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Mathematical Logic

Lecture 6

13.11.2023

Theorem 2. Let ρ be an equivalence relation on the set A. Then for any $x, y \in A$, the following statements are equivalent:

(i) $x \rho y$

(ii) $y \in \rho\langle x \rangle \overset{def}{=} \{ x' \in A \mid x \rho x' \}$.

(iii) $\rho\langle x \rangle = \rho\langle y \rangle$

(iv) $\rho\langle x \rangle \cap \rho\langle y \rangle \neq \emptyset$

Remarks. 1). The conditions (i)–(iv) say that the set $\{ \rho\langle x \rangle \mid x \in A \}$ is a partition of A

because: (1) $\bigcup_{x \in A} \rho\langle x \rangle = A$, because $x \in \rho\langle x \rangle$ by reflexivity

(2) if $\rho\langle x \rangle \neq \rho\langle y \rangle \Rightarrow \rho\langle x \rangle \cap \rho\langle y \rangle = \emptyset$

Def. We denote $\boxed{A/\rho = \{ \rho\langle x \rangle \mid x \in A \}}$.

This partition is called the quotient (factor) set of A w.r.t. ρ. (modulo ρ)

Not. $\rho\langle x \rangle =: [x]_\rho$ the class of x modulo ρ.

Rmk 2) Consider the relation ρ_{π_1} of Lecture 5.

$\rho_{\pi_1}\langle 1 \rangle = \{1, 2\} = \rho_{\pi_1}\langle 2 \rangle$.

$\rho_{\pi_1}\langle 3 \rangle = \{3, 4, 5\} = \rho_{\pi_1}\langle 4 \rangle = \rho_{\pi_1}\langle 5 \rangle$

$\rho_{\pi_1}\langle 6 \rangle = \{6\}$. We get $A/\rho_{\pi_1} = \{\{1,2\}, \{3,4,5\}, \{6\}\} = \pi_1$

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3) Given a relation $\rho = (A, A, R)$, we may always consider the set of sections $\{\rho\langle x\rangle \mid x \in A\}$.

Then: ρ is an equivalence $\Longleftrightarrow$ this set is a partition of A.

Proof of Thm 2 $\quad$ (i) $\Longleftrightarrow$ (ii) by the def of $\rho\langle x\rangle$.

(i) $\Longrightarrow$ (iii) Assume that $x \rho y$. Because of symmetry, it is enough to prove that $\rho\langle x\rangle \subseteq \rho\langle y\rangle$.

Let $z \in \rho\langle x\rangle$. Then we have $x \rho z$.

From $x\rho y$, we get $y \rho x$ (by (S))

By transitivity, $y \rho z$, hence $z \in \rho\langle y\rangle$.

(iii) $\Longrightarrow$ (i) Assume that $\rho\langle x\rangle = \rho\langle y\rangle$.

By (R) we have $y \in \rho\langle y\rangle$, hence $y \in \rho\langle x\rangle$

hence $x \rho y$.

(i) $\Longrightarrow$ (iv) Assume that $x \rho y$, so $y \in \rho\langle x\rangle$.

But $y \in \rho\langle y\rangle$. Then $y \in \rho\langle x\rangle \cap \rho\langle y\rangle \neq \emptyset$

(iv) $\Longrightarrow$ (i) Assume that $\exists z \in \rho\langle x\rangle \cap \rho\langle y\rangle$,

so $x \rho z$ and $y \rho z$. By (S), we have $z \rho y$.

By (T) it follows that $x \rho y$.

Remarks These two theorems say that the concepts of equivalence and partition are essentially the same. Moreover:

- If π is a partition, then ρ_π is an equiv. rel., and we have $A/\rho_\pi = \pi$
- If ρ is an equiv. rel., then A/ρ is a partition, and we have $\rho_{A/\rho} = \rho$.

Functions and equivalence relations

Def 1. Let $f : A \to B$ be a function.
The kernel of f (denoted $\ker f$) is the relation on A defined as follows:

$$\forall x, y \in A \quad x \ker f\, y \overset{def}{\iff} f(x) = f(y)$$

Prop. 1 Let $f : A \to B$ be a function. Then
1). $\ker f$ is an equivalence relation on A
2). $A/\ker f = \{ f^{-1}(b) \mid b \in \operatorname{Im} f \}$.

Proof 1). (R). $x \ker f\, x \iff f(x) = f(x)$ true $\forall x \in A$

(T): Assume $x \ker f\, y$ and $y \ker f\, z$. Then:
$f(x) = f(y)$ and $f(y) = f(z) \implies f(x) = f(z)$
$\implies x \ker f\, z$.

(S). Assume $x \ker f\, y \implies f(x) = f(y) \implies f(y) = f(x) \implies$
$\implies y \ker f\, x$.

2). By def, $A/\ker f = \{ (\ker f)\langle x \rangle \mid x \in A \}$.
Let $x \in A$. Let $b := f(x) \in \operatorname{Im} f$
We only need to prove that $(\ker f)\langle x \rangle = f^{-1}(b)$
Indeed let $y \in A$. We have
$\underline{y \in (\ker f)\langle x \rangle} \iff x \ker f\, y \iff f(x) = f(y) \iff f(y) = b \iff \underline{y \in f^{-1}(b)}$

Def 2 Let ρ be an equivalence relation on A.
The canonical projection associated to ρ is
the function $\begin{cases} p_\rho : A \longrightarrow A/\rho \\ p_\rho(x) = \rho\langle x\rangle = [x]_\rho \end{cases}$

Proposition 2. The canonical projection $p_\rho : A \to A/\rho$ has the following properties:

1). p_ρ is surjective. (i.e. $\operatorname{Im} p_\rho = A/\rho$)

2). $\ker p_\rho = \rho$

Proof 1). We have $A/\rho = \{\rho\langle x\rangle \mid x \in A\}$.

For any $\rho\langle x\rangle \in A/\rho$, where $x \in A$, we have $p_\rho(x) = \rho\langle x\rangle$,

hence p_ρ is surjective.

2). Let $x, y \in A$. We have:

$$x \ker p_\rho\, y \underset{\ker p_\rho}{\overset{\text{def of}}{\Longleftrightarrow}} p_\rho(x) = p_\rho(y) \underset{p_\rho}{\overset{\text{def of}}{\Longleftrightarrow}} \rho\langle x\rangle = \rho\langle y\rangle \overset{\text{Th 2}}{\Longleftrightarrow} x\rho y$$

then $\ker p_\rho = \rho$.

2 iota.

Theorem (the 1st factorization theorem) Let $f : A \to B$ be a function.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "p_{\ker f}"'] & B \\ A/\ker f \arrow[r, dashed, "\bar{f}"'] & \operatorname{Im} f \arrow[u, hook, "i"'] \end{tikzcd}$$

canonical inclusion

$i(y) = y$

Then $\exists!$ bijective function

$\bar{f} : A/\ker f \longrightarrow \operatorname{Im} f$

s.t. the diagram is commutative, i.e.

$$f = i \circ \bar{f} \circ p_{\ker f}$$

(this is the canonical decomposition of f)

Proof (!) (uniqueness of $\bar{f}$) We assume that $\bar{f}$ exists and we prove that it is unique.

We have $\forall x \in A$:

$$f(x) = (i \circ \bar{f} \circ p_{\ker f})(x) = i(\bar{f}(p_{\ker f}(x))) =$$

$$= \bar{f}((\ker f)\langle x \rangle)$$

Hence $\boxed{\bar{f}((\ker f)\langle x \rangle) = f(x)}$ is uniquely defined $\forall x \in A$

($\exists$). Let $\begin{cases} \bar{f} : A/\ker f \longrightarrow \operatorname{Im} f \\ \bar{f}((\ker f)\langle x \rangle) = f(x) \in \operatorname{Im} f \end{cases}$

- the def of $\bar{f}$ is given by using the representative $x \in (\ker f)\langle x \rangle$. We have to show that the def of $\bar{f}$ does not depend on the choice of representatives.

Indeed, let $y \in (\ker f)\langle x\rangle$, i.e. $x \ker f\, y$

here $(\ker f)\langle x\rangle = (\ker f)\langle y\rangle$.

Then $\bar{f}\big((\ker f)\langle y\rangle\big) = f(y) = f(x)$

- we show that $\bar{f}$ is injective.

Let $x, y \in A$ s.t. $\bar{f}\big((\ker f)\langle x\rangle\big) = \bar{f}\big((\ker f)\langle y\rangle\big)$

$\overset{\text{def of}}{\underset{\bar{f}}{\Longrightarrow}} f(x) = f(y) \Longrightarrow x \ker f\, y \Longrightarrow$

$\overset{\text{Th 2}}{\Longrightarrow} (\ker f)\langle x\rangle = (\ker f)\langle y\rangle$.

- we show that $\bar{f}$ is surjective.

Let $b \in \operatorname{Im} f$. Then $\exists\, x \in A$ s.t. $f(x) = b$

Then $\bar{f}\big((\ker f)\langle x\rangle\big) = b$.

Hence $\bar{f}$ is surjective.

- we show that the diagram is commutative:

Let $x \in A$. We have:

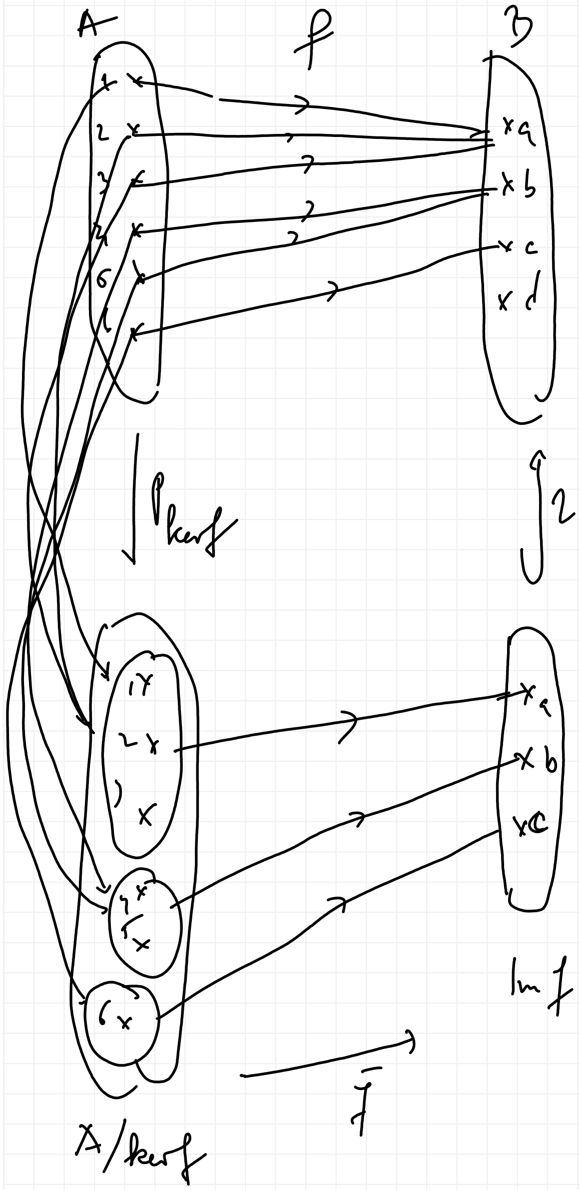
$$(\iota \circ \bar{f} \circ p_{\ker f})(x) = \iota\big(\bar{f}(p_{\ker f}(x))\big) =$$

$$= \bar{f}\big((\ker f)\langle x\rangle\big) \overset{\text{def of}}{\underset{\bar{f}}{=}} f(x)$$

Hence $\iota \circ \bar{f} \circ p_{\ker f} = f$.

$$\boxed{\text{Homework} \quad 51, 52 \quad 55 - 64}$$

Example Consider the function $f : \{1,2,3,4,5,6\} \longrightarrow \{a,b,c,d\}$



f	1	2	3	4	5	6
$f(x)$	a	a	a	b	b	c.

Apply the 1st factorization theorem (ie determine all the objects from the statement of the theorem)

* $\operatorname{Im} f = \{a, b, c\}$.

$f^{-1}(a) = \{1, 2, 3\}$.

$f^{-1}(b) = \{4, 5\}$

$f^{-1}(c) = \{6\}$

* $A/\ker f = \{\{1,2,3\}, \{4,5\}, \{6\}\}$.

* $\ker f = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3), (4,4), (4,5), (5,5), (5,4), (6,6)\}$

$x \in A$	1	2	3	4	5	6
$f(x) \in B$	a	a	a	b	b	c
* $p_{\ker f}(x)$	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{4,5\}$	$\{4,5\}$	$\{6\}$

$\operatorname{Im} f \ni y$	a	b	c
$i(y) \in B$	a	b	c

$(\ker f)(x) \in A/\ker f$	$\{1, 2, 3\}$	$\{4, 5\}$	$\{6\}$
* $\bar{f}((\ker f)(x))$	a	b	c