UCB Math 128B, Spring 2016: Programming Assignment 4

Due April 28

In this programming assignment, you will implement a numerical method for calculating *minimal surfaces*. These are surfaces with minimum area (at least locally) for a given boundary shape. They arise in many physical applications, for example as the surface formed by a soap film between a given wire frame.

We will solve on the unit square domain

$$\Omega = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\} \tag{1}$$

On the boundary $\partial\Omega$ of the square, we will impose the shape (the z-coordinate)

$$z(x, y) = 3y(1 - y) \tag{2}$$

Inside the domain, the shape $z(x, y)$ satisfies the differential equation

$$(1 + z_x^2)z_{yy} - 2z_x z_y z_{xy} + (1 + z_y^2)z_{xx} = 0 \tag{3}$$

All derivatives will be approximated by second-order finite differences, on a regular grid of size $n \times n$ with spacing $h = k = 1/n$. Equation (3) is nonlinear, and we will solve it using a type of fixed point iteration. First, set $z^{(0)}(x, y) = 0$. Then iterate by solving

$$(1 + [z_x^{(k-1)}]^2)z_{yy}^{(k)} - 2z_x^{(k-1)} z_y^{(k-1)} z_{xy}^{(k)} + (1 + [z_y^{(k-1)}]^2)z_{xx}^{(k)} = 0 \tag{4}$$

for $z^{(k)}(x, y)$, until the differences between two iterates, $\|z^{(k)} - z^{(k-1)}\|_\infty$, is less than 10^{-8}. Note that since $z^{(k-1)}$ is a given function at step k, equation (4) is linear in $z^{(k)}$ and can be solved as a standard linear finite difference problem.

1. Implement the scheme (4) in MATLAB, and solve the problem on a grid of size $n = 40$. If the arrays `x,y` contain the grid coordinates as produced by `ndgrid`, and `z` the solution, then the shape can be plotted in a 3D plot with the commands

```
surf(x', y', z'), axis equal, shading interp
cameramenu, lighting phong, camlight right
```

Plot the resulting solution, and report the convergence of the fixed points iterations (the differences between two iterates as above).

2. Perform a convergence study as follows. Solve the problem for a sequence of grids, corresponding to $n = 5$, 10, 20, 40, and 80. Consider the solution on the finest grid ($n = 80$) the "exact solution", and compute max-norm errors of the solution on the remaining four grids. Plot these errors vs. $h = 1/n$ in a log-log plot, and estimate the slope of the curve.