COURSES 8+9

Vector spaces, subspaces

Let $(K,+,\cdot)$ be a field. Throughout this course this condition on K will always be valid.

Definition 1. Let K be a field. A **vector space over** K (or a K**-vector space**) is an Abelian group $(V,+)$ together with an external operation

$$\cdot : K \times V \to V\,, \quad (k,v) \mapsto k \cdot v\,,$$

satisfying the following axioms: for any $k, k_1, k_2 \in K$ and any $v, v_1, v_2 \in V$,
(L_1) $k \cdot (v_1 + v_2) = k \cdot v_1 + k \cdot v_2$;
(L_2) $(k_1 + k_2) \cdot v = k_1 \cdot v + k_2 \cdot v$;
(L_3) $(k_1 \cdot k_2) \cdot v = k_1 \cdot (k_2 \cdot v)$;
(L_4) $1 \cdot v = v$.
In this context, the elements of K are called **scalars**, the elements of V are called **vectors** and the external operation is called **scalar multiplication**. Sometimes a vector space is also called **linear space**.

We denote the fact that V is a vector space over K either by ${}_KV$ or by $(V,K,+,\cdot)$, since for a given field K, the addition on V and the external operation are the operations that determine the vector space structure of V.

Remark 2. In the definition of a vector space appear four operations, two denoted by the same symbol $+$ and two denoted by the same symbol $\cdot$. Of course, most of the time they are not the same, but we denote them identically for the sake of simplicity of writing. The nature of the elements involved when using these symbols tells us which is the operation. More precisely, if $+$ appears between two vectors, then it is the addition from V, if it appears between two scalars, it is the addition from K; if $\cdot$ appears between a scalar and a vector, then it is the scalar multiplication, otherwise, it appears between to scalars, hence it is the multiplication from K.

Examples 3. (a) If $V = \{0\}$ is a single element set, then we know that there is a unique Abelian group structure on V, defined by $0 + 0 = 0$. There is also a unique scalar multiplication, namely

$$k \cdot 0 = 0,\ \forall k \in K.$$

Thus, V is a vector space, called the **zero (null) vector space** and denoted by $\{0\}$.
(b) Let $n \in \mathbb{N}^*$ and

$$K^n = \{(x_1,\ldots,x_n) \mid x_i \in K,\ i = \{1,\ldots,n\}\}.$$

Define for any $(x_1,\ldots,x_n),(y_1,\ldots,y_n) \in K^n$ and for any $k \in K$,

$$(x_1,\ldots,x_n) + (y_1,\ldots,y_n) = (x_1 + y_1,\ldots,x_n + y_n)\,,$$

$$k \cdot (x_1,\ldots,x_n) = (kx_1,\ldots,kx_n)\,.$$

Then K^n is a K-vector space.

For $n = 1$, we get that ${}_KK$ is a vector space (in particular, ${}_{\mathbb{Q}}\mathbb{Q}$, ${}_{\mathbb{R}}\mathbb{R}$ and ${}_{\mathbb{C}}\mathbb{C}$ are vector spaces).

(c) Let A be a subfield of the field K. Then K is a vector space over A, where the addition and the scalar multiplication are just the addition and the multiplication of elements in the field K.

In particular, ${}_{\mathbb{Q}}\mathbb{R}$, ${}_{\mathbb{Q}}\mathbb{C}$ and ${}_{\mathbb{R}}\mathbb{C}$ are vector spaces.
(d) Let V_2 be the set of all vectors (in the classical sense) in the plane with a fixed origin O. Then V_2 is a vector space over $\mathbb{R}$ (or a *real vector space*), where the addition is the usual addition of two vectors by the parallelogram rule and the external operation is the usual scalar multiplication of vectors by real scalars.

If we consider two coordinate axes Ox and Oy in the plane, each vector in V_2 is perfectly determined by the coordinates of its ending point. Therefore, the addition of vectors and the scalar multiplication of vectors by real numbers become:

$$(x, y) + (x', y') = (x + x', y + y'),$$

$$k \cdot (x, y) = (k \cdot x, k \cdot y).$$

Thus, one can identify the vector space $(V_2, \mathbb{R}, +, \cdot)$ with the vector space $(\mathbb{R}^2, \mathbb{R}, +, \cdot)$.

Similarly, one can consider the real vector space V_3 of all vectors in the space with a fixed origin and this vector space can be seen as the real vector space $\mathbb{R}^3$.
(e) Let $m, n \in \mathbb{N}^*$. The Abelian group $(M_{m,n}(K), +)$ of the $m \times n$ matrices over K is a K-vector space with the scalar multiplication

$$\alpha(a_{ij}) = (\alpha a_{ij}) \ (\alpha \in K,\ (a_{ij}) \in M_{m,n}(K)).$$

Let us notice that for $n \times n$ square matrices, besides the K-vector space structure, $M_n(K)$ also has a ring structure. Moreover, there is a certain connection between the scalar multiplication and the matrix multiplication given by

$$\alpha(A\,B) = (\alpha A)B = A(\alpha B),\ \forall \alpha \in K,\ \forall A, B \in M_n(K).$$

(f) $(K[X], K, +, \cdot)$ is a vector space, where the addition is the usual addition of polynomials and the scalar multiplication is defined as follows: if $f = a_0 + a_1 X + \cdots + a_n X^n \in K[X]$,

$$kf = (ka_0) + (ka_1)X + \cdots + (ka_n)X^n,\ \forall k \in K.$$

As in the previous example, $K[X]$ has also a ring structure which is connected to the vector space structure by the condition

$$\alpha(fg) = (\alpha f)g = f(\alpha g),\ \forall \alpha \in K,\ \forall f, g \in K[X].$$

(g) Let A be a non-empty set. Denote $K^A = \{f \mid f : A \to K\}$. Then $(K^A, K, +, \cdot)$ is a vector space, where the addition and the scalar multiplication are defined as follows: for any $f, g \in K^A$, for any $k \in K$, we have $f + g \in K^A$, $kf \in K^A$, where

$$(f + g)(x) = f(x) + g(x),\ (kf)(x) = kf(x), \forall x \in A.$$

As a particular case, we obtain the vector space $(\mathbb{R}^{\mathbb{R}}, \mathbb{R}, +, \cdot)$ of real functions of a real variable.
(i) If V_1 and V_2 are K-vector spaces, one defines on the Cartesian product $V_1 \times V_2$ the following operations: for any $(x_1, x_2), (x'_1, x'_2) \in V_1 \times V_2$ and $\alpha \in K$

$$(x_1, x_2) + (x'_1, x'_2) = (x_1 + x'_1, x_2 + x'_2),$$

$$\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2).$$

This way $V_1 \times V_2$ becomes a K- vector space, called the **direct product** of ${}_K V_1$ and ${}_K V_2$.

Next we give some computation rules in a vector space. Notice that we denote by 0 both the zero scalar and the zero vector.

Theorem 4. Let V be a vector space over K. Then for any $k, k', k_1, \dots, k_n \in K$ and for any $v, v', v_1, \dots, v_n \in V$ we have:

(i) $k \cdot 0 = 0 \cdot v = 0$;

(ii) $k(-v) = (-k)v = -kv, \ \ (-k)(-v) = kv$;

(iii) $k(v - v') = kv - kv', \ \ (k - k')v = kv - k'v$;

(iv) $(k_1 + \dots + k_n)v = k_1 v + \dots + k_n v, \ \ k(v_1 + \dots + v_n) = kv_1 + \dots + kv_n$.

Proof. □

Theorem 5. Let V be a vector space over K and let $k \in K$ and $v \in V$. Then

$$kv = 0 \Leftrightarrow k = 0 \text{ or } v = 0\,.$$

Proof. □

Definition 6. Let V be a vector space over K and let $S \subseteq V$. Then S is a **subspace** of V if:
(1) S is closed with respect to the addition of V and to the scalar multiplication, that is,

$$\forall x, y \in S\,, \quad x + y \in S\,,$$

$$\forall k \in K\,,\ \forall x \in S\,,\ kx \in S\,.$$

(2) S is a vector space over K with respect to the induced operations of addition and scalar multiplication.

We denote by $S \leq_K V$ the fact that S is a subspace of the vector space V over K.

Remark 7. If $S \leq_K V$ then S contains the zero vector of V, i.e. $0 \in S$.

We have the following **characterization theorem for subspaces**.

Theorem 8. Let V be a vector space over K and let $S \subseteq V$. The following conditions are equivalent:

1) $S \leq_K V$.

2) The following conditions hold for S:

α) $0 \in S$;

β) $\forall x, y \in S\,,\ x + y \in S$;

γ) $\forall k \in K\,,\ \forall x \in S\,,\ kx \in S\,.$

3) The following conditions hold for S:

α) $0 \in S$;

δ) $\forall k_1, k_2 \in K\,,\ \forall x, y \in S\,,\ k_1 x + k_2 y \in S\,.$

Proof. ☐

Remark 9. (1) One can replace α) in the previous theorem with $S \neq \emptyset$.
(2) If $S \leq_K V$, $k_1, \dots, k_n \in K$ and $x_1, \dots, x_n \in S$ then $k_1x_1 + \cdots + k_nx_n \in S$.

Examples 10. (a) Every non-zero vector space V over K has two subspaces, namely $\{0\}$ and V. They are called the **trivial subspaces**.
(b) Let

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\},$$
$$T = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}.$$

Then S and T are subspaces of the real vector space $\mathbb{R}^3$.

(c) Let $n \in \mathbb{N}$ and let

$$K_n[X] = \{f \in K[X] \mid \deg f \leq n\}.$$

Then $K_n[X]$ is a subspace of the polynomial vector space $K[X]$ over K.
d) Let $I \subseteq \mathbb{R}$ be an interval. The set $\mathbb{R}^I = \{f \mid f : I \to \mathbb{R}\}$ is a $\mathbb{R}$-vector space with respect to the following operations

$$(f + g)(x) = f(x) + g(x),\ (\alpha f)(x) = \alpha f(x)$$

with $f, g \in \mathbb{R}^I$ and $\alpha \in \mathbb{R}$. The subsets

$$C(I, \mathbb{R}) = \{f \in \mathbb{R}^I \mid f \text{ continuous on } I\},\ D(I, \mathbb{R}) = \{f \in \mathbb{R}^I \mid f \text{ derivable on } I\}$$

are subspaces of $\mathbb{R}^I$ since they are nonempty and

$$\alpha, \beta \in \mathbb{R},\ f, g \in C(I, \mathbb{R}) \Rightarrow \alpha f + \beta g \in C(I, \mathbb{R});$$
$$\alpha, \beta \in \mathbb{R},\ f, g \in D(I, \mathbb{R}) \Rightarrow \alpha f + \beta g \in D(I, \mathbb{R}).$$

Theorem 11. Let I be a nonempty set, V be a vector space over K and let $(S_i)_{i \in I}$ be a family of subspaces of V. Then $\bigcap_{i \in I} S_i \leq_K V$.

Proof. ☐

Remark 12. In general, the union of two subspaces is not a subspace.
For instance, ...

Next, we will see how to complete a subset of a vector space to a subspace in a minimal way.

Definition 13. Let V be a vector space and let $X \subseteq V$. We denote

$$\langle X \rangle = \bigcap\{S \leq_K V \mid X \subseteq S\}$$

and we call it the **subspace generated** (or **spanned**) **by** X. The set X is the **generating set** of $\langle X \rangle$. If $X = \{x_1, \dots, x_n\}$, we denote $\langle x_1, \dots, x_n \rangle = \langle \{x_1, \dots, x_n\} \rangle$.

Remarks 14. (1) $\langle X\rangle$ is the smallest subspace of V (with respect to $\subseteq$) which contains X.
(2) Notice that $\langle\emptyset\rangle = \{0\}$.
(3) If V is a K-vector space, then:
  (i) If $S \leq_K V$ then $\langle S\rangle = S$.
  (ii) If $X \subseteq V$ then $\langle\langle X\rangle\rangle \subseteq \langle X\rangle$.
  (iii) If $X \subseteq Y \subseteq V$ then $\langle X\rangle \subseteq \langle Y\rangle$.

Definition 15. A K-vector space V is **finitely generated** if there exist $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in V$ such that $V = \langle x_1, \ldots, x_n\rangle$. The set $\{x_1, \ldots, x_n\}$ is also called **system of generators** for V.

Definition 16. Let V be a K-vector space. A finite sum of the form

$$k_1x_1 + \cdots + k_nx_n\,,$$

with $k_1, \ldots, k_n \in K$ and $x_1, \ldots, x_n \in V$, is called a **linear combination** of the vectors $x_1, \ldots, x_n$.

Let us show how the elements of a generated subspace look like.

Theorem 17. Let V be a vector space over K and let $\emptyset \neq X \subseteq V$. Then

$$\langle X\rangle = \{k_1x_1 + \cdots + k_nx_n \mid k_i \in K\,,\ x_i \in X\,, i = 1, \ldots, n\,,\ n \in \mathbb{N}^*\}\,,$$

i.e. $\langle X\rangle$ is the set of all finite linear combinations of vectors of V.

Proof. □

Corollary 18. Let V be a vector space over K and $x_1, \ldots, x_n \in V$. Then

$$\langle x_1, \ldots, x_n\rangle = \{k_1x_1 + \cdots + k_nx_n \mid k_i \in K\,,\ x_i \in X\,, i = 1, \ldots, n\}\,.$$

Remark 19. Notice that a linear combination of linear combinations is again a linear combination.

Examples 20. (a) Consider the real vector space $\mathbb{R}^3$. Then

$$\langle(1,0,0),(0,1,0),(0,0,1)\rangle = \{k_1(1,0,0) + k_2(0,1,0) + k_3(0,0,1) \mid k_1,k_2,k_3 \in \mathbb{R}\} =$$

$$= \{(k_1,0,0) + (0,k_2,0) + (0,0,k_3) \mid k_1,k_2,k_3 \in \mathbb{R}\} = \{(k_1,k_2,k_3) \mid k_1,k_2,k_3 \in \mathbb{R}\} = \mathbb{R}^3\,.$$

Hence $\mathbb{R}^3$ is generated by the three vectors $(1,0,0), (0,1,0), (0,0,1)$.
(b) More generally, the subspaces of $\mathbb{R}^3$ are the trivial subspaces, the lines containing the origin and the planes containing the origin.

If $S, T \leq_K V$, the smallest subspace of V which contains the union $S \cup T$ is $\langle S \cup T \rangle$. We will show that this subspace is the sum of the given subspaces.

Definition 21. Let V be a vector space over K and let $S, T \leq_K V$. Then we define the **sum** of the subspaces S and T as the set

$$S + T = \{s + t \mid s \in S\,,\ t \in T\}\,.$$

If $S \cap T = \{0\}$, then $S + T$ is denoted by $S \oplus T$ and is called the **direct sum** of the subspaces S and T.

Remarks 22. a) If V is a K-vector space, $V_1, V_2 \leq_K V$, then $V = V_1 \oplus V_2$ if and only if

$$V = V_1 + V_2 \text{ and } V_1 \cap V_2 = \{0\}.$$

Under these circumstances, we say that V_i $(i = 1, 2)$ is a **direct summand** of V.
b) If $V_1, V_2, V_3 \leq_K V$ and $V = V_1 \oplus V_2 = V_1 \oplus V_3$, we cannot deduce that $V_2 = V_3$.

c) The property of a subspace of being a direct summand is transitive. (during the seminar)

Theorem 23. Let V be a vector space over K and let $S, T \leq_K V$. Then

$$S + T = \langle S \cup T \rangle\,.$$

Proof. □

Remarks 24. (1) Actually, a more general result can be proved: if $S_1, \dots, S_n$ are subspaces of a K-vector space V then

$$S_1 + \cdots + S_n = \langle S_1 \cup \cdots \cup S_n \rangle.$$

(2) Moreover, if $X_i \subseteq V$ $(i = 1, \dots, n)$, then $\langle X_1 \cup \cdots \cup X_n \rangle = \langle X_1 \rangle + \cdots + \langle X_n \rangle$.

Linear maps

Definition 25. Let V and V' be vector spaces over K. The map $f : V \to V'$ is called a **(vector space) homomorphism** or a **linear map** (or a **linear transformation**) if

$$f(x + y) = f(x) + f(y)\,,\ \forall x, y \in V\,,$$

$$f(kx) = kf(x),\ \ \forall k \in K,\ \forall x \in V\,.$$

The **(vector space) isomorphism**, **endomorphism** and **automorphism** are defined as usual.

We will mainly use the name *linear map* or *K-linear map*.

Remarks 26. (1) When defining a linear map, we consider vector spaces over the same field K.
(2) If $f : V \to V'$ is a K-linear map, then the first condition from its definition tells us that f is a group homomorphism between $(V, +)$ and $(V', +)$. Thus we have

$$f(0) = 0' \text{ and } f(-x) = -f(x),\ \forall x \in V.$$

We denote by $V \simeq V'$ the fact that two vector spaces V and V' are isomorphic and

$$Hom_K(V, V') = \{f : V \to V' \mid f \text{ is a } K\text{-linear map}\},$$

$$End_K(V) = \{f : V \to V \mid f \text{ is a } K\text{-linear map}\},$$

$$Aut_K(V) = \{f : V \to V \mid f \text{ is a } K\text{-isomorphism}\}.$$

Theorem 27. Let V, V' be K-vector spaces. Then $f : V \to V'$ is a linear map if and only if

$$f(k_1v_1 + k_2v_2) = k_1f(v_1) + k_2f(v_2),\ \forall k_1, k_2 \in K,\ \forall v_1, v_2 \in V.$$

Proof. □

One can easily prove by way of induction the following:

Corollary 28. If $f : V \to V'$ is a linear map, then

$$f(k_1v_1 + \cdots + k_nv_n) = k_1f(v_1) + \cdots + k_nf(v_n),\ \forall v_1, \ldots, v_n \in V,\ \forall k_1, \ldots, k_n \in K.$$

Examples 29. (a) Let V and V' be K-vector spaces and let $f : V \to V'$ be defined by $f(x) = 0'$, for any $x \in V$. Then f is a K-linear map, called the **trivial linear map**.
(b) Let V be a vector space over K. Then the identity map $1_V : V \to V$ is an automorphism of V.
(c) Let V be a vector space and $S \leq_K V$. Define $i : S \to V$ by $i(x) = x$, for any $x \in S$. Then i is a K-linear map, called the **inclusion linear map**.
(d) Let us consider $\varphi \in \mathbb{R}$. The map

$$f : \mathbb{R}^2 \to \mathbb{R}^2,\ f(x, y) = (x \cos\varphi - y \sin\varphi, x \sin\varphi + y \cos\varphi),$$

i.e. the plane rotation with the rotation angle φ, is a linear map.
(e) If $a, b \in \mathbb{R}$, $a < b$, $I = [a, b]$, and $C(I, \mathbb{R}) = \{f : I \to \mathbb{R} \mid f \text{ continuous on } I\}$, then

$$F : C(I, \mathbb{R}) \to \mathbb{R},\ F(f) = \int_a^b f(x)dx$$

is a linear map.

As in the case of group homomorphisms, we have the following:

Theorem 30. Let V, V', V'' be K-vector spaces.
(i) If $f : V \to V'$ and $g : V' \to V''$ are K-linear maps (isomorphisms) then $g \circ f : V \to V''$ is a K-linear map (isomorphism).
(ii) If $f : V \to V'$ is an isomorphism of K-vector spaces then $f^{-1} : V' \to V$ is again an isomorphism of K-vector spaces.

Proof. □

Definition 31. Let $f : V \to V'$ be a K-linear map. Then the set

$$\mathrm{Ker} f = \{x \in V \mid f(x) = 0'\}$$

is called the **kernel** of the K-linear map f and the set

$$\mathrm{Im} f = \{f(x) \mid x \in V\}$$

is called the **image** of the K-linear map f.

Theorem 32. Let $f : V \to V'$ be a K-linear map. Then we have
1) $\mathrm{Ker} f \leq_K V$ and $\mathrm{Im} f \leq_K V'$.
2) f is injective if and only if $\mathrm{Ker} f = \{0\}$.

Proof. □

Theorem 33. Let $f : V \to V'$ be a K-linear map and let $X \subseteq V$. Then

$$f(\langle X \rangle) = \langle f(X) \rangle\,.$$

Proof. □

Theorem 34. Let V and V' be vector spaces over K. For any $f, g \in Hom_K(V, V')$ and for any $k \in K$, we consider $f + g, k \cdot f \in Hom_K(V, V')$,

$$(f + g)(x) = f(x) + g(x),\ \forall x \in V\,,$$

$$(kf)(x) = kf(x),\ \forall x \in V\,.$$

The above equalities define an addition and a sclar multiplication on $Hom_K(V, V')$ and $Hom_K(V, V')$ is a vector space over K.

Proof. □

Corollary 35. If V is a K-vector space, then $End_K(V)$ is a vector space over K.

Remarks 36. a) Let V be a K-vector space. From Theorem 30 one deduces that $End_K(V)$ is a subgroupoid of $(V^V, \circ)$ and from Example 29 (b) it follows that $(End_K(V), \circ)$ is a monoid. Moreover, the endomorphism composition $\circ$ is distributive with respect to endomorphism addition $+$, thus $End_K(V)$ also has a unitary ring structure, $(End_K(V), +, \circ)$.

b) The set $Aut_K(V)$ is the group of the units of $(End_K(V), \circ)$.