**4.3. Theorem** (characterization of compact sets in $\mathbb{R}^n$). Given a set $A \subseteq \mathbb{R}^n$, the following assertions are equivalent:

1° $A$ is compact

2° Every infinite subset of $A$ has a limit point in $A$.

3° $A$ is sequentially compact (i.e., every sequence of points in $A$ has a convergent subsequence converging to some point in $A$).

4° $A$ is bounded and closed.

**Proof.** $\boxed{2° \Rightarrow 3°}$ Assume that every infinite subset of $A$ has a limit point in $A$.

Let $(x_k)$ be an arbitrary seq. of points in $A$, and let

$$A_0 := \{x_1, x_2, \dots\}.$$

$x_k = (-1)^k \Rightarrow A_0 = \{-1, 1\}$.

Case 1 $A_0$ is finite $\Rightarrow$ at least one of the terms of the sequence is repeated infinitely many times

$\Rightarrow$ the seq. $(x_k)$ has a constant subsequence, and this subsequence is convergent

Case 2 $A_0$ is infinite $\overset{2^\circ}{\Longrightarrow}$ $\exists\, x \in A$ s.t. $x \in A_0'$ $\Rightarrow \forall V \in \mathcal{V}(x) : V \cap A_0 \setminus \{x\} \neq \phi$

$\Updownarrow$ (Homework)

$\forall V \in \mathcal{V}(x) : V \cap A_0$ is infinite

$V = B(x, 1) \Rightarrow V \cap A_0$ is infinite $\Rightarrow \exists\, x_{k_1} \in B(x, 1) \Rightarrow \| x_{k_1} - x \| < 1$

$V = B(x, \frac{1}{2}) \Rightarrow V \cap A_0$ is infinite $\Rightarrow \exists\, x_{k_2} \in B(x, \frac{1}{2})$ with $k_2 > k_1 \Rightarrow \| x_{k_2} - x \| < \frac{1}{2}$

$V = B(x, \frac{1}{3}) \Rightarrow V \cap A_0$ is infinite $\Rightarrow \exists\, x_{k_3} \in B(x, \frac{1}{3})$ with $k_3 > k_2 \Rightarrow \| x_{k_3} - x \| < \frac{1}{3}$

$\vdots$

$\Rightarrow$ we construct inductively a subsequence $(x_{k_j})_{j \geq 1}$ of $(x_k)$ s.t.

$$\| x_{k_j} - x \| < \frac{1}{j} \quad \forall\, j \geq 1$$

$$\Downarrow$$

$$\lim_{j \to \infty} \| x_{k_j} - x \| = 0 \quad \Longrightarrow \quad \lim_{j \to \infty} x_{k_j} = x.$$

$\boxed{3^\circ \Rightarrow 4^\circ}$ Assume that is $A$ is sequentially compact.

$\cancel{X}$ A is closed $\Leftrightarrow$ $\forall\, (x_k)$ convergent sequence of points in $A$ we have $\lim_{k \to \infty} x_k \in A$

Let $(x_k)$ be a convergent seq. of points in $A$, and let $x := \lim_{k \to \infty} x_k$.

$$\left.\begin{array}{l} A = \text{sequentially compact} \Rightarrow \exists (x_{k_j})_{j \geq 1} \text{ subseq. of } (x_k) \text{ s.t. } \lim_{j \to \infty} x_{k_j} \in A \\ \text{But } \lim_{j \to \infty} x_{k_j} = \lim_{k \to \infty} x_k = x \end{array}\right\} \Rightarrow x \in A$$

$\cancel{3}$ A is bounded $\Leftrightarrow$ $\exists\, r>0$ s.t. $A \subseteq \overline{B}(0_n, r)$.

Assume that A is not bounded $\Rightarrow$ $\forall\, r>0$ : $A \not\subseteq \overline{B}(0_n, r)$ $\Leftrightarrow$ $\forall\, r>0$ $\exists\, x \in A$ s.t. $\|x\|>r$.

$r = k \in \mathbb{N}$ $\Rightarrow$ $\forall\, k \geq 1$ $\exists\, x_k \in A$ s.t. $\|x_k\| > k$

$\Rightarrow$ $(x_k)$ is a seq. in A $\underset{\substack{\uparrow \\ A \text{ seq. compact}}}{\Longrightarrow}$ $\exists\, (x_{k_j})_{j \geq 1}$ subseq. of $(x_k)$ and $\exists\, x \in A$ s.t.

$\lim\limits_{j\to\infty} x_{k_j} = x$ $\Rightarrow$ $\lim\limits_{j\to\infty} \|x_{k_j}\| = x$ $\Rightarrow$ the seq. $(\|x_{k_j}\|)$ is convergent $\Rightarrow$

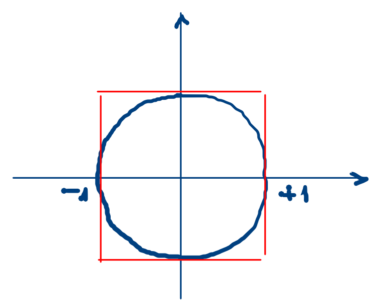
$\Rightarrow$ the seq. $(\|x_{k_j}\|)$ is bounded $\Rightarrow\!\!\!\!\!\!\times\!\!\Leftarrow$ $\|x_{k_j}\| > k_j$ $\forall\, j \geq 1$

$(k_j)$ is a strictly increasing seq. of natural numbers

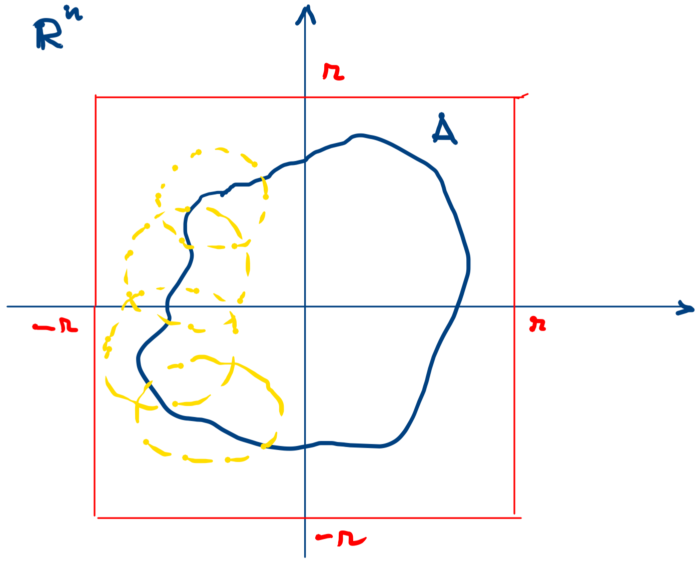
$\boxed{4^\circ \Rightarrow 1^\circ}$ Assume that A is bounded and closed.

$\Downarrow$

$\exists\, r>0$ s.t. $A \subseteq \overline{B}(0_n, r) \subseteq [-r, r]^n$




Let $(G_i)$ be an arbitrary open cover of A




$$A \subseteq \bigcup_{i \in I} G_i$$

$\{G_i \mid i \in I\} \cup \{\underbrace{\mathbb{R}^n \setminus A}_{\text{open}}\}$ – open cover of the closed cell $[-r,r]^n$

compact set

$\Rightarrow \exists J \subseteq I$, $J$ = finite s.t.

$$\left.\begin{array}{l} [-r,r]^n \subseteq \left(\bigcup_{i \in J} G_i\right) \cup (\mathbb{R}^n \setminus A) \\ A \subseteq [-r,r]^n \end{array}\right\} \Rightarrow$$

$$\Rightarrow A \subseteq \left(\bigcup_{i \in J} G_i\right) \cup (\mathbb{R}^n \setminus A)$$

$\Rightarrow A \subseteq \bigcup_{i \in J} G_i \Rightarrow A$ is compact.

$1^\circ \Leftrightarrow 3^\circ$ Hausdorff's Theorem

$1^\circ \Leftrightarrow 4^\circ$ The Heine – Borel Theorem

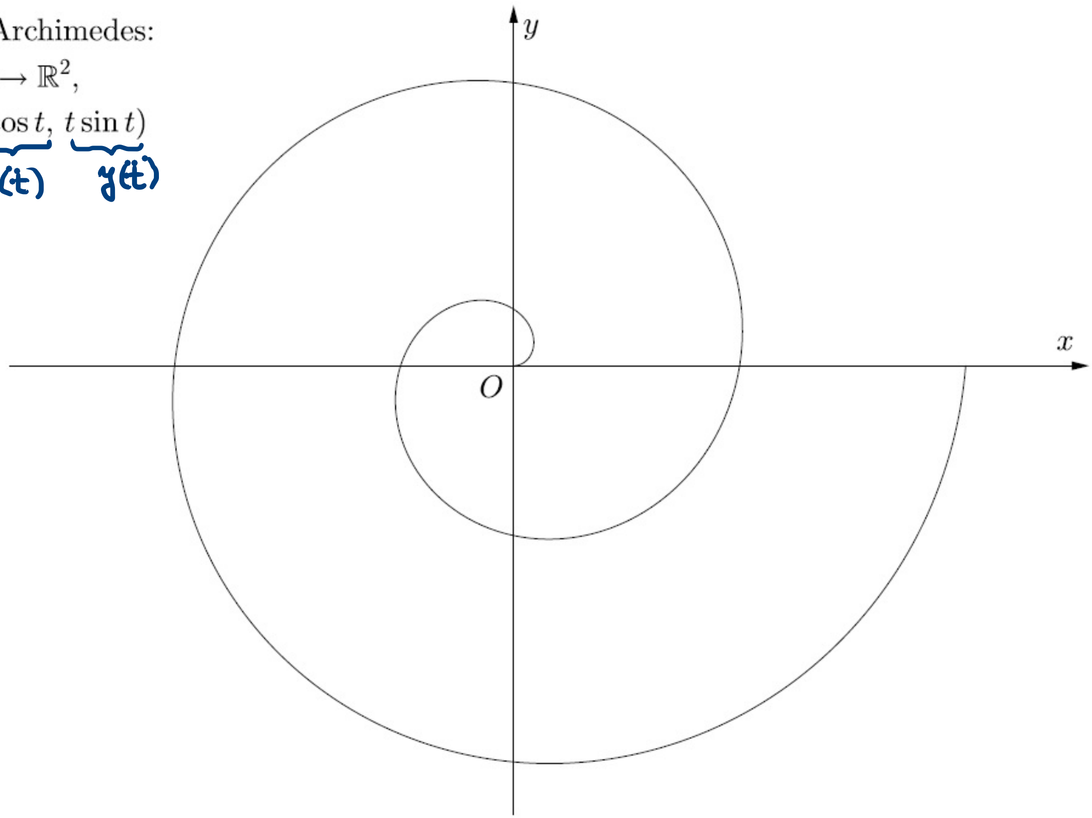
## 5. Limits of vector functions of vector variable

**5.1. Def.** If $A \subseteq \mathbb{R}$, then any function $f : A \to \mathbb{R}^m$ will be called vector function of one real variable.

**Examples.** a) Spiral of Archimedes:

$f : [0, 4\pi] \to \mathbb{R}^2$,

$f(t) = (\underbrace{t\cos t}_{x(t)},\ \underbrace{t\sin t}_{y(t)})$

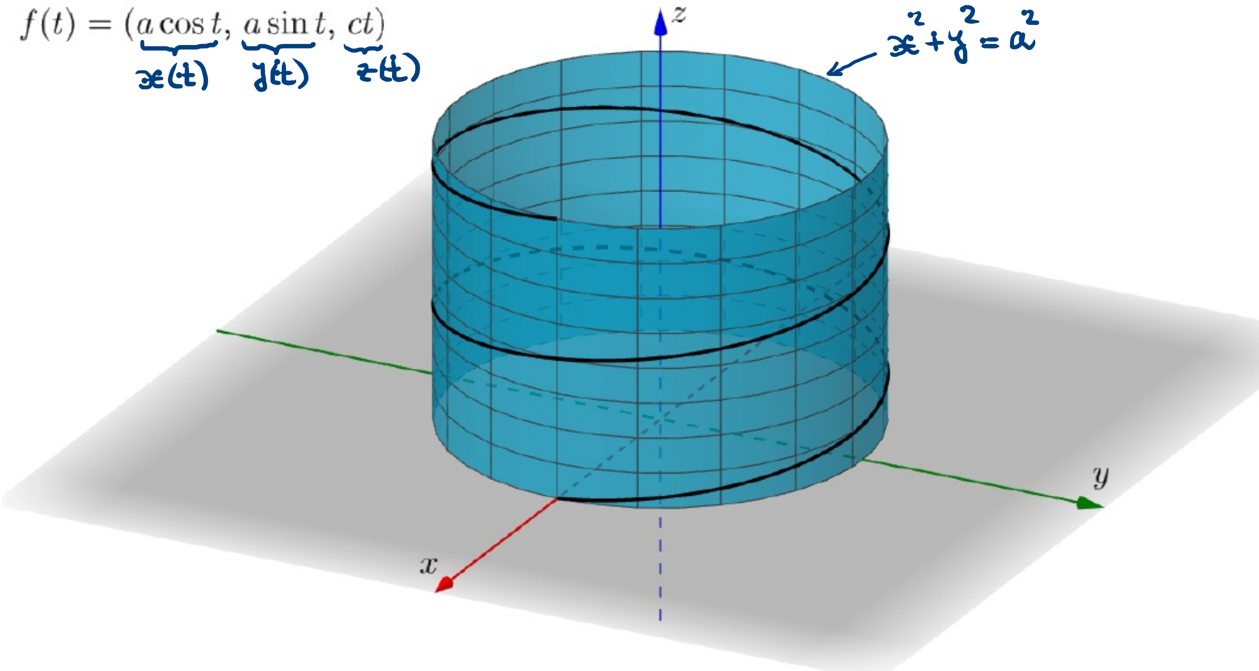


b) Helix:

$f : [0, 4\pi] \to \mathbb{R}^3$,

$f(t) = (\underbrace{a\cos t}_{x(t)}, \underbrace{a\sin t}_{y(t)}, \underbrace{ct}_{z(t)})$

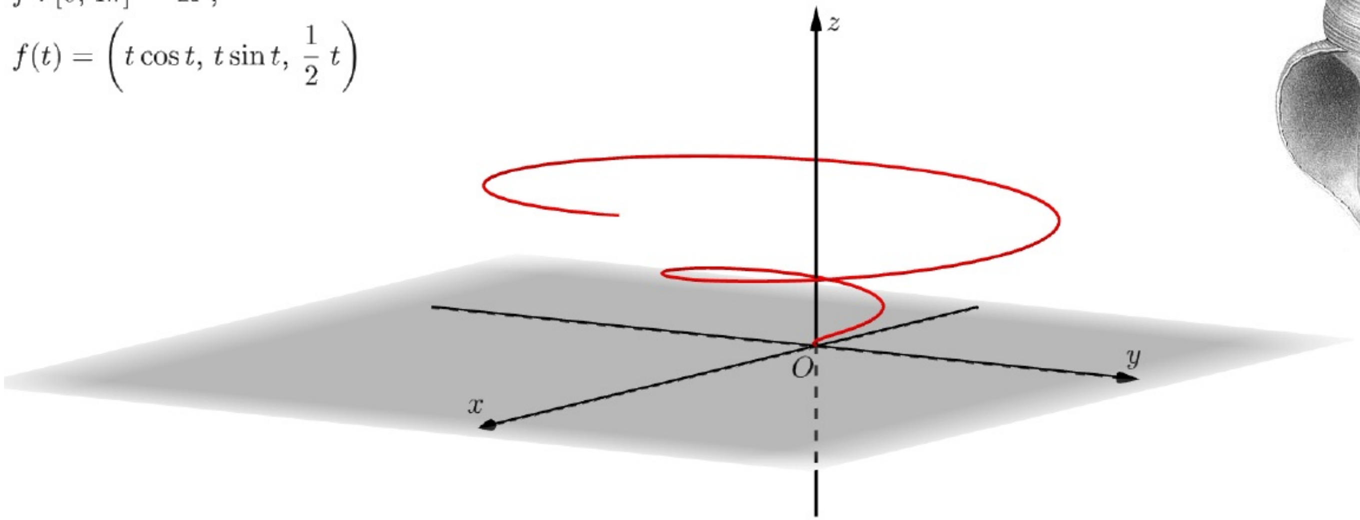
$x^2 + y^2 = a^2$

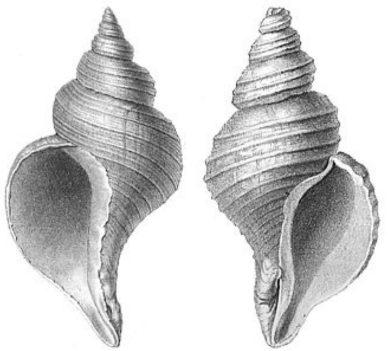



c) Conical spiral:

$f : [0, 4\pi] \to \mathbb{R}^3$,

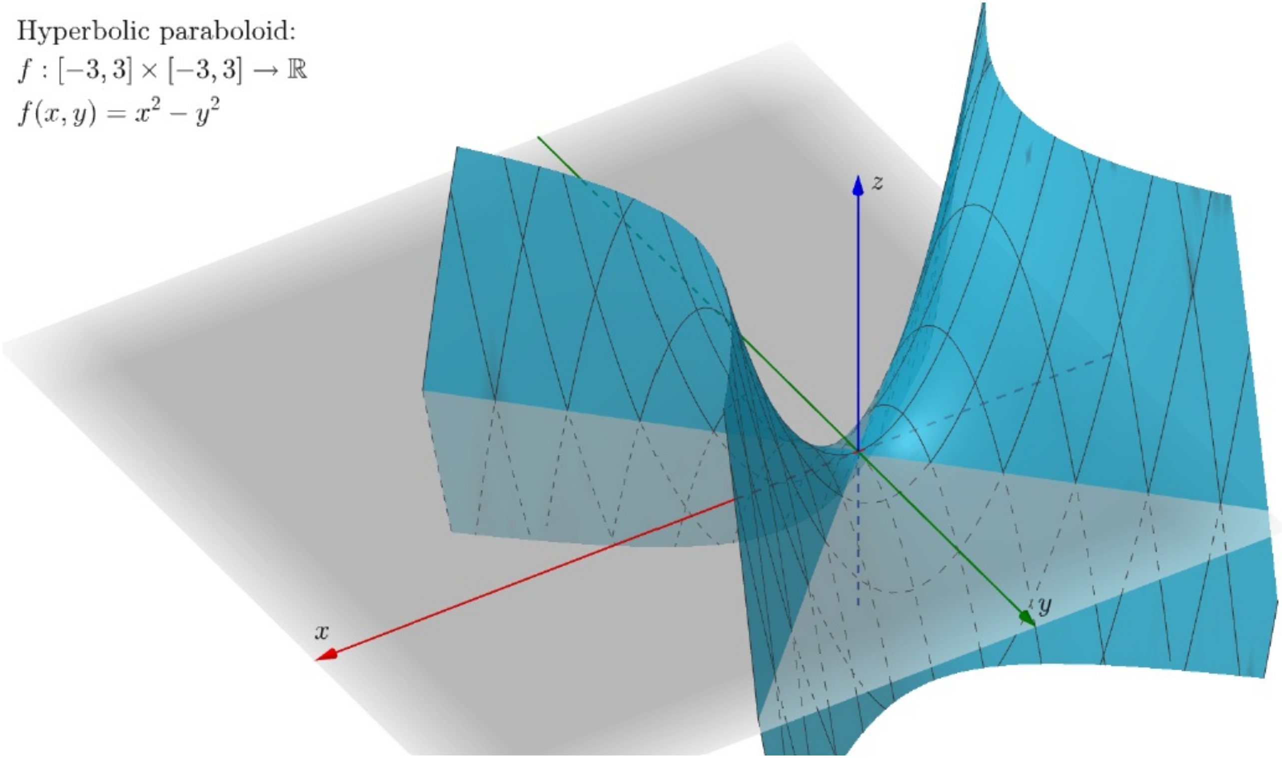
$$f(t) = \left(t\cos t,\ t\sin t,\ \frac{1}{2}\, t\right)$$

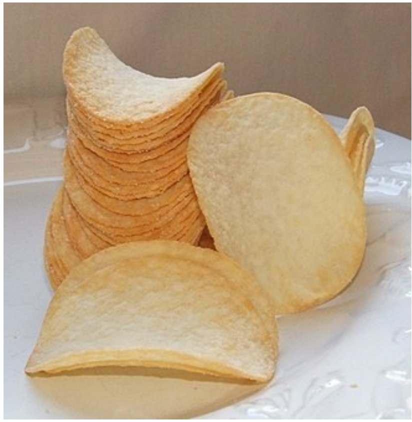


If $A \subseteq \mathbb{R}^n$, then any function $f: A \to \mathbb{R}$ will be called real valued function of vector variable

$f: \mathbb{R}^2 \to \mathbb{R}$ $\qquad f(x, y) = x^2 - y^2$



If $A \subseteq \mathbb{R}^n$, then any function $f : A \to \mathbb{R}^m$ will be called a vector function of vector variable.

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ $\qquad$ $f(u, v) = (a \sin u \cos v,\ a \sin u \sin v,\ a \cos u)$

↳ a parametrization of a sphere by using the spherical coordinates

**5.2. Definition.** Let $A \subseteq \mathbb{R}^n$, $f: A \to \mathbb{R}^m$. We define the functions $f_1, \dots, f_m : A \to \mathbb{R}$ as follows: let $x \in A \Rightarrow f(x) =: y \in \mathbb{R}^m$, so $y$ has the form $y = (y_1, \dots, y_m)$

We set $\quad f_1(x) := y_1, \dots, f_m(x) := y_m$

$f_1, \dots, f_m$ = the (scalar) components of the vector function $f$

$\Rightarrow \quad \forall x \in A: \quad f(x) = (f_1(x), \dots, f_m(x))$

We write $f = (f_1, \dots, f_m) : A \to \mathbb{R}^m$ whenever we want to emphasize the components of the vector function $f$.

$$f : \mathbb{R}^3 \to \mathbb{R}^2 \qquad f(x,y,z) = \Big( \underbrace{x - 2y + \sin(x-z)}_{f_1(x,y,z)}, \underbrace{e^x + \sin y + \cos z}_{f_2(x,y,z)} \Big)$$

**5.3. Definition.** Let $A \subseteq \mathbb{R}^n$, $f : A \to \mathbb{R}^m$, $a \in A'$, $b \in \mathbb{R}^m$. One says that $f$ has the limit $b$ at the point $a$ if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \text{ s.t. } \forall x \in A \setminus \{a\} \text{ with } \|x - a\| < \delta \text{ we have } \|f(x) - b\| < \varepsilon$$

It can be easily proved that $f$ can have at most one limit at $a$. If the limit exists, then we write $\lim\limits_{x \to a} f(x) = b$.

## 5.4 Theorem (Heine's theorem, characterization of limits by means of sequences)

Let $A \subseteq \mathbb{R}^n$, $f: A \to \mathbb{R}^m$, $a \in A'$, $b \in \mathbb{R}^m$. Then

(*)

$$\lim_{x \to a} f(x) = b \iff \forall (x_k) \text{ seq. in } A \setminus \{a\} \text{ s.t. } \lim_{k\to\infty} x_k = a \text{ we have } \lim_{k\to\infty} f(x_k) = b$$

**Proof.** $\boxed{\Rightarrow}$ Assume that $\lim_{x \to a} f(x) = b$. Let $(x_k)$ be a seq. in $A \setminus \{a\}$ s.t. $\lim_{k\to\infty} x_k = a$

Let $\varepsilon > 0$ $\Rightarrow \exists \delta > 0$ s.t. $\forall x \in A \setminus \{a\}$ with $\|x - a\| < \delta$ we have $\|f(x) - b\| < \varepsilon$

Since $(x_k) \longrightarrow a \Rightarrow \exists k_0 \in \mathbb{N}$ s.t. $\forall k \geq k_0$: $\|x_k - a\| < \delta$ $\Rightarrow$

$\Rightarrow \forall k \geq k_0$: $\|f(x_k) - b\| < \varepsilon$ $\Rightarrow$ $\lim_{k\to\infty} f(x_k) = b$.

$\boxed{\Leftarrow}$ Assume that (*) holds, but $f$ does not have the limit $b$ at the point $a$

$\exists \varepsilon > 0$ s.t. $\forall \delta > 0$ $\exists x \in A \setminus \{a\}$ with $\|x - a\| < \delta$ s.t. $\|f(x) - b\| \geq \varepsilon$

Taking $\delta = \frac{1}{k}$ $\Rightarrow \forall k \geq 1$ $\exists x_k \in A \setminus \{a\}$ with $\|x_k - a\| < \frac{1}{k}$ s.t. $\|f(x_k) - b\| \geq \varepsilon$ $\Rightarrow\Leftarrow$

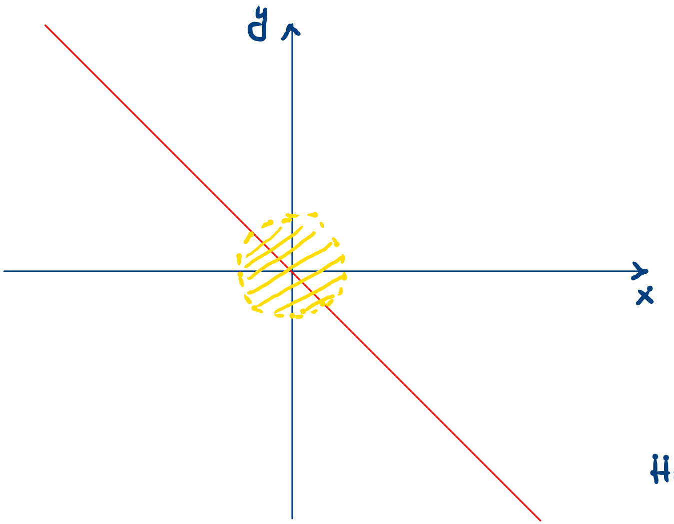
$\Rightarrow (x_k)$ is a seq. in $A \setminus \{a\}$, $(x_k) \longrightarrow a$ $\overset{(*)}{\Rightarrow}$ $\lim_{k\to\infty} f(x_k) = b \Rightarrow \lim_{k\to\infty} \|f(x_k) - b\| = 0$

$\Rightarrow \lim_{x \to a} f(x) = b$.

**Example.** Prove that $\nexists \lim_{(x,y)\to(0,0)} \frac{x}{x+y} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x}{x+y}$

$f: A \to \mathbb{R}$, $f(x,y) = \frac{x}{x+y}$ where $A = \{(x,y)\in\mathbb{R}^2 \mid x+y \neq 0\}$

$\hookrightarrow$ $\mathbb{R}^2$ without the points on the line $y = -x$ (second bisector)




$a = (0,0) \notin A$, but $a \in A'$

$$\left.\begin{array}{l} \left(\frac{1}{k}, \frac{1}{k}\right) \xrightarrow{k\to\infty} (0,0); \quad f\left(\frac{1}{k}, \frac{1}{k}\right) = \frac{1}{2} \xrightarrow{k\to\infty} \frac{1}{2} \\ \left(\frac{1}{k}, 0\right) \xrightarrow{k\to\infty} (0,0); \quad f\left(\frac{1}{k}, 0\right) = 1 \xrightarrow{k\to\infty} 1 \end{array}\right\} \Rightarrow$$

$$\overset{\text{Heine's Thm}}{\Longrightarrow} \nexists \lim_{(x,y)\to(0,0)} f(x,y)$$

**5.5. Theorem.** Let $A \subseteq \mathbb{R}^n$, $f: A \to \mathbb{R}^m$, $a \in A'$, $b \in \mathbb{R}^m$. Then

$$\lim_{x\to a} f(x) = b \iff \lim_{x\to a} \|f(x) - b\| = 0$$

**5.6. Theorem.** Let $A \subseteq \mathbb{R}^n$, $f = (f_1, \dots, f_m): A \to \mathbb{R}^m$, $a \in A'$, $b = (b_1, \dots, b_m) \in \mathbb{R}^m$.

Then $\lim_{x\to a} f(x) = b \iff \lim_{x\to a} f_i(x) = b_i \quad \forall i = \overline{1,m}$.

## 6. Continuity of vector functions of vector variable

**6.1. Definition.** Let $A \subseteq \mathbb{R}^n$, $f : A \to \mathbb{R}^m$, $a \in A$. One says that $f$ is continuous at $a$ if

$$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 \ \text{s.t.} \ \forall\, x \in A \ \text{with} \ \|x - a\| < \delta \ \text{we have} \ \|f(x) - f(a)\| < \varepsilon.$$

**Remark** If $a$ is an isolated point of $A$ $\Rightarrow$ every function is continuous at $a$

**6.2. Theorem** (characterization of continuity by means of sequences). Let $A \subseteq \mathbb{R}^n$, $f : A \to \mathbb{R}^m$, $a \in A$. Then $f$ is continuous at $a$ $\Leftrightarrow$ $\forall\, (x_k)$ seq. in $A$ s.t. $\lim\limits_{k\to\infty} x_k = a$ : $\lim\limits_{k\to\infty} f(x_k) = f(a)$

**6.3. Theorem.** Let $A \subseteq \mathbb{R}^n$, $f : A \to \mathbb{R}^m$, $a \in A \cap A'$. Then

$$f \text{ is continuous at } a \iff \lim_{x \to a} f(x) = f(a).$$

**6.4. Theorem.** Let $A \subseteq \mathbb{R}^n$, $f : A \to \mathbb{R}^m$, $f = (f_1, \ldots, f_m)$, $a \in A$. Then

$$f \text{ is continuous at } a \iff f_i : A \to \mathbb{R} \text{ is continuous at } a, \ \forall\, i = \overline{1, m}$$

**6.5 Theorem.** If $A$ is a compact subset of $\mathbb{R}^n$

$f: A \to \mathbb{R}^m$ is continuous on $A$

$\Longrightarrow$ $f(A)$ is compact in $\mathbb{R}^m$

**Proof.** $\cancel{?}$ $f(A)$ is compact $\Longleftrightarrow$ $f(A)$ is sequentially compact $\quad \boxed{f(A) = \{ f(x) \mid x \in A \}}$

$\Longleftrightarrow$ every seq. in $f(A)$ has a convergent subsequence whose limit belongs to $f(A)$

Let $(y_k)$ be an arbitrary seq. in $f(A)$ $\Longrightarrow$ $\forall\, k \geq 1$ $\exists\, x_k \in A$ s.t. $y_k = f(x_k)$

$\Longrightarrow$ $(x_k)$ is a seq. in $A$ $\underset{\text{A compact}}{\Longrightarrow}$ $\exists\, (x_{k_j})_{j\geq 1}$ subseq. of $(x_k)$ and $\exists\, x \in A$ s.t.

$$\lim_{j\to\infty} x_{k_j} = x$$

$\Downarrow$ $f$ is continuous on $A$, hence at $x$

$$\lim_{j\to\infty} \underbrace{f(x_{k_j})}_{y_{k_j}} = f(x)$$

$$\Longrightarrow \lim_{j\to\infty} y_{k_j} = f(x) \in f(A)$$