1}{z^2+1} dz$; c) $\displaystyle\int_\gamma (z-1)e^{\frac{1}{z-1}} dz$.

6. Let $\gamma : [0,1] \to \mathbb{C}$, $\gamma(t) = 2e^{it}, t \in [0, 2\pi]$. Compute the integrals:
a) $\displaystyle\int_\gamma \frac{\cos z}{z - \frac{\pi}{2}} dz$; **b)** $\displaystyle\int_\gamma \frac{1}{z^3 - 3z^2} dz$; **c)** $\displaystyle\int_\gamma z \cdot \cos \frac{1}{z-1} dz$.

7. Let $f, g \in \mathcal{H}(\mathbb{C})$ be such that $f(x) \cdot g(x) = 0$, $\forall x \in \mathbb{R}$. Prove that $f \equiv 0$ or $g \equiv 0$.

8. Let $f \in \mathcal{H}(\mathbb{C})$ be such that $|f(z)| \le |z|$, $\forall z \in \mathbb{C}$. Prove that there exits $\alpha \in \mathbb{C}$ such that $f(z) = \alpha z$, $\forall z \in \mathbb{C}$.

9. Find $\displaystyle\max_{z \in \overline{U}(0,1)} \left| z^3 \mathrm{e}^{-z^2 - z} \right|$.