Seminar 9 (1) Analiză complexă

Funcții omografice

(1) a) Să se determine transformarea omografică $f:\mathbb{C}_\infty \to \mathbb{C}_\infty$ a.î. $f(i)=\infty$, $f(-1)=0$, $f(\infty)=1$.

b) Fie $D=T(0,1,i)$ și $f:\mathbb{C}_\infty \to \mathbb{C}_\infty$ transformarea de la a). Să se determine imaginea domeniului D prin transformarea f ($f(D)=?$).

c) Să se determine coeficientul de deformare liniară și unghiul de rotație al tangentei în $z_0=2i$ prin transformarea f de la a).

Soluție: a) f conservă biraportul, fiind omografică:

$$(z,i,-1,\infty)=(f(z),\infty,0,1) \Leftrightarrow$$

$$\Leftrightarrow \frac{z-i}{z-\infty}:\frac{-1-i}{-1-\infty}=\frac{f(z)-\infty}{f(z)-1}:\frac{0-\infty}{0-1} \Leftrightarrow$$

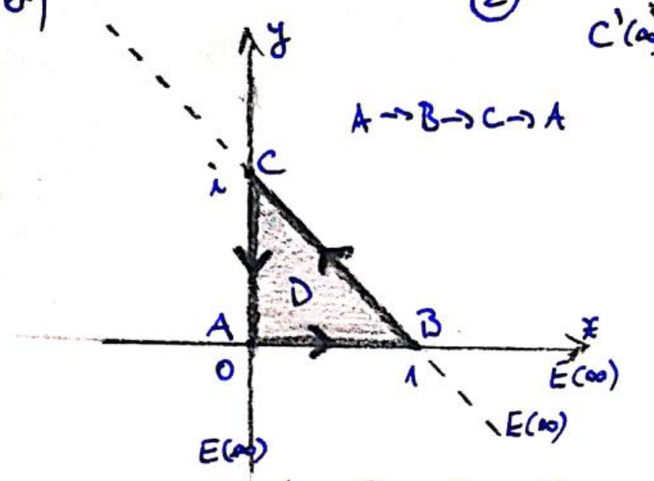
$$\Leftrightarrow \frac{z-i}{-1-i}\cdot\underbrace{\frac{-1-\infty}{z-\infty}}_{=1}=\underbrace{\frac{f(z)-\infty}{0-\infty}}_{=1}\cdot\frac{0-1}{f(z)-1} \Leftrightarrow$$

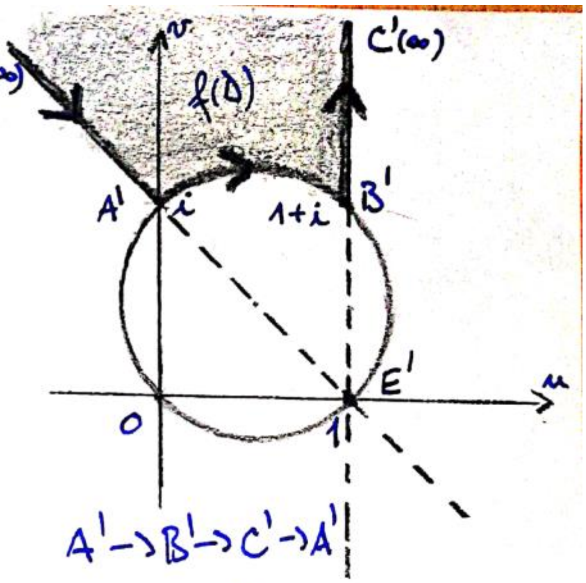
$$\Leftrightarrow \frac{z-i}{-1-i}=\frac{-1}{f(z)-1} \Leftrightarrow (f(z)-1)(z-i)=1+i$$

$$\Leftrightarrow f(z)(z-i)=z+1 \Leftrightarrow f(z)=\frac{z+1}{z-i},\quad z\in\mathbb{C}_\infty.$$

Verificare: $f(i)=\frac{i+1}{i-i}=\infty$, $f(-1)=\frac{-1+1}{-1-i}=0$, $f(\infty)=\frac{\infty+1}{\infty-i}=1$.

b) ②





	A	B	C	E
z	0	1	i	∞
$f(z)$	i	$1+i$	∞	1
	A'	B'	C'	E'

f păstrează sensul de parcurs dat pe ∂D.

$$f(1) = \frac{2}{1-i} = \frac{2(1+i)}{2} = 1+i$$

f omeomorfism de la $\mathbb{C}_\infty$ la $\mathbb{C}_\infty$ $\Rightarrow$

$$\Rightarrow f(\partial D) = \partial f(D) \text{ și } \mathbb{C}_\infty = f(\mathbb{C}_\infty) = f(D \cup \partial D \cup \operatorname{ext} D)$$
$$= f(D) \cup \partial f(D) \cup \operatorname{ext} f(D).$$

c) $f'(z) = -\dfrac{1+i}{(z-i)^2}$, $z \in \mathbb{C} \setminus \{i\}$.

$$f'(2i) = -\frac{1+i}{i^2} = 1+i \Rightarrow |f'(z_0)| = \sqrt{2}, \ \arg f'(z_0) = \frac{\pi}{4}$$

$\Rightarrow$ coeficientul de deformare liniară în $z_0 = 2i$ este $\sqrt{2}$ și unghiul de rotație al tangentei în $z_0 = 2i$ este $\frac{\pi}{4}$.

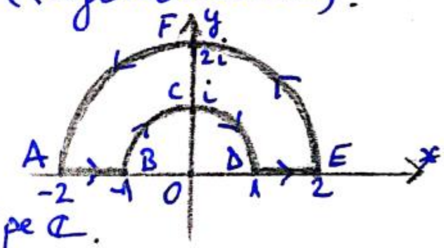
Integrale complexe (3)

(2) Să se calculeze integrala $\int_\gamma \bar{z}\, dz$, unde γ e drumul de suport:

a) $\{\gamma\} = [z_1, z_2]$, unde $z_1, z_2 \in \mathbb{C}$, $z_1 \neq z_2$.

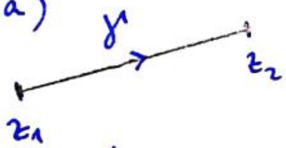
b) $\{\gamma\} = \partial U(z_0, r)$, cercul de centru $z_0 \in \mathbb{C}$ și rază $r > 0$ parcurs în sens direct (trigonometric).

c) $\{\gamma\} = \overset{\frown}{ABCDEFA}$:



$f(z) = \bar{z}$, $z \in \mathbb{C}$ $\Rightarrow$ f continuă pe $\mathbb{C}$.

Soluție: a)



$\gamma(t) = (1-t)z_1 + t z_2$, $t \in [0,1]$

$\{\gamma\} = [z_1, z_2]$.

$$\int_\gamma f(z)\, dz = \int_0^1 f(\gamma(t))\, d\gamma(t) = \int_0^1 f(\gamma(t)) \cdot \gamma'(t)\, dt =$$

$$= \int_0^1 \overline{\gamma(t)} \cdot \gamma'(t)\, dt = \int_0^1 ((1-t)\bar{z}_1 + t\bar{z}_2) \cdot (z_2 - z_1)\, dt$$

$$= (z_2 - z_1)\left(\bar{z}_1 \int_0^1 (1-t)\, dt + \bar{z}_2 \int_0^1 t\, dt\right) =$$

$$= (z_2 - z_1)\left(\bar{z}_1 \cdot \left(t - \frac{t^2}{2}\right)\Big|_0^1 + \bar{z}_2 \cdot \frac{t^2}{2}\Big|_0^1\right)$$

$$= \frac{1}{2}(z_2 - z_1)(\bar{z}_2 + \bar{z}_1).$$

Notație: $\int_{[z_1,z_2]} \bar{z}\, dz = \int_\gamma \bar{z}\, dz$.

b)

$\gamma(t) = z_0 + r e^{2\pi i t}$, $t \in [0,1] \Rightarrow \{\gamma\} = \partial U(z_0, r)$ cercul parcurs în sens direct (trigonometric).

$|\gamma(t) - z_0| = |r(\cos(2\pi t) + i \sin(2\pi t))| = r$, $t \in [0,1]$.

$\gamma(0) = \gamma(1) = z_0 + r$.

$$\int_\gamma \bar{z}\, dz = \int_0^1 \overline{\gamma(t)}\, d\gamma(t) = \int_0^1 \overline{\gamma(t)} \cdot \gamma'(t)\, dt =$$

$$= \int_0^1 \left(\bar{z}_0 + r e^{-2\pi i t}\right) \cdot \left(r e^{2\pi i t} \cdot (2\pi i)\right) dt =$$

$$= (2\pi i r)\left(\bar{z}_0 \int_0^1 e^{2\pi i t}\, dt + r \int_0^1 1\, dt\right) =$$

$$= (2\pi i r)\left(\bar{z}_0 \cdot \frac{1}{2\pi i} \cdot e^{2\pi i t}\Big|_0^1 + r\right) = (2\pi i r)\left(\bar{z}_0 \cdot \frac{1}{2\pi i}(1-1) + r\right)$$

$$= 2\pi i r^2 \Rightarrow$$

$$\Rightarrow \int_\gamma \bar{z}\, dz = 2\pi i r^2.$$

Notație: $\displaystyle\int_\gamma \bar{z}\, dz = \int_{\partial U(z_0, r)} \bar{z}\, dz$.

c) $\gamma_1(t) = (1-t)\cdot(-2) + t\cdot(-1)$, $t \in [0,1] \Rightarrow \{\gamma_1\} = [AB]$;

$\gamma_2(t) = e^{\pi i t}$, $t \in [0,1] \Rightarrow \{\gamma_2\} = \widehat{DCB}$;

$\gamma_3(t) = (1-t)\cdot 1 + t \cdot 2$, $t \in [0,1] \Rightarrow \{\gamma_3\} = [DE]$;

$\gamma_4(t) = 2 \cdot e^{\pi i t}$, $t \in [0,1] \Rightarrow \{\gamma_4\} = \widehat{EFA}$.

$\gamma = \gamma_1 \cup \gamma_2^- \cup \gamma_3 \cup \gamma_4$.

(5)

$$\int_\gamma f = \int_{\gamma_1} f + \int_{\gamma_2^-} f + \int_{\gamma_3} f + \int_{\gamma_4} f$$

$$= \int_{\gamma_1} \bar{z}\,dz - \int_{\gamma_2} \bar{z}\,dz + \int_{\gamma_3} \bar{z}\,dz + \int_{\gamma_4} \bar{z}\,dz.$$

$$\int_{\gamma_1} \bar{z}\,dz = \int_{[-2,-1]} \bar{z}\,dz \overset{a)}{=} \frac{1}{2}(-1-(-2))(-1+(-2)) = -\frac{3}{2}.$$

$$\int_{\gamma_2} \bar{z}\,dz = \int_0^1 e^{-\pi i t} \cdot e^{\pi i t} \cdot (\pi i)\,dt = \pi i \int_0^1 1\,dt = \pi i.$$

$$\int_{\gamma_3} \bar{z}\,dz = \int_{[1,2]} \bar{z}\,dz \overset{a)}{=} \frac{1}{2}(2-1)(2+1) = \frac{3}{2}.$$

$$\int_{\gamma_4} \bar{z}\,dz = \int_0^1 2e^{-\pi i t} \cdot 2e^{\pi i t} \cdot (\pi i)\,dt = 4\pi i.$$

$$\int_\gamma \bar{z}\,dz = -\frac{3}{2} - \pi i + \frac{3}{2} + 4\pi i = 3\pi i.$$

(3) Calculați integrala $\displaystyle\int_{\partial T} \frac{dz}{(z^2+4)^2(z^2+16)^3}$, unde $T = T(-1, 1, i)$.

Soluție: Fie $f : \mathbb{C} \setminus A \to \mathbb{C}$,

$$f(z) = \frac{1}{(z^2+4)^2(z^2+16)^3},\ z \in \mathbb{C} \setminus A,$$

unde $A = \{z \in \mathbb{C} : (z^2+4)^2(z^2+16)^3 = 0\}$.

6

$z \in A \iff z^2+4=0$ sau $z^2+16=0$

$\iff z \in \sqrt{-4}$ sau $z \in \sqrt{-16}$

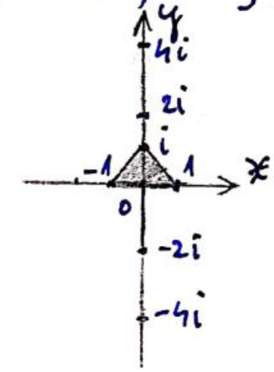
$\iff z \in \{-2i, 2i, -4i, 4i\}$.

$\partial T = \lambda_1 \cup \lambda_2 \cup \lambda_3$,

$\{\lambda_1\} = [-1, 1]$

$\{\lambda_2\} = [1, i]$

$\{\lambda_3\} = [i, -1]$.



$$\left.\begin{array}{l} \overline{T} = \overline{T(-1,1,i)} \subset \mathbb{C} \setminus A \\ f \in \mathcal{H}(\mathbb{C}\setminus A) \end{array}\right\} \xrightarrow[\text{Cauchy-Goursat}]{\text{Teorema}} \int_{\partial T} f = 0.$$

(4) Calculați integrala $\int_\gamma \frac{\cos z}{(z-2i)^n}\, dz$, $n \in \mathbb{N}$, unde γ este un contur din $U(0,1)$.

Soluție: Fie $f_n(z) = \frac{\cos z}{(z-2i)^n}$, $\forall z \in \mathbb{C}\setminus\{2i\}$, unde $n \in \mathbb{N}$.

$$\left.\begin{array}{l} f_n \in \mathcal{H}(\mathbb{C}\setminus\{2i\}),\ n \in \mathbb{N} \\ \{\gamma\} \subset U(0,1) \subset \mathbb{C}\setminus\{2i\} \\ U(0,1) \text{ simplu conex} \end{array}\right\} \xrightarrow[\text{Cauchy}]{\text{T. fundamen.}} \int_\gamma f_n = 0,\ \forall n \in \mathbb{N}.$$

5) Calculați $\int_\gamma \frac{\sin(z^3-z)}{z^3-z}\,dz$, unde $\gamma(t)=2e^{2\pi i t}$, $t\in[0,1]$ ($\{\gamma\}$ e $\partial U(0,2)$, cercul e parcurs în sens direct).

Soluție:

Observăm că: $z^3-z=0 \iff z(z-1)(z+1)=0$
$\iff z\in\{-1,0,1\}$.

Pentru $z_0\in\{-1,0,1\}$ avem:

$$\lim_{z\to z_0}\frac{\sin(z^3-z)}{z^3-z} \overset{\zeta=z^3-z}{=\!=\!=} \lim_{\zeta\to 0}\frac{\sin\zeta}{\zeta}$$

$$=\lim_{\zeta\to 0}\frac{\sin\zeta-\sin 0}{\zeta-0}=\sin' 0=\cos 0=1.$$

Fie $f(z)=\begin{cases}\frac{\sin(z^3-z)}{z^3-z}, & z\in\mathbb{C}\setminus\{-1,0,1\}\\ 1, & z\in\{-1,0,1\}.\end{cases}$

Avem $f\in\mathcal{H}(\mathbb{C}\setminus\{-1,0,1\})\cap C(\mathbb{C})$.

$\mathbb{C}$ este stelat în raport cu orice punct

$\xrightarrow[\text{dom. și primitivă}]{\text{T. de leg. între}}$ f admite primitive pe $\mathbb{C}$

$\xrightarrow{\text{(Leibniz-Newton)}}$ $\int_\gamma f=\int_\gamma \frac{\sin(z^3-z)}{z^3-z}\,dz=0.$

(6) Calculați $\int_\gamma \frac{\sin z}{\cos(iz)(z+2i)(z-5i)^2}\,dz$, unde γ este un contur în $U(0,1)$.

Soluție:

Fie $A = \{z \in \mathbb{C} : \cos(iz)(z+2i)(z-5i)^2 = 0\}$.

$\cos(iz) = 0 \iff \frac{e^{i\cdot iz} + e^{-i\cdot iz}}{2} = 0 \iff e^{-z} + e^{z} = 0$

$\iff e^{2z} = -1 \underset{z=x+iy}{\iff} e^{2x}(\cos(2y) + i\sin(2y)) = \cos\pi + i\sin\pi$

$\iff \begin{cases} 2x = 0 \\ 2y = \pi \ (\mathrm{mod}\ 2\pi) \end{cases} \iff \begin{cases} x_k = 0 \\ 2y_k = \pi + 2k\pi, \ k \in \mathbb{Z} \end{cases}$

$\iff z_k = i\left(\frac{\pi}{2} + k\pi\right), \ k \in \mathbb{Z}$.

$A = \{-2i, 5i\} \cup \left\{i\left(\frac{\pi}{2} + k\pi\right) : k \in \mathbb{Z}\right\}$.

Fie $f(z) = \frac{\sin z}{\cos(iz)(z+2i)(z-5i)^2}$, $z \in \mathbb{C} \setminus A$.

$A \cap U(0,1) = \emptyset$, pentru că $\left|i\left(\frac{\pi}{2} + k\pi\right)\right| \geq \frac{\pi}{2} > 1, \ \forall k \in \mathbb{Z}$.

Deci, $f \in \mathcal{H}(U(0,1)) \xrightarrow[U(0,1)\text{ simplu conexă}]{\text{T. fun. Cauchy}} \int_\gamma f = 0$.