Voting Protocols

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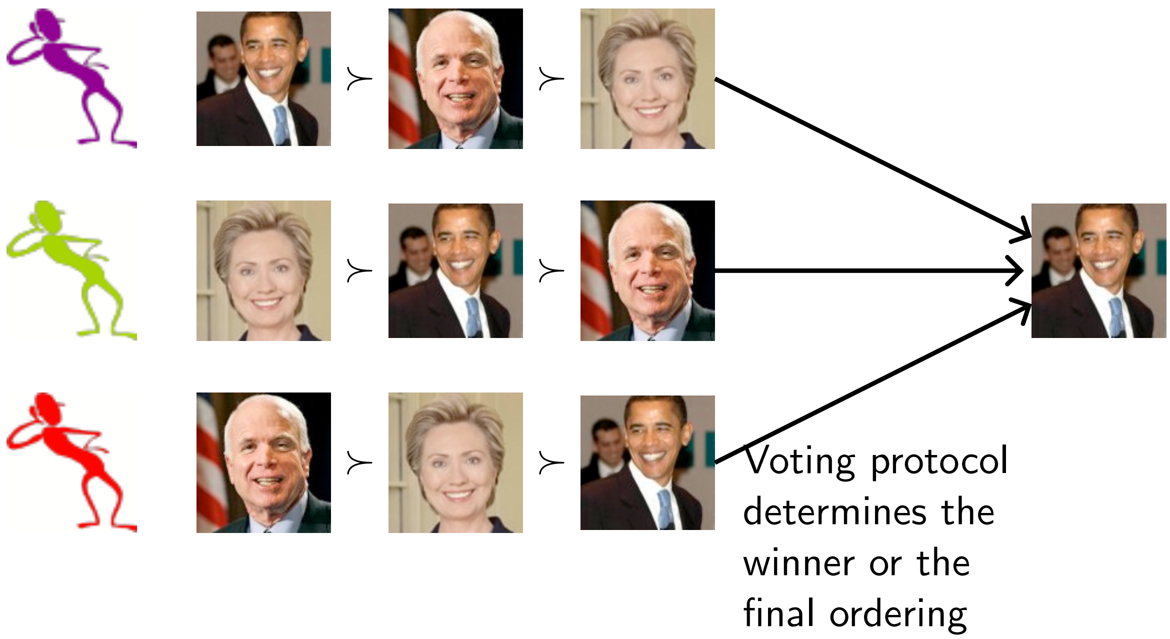
Introduction

- Social choice: preference aggregation
- Our settings
  - A set of agents have preferences over a set of alternatives
  - Taking preferences of all agents, the mechanism outputs a social preference over the set of alternatives or output a single winner
  - Hope to satisfy some desired properties
- Voting protocols are examples of social choice mechanisms

Voting



Example Voting Protocols

- Plurality Voting
  - Each voter cast a single vote.
  - The candidate with the most votes is selected.
- Plurality with Runoff (President election in France)
  - Each voter cast a single vote in the first round.
  - A winner is selected among the top two candidates using the plurality rule in the second round.
- Approval Voting (Mathematical Association of America, etc.)
  - Each voter can cast a single vote for as many candidates as he wants.
  - The candidate with the most votes is selected.
- Single Transferable Vote (Instant Runoff) (Electoral Reform Society in UK)
  - Each candidate votes for their most-preferred candidate
  - The candidate with the fewest votes is eliminated if there is no majority winner
  - Each voter who voted for the eliminated candidate transfers their vote to their most-preferred candidate among the remaining candidates

Voting Paradox: The No-Show Paradox

35 agents: $a \succ c \succ b$
33 agents: $b \succ a \succ c$
32 agents: $c \succ b \succ a$

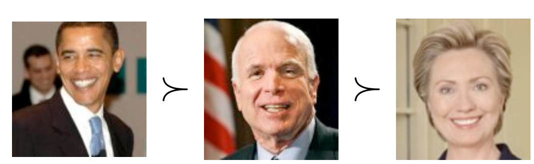
- Which alternative is elected under plurality with runoff?
- Now suppose 4 agents of the first preference do not show up. Which alternative is elected under plurality with runoff?

Pairwise Elections

2 prefer Obama to McCain

2 prefer McCain to Hillary

2 prefer Obama to Hillary



More Voting Protocols

- Pairwise elimination
  - Pair candidates with a schedule
  - The candidate who is preferred by a minority of voters is deleted
  - Repeat until only one candidate is left
- Borda Voting (Election in Slovenia; granting awards in sports.)
  - Each voter submits a full ordering on the m candidates
  - Candidates of an ordering get score $(m-1, m-2, ..., 0)$
  - The candidate with the highest score is selected
- Slater
  - The overall ordering that is inconsistent with as few pairwise elections as possible is selected.
  - NP-hard
- Kemeney
  - The overall ordering that is inconsistent with as few votes on pairs of candidates as possible.
  - NP-hard
- ... and many other voting rules

Positional Scoring Rules

- A positional scoring rule is given by a scoring vector $s =< s_1, \ldots, s_m >$ with $s_1 \geq s_2 \geq \cdots \geq s_m$ and $s_1 > s_m$.
- Each candidate receives s_i points for every voter putting it at the ith position.
- Borda rule, plurality rule, and veto rule are all positional scoring rules.

Vote for a Finale Cake

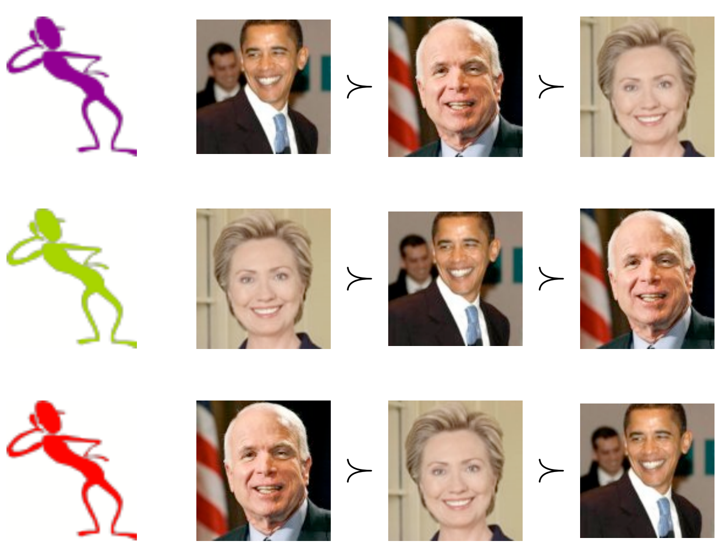
- Chocolate Symphony
  - Three tiers of Valrhona chocolate mousse (bittersweet, milk and white) with chocolate cak
- Tiramisu
  - Coffee-soaked ladyfingers with rich mascarpone mousse, topped with Valrhona cocoa powde
- Blueberry Cheesecake
  - Creamy cheesecake marbled with blueberry pure with a graham cracker crust
- Boston Cream Cake
  - Finale-style Boston original, with moist yellow cake, Bavarian cream and fine chocolate ganach

What is the Perfect Voting Protocol?

Condorcet Condition

- A candidate is a Condorcet winner if it wins all its pairwise elections.
- A voting protocol satisfies the Condorcet condition, if the Condorcet winner, if exists, must be elected by the protocol.
- Condorcet winner may not exist.
- Many voting protocols do not satisfy the Condorcet condition.

Condorcet Circle



2 prefer Obama to McCain

2 prefer McCain to Hillary

2 prefer Hillary to Obama

?

An Example of Condorcet Condition

499 agents: $a \succ b \succ c$
3 agents: $b \succ c \succ a$
498 agents: $c \succ b \succ a$

- Which alternative is the Condorcet winner if exists?
- Which alternative does the plurality voting select?
- Which alternative does the Single Transferable Vote select?

Positional Scoring Rules Violate Condorcet Condition

3 agents: $a \succ b \succ c$
2 agents: $b \succ c \succ a$
1 agent: $b \succ a \succ c$
1 agent: $c \succ a \succ b$

- a is the Condorcet winner
- Any positional scoring rule with $s_2 > s_3$ will elect b.
  - A: $3s_1 + 2s_2 + 2s_3$
  - B: $3s_1 + 3s_2 + s_3$
  - C: $s_1 + 2s_2 + 4s_3$

If $|A| \geq 3$, no positional scoring rule satisfies the Condorcet condition.

Voting Paradox: Sensitivity to A Losing Candidate

35 agents: $a \succ c \succ b$
33 agents: $b \succ a \succ c$
32 agents: $c \succ b \succ a$

- Which alternative is the winner under plurality voting?
- Which alternative is the winner under Borda voting?
- What happens if c drops off?

Notations

- N: a set of individuals, $|N| = n$
- A: a set of alternatives, $|A| = m$
- $\succ_i$: agent i's preference over A (e.g. $a_1 \succ_i a_3 \succ_i a_5$)
- L: the set of total orders, $\succ \in L$
- L^n: the set of preference profiles, $[\succ] \in L^n$
- A social welfare function is a function $W : L^n \rightarrow L$
- $\succ_W$: the preference ordering selected by W
- A social choice function is a function $C : L^n \rightarrow A$

Social Welfare Function: Pareto Efficiency

- A social welfare function W is Pareto efficient if for any $a_1, a_2 \in A$, $\forall a_1 \succ_i a_2$ implies that $a_1 \succ_W a_2$.

- It means that when all agents agree on the ordering of two alternatives, the social welfare function must select the ordering.

Social Welfare Function: Independence of Irrelevant Alternatives (IIA)

- A social welfare function W is independent of irrelevant alternatives if, for any $a_1, a_2 \in A$ and any two preference profiles $[\succ'], [\succ''] \in L^n$, $\forall i$

$$(a_1 \succ'_i a_2 \text{ if and only if } a_1 \succ''_i a_2) \Rightarrow$$
$$(a_1 \succ_{W([\succ'])} a_2 \text{ if and only if } a_1 \succ_{W([\succ''])} a_2).$$

- IIA means that if (1) W ranks a_1 ahead of a_2 now, and (2) we change the preferences without change the relative preferences between a_1 and a_2, then a_1 is still ranked ahead of a_2.
- An example with plurality voting protocol

$$\begin{array}{rll} \text{499 agents: } & a \succ b \succ c & \quad a \succ b \succ c \\ \text{3 agents: } & b \succ c \succ a & \Rightarrow b \succ c \succ a \\ \text{498 agents: } & c \succ b \succ a & \quad b \succ a \succ c \end{array}$$

- None of our rules satisfy IIA

Social Welfare Function: Nondictatorship

- We do not have a dictator if there does not exist an i such that $\forall a_1, a_2$,

$$a_1 \succ_i a_2 \Rightarrow a_1 \succ_W a_2$$

- Nondictatorship means that there does not exist a voter such that the social welfare function W always outputs the voter's preference

Arrow's Impossibility Results (1951)

- If $|A| \geq 3$, any social welfare function W can not simultaneously satisfy
  - Pareto efficiency
  - Independence of irrelevant alternatives
  - Nondictatorship
- Most influential result in social choice theory
- Read the proof

Arrow's Impossibility Results

- A surprising result!
- As a characterization result: Any voting protocol for $|A| \geq 3$ alternatives satisfies the Pareto efficiency and IIA if and only if it is dictatorial.
- The importance of Arrow's Theorem is not only due to the result itself but also due to its method.

Maybe asking for a complete ordering is too much? Let's consider social choice functions.

Social Choice Function: Weak Pareto Efficiency

- A social choice function C is weakly Pareto efficient if for any preference profile $[\succ] \in L^n$, if there exist a pair of alternatives a_1 and a_2 such that $\forall i \in N$, $a_1 \succ_i a_2$, then $C(\succ) \neq a_2$.
- It means that a dominated alternative can not be selected.
- Weak Pareto efficiency implies unanimity: If a_1 is the top choice for all agents, we must have $C[\succ] = a_1$.
- Pareto efficient rules satisfy weak Pareto efficiency. But the reverse is not true.

Social Choice Function: Strong Monotonicity

- A social choice function C is strongly monotonic, if for any preference profile $[\succ]$ with $C[\succ] = a$, then for any other preference profile $[\succ']$ with the property that

$$\forall i \in N, \forall a' \in A, a \succ'_i a' \text{ if } a \succ_i a',$$

it must be that $C[\succ'] = a$.
- Strong monotonicity means that if
  - The current winner is a
  - We change the preference profile in the way such that if alternative a' ranks below a previously it is still below a in the new preference

  Then, a is the winner for the new preference profile.
- An example with STV

$$\begin{array}{lllll} \text{9 agents:} & a \succ b \succ c & & \text{12 agents:} & a \succ b \succ c \\ \text{9 agents:} & b \succ c \succ a & \Rightarrow & \text{6 agents:} & b \succ c \succ a \\ \text{7 agents:} & c \succ a \succ b & & \text{7 agents:} & c \succ a \succ b \end{array}$$

- None of our rules satisfy strong monotonicity

Social Choice Function: Nondictatorship

- A social choice function C is nondictatorial if there does not exist an agent i such that C always outputs the top choice of i.

Muller-Satterthwaite's Impossibility Results (1977)

- If $|A| \geq 3$, any social choice function C can not simultaneously satisfy
  - Weak Pareto efficiency (unanimity)
  - Strong monotonicity
  - Nondictatorship
- Social choice functions are no simpler than social welfare functions
- Intuition: We can repeatedly probe a social choice function for given pairs of alternatives, and then construct a full social welfare ordering.

Two Alternatives

- When there are only two alternatives, all voting protocols we've seen coincide, and they seem to do the "right thing".
- Can we formalize this intuition?

Anonymity and Neutrality

- Anonymity: C is anonymous if $C(\succ_1, \succ_2, \ldots, \succ_n) = C(\succ_{\pi(1)}, \succ_{\pi(2)}, \ldots, \succ_{\pi(3)})$ for any permutation π of the voters.
- Neutrality: W is neutral if $C(\pi([\succ])) = \pi(C([\succ]))$ for any permutation π of the alternatives.

Any anonymous voting protocol is nondictatorial.

Positive Responsiveness

A voting protocol satisfies positive responsiveness if whenever some voter raises a (possibly tied) winner a in her preference ordering, a becomes the unique winner.

- This defines some notion of monotonicity.

May's Theorem (1952)

- A voting procedure for two alternatives satisfies
  - anonymity
  - neutrality
  - positive responsiveness

  if and only if it is the plurality rule.
- We now fully characterize the plurality rule.

Characterization Theorems of Positional Scoring Rules

- When $|A| \geq 3$, different voting protocols are really different. We need to under stand their properties to choose one.
- Positional Scoring Rules: $s =< s_1, \ldots, s_m >$ with $s_1 \geq s_2 \geq \dot{\geq} s_m$ and $s_1 > s_m$.
- Generalized Positional Scoring Rules: same but without the constraints that $s_1 \geq s_2 \geq \dot{\geq} s_m$ and $s_1 > s_m$.

Reinforcement (Consistency)

A voting protocol satisfies reinforcement if, whenever we split the voters into two groups and some alternative would win in both groups, then it will also win for the original set of voters.

Continuity

A voting protocol is continuous if, whenever the set of voters N elects a unique winner a, then for any other set of voters N' there exists a number k such that N' together with k copies of N will also elect only a.

Young's Theorem (1975)

- A voting procedure satisfies
  - anonymity
  - neutrality
  - reinforcement
  - continuity

  if and only if it is a generalized positional scoring rule.