Quadrature formulas (1p.)

A

1. Find the degree of exactness of the following quadrature formula: $\int_{-1}^{1} f(x)\,dx \approx f(-1)+f(1)$.
2. Let $\int_{-1}^{1} f(x)\,dx \approx af(-1)+bf(0)+cf(1)$. Find the values of a, b, c for which the degree of exactness d is maximum.
3. Find $n \in \mathbb{N}$ such that $\int_0^2 \frac{1}{x+4}\,dx$ is approximated by the repeated Simpson formula with precision $\epsilon = 10^{-5}$.
4. Approximate
$$\int_0^2 \frac{2}{x^2+4}\,dx$$
using the repeated trapezium formula with $n = 2$.

B

1. Find the degree of exactness of the following quadrature formula: $\int_{-1}^{1} f(x)\,dx \approx f\left(-\frac{1}{\sqrt{3}}\right)+f\left(\frac{1}{\sqrt{3}}\right)$.
2. Let $\int_{0}^{1} f(x)\,dx \approx af(0)+bf(0.5)+cf(1)$. Find the values of a, b, c for which the degree of exactness d is maximum.
3. Find $n \in \mathbb{N}$ such that $\int_0^2 \frac{1}{x+4}\,dx$ is approximated by the repeated trapezium formula with precision $\epsilon = 10^{-5}$.
4. Approximate
$$\int_0^2 \frac{2}{x^2+4}\,dx$$
using the repeated Simpson formula with $n = 2$.

Direct and iterative methods for solving linear systems (0.75p)

A

Consider the system $A\mathbf{x} = b$, where

$$A = \begin{pmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 1 & 3 & 4 & 4 \\ 0 & 2 & 1 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 4 \\ 15 \\ 3 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

Solve the system using:

1. Gauss elimination with **partial pivoting**;
2. 2 iterations of Jacobi method with $\mathbf{x}^{(0)} = (0\ 0\ 0\ 0)^T$;
3. 2 iterations of Gauss-Seidel method with $\mathbf{x}^{(0)} = (0\ 0\ 0\ 0)^T$.

$\boxed{\textbf{B}}$

Consider the system $A\mathbf{x} = b$, where

$$A = \begin{pmatrix} 2 & 0 & 4 & 1 \\ 0 & 2 & 4 & 1 \\ 2 & 4 & 3 & 0 \\ 1 & 2 & 0 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} -2 \\ -4 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

Solve the system using:

1. Gauss elimination with **partial pivoting**;
2. 2 iterations of Jacobi method with $\mathbf{x}^{(0)} = (1\ 1\ 1\ 1)^T$;
3. 2 iterations of Gauss-Seidel method with $\mathbf{x}^{(0)} = (1\ 1\ 1\ 1)^T$.

Below you have an example of how we apply iterative methods to solve a linear system (i.e., to obtain an approximation of the solution). Please, read Course 9 for the theory.

Example. Determine the approximate solution for the system

$$\begin{cases} 5x_1 + x_2 - x_3 = 7 \\ x_1 + 5x_2 + x_3 = 7 \\ x_1 + x_2 + 5x_3 = 7 \end{cases}$$

with the initial approximation $x^{(0)} = (0, 0, 0)^T$ using

a) Jacobi method in 2 steps ;
To apply the method, we have to express the unknown x_j from the equation j with respect to the other unknowns. So, we have

$$\begin{cases} x_1 = \dfrac{7 - x_2 + x_3}{5} \\ x_2 = \dfrac{7 - x_1 - x_3}{5} \\ x_3 = \dfrac{7 - x_1 - x_2}{5} \end{cases} \tag{1}$$

Now, the Jacobi method consists in expressing $x^{(k)}$ (the unknown x at step k) using the previous approximations $x^{(k-1)}$. We have:

$$\begin{cases} x_1^{(1)} = \dfrac{7 - x_2^{(0)} + x_3^{(0)}}{5} = \dfrac{7 - 0 + 0}{5} = \dfrac{7}{5} = 1.4 \\ x_2^{(1)} = \dfrac{7 - x_1^{(0)} - x_3^{(0)}}{5} = \dfrac{7 - 0 - 0}{5} = \dfrac{7}{5} = 1.4 \\ x_3^{(1)} = \dfrac{7 - x_1^{(0)} - x_2^{(0)}}{5} = \dfrac{7 - 0 - 0}{5} = \dfrac{7}{5} = 1.4 \end{cases}$$

Next, on the second iteration we have:

$$\begin{cases} x_1^{(2)} = \dfrac{7 - x_2^{(1)} + x_3^{(1)}}{5} = \dfrac{7 - \frac{7}{5} + \frac{7}{5}}{5} = \dfrac{7}{5} = 1.4 \\ x_2^{(2)} = \dfrac{7 - x_1^{(1)} - x_3^{(1)}}{5} = \dfrac{7 - \frac{7}{5} - \frac{7}{5}}{5} = \dfrac{21}{25} = 0.84 \\ x_3^{(2)} = \dfrac{7 - x_1^{(1)} - x_2^{(1)}}{5} = \dfrac{7 - \frac{7}{5} - \frac{7}{5}}{5} = \dfrac{21}{25} = 0.84 \end{cases}$$

b) Gauss-Seidel method in 2 steps ;

The difference between Jacobi and Gauss-Seidel is that in this case, we have to replace the unknowns with their most recent approximations. So, if we are at the step k, when we compute $x_3^{(k)}$, we won't use x_1 and x_2 from the previous step ($x_1^{(k-1)}$, $x_2^{(k-1)}$), but instead we will use their values from the current step, since we have already determined them. Using again (1), we obtain

$$\begin{cases} x_1^{(1)} = \dfrac{7 - x_2^{(0)} + x_3^{(0)}}{5} = \dfrac{7 - 0 + 0}{5} = \dfrac{7}{5} = 1.4 \\ x_2^{(1)} = \dfrac{7 - x_1^{(1)} - x_3^{(0)}}{5} = \dfrac{7 - \frac{7}{5} - 0}{5} = \dfrac{28}{25} = 1.12 \\ x_3^{(1)} = \dfrac{7 - x_1^{(1)} - x_2^{(1)}}{5} = \dfrac{7 - \frac{7}{5} - \frac{28}{25}}{5} = \dfrac{112}{125} = 0.896 \end{cases}$$

Next, we have:

$$\begin{cases} x_1^{(2)} = \dfrac{7 - x_2^{(1)} + x_3^{(1)}}{5} = \dfrac{7 - \frac{28}{25} + \frac{112}{125}}{5} = \dfrac{847}{625} = 1.3552 \\ x_2^{(2)} = \dfrac{7 - x_1^{(2)} - x_3^{(1)}}{5} = \dfrac{7 - \frac{847}{625} - \frac{112}{125}}{5} = \dfrac{2968}{3125} = 0.94976 \\ x_3^{(2)} = \dfrac{7 - x_1^{(2)} - x_2^{(2)}}{5} = \dfrac{7 - \frac{847}{625} - \frac{2968}{3125}}{5} = \dfrac{14672}{15625} = 0.939008 \end{cases}$$