Seminar 4

① Let $f: \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $f(x,y) = x \cdot \sqrt{x^2+y^2}$. Determine the first order partial derivatives of f, $\nabla f(3,4)$ and $df(3,4)$

Solution: We have

$$\frac{\partial f}{\partial x}(x,y) = \sqrt{x^2+y^2} + x\,\frac{1}{\cancel{2}\sqrt{x^2+y^2}}\cdot\cancel{2}x = \sqrt{x^2+y^2} + \frac{x^2}{\sqrt{x^2+y^2}}$$

$$= \frac{x^2+y^2+x^2}{\sqrt{x^2+y^2}} = \frac{2x^2+y^2}{\sqrt{x^2+y^2}}$$

$$\frac{\partial f}{\partial y} = x\cdot\frac{1}{\cancel{2}\sqrt{x^2+y^2}}\cdot\cancel{2}y = \frac{xy}{\sqrt{x^2+y^2}}$$

$\forall\, (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0}\frac{1}{x-0}\cdot[f(x,0)-f(0,0)]$$

$$= \lim_{x\to 0}\frac{\cancel{x}\cdot|x|}{\cancel{x}} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0}\frac{1}{y-0}[f(0,y)-f(0,0)] = \lim_{y\to 0}\frac{0\cdot 0}{y} = \underline{\underline{0}}$$

Therefore f has both partial derivatives at every point in $\mathbb{R}^2$ and

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \dfrac{2x^2+y^2}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

∇ = differential operator NABLA

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right)$$

↳ the gradiant of f at (x,y)

$$\nabla f(3,4) = \left(\frac{\partial f}{\partial x}(3,4), \frac{\partial f}{\partial y}(3,4)\right) = \left(\frac{34}{5}, \frac{12}{5}\right)$$

$$\frac{\partial f}{\partial x}(3,4) = \frac{2\cdot 9+16}{\sqrt{9+16}} = \frac{34}{5}$$

$$\frac{\partial f}{\partial y}(3,4) = \frac{12}{5}$$

$$df(3,4) \in L(\mathbb{R}^2, \mathbb{R})$$

$$df(3,4)(h_1,h_2) = h_1 \cdot \frac{\partial f}{\partial x}(3,4) + h_2 \cdot \frac{\partial f}{\partial y}(3,4)$$

$$= h_1 \cdot \frac{34}{5} + h_2 \cdot \frac{12}{5} = \frac{34h_1 + 12h_2}{5}$$

(2) (Homework) Let $A = \{(x,y) \in \mathbb{R}^2 \mid y \neq 0\}$ and let $f: A \to \mathbb{R}$, $f(x,y) = \operatorname{arctg}\frac{x}{y}$. Det the first order partial derivatives of f, $\nabla f(1,1)$, $df(1,1)$

(3) Let $f: \mathbb{R}^2 \to \mathbb{R}^3$ be the function defined by $f(x,y) = (\underbrace{x^2-y}_{f_1(x,y)}, \underbrace{xy+y^2}_{f_2(x,y)}, \underbrace{e^{x^2-y^2}}_{f_3(x,y)})$. Determine $df(1,1)$

Solution: We have $df(1,1) \in L(\mathbb{R}^2, \mathbb{R}^3)$

$$\frac{\partial f}{\partial x}(x,y) = \left(2x, y, 2x \cdot e^{x^2-y^2}\right)$$

$$\frac{\partial f}{\partial y}(x,y) = \left(-1, x+2y, -2y e^{x^2-y^2}\right)$$

It will be proved in the next lecture that

$$[df(a,b)] = J(f)(a,b) \leftarrow \text{the Jacobi matrix of } f \text{ at } (a,b)$$

$$J(f)(a,b) = \begin{array}{cl} \begin{pmatrix} \frac{\partial f_1}{\partial x}(a,b) & \frac{\partial f_1}{\partial y}(a,b) \\ \frac{\partial f_2}{\partial x}(a,b) & \frac{\partial f_2}{\partial y}(a,b) \\ \frac{\partial f_3}{\partial x}(a,b) & \frac{\partial f_3}{\partial y}(a,b) \end{pmatrix} & \begin{array}{l} \leftarrow f_1 \\ \leftarrow f_2 \\ \leftarrow f_3 \end{array} \\ \begin{array}{cc} \uparrow \qquad\qquad & \qquad\qquad \uparrow \\ x \qquad\qquad & \qquad\qquad y \end{array} & \end{array}$$

$$J(f)(x,y) = \begin{pmatrix} 2x & -1 \\ y & x+2y \\ 2x \cdot e^{x^2-y^2} & -2y \cdot e^{x^2-y^2} \end{pmatrix}$$

$$J(f)(1,1) = \begin{pmatrix} 2 & -1 \\ 1 & 3 \\ 2 & -2 \end{pmatrix} = [df(1,1)]$$

$$df(1,1)(h_1,h_2) = \underbrace{\begin{pmatrix} 2 & -1 \\ 1 & 3 \\ 2 & -2 \end{pmatrix}}_{J(f)(1,1)} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 2h_1 - h_2 \\ h_1 + 3h_2 \\ 2h_1 - 2h_2 \end{pmatrix}$$

$$df(1,1)(h_1,h_2) = (2h_1 - h_2,\ h_1 + 3h_2,\ 2h_1 - 2h_2)$$

<u>Rem</u>: $\frac{\partial f}{\partial x}(x,y) = \left(\frac{\partial f_1}{\partial x}(x,y),\ \frac{\partial f_2}{\partial x}(x,y),\ \frac{\partial f_3}{\partial x}(x,y) \right) = (2x, y, 2xe$

$$\frac{\partial f}{\partial y}(x,y) = \left(\frac{\partial f_1}{\partial y}(x,y),\ \frac{\partial f_2}{\partial y}(x,y),\ \frac{\partial f_3}{\partial y}(x,y) \right) = (-1, x+2y, \dots)$$

4) (Homework) Let $f:(0,\infty)^2 \to \mathbb{R}^3$, $f(x,y) = \left(\operatorname{arctg}\frac{x}{y}, \frac{1}{xy}, x^y + y^x\right)$. Det. the first order partial derivatives of f and $df(1,1)$

5) Let $A = \{(x,y,z) \in \mathbb{R}^3 \mid xy + 2z > 0\}$ and let $f: A \to \mathbb{R}^2$ be the function $f(x,y,z) = \left(\ln(xy+2z), \sin(xy+yz+zx)\right)$

Det. the first order partial derivativeatives of f and $df(2,-1,2)$

Sol.

$$\frac{\partial f}{\partial x}(x,y,z) = \left(\frac{1}{xy+2z}\cdot y, \cos(xy+yz+zx)\cdot(y+z)\right)$$
$$= \left(\frac{y}{xy+2z}, (y+z)\cos(xy+yz+zx)\right)$$

$$\frac{\partial f}{\partial y}(x,y,z) = \left(\frac{1}{xy+2z}\cdot x, \cos(xy+yz+zx)\cdot(x+z)\right)$$
$$= \left(\frac{x}{xy+2z}, (x+z)\cos(xy+yz+zx)\right)$$

$$\frac{\partial f}{\partial z}(x,y,z) = \left(\frac{2}{xy+2z}, (y+x)\cos(xy+yz+zx)\right)$$

$$[df(x,y,z)] = J(f)(x,y,z) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(x,y,z) & \frac{\partial f_1}{\partial y}(x,y,z) & \frac{\partial f_2}{\partial z}(x,y,z) \\ \frac{\partial f_2}{\partial x}(x,y,z) & \frac{\partial f_2}{\partial y}(x,y,z) & \frac{\partial f_3}{\partial z}(x,y,z) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{y}{xy+2z} & \frac{x}{xy+2z} & \frac{2}{xy+2z} \\ (y+z)\cos(xy+yz+zx) & (z+x)\cos(xy+yz+zx) & (x+y)\cos(xy+yz+zx) \end{pmatrix}$$

$df(2,-1,2) \in L(\mathbb{R}^3, \mathbb{R}^2)$

$$[df(2,-1,2)] = J(f)(2,-1,2) = \begin{pmatrix} \frac{-1}{2} & 1 & 1 \\ 1 & 4 & 1 \end{pmatrix}$$

$$df(2,-1,2)(h_1,h_2,h_3) = \begin{pmatrix} -\frac{1}{2} & 1 & 1 \\ 1 & 4 & 1 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2}h_1 + h_2 + h_3 \\ h_1 + 4h_1 + h_3 \end{pmatrix}$$

Alternatively, $df(2,-1,2) = \left(-\frac{1}{2}h_1 + h_2 + h_3,\ h_1 + 4h_1 + h_3\right)$

⑥ Det $\alpha \in \mathbb{R}$ s.t. the function $f : \mathbb{R} \times (0;\infty) \to \mathbb{R}$, $f(x,y) = y^\alpha \cdot e^{-\frac{x^2}{4y}}$ satisfies $x^2 \cdot \frac{\partial f}{y}(x,y) = \frac{d}{\partial x}\left(x^2 \cdot \frac{\partial f}{\partial x}(x,y)\right)$

Sol:

$$\frac{\partial f}{\partial y}(x,y) = \alpha y^{\alpha-1} \cdot e^{\frac{-x^2}{4y}} + y^\alpha \cdot e^{-\frac{x^2}{4y}} \cdot \cancel{-}\frac{x^2}{4} \cdot \cancel{-}\frac{1}{y^2}$$

$$= \alpha y^{\alpha-1} \cdot e^{\frac{-x^2}{4y}} + \frac{x^2 \cdot y^{\alpha-2}}{4} \cdot e^{\frac{-x^2}{4y}} \quad \Big| \cdot x^2$$

$$\Rightarrow x^2 \cdot \frac{\partial f}{\partial y}(x,y) = \underbrace{e^{-\frac{x^2}{4y}}\left(\alpha x^2 y^{\alpha-1} + \frac{x^4}{4} y^{\alpha-2}\right)} \qquad (1)$$

$$\frac{\partial f}{\partial x}(x,y) = y^{\alpha} \cdot e^{-\frac{x^2}{4y}} \cdot \left(-\frac{1}{4y}\right) 2x = -\frac{x}{2} y^{\alpha-1} \cdot e^{-\frac{x^2}{4y}} \quad \Big| \cdot x^2$$

$$\Rightarrow x^2 \frac{\partial f}{\partial x}(x,y) = -\frac{x^3}{2} y^{\alpha-1} \cdot e^{-\frac{x^2}{4y}}$$

$$\Rightarrow \frac{\partial}{\partial x}\left(x^2 \frac{\partial f}{\partial x}(x,y)\right) = -\frac{y^{\alpha-1}}{2}\left(3x^2 \cdot e^{\frac{-x^2}{4y}} + x^3 \cdot e^{\frac{-x^2}{4y}} \cdot \left(-\frac{x}{2y}\right)\right)$$

$$\Rightarrow \frac{\partial}{\partial x}\left(x^2 \cdot \frac{\partial f}{\partial x}(x,y)\right) = \underbrace{e^{-\frac{x^2}{4y}}\left(-\frac{3}{2} x^2 y^{\alpha-1} + \frac{1}{4} x^4 y^{\alpha-2}\right)}_{}\ (2)$$

By comparing $(1)=(2) \Rightarrow \boxed{\alpha = -\frac{3}{2}}$

⑦ Let $r>0$, let $A = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2<r^2\}$, and let $f: A \to \mathbb{R}$, $f(x,y) = 2 \cdot \ln \frac{r\sqrt{8}}{r^2-x^2-y^2}$ $(*)$. Prove that f satisfies:

$$\frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y) = e^{f(x,y)} \qquad \forall (x,y) \in A$$

Sol: $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$; $\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$

$(*) = 2\ln(r\sqrt{8}) - 2\ln(r^2-x^2-y^2)$

$$\frac{\partial f}{\partial x}(x,y) = -2 \cdot \frac{1}{r^2-x^2-y^2} \cdot (-2x) = \frac{4x}{r^2-x^2-y^2}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{4y}{r^2-x^2-y^2} \quad \text{(by the symmetry of } x \text{ and } y\text{)}$$

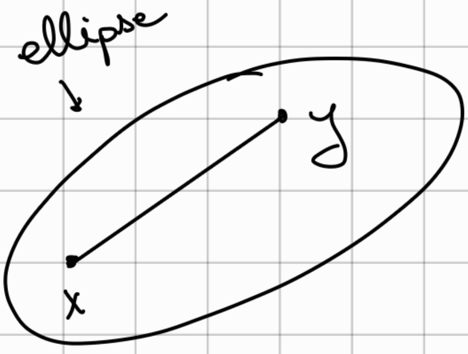
$$\frac{\partial^2 f}{\partial x^2}\left(\frac{\partial f}{\partial x}\right) = \left(\frac{4x}{r^2-x^2-y^2}\right)'_x = 4\frac{r^2-x^2-y^2-x(-2x)}{(r^2-x^2-y^2)^2} = 4\frac{r^2-y^2+x^2}{(r^2-x^2-y^2)^2}$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = 4\frac{r^2+y^2-x^2}{(r^2-x^2-y^2)^2}$$

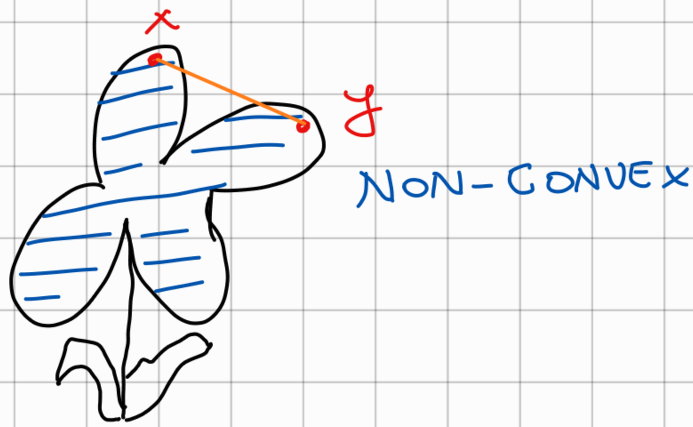
$$\Rightarrow \frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y) = \frac{8r^2}{(r^2-x^2-y^2)^2} \overset{?}{=} e^{f(x,y)}$$

$$(*) = \ln\left(\frac{8r^2}{(r^2-x^2-y^2)}\right)^2 \Rightarrow e^{f(x,y)} = \frac{d^2 f}{d x^2}(x,y)$$

<u>Def</u>: A set $A \subseteq \mathbb{R}^n$ is called CONVEX if $\forall\, x, y \in A$ $\forall t \in [0,1]$ one has $(1-t)x + ty \in A$



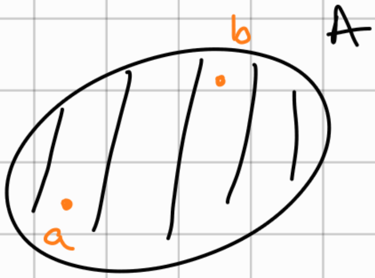
convex set



⑧ Let $A \subseteq \mathbb{R}^n$ be a compact convex set and let $f: A \to \mathbb{R}$ -cont. Prove that $f(A)$ is compact

<u>Sol</u>: $\left.\begin{array}{l} A \text{ compact} \\ f - \text{cont on } A \end{array}\right\} \overset{\text{Wierstrass}}{\Rightarrow}$ f is bounded and reaches its bounds

$\exists\, a, b \in A$ s.t. $f(a) = \min f(A)$
$f(b) = \max f(A)$



$$\Rightarrow f(A) \subseteq [f(a), f(b)]. \quad (1)$$

Let $g: [0;1] \to \mathbb{R}$, $g(t) = f((1-t)a + tb))$
Since f - cont on $A \Rightarrow g$ - cont on $[0;1]$
$\Rightarrow \operatorname{Im}(g) = g([0;1]) = $ an interval

$$\left.\begin{array}{l} g(0)=f(a)\in \mathrm{Im}(g) \\ g(1)=f(b)\in \mathrm{Im}(g) \end{array}\right\}$$

$$\left.\begin{array}{l} \Longrightarrow [f(a), f(b)] \subseteq \mathrm{Im}(g) \\ \text{But } g=f\big|_{[a,b]} \Rightarrow \mathrm{Im}(g)\subseteq f(A) \end{array}\right\} \Rightarrow [f(a),f(b)]\subseteq f(A) \quad (2)$$

By (1), (2) $\Rightarrow$ $\boxed{f(A)=[f(a), f(b)]}$

⑨ Let $f:[a,b]\to\mathbb{R}$ and let $G_f=\{(x,f(x))\in\mathbb{R}^2 \mid x\in[a,b]\}$ be the graph of f. Prove that

$$\underset{\text{on }[a,b]}{f\text{-cont}} \iff G_f\text{-compact set in } \mathbb{R}^2$$

<u>Sol</u>: $\boxed{\Longrightarrow}$ Assume f-cont on $[a,b]$

? G_f-Compact $\iff$ every seq. in G_f has a subseq. converging to some point in G_f

Let $((x_k, f(x_k)))_{k\geq 1}$ seq in G_f

$$\Downarrow$$

$(x_k)_{k\geq 1}$ is a seq of points in $[a,b]$

$\xRightarrow[\text{Lemma}]{\text{Cesaro's}}$ $\exists (x_{k_j})_{j\geq 1}$ subseq. of x_k and $\exists\, x\in[a,b]$ s.t.

$$\left.\begin{array}{l} \lim\limits_{j\to\infty} x_{k_j}=x \\ \Downarrow f\text{-cont} \\ \lim\limits_{j\to\infty} f(x_{k_j})=f(x) \end{array}\right\} \Rightarrow \lim_{j\to\infty}(x_{k_j}-f(x_{k_j}))=(x, f(x)\in G_f$$

(2) (Homework) Let $A = \{(x,y) \in \mathbb{R}^2 \mid y \neq 0\}$ and let $f: A \to \mathbb{R}$, $f(x,y) = \operatorname{arctg} \frac{x}{y}$. Det the first order partial derivatives of f, $\nabla f(1,1)$, $df(1,1)$

$$\frac{\partial f}{\partial x}(x,y) = \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \frac{1}{y} = \frac{1}{y + \frac{x^2}{y^{\cancel{2}}} \cdot \cancel{y}} = \frac{1}{\frac{y^2+x^2}{y}} = \frac{y}{x^2+y^2}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{1}{1+\left(\frac{x^2}{y^2}\right)} \cdot x \cdot -\frac{1}{y^2} = \frac{-x}{y^2+x^2}$$

$$\frac{\partial f}{\partial x}(1,1) = \frac{1}{2}$$

$$\frac{\partial f}{\partial y}(1,1) = \frac{-1}{2}$$

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) \right)$$

So $\nabla f(1,1) = \left(\frac{1}{2}, -\frac{1}{2} \right)$

$$df(1,1)(h_1, h_2) = h_1 \cdot \frac{\partial f}{\partial x}(1,1) + h_2 \cdot \frac{\partial f}{\partial y}(1,1)$$

$$= h_1 \cdot \frac{1}{2} - h_2 \cdot \frac{1}{2} = \frac{h_1 - h_2}{2}$$

(4) (Homework) Let $f: (0,\infty)^2 \to \mathbb{R}^3$, $f(x,y) = \left(\operatorname{arctg} \frac{x}{y}, \frac{1}{xy}, x^y + y^x\right)$

Det. the first order partial derivatives of f and $df(1,1)$

$$\frac{\partial f}{\partial x}(x,y) = \left(\frac{y}{x^2+y^2}, \overbrace{\frac{1}{y} \cdot \left(-\frac{1}{x^2}\right)}^{-\frac{1}{x^2 y}}, y x^{y-1} + y^x \cdot \ln y \right)$$

$$\frac{\partial f}{\partial y}(x,y) = \left(\frac{-x}{x^2+y^2}, \overbrace{\frac{1}{x} \cdot \left(-\frac{1}{y^2}\right)}^{-\frac{1}{xy^2}}, x^y \cdot \ln x + x y^{x-1} \right)$$

$$\frac{\partial f}{\partial x}(1,1) = \left(\frac{1}{2}, -\frac{1}{2}, 1 \right) \qquad \frac{\partial f}{\partial y}(1,1) = \left(\frac{-1}{2}, -\frac{1}{2}, 1 \right)$$

$$[df(1,1)] = J(f)(1,1) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \\ 1 & 1 \end{pmatrix}$$

$$[df(1,1)] = J(f)(1,1) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} \frac{h_1 - h_2}{2} \\ \frac{-h_1 - h_2}{2} \\ \frac{h_1 + h_2}{2} \end{pmatrix}$$