Aplicații ale Teoremei reziduurilor

Să se calculeze integralele următoare:

① $J = \int_\gamma \frac{e^{\frac{1}{z}}}{z}\, dz$, $\gamma(t) = e^{2\pi i t}$, $t \in [0,1]$.

Soluție: Fie $f(z) = \frac{e^{\frac{1}{z}}}{z}$, $z \in \mathbb{C}^*$.

$z_0 = 0$ este punct singular izolat pentru f.

$\{\gamma\} = \partial U(0,1) \subset \mathbb{C}^*$ și $\gamma \underset{\mathbb{C}}{\sim} 0$, deci putem aplica Teorema reziduurilor:

$$J = \int_\gamma f(z)\, dz = 2\pi i \operatorname{Rez}(f; 0).$$

Avem dezvoltarea în serie Laurent:

$$\frac{1}{z} e^{\frac{1}{z}} = \frac{1}{z}\left(1 + \frac{1}{1!}\cdot\frac{1}{z} + \frac{1}{2!}\cdot\frac{1}{z^2} + \dots + \frac{1}{n!}\cdot\frac{1}{z^n} + \dots\right)$$

$$= \frac{1}{z} + \frac{1}{1!}\cdot\frac{1}{z^2} + \frac{1}{2!}\cdot\frac{1}{z^3} + \dots + \frac{1}{n!}\cdot\frac{1}{z^{n+1}} + \dots, \ \forall z \in \mathbb{C}^*.$$

Deci, $\operatorname{Rez}(f; 0) = a_{-1} = 1 \Rightarrow J = 2\pi i$.

② $J_r = \int_{\gamma_r} \frac{e^{\frac{1}{z-a}}}{z}\, dz$, unde $\gamma_r(t) = r e^{2\pi i t}$, $t \in [0,1]$, $a \in \mathbb{C}^*$, $r \in (0,\infty) \setminus \{|a|\}$.

Soluție: Fie $a \in \mathbb{C}^*$ și $f(z) = \frac{e^{\frac{1}{z-a}}}{z}$, $z \in \mathbb{C} \setminus \{0, a\}$

$z_0 = 0$ este un pol de ordinul I, pentru că $f(z) = \frac{g(z)}{z-0}$, $z \in \mathbb{C} \setminus \{0, a\}$, unde $g(z) = e^{\frac{1}{z-a}}$, $z \in \mathbb{C} \setminus \{a\}$, $g \in \mathcal{H}(\mathbb{C} \setminus \{a\})$, $g(0) = e^{-\frac{1}{a}} \neq 0$.

Deci, $\operatorname{Rez}(f; 0) = \lim_{z \to 0} z \cdot f(z) = e^{-\frac{1}{a}}$.

$\lim_{z \to a} f(z) = \lim_{z \to a} \frac{1}{z} \cdot e^{\frac{1}{z-a}} = \frac{1}{a} \lim_{z \to a} e^{\frac{1}{z-a}}$ nu există:

$$\lim_{n \to \infty} e^{\frac{1}{\frac{1}{n} + a - a}} = \lim_{n \to \infty} e^{n} = \infty$$

$$\lim_{n \to \infty} e^{\frac{1}{\frac{1}{2n\pi i} + a - a}} = \lim_{n \to \infty} e^{2n\pi i} = 1.$$

Cazul I: $|a| < r \Rightarrow a \in U(0, r) = (\gamma_r)$ (the interior of γ_r).

Teorema reziduurilor $\Rightarrow J_r = 2\pi i \left(\operatorname{Rez}(f; 0) + \operatorname{Rez}(f; a) \right)$.

Cazul II: $|a| > r \Rightarrow a \notin (\gamma_r) \Rightarrow J_r = 2\pi i \cdot \operatorname{Rez}(f; 0)$.

$z_1 = a$ este punct esențial izolat pentru f.

Pentru a găsi $\operatorname{Rez}(f; a)$, considerăm dezvoltarea în serie Laurent în jurul lui $z_1 = a$:

$$e^{\frac{1}{z-a}} = 1 + \frac{1}{1!}\cdot\frac{1}{z-a} + \frac{1}{2!}\cdot\frac{1}{(z-a)^2} + \dots + \frac{1}{n!}\cdot\frac{1}{(z-a)^n} + \dots, \forall z \in \mathbb{C}\setminus\{a\}$$

$$\frac{1}{z} = \frac{1}{(z-a)+a} = \frac{1}{a}\cdot\frac{1}{1+\frac{z-a}{a}}, \forall z \in \mathbb{C}^*$$

$$\frac{1}{1+y} = 1 - y + y^2 - \dots + (-y)^n + \dots, \forall y \in U(0,1)$$

$$\Rightarrow \frac{1}{z} = \frac{1}{a}\left(1 - \frac{1}{a}\cdot(z-a) + \frac{1}{a^2}(z-a)^2 - \dots + \frac{(-1)^n}{a^n}\cdot(z-a)^n + \dots\right),$$

$$\forall z \in \mathbb{C}^*, \left|\frac{z-a}{a}\right| < 1$$

$$\Rightarrow \frac{1}{z} = \frac{1}{a} - \frac{1}{a^2}(z-a) + \frac{1}{a^3}(z-a)^2 - \dots + \frac{(-1)^n}{a^{n+1}}(z-a)^n + \dots,$$

$$\forall z \in U(a,|a|).$$

$$f(z) = \sum_{-\infty}^{\infty} a_n (z-a)^n, \forall z \in \dot{U}(a,|a|), \text{ unde}$$

$$a_{-1} = \frac{1}{1!}\cdot\frac{1}{a} - \frac{1}{2!}\cdot\frac{1}{a^2} + \dots + \frac{1}{n!}\cdot\frac{(-1)^{n-1}}{a^n} + \dots$$

$$= 1 - \left(1 + \frac{1}{1!}\left(-\frac{1}{a}\right) + \frac{1}{2!}\left(-\frac{1}{a}\right)^2 + \dots + \frac{1}{n!}\cdot\left(-\frac{1}{a}\right)^n + \dots\right)$$

$$= 1 - e^{-\frac{1}{a}}.$$

Deci, $$J_r = \begin{cases} 2\pi i\left(e^{-\frac{1}{a}} + 1 - e^{-\frac{1}{a}}\right) = 2\pi i, & |a| < r \\ 2\pi i\, e^{-\frac{1}{a}}, & |a| > r \end{cases}$$

③ $$I_r = \int_{\gamma_r} \frac{1}{(z-a)^m(z-b)^n}\,dz, \; m,n \in \mathbb{N}^*, \; a,b \in \mathbb{C}\setminus\partial U(0,r),$$
$$a \neq b, \; r > 0, \; \gamma_r(t) = r e^{2\pi i t}, \; t \in [0,1].$$

Soluție: Fie $f(z) = \frac{1}{(z-a)^m(z-b)^n}$, $z \in \mathbb{C}\setminus\{a,b\}$.

$f \in \mathcal{H}(\mathbb{C}\setminus\{a,b\})$, $z_1 = a$ și $z_2 = b$ sunt poli pentru f.

$f(z)=\frac{g(z)}{(z-a)^m}$, $z\in\mathbb{C}\setminus\{a,b\}$, unde $g(z)=\frac{1}{(z-b)^n}$, $z\in\mathbb{C}\setminus\{b\}$, $g\in\mathcal{H}(\mathbb{C}\setminus\{b\})$, $g(a)\neq 0$.

Deci, $z_1=a$ este pol de ordinul m și

$$Rez(f;a)=\frac{1}{(m-1)!}\lim_{z\to a}\left[(z-a)^m\cdot f(z)\right]^{(m-1)}=\frac{1}{(m-1)!}\lim_{z\to a}\left[(z-b)^{-n}\right]^{(m-1)}$$

$$=\frac{(-n)\cdot(-n-1)\cdot\ldots\cdot(-n-m+2)}{(m-1)!}\cdot\frac{1}{(a-b)^{m+n-1}}$$

$$=\frac{(-1)^{m-1}\cdot(m+n-2)!}{(m-1)!\,(n-1)!}\cdot\frac{1}{(a-b)^{m+n-1}}.$$

Similar, deducem

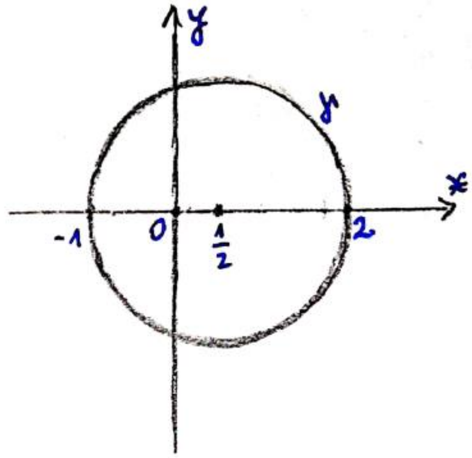
$$Rez(f;b)=\frac{(-1)^{n-1}\cdot(m+n-2)!}{(m-1)!\,(n-1)!}\cdot\frac{1}{(b-a)^{m+n-1}}$$

$$=\frac{(-1)^{m}\cdot(m+n-2)!}{(m-1)!\,(n-1)!}\cdot\frac{1}{(a-b)^{m+n-1}}.$$

Teorema reziduurilor implică

$$I_r=\begin{cases}\frac{2\pi i\,(-1)^{m-1}\cdot(m+n-2)!}{(m-1)!\,(n-1)!}\cdot\frac{1}{(a-b)^{m+n-1}}, & |a|<r,\ |b|>r\\ \frac{2\pi i\,(-1)^{m}\,(m+n-2)!}{(m-1)!\,(n-1)!}\cdot\frac{1}{(a-b)^{m+n-1}}, & |a|>r,\ |b|<r\\ 0, & (|a|>r,\ |b|>r)\ \text{sau}\ (|a|<r,\ |b|<r).\end{cases}$$

(4) $I = \int_\gamma \frac{\sin z}{z \cos z}\, dz$, unde $\gamma(t) = \frac{1}{2} + \frac{3}{2} e^{2\pi i t}$, $t \in [0,1]$.



Soluție:

$\cos z = 0 \iff \frac{e^{iz}+e^{-iz}}{2} = 0$

$\iff e^{2iz} = -1 \overset{z=x+iy}{\iff}$

$\iff e^{-2y}(\cos(2x) + i\sin(2x)) = 1(\cos\pi + i\sin\pi)$

$\iff \begin{cases} -2y = 0 \\ 2x = \pi \ (\mathrm{mod}\, 2\pi) \end{cases} \iff z \in \{ \frac{\pi}{2} + k\pi : k \in \mathbb{Z} \}$.

Rădăcina lui cos din interiorul lui γ este

$z_1 = \frac{\pi}{2}$.

Fie $r > \frac{3}{2}$ a.î. $f(z) = \frac{\sin z}{z \cos z}$, $z \in U(\frac{1}{2}, r) \setminus \{0, z_1\}$, este bine definită. Fie $g(z) = \begin{cases} \frac{\sin z}{z}, & z \in U(\frac{1}{2}, r) \\ 1, & z = 0 \end{cases}$.

$f \in \mathcal{H}(U(\frac{1}{2}, r) \setminus \{0, z_1\})$, $f(z) = \frac{g(z)}{\cos z}$, $z \in U(\frac{1}{2}, 1) \setminus \{0, z_1\}$

$g \in \mathcal{H}(U(\frac{1}{2}, r))$, $g(z_1) \neq 0$ și z_1 este zero simplu pentru cos.

Deci, $z_1 = \frac{\pi}{2}$ este pol de ordinul I pentru f, iar $z_0 = 0$ este punct eliminabil.

Deci, $Rez(f;0)=0$ și

$$Rez\left(f;\frac{\pi}{2}\right)=\lim_{z\to\frac{\pi}{2}}\left(z-\frac{\pi}{2}\right)\cdot f(z)=\lim_{z\to\frac{\pi}{2}}\frac{\sin z}{z}\cdot\frac{1}{\frac{\cos z-\cos\frac{\pi}{2}}{z-\frac{\pi}{2}}}=$$

$$=\frac{1}{\frac{\pi}{2}}\cdot\frac{1}{\cos'\frac{\pi}{2}}=\frac{2}{\pi}\cdot\frac{1}{-\sin\frac{\pi}{2}}=-\frac{2}{\pi}.$$

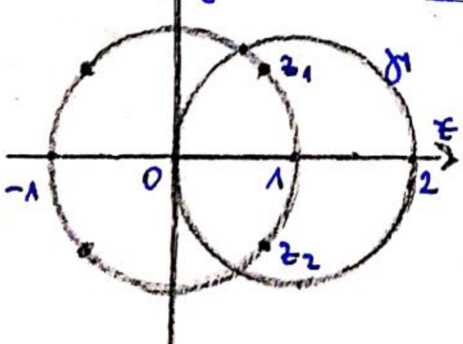
Teorema reziduurilor implică

$$I=2\pi i\left(Rez(f;0)+Rez\left(f;\frac{\pi}{2}\right)\right)$$
$$=2\pi i\left(0-\frac{2}{\pi}\right)=-4i.$$

(5) $I=\int_\gamma\frac{1}{1+z^4}\,dz$, $\gamma(t)=1+e^{2\pi it}$, $t\in[0,1]$.

Soluție: $1+z^4=0 \iff z\in\sqrt[4]{-1}\iff z\in\sqrt[4]{\cos\pi+i\sin\pi}$

$\iff z\in\left\{\cos\left(\frac{\pi}{4}+\frac{k\pi}{2}\right)+i\sin\left(\frac{\pi}{4}+\frac{k\pi}{2}\right):k=\overline{0,3}\right\}$,



Rădăcinile ecuației $1+z^4=0$ care sunt în (γ) sunt $z_1=\frac{\sqrt2}{2}+i\frac{\sqrt2}{2}$ și $z_2=\frac{\sqrt2}{2}-i\frac{\sqrt2}{2}$.

Fie $r>1$ a.î. $f(z)=\frac{1}{1+z^4}$, $z\in U(1,r)\setminus\{z_1,z_2\}$ bine definită. $f\in \mathcal{H}(U(1,r)\setminus\{z_1,z_2\})$ și z_1, z_2 sunt poli de ordinul I, fiind zerouri de ordinul I pentru numitor.

Deci, $\text{Rez}(f;z_1)=\lim\limits_{z\to z_1}(z-z_1)f(z)=\lim\limits_{z\to z_1}\frac{1}{\frac{1+z^4}{z-z_1}}$

$$=\frac{1}{4z_1^3}=-\frac{z_1}{4} \text{ și}$$

$$\text{Rez}(f;z_2)=-\frac{z_2}{4}.$$

Teorema reziduurilor implică

$J=2\pi i\left(\text{Rez}(f;z_1)+\text{Rez}(f;z_2)\right)=\frac{\pi i}{2}(-z_1-z_2)=-\frac{\pi i}{\sqrt{2}}$.

⑥ $J=\int_0^{2\pi}\frac{1}{2+\sin x}\,dx$.

Soluție: $J=\int_\gamma \frac{1}{2+\frac{z-\frac{1}{z}}{2i}}\cdot\frac{1}{iz}\,dz$, unde $\gamma(t)=e^{2\pi i t}, t\in[0,1]$.

$$J=\int_\gamma \frac{2iz}{z^2+4iz-1}\cdot\frac{1}{iz}\,dz=\int_\gamma\frac{2}{z^2+4iz-1}\,dz.$$

Teorema reziduurilor $\Rightarrow J=2\pi i\sum\limits_{|z|<1}\text{Rez}(f;z)$, unde $f(z) = \frac{2}{z^2+4iz-1}$, $z \in \mathbb{C} \setminus \{z_1, z_2\}$,

$z_{1,2}$ sunt rădăcinile ecuației $z^2+4iz-1=0$

$\Delta = -16+4 = -12$

$z_{1,2} = \frac{-4i \pm 2i\sqrt{3}}{2}$

$z_{1,2} = (-2 \pm \sqrt{3})i$.

Deoarece doar $z_1 = (-2+\sqrt{3})i \in U(0,1)$, avem

$J = 2\pi i \, \text{Rez}(f; (-2+\sqrt{3})i)$.

$z_1 = (-2+\sqrt{3})i$ este pol de ordinul $\underline{I}$ pentru f,

deci $\text{Rez}(f; z_1) = \lim_{z \to z_1} (z-z_1) \cdot f(z) =$

$= \lim_{z \to z_1} \frac{2}{\frac{z^2+4iz-1}{z-z_1}} = \lim_{z \to z_1} \frac{2}{z-z_2}$

$= \frac{2}{z_1 - z_2} = \frac{2}{2\sqrt{3}i} = \frac{-i}{\sqrt{3}}$.

$J = 2\pi i \cdot \frac{-i}{\sqrt{3}} = \frac{2\sqrt{3}\pi}{3}$.