# Seminar 10

**1.** *Let $R$ be a ring. An element $a \in R$ is called idempotent if $a^2 = a$.*
Determine the idempotents of the ring $\mathbb{Z}_{12}$, and write down 4 idempotents of the ring $M_2(\mathbb{Z})$.

**2.** *Let $R$ be a ring. An element $a \in R$ is called nilpotent if there exists $n \in \mathbb{N}$ such that $a^n = 0$.*

Determine the nilpotent elements of the ring $\mathbb{Z}_{12}$, and write down 2 nilpotent elements of the ring $M_2(\mathbb{Z})$.

**3.** Are the following functions ring homomorphism between the corresponding rings:
$(i)$ $f : \mathbb{Z} \to \mathbb{Z}_n$ defined by $f(x) = \widehat{x}$;
$(ii)$ $f : \mathbb{R} \to M_2(\mathbb{R})$ defined by $f(x) = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$;
$(iii)$ $g : M_2(\mathbb{R}) \to \mathbb{R}$ defined by $g(A) = \det(A)$?

**4.** Let $f : \mathbb{Z}_{12} \to \mathbb{Z}_4$ be defined by $f(\widehat{x}) = \overline{x}$. Prove that $f$ is well defined (that is, $f$ is a function) and $f$ is a ring homomorphism.

**5.** Consider the fields $(\mathcal{M}, +, \cdot)$ and $(\mathbb{Q}(\sqrt{2}), +, \cdot)$, where $\mathcal{M} = \left\{ \begin{pmatrix} a & b \\ 2b & a \end{pmatrix} \middle| a, b \in \mathbb{Q} \right\}$ and $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$. Show that the fields $(\mathcal{M}, +, \cdot)$ and $(\mathbb{Q}(\sqrt{2}), +, \cdot)$ are isomorphic.

**6.** Consider the field $(\mathcal{M}, +, \cdot)$, where $\mathcal{M} = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \middle| a, b \in \mathbb{R} \right\}$. Show that the fields $(\mathcal{M}, +, \cdot)$ and $(\mathbb{C}, +, \cdot)$ are isomorphic.

**7.** For $a \in \mathbb{Z}$, let $t_a : \mathbb{Z} \to \mathbb{Z}$ be defined by $t_a(x) = ax$. Using the result $\mathrm{End}(\mathbb{Z}, +) = \{t_a \mid a \in \mathbb{Z}\}$, show that $\mathrm{End}(\mathbb{Z}, +, \cdot) = \{t_0, t_1\}$ and $\mathrm{Aut}(\mathbb{Z}, +, \cdot) = \{t_1\}$.

**8.** Consider the field $(\mathbb{Q}(\sqrt{2}), +, \cdot)$, where $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$. Determine $\mathrm{Aut}(\mathbb{Q}(\sqrt{2}), +, \cdot)$.