Seminar 1

1. Which ones of the usual symbols of addition, subtraction, multiplication and division define an operation (composition law) on the numerical sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$?

2. What algebraic structures with one operation (groupoid, semigroup, monoid or group) are the numerical sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ together with addition or multiplication?

3. Give examples of:
(i) a groupoid which is not a semigroup.
(ii) a semigroup which is not a monoid.
(iii) a monoid which is not a group.

4. Give example of a groupoid with identity element in which there exists an element having two different symmetric elements.

5. Let $A = \{a_1, a_2, a_3\}$ be a set. Determine the number of:
(i) operations on A;
(ii) commutative operations on A;
(iii) operations on A with identity element.
Generalization for a set A with n elements ($n \in \mathbb{N}^*$).

6. Let "$*$" be the operation on $\mathbb{R}$ defined by:

$$x * y = x + y + xy.$$

Show that:
(i) $(\mathbb{R}, *)$ is a commutative monoid.
(ii) The interval $[-1, \infty)$ is a stable subset of $(\mathbb{R}, *)$.

7. Let "$*$" be the operation on $\mathbb{N}$ defined by $x * y = \text{g.c.d.}(x, y)$.
(i) Prove that $(\mathbb{N}, *)$ is a commutative monoid.
(ii) Show that $D_n = \{x \in \mathbb{N} \mid x/n\}$ ($n \in \mathbb{N}^*$) is a stable subset of $(\mathbb{N}, *)$ and $(D_n, *)$ is a commutative monoid.
(iii) Fill in the table of the operation "$*$" on D_6.

8. Determine the finite stable subsets of $(\mathbb{Z}, \cdot)$.

9. Let A be a set and let $\mathcal{P}(A)$ be the power set of A (that is, the set of all subsets of A). What algebraic structure with one operation (groupoid, semigroup, monoid or group) is $\mathcal{P}(A)$ together with the operation "$\cup$" or "$\cap$"?

10. Let $(A, \cdot)$ be a groupoid and $X, Y \subseteq A$. Let "$\cdot$" be the operation on the power set $\mathcal{P}(A)$ defined by:

$$X \cdot Y = \{x \cdot y \mid x \in X, y \in Y\}.$$

Show that:
(i) If $(A, \cdot)$ is commutative, then $(\mathcal{P}(A), \cdot)$ is commutative.
(ii) If $(A, \cdot)$ is a semigroup, then $(\mathcal{P}(A), \cdot)$ is a semigroup.
(iii) If $(A, \cdot)$ is a monoid, then $(\mathcal{P}(A), \cdot)$ is a monoid.
(iv) If $(A, \cdot)$ is a group, then in general $(\mathcal{P}(A), \cdot)$ is not a group (for $A \neq \emptyset$).