Lecture 8

Linear differential equations

(1) $y^{(n)} + a_1(x) \cdot y^{(n-1)} + \ldots + a_n(x) \cdot y = f(x)$ $\quad a_i \in C(I)$, $f \in C(I)$

$f \equiv 0$ homogeneous linear diff. eq.

$f \not\equiv 0$ nonhomogeneous linear diff. eq.

(2) $y^{(n)} + a_1(x) \cdot y^{(n-1)} + \ldots + a_n(x) \cdot y = 0$

S_0 – the sol. set of (2)

S – the sol set of (1)

$L : C^n(I) \to C(I)$ linear operator.

$$y \mapsto Ly = y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n \cdot y$$

(2) $\Leftrightarrow$ $Ly = 0$ $\qquad S_0 = \ker L \qquad \dim S_0 = n$

(1) $\Leftrightarrow$ $Ly = f$. $\qquad S = S_0 + \{y_p\}$

where y_p is a particular sol. of (1).

$\{y_1, \dots, y_n\}$ is a fundamental system of sol. $\Leftrightarrow$

$\Leftrightarrow$ $y_i \in S_0$, $i = \overline{1,n}$, and $\{y_1, \dots, y_n\}$ is a linear indip. system of functions

$\Leftrightarrow$ $y_i \in S_0$, $i = \overline{1,n}$, and $W(x; y_1, \dots, y_n) \neq 0$

$$W(x; y_1, \dots, y_n) = \begin{vmatrix} y_1 & \dots & y_n \\ y_1' & & y_n' \\ \vdots & & \vdots \\ y_1^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix}$$ the wronskian

$$S_0 = \ker L = \{ c_1 y_1 + \dots + c_n y_n \mid c_1, \dots, c_n \in \mathbb{R} \}$$

The general sol. (1)

$$y = y_0 + y_p \qquad \boxed{y_0 = c_1 y_1 + \dots + c_n y_n, \ c_1, \dots, c_n \in \mathbb{R}}$$

y_0 — the general sol. of (2)

y_p - is a particular sol of (1), which can be found using the variation of the constants method.

Variation of the constants method

$\{y_1, \ldots, y_n\}$ a fundam. system of sol. for (2).

$$\boxed{y_0 = c_1 y_1 + \ldots + c_n y_n, \quad c_1, \ldots, c_n \in \mathbb{R}}$$

$y_p = ?$ sol of (1).

we look for y_p of the form

$$y_p(x) = c_1(x) \cdot y_1 + \ldots + c_n(x) \cdot y_n$$

$$y_p'(x) = \underline{c_1' \cdot y_1} + c_1 \cdot y_1' + \underline{c_2' \cdot y_2} + c_2 \cdot y_2' + \ldots + \underline{c_n' \cdot y_n} + c_n \cdot y_n'$$

we impose the condition

$$\boxed{c_1' y_1 + c_2' \cdot y_2 + \ldots + c_n' \cdot y_n = 0}$$

$$\Rightarrow y_p'(x) = c_1 \cdot y_1' + c_2 \cdot y_2' + \ldots + c_n \cdot y_n'$$

$$y_p''(x) = \underline{c_1' \cdot y_1'} + c_1 \cdot y_1'' + \underline{c_2' \cdot y_2'} + c_2 \cdot y_2'' + \ldots + \underline{c_n' \cdot y_n'} + c_n \cdot y_n''$$

we impose the condition

$$\boxed{c_1' y_1' + c_2' y_2' + \ldots + c_n' y_n' = 0}$$

$$\Rightarrow y_p''(x) = c_1 \cdot y_1'' + c_2 \cdot y_2'' + \dots + c_n \cdot y_n''$$

$\vdots$

On each step we impose the condition

$$\boxed{c_1' y_1^{(k)} + \dots + c_n' \cdot y_n^{(k)} = 0, \quad k = 0, 1, \dots, n-2}$$

$$\Rightarrow \boxed{y_p^{(k+1)}(x) = c_1 \cdot y_1^{(k+1)} + \dots + c_n \cdot y_n^{(k+1)}, \quad k = 0, 1, \dots, n-2.}$$

$$k = n-2 \Rightarrow y_p^{(n-1)}(x) = c_1 \cdot y_1^{(n-1)} + \dots + c_n \cdot y_n^{(n-1)}$$

$$\Rightarrow y_p^{(n)}(x) = c_1' \cdot y_1^{(n-1)} + c_1 \cdot y_1^{(n)} + \dots + c_n' \cdot y_n^{(n-1)} + c_n \cdot y_n^{(n)}$$

replace y_p in (1) (y_p is a sol. of (1))

$$y_p^{(n)} + a_1 \cdot y_p^{(n-1)} + \dots + a_n \cdot y_p = f(x)$$

$$\underline{c_1' \cdot y_1^{(n-1)}} + c_1 \cdot y_1^{(n)} + \dots + \underline{c_n' \cdot y_n^{(n-1)}} + c_n \cdot y_n^{(n)} +$$

$$+ a_1 \left(c_1 \cdot y_1^{(n-1)} + \dots + c_n \cdot y_n^{(n-1)} \right) +$$

$$+ a_2 \cdot \left(c_1 \cdot y_1^{(n-2)} + \dots + c_n \cdot y_n^{(n-2)} \right) + \dots +$$

$$+ a_{n-1} \left(c_1 \cdot y_1' + \dots + c_n \, y_n' \right) +$$

$$+ a_n \cdot \left(c_1 \cdot y_1 + \dots + c_n \cdot y_n \right) = f(x)$$

$$\Rightarrow c_1' \cdot y_1^{(n-1)} + \dots + c_n' \cdot y_n^{(n-1)} +$$

$$+ c_1 \cdot \Big(\underbrace{y_1^{(n)} + a_1 \cdot y_1^{(n-1)} + \dots + a_{n-1} \cdot y_1' + a_n y_1}_{Ly_1 = 0} \Big) + \dots +$$

$$+ c_n \cdot \Big(\underbrace{y_n^{(n)} + a_1 y_n^{(n-1)} + \dots + a_{n-1} \cdot y_n' + a_n \cdot y_n}_{Ly_n = 0} \Big) = f(x)$$

$$\Rightarrow \boxed{c_1' \, y_1^{(n-1)} + \dots + c_n' \cdot y_n^{(n-1)} = f(x)}$$

$\Rightarrow$ we get the system:

$$(3)\quad \begin{cases} c_1'\cdot y_1 + \dots + c_n'\cdot y_n = 0 \\ c_1'\cdot y_1' + \dots + c_n' y_n' = 0 \\ \dots\dots\dots \\ c_1' y_1^{(n-2)} + \dots + c_n'\cdot y_n^{(n-2)} = 0 \\ c_1'\cdot y_1^{(n-1)} + \dots + c_n' y_n^{(n-1)} = f \end{cases}$$

$c_1', c_2', \dots, c_n'$ unknowns

coefficient matrix of (3)

$$A = \begin{pmatrix} y_1 & \dots & y_n \\ y_1' & & y_n' \\ \vdots & & \vdots \\ y_1^{(n-1)} & \dots & y_n^{(n-1)} \end{pmatrix}$$

$\left.\begin{array}{l} \det A = W(x; y_1, \dots, y_n) \\ \{y_1, \dots, y_n\} \text{ is fundam system of sol.} \end{array}\right\} \Rightarrow \det A \neq 0, \forall x \in I$

$\Rightarrow$ syst (3) has an unique

$\Rightarrow c_1'(x), \dots, c_n'(x) \xRightarrow{\int} c_1(x), \dots, c_n(x) \Rightarrow y_p(x).$

Linear differential equations with constant coefficients

$a_i(x) \equiv a_i \ , \ a_i \in \mathbb{R} \ , \ i = \overline{1,n}$

(4) $\quad y^{(n)} + a_1 \cdot y^{(n-1)} + \dots + a_{n-1} \cdot y' + a_n \cdot y = f \qquad \underline{a_1, \dots, a_n \in \mathbb{R}}$

(5) $\quad y^{(n)} + a_1 \cdot y^{(n-1)} + \dots + a_{n-1} \cdot y' + a_n \cdot y = 0$

(5) homogeneous eq.

The general sol. of (4)

$y = y_0 + y_p \qquad y_0$ - the gen. sol. of (5)

y_p - a partic. sol. of (4).

$$\boxed{y_0 = ?}$$

The homogeneous case

(5) $\quad y^{(n)} + a_1 \cdot y^{(n-1)} + \ldots + a_{n-1} \cdot y' + a_n \cdot y = 0$, $\quad \underline{a_1, \ldots, a_n \in \mathbb{R}}$

we try to find sol. of the form:

$$\left.\begin{aligned} y(x) &= e^{rx} \\ y'(x) &= r \cdot e^{rx} \\ y''(x) &= r^2 \cdot e^{rx} \\ &\vdots \\ y^{(n)}(x) &= r^n \cdot e^{rx} \end{aligned}\right\} \overset{(5)}{\Longrightarrow} r^n \cdot e^{rx} + a_1 \cdot r^{n-1} \cdot e^{rx} + \ldots + a_{n-1} \cdot r e^{rx} + a_n \cdot e^{rx} = 0 \;/ : e^{rx}$$

$$\Rightarrow (6) \quad \boxed{r^n + a_1 \cdot r^{n-1} + \ldots + a_{n-1} \cdot r + a_n = 0}$$ the characteristic equation

$P(r) = r^n + a_1 r^{n-1} + \ldots + a_{n-1} \cdot r + a_n$ the characteristic polynomial

1. The case when (6) has n simple real roots

$$\underbrace{r_1, \dots, r_n \in \mathbb{R} \quad (r_i \neq r_j,\ i \neq j)}$$

$$\Rightarrow \boxed{y_1(x) = e^{r_1 x}, \dots, y_n(x) = e^{r_n x}} \text{ are sol. of (5)}$$

$$W(x; y_1, \dots, y_n) = \begin{vmatrix} y_1 & \dots & y_n \\ y_1' & & y_n' \\ \vdots & & \vdots \\ y_n^{(n-1)} & & y_n^{(n-1)} \end{vmatrix} =$$

$$= \begin{vmatrix} e^{r_1 x} & \dots & e^{r_n x} \\ r_1 e^{r_1 x} & & r_n \cdot e^{r_n x} \\ \vdots & & \vdots \\ r_1^{n-1} \cdot e^{r_1 x} & & r_n^{n-1} \cdot e^{r_n x} \end{vmatrix} = e^{r_1 x} \cdot e^{r_2 x} \dots e^{r_n x} \cdot \begin{vmatrix} 1 & \dots & 1 \\ r_1 & & r_n \\ r_1^2 & & r_n^2 \\ \vdots & & \vdots \\ r_1^{n-1} & \dots & r_n^{n-1} \end{vmatrix}$$

$$= \underbrace{e^{x(r_1 + \dots + r_n)}}_{>0} \cdot \underbrace{\prod_{1 \le i < j \le n} (r_i - r_j)}_{\neq 0} \neq 0$$

$$\Rightarrow \{y_1, \dots, y_n\} \text{ fundam. syst. of sol.}$$

2. Case of multiple roots

Proposition: If $r \in \mathbb{R}$ is a sol. of (6) with the multiplicity $\mu > 1$. Then:

$$y_1(x) = e^{rx}$$
$$y_2(x) = x.e^{rx}$$
$$\vdots$$
$$y_\mu(x) = x^{\mu-1}.e^{rx}$$

are sol. of (5).

Proof. y_1 is a sol. ✓

let's prove $y_2(x) = x.e^{rx}$ is a sol. of (5).

r is a sol. of (6) with multiplicity $\mu \Leftrightarrow$

$$\Leftrightarrow \begin{cases} P(r) = 0 \\ P'(r) = 0 \\ \vdots \\ P^{(\mu-1)}(r) = 0 \\ P^{(\mu)}(r) \neq 0 \end{cases}$$

$P(r) = r^n + a_1 r^{n-1} + \dots + a_{n-1} r + a_n$

Charact. polynomial.

$$y_2(x) = x \cdot e^{rx}$$

$$y_2'(x) = e^{rx} + x \cdot r e^{rx}$$

$$y_2''(x) = r \cdot e^{rx} + r e^{rx} + x \cdot r^2 e^{rx} = 2r e^{rx} + x \cdot r^2 \cdot e^{rx}$$

$$\vdots$$

$$y_2^{(k)}(x) = k \cdot r^{k-1} \cdot e^{rx} + x \cdot r^k \cdot e^{rx} \quad , \quad k = 1, \dots, n$$

$$y_2^{(n)} + a_1 \, y_2^{(n-1)} + \dots + a_{n-1} \cdot y_2' + a_n y_2 =$$

$$= n \cdot r^{n-1} \cdot e^{rx} + x \cdot r^n \cdot e^{rx} +$$

$$+ a_1 \left((n-1) \cdot r^{n-2} \cdot e^{rx} + x \cdot r^{n-1} \cdot e^{rx} \right) + \dots +$$

$$+ a_{n-1} \left(e^{rx} + x \cdot r \cdot e^{rx} \right) +$$

$$+ a_n \cdot x \cdot e^{rx} =$$

$$= e^{rx} \left(n \cdot r^{n-1} + a_1 (n-1) r^{n-2} + \dots + a_{n-1} \right) +$$

$$+ x \cdot e^{rx} \left(r^n + a_1 r^{n-1} + \dots + a_{n-1} \cdot r + a_n \right) =$$

$$= e^{rx} \cdot P'(r) + x \cdot e^{rx} \cdot P(r) = 0 \, .$$

3. Case of complex roots

Proposition

If $r = \alpha + i\beta \in \mathbb{C}$ is a complex root of (6) then:

$y_1(x) = e^{\alpha x} \cdot \cos \beta x$, $y_2(x) = e^{\alpha x} \sin \beta x$

are sol. of (5).

Lemma: $y = u(x) + i \cdot v(x)$, $y : I \to \mathbb{C}$ is a sol of (5) $\Leftrightarrow$

$\Leftrightarrow u(x), v(x) : I \to \mathbb{R}$ are sol of (5).

$r = \alpha + i\beta$ is a sol of (6) $\Rightarrow y(x) = e^{rx}$ is a sol of (5)

$y(x) = e^{rx} = e^{(\alpha + i\beta)x} = e^{\alpha x + i\beta x} = e^{\alpha x} \cdot e^{i\beta x} =$

$$\boxed{e^{it} = \cos t + i \sin t}$$

$$= \underbrace{e^{\alpha x} \cos \beta x}_{y_1} + i \underbrace{e^{\alpha x} \cdot \sin \beta x}_{y_2}$$

The algorithm of solving linear homog. diff. eq. with const. coeff

1. write the charact. eq.:

$$r^n + a_1 r^{n-1} + \dots + a_{n-1} \cdot r + a_n = 0$$

2. solve the charact. eq.

$r_1, \dots, r_s$

- if $r \in \mathbb{R}$ is a root with multiplicity μ then:

$$y_1(x) = e^{rx},\ y_2(x) = x e^{rx},\ \dots,\ y_\mu(x) = x^{\mu-1} e^{rx}$$

- if $r = \alpha \pm i\beta \in \mathbb{C}$ with multiplicity μ then:

$$y_1(x) = e^{\alpha x} \cos \beta x \qquad y_2(x) = e^{\alpha x} \sin \beta x$$

$$\vdots$$

$$y_{2\mu-1}(x) = x^{\mu-1} e^{\alpha x} \cos \beta x,\ y_{2\mu}(x) = x^{\mu-1} \cdot e^{\alpha x} \sin \beta x$$

$\Rightarrow$ $y_1(x), \dots, y_n(x)$ the fundam. system of sol.