Lecture 2

March 3, 2022

2.4. Theorem. If $A \subseteq \mathbb{R}^n$, then

1° $\operatorname{int} A \subseteq A \subseteq \operatorname{cl} A$;

2° $\operatorname{cl} A = \mathbb{R}^n \setminus \operatorname{int}(\mathbb{R}^n \setminus A)$;

3° $\operatorname{bd} A = (\operatorname{cl} A) \cap \operatorname{cl}(\mathbb{R}^n \setminus A)$;

4° $\operatorname{cl} A = A \cup A'$.

Proof $\longrightarrow$ seminar

2.5. Definition A set $A \subseteq \mathbb{R}^n$ is called open if $A \in \mathcal{V}(x)$ $\forall x \in A$

$\iff \forall x \in A \ \exists r > 0$ s.t. $B(x, r) \subseteq A$

A is called closed if $\mathbb{R}^n \setminus A$ is open

Examples. Every open ball in $\mathbb{R}^n$ is an open set, while every closed ball in $\mathbb{R}^n$ is a closed set.

Proof $\longrightarrow$ seminar

2.6. Theorem If $A \subseteq \mathbb{R}^n$, then $\begin{cases} \operatorname{int} A \text{ is an open set} \\ \operatorname{cl} A \text{ is a closed set.} \end{cases}$

2.7. Theorem (characterization of open sets)

A set $A \subseteq \mathbb{R}^n$ is open $\iff$ $A = \operatorname{int} A$

2.8. Theorem (characterization of closed sets)

A set $A \subseteq \mathbb{R}^n$ is closed $\iff$ $A = \operatorname{cl} A$

3. Sequences in $\mathbb{R}^n$

3.1. Definition (convergent sequences in $\mathbb{R}^n$). A sequence of points in $\mathbb{R}^n$ is a function $f: \mathbb{N} \to \mathbb{R}^n$. If $f(k) = x_k$, then the sequence will be denoted by

$(x_k)_{k \in \mathbb{N}}$, $(x_k)_{k \geq 1}$, (x_k)

Since $x_k \in \mathbb{R}^n \Rightarrow x_k$ has the form $x_k = (x_{k1}, x_{k2}, \dots, x_{kn})$

Examples.

$$\left(\left(\frac{1}{k^2+1}, \frac{k^2}{k^2+1}\right)\right)_{k \geq 1}$$ is a sequence in $\mathbb{R}^2$

$$\left(\left(\frac{k}{k+1}, k \sin\frac{\pi}{k}, \left(1+\frac{1}{k}\right)^k\right)\right)_{k \in \mathbb{N}}$$ is a sequence in $\mathbb{R}^3$

A sequence (x_k) is called <u>convergent</u> if $\exists\, x \in \mathbb{R}^n$ s.t.

$$\forall \varepsilon>0 \;\; \exists\, k_0 \in \mathbb{N} \text{ s.t. } \forall k \geq k_0 : \; \|x_k - x\| < \varepsilon \iff \big|\, \|x_k - x\| - 0 \,\big| < \varepsilon$$

If (x_k) is convergent $\Rightarrow$ the point x with the above property is unique.

It is called the <u>limit</u> of (x_k) and will be denoted by $\lim\limits_{k\to\infty} x_k$.

We write also $(x_k) \longrightarrow x$.

<u>Remark</u> $(x_k) \longrightarrow x$ (in $\mathbb{R}^n$) $\iff$ $(\|x_k - x\|) \longrightarrow 0$ (in $\mathbb{R}$)

<u>3.2. Theorem</u>. Let (x_k) be a sequence in $\mathbb{R}^n$, $x_k = (x_{k1}, \dots, x_{kn})$ and let $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n) \in \mathbb{R}^n$. Then

$$\lim_{k\to\infty} x_k = \bar{x} \iff \lim_{k\to\infty} x_{kj} = \bar{x}_j \quad \forall\, j \in \{1, \dots, n\}.$$

$$\begin{array}{l}
x_1 = (x_{11}, x_{12}, \dots, x_{1n}) \\
x_2 = (x_{21}, x_{22}, \dots, x_{2n}) \\
\dots \\
x_k = (x_{k1}, x_{k2}, \dots, x_{kn}) \\
\dots \\
\downarrow \quad\;\; \downarrow \qquad \downarrow \qquad\qquad \downarrow \\
\bar{x} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)
\end{array}$$

<u>Ex</u>

$$\lim_{k\to\infty} \left(\frac{1}{k^2+1}, \frac{k^2}{k^2-1} \right) = (0, 1)$$

$$\lim_{k\to\infty} \left(\frac{k}{k+1}, k \sin\frac{\pi}{k}, \left(1+\frac{1}{k}\right)^k \right) = (1, \pi, e)$$

$$k \sin\frac{\pi}{k} = \frac{\sin\frac{\pi}{k}}{\frac{\pi}{k}} \cdot \pi$$

3.3. Definition (Cauchy sequences) A seq. (x_k) in $\mathbb{R}^n$ is called a Cauchy sequence if

$$\forall \varepsilon > 0 \quad \exists k_0 \in \mathbb{N} \text{ s.t. } \quad \forall k, l \geq k_0 \; : \quad \| x_k - x_l \| < \varepsilon$$

3.4. Theorem. Let (x_k) be a seq. in $\mathbb{R}^n$, $x_k = (x_{k1}, \dots, x_{kn})$. Then

$$\underset{(\text{in } \mathbb{R}^n)}{(x_k) \text{ is a Cauchy sequence}} \iff \underset{(\text{in } \mathbb{R})}{(x_{kj})_{k \geq 1} \text{ is a Cauchy sequence}} \quad \forall j = \overline{1,n}$$

3.5. Theorem (Cauchy) A seq. of points in $\mathbb{R}^n$ is convergent $\iff$ it is a Cauchy sequence.

3.6. Theorem (characterization of adherent points by means of sequences).

Let $A \subseteq \mathbb{R}^n$ and let $x \in \mathbb{R}^n$. Then

$$x \in \operatorname{cl} A \iff \exists (x_k) \text{ sequence of points in } A \text{ s.t. } x = \lim_{k \to \infty} x_k$$

Proof $\boxed{\Rightarrow}$ Assume that $x \in \operatorname{cl} A \Rightarrow \forall V \in \mathcal{V}(x) : V \cap A \neq \emptyset$

$B(x, \frac{1}{k}) \in \mathcal{V}(x) \Rightarrow B(x, \frac{1}{k}) \cap A \neq \emptyset \quad \forall k \geq 1 \Rightarrow \forall k \geq 1 \quad \exists x_k \in A$ and $x_k \in B(x, \frac{1}{k})$

$\Rightarrow (x_k)$ is a sequence in A and $\quad \|x_k - x\| < \frac{1}{k} \quad \forall k \geq 1$

$$\Downarrow$$

$$\|x_k - x\| \longrightarrow 0 \quad \Rightarrow \quad (x_k) \longrightarrow x$$

$\boxed{\Leftarrow}$ Assume that $\exists\ (x_k)$ seq. in A s.t. $(x_k) \to x$.

Let $V \in \mathcal{V}(x)$ arbitrarily chosen $\Rightarrow \exists\ \varepsilon > 0$ s.t. $B(x, \varepsilon) \subseteq V$

Since $(x_k) \to x \Rightarrow \exists\ k_0 \in \mathbb{N}$ s.t. $\forall\ k \geq k_0 : \ \|x_k - x\| < \varepsilon$

$$\Rightarrow \left.\begin{array}{l} \forall\ k \geq k_0 : \ x_k \in B(x,\varepsilon) \subseteq V \\ \qquad\qquad\quad x_k \in A \end{array}\right\} \Rightarrow \forall\ k \geq k_0 : \ x_k \in V \cap A \Rightarrow V \cap A \neq \emptyset$$

$\Rightarrow \ x \in \operatorname{cl} A$

3.7. Theorem (characterization of limit points by means of sequences).

Let $A \subseteq \mathbb{R}^n$ and let $x \in \mathbb{R}^n$. Then

$$x \in A' \iff \exists (x_k) \text{ seq. in } A \setminus \{x\} \text{ s.t. } (x_k) \to x.$$

Proof Follows by $\boxed{T3.6}$ taking into account that

$$x \in A' \iff \forall\ V \in \mathcal{V}(x) : \ V \cap (A \setminus \{x\}) \neq \emptyset \iff x \in \operatorname{cl}(A \setminus \{x\})$$

3.8. Theorem (characterization of closed sets by means of sequences).

A set $A \subseteq \mathbb{R}^n$ is closed $\iff$ every convergent sequence of points in A has the limit in A, too.

4. Compact sets in $\mathbb{R}^n$

4.1. Definition. A family $(A_i)_{i\in I}$ of subsets of $\mathbb{R}^n$ is called a cover of a given set $A \subseteq \mathbb{R}^n$ if

$$A \subseteq \bigcup_{i\in I} A_i$$

If each set A_i $(i \in I)$ is open, then the cover $(A_i)_{i\in I}$ is called open.

A set $A \subseteq \mathbb{R}^n$ is called compact if each of its open covers has a finite subcover.

A is compact $\Longleftrightarrow$ $\forall (A_i)_{i\in I}$ open cover of A $\exists J \subseteq I$, J = finite set s.t

$$A \subseteq \bigcup_{i\in J} A_i$$

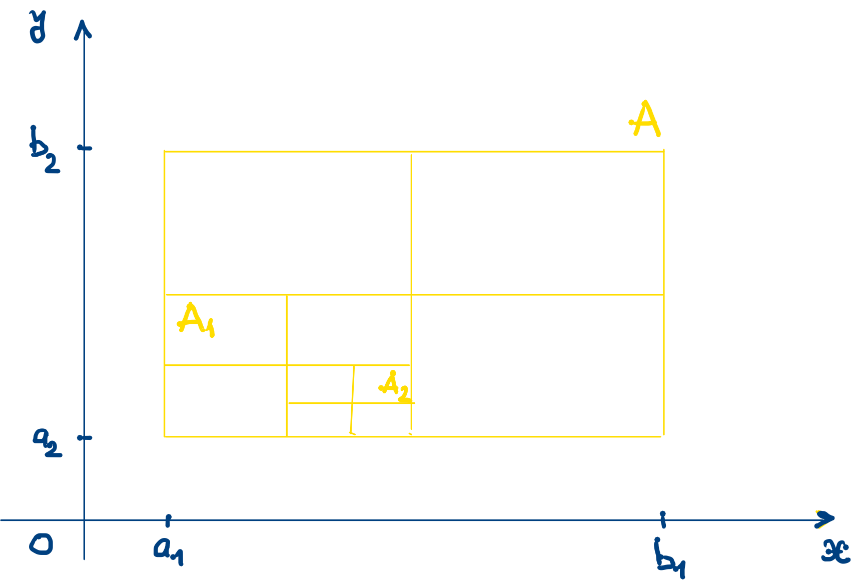
Examples a) If A is finite $\Rightarrow$ A is compact.

b) A set $A \subseteq \mathbb{R}^n$ is called a closed cell if it has the form $A = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$, where $a_1 < b_1$, $a_2 < b_2$, $\dots$, $a_n < b_n$ are real numbers.

Every closed cell in $\mathbb{R}^n$ is a compact set.

<u>Proof</u>. $n=2$. $A=[a_1,b_1]\times[a_2,b_2]$ rectangle $\quad l_1:=b_1-a_1$

$l_2:=b_2-a_2$



Assume that A is not compact $\Rightarrow$

$\Rightarrow \exists$ an open cover $(G_i)_{i\in I}$ of A, having no finite subcover, i.e., A cannot be covered by finitely many sets G_i $(i\in I)$.

Drawing parallels to the coordinate axes through the centre of A, we divide it into $4=2^2$ smaller rectangles

Then at least one of these smaller rectangles cannot be covered by finitely many sets G_i $(i\in I)$

Let A_1 be this rectangle, $\quad A_1=[a_{11},b_{11}]\times[a_{21},b_{21}]$ $\qquad b_{11}-a_{11}=\dfrac{l_1}{2}$

$b_{21}-a_{21}=\dfrac{l_2}{2}$

Drawing parallels to the axes of coordinate through the centre of A_1 we divide it into 4 smaller rectangles. At least one of these rectangles cannot be covered by finitely many sets G_i $(i \in I)$. Let A_2 be this rectangle

$$A_2 = [a_{12}, b_{12}] \times [a_{22}, b_{22}] \qquad b_{12} - a_{12} = \frac{l_1}{2^2} \qquad b_{22} - a_{22} = \frac{l_2}{2^2}$$

$$A_2 \subseteq A_1 \subseteq A \quad \Rightarrow \quad \begin{aligned} a_1 \leq a_{11} \leq a_{12} \leq b_{12} \leq b_{11} \leq b_1 \\ a_2 \leq a_{21} \leq a_{22} \leq b_{22} \leq b_{21} \leq b_2 \end{aligned}$$

Continuing this reasoning, we construct inductively a sequence $(A_k)_{k\geq 1}$ of rectangles with the following properties

- $A_k = [a_{1k}, b_{1k}] \times [a_{2k}, b_{2k}] \qquad b_{1k} - a_{1k} = \dfrac{l_1}{2^k} \qquad b_{2k} - a_{2k} = \dfrac{l_2}{2^k}$

- $A \supseteq A_1 \supseteq A_2 \supseteq \ldots \supseteq A_k \supseteq A_{k+1} \supseteq \ldots \qquad (*)$

- each A_k cannot be covered by finitely many sets G_i $(i \in I)$

By $(*)$ $\Rightarrow$ $(a_{1k})_{k\geq 1}$, $(a_{2k})_{k\geq 1}$ are non-decreasing sequences of real numbers

$(b_{1k})_{k\geq 1}$, $(b_{2k})_{k\geq 1}$ — " — non-increasing ——— " ———

$\Rightarrow$ $\left([a_{1k}, b_{1k}]\right)_{k\geq 1}$, $\left([a_{2k}, b_{2k}]\right)_{k\geq 1}$ are nested sequences of closed intervals

$\Rightarrow$ $\bigcap_{k=1}^{\infty} [a_{1k}, b_{1k}] = \{x_1^*\}$ and $\bigcap_{k=1}^{\infty} [a_{2k}, b_{2k}] = \{x_2^*\}$

Let $x^* := (x_1^*, x_2^*)$ $\Rightarrow$ $x^* \in A_k$ $\forall\, k = 1, 2, \ldots$

$\Rightarrow$ $x^* \in A \subseteq \bigcup_{i\in I} G_i$ $\Rightarrow$ $\exists\, i^* \in I$ s.t. $\left.\begin{array}{l} x^* \in G_{i^*} \\ G_{i^*} \text{ is open} \end{array}\right\}$ $\Rightarrow$ $\exists\, r > 0$ s.t. $B(x^*, r) \subseteq G_{i^*}$

Choose $k \in \mathbb{N}$ sufficiently large s.t. $\dfrac{\sqrt{l_1^2 + l_2^2}}{2^k} < r$

Let $x := (x_1, x_2) \in A_k$ arbitrarily chosen

$$\|x - x^*\| = \sqrt{(x_1 - x_1^*)^2 + (x_2 - x_2^*)^2} \leq \sqrt{\frac{l_1^2 + l_2^2}{2^{2k}}} = \frac{\sqrt{l_1^2 + l_2^2}}{2^k} < r$$

$x_1, x_1^* \in [a_{1k}, b_{1k}]$ $\Rightarrow$ $|x_1 - x_1^*| \leq b_{1k} - a_{1k} = \dfrac{l_1}{2^k}$ $\Rightarrow$ $(x_1 - x_1^*)^2 \leq \dfrac{l_1^2}{2^{2k}}$

Analogously $(x_2 - x_2^*)^2 \leq \dfrac{l_2^2}{2^{2k}}$

$\Rightarrow$ $A_k \subseteq B(x^*, r) \subseteq G_{i^*}$ $\Rightarrow\Leftarrow$ A_k cannot be covered by finitely many sets G_i $(i \in I)$

The obtained contradiction shows that A is compact.

4.2. Definition (bounded sets). A set $A \subseteq \mathbb{R}^n$ is called <u>bounded</u> if

$\exists\, a \in \mathbb{R}^n \;\; \exists\, r > 0$ s.t. $A \subseteq \overline{B}(a, r)$

$\iff \exists\, r > 0$ s.t. $A \subseteq \overline{B}(0_n, r)$

$\iff$ A is contained in a closed cell in $\mathbb{R}^n$.

4.3. Theorem (characterization of compact sets in $\mathbb{R}^n$). Given a set $A \subseteq \mathbb{R}^n$, the following assertions are equivalent:

1° A is compact

2° Every infinite subset of A has a limit point in A.

3° A is sequentially compact (i.e., every sequence of points in A has a convergent subsequence converging to some point in A).

4° A is bounded and closed.

<u>Proof</u>. $\boxed{1^\circ \Rightarrow 2^\circ}$ Assume that A is compact. Let A_0 be an arbitrary infinite subset of A.

Suppose, by reductio ad absurdum, that A_0 has no limit point in A

$\Rightarrow \quad \forall x \in A : \quad x \notin A_0'$

$\Rightarrow \quad \forall x \in A \quad \exists V_x \in \mathcal{V}(x)$ s.t. $V_x \cap A_0 \setminus \{x\} = \phi$

$\Downarrow$

$\exists r_x > 0$ s.t. $B(x, r_x) \subseteq V_x$

$\Rightarrow B(x, r_x) \cap A_0 \setminus \{x\} = \phi$

$\Downarrow$

$B(x, r_x) \cap A_0 \subseteq \{x\}$

The family of open balls $\left(B(x, r_x)\right)_{x \in A}$ is an open cover of A $\Rightarrow$

$\uparrow$ A is compact

$\Rightarrow$ this open cover has a finite subcover

$\Rightarrow \quad \exists x_1, x_2, \ldots, x_p \in A$ s.t. $A \subseteq \bigcup_{k=1}^{p} B(x_k, r_{x_k})$

$$\Rightarrow \quad A_0 = A_0 \cap A \subseteq A_0 \cap \left(\bigcup_{k=1}^{p} B(x_k, r_{x_k}) \right) = \bigcup_{k=1}^{p} \underbrace{\left(A_0 \cap B(x_k, r_{x_k}) \right)}_{\subseteq \{x_k\}} \subseteq \{x_1, \ldots, x_p\}$$

$\Rightarrow \quad A_0 \subseteq \{x_1, x_2, \ldots, x_p\}$ $\quad \Rightarrow\Leftarrow \quad$ A_0 is infinite