Seminar 12

1. Let $n \in \mathbb{N}$, $n \geq 2$. Prove the ring isomorphism

$$\mathbb{Z}[X]/(n) \cong \mathbb{Z}_n[X]$$

by using the first isomorphism theorem.

2. Prove the ring isomorphism

$$\mathbb{Q}[X]/(X+1) \cong \mathbb{Q}$$

by using the first isomorphism theorem.

3. Prove the ring isomorphism

$$\mathbb{R}[X]/(X^2+1) \cong \mathbb{C}$$

by using the first isomorphism theorem.

4. Let

$$R = \left\{ \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \middle| a, b \in \mathbb{Q} \right\}, \quad I = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \middle| a \in \mathbb{Q} \right\}.$$

Show that R is a subring of the ring $\mathbb{M}_2(\mathbb{Q})$, I is an ideal of R and $R/I \cong \mathbb{Q}$.

5. Determine the factor rings of the ring $\mathbb{Z}_{12}$ by using the third isomorphism theorem.

6. Determine the characteristic of the ring $\mathbb{Z}_4 \times \mathbb{Z}_6$. Generalization for the ring $\mathbb{Z}_m \times \mathbb{Z}_n$ $(m, n \in \mathbb{N},\ m, n \geq 2)$.

7. Give examples of:
(i) Infinite ring having finite characteristic.
(ii) Commutative ring with identity which is not a field but has a prime characteristic.

8. Let R be a unitary commutative ring with $1 \neq 0$ and $\mathrm{char}(R) = p$ for some prime p. Prove that:

$$(a+b)^p = a^p + b^p, \quad \forall a, b \in R.$$