Mathematical Logic
Lecture 11
18.12.2023

Chapter 8. Cardinal numbers

We will give the 'naive' definition of the cardinality of a set, due to Georg Cantor. ≈ 1870 →

Def. The set A, B are called <u>equipotent</u> if $\exists f : A \to B$ a bijective function. <u>Not</u> $A \sim B$.

Rem. The rel "$\sim$" is an equivalence.

(R) $A \sim A$ because $1_A : A \to A$ is bijective.

(T) Assume $A \sim B$, $B \sim C$. Let $f : A \to B$, $g : B \to C$ bijective functions. Then $g \circ f : A \to C$ is also bijective, hence $A \sim C$.

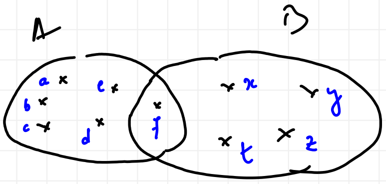
(S) Assume $A \sim B$. Let $f : A \to B$ bijective. Then $f^{-1} : B \to A$ is also bijective, hence $B \sim A$.

Def The <u>cardinal number</u> of the set A is the equipotence class of A:
$$|A| \overset{def}{:=} \{ B \text{ set} \mid A \sim B \}.$$

We also say that the set A is a <u>representative</u> of the cardinal number $\alpha = |A|$.

Operations with cardinal numbers

1). <u>Addition</u> we want: $|A| + |B| = |A \cup B|$, if $A \cap B = \emptyset$



Let $\alpha_i = |A_i|$, $i \in I$, be a family of cardinal numbers.

The problem is that we might not have that $A_i \cap A_j = \emptyset$ $\forall i \neq j$.

The idea is to replace the sets A_i by other sets A_i', $i \in I$, such that:
$$\begin{cases} A_i \sim A_i' & \forall i \in I \\ A_i' \cap A_j' = \emptyset & \text{for } \forall i,j \in I,\ i \neq j. \end{cases}$$
(the sets are pairwise disjoint)

Let $A_i' := A_i \times \{i\} = \{(a_i, i) \mid a_i \in A_i\}$.

obviously: $\begin{cases} A_i \sim A_i' \ ; \ a_i \longleftrightarrow (a_i, i) \text{ is a bijection} \\ A_i' \cap A_j' = \emptyset \text{ because } (a_i, i) \neq (a_j, j), \text{ for } i \neq j \end{cases}$

By definition: $$\underbrace{\sum_{i \in I} \alpha_i \overset{def}{=} \left| \bigcup_{i \in I} A_i' \right|}_{} = \left| \{(a_i, i) \mid a_i \in A_i, i \in I\} \right|$$

the disjoint union of the family $(A_i)_{i \in I}$

In the above example

$A' = \{(a,1),(b,1),(c,1),(d,1),(e,1),(f,1)\}$

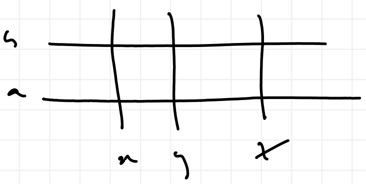
$B' = \{(x,2),(y,2),(z,2),(t,2),(f,2)\}$ $\Big\} \Rightarrow A' \cap B' = \emptyset$

even if $A \cap B = \{f\}$

2) Multiplication

$2 \cdot 3 = ?$ $\quad A = \{a, b\}$ $\quad A \times B = \{(a,x),(a,y),(a,z),(b,x),(b,y),(b,z)\}$

$B = \{x, y, z\}$ $\quad |A| \cdot |B| \overset{def}{=} |A \times B|$



Let $\alpha_i = |A_i|$, $i \in I$ be a family of cardinal numbers

We define $\prod_{i \in I} \alpha_i \overset{def}{=} \left| \prod_{i \in I} A_i \right|$, where

$$\prod_{i \in I} A_i \overset{def}{=} \{(a_i)_{i \in I} \mid a_i \in A_i \ \forall i \in I\}$$

is called the <u>generalized cartesian product</u> of the family $(A_i)_{i \in I}$

3. Exponentiation

Let $\alpha = |A|$, $\beta = |B|$

we define $\beta^\alpha \overset{def}{=} |B^A|$, where $B^A = Hom(A, B) := \{f \mid f : A \to B\}$

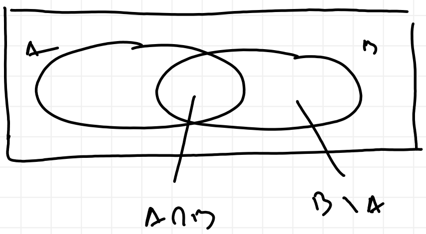
the set of functions with domain A and codomain B.

Remark We have given these definitions by using representatives for each cardinal. These definitions do not depend on the choice of representatives.

(e.g. If $A \sim A'$, $B \sim B'$, then $B^A \sim B'^{A'}$!)

Prop. 1 $|A \cup B| + |A \cap B| = |A| + |B|$

Proof.



We have: • $A \cup B = A \cup (B \setminus A)$, where $A \cap (B \setminus A) = \emptyset$

• $B = (A \cap B) \cup (B \setminus A)$, where $(A \cap B) \cap (B \setminus A) = \emptyset$

We obtain:

$$|A \cup B| + |A \cap B| = \underbrace{|A| + |B \setminus A| + |A \cap B|} = |A| + |B|$$

Prop 2 (properties of the operations)

1) Addition and multiplication are associative

2) Addition and multiplication are commutative.

3) The multiplication is distributive w.r.t. addition:

$$\left(\sum_{i \in I} \alpha_i\right)\left(\sum_{j \in J} \beta_j\right) = \sum_{(i,j) \in I \times J} \alpha_i \cdot \beta_j$$

4).
$$\left(\prod_{j \in J} \beta_j\right)^{\alpha} = \prod_{j \in J} \beta_j^{\alpha}$$

5)
$$\beta^{\sum_{i \in I} \alpha_i} = \prod_{i \in I} \beta^{\alpha_i}$$

6)
$$\gamma^{\alpha\beta} = \left(\gamma^{\beta}\right)^{\alpha}$$

Proof. 1), 2) omitted.

4), 5) are based on the the so-called "universal properties" of the cartesian product and of the disjoint union.

3). Let $\alpha_i = |A_i|$, $i \in I$ and $\beta_j = |B_j|$, $j \in J$.

• The left hand side: $\left(\sum_{i \in I} \alpha_i\right)\left(\sum_{j \in J} \beta_j\right) = |X|$, where $X := \left(\bigcup_{i\in I} (A_i \times \{i\})\right) \times \left(\bigcup_{j\in J} (B_j \times \{j\})\right) =$

$= \{ ((a_i, i), (b_j, j)) \mid a_i \in A_i, i \in I, b_j \in B_j, j \in J \}$

- The right hand side: $\sum_{(i,j)\in I\times J} \alpha_i \beta_j = |Y|$

where $Y := \bigcup_{(i,j)\in I\times J} ((A_i \times B_j) \times \{(i,j)\}) =$

$= \{ ((a_i, b_j), (i,j)) \mid a_i \in A_i, b_j \in B_j, i \in I, j \in J \}$

Then, the map $\varphi : X \to Y$, $\varphi(((a_i, i), (b_j, j))) = ((a_i, b_j), (i,j))$

is obviously bijective, hence $|X| = |Y|$

c) The left hand side: $\gamma^{\alpha\beta} = |\mathrm{Hom}(A\times B, C)|$

The right hand side: $(\gamma^\beta)^\alpha = |\mathrm{Hom}(A, \mathrm{Hom}(B, C))|$

where $\alpha = |A|$, $\beta = |B|$, $\gamma = |C|$

To show the equality, we will define the functions

$$f \in \mathrm{Hom}(A\times B, C) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathrm{Hom}(A, \mathrm{Hom}(B, C)) \ni g$$

and tell $\psi = \varphi^{-1}$

- let $f : A\times B \to C$. We want to define $\varphi(f) : A \to \mathrm{Hom}(B, C)$

i.e. $\varphi(f)(a) : B \to C$

i.e. $\varphi(f)(a)(b) \in C \quad \forall a \in A, b \in B$

We define $\underline{\varphi(f)(a)(b) \overset{def}{=} f(a,b)} \in C \quad \forall a \in A, b \in B$.

- let $g : A \to \mathrm{Hom}(B, C)$, so $g(a) : B \to C$, so $g(a)(b) \in C$

We want to define $\psi(g) : A\times B \to C$

let $\underline{\psi(g)(a,b) \overset{def}{=} g(a)(b)} \in C$

We see that $\varphi(f) = g \Leftrightarrow \psi(g) = f$

hence $\psi \circ \varphi = \mathbb{1}$, $\varphi \circ \psi = \mathbb{1}$, i.e. $\psi = \varphi^{-1}$ so φ, ψ are bij.

<u>Theorem (Cantor)</u> Let A be a set. Then $\boxed{|\mathcal{P}(A)| = 2^{|A|}}$

<u>Proof</u> We have $\mathcal{P}(A) = \{X \mid X \subseteq A\}$

$2^{|A|} = |\mathrm{Hom}(A, \{0,1\})|$

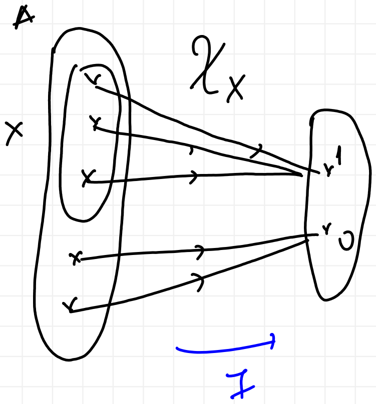
$\boxed{\chi \text{ chi}}$

We need function $X \in \mathcal{P}(A) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathrm{Hom}(A, \{0,1\}) \ni f$ s.t. $\psi = \varphi^{-1}$

We will use the <u>characteristic function of a subset</u> X of A

Let $X \subseteq A$, i.e. $X \in \mathcal{P}(A)$.

Let $\chi_X : A \to \{0,1\}$, $\chi_X(a) \overset{def}{=} \begin{cases} 1, & \text{if } a \in X \\ 0, & \text{if } a \notin X \end{cases}$



We define $\varphi(X) \overset{def}{=} \chi_X : A \to \{0,1\}$,

and $\psi(f) \overset{def}{=} f^{-1}(1) = \{a \in A \mid f(a) = 1\}$

We have:

- $(\psi \circ \varphi)(X) = \psi(\varphi(X)) = \psi(\chi_X) = \chi_X^{-1}(1) = X = \mathbb{1}_{\mathcal{P}(A)}(X)$
- $(\varphi \circ \psi)(f) = \varphi(\psi(f)) = \varphi(f^{-1}(1)) = \chi_{f^{-1}(1)} \overset{(*)}{=\!=} f = \mathbb{1}(f)$

(*) we have
$\chi_{f^{-1}(1)}(a) = \begin{cases} 1, & \text{if } a \in f^{-1}(1) \Leftrightarrow f(a) = 1 \\ 0, & \text{if } a \notin f^{-1}(1) \Leftrightarrow f(a) \neq 1 \Leftrightarrow f(a) = 0 \end{cases} = f(a)$.

Ordering cardinal numbers

<u>Def</u> Let $\alpha = |A|$, $\beta = |B|$. We say that $\alpha \leq \beta \overset{def}{\Leftrightarrow} \exists f : A \to B$ injective function

Remarks 1) The definition does not depend on the choice of representatives.

2). the rel. „≤" is an order relation:

(R). $\alpha \leq \alpha$ because $1_A : A \to A$ is injective

(T) If $\alpha \leq \beta$, $\beta \leq \gamma$, and $f: A \to B$, $g: B \to C$ are injective then $g \circ f : A \to C$ is also injective, hence $\alpha \leq \gamma$

(A) Assume $\alpha \leq \beta$ and $\beta \leq \alpha$. Let $f: A \to B$ and $g: B \to A$ be injective functions. In order to show that $\alpha = \beta$, we need a bijective function $h: A \to B$. The existence of h is a consequence of Cantor-Berstein-Schröder

If $A_2 \subseteq A_1 \subseteq A_0$ and $A_2 \sim A_0$, then $A_1 \sim A_0$

3). The order rel. „≤" is total:

If $\alpha = |A|$ and $\beta = |B|$ then $\exists f: A \to B$ inj or $\exists g: B \to A$ inj., i.e. $\alpha \leq \beta$ or $\beta \leq \alpha$

The proof uses Zorn's lemma (i.e. requires the Axiom of Choice)

Theorem (Cantor) $\boxed{\alpha < 2^\alpha}$

Proof. Let $\alpha = |A|$. We know that $2^\alpha = |\mathcal{P}(A)|$

The function $\varphi: A \to \mathcal{P}(A)$, $\varphi(a) = \{a\}$ is obviously injective, hence $\alpha \leq 2^\alpha$

We need to prove that there is no bijective function $\psi: A \to \mathcal{P}(A)$

Assume by contradiction, that $\psi: A \to \mathcal{P}(A)$ is a bij. function

Let $X = \{a \in A \mid a \notin \psi(a)\} \in \mathcal{P}(A)$

Because ψ is bij, $\exists x \in A$ s.t. $\psi(x) = X$.

We analyze two cases:

Case 1 $x \in X \Rightarrow x \in \psi(x) \Rightarrow x$ does not satisfy the cond in the def of $X \Rightarrow x \notin X$

Case 2 $x \notin X \Rightarrow x \notin \psi(x) \Rightarrow x$ satisfies the cond in the def of $X \Rightarrow x \in X$

We get contrad., hence ψ cannot be bijective

| HW. ex 94 |