# Lecture 5

## Mathematical models given by first order differential equations

### 1) Radioactive decay

The Rutherford Law: The desintegration rate of a radioactive substance is proportional with the quantity of the substance present at that time

$R(t)$ – the quantity of the radioactive substance at the moment $t>0$.

$R_0$ – the initial quantity of the radioactive substance at initial moment $t_0=0$

$\boxed{R(0)=R_0}$ the initial condition $(R_0>0)$

$t \longrightarrow t+\Delta t$

$R(t) \qquad R(t+\Delta t)$

the change rate $\qquad \dfrac{R(t+\Delta t)-R(t)}{\Delta t} \xrightarrow[\Delta t \to 0]{} R'(t)$

$R'(t)$ – the change rate of the subst. quantity

$R' \sim R$ $\quad \begin{cases} R'(t) = -k \cdot R(t) & , k > 0 \\ R(0) = R_0 \end{cases}$

$R(t)$ is a decreasing function $\Rightarrow R'(t) < 0$

$k$ - the desintegration constant

$R' = -k \cdot R$ a homogeneous linear first order diff. eq.

$$\frac{dR}{dt} = -k \cdot R \Rightarrow \frac{dR}{R} = -k \cdot dt \Rightarrow \int \frac{dR}{R} = \int -k \cdot dt \Rightarrow$$

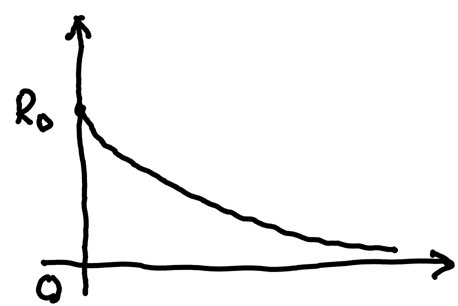
$$\Rightarrow \ln R = -kt + \ln c \Rightarrow R = c \cdot e^{-kt}$$

$$\Rightarrow \boxed{R(t) = c \cdot e^{-kt}, c \in \mathbb{R}}$$ the gen. sol.

$R(0) = R_0 \Rightarrow c = R_0$

$\Rightarrow$ the model solution:

$$\boxed{R(t) = R_0 \cdot e^{-kt}}$$



$\lim_{t \to +\infty} R(t) = 0 \Rightarrow$ in time the substance will disapear.

$T_{1/2}$ – the half-life time of a radioactive substance is the length of the time that it takes to decay to the half of its original size.

$t_0 = 0 \rightarrow R_0$

$t = T_{1/2} \rightarrow \frac{R_0}{2} \Rightarrow R(T_{1/2}) = \frac{R_0}{2}$

$R_0 \cdot e^{-k T_{1/2}} = \frac{R_0}{2} \quad ) : R_0$

$\Rightarrow e^{-k \cdot T_{1/2}} = \frac{1}{2} \Rightarrow -k \cdot T_{1/2} = \underset{\underset{\ln 2^{-1}}{\Uparrow}}{\ln \frac{1}{2}} \Rightarrow -k \cdot T_{1/2} = -\ln 2$

$\Rightarrow \boxed{k \cdot T_{1/2} = \ln 2}$

$\Downarrow$ $\Downarrow$

$\boxed{k = \frac{\ln 2}{T_{1/2}}}$ $\boxed{T_{1/2} = \frac{\ln 2}{k}}$

$T_{1/2}$ known

$$R(t) = R_0 \cdot e^{-k \cdot t} = R_0 \cdot e^{-\frac{\ln 2}{T_{1/2}} \cdot t} = R_0 \cdot \left(e^{\ln 2}\right)^{-\frac{t}{T_{1/2}}}$$

$$\boxed{R(t) = R_0 \cdot 2^{-\frac{t}{T_{1/2}}}}$$

2) Radiocarbon Dating ( Willard Libbi 1950 , in 1960 Nobel Prize)

- the method finds an approximating age of some some fossilized matter.
- the theory of the radiocarbon dating is based on the fact that radioisotop $C^{14}$ is produced in the atmosphere by the action of cosmic radiation

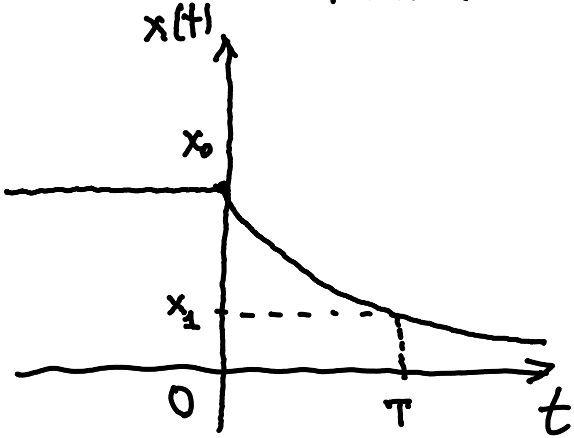
$C^{14}$ is a radioactive substance with half-life.

$$\boxed{T_{1/2} \simeq 5730 \text{ years}} \Rightarrow \boxed{k = \frac{\ln 2}{5730} \text{ years}^{-1}}$$

- the ratio of the $C^{14}$ to the stable isotope $C^{12}$ in the atmosphere appears to be constant and as a consequence the amount of the ratio $C^{14}/C^{12}$ present in the living body is also constant. (the same as in the atmosphere)
- when the living body dies, the absorbtion of the $C^{14}$ ceases and the existing amount of $C^{14}$ decays.

$x(t)$ – the amount of $C^{14}/C^{12}$ at the moment $t>0$

$x_0$ – the initial amount of $C^{14}/C^{12}$ at the initial moment $t=0$.




$t=0$ is the moment when the body dies

$$\begin{cases} x' = -k\cdot x \\ x(0) = x_0 \end{cases}$$

$$x(t) = x_0 \cdot e^{-kt}$$

at the moment $T>0$ it is measured the amount of $C^{14}/C^{12}$ from the remains $\Rightarrow x_1$

$\boxed{x(T)=x_1}$ $\quad T=?$

$x(t)=x_0 e^{-kt}$ $\qquad x(T)=x_1 \Rightarrow x_0 e^{-kT}=x_1$

$\Rightarrow e^{-kT}=\dfrac{x_1}{x_0} \Rightarrow -kT=\ln\dfrac{x_1}{x_0} \Rightarrow T=-\dfrac{1}{k}\cdot\ln\dfrac{x_1}{x_0}$

$$\Rightarrow \boxed{T=\frac{1}{k}\ln\frac{x_0}{x_1}}$$

## 3) The Thermal Cooling

The Cooling Newton's Law:

The rate of change of the surface temperature of an object is proportional to the difference between the object temperature and the temperature of its surrounding (called the ambient temp.) at the same time.

$T(t)$ – the object temperature at the moment $t > 0$.

$T_0$ – the initial object temp. at the initial moment $t = 0$.

$\Rightarrow \boxed{T(0) = T_0}$ the initial cond.

$T_A$ – the ambiental temp. (const. value)

$T'(t) \sim T(t) - T_A$

$$\begin{cases} T'(t) = -k \cdot (T(t) - T_A), & k > 0 \\ T(0) = T_0 \end{cases}$$

$k$ – the cooling constant

Why $k > 0$

if $T(t) < T_A \Rightarrow T(t)$ increase $\Rightarrow T'(t) > 0$

$$\underbrace{T'}_{>0} = \underbrace{-k}_{<0} \cdot (\underbrace{T - T_A}_{<0})$$

$$\Downarrow$$

$$k > 0$$

if $T(t) > T_A \Rightarrow T(t)$ is decreasing $\Rightarrow T' < 0$

$$\underbrace{T'}_{<0} = \underbrace{-k}_{<0} \underbrace{(T - T_A)}_{>0}$$

$$\Downarrow$$

$$k > 0$$

$$T' = -k(T - T_A)$$

separable diff. eq

$T' + kT = k \cdot T_A$
nonhomogeneous linear first order diff. eq.

$$\frac{dT}{dt} = -k \cdot (T - T_A) \Rightarrow \int \frac{dT}{T - T_A} = \int -k \cdot dt =$$

$$\Rightarrow \ln(T - T_A) = -kt + \ln c \Rightarrow$$

$$\Rightarrow \quad T - T_A = c \cdot e^{-kt}$$

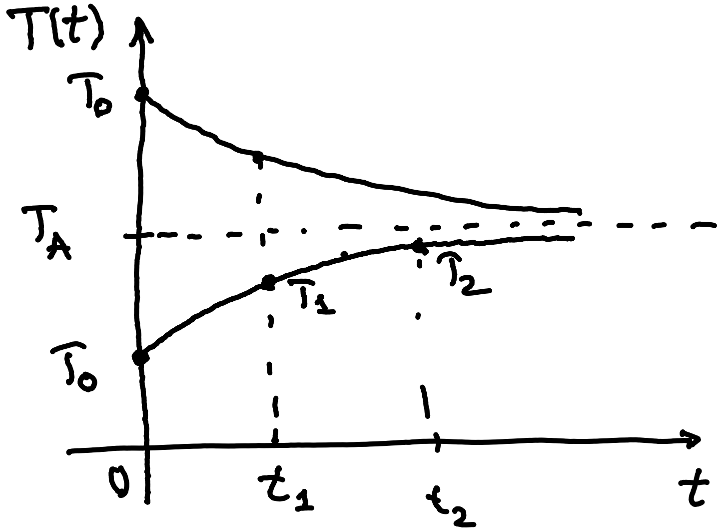
$$\Rightarrow \boxed{T(t) = T_A + c \cdot e^{-kt}, \; c \in \mathbb{R}}$$ the gen. sol.

$T(0) = T_0 \Rightarrow T_A + c = T_0 \Rightarrow c = T_0 - T_A$

$\Rightarrow$ the model solution:

$$\boxed{T(t) = (T_0 - T_A)e^{-kt} + T_A}$$

$$T(t) \xrightarrow[t \to +\infty]{} T_A$$




when $T_0 = T_A \Rightarrow$

$\Rightarrow T(t) \equiv T_A$

$\underline{k = ?}$.

$t = 0 \quad T_0$ $\qquad T(0) = T_0$

$t_1 > 0 \rightarrow T_1 \qquad \boxed{T(t_1) = T_1.} \qquad \underline{\underline{k = ?}}$

$$\Downarrow$$

$$(T_0 - T_A)e^{-kt_1} + T_A = T_1$$

$$(T_0 - T_A)e^{-kt_1} = T_1 - T_A$$

$$\Rightarrow e^{-kt_1} = \frac{T_1 - T_A}{T_0 - T_A} \Rightarrow -kt_1 = \ln \frac{T_1 - T_A}{T_0 - T_A} \Rightarrow$$

$$\Rightarrow k = -\frac{1}{t_1} \ln \frac{T_1 - T_A}{T_0 - T_A} \rightarrow \boxed{k = \frac{1}{t_1} \ln \frac{T_0 - T_A}{T_1 - T_A}}$$