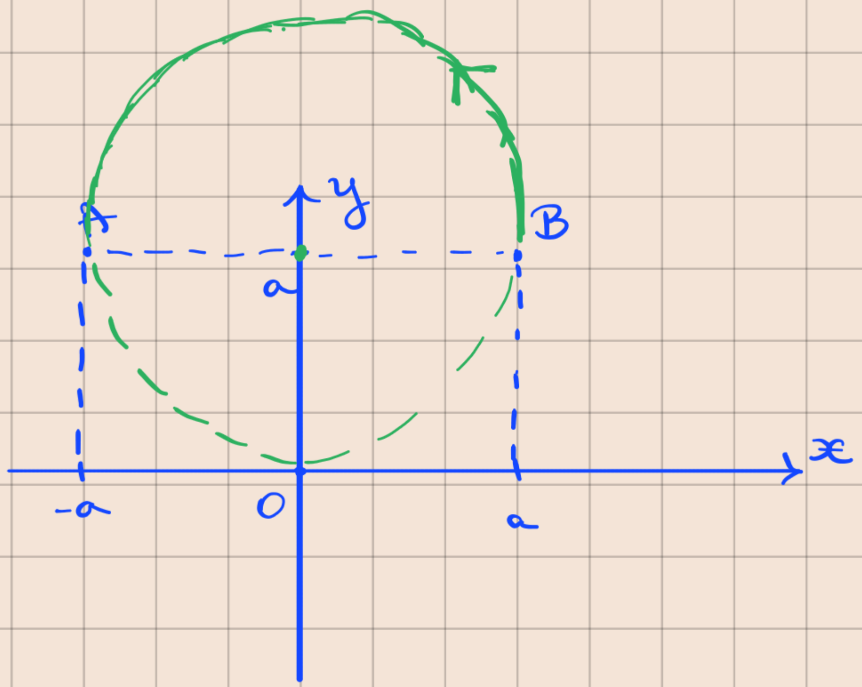
Seminar 12 29.05.2025

1 Given $a>0$, consider the points $A(-a,a)$, $B(a,a)$ and the vector field
$$\vec{F}(x,y) = [3a(x^2+y^2) - y^3]\,\vec{i} + 3xy(2a-y)\,\vec{j}$$
Compute $\int_\gamma \vec{F}\,d\vec{r}$ where γ is the param. path whose image is the semicircle of diameter AB not containing the origin, traced counterclockwise.

Sol1:



$x^2+(y-a)^2 = a^2$ (eq. of the green circle)

$$\begin{cases} x = a\cos t \\ y = a\sin t + a \end{cases} \Leftrightarrow \begin{cases} x = a\cos t \\ y = a(\sin t + 1) \end{cases} \quad t\in[0;\pi]$$

$$\int_\gamma \vec{F}\cdot d\vec{r} = \int_\gamma [3a(x^2+y^2) - y^3]\,dx + 3xy(2a-y)\,dy$$

$$= \int_0^\pi [3a(a^2\cos^2 t + a^2(\underset{\shortparallel}{\sin^2 t} + 2\sin t + 1) - a^3(\sin^3 t + 3\sin^2 t + 3\sin t)] \cdot (-a\sin t)\,dt + \int_0^\pi 3a\cos t\cdot a(\sin t + 1)\big(2a - a(\sin t + 1)\big)\cdot a\cos t\,dt = \ldots$$

Sol 2: We check if $\vec{F}$ is a conservative vector field or not. $\overset{\text{Poincaré}}{\Longleftrightarrow}$ $\dfrac{\partial F_1}{\partial y} = \dfrac{\partial F_2}{\partial x} \Longleftrightarrow 3a \cdot 2y - 3y^2 = 6ay - 3y^2$

$F_1(x,y) = 3a(x^2+y^2) - y^3$ $\Longleftrightarrow 6ay - 3y^2 = 6ay - 3y^2$ ✓

$F_2(x,y) = 3xy(2a-y)$
$= 6axy - 3xy^2$

$$\int_\gamma \vec{F} \cdot d\vec{r} = U\Big|_B^A = U(A) - U(B) = U(-a,a) - U(a,a),$$

where U is a scalar potential for $\vec{F}$

$$\nabla U = \vec{F}$$

$$\nabla U = \vec{F} \Longleftrightarrow \frac{\partial U}{\partial x} = F_1 \,;\, \frac{\partial U}{\partial y} = F_2$$

- $\dfrac{\partial U}{\partial x}(x,y) = F_1(x,y) = 3ax^2 + 3ay^2 - y^3$

$$U(x,y) = \int (3ax^2 + 3ay^2 - y^3)\,dx$$

$$= ax^3 + 3axy^2 - xy^3 + \varphi(y)$$

$$\Rightarrow \frac{\partial U}{\partial y}(x,y) = 6axy - 3xy^2 + \varphi'(y)$$

- But $\dfrac{\partial U}{\partial y}(x,y) = F_2(x,y) = 6axy - 3xy^2$ $\Bigg\} \Rightarrow \varphi'(y) = 0$

$\Rightarrow \varphi'(y)$ is a CONSTANT c

$$U(x,y) = ax^3 + 3axy^2 - xy^3 + c$$

$$\int_\gamma \vec{F} \cdot d\vec{r} = U(-a,a) - U(a,a) = -a^4 - 3a^4 + a^4 + \cancel{c} - (a^4 + 3a^4 - a^4 + \cancel{c})$$

$= -3a^4 - 3a^4 = \underset{\sim\sim}{-6a^4}$

$\boxed{2}$ Given the vector field $\vec{F}(x,y) = (\underbrace{y^3 + 6xy^2 + 3x^2}_{F_1(x,y)})\vec{i} + (\underbrace{3xy^2 + 6x^2y + 2y}_{F_2(x,y)})\cdot\vec{j}$, compute :

$$\int_{(1,1)}^{(2,0)} \vec{F}\cdot d\vec{r}$$

Solution: We check first if the work of $\vec{F}$ does not depend on the path of integration

$$\overset{\text{Poincaré}}{\Longleftrightarrow} \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \Longleftrightarrow 3y^2 + 12xy = 3y^2 + 12xy \checkmark$$

$\overset{\text{Poincaré}}{\Longrightarrow} \vec{F}$ is a conservative vector field $\overset{\text{Leibniz-Newton}}{\Longrightarrow}$

$\int_{(1,1)}^{(2,0)} \vec{F}\cdot d\vec{r} = U(2,0) - U(1,1)$, where U is a scalar potential for $\vec{F}$

- $\frac{\partial U}{\partial x}(x,y) = F_1(x,y) = y^3 + 6xy^2 + 3x^2$

$$U(x,y) = \int (y^3 + 6xy^2 + 3x^2)\,dx = y^3x + 3y^2\cdot x^2 + x^3 + \varphi(y)$$

$$\left.\begin{array}{l} \Rightarrow \frac{\partial U}{\partial y} = 3xy^2 + 6yx^2 + \varphi'(y) \\ \cdot \text{But } \frac{\partial U}{\partial y} = F_2(x,y) = 3xy^2 + 6x^2y + 2y \end{array}\right\} \Rightarrow \varphi'(y) = 2y$$

$$\Downarrow$$

$$\underline{\varphi(y) = y^2 + \mathscr{C}}$$

$$\Rightarrow U(x,y) = xy^3 + 3x^2y^2 + x^3 + y^2 + \mathscr{C}$$

$$\int_{(1,1)}^{(2,0)} \vec{F}\cdot d\vec{r} = (0 + 0 + 8 + 0 + \cancel{\mathscr{C}}) - (6 + \cancel{\mathscr{C}}) = \underline{\underline{2}}$$

3 Compute $I = \int_{(1,0,1)}^{(1,1,\frac{\pi}{2})} \underbrace{2xz}_{F_1(x,y,z)}\,dx + \underbrace{\sin z}_{F_2(x,y,z)}\,dy + (\underbrace{x^2+2z+y\cdot\cos z}_{F_3(x,y,z)})\,dz$

Solution We check first if the integral does not depend on the path of integration

$$\Longleftrightarrow \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \;;\; \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y} \;;\; \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$$

$$\Longleftrightarrow \quad 0 = 0 \checkmark \;;\; \cos z = \cos z \checkmark \;;\; 2x = 2x \checkmark$$

$\Longrightarrow I = U(1,1,\frac{\pi}{2}) - U(1,0,1)$ where U is a scalar potential of the vector field $\vec{F} = F_1\,\vec{i} + F_2\,\vec{j} + F_3\,\vec{k}$

- $\frac{\partial U}{\partial x}(x,y,z) = F_1(x,y,z) = 2xz \Longrightarrow U(x,y,z) = \int 2xz\,dx$

$$= 2z\cdot\frac{x^2}{2} + \varphi(y,z) = x^2\cdot z + \varphi(y,z)$$

$$\Longrightarrow \frac{\partial U}{\partial y}(x,y,z) = \frac{\partial \varphi}{\partial y}(y,z)$$

- But $\frac{\partial U}{\partial y}(x,y,z) = F_2(x,y,z) = \sin z$

$$\Longrightarrow \frac{\partial \varphi}{\partial y}(y,z) = \sin z$$

$$\Downarrow \int$$

$$\varphi(y,z) = \int \sin z\,dy = y\cdot\sin z + \psi(z)$$

$$\Longrightarrow U(x,y,z) = x^2\cdot z + y\cdot\sin z + \psi(z)$$

$$\Longrightarrow \frac{\partial U}{\partial z}(x,y,z) = x^2 + y\cdot\cos z + \psi'(z)$$

- But $\frac{\partial U}{\partial z}(x,y,z) = F_3(x,y,z) = x^2 + 2z + y\cdot\cos z$

$$\Longrightarrow \psi'(z) = 2z$$

$$\Downarrow$$

$$\psi(z) = \int 2z = 2\cdot\frac{z^2}{2} + \mathcal{C} = z^2 + \mathcal{C}$$

$\Rightarrow U(x,y,z) = x^2 \cdot z + y \cdot \sin z + z^2 + \mathscr{C}$

$\Rightarrow I = \left(\frac{\pi}{2} + \cancel{1} + \frac{\pi^2}{4} + \cancel{\mathscr{C}}\right) - \left(\cancel{1} + 1 + \cancel{\mathscr{C}}\right) = \underline{\underline{\frac{\pi}{2} + \frac{\pi^2}{4} - 1}}$

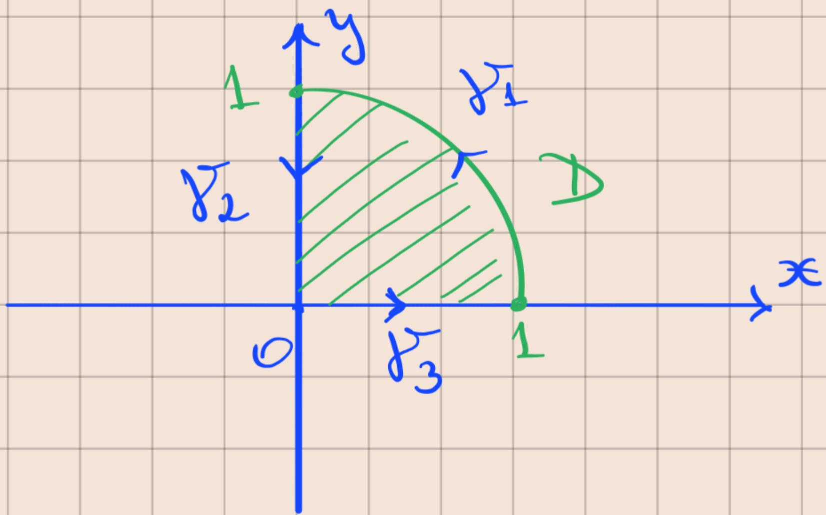
$\boxed{4}$ Consider the set $D = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x^2 + y^2 \leq 1\}$ and the vector field $\vec{F}(x,y) = (x^2 y)\vec{i} + (2xy)\vec{j}$.

Compute $\oint_{\partial D} \vec{F} \cdot d\vec{r}$

a) Using a param. of ∂D

b) Using Green's formula

Solution:



a) $\partial D = \gamma_1 + \gamma_2 + \gamma_3$

$$\oint_{\partial D} \vec{F} \cdot d\vec{r} = \underbrace{\int_{\gamma_1} \vec{F} \cdot d\vec{r}}_{I_1} + \underbrace{\int_{\gamma_2} \vec{F} \cdot d\vec{r}}_{I_2} + \underbrace{\int_{\gamma_3} \vec{F} \cdot d\vec{r}}_{I_3}$$

$\gamma_1 : \begin{cases} x = \cos t \\ y = \sin t \end{cases} \quad t \in [0; \frac{\pi}{2}]$

$$I_1 = \int_0^{\frac{\pi}{2}} \cos^2 t \cdot \sin t \cdot (-\sin t)\,dt + \int_0^{\frac{\pi}{2}} 2\cos t \sin t \cos t\,dt$$

$$\left(I_1 = \int_{\gamma_1} x^2 y\,dx + 2xy\,dy \right)$$

$$= -\frac{1}{4}\int_0^{\frac{\pi}{2}} 4\sin^2 t \cdot \cos^2 t \; dt + 2\int_0^{\frac{\pi}{2}} \cos^2 t \sin t \; dt$$

$$= -\frac{1}{4}\int_0^{\frac{\pi}{2}} \sin^2 2t \; dt - 2 \cdot \frac{\cos^3 t}{3}\Big|_0^{\frac{\pi}{2}}$$

$$= -\frac{1}{4}\int_0^{\frac{\pi}{2}} \frac{1-\cos 4t}{2} \, dt + \frac{2}{3}$$

$$= \frac{2}{3} - \frac{1}{8}\left(t\Big|_0^{\frac{\pi}{2}} - \frac{\sin 4t}{4}\Big|_0^{\frac{\pi}{2}} \right)$$

$$= \frac{2}{3} - \frac{1}{8} \cdot \frac{\pi}{2} = \frac{2}{3} - \frac{\pi}{16}$$

$$\gamma_2 : \begin{cases} x = 0 \\ y = 1-t \end{cases} \quad t \in [0;1]$$

$$I_2 = \int_{\gamma_2} x^2 y \, dx + 2xy \, dy \overset{\boxed{x=0}}{=} 0$$

$$\gamma_3 : \begin{cases} x = t \\ y = 0 \end{cases} \qquad I_3 = \int_{\gamma_3} x^2 y \, dx + 2xy \, dy = 0$$

$$t \in [0;1)$$

$$\boxed{\oint_{\partial D} \vec{F} \cdot d\vec{r} = \frac{2}{3} - \frac{\pi}{16}}$$

b) $$\oint_{\partial D} F_1(x,y)\,dx + F_2(x,y)\,dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\,dy$$

$$= \iint_D (2y - x^2)\,dx\,dy = \int_0^1 \int_0^{\frac{\pi}{2}} (2\rho \sin\theta - \rho^2 \cos^2\theta)\,\rho \, d\rho \, d\theta$$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

$\theta \in [0; \frac{\pi}{2}]$

$\rho \in [0; 1]$

$$= 2 \cdot \int_0^1 \rho^2 d\rho \cdot \int_0^{\frac{\pi}{2}} \sin\theta \, d\theta - \int_0^1 \rho^3 d\rho \cdot \int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta$$

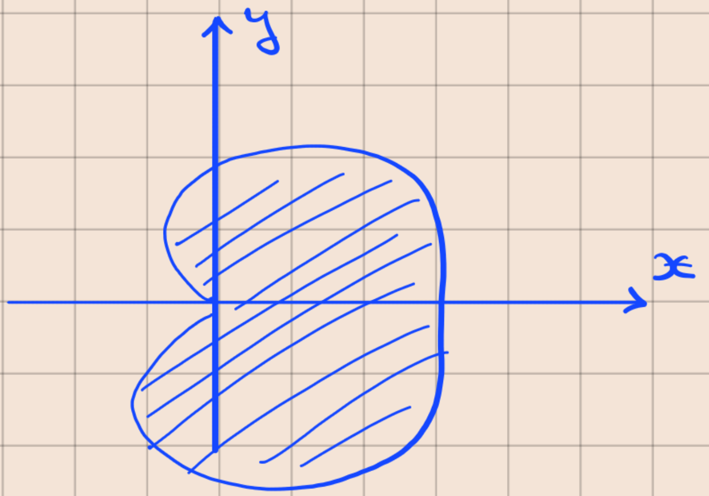
$$= \frac{2}{3} - \frac{1}{4} \int_0^{\frac{\pi}{2}} \frac{1+\cos 2t}{2} \, d\theta$$

$$= \frac{2}{3} - \frac{1}{8}\left(\theta\Big|_0^{\frac{\pi}{2}} + \frac{\sin 2\theta}{2}\Big|_0^{\frac{\pi}{2}}\right) = \frac{2}{3} - \frac{\pi}{16}$$

$\boxed{5}$ Determine the area of the region inside the cardioid $x^2+y^2 = 2a(x+\sqrt{x^2+y^2})$, $a \geq 0$

a) Using a double integral

b) Using a corollary of Green's formula



a) $\text{area}(D) = \iint dx\,dy$

$\rho = 2a(\cos\theta + 1)$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

$\rho \in [0; 2a(1+\cos\theta)]$

$\theta \in [0; 2\pi]$

$$\text{area}(D) = \int_{\theta=0}^{\theta=2\pi} \left(\int_{\rho=0}^{\rho=2a(1+\cos\theta)} \rho \, d\rho \right) d\theta$$

$$= \int_0^{2\pi} \frac{\rho^2}{2}\Big|_0^{2a(1+\cos\theta)} d\theta = \int_0^{2\pi} 2a^2(1+2\cos\theta+\cos^2\theta)\,d\theta$$

$$= 2a^2 \int_0^{2\pi} (1+2\cos\theta + \underbrace{\cos^2\theta}_{=\frac{1+\cos 2\theta}{2}})\,d\theta = 2a^2\left(\theta\Big|_0^{2\pi} + 2\underbrace{\sin\theta\Big|_0^{2\pi}}_{0} + \frac{1}{2}\theta\Big|_0^{2\pi}\right.$$

$$\left. + \frac{1}{2}\cdot\frac{1}{2}\cdot\underbrace{\sin 2\theta\Big|_0^{2\pi}}_{0}\right) = 2a^2(2\pi+\pi) = \underline{\underline{6\pi a^2}}$$

b) $\text{area}(D) = \frac{1}{2}\oint_{\partial D} x\,dy - y\,dx$

$$\partial D: \begin{cases} x = 2a(1+\cos\theta)\cdot\cos\theta \\ y = 2a(1+\cos\theta)\cdot\sin\theta \end{cases} \qquad \theta\in[0;2\pi]$$

$$= \frac{1}{2}\int_0^{2\pi} 2a(\cos\theta+\cos^2\theta)\cdot 2a(\cos\theta+\cos^2\theta-\sin^2\theta)\,d\theta$$

$$-\frac{1}{2}\int_0^{2\pi} 2a(\sin\theta+\sin\theta\cos\theta)\cdot 2a(-\sin\theta-2\cos\theta\sin\theta)\,d\theta$$

$$= 2a^2\int_0^{2\pi}\big(\cos^2\theta+\cos^3\theta+\cos^3\theta+\cos^4\theta-\sin^2\theta\cos\theta$$
$$-\sin^2\theta\cos^2\theta\big)\,d\theta$$

$$+2a^2\int_0^{2\pi}\big(\underbrace{\sin^2\theta}_{1}+\sin^2\theta\cos\theta+\sin^2\theta+\cos\theta+2\sin^2\theta\cos^2\theta\big)\,d\theta$$

$$= 2a^2\cdot\int_0^{2\pi}\big(1+\underbrace{\cos^4\theta+\sin^2\theta\cos^2\theta}_{\cos^2\theta}+\underbrace{2\sin^2\theta\cos\theta+2\cos^3\theta}_{2\cos\theta}\big)\,d\theta$$

$$= 2a^2\cdot\int_0^{2\pi}(1+\underbrace{\cos^2\theta}_{\frac{1+\cos 2\theta}{2}}+2\cos\theta)\,d\theta$$

$$= 2a^2\left(\theta\Big|_0^{2\pi} + 2\sin\theta\Big|_0^{2\pi} + \frac{\theta}{2}\Big|_0^{2\pi} + \frac{1}{2}\frac{\sin 2\theta}{2}\Big|_0^{2\pi}\right)$$

$$\ldots = \underline{\underline{6\pi a^2}}$$