Seminar 2 - 2025

1. Consider the English alphabet with 21 consonants and 5 vowels (lowercase letters). In how many ways can we choose 6 letters such that we get 4 distinct consonants and 2 distinct vowels, if: a) the letters are not ordered; b) the letters are ordered? Examples: a) {s,e,a,r,c,h}; b) (a,r,c,h,e,s), (c,a,s,h,e,r), (c,h,a,s,e,r).

2. Two numbers are obtained by rolling two dice. Compute the probabilities of the following events:
a) A: "the numbers are equal".
b) B: "the sum of the numbers is even".
c) C: "the sum of the numbers is at most equal to 10."

3. For this problem, we assume that the birthday of every person falls equally likely in any month of the year (i.e., the probability that a person was born in a certain month is $\frac{1}{12}$). Compute the probability that
a) in a group of 5 persons there are at least 2 persons that celebrate their birthdays in the same month?
b) in a group of 5 persons all the birthdays fall in at most two months?

4. For this problem, we assume that the birthday of every person falls equally likely on any day of the year 2003 (i.e., the probability that a person was born in a certain day is $\frac{1}{365}$). Which is the minimum number n such that the probability of the event "at least two persons share a birthday in an arbitrary group of n persons born in 2003" is at least 50%?

5. A person sends 10 memes by choosing for each meme a recipient from a list of 20 friends. Compute the probability that the first friend in the list receives exactly 5 memes?

6. 5 balls numbered from 1 to 5 are randomly placed on a line. Compute the probability that:
a) the first and the last balls have even numbers;
b) the first two balls have odd numbers;
c) the balls with even numbers are next to each other;
d) at least two balls that are placed next to each other have the same parity.

7. In how many ways can we split the following marbles among 3 persons:
a) 1 red marble, 1 blue marble, 1 green marble, 1 yellow marble and 1 orange marble?
b) 5 red marbles;
c) 5 red marbles and 3 blue marbles;
d) 5 red marbles, 3 blue marbles and 4 green marbles.

1. Consider the English alphabet with 21 consonants and 5 vowels (lowercase letters). In how many ways can we choose 6 letters such that we get 4 distinct consonants and 2 distinct vowels, if: a) the letters are not ordered; b) the letters are ordered? Examples: a) {s,e,a,r,c,h}; b) (a,r,c,h,e,s), (c,a,s,h,e,r), (c,h,a,s,e,r).

Sol. : 21 c , 5 v

a) letters are not ordered

$$C_{21}^{4} \cdot C_{5}^{2}$$

b) letters are ordered

$$\underbrace{A_{21}^{4}}_{\text{\# of ways to choose the consonants (where order matters}} \cdot \underbrace{A_{5}^{2}}_{\text{\# of ways to choose the vowel}} \cdot \underbrace{C_{6}^{4}}_{\text{\# of ways to choose the positions of the consonants among the 6 positions}}$$

Ex. : Say that positions 1 2 3 5 have consonants
Say we chose b, c, d, f

b c d _ f _

b c f _ d

b d c f

b d f c

The answer $C_{21}^{4} \cdot 4! \cdot C_{5}^{2} \cdot 2! \cdot C_{6}^{4}$

2. Two numbers are obtained by rolling two dice. Compute the probabilities of the following events:

a) A:"the numbers are equal".

b) B:"the sum of the numbers is even".

c) C:"the sum of the numbers is at most equal to 10."

a) $P(A) = \frac{6}{36} = \frac{1}{6}$

b) $P(B) = \frac{3 \cdot 3 + 3 \cdot 3}{36} = \frac{18}{36} = \frac{1}{2}$

c) $P(C) = 1 - P(\bar{C}) = 1 - \frac{3}{36} = 1 - \frac{1}{12} = \frac{11}{12}$

3. For this problem, we assume that the birthday of every person falls equally likely in any month of the year (i.e., the probability that a person was born in a certain month is $\frac{1}{12}$). Compute the probability that
a) in a group of 5 persons there are at least 2 persons that celebrate their birthdays in the same month?
b) in a group of 5 persons all the birthdays fall in at most two months?

a) $P($"at least 2 people celebrate their birthday in the same month"$) =$

$= 1 - P($"everyone has different birth months"$) =$

$$= 1 - \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{12^5} = 1 - \frac{A_{12}^5}{12^5}$$

b) $P($"all birthdays fall in at most two months"$) =$

$= P($"all birthdays are in the same month"$) +$

$+ P($"all birthdays are in exactly two months"$) =$

$$= \frac{12}{12^5} + \frac{C_{12}^2 \cdot 2^5}{12^5}$$

4. For this problem, we assume that the birthday of every person falls equally likely on any day of the year 2003 (i.e., the probability that a person was born in a certain day is $\frac{1}{365}$). Which is the minimum number n such that the probability of the event "at least two persons share a birthday in an arbitrary group of n persons born in 2003" is at least 50%?

P("at least two persons share a birthday in an arbitrary group of n perssons born in 2003") =

= 1 − P("everyone has distinct birthdays") =

$$= 1 - \frac{A_{365}^{n}}{365^{n}}$$

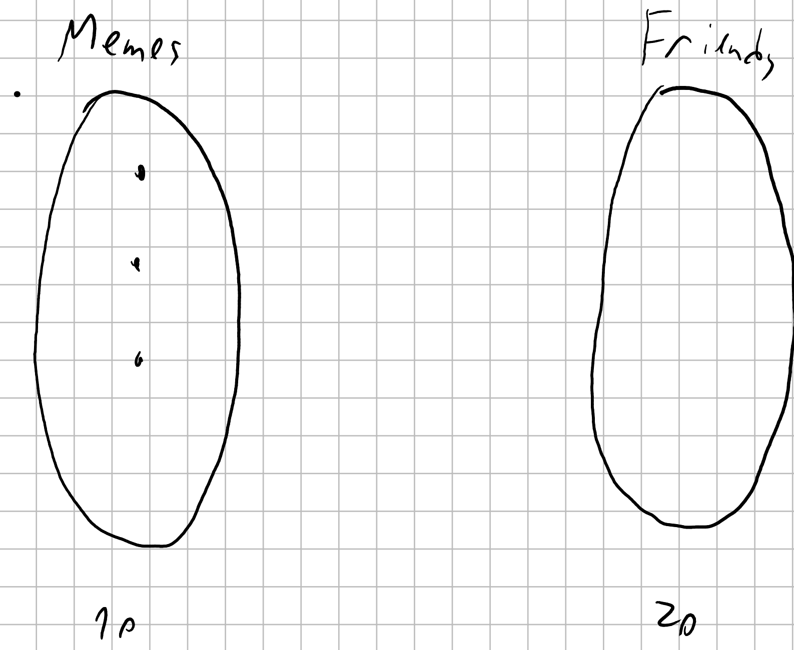
We have to find min n so that

$$\frac{A_{365}^{n}}{365^{n}} \leq 0.5$$

Using Octave the answer is 23.

5. A person sends 10 memes by choosing for each meme a recipient from a list of 20 friends. Compute the probability that the first friend in the list receives exactly 5 memes?

Sol.:



$|A| = n$ $|B| = m$

What is the number of functions from A to B?

m^n

$$P(\text{"the first friend receives exactly 5 memes"}) = \frac{C_{10}^{5} \cdot 19^{5}}{20^{10}}$$

6. 5 balls numbered from 1 to 5 are randomly placed on a line. Compute the probability that:
a) the first and the last balls have even numbers;
b) the first two balls have odd numbers;
c) the balls with even numbers are next to each other;
d) at least two balls that are placed next to each other have the same parity.

1, 2, 3, 4, 5 2 evens, 3 odds

a) $P($"1st and 5th have even numbers"$) =$

$$= \frac{\underbrace{C_2^2}_{\text{ways to choose 2 objects out of 2 objects}} \cdot \underbrace{2!}_{\text{ways to arrange the evens}} \cdot \underbrace{3!}_{\text{ways to arrange the odds}}}{\underbrace{5!}_{\text{total number of ways to arrange 5 objects in a line}}} = \frac{2!}{4 \cdot 5} = \frac{1}{10}$$

b) $P($"1st and 2nd have odd numbers"$) =$

$$= \frac{C_3^2 \cdot A_2^2 \cdot 3!}{5!} = \frac{3 \cdot 2 \cdot}{4 \cdot 5} = \frac{3}{10}$$

c) $P($"the balls with even numbers are next to each other"$) = \frac{A_2^2 \cdot 4!}{5!} = \frac{2}{5}$

d) $P($"at least two balls that are placed next to each other have the same parity"$) =$

$= 1 - P($"no two balls next to each other have the same parity"$) = 1 - \frac{2! \cdot 3!}{5!} =$

$$= 1 - \frac{2!}{4 \cdot 5} = \frac{9}{10}$$

7. In how many ways can we split the following marbles among 3 persons:

a) 1 red marble, 1 blue marble, 1 green marble, 1 yellow marble and 1 orange marble?

b) 5 red marbles;

c) 5 red marbles and 3 blue marbles;

d) 5 red marbles, 3 blue marbles and 4 green marbles.

Sol.: a) $f : \{\text{marbles}\} \to \{\text{persons}\}$

$\Rightarrow 3^5$

b) In how many ways can we distribute n identical objects among k non-identical boxes?

Ex.: 5 marbles, 3 people

$4+1+0$

$2+2+1$

$3+0+5$

Instead of distributing n objects into k boxes. We line up the n objects and place separators between some of them to signal when we change the box

$4+1+0$ o o o o | o |

$2+2+1$ o o | o o | o

$0+4+1$ $\quad | \circ \circ \circ \circ | \circ$

So the problem becomes equivalent to:
In how many ways can we place n objects (or $k-1$ separators) on $n+k-1$ positions?

$$\Rightarrow C_{n+k-1}^{k-1} = C_{n+k-1}^{n}$$

c) We want to split 5 reds and 3 blues among 3 people

Distributing the reds and the blues are independent actions. So the total # of ways to split 5 reds and 3 blues is:

$$\underbrace{C_{5+3-1}^{2}}_{\text{\# of ways to split the reds}} \cdot \underbrace{C_{3+3-1}^{2}}_{\text{\# of ways to split the blues}}$$

d) Same thing, just with more colours

$$C_{5+3-1}^{2} \cdot C_{3+3-1}^{2} \cdot C_{3+4-1}^{2}$$

- Special Mathematics (Daniela Roșca)

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