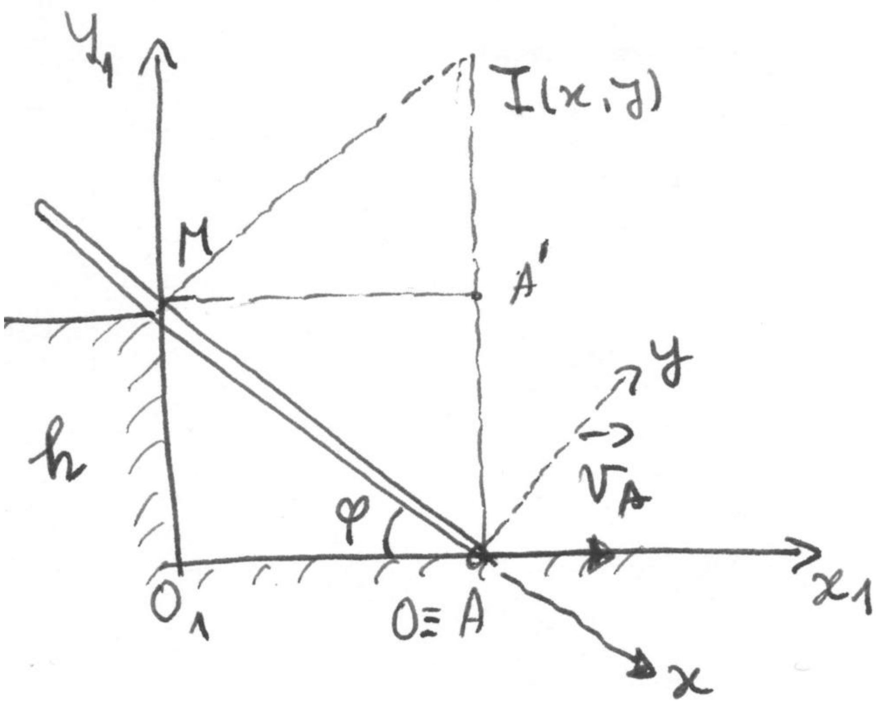
THEORETICAL MECHANICS – KINEMATICS
(midterm exam – 19.04.2019)

II. On a step of height h is leaning continuously a rigid bar AB. The extremity A of the bar is moving on the horizontal axis $O_1x_1$with the velocity v_A (Fig.1). Find the space and body centrode and the instantaneous angular velocity of the bar.



$$\begin{cases} x_{10} = h\,\mathrm{ctg}\,\varphi \\ y_{10} = 0 \end{cases}$$

Space centrode:

$$\begin{cases} x_1 = x_{10} - \dfrac{dy_{10}}{d\varphi} \\ y_1 = y_{10} + \dfrac{dx_{10}}{d\varphi} \end{cases}$$

Theoretical Mechanics

$$\begin{cases} x_1 = h\,\mathrm{ctg}\,\varphi \\ y_1 = -h \cdot \dfrac{1}{\sin^2\varphi} = -h\,\dfrac{\cos^2\varphi + \sin^2\varphi}{\sin^2\varphi} = -h(\mathrm{ctg}^2\varphi + 1) \end{cases}$$

$$\Rightarrow \quad y_1 = -\frac{x_1^2}{h} + h \quad \text{- parabola}$$

Body centrode: $\begin{cases} x = \dfrac{dx_{10}}{d\varphi}\sin\varphi - \dfrac{dy_{10}}{d\varphi}\cos\varphi \\ y = \dfrac{dx_{10}}{d\varphi}\cos\varphi + \dfrac{dy_{10}}{d\varphi}\sin\varphi \end{cases}$

$$\begin{cases} x = \dfrac{-h}{\sin^2\varphi}\cdot\sin\varphi \\ y = \dfrac{-h}{\sin^2\varphi}\cdot\cos\varphi \end{cases} \Rightarrow \begin{cases} \dfrac{x}{1} = -\dfrac{h}{\sin\varphi} \\ y = -h\,\dfrac{\mathrm{ctg}\,\varphi}{\sin\varphi} \end{cases} \Leftrightarrow \begin{cases} x^2 = h^2\left(\dfrac{\cos^2\varphi + \sin^2\varphi}{\sin^2\varphi}\right) \\ y^2 = \dfrac{h^2\,\mathrm{ctg}^2\varphi}{\sin^2\varphi} \end{cases}$$

$$\Leftrightarrow \begin{cases} x^2 = h^2(\mathrm{ctg}^2\varphi + 1) \\ y^2 = h^2 \dfrac{\mathrm{ctg}^2\varphi}{\sin^2\varphi} = h^2 \cdot \dfrac{\mathrm{ctg}^2\varphi}{\frac{h^2}{x^2}} = x^2 \mathrm{ctg}^2\varphi \Rightarrow y^2 = x^2\left(\dfrac{x^2}{h^2} - 1\right) \quad \text{(body centrode)} \\ \sin\varphi = \dfrac{-h}{x} \end{cases}$$

$$\vec{\omega} \perp \vec{IA}$$

$$\vec{v}_A = \vec{\omega} \times \vec{IA} \Rightarrow v_A = |\vec{\omega} \times \vec{IA}| = \omega \cdot IA \Rightarrow \omega = \frac{v_A}{IA} = \frac{v_A \sin^2\varphi}{h}$$

$$MA^2 = AA' \cdot IA \Rightarrow IA = \frac{MA^2}{AA'} = \frac{h}{\sin^2\varphi}$$

$$\frac{\sin\varphi}{1} = \frac{h}{MA} \Rightarrow MA = \frac{h}{\sin\varphi}$$

III. Consider a rigid body. In the mobile frame $Oxyz$ three points of the body $O(0,0,0)$, $A(1,1,0)$ and $B(1,1,1)$ have the velocities $\vec{v}_O(2,1,-3)$, $\vec{v}_A(0,3,-1)$ and $\vec{v}_B(-1,2,-1)$. Find the equations of the instantaneous helical axis, the translation velocity $\vec{v}_{tr}$ and the angular velocity $\vec{\omega}$.

Let be

$\vec{\omega}(p,q,r)$ $\qquad$ $\vec{v}_A = \vec{v}_0 + \vec{\omega} \times \vec{r}_A$

$$(0,3,-1) = (2,1,-3) + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ p & q & r \\ 1 & 1 & 0 \end{vmatrix} = (2,1,-3) + (-r, r, p-q) \Rightarrow \begin{cases} r = 2 \\ p - q = 2 \end{cases}$$

$$\vec{v}_B = \vec{v}_0 + \vec{\omega} \times \vec{r}_B$$

$$(-1,2,-1) = (2,1,-3) + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = (2,1,-3) + (q-r, r-p, p-q)$$

$$\Rightarrow \begin{cases} q - r = -3 \\ r - p = 1 \\ p - q = 2 \end{cases}$$

Thus $p = 1,\ r = 2,\ q = -1 \Rightarrow \vec{\omega}(1, -1, 2) \Rightarrow \omega = \sqrt{6}$

instantaneous helical axis:

$$\frac{v_{0x} + qz - ry}{p} = \frac{v_{0y} + rx - pz}{q} = \frac{v_{0z} + py - qx}{r}$$

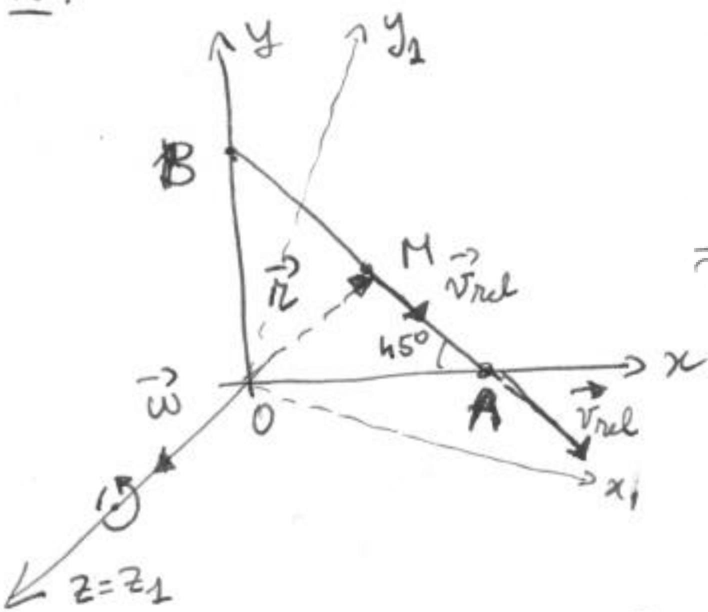
$$\frac{2 - z - 2y}{1} = \frac{1 + 2x - z}{-1} = \frac{-3 + y + z}{2}$$

Translation velocity:

$$v_{tr} = \vec{v}_0 \cdot \frac{\vec{\omega}}{\omega} = (2, 1, -3)\left(\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right) = \frac{-5}{\sqrt{6}}$$

IV. An isosceles right triangle $OAB, m(\hat{O}) = \frac{\pi}{2}$, rotates in his plane about the fixed point O with the angular velocity $\omega = \text{const}$. A material point M moves uniformly along the side $AB = c$ (from B to A) with the speed $v_M = \frac{c\omega}{2\pi}$. Find the absolute velocity and acceleration of M when it reaches the point A .

IV.



$$v_{rel} = \frac{c \cdot \omega}{2\pi}$$

$$\vec{v}_{rel} = \frac{c\omega}{2\pi} \cdot \frac{\sqrt{2}}{2} \cdot \vec{i} - \frac{c\omega}{2\pi} \cdot \frac{\sqrt{2}}{2} \cdot \vec{j}$$

$$\vec{v}_{transp} = \vec{\omega} \times \vec{OA} = \omega \cdot \vec{k} \times \frac{c\sqrt{2}}{2} \vec{i} = c\omega \cdot \frac{\sqrt{2}}{2} \cdot \vec{j}$$

$$\vec{v}_M \Big|_{M=A} = \vec{v}_{transp} + \vec{v}_{rel}$$

$$= \frac{c\omega}{2\pi} \frac{\sqrt{2}}{2} \cdot \vec{i} + c\omega \cdot \frac{\sqrt{2}}{2} \left(1 - \frac{1}{2\pi}\right) \cdot \vec{j}$$

$$\vec{a}_{transp} = \vec{\omega} \times (\vec{\omega} \times \vec{OA}) = -\omega^2 \cdot \vec{OA} = -\omega^2 c \frac{\sqrt{2}}{2} \cdot \vec{i}$$

$$\vec{a}_{rel} = \frac{\partial \vec{v}_{rel}}{\partial t} = 0$$

$$\vec{a}_{Coriolis} = 2\,\vec{\omega} \times \vec{v}_{rel} = \cancel{2}\,\omega \cdot \frac{c\omega}{\cancel{2}\pi}\frac{\sqrt{2}}{2} \cdot \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{vmatrix} = \frac{c\omega^2}{\pi} \cdot \frac{\sqrt{2}}{2}\left(\vec{i} + \vec{j}\right)$$

$$\vec{a}_M = \vec{a}_{transport} + \vec{a}_{rel} + \vec{a}_{coriolis} = \frac{c\omega^2\sqrt{2}}{2}\left(\frac{1}{\pi} - 1\right)\cdot\vec{i} + \frac{c\omega^2}{\pi}\cdot\frac{\sqrt{2}}{2}\cdot\vec{j}$$