13.05.2013.

Фуријеови редови

$f:\mathbb{R}\to\mathbb{R}$ 2π-периодична

$\int_{-\pi}^{\pi} f^2(x)dx < +\infty \iff f^2(x)\in\mathcal{R}[-\pi,\pi]$

$S(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos nx + b_n\sin nx$ Фуријеов ред фје f

$$\left.\begin{aligned} a_n&=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad n\in\mathbb{N}_0\\ b_n&=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx, \quad n\in\mathbb{N}\end{aligned}\right\}\ \text{Фуријеови коеф.}$$

Ⓣ $f:\mathbb{R}\to\mathbb{R}$ 2π-периодична
део по део глатка

$\Rightarrow$ Фуријеов ред конвертира за $\forall x\in\mathbb{R}$ и важи:

$$S(x)=\frac{f(x+)-f(x-)}{2}$$

Специјално, ако f непр. у тачки x, онда $S(x)=f(x)$

Парсевалова једнакост

$$\frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_n^2+b_n^2)=\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx$$

f глатка на (α,β)
$\Rightarrow f$ непр. на (α,β)

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos nx + b_n\sin nx$$

$$f'(x)=\sum_{n=1}^{\infty}\left(-na_n\sin nx + nb_n\cos nx\right) \quad \text{за } \forall x\in(\alpha,\beta)$$

$F:(\alpha,\beta)\to\mathbb{R}$
$F'(x)=f(x)$

$$F(x)=\frac{a_0}{2}x+\sum_{n=1}^{\infty}\left(\frac{a_n}{n}\sin nx-\frac{b_n}{n}\cos nx\right)+C$$

① Развити у Ф.ред $f(x)=\left|\cos\frac{x}{2}\right|$ и на основу тога наћи $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{4n^2-1}$

$$f(x+2\pi)=\left|\cos\frac{x+2\pi}{2}\right|=\left|\cos\left(\frac{x}{2}+\pi\right)\right|=\left|-\cos\frac{x}{2}\right|=\left|\cos\frac{x}{2}\right|=f(x)$$

Не мора 2π да буде основни (најмањи) период, битно да је f 2π - периодична

f је део по део глатка (није диф. кад $\cos\frac{x}{2}=0$)

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{f(x)\cos nx}_{\text{парна фја}}\,dx=\frac{2}{\pi}\int_{0}^{\pi}\left|\cos\frac{x}{2}\right|\cos nx\,dx=\frac{2}{\pi}\int_{0}^{\pi}\cos\frac{x}{2}\cos nx\,dx$$

$\cos\frac{x}{2}\geqslant 0$ на $[0,\pi]$

$$=\frac{1}{\pi}\int_{0}^{\pi}\left(\cos\left(\frac{x}{2}+nx\right)+\cos\left(\frac{x}{2}-nx\right)\right)dx$$

$$=\frac{1}{\pi}\cdot\frac{1}{\frac{1}{2}+n}\sin\left(\frac{x}{2}+nx\right)\Big|_0^{\pi}+\frac{1}{\pi}\cdot\frac{1}{\frac{1}{2}-n}\sin\left(\frac{x}{2}-nx\right)\Big|_0^{\pi}$$

$$=\frac{1}{\pi\left(\frac{1}{2}+n\right)}\sin\left(\frac{\pi}{2}+n\pi\right)+\frac{1}{\pi\left(\frac{1}{2}-n\right)}\sin\left(\frac{\pi}{2}-n\pi\right)$$

$$=\frac{1}{\pi\left(\frac{1}{2}+n\right)}(-1)^n+\frac{1}{\pi\left(\frac{1}{2}-n\right)}(-1)^n$$

$$=\frac{\frac{1}{2}\cancel{-n}+\frac{1}{2}\cancel{+n}}{\pi\left(\frac{1}{4}-n^2\right)}(-1)^n$$

$$=\frac{(-1)^n}{\pi\left(\frac{1}{4}-n^2\right)}$$

$$=\frac{4}{\pi}\cdot\frac{(-1)^{n-1}}{4n^2-1}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{\left|\cos\frac{x}{2}\right|\sin nx}_{\text{непарна}}\,dx=0$$

$$a_0=\frac{4}{\pi}\,\frac{(-1)^{-1}}{-1}=\frac{4}{\pi}$$

$$\boxed{S(x)=\frac{2}{\pi}+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{4n^2-1}\cos nx}$$

Специјално, за $x=0$

$$S(0)=\frac{2}{\pi}+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{4n^2-1}$$

$$f(0)=1$$

$$S(0)=\frac{f(0^+)+f(0^-)}{2}=f(0)$$

зато што је f непр. у нули

$$\Rightarrow \frac{2}{\pi}+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{4n^2-1}=1$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{4n^2-1}=\frac{1-\frac{2}{\pi}}{\frac{4}{\pi}}$$

$$\boxed{\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{4n^2-1}=\frac{\pi}{4}-\frac{1}{2}}$$

Напомена

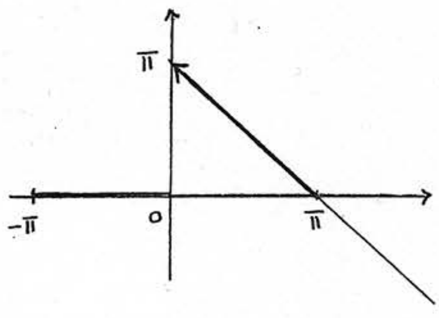
Ако је f парна $\Rightarrow b_n=0$, $\forall n\in\mathbb{N}$

f непарна $\Rightarrow a_n=0$, $\forall n\in\mathbb{N}_0$

2

$$f(x)=\begin{cases}0, & x\le 0\\ \pi-x, & x>0\end{cases}$$

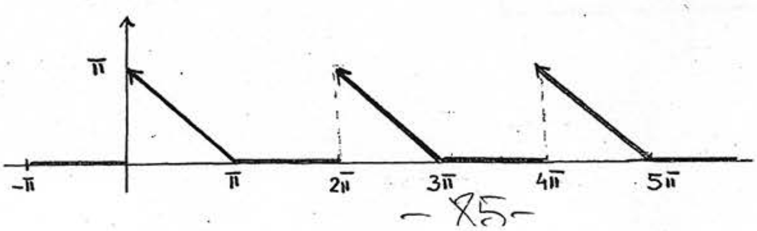
Развити f у Ф.ред на $(-\pi,\pi)$ и наћи $A=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$, $B=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}$



f није периодична!

$f^*(x)=f(x) \qquad x\in[-\pi,\pi)$

$f^*(x+2\pi)=f^*(x) \qquad \forall x\in\mathbb{R}$



f^* 2π - периодична
део по део глатка

$S(x)$ Фуријеов ред фје f^* на $\mathbb{R}$
$S(x)$ Фуријеов ред фје f на $(-\pi,\pi)$

- $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f^*(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} \underbrace{f^*(x)}_{=0}\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} \underbrace{f^*(x)}_{\pi-x}\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi}(\pi-x)\cos nx\,dx = \left\{\begin{matrix} u=\pi-x & dv=\cos nx\,dx \\ du=-dx & v=\frac{1}{n}\sin nx\end{matrix}\right\}$$

$$= \frac{1}{\pi}\left(\frac{1}{n}(\pi-x)\sin nx\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\sin nx\,dx\right)$$

важи за $n\neq 0$

$$= \frac{1}{\pi}\cdot\frac{1}{n}\left(-\frac{1}{n}\right)\cos nx\Big|_0^{\pi}$$

$$= -\frac{1}{n^2\pi}\left((-1)^n-1\right)$$

$$= \frac{(-1)^{n+1}+1}{n^2\pi}$$

Пошто претходно важи за $n\neq 0$, морамо посебно да израчунамо a_0

- $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f^*(x)\,dx = \frac{1}{\pi}\int_0^{\pi}(\pi-x)\,dx = \frac{1}{\pi}\left(\pi x-\frac{x^2}{2}\right)\Big|_0^{\pi} = \frac{1}{\pi}\cdot\frac{\pi^2}{2} = \frac{\pi}{2}$$

- $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f^*(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi}(\pi-x)\sin nx\,dx = \left\{\begin{matrix} u=\pi-x & dv=\sin nx\,dx \\ du=-dx & v=-\frac{1}{n}\cos nx\end{matrix}\right\}$$

$$= \frac{1}{\pi}\left(-\frac{1}{n}(\pi-x)\cos nx\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\cos nx\,dx\right)$$

$$= \frac{1}{\pi}\left(\frac{\pi}{n} - \frac{1}{n}\cdot\frac{1}{n}\sin nx\Big|_0^{\pi}\right)$$

$$= \frac{1}{\pi}\cdot\frac{\pi}{n}$$

$$= \frac{1}{n}$$

$$\boxed{S(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}+1}{n^2\pi}\cos nx + \frac{1}{n}\sin nx\right]}$$

$$\frac{(-1)^{n+1}+1}{n^2\pi} = \begin{cases} \frac{2}{n^2\pi}, & n=2k-1 \\ 0, & n=2k \end{cases}$$

$$S(0) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}+1}{n^2\pi} = \frac{\pi}{4} + \sum_{k=1}^{\infty} \frac{2}{(2k-1)^2\pi} = \frac{\pi}{4} + \frac{2}{\pi} \underbrace{\sum_{n=1}^{\infty} \frac{1}{(2k-1)^2}}_{A}$$

$$S(0) = \frac{f(0^+)+f(0^-)}{2} = \frac{\pi+0}{2} = \frac{\pi}{2}$$

$$\Rightarrow \quad \frac{\pi}{2} = \frac{\pi}{4} + \frac{2}{\pi}A$$

$$A = \frac{\pi}{4} \cdot \frac{\pi}{2}$$

$$\boxed{A = \frac{\pi^2}{8}}$$

$\cos n\pi = (-1)^n$
$\sin n\pi = 0$

$$\cos\frac{n\pi}{2} = \begin{cases} (-1)^k, & n=2k \\ 0, & n=2k-1 \end{cases}$$

$$\sin\frac{n\pi}{2} = \begin{cases} 0, & n=2k \\ (-1)^{k+1}, & n=2k-1 \end{cases}$$

Заменићемо $x=\frac{\pi}{2}$ да бисмо добили B

$$S\left(\frac{\pi}{2}\right) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left[\underbrace{\underset{\substack{\parallel \\ 0 \\ \text{ако } n=2k}}{\frac{(-1)^{n+1}+1}{n^2\pi}} \; \underset{\substack{\parallel \\ 0 \\ \text{ако } n=2k-1}}{\cos\frac{n\pi}{2}}}_{\text{увек једнако нули}} + \frac{1}{n}\sin\frac{n\pi}{2} \right]$$

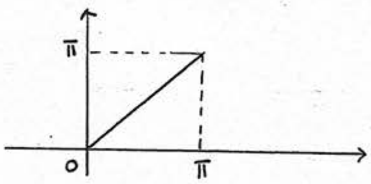
$$\left.\begin{aligned} S\left(\frac{\pi}{2}\right) &= \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{1}{n}\sin\frac{n\pi}{2} = \frac{\pi}{4} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} = \frac{\pi}{4} + B \\ S\left(\frac{\pi}{2}\right) &\underset{\substack{\uparrow \\ f \text{ непр. у } \frac{\pi}{2}}}{=} f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \end{aligned}\right\} \Rightarrow \frac{\pi}{4} + B = \frac{\pi}{2}$$

$$\boxed{B = \frac{\pi}{4}}$$

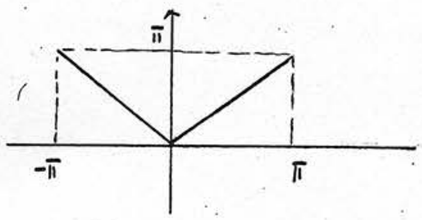
③ $f(x) = x$ развити у Ф. ред на $[0,\pi]$

а) по косинусима
б) по синусима

Одредити: $A = \sum_{n=1}^{\infty} \frac{1}{n^2}$ $\qquad B = \sum_{n=1}^{\infty} \frac{1}{n^4}$ $\qquad C = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1}$

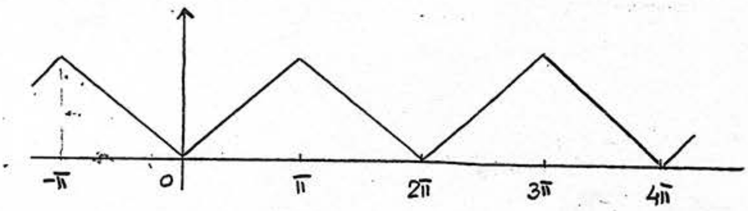


а) Да бисмо развили фју по косинусима, треба да буде парна



$f^*(x) = |x|$ на $[-\pi,\pi]$

$f^*(x+2\pi) = f^*(x)$ $\qquad x \in \mathbb{R}$



$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx = \left\{ \begin{array}{ll} u = x & dv = \cos nx\,dx \\ du = dx & v = \frac{1}{n}\sin nx \end{array} \right\}$$

парна

$$= \frac{2}{\pi}\left(\frac{1}{n} x \sin nx \Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi} \sin nx\,dx \right)$$

$$= \frac{2}{\pi}\left(-\frac{1}{n}\right)\left(-\frac{1}{n}\right) \cos nx \Big|_0^{\pi}$$

$$= \frac{2}{n^2\pi}\left((-1)^n - 1\right)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{1}{\pi}x^2 \Big|_0^{\pi} = \pi$$

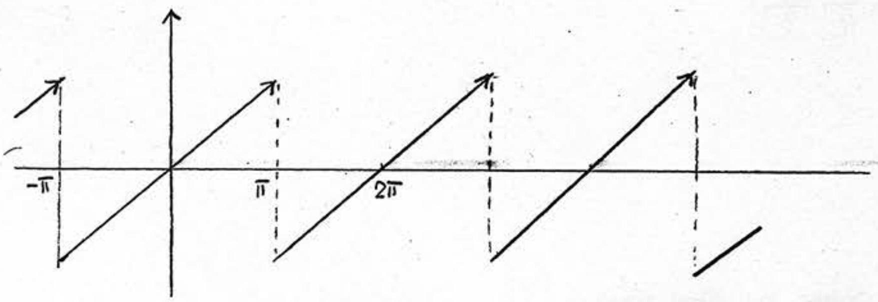
$b_n = 0$ $\qquad$ јер је f^* парна

$$S_1(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n-1}{n^2}\cos nx$$

$$\frac{(-1)^n-1}{n^2} = \begin{cases} 0 \ , & n=2k \\ -\frac{2}{n^2} \ , & n=2k-1 \end{cases}$$

$$\boxed{S_1(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2}}$$

б) Да бисмо развили фју по синусима, треба да буде непарна



$f^*(x) = x \qquad x\in[-\pi,\pi)$

$f^*(x+2\pi) = f^*(x) \qquad x\in\mathbb{R}$

$a_n = 0 \qquad$ јер је f^* непарна

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx = \begin{cases} u=x & dv=\sin nx\,dx \\ du=dx & v=-\frac{1}{n}\cos nx \end{cases}$$

$$= \frac{2}{\pi}\left(-\frac{1}{n}\,x\cos nx\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\,dx\right)$$

$$= \frac{2}{\pi}\left(-\frac{1}{n}\,\pi(-1)^n + \frac{1}{n^2}\sin nx\Big|_0^{\pi}\right)$$

$$= \frac{2}{n}(-1)^{n+1}$$

$$\boxed{S_2 = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx}$$

Парсевалова једнакост

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2+b_n^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx$$

Примењујемо на $S_2(x)$:

$$a_0 = 0$$
$$a_n = 0$$
$$b_n = \frac{2}{n}(-1)^{n+1}$$

$$\sum_{n=1}^{\infty}\frac{4}{n^2} = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2}{\pi}\frac{x^3}{3}\Big|_0^{\pi} = \frac{2\pi^2}{3}$$

$$A = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{4}\cdot\frac{2\pi^2}{3}$$

$$\boxed{A = \frac{\pi^2}{6}}$$

⌈ довољно је знати једну од три суме

$$\sum_{n=1}^{\infty}\frac{1}{n^m} \qquad \sum_{n=1}^{\infty}\frac{1}{(2n-1)^m} \qquad \sum_{n=1}^{\infty}\frac{1}{(2n)^m}$$

за фиксирано $m>1$
и могу се одредити и друге две суме ⌋

$$B = \sum_{n=1}^{\infty}\frac{1}{n^4} = \sum_{k=1}^{\infty}\frac{1}{(2k)^4} + \sum_{k=1}^{\infty}\frac{1}{(2k-1)^4}$$

$$= \frac{1}{16}B + \sum_{k=1}^{\infty}\frac{1}{(2k-1)^4}$$

$$\frac{15}{16}B = \underbrace{\sum_{k=1}^{\infty}\frac{1}{(2k-1)^4}}$$

ово налазимо применом Парсевала на S_1

$$a_0 = \pi$$

$$a_{2k-1} = -\frac{4}{\pi}\cdot\frac{1}{(2k-1)^2}$$

$$a_{2k} = 0$$
$$b_n = 0$$

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{\pi^2}{2} + \frac{16}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^4} = \frac{1}{\pi} \int_{-\pi}^{\pi} |x|^2 \, dx = \frac{2\pi^2}{3}$$

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^4} = \frac{\pi^2}{16} \left(\frac{2\pi^2}{3} - \frac{\pi^2}{2} \right)$$

$$\frac{15}{\cancel{16}} B = \frac{\pi^2}{\cancel{16}} \cdot \frac{\pi^2}{6}$$

$$B = \frac{\pi^4}{15 \cdot 6}$$

$$\boxed{B = \frac{\pi^4}{90}}$$

$$S_2\left(\frac{\pi}{2}\right) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi}{2}$$

$$\sin \frac{n\pi}{2} = \begin{cases} 0, & n = 2k \\ (-1)^{k+1}, & n = 2k-1 \end{cases}$$

$$\left.\begin{aligned} & S_2\left(\frac{\pi}{2}\right) = 2 \sum_{k=1}^{\infty} \frac{(-1)^{2k-1+1}}{2k-1} (-1)^{k+1} = 2 \sum_{n=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} = 2C \\ & S_2\left(\frac{\pi}{2}\right) \underset{\substack{\uparrow \\ f \text{ непр. у } \frac{\pi}{2}}}{=} f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \end{aligned}\right\} \Rightarrow \quad 2C = \frac{\pi}{2}$$

$$\boxed{C = \frac{\pi}{4}}$$

15.05.2013.

① $f(x)=x^2$ развити у Ф-ред на $(0,\pi)$

а) по косинусима

б) по синусима

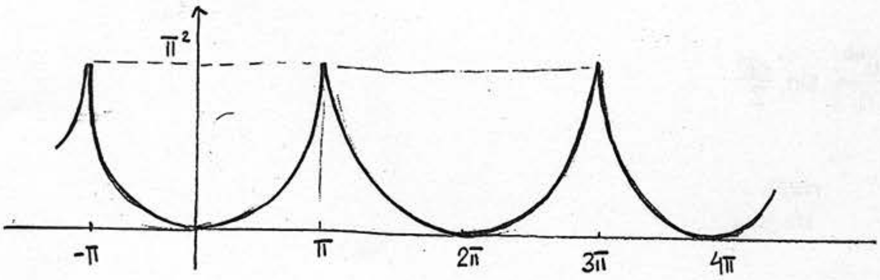
Одредити:
$$A=\sum_{n=1}^{\infty}\frac{1}{n^2}\qquad B=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\qquad C=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

а)

$f_1(x)=x^2 \qquad x\in[-\pi,\pi]$

$f_1(x+2\pi)=f_1(x) \qquad x\in\mathbb{R}$

f_1 парна
2π-периодична
део по део глатка
непрекидна



$b_n=0$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\cos nx\,dx=\frac{2}{\pi}\int_0^{\pi}x^2\cos nx\,dx=\left\{\begin{matrix}u=x^2 & dv=\cos nx\,dx\\ du=2x\,dx & v=\frac{1}{n}\sin nx\end{matrix}\right\}$$

$n\neq 0$
$$=\frac{2}{\pi}\left(\frac{1}{n}x^2\sin nx\Big|_0^{\pi}-\frac{2}{n}\int_0^{\pi}x\sin nx\,dx\right)$$
$$=\frac{2}{\pi}\left(-\frac{2}{n}\right)\int_0^{\pi}x\sin nx\,dx=\left\{\begin{matrix}u=x & dv=\sin nx\,dx\\ du=dx & v=-\frac{1}{n}\cos nx\end{matrix}\right\}$$
$$=-\frac{4}{n\pi}\left(-\frac{1}{n}x\cos nx\Big|_0^{\pi}+\frac{1}{n}\int_0^{\pi}\cos nx\,dx\right)$$
$$=\frac{4}{n^2\pi}\pi\cos n\pi-\frac{4}{n^2\pi}\cdot\frac{1}{n}\sin nx\Big|_0^{\pi}$$
$$=\frac{4}{n^2}(-1)^n$$

Пошто ово важи за $n\neq 0$, посебно рачунамо a_0

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\,dx=\frac{2}{\pi}\cdot\frac{x^3}{3}\Big|_0^{\pi}=\frac{2\pi^2}{3}$$

$$\boxed{S_1(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx}$$

$S_1(x) = f_1(x)$ за $\forall x \in \mathbb{R}$

$S_1(x) = f(x)$ за $x \in (0,\pi)$

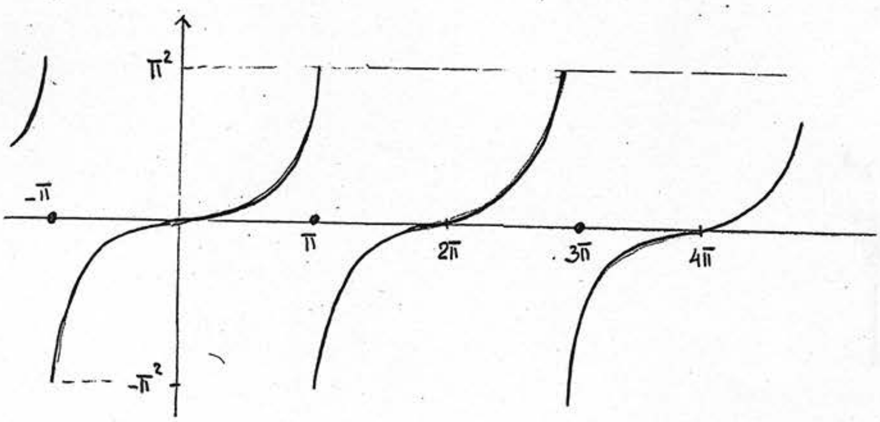
б)

$f_2(x) = x^2$ $\quad x \in [0,\pi)$

$f_2(x) = -x^2$ $\quad x \in (-\pi, 0)$

$f_2(k\pi) = 0$ $\quad \forall k \in \mathbb{Z}$

$f_2(x+2\pi) = f_2(x)$ $\quad \forall x \in \mathbb{R}$



f_2 : непарна

деo по део глатка (⇒ део по део непр.)

2π - периодична

$a_n = 0 \quad n \in \mathbb{N}_0$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f_2(x)\sin nx}_{\substack{\text{парна} \\ \text{(производ две непарне)}}}\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx\,dx = \left\{ \begin{matrix} u = x^2 & dv = \sin nx\,dx \\ du = 2x\,dx & v = -\frac{1}{n}\cos nx \end{matrix} \right\}$$

$$= \frac{2}{\pi}\left(-\frac{1}{n}x^2\cos nx \Big|_0^{\pi} + \frac{2}{n}\int_0^{\pi} x\cos nx\,dx \right)$$

$$= -\frac{2\pi^2}{\pi n}\cos n\pi + \frac{4}{n\pi}\left(x\,\frac{1}{n}\sin nx \Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin nx\,dx \right)$$

$$= -\frac{2\pi}{n}(-1)^n + \frac{4}{n\pi}\left(-\frac{1}{n}\right)\left(-\frac{1}{n}\right)\cos nx \Big|_0^{\pi}$$

$$= \frac{2\pi}{n}(-1)^{n+1} + \frac{4}{n^3\pi}\left((-1)^n - 1\right)$$

$$\boxed{S_2(x) = \sum_{n=1}^{\infty}\left(\frac{2\pi}{n}(-1)^{n+1} + \frac{4}{n^3\pi}\left((-1)^n - 1\right)\right)\sin nx}$$

$S_2(x) = f_2(x)$ за $\forall x \in \mathbb{R}$

(зато што смо изабрали f_2 т.д. $\dfrac{f_2(x^+)+f_2(x^-)}{2} = f_2(x)$, $\forall x \in \mathbb{R}$)

$S_2(x) = f(x)$ $x \in (0,\pi)$

$$S_1(\pi) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} (-1)^n = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$S_1(\pi) = \frac{\pi^2}{3} + 4A$$

$$S_1(\pi) = f_1(\pi) = \pi^2$$

непрекидност f_1

$$\Rightarrow \quad 4A + \frac{\pi^2}{3} = \pi^2$$

$$A = \frac{1}{4}\,\frac{2\pi^2}{3}$$

$$\boxed{A = \frac{\pi^2}{6}}$$

$$S_1(0) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \frac{\pi^2}{3} - 4B$$

$$S_1(0) = f_1(0) = 0$$

$$\Rightarrow \quad 4B = \frac{\pi^2}{3}$$

$$\boxed{B = \frac{\pi^2}{12}}$$

$$A = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{4}A + \underbrace{\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}}_{C}$$

$$\frac{3}{4}A = C$$

$$C = \frac{3}{4}\cdot\frac{\pi^2}{6}$$

$$\boxed{C = \frac{\pi^2}{8}}$$

② Развити у Ф. ред следеће ф-је на $(-\pi,\pi)$

а) $f(x)=x^2+1$

б) $f(x)=\sin^2 x$

в) $f(x)=\left|\sin\frac{x}{2}\right|$

г) $f(x)=3x^2$

а)
$$x^2+1 = 1+\underbrace{\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx}_{S_1(x) \text{ из претх. зад.}} \qquad \text{за } x\in(-\pi,\pi)$$

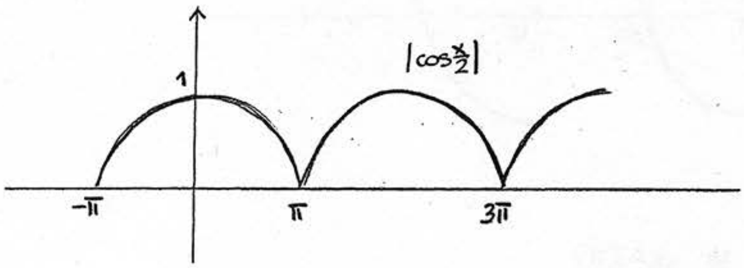
б)
$$\sin^2 x = \frac{1-\cos 2x}{2}$$

Због јединствености Фуријеових коеф.

$$\Rightarrow \quad a_0 = 1$$
$$a_2 = -\frac{1}{2}$$
$$a_n = 0 \qquad n\in\mathbb{N}\setminus\{2\}$$
$$b_n = 0 \qquad n\in\mathbb{N}$$

в) Претходни час, ① зад.

$$\left|\cos\frac{x}{2}\right| = \frac{2}{\pi}+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{4n^2-1}\cos nx \qquad \forall x\in\mathbb{R}$$



$$f(x)=\left|\sin\frac{x}{2}\right| = \left|\cos\left(\frac{x}{2}+\frac{\pi}{2}\right)\right| = \left|\cos\frac{x+\pi}{2}\right| = \frac{2}{\pi}+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{4n^2-1}\cos(nx+n\pi)$$

$$\cos(nx+n\pi) = \cos nx\cdot\underbrace{\cos n\pi}_{(-1)^n} - \sin nx\cdot\underbrace{\sin n\pi}_{0} = (-1)^n\cos nx$$

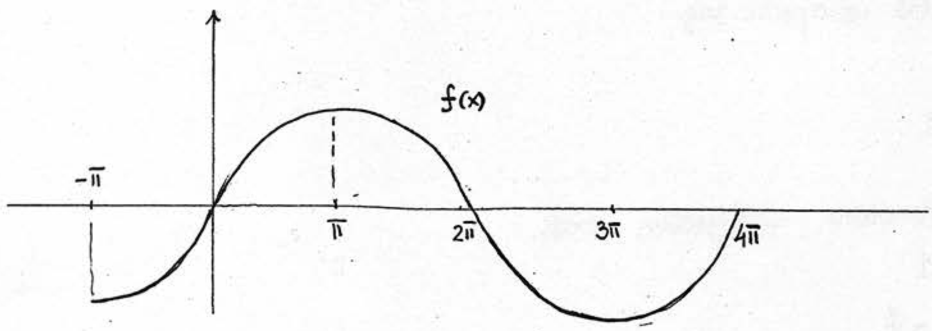
$$\boxed{\left|\sin\frac{x}{2}\right| = \frac{2}{\pi}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos nx} \qquad \forall x\in\mathbb{R}$$

и)

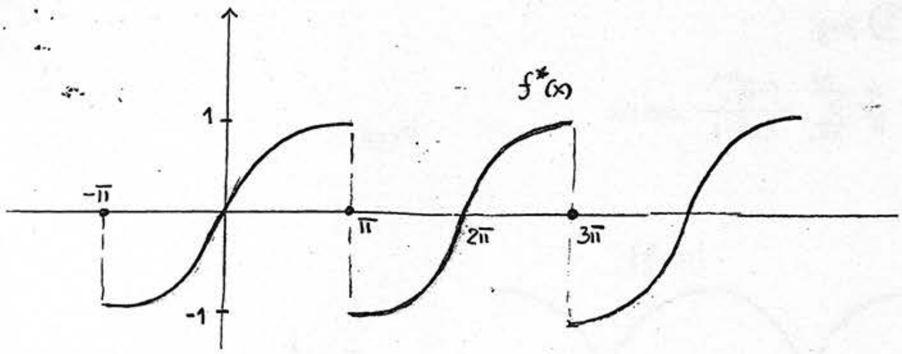
$$3x^2 = 3\left(\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx\right)$$

$$3x^2 = \pi^2 + 12\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx \qquad x\in(-\pi,\pi)$$

(3) $f(x) = \sin\frac{x}{2}$ развити у Ф.ред на $(-\pi,\pi)$ и одредити $\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}$



f је 4π-периодична, глатка



$f^*(x) = \sin\frac{x}{2}$ за $x\in(-\pi,\pi)$

$f^*(-\pi) = f^*(\pi) = 0$

$f^*(x+2\pi) = f^*(x) \qquad \forall x\in\mathbb{R}$

f^* непарна
део по део глатка
2π-периодична

$a_n = 0 \qquad n\in\mathbb{N}_0$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin\frac{x}{2}\sin nx\,dx \underset{\text{парна}}{=} \frac{1}{\pi}\int_{0}^{\pi}2\sin\frac{x}{2}\sin nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{\pi}\left(\cos\left(\frac{2n-1}{2}x\right) - \cos\left(\frac{2n+1}{2}x\right)\right)dx$$

$$\left[\; 2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta) \;\right]$$

$$b_n = \frac{1}{\pi}\cdot\frac{2}{2n-1}\sin\frac{2n-1}{2}x\Big|_0^{\pi} - \frac{1}{\pi}\frac{2}{2n+1}\sin\frac{2n+1}{2}x\Big|_0^{\pi}$$

$$= \frac{1}{\pi}\cdot\frac{2}{2n-1}\sin\left(n\pi-\frac{\pi}{2}\right) - \frac{1}{\pi}\frac{2}{2n+1}\sin\left(n\pi+\frac{\pi}{2}\right)$$

$$= \frac{2}{\pi(2n-1)}(-1)^{n+1} - \frac{2}{\pi(2n+1)}(-1)^n$$

$$= \frac{2}{\pi}(-1)^{n+1}\frac{2n\cancel{+1}+2n\cancel{-1}}{(2n-1)(2n+1)}$$

$$= \frac{8n}{(4n^2-1)\pi}(-1)^{n+1}$$

$$\boxed{S(x) = -\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n n}{4n^2-1}\sin nx}$$

$$S(x) = f^*(x) \qquad \forall x\in\mathbb{R}$$

$$\int f^*(x)\,dx = \int\left(-\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n n}{4n^2-1}\sin nx\right)dx = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}\cos nx + C$$

Фуријеов ред може да се интеграли

$$\int \sin\frac{x}{2}\,dx = -2\cos\frac{x}{2} + C_1$$

За $x\in(-\pi,\pi)$ важи:

$$\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}\cos nx + C = -2\cos\frac{x}{2}$$

$$\cos\frac{x}{2} = \underbrace{-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}\cos nx - \frac{1}{2}C}$$

Фуријеов ред за фју $\cos\frac{x}{2}$, одавде можемо наћи const C

$$\frac{a_0}{2} = -\frac{1}{2}C \qquad \text{због јединствености Ф. коеф.}$$

$$C = -a_0$$

$$C = -a_0 = -\frac{1}{\pi}\int_{-\pi}^{\pi} \cos\frac{x}{2}\,dx = -\frac{2}{\pi}\int_{0}^{\pi}\cos\frac{x}{2}\,dx = -\frac{2}{\pi}\,2\sin\frac{x}{2}\Big|_0^{\pi} \boxed{= -\frac{4}{\pi}}$$

$$\cos\frac{x}{2} = -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}\cos nx - \frac{1}{2}\left(-\frac{4}{\pi}\right) \quad , \ x\in(-\pi,\pi)$$

Специјално, за $x=0$

$$1 = -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1} + \frac{2}{\pi}$$

$$\boxed{\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1} = \frac{1}{2}-\frac{\pi}{4}}$$

16.05.2013.

f 2ℓ-периодична

$f^2 \in \mathcal{R}[-\ell,\ell]$

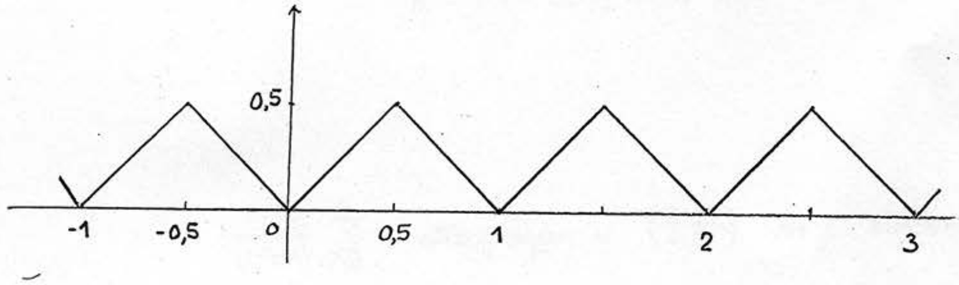
$$S(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell}$$

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx \qquad n\in\mathbb{N}_0$$

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx \qquad n\in\mathbb{N}$$

(1) Развити у Ф.ред фју $f(x) = (x)$

← растојање од x до најближег целог броја



$f(x) = x$ за $x\in[0,\frac{1}{2}]$

$f(x) = -x$ за $x\in[-\frac{1}{2},0]$

$f(x+1) = f(x)$ периодична са периодом $T=1$

$2\ell = 1$

$\ell = \frac{1}{2}$

$$a_n = \frac{1}{\frac{1}{2}}\int_{-1/2}^{1/2}(x)\cos\frac{n\pi x}{\frac{1}{2}}\,dx \underset{\uparrow}{=} 2\cdot 2\int_0^{1/2} x\cos(2n\pi x)\,dx = \left\{\begin{matrix} u=x & dv=\cos(2n\pi x)dx \\ du=dx & v=\frac{1}{2n\pi}\sin(2n\pi x)\end{matrix}\right\}$$

$f(x)$ парна

$$= 4\left(\underbrace{\frac{1}{2n\pi}x\sin(2n\pi x)\Big|_0^{1/2}}_{=0} - \frac{1}{2n\pi}\int_0^{1/2}\sin(2n\pi x)\,dx\right)$$

$$= -\frac{2}{n\pi}\cdot\frac{1}{2n\pi}\left(-\cos(2n\pi x)\right)\Big|_0^{1/2}$$

$$= \frac{1}{n^2\pi^2}\left(\cos n\pi - 1\right)$$

$$= \frac{1}{n^2\pi^2}\left((-1)^n - 1\right)$$

Претходно не важи за $n=0$, то посебно рачунамо

$$a_0 = 2\int_{-1/2}^{1/2} (x)\,dx = 4\int_0^{1/2} x\,dx = 2x^2\Big|_0^{1/2} = \frac{2}{4} = \frac{1}{2}$$

$b_n = 0$ јер је $f(x)$ парна

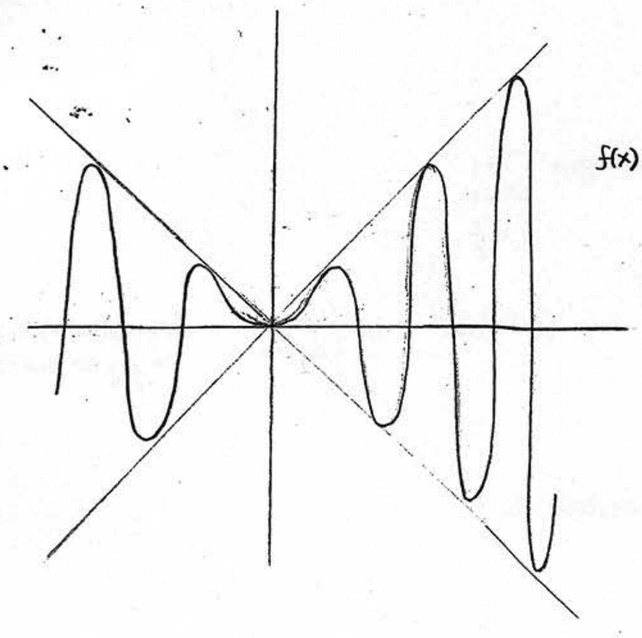
$$S(x) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2}\left((-1)^n - 1\right) \cos(2n\pi x)$$

$$(-1)^n - 1 = \begin{cases} -2, & n = 2k-1 \\ 0, & n = 2k \end{cases}$$

$$S(x) = \frac{1}{4} - \frac{2}{\pi^2}\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \cos\left(2\pi(2k-1)x\right)$$

$S(x) = f(x)$ за $x \in \mathbb{R}$ зато што је f непрекидна

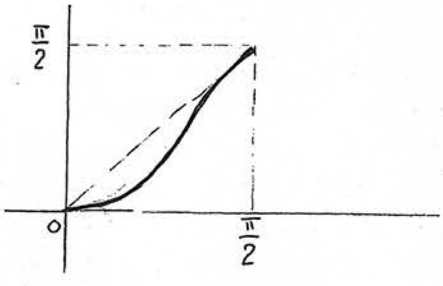
② Развити $f(x) = x\sin x$ на $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ и израчунати $\sum_{n=1}^{\infty} \frac{4n^2+1}{(4n^2-1)^2}$



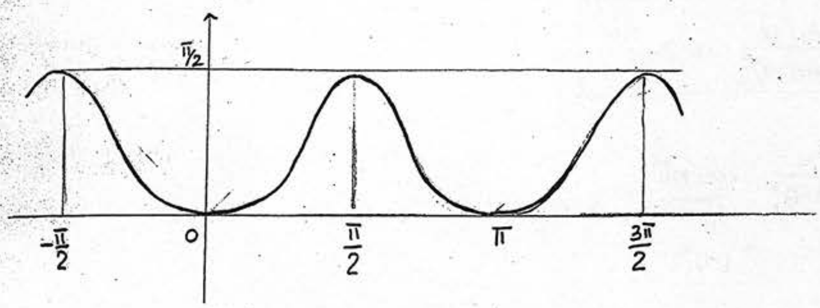
$f(x) = x\sin x$

$|x\sin x| \leq |x|$

$-x \leq x\sin x \leq x$ за $x > 0$



f је парна



$\ell = \frac{\pi}{2}$

$\tilde{f}(x) = f(x), \qquad x \in (-\frac{\pi}{2}, \frac{\pi}{2}]$

$\tilde{f}(x+\pi) = \tilde{f}(x), \qquad x \in \mathbb{R}$

$\tilde{f}$ π-периодична
непрекидна
део по део глатка
парна

$b_n = 0$

$$a_n = \frac{1}{\frac{\pi}{2}} \int_{-\pi/2}^{\pi/2} \tilde{f}(x) \cos \frac{n\pi x}{\frac{\pi}{2}} \, dx = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} \underbrace{x \sin x \cdot \cos(2nx)}_{\text{парна}} \, dx$$

$$= \frac{4}{\pi} \int_0^{\pi/2} x \sin x \cos 2nx \, dx$$

$$\boxed{2 \sin\alpha \cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi/2} x\left(\sin(2n+1)x + \sin(1-2n)x\right) dx = \left\{ \begin{matrix} u = x & dv = (\sin(2n+1)x - \sin(2n-1)x)\,dx \\ du = dx & v = -\frac{1}{2n+1}\cos(2n+1)x + \frac{1}{2n-1}\cos(2n-1)x \end{matrix} \right\}$$

$$= \frac{2}{\pi}\left(\left(-\frac{1}{2n+1} x \cos(2n+1)x + \frac{1}{2n-1} x \cos(2n-1)x\right)\Big|_0^{\pi/2} + \frac{1}{2n+1}\int_0^{\pi/2} \cos(2n+1)x \, dx - \frac{1}{2n-1}\int_0^{\pi/2} \cos(2n-1)x \, dx \right)$$

$$= \frac{2}{\pi}\left(-\frac{1}{2n+1}\frac{\pi}{2}\underbrace{\cos\left(n\pi+\frac{\pi}{2}\right)}_{=0} + \frac{1}{2n-1}\cdot\frac{\pi}{2}\underbrace{\cos\left(n\pi-\frac{\pi}{2}\right)}_{=0} + \frac{1}{(2n+1)^2}\sin(2n+1)x\Big|_0^{\pi/2} - \frac{1}{(2n-1)^2}\sin(2n-1)x\Big|_0^{\pi/2} \right)$$

$$= \frac{2}{\pi}\cdot\frac{1}{(2n+1)^2}\sin\left(n\pi+\frac{\pi}{2}\right) - \frac{2}{\pi}\cdot\frac{1}{(2n-1)^2}\sin\left(n\pi-\frac{\pi}{2}\right)$$

$$= \frac{2}{\pi}\cdot\frac{1}{(2n+1)^2}(-1)^n - \frac{2}{\pi}\cdot\frac{1}{(2n-1)^2}(-1)^{n-1}$$

$$= \frac{2(-1)^n}{\pi}\cdot\frac{4n^2 \cancel{-4n} + 1 + 4n^2 \cancel{+4n} + 1}{(2n+1)^2(2n-1)^2}$$

$$= \frac{4(-1)^n}{\pi}\cdot\frac{4n^2+1}{(4n^2-1)^2}$$

Специјално, $a_0 = \frac{4}{\pi}$

$$\boxed{S(x) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty}(-1)^n \frac{4n^2+1}{(4n^2-1)^2}\cos 2nx}$$

$$S\left(\frac{\pi}{2}\right) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty}(-1)^n \frac{4n^2+1}{(4n^2-1)^2}\underbrace{\cos n\pi}_{\substack{\parallel \\ (-1)^n}}$$

$$S\left(\frac{\pi}{2}\right) = \frac{2}{\pi} + \frac{4}{\pi}\underbrace{\sum_{n=1}^{\infty}\frac{4n^2+1}{(4n^2-1)^2}}_{A}$$

$$S\left(\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\sin\frac{\pi}{2} = \frac{\pi}{2}$$ због непрекидности f

$$\Rightarrow \quad \frac{\pi}{2} = \frac{2}{\pi} + \frac{4}{\pi}A$$

$$A = \frac{\pi}{4}\left(\frac{\pi}{2} - \frac{2}{\pi}\right)$$

$$\boxed{A = \frac{\pi^2}{8} - \frac{1}{2}}$$

3. а) Доказати једнакост

$$\frac{\pi}{2a}\cdot\frac{\operatorname{ch} at}{\operatorname{sh} a\pi} = \frac{1}{2a^2} + \sum_{n=1}^{\infty}(-1)^n\frac{\cos nt}{a^2+n^2}$$

$t \in (-\pi, \pi)$
$a \neq 0$

б) Израчунати $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n \cos nt}{n^2}$ за $t \in (-\pi, \pi)$

а) $f^*(t) = \operatorname{ch} at \qquad t \in (-\pi, \pi)$
$f^*(t+2\pi) = f^*(t) \qquad t \in \mathbb{R}$

f^* 2π – периодична
део по део глатка
парна

$b_n = 0$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\operatorname{ch} at \cos nt\, dt = \frac{2}{\pi}\int_0^{\pi}\operatorname{ch} at \cos nt\, dt = \left\{\begin{array}{ll} u = \cos nt & dv = \operatorname{ch} at\, dt \\ du = -n \sin nt\, dt & v = \frac{1}{a}\operatorname{sh} at \end{array}\right\}$$

сме јер $a \neq 0$

$$a_n = \frac{2}{\pi}\left(\frac{1}{a}\,\mathrm{sh}at\cdot\cos nt\Big|_0^{\pi} + \frac{n}{a}\int_0^{\pi}\mathrm{sh}at\ \sin nt\,dt\right) = \begin{Bmatrix} u=\sin nt & dv=\mathrm{sh}at\,dt \\ du=n\cos nt\,dt & v=\frac{1}{a}\mathrm{ch}at \end{Bmatrix}$$

$$= \frac{2}{\pi}\left(\frac{1}{a}\,\mathrm{sh}a\pi\cos n\pi + \frac{n}{a}\left(\underbrace{\frac{1}{a}\,\mathrm{ch}at\sin nt\Big|_0^{\pi}}_{=0} - \frac{n}{a}\int_0^{\pi}\mathrm{ch}at\cos nt\,dt\right)\right)$$

$$= \frac{2}{\pi}\cdot\frac{(-1)^n}{a}\,\mathrm{sh}a\pi - \frac{n^2}{a^2}\cdot\underbrace{\frac{2}{\pi}\int_0^{\pi}\mathrm{ch}at\cos nt\,dt}_{=a_n}$$

$$a_n = \frac{2(-1)^n}{a\pi}\,\mathrm{sh}a\pi - \frac{n^2}{a^2}a_n$$

$$\frac{n^2+a^2}{a^2}a_n = \frac{2(-1)^n}{a\pi}\,\mathrm{sh}a\pi$$

$$a_n = \frac{a^2}{n^2+a^2}\cdot\frac{2(-1)^n}{a\pi}\,\mathrm{sh}a\pi$$

$$a_n = \frac{2a}{\pi}\cdot\frac{(-1)^n}{n^2+a^2}\,\mathrm{sh}a\pi$$

Специјално, $a_0 = \frac{2a}{\pi}\cdot\frac{1}{a^2}\,\mathrm{sh}a\pi = \frac{2}{a\pi}\,\mathrm{sh}a\pi$

$$S(t) = \frac{\mathrm{sh}a\pi}{a\pi} + \frac{2a\,\mathrm{sh}a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+a^2}\cos nt$$

$S(t) = \mathrm{ch}at$ за $t\in(-\pi,\pi)$

За $t\in(-\pi,\pi)$ важи:

$$\mathrm{ch}at = \frac{\mathrm{sh}a\pi}{a\pi} + \frac{2a\,\mathrm{sh}a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+a^2}\cos nt \quad \Big/\cdot\frac{\pi}{2a\,\mathrm{sh}a\pi}$$

$$\mathrm{ch}at\,\frac{\pi}{2a\,\mathrm{sh}a\pi} = \frac{\cancel{\mathrm{sh}a\pi}}{a\cancel{\pi}}\cdot\frac{\cancel{\pi}}{2a\,\cancel{\mathrm{sh}a\pi}} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+a^2}\cos nt$$

$$\frac{\pi}{2a}\cdot\frac{\mathrm{ch}at}{\mathrm{sh}a\pi} = \frac{1}{2a^2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+a^2}\cos nt$$

б)
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+a^2} \cos nt$$

$$\left. \begin{array}{l} \left| \frac{(-1)^n}{n^2+a^2} \cos nt \right| \leq \frac{1}{n^2+a^2} \leq \frac{1}{n^2} \\ \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{конв.} \end{array} \right\} \Rightarrow \begin{array}{l} \text{Вајерштрас} \\ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+a^2} \cos nt \quad \text{р.к. по } a \in \mathbb{R} \end{array}$$

Због равномерне конвергенције важи:

$$\lim_{a\to 0} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+a^2} \cos nt = \sum_{n=1}^{\infty} \lim_{a\to 0} \frac{(-1)^n}{n^2+a^2} \cos nt = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nt$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nt = \lim_{a\to 0} \left(\frac{\pi}{2a} \frac{\operatorname{ch} at}{\operatorname{sh} a\pi} - \frac{1}{2a^2} \right) \qquad \text{према делу под а)}$$

$$= \lim_{a\to 0} \frac{a\pi \operatorname{ch} at - \operatorname{sh} a\pi}{2a^2 \operatorname{sh} a\pi}$$

$$\overset{\text{"}\frac{0}{0}\text{"}}{=} \frac{1}{2} \lim_{a\to 0} \frac{\pi \operatorname{ch} at + a\pi t \operatorname{sh} at - \pi \operatorname{ch} a\pi}{2a \operatorname{sh} a\pi + a^2 \pi \operatorname{ch} a\pi}$$

$$= \frac{\pi}{2} \lim_{a\to 0} \frac{\operatorname{ch} at + at \operatorname{sh} at - \operatorname{ch} a\pi}{2a \operatorname{sh} a\pi + a^2 \pi \operatorname{ch} a\pi}$$

$$\overset{\text{"}\frac{0}{0}\text{"}}{=} \frac{\pi}{2} \lim_{a\to 0} \frac{t \operatorname{sh} at + t \operatorname{sh} at + at^2 \operatorname{ch} at - \pi \operatorname{sh} a\pi}{2 \operatorname{sh} a\pi + 2a\pi \operatorname{ch} a\pi + 2a\pi \operatorname{ch} a\pi + a^2\pi^2 \operatorname{sh} a\pi}$$

$$= \frac{\pi}{2} \lim_{a\to 0} \frac{2t \operatorname{sh} at + at^2 \operatorname{ch} at - \pi \operatorname{sh} a\pi}{\operatorname{sh} a\pi (2+a^2\pi^2) + 4a\pi \operatorname{ch} a\pi}$$

$$\overset{\text{"}\frac{0}{0}\text{"}}{=} \frac{\pi}{2} \lim_{a\to 0} \frac{2t^2 \operatorname{ch} at + at^3 \operatorname{sh} at + t^2 \operatorname{ch} at - \pi^2 \operatorname{ch} a\pi}{2a\pi^2 \operatorname{sh} a\pi + \pi(2+a^2\pi^2) \operatorname{ch} a\pi + 4\pi \operatorname{ch} a\pi + 4a\pi^2 \operatorname{sh} a\pi}$$

$$= \frac{\pi}{2} \lim_{a\to 0} \frac{3t^2 \operatorname{ch} at + at^3 \operatorname{sh} at - \pi^2 \operatorname{ch} a\pi}{\underbrace{6a\pi^2 \operatorname{sh} a\pi}_{\downarrow\, 0} + \pi (6+\underbrace{a^2\pi^2}_{\downarrow\, 0}) \underbrace{\operatorname{ch} a\pi}_{\downarrow\, 1}}$$

$$= \frac{\pi}{2} \cdot \frac{3t^2 - \pi^2}{6\pi}$$

$$= \frac{1}{2} \cdot \frac{t^2}{2} - \frac{1}{2} \cdot \frac{\pi^2}{6}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nt = \frac{t^2}{4} - \frac{\pi^2}{12}}$$