Seminar 2

Solvable first order differential equations

1) Linear diff. equations

General form: $\boxed{y' + P(x).y = Q(x)}$, P, Q are cont. functions.

$$L : C^1[a,b] \to C[a,b] \quad \text{linear operator}$$
$$y \mapsto y' + P(x).y$$

$Ly = Q$ nonhomogeneous equation

$Ly = 0$ homogeneous equation

general solution: $y = y_0 + y_p$ where:

y_0 – the general sol. of the homogeneous eq.
$$y' + P.y = 0$$

y_p – a particular sol. of the nonhomogeneous eq $y' + P.y = Q$.

can be found using the variation of the constant method (Lagrange method)

Exercise 1: Find the gen. sol. of the following eq.:

a) $y' + y \cdot tg(x) = \frac{1}{\cos(x)}$

b) $y' + \frac{2}{x} \cdot y = x^3$

c) $y' + 2x \cdot y = 2x e^{-x^2}$

d) $xy' - y + x = 0$

e) $y' - y = \sin x$

f) $y' + \frac{x}{1-x^2} \cdot y = x + \arcsin x$

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a) $y' + y \cdot tg x = \frac{1}{\cos x}$ $\qquad y = y_0 + y_p$

$y' + y \cdot tg x = 0$ the homogeneous eq.

$\boxed{y' = -tg x \cdot y}$ sep. diff. eq.

$\frac{dy}{dx} = -tg x \cdot y \Rightarrow \frac{dy}{y} = -tg x \cdot dx \Rightarrow \int \frac{dy}{y} = -\int tg x \, dx$

$\ln y = \ln|\cos x| + \ln c$.

$-\int tg x\, dx = -\int \frac{\sin x}{\cos x} dx =$

$= \int \frac{(\cos x)'}{\cos x} dx$

$\boxed{y_0(x) = c \cdot \cos x \quad , c \in \mathbb{R}}$

the gen. sol. of the homog. eq.

$y_p(x) = ?$ $\quad y' + y \cdot tg x = \frac{1}{\cos x}$:

$y_p(x) = c(x) \cdot \cos x$: $\quad y_p' + y_p \cdot tg x = \frac{1}{\cos x}$

$c'(x) \cdot \cos x + \cancel{c(x)(-\sin x)} + \cancel{c(x) \cdot \cos x \cdot tg x} = \frac{1}{\cos x}$

$\frac{\sin x}{\cancel{\cos x}}$

$\Rightarrow c'(x) \cdot \cos x = \frac{1}{\cos x} \Rightarrow c'(x) = \frac{1}{\cos^2 x} \Rightarrow$

$\Rightarrow c(x) = \int \frac{1}{\cos^2 x} dx = tg x$

$\Rightarrow y_p(x) = c(x) \cdot \cos x = tg x \cdot \cos x = \frac{\sin x}{\cancel{\cos x}} \cdot \cancel{\cos x} = \sin x$.

the general sol: $y = y_0 + y_p$

$\boxed{y(x) = c \cdot \cos x + \sin x \quad , c \in \mathbb{R}}$

b) $y' + \underbrace{\frac{2}{x}}_{P(x)} \cdot y = \underbrace{x^3}_{Q(x)}$ nonhomogeneous diff. eq.

$\left(y(x) \equiv 0 \text{ sing. sol.} \right)$

$\boxed{y' + \frac{2}{x} \cdot y = 0}$ homog. eq. $\Rightarrow y' = -\frac{2}{x} \cdot y \Rightarrow \frac{dy}{dx} = -\frac{2}{x} \cdot y$

$\Rightarrow \frac{dy}{y} = -\frac{2}{x} dx \Rightarrow \int \frac{dy}{y} = -\int \frac{2}{x} dx$

$\ln y = -2 \ln x + \ln c$

$y(x) = c \cdot x^{-2} \Rightarrow \boxed{y_0(x) = \frac{c}{x^2}, \ c \in \mathbb{R}}$;

$y_p(x) = ?$ $\boxed{y_p(x) = \frac{c(x)}{x^2}}$: $y_p' + \frac{2}{x} \cdot y_p = x^3$

$\Rightarrow$ $\left(c(x) \cdot \frac{1}{x^2} \right)' + \frac{2}{x} \cdot c(x) \cdot \frac{1}{x^2} = x^3$

$c'(x) \cdot \frac{1}{x^2} + \cancel{c(x) \cdot \frac{(-2)}{x^3}} + \cancel{\frac{2}{x} \cdot c(x) \cdot \frac{1}{x^2}} = x^3$

$\left(\frac{1}{x^2} \right)' = (x^{-2})' = (-2) \cdot x^{-2-1}$

$\Rightarrow c'(x) \cdot \frac{1}{x^2} = x^3 \Rightarrow c'(x) = x^5 \Rightarrow c(x) = \int x^5 dx = \frac{x^6}{6}$

$\Rightarrow \left| y_p(x) = \frac{c(x)}{x^2} = \frac{x^6}{6} \cdot \frac{1}{x^2} = \frac{x^4}{6} \right] \Rightarrow$ the general sol.

$\boxed{y(x) = \frac{c}{x^2} + \frac{x^4}{6}, \ c \in \mathbb{R}}$

c) $\boxed{y' + 2x \cdot y = 2x \cdot e^{-x^2}}$ $\qquad \boxed{y = y_0 + y_p}$?

$y' + 2x \cdot y = 0$ homog. eq.

$y' = -2x \cdot y \Rightarrow \frac{dy}{dx} = -2x \cdot y \Rightarrow \int \frac{dy}{y} = \int -2x \cdot dx \Rightarrow$

$\Rightarrow \ln y = -x^2 + \ln c \Rightarrow \boxed{y_0(x) = c \cdot e^{-x^2}, \ c \in \mathbb{R}}$

$e^{\ln y} = e^{-x^2 + \ln c}$

$\underbrace{e^{-x^2} \cdot e^{\ln c}}$

$y_p(x) = ?$ $\qquad y_p(x) = c(x) \cdot e^{-x^2}$

$y_p' + 2x \cdot y_p = 2x \cdot e^{-x^2} \Rightarrow c'(x) \cdot e^{-x^2} + \cancel{c(x) \cdot e^{-x^2}(-2x)} +$

$+ \cancel{2x \cdot c(x) \cdot e^{-x^2}} = 2x \cdot e^{-x^2}$

$\Rightarrow c'(x) \cdot \cancel{e^{-x^2}} = 2x \cancel{e^{-x^2}}$

$\Rightarrow c'(x) = 2x \Rightarrow c(x) = x^2$

$\Rightarrow y_p(x) = c(x) \cdot e^{-x^2} = x^2 \cdot e^{-x^2}$

the general solution: $y = y_0 + y_p$

$$\boxed{y(x) = c \cdot e^{-x^2} + x^2 \cdot e^{-x^2}, \ c \in \mathbb{R}}$$

Exercise 2 : Solve the following (IVPs) :

a) $\begin{cases} xy' + y = e^x \\ y(a) = b \end{cases}$, $a, b \in \mathbb{R}$, $a \neq 0$

b) $\begin{cases} y' + (2x^2 - 1) . y = 2x^2 - 1 \\ y(1) = 1 - \frac{1}{e} \end{cases}$

...

c) $\begin{cases} y' - 2y = -x^2 \\ y(0) = \frac{1}{4} \end{cases}$

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a) $xy' + y = e^x \mid \cdot \frac{1}{x}$

$\boxed{y' + \frac{1}{x} y = \frac{e^x}{x}}$

$y' + \frac{1}{x} y = 0 \Rightarrow y' = -\frac{1}{x} y$

$\frac{dy}{dx} = -\frac{1}{x} y \Rightarrow \int \frac{dy}{y} = \int -\frac{1}{x} dx$

$\Rightarrow \ln y = -\ln x + \ln c \Rightarrow \boxed{y_0(x) = c . x^{-1} = \frac{c}{x}, \quad c \in \mathbb{R}}$

$y_p(x) = ?$ $\quad \boxed{y_p(x) = \frac{c(x)}{x}}$

$\Rightarrow y_p' + \frac{1}{x} y_p = \frac{e^x}{x} \Rightarrow \frac{c'(x)}{x} - \cancel{\frac{c(x)}{x^2}} + \cancel{\frac{1}{x} . \frac{c(x)}{x}} = \frac{e^x}{x}$

$\Rightarrow \frac{c'(x)}{x} = \frac{e^x}{x} \Rightarrow c'(x) = e^x \Rightarrow c(x) = e^x$

$\Rightarrow y_p(x) = \frac{e^x}{x} \Rightarrow$ the general sol : $y = y_0 + y_p$

$\boxed{y(x) = \frac{c}{x} + \frac{e^x}{x}, \quad c \in \mathbb{R}}$ ⋮

$y(a)=b \Rightarrow \frac{c+e^a}{a}=b \Rightarrow c+e^a=ab \Rightarrow \boxed{c=ab-e^a}$

$\Rightarrow$ the IVP solution:

$$\boxed{y(x)=\frac{ab-e^a+e^x}{x}}$$

2. The Bernoulli equations

the general form: $\boxed{y'+P(x).y=Q(x).y^{\alpha}}$, $\alpha\neq 0$ and $\alpha\neq 1$

subst: $z=y^{1-\alpha}$ $\qquad z(x)=y(x)^{1-\alpha}$

$$\Downarrow$$

$$\boxed{y=z^{\frac{1}{1-\alpha}}} \Rightarrow y'=\frac{1}{1-\alpha}\cdot z^{\frac{1}{1-\alpha}-1}\cdot z'$$

$$y'=\frac{1}{1-\alpha}\cdot z^{\frac{\alpha}{1-\alpha}}\cdot z'$$

$$\Rightarrow \frac{1}{1-\alpha}\cdot z^{\frac{\alpha}{1-\alpha}}\cdot z'+P\cdot z^{\frac{1}{1-\alpha}}=Q\cdot z^{\frac{\alpha}{1-\alpha}} \quad \Big|\,(1-\alpha)\cdot z^{-\frac{\alpha}{1-\alpha}}$$

$$\Rightarrow z'+(1-\alpha)\cdot P\cdot z^{\frac{1}{1-\alpha}-\frac{\alpha}{1-\alpha}}=(1-\alpha)\cdot Q.$$

$$\Rightarrow \boxed{z'+(1-\alpha)\cdot P\cdot z=(1-\alpha)\,Q}$$ nonhomogeneous linear eq.

$\Rightarrow z=z_0+z_p \Rightarrow y=(z_0+z_p)^{\frac{1}{1-\alpha}}$

Exercise 3 Solve the eq.:

a) $xy' = 2x^2\sqrt{y} + 4y$

b) $y' = y\cos x + y^2\cos x$

c) $xy' + y + x^5 y^3 \cdot e^x = 0$

d) $y' + \frac{y}{x} = \frac{1}{x^2y^2}$

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a) $xy' - 4y = 2x^2\sqrt{y} \quad |:x$

$$\boxed{y' - \frac{4}{x}y = 2x \cdot \sqrt{y}} \qquad \sqrt{y} = y^{\frac{1}{2}} \quad \alpha = \frac{1}{2}$$

subst: $z = y^{1-\alpha} = y^{1-\frac{1}{2}} = y^{\frac{1}{2}} = \sqrt{y}$

$z = \sqrt{y} \Rightarrow \boxed{y = z^2}$ $\qquad y(x) = z(x)^2$

$$\boxed{y'(x) = 2 \cdot z(x) \cdot z'(x)}$$

$\Rightarrow \underbrace{2 \cdot z} \cdot z' - \frac{4}{x} \cdot z^2 = 2x \cdot z \quad |:2z$

$$\boxed{z' - \frac{2}{x}z = x}$$

$$z' - \frac{2}{x} z = 0 \Rightarrow z' = \frac{2}{x} z \Rightarrow \frac{dz}{dx} = \frac{2}{x} \cdot z \Rightarrow \int \frac{dz}{z} = \int \frac{2}{x} \cdot dx$$

$$\Rightarrow \ln z = 2 \cdot \ln x + \ln c \Rightarrow \boxed{z_0(x) = c \cdot x^2, \ c \in \mathbb{R}}$$

$$z_p(x) = c(x) \cdot x^2$$

$$z_p' - \frac{2}{x} \cdot z_p = x \Rightarrow c'(x) \cdot x^2 + \cancel{c(x) \cdot 2x} - \cancel{\frac{2}{x} \cdot c(x) \cdot x^2} = x$$

$$\ldots \Rightarrow c'(x) \cdot x^2 = x \Rightarrow c'(x) = \frac{1}{x} \Rightarrow c(x) = \ln|x|$$

$$\Rightarrow z_p(x) = c(x) \cdot x^2 \Rightarrow z_p(x) = x^2 \cdot \ln|x|$$

the gen. sol. of the eq in z: $z = z_0 + z_p$

$$\boxed{z(x) = c \cdot x^2 + x^2 \cdot \ln|x|, \ c \in \mathbb{R}}$$

the gen sol of the eq in y:

$$y = z^2 \Rightarrow \boxed{y(x) = \left(c \cdot x^2 + x^2 \cdot \ln|x|\right)^2, \ c \in \mathbb{R}}$$

b) $y' = y\cos x + y^2 \cdot \cos x \quad \Rightarrow \quad y' - y \cdot \cos x = \cos x \cdot y^2 \qquad \underline{\underline{\alpha = 2}}$

$z = y^{1-\alpha} = y^{1-2} = y^{-1} \Rightarrow z = \frac{1}{y} \Rightarrow \boxed{y = \frac{1}{z}}$

$\Rightarrow y' = -\frac{1}{z^2} \cdot z'$

$\Rightarrow -\frac{1}{z^2} \cdot z' - \frac{1}{z} \cdot \cos x = \cos x \cdot \frac{1}{z^2} \quad \Big| (-z^2)$

$\Rightarrow \boxed{z' + \cos x \cdot z = -\cos x}$

$z' + \cos x \, z = 0$

$z' = -\cos x \cdot z$

$\frac{dz}{dx} = -\cos x \cdot z$

$\int \frac{dz}{z} = \int -\cos x \cdot dx$

$\ln z = -\sin x + \ln c$

$\boxed{z_0(x) = c \cdot e^{-\sin x}, \quad c \in \mathbb{R}}$

$z_p = ? \quad z_p(x) = c(x) \cdot e^{-\sin x}$

$z_p' + \cos x \cdot z_p = -\cos x$

$c'(x) \cdot e^{-\sin x} + \cancel{c(x) \cdot e^{-\sin x} \cdot (-\cos x)} +$

$+ \cancel{\cos x \cdot c(x) \cdot e^{-\sin x}} = -\cos x$

$c'(x) \cdot e^{-\sin x} = -\cos x \,/\cdot e^{\sin x}$

$c'(x) = -\cos x \cdot e^{\sin x}$

$c(x) = -\int \cos x \cdot e^{\sin x} dx =$

$c(x) = -e^{\sin x}$

$\Rightarrow z_p(x) = c(x) \cdot e^{-\sin x} = -e^{\sin x} \cdot e^{-\sin x} = -1.$

$\Rightarrow$ the gen. sol. of the eq in z:

$z = z_0 + z_p \Rightarrow \boxed{z(x) = c \cdot e^{-\sin x} - 1, \; c \in \mathbb{R}}$

the gen. sol of the eq in y:

$y = \frac{1}{z} \Rightarrow \boxed{y(x) = \frac{1}{c \cdot e^{-\sin x} - 1}, \; c \in \mathbb{R}}$