Seminar 3

13.03.2025

① Determine the following limits:

a) $\lim\limits_{(x,y)\to(0,2)} \dfrac{\sin(xy)}{x} = 2$

b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{e^{xy}-1}{e^x-1} = \lim\limits_{(x,y)\to(0,0)} \underbrace{\dfrac{e^{xy}-1}{xy}}_{\to 1} \cdot \underbrace{\dfrac{x}{e^x-1}}_{\to 1} \cdot \underbrace{y}_{\to 0} = 0$

c) $\lim\limits_{(x,y)\to(\frac{\pi}{4},\frac{\pi}{4})} \dfrac{x-y}{\sin x - \sin y} = \lim\limits_{(x,y)\to(\frac{\pi}{4},\frac{\pi}{4})} \dfrac{x-y}{2\cos\frac{x+y}{2}\cdot\sin\frac{x-y}{2}}$

$= \lim\limits_{(x,y)\to(\frac{\pi}{4},\frac{\pi}{4})} \underbrace{\dfrac{\frac{x-y}{2}}{\sin\frac{x-y}{2}}}_{\to 1} \cdot \underbrace{\dfrac{1}{\cos\frac{x+y}{2}}}_{\to 0} = \dfrac{2}{\sqrt{2}} = \dfrac{2\sqrt{2}}{2} = \sqrt{2}$

d) $\lim\limits_{(x,y)\to(a,a)} \dfrac{\frac{x+y}{2} - \frac{2xy}{x+y}}{\frac{x+y}{2} - \sqrt{xy}}$, $a>0$

arithm. mean ← $\frac{x+y}{2}$; $\frac{2xy}{x+y}$ → harmonic mean; $\sqrt{xy}$ → geom. mean; $\left(\frac{0}{0}\right)$

$= \lim\limits_{(x,y)\to(a,a)} \dfrac{(x+y)^2-4xy}{\cancel{2}(x+y)} \cdot \dfrac{\cancel{2}}{x+y-2\sqrt{xy}}$

$= \lim\limits_{(x,y)\to(a,a)} \dfrac{(x+y)^2-4xy}{(x+y)\cdot(x+y-2\sqrt{xy})}$

"Separate the good boys from the bad boys"

$= \dfrac{1}{x+y} \cdot \lim\limits_{(x,y)\to(a,a)} \dfrac{(x-y)^2}{(x+y)(x+y-2\sqrt{xy})}$

$$= \frac{1}{x+y} \cdot \lim_{(x,y)\to(a,a)} \frac{(x-y)^2}{(\sqrt{x}-\sqrt{y})^2} = \frac{1}{2a} \lim_{(x,y)\to(a,a)} \left(\frac{x-y}{\sqrt{x}-\sqrt{y}}\right)^2$$

$$= \frac{1}{2a} \cdot \lim_{(x,y)\to(a,a)} \left(\frac{(\sqrt{x})^2-(\sqrt{y})^2}{\sqrt{x}-\sqrt{y}}\right)^2 = \frac{1}{2a} \lim \left(\frac{\cancel{(\sqrt{x}-\sqrt{y})}(\sqrt{x}+\sqrt{y})}{\cancel{\sqrt{x}-\sqrt{y}}}\right)^2$$

$$= \frac{1}{2a} \lim (\sqrt{x}+\sqrt{y})^2 = \frac{1}{2a} \cdot (2\sqrt{a})^2 = \frac{1}{2a} \cdot 4a = \underline{\underline{2}}$$

e) $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} \to$ **does not exist**

$$\frac{\frac{1}{k^2}-0}{\frac{1}{k^2}+0} = 1$$

Let $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$

$$\left.\begin{aligned}
&\lim_{k\to\infty} \left(\tfrac{1}{k}, \tfrac{1}{k}\right) = (0,0) \\
&\lim_{k\to\infty} f\left(\tfrac{1}{k}, \tfrac{1}{k}\right) = \lim_{k\to\infty} 0 = 0 \\
&\lim_{k\to\infty} \left(\tfrac{1}{k}, 0\right) = (0,0) \\
&\lim_{k\to\infty} f\left(\tfrac{1}{k}, 0\right) = 1
\end{aligned}\right\} \Rightarrow \nexists \lim_{(x,y)\to(0,0)} f(x,y)$$

f) $\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = 0$

$$0 \le \left| \frac{x^3-y^3}{x^2+y^2} - 0 \right| = \frac{|x^3-y^3|}{x^2+y^2} \le \frac{|x^3|+|y^3|}{x^2+y^2} = \underbrace{\frac{x^2}{x^2+y^2}}_{\le 1} \cdot |x| + \underbrace{\frac{y^2}{x^2+y^2}}_{\le 1} \cdot |y|$$

$\leq |x| + |y|$

$$0 \leq \left| \frac{x^3 - y^3}{x^2 + y^2} - 0 \right| \leq |x| + |y|$$

$$\searrow \quad \downarrow \quad \swarrow$$

$$0$$

g) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3 - y^3}{xy}$

Let $f(x,y) = \dfrac{x^3 - y^3}{xy}$

$$\frac{\frac{1}{k^3} - \frac{1}{k^6}}{\frac{1}{k^3}} = 1 - \frac{1}{k^6} = 1$$

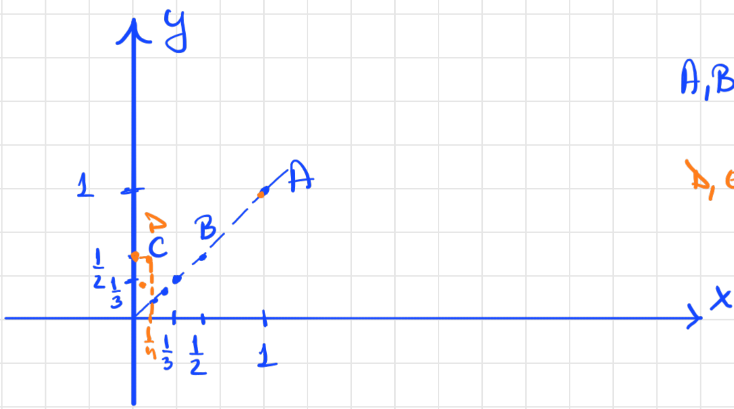
$\lim\limits_{k\to\infty} \left(\frac{1}{k}, \frac{1}{k}\right) = (0,0)$

$\lim\limits_{k\to\infty} f\left(\frac{1}{k}, \frac{1}{k}\right) = 0$

$\lim\limits_{k\to\infty} \left(\frac{1}{k}, \frac{1}{k^2}\right) = (0,0)$

$\lim\limits_{k\to\infty} f\left(\frac{1}{k}, \frac{1}{k^2}\right) = 1$

$\Rightarrow \nexists \lim\limits_{(x,y)\to(0,0)} f(x,y)$



A, B, C ∈ the first bisector

D, ∈ ∈ $f(x) = x^2$

② Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a continuous function and let B be an open subset of $\mathbb{R}^m$. Prove that the set $f^{-1}(B) = \{x \in \mathbb{R}^n \mid f(x) \in B\}$ is open.

Solution: ? $f^{-1}(B)$ is open $\Leftrightarrow \forall a \in f^{-1}(B), \exists \delta > 0$ s.t. $B(a, \delta) \subseteq f^{-1}(B)$

Let $a \in f^{-1}(B)$ arbitrary fixed

$\Downarrow$

$\left.\begin{array}{l} f(a) \in B \\ B \text{ is open} \end{array}\right\} \Rightarrow \exists \varepsilon > 0$ s.t. $B(f(a), \varepsilon) \subseteq B$

f is continuous at $a \Rightarrow \exists \delta > 0$ s.t. $\forall x \in \mathbb{R}^n$ with $\underbrace{\|x-a\| < \delta}_{\text{translation}} : \underbrace{\|f(x) - f(a)\| < \varepsilon}_{\text{translation}}$

$\Rightarrow \forall x \in B(a, \delta) : f(x) \in B(f(a), \varepsilon) \subseteq B$

$\Rightarrow \forall x \in B(a, \delta) : x \in f^{-1}(B)$

$\Rightarrow B(a, \delta) \subseteq f^{-1}(B)$

$\Downarrow$

Q.E.D.

③ Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a continuous function and let $B \subseteq \mathbb{R}^m$ be a closed set. Prove that the set $f^{-1}(B) = \{x \in \mathbb{R}^n \mid f(x) \in B\}$ is closed.

Solution: ~~?~~ $f^{-1}(B)$ is closed $\Leftrightarrow$ the limit of every convergent seq. of points in $f^{-1}(B)$ belongs to $f^{-1}(B)$

Let (x_k) be a converg. seq. of points in $f^{-1}(B)$ and let $a := \lim_{k\to\infty} x_k$

$$\left.\begin{array}{l} a := \lim_{k\to\infty} x_k \\ f \text{ is cont. at } a \end{array}\right\} \Rightarrow \lim_{k\to\infty} f(x_k) = f(a),$$

$$x_k \in f^{-1}(B) \Rightarrow f(x_k) \in B$$

$$\left.\begin{array}{l} \lim_{k\to\infty} f(x_k) = f(a) \\ f(x_k) \in B \end{array}\right\} \xRightarrow{B \text{ is closed}} f(a) \in B \Downarrow a \in f^{-1}(B)$$

④ Let $A = \{(x,y) \in \mathbb{R}^2 \mid y \neq 0\}$ and let $f : A \to \mathbb{R}$, $f(x,y) = \operatorname{arctg}\left(\frac{x}{y}\right)$. Deterine the first order derivative of f.

Solution: $\frac{\partial f}{\partial x}(x,y) = f'_x(x,y)$ $\qquad$ ∂ - curly d

$$= \frac{1}{1+u^2} \cdot u' = \frac{\frac{1}{y}}{1+\frac{x^2}{y^2}} = \frac{1}{\cancel{y}} \cdot \frac{y^{\cancel{2}}}{y^2+x^2} = \frac{y}{y^2+x^2} = \frac{y}{x^2+y^2}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{-\frac{1}{y^2}\cdot x}{1+\frac{x^2}{y^2}} = -\frac{x}{\cancel{y^2}} \cdot \frac{\cancel{y^2}}{x^2+y^2} = \frac{-x}{x^2+y^2}$$

⑤ Let $f : \mathbb{R}^2 \to \mathbb{R}^3$, $f(x,y) = (\underbrace{x^2-y}_{f_1(x,y)}, \underbrace{xy+y^2}_{f_2(x,y)}, \underbrace{e^{x^2-y^2}}_{f_3(x,y)})$. Determine the first order partial derivatives of f at $(1,1)$

Solution: $\frac{\partial f}{\partial x}(x,y) = \left(\frac{\partial f_1}{\partial x}(x,y), \frac{\partial f_2}{\partial x}(x,y), \frac{\partial f_3}{\partial x}(x,y)\right)$

$$= \left(2x, y, e^{x^2-y^2}\cdot 2x\right)$$

$$\frac{\partial f}{\partial y}(x,y) = \left(\frac{\partial f_1}{\partial y}(x,y), \frac{\partial f_2}{\partial y}(x,y), \frac{\partial f_3}{\partial y}(x,y)\right)$$

$$= \left(-1, x+2y, -2y\cdot e^{x^2-y^2}\right)$$

$$\frac{\partial f}{\partial x}(1,1) = (2,1,2) \; ; \; \frac{\partial f}{\partial y}(1,1) = (-1,3,-2)$$

⑥ Prove that $f: \mathbb{R}\times(0,\infty) \to \mathbb{R}$, $f(x,y) = \ln\left(2\sqrt{x^2+y^2} - x\right) - \ln y$ satisfies $x\cdot\frac{\partial f}{\partial x}(x,y) + y\cdot\frac{\partial f}{\partial y}(x,y) = 0$ (Not: S) $\forall x,y \in \mathbb{R}\times(0,\infty)$

Solution: $$\frac{\partial f}{\partial x}(x,y) = \frac{1}{2\sqrt{x^2+y^2}-x}\cdot\left(2\sqrt{x^2+y^2}-x\right)'$$

$$= \frac{1}{2\sqrt{x^2+y^2}-x}\cdot\left(2\cdot\frac{1}{2\sqrt{x^2+y^2}}\cdot 2x - x\right) = \frac{2x-\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}\cdot\left(2\sqrt{x^2+y^2}-x\right)}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{1}{2\sqrt{x^2+y^2}-x}\cdot\left(2\sqrt{x^2+y^2}-x\right)' - \frac{1}{y}$$

$$= \frac{2\cdot\frac{1}{2\sqrt{x^2+y^2}}\cdot 2y}{2\sqrt{x^2+y^2}-x} - \frac{1}{y} = \frac{\frac{y}{2y}}{\sqrt{x^2+y^2}}\cdot\frac{1}{2\sqrt{x^2+y^2}-x} - \frac{1}{y}$$

$$= \frac{2y^2 - \sqrt{x^2+y^2}\left(2\sqrt{x^2+y^2}-x\right)}{y\sqrt{x^2+y^2}\left(2\sqrt{x^2+y^2}-x\right)}$$

$$S = \frac{2x^2 - x\sqrt{x^2+y^2} + 2y^2 - 2(x^2+y^2) + x\sqrt{x^2+y^2}}{\dots} = \underline{\underline{0}}$$

④

⑦ Let $f: (0;\infty)\times\mathbb{R}^2 \to \mathbb{R}$, $f(x,y,z) = (xy+z^2)\cdot\cos\frac{yz}{x^2}$. Prove that $x\cdot\frac{\partial f}{\partial x}(x,y,z) + y\cdot\frac{\partial f}{\partial y}(x,y,z) + z\cdot\frac{\partial f}{\partial z}(x,y,z) = 2\cdot f(x,y,z)$ $\forall (x,y,z) \in (0;\infty)\times\mathbb{R}^2$

Sol: $$\frac{\partial f}{\partial x}(x,y,z) = (xy+z^2)'\cdot\cos\frac{yz}{x^2} + (xy+z^2)\cdot\left(\cos\frac{yz}{x^2}\right)'$$
$$= y\cdot\cos\frac{yz}{x^2} + (xy+z^2)\cdot\left(-\sin\frac{yz}{x^2}\cdot yz\cdot(-2)\cdot x^{-3}\right)$$

$$= y \cdot \cos\frac{yz}{x^2} + \frac{2yz(xy+z^2)}{x^3} \cdot \sin\frac{yz}{x^2} \quad \Big| \cdot x$$

$$\frac{\partial f}{\partial y}(x,y,z) = x \cdot \cos\frac{yz}{x^2} + (xy+z^2) \cdot \left(-\sin\frac{yz}{x^2} \cdot \frac{z}{x^2}\right)$$

$$= x \cdot \cos\frac{yz}{x^2} - \frac{z(xy+z^2)}{x^2} \cdot \sin\frac{yz}{x^2} \quad \Big| \cdot y$$

$$\frac{\partial f}{\partial z}(x,y,z) = 2z \cdot \cos\frac{yz}{x^2} - \frac{y(xy+z^2)}{x^2} \cdot \sin\frac{yz}{x^2} \quad \Big| \cdot z$$

$S =$