SEMINARS 9+10

1) Which of the following subsets is a subspace in the space mentioned nearby:

a) $A = \{(x,y) \in \mathbb{R}^2 \mid ax+by=0\},\ (a,b \in \mathbb{R}$ are given) in ${}_{\mathbb{R}}\mathbb{R}^2$;

b) $D = [-1,1]$ in ${}_{\mathbb{R}}\mathbb{R}$;

b') $D' = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$ in ${}_{\mathbb{R}}\mathbb{R}^2$;

b") $D'' = \{(x_1,\dots,x_n) \in \mathbb{R}^n \mid x_1^2+\dots+x_n^2 \leq 1\}$ in ${}_{\mathbb{R}}\mathbb{R}^n$;

c) $P_n(\mathbb{R}) = \{f \in \mathbb{R}[X] \mid \mathrm{grad} f \leq n\}$ in ${}_{\mathbb{R}}\mathbb{R}[X]$ ($n \in \mathbb{N}$ is given);

d) $B = \{f \in \mathbb{R}[X] \mid \mathrm{grad} f = n\}$ in ${}_{\mathbb{R}}\mathbb{R}[X]$ ($n \in \mathbb{N}$ is given)?

2) Let V be a K-vector space, $A \leq_K V$ and $C_V A = V \setminus A$.

i) Is $C_V A$ a subspace in ${}_K V$?

ii) What about $C_V A \cup \{0\}$?

3) Let V be a K-vector space, $S \leq_K V$ and $x,y \in V$. We denote $\langle S,x\rangle = \langle S \cup \{x\}\rangle$. Show that if $x \in V \setminus S$ and $x \in \langle S,y\rangle$ then $y \in \langle S,x\rangle$.

4) Let V be a K-vector space and $\alpha,\beta,\gamma \in K,\ x,y,z \in V$ such that $\alpha\gamma \neq 0$ and $\alpha x+\beta y+\gamma z=0$. Show that $\langle x,y\rangle = \langle y,z\rangle$.

5) Is the real vector space $\mathbb{R}_3[X] = \{f \in \mathbb{R}[X] \mid \deg f \leq 3\}$ generated by the set

$$\{f_1 = 3X+2,\ f_2 = 4X^2-X+1,\ f_3 = X^3-X^2+3\}\,?$$

Why?

6) Let V,V' be K-vector spaces, $f : V \to V'$ a linear map, $A \leq_K V$ and $A' \leq_K V'$. Show that:

a) $f(A) = \{f(a) \in V' \mid a \in A\} \leq_K V'$;

b) $\overset{-1}{f}(A') = \{x \in V \mid f(x) \in A'\} \leq_K V$.

7) In the $\mathbb{R}$-vector space $\mathbb{R}^{\mathbb{R}} = \{f \mid f : \mathbb{R} \to \mathbb{R}\}$ we consider

$$\mathbb{R}^{\mathbb{R}}_o = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is odd}\},\ \mathbb{R}^{\mathbb{R}}_e = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is even}\}.$$

Show that $\mathbb{R}^{\mathbb{R}}_o$ şi $\mathbb{R}^{\mathbb{R}}_e$ are subspaces of $\mathbb{R}^{\mathbb{R}}$ and $\mathbb{R}^{\mathbb{R}} = \mathbb{R}^{\mathbb{R}}_o \oplus \mathbb{R}^{\mathbb{R}}_e$.

8) Show that the property of being a direct summand is transitive.

9) Let us consider:
a) $f_1 : \mathbb{R}^2 \to \mathbb{R}^2$, $f_1(x,y) = (-x,y)$ (the symmetry with respect to Oy);
b) $f_2 : \mathbb{R}^2 \to \mathbb{R}^2, f_2(x,y) = (x,-y)$ (the symmetry with respect to Ox);
c) $f_3 : \mathbb{R}^2 \to \mathbb{R}^2, f_3(x,y) = (x\cos\varphi - y\sin\varphi, x\sin\varphi + y\cos\varphi)$, $\varphi \in \mathbb{R}$, (the plane rotation of angle φ);
d) $f_4 : \mathbb{R}^2 \to \mathbb{R}^3, f_4(x,y) = (x+y, 2x-y, 3x+2y)$.
Show that $f_1,\ f_2,\ f_3,\ f_4$ are $\mathbb{R}$-linear maps. Are they isomorphisms? Are they endomorphisms? Are they automorphisms?

10) Can you find an $\mathbb{R}$-linear map $f : \mathbb{R}^3 \to \mathbb{R}^2$ such that

$$f(1,0,3) = (1,1) \text{ şi } f(-2,0,-6) = (2,1)\,?$$