Training problems for the midterm test

1. Let $f : \mathbb{C} \to \mathbb{C}$,

$$f(z) = |z-1|^2(\overline{z}-1) - 2\overline{z}, \quad z \in \mathbb{C}.$$

a) Compute $f(1+i)$ and $f(\frac{1}{i})$.
b) Find the set $S = \{z \in \mathbb{C} : f \text{ is differentiable at } z\}$ and represent it graphically in the complex plane.
c) Compute $f'(z)$ for $z \in S$.
d) Represent graphically in the complex plane the set

$$\{z \in S : |f'(z)| < |z+1|^2 - 4\}.$$

2. Let $f : \mathbb{C} \to \mathbb{C}$, $f(z) = e^{-4(x^2+y^2)} \cdot (x - iy)$, $z = x + iy \in \mathbb{C}$. Represent graphically in the complex plane the set of all $z \in \mathbb{C}$ such that f is differentiable at z and compute the derivative at these points.

3. Let $f : \mathbb{C} \setminus \{1\} \to \mathbb{C}$, $f(z) = \dfrac{3z + i - 2}{z-1}$, $z \in \mathbb{C} \setminus \{1\}$. Represent graphically in the complex plane the set $A \cap B$, where

$$A = \left\{ z \in \mathbb{C} \setminus \{1\} : |f'(z)| > \sqrt{2} \right\}$$
$$B = \left\{ z \in \mathbb{C} \setminus \{0\} : 0 < \arg z < \frac{\pi}{2} \right\}.$$

4. Find $\lim\limits_{n\to\infty} z_n$, where the sequence $(z_n)_{n\in\mathbb{N}^*}$ is given by

$$z_n = \left(\frac{1}{2} + \frac{i\pi}{n}\right)^n + \left(1 + \frac{\pi}{in}\right)^n, \; n \in \mathbb{N}^*.$$

5. Find $\lim\limits_{n\to\infty} z_n$, where the sequence $(z_n)_{n\in\mathbb{N}^*}$ is given by

$$z_n = n^2 \cdot \left(\frac{1}{2} + \frac{2i}{3}\right)^n + \left(1 + \frac{\pi}{2in}\right)^{3n}, \; n \in \mathbb{N}^*.$$

6. Solve in $\mathbb{C}$ the equation $\sin z = i$.

7. Solve in $\mathbb{C}$ the equation: $3\cos z - 5i \sin z = 4i$.

8. Find all $f \in \mathcal{H}(\mathbb{C})$ such that

$$|f(z)| = e^{xy}, \forall z = x + iy \in \mathbb{C}.$$