Seminar 8

1. Show that the sets $\mathbb{Z}_n$ (residue classes modulo n), $M_n(\mathbb{R})$ (matrices $n \times n$) and $\mathbb{R}[X]$ (polynomials) form rings together with the corresponding addition and multiplication. Are they commutative rings, integral domains, division rings or fields? Generalization.

2. Show that the set $\mathbb{R}^{\mathbb{R}}$ of functions $f : \mathbb{R} \to \mathbb{R}$ forms a ring together with the addition and the multiplication defined by: $\forall f, g \in \mathbb{R}^{\mathbb{R}}$, $(f+g)(x) = f(x) + g(x)$, $(f \cdot g)(x) = f(x) \cdot g(x)$, $\forall x \in \mathbb{R}$. Is it a commutative ring, integral domain, division ring or field? Generalization.

3. Let $(G, +)$ be an abelian group. Show that $(\mathrm{End}(G), +, \circ)$ is a ring with identity.

4. Let $(R, +, \cdot)$ be a ring. Consider on the set $\mathbb{Z} \times R$ the addition and the multiplication defined by:

$$\begin{aligned} (m, a) + (n, b) &= (m + n, a + b), \\ (m, a) \cdot (n, b) &= (mn, ab + na + mb), \end{aligned}$$

$\forall (m, a), (n, b) \in \mathbb{Z} \times R$. Show that $(\mathbb{Z} \times R, +, \cdot)$ is a ring with identity.

5. Let $n \in \mathbb{N}$, $n \geq 2$ and $\widehat{0} \neq \widehat{a} \in \mathbb{Z}_n$. Prove that:

$$\widehat{a} \text{ is invertible in the ring } (\mathbb{Z}_n, +, \cdot) \Longleftrightarrow (a, n) = 1.$$

When is $(\mathbb{Z}_n, +, \cdot)$ a field?

6. Solve the following equations in the ring $(\mathbb{Z}_{12}, +, \cdot)$: $\widehat{4}x + \widehat{5} = \widehat{9}$ and $\widehat{5}x + \widehat{5} = \widehat{9}$.

7. Solve the following system of equations in the ring $(\mathbb{Z}_{12}, +, \cdot)$:

$$\begin{cases} \widehat{3}x + \widehat{4}y = \widehat{11} \\ \widehat{4}x + \widehat{9}y = \widehat{10} \end{cases}.$$

8. Solve the equation $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} X = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ in the ring $(M_2(\mathbb{C}), +, \cdot)$.

9. Let $\mathcal{M} = \left\{ \begin{pmatrix} a & b \\ 2b & a \end{pmatrix} \middle| a, b \in \mathbb{Q} \right\} \subseteq M_2(\mathbb{Q})$. Show that $\mathcal{M}$ is a stable subset of the ring $(M_2(\mathbb{Q}), +, \cdot)$ and $(\mathcal{M}, +, \cdot)$ is a field.

10. Let $\mathcal{M} = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \middle| a, b \in \mathbb{R} \right\} \subseteq M_2(\mathbb{R})$. Show that $\mathcal{M}$ is a stable subset of the ring $(M_2(\mathbb{R}), +, \cdot)$ and $(\mathcal{M}, +, \cdot)$ is a field.