

Local Connection Game

Motivations



often built and maintained by
self-interested agents



Introduction

- Introduced in [FLMPS'03]
- A LCG is a game that models the creation of networks
- two competing issues: players want
 - to minimize the cost they incur in building the network
 - to ensure that the network provides them with a high quality of service
- Players are nodes that:
 - pay for the links
 - benefit from short paths

[FLMPS'03]:

A. Fabrikant, A. Luthra, E. Maneva, C.H. Papadimitriou, S. Shenker,
On a network creation game, PODC'03

The model

- n players: nodes in a graph to be built
- Strategy for player u : a set of undirected edges that u will build (all incident to u)
- Given a strategy vector S , the constructed network will be $G(S)$
 - there is the undirected edge (u,v) if it is bought by u or v (or both)
- player u 's goal:
 - to make the distance to other nodes small
 - to pay as little as possible

The model

- Each edge costs α
- $\text{dist}_{G(S)}(u,v)$: length of a shortest path (in terms of number of edges) between u and v
- Player u aims to minimize its cost:

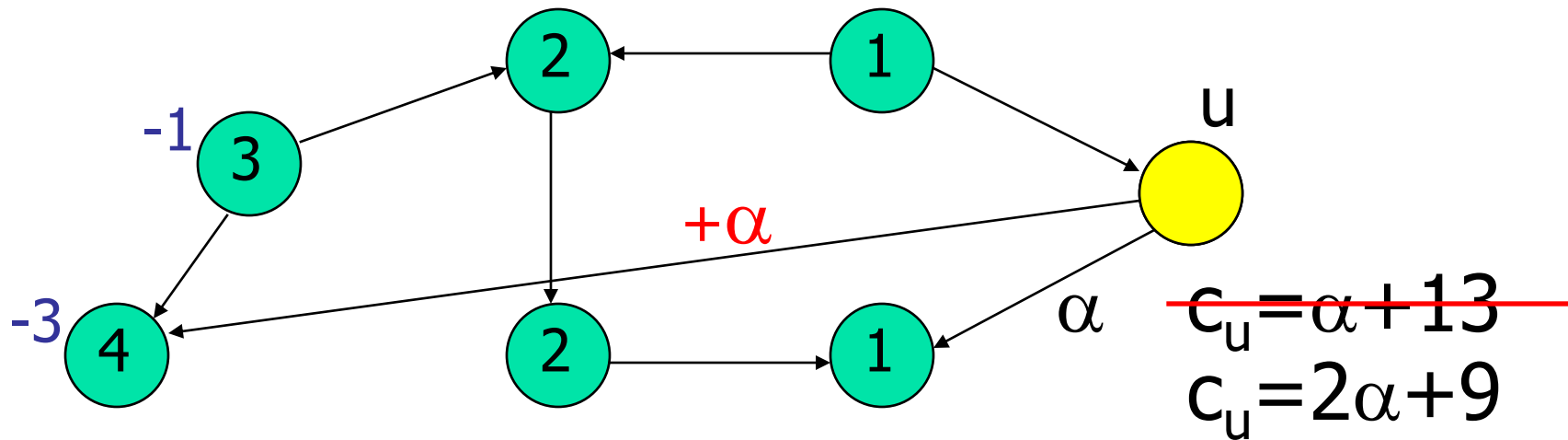
$$\text{cost}_u(S) = \underbrace{\alpha n_u}_{\text{building cost}} + \underbrace{\sum_v \text{dist}_{G(S)}(u,v)}_{\text{usage cost}}$$

- n_u : number of edges bought by node u

Remind

- We use Nash equilibrium (NE) as the solution concept
- To evaluate the overall quality of a network, we consider the *social cost*, i.e. the sum of all players' costs
- a network is *optimal* or *socially efficient* if it minimizes the social cost
- A graph $G=(V,E)$ is *stable* (for a value α) if there exists a strategy vector S such that:
 - S is a NE
 - S forms G

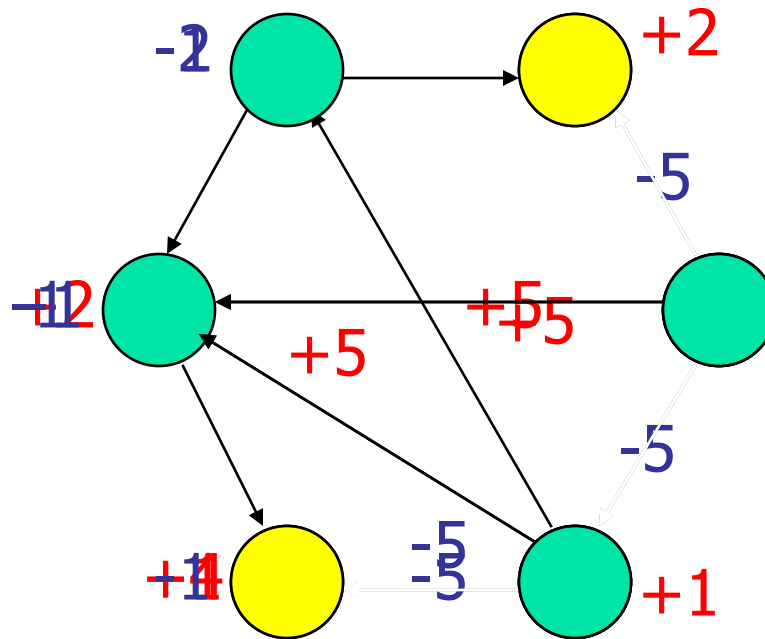
Example



(Convention: arrow *from* the node buying the link)

Example

- Set $\alpha=5$, and consider:



That's a stable network!

Some simple observations

- In $SC(S)$ each term $\text{dist}_{G(S)}(u,v)$ contributes to the overall quality twice
- In a stable network each edge (u,v) is bought at most by one player
- Any stable network must be connected
 - Since the distance $\text{dist}(u,v)$ is infinite whenever u and v are not connected

Social cost of a stable network $G(S)=(V,E)$:
$$SC(S) = \alpha |E| + \sum_{u,v} \text{dist}_{G(S)}(u,v)$$

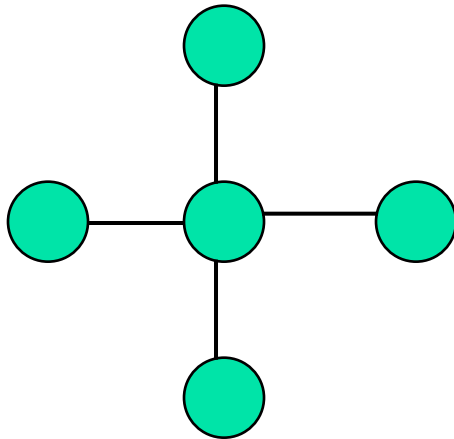
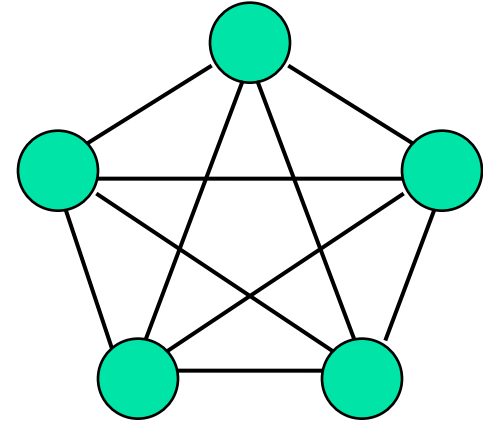
Our goal

- to bound the efficiency loss resulting from stability
- In particular:
 - To bound the Price of Stability (PoS)
 - To bound the Price of Anarchy (PoA)

*How does an optimal
network look like?*

Some notation

K_n : complete graph
with n nodes



A **star** is a tree
with height at most 1
(when rooted at its
center)

Lemma

If $\alpha \leq 2$ then the complete graph is an optimal solution, while if $\alpha \geq 2$ then any star is an optimal solution.

proof

Let $G=(V,E)$ be a network with $|E|=m$ edges

$$SC(G) \geq \alpha m + 2m + 2(n(n-1) - 2m) = \underbrace{(\alpha - 2)m + 2n(n-1)}_{LB(m)}$$

Notice: $LB(m)$ is equal to $SC(K_n)$ when $m = n(n-1)/2$ and to SC of any star when $m = n-1$

proof

$G=(V,E)$: optimal solution;
 $SC(G)=OPT$

$$LB(m)=(\alpha-2)m + 2n(n-1)$$

$$OPT \geq \min_m LB(m) \geq \begin{cases} \xrightarrow[\min m]{\alpha \geq 2} LB(n-1) = SC \text{ of any star} \\ \xrightarrow[\max m]{\alpha \leq 2} LB(n(n-1)/2) = SC(K_n) \end{cases}$$



*Are the complete graph
and stars stable?*

Lemma

If $\alpha \leq 1$ the complete graph is stable, while if $\alpha \geq 1$ then any star is stable.

proof

$\alpha \leq 1$

a node v cannot improve
by saving k edges

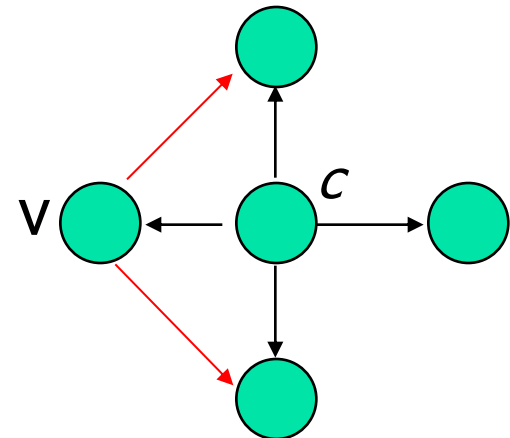
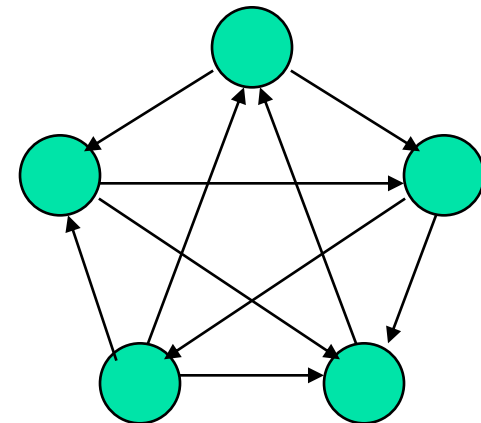
$\alpha \geq 1$

c has no interest to deviate

v buys k more edges...

...pays αk more...

...saves (w.r.t
distances) k ...



Theorem

For $\alpha \leq 1$ and $\alpha \geq 2$ the PoS is 1. For $1 < \alpha < 2$ the PoS is at most $4/3$

proof

$\alpha \leq 1$ and $\alpha \geq 2$...trivial!

$1 < \alpha < 2$... K_n is an optimal solution, any star T is stable...

$$\text{PoS} \leq \frac{SC(T)}{SC(K_n)} = \frac{(\alpha-2)(n-1) + 2n(n-1)}{\alpha n(n-1)/2 + n(n-1)} \leq \frac{-1(n-1) + 2n(n-1)}{n(n-1)/2 + n(n-1)}$$

maximized when $\alpha \rightarrow 1$

$$= \frac{2n-1}{(3/2)n} = \frac{4n-2}{3n} < 4/3$$



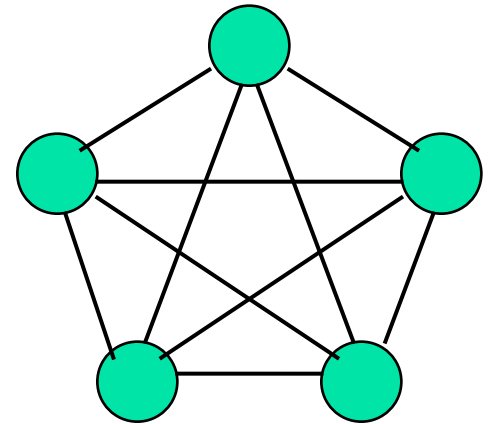
What about price of Anarchy?

...for $\alpha < 1$ the complete graph is the
only stable network,
(try to prove that formally)
hence $PoA = 1$...

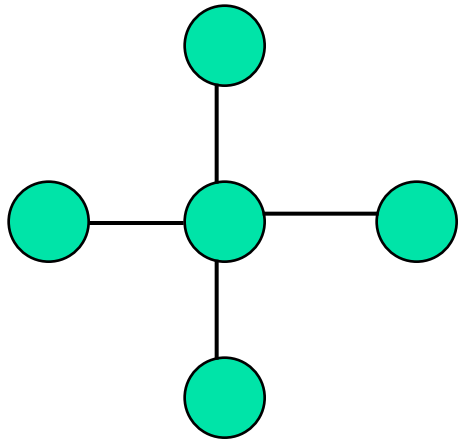
...for larger value of α ?

Some more notation

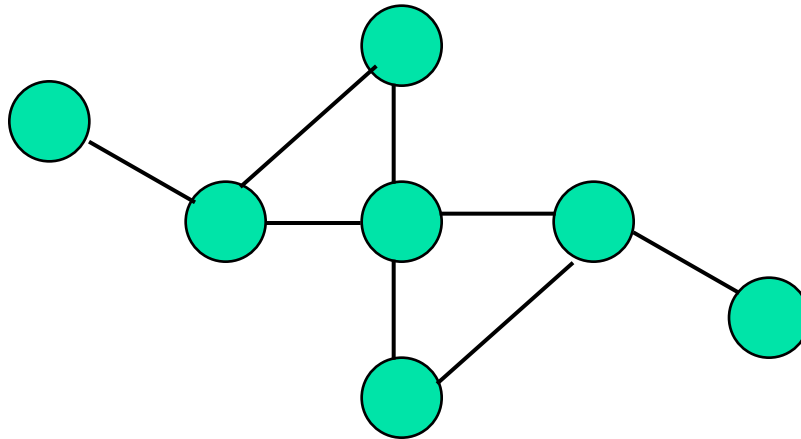
The **diameter** of a graph G is the maximum distance between any two nodes



diam=1



diam=2



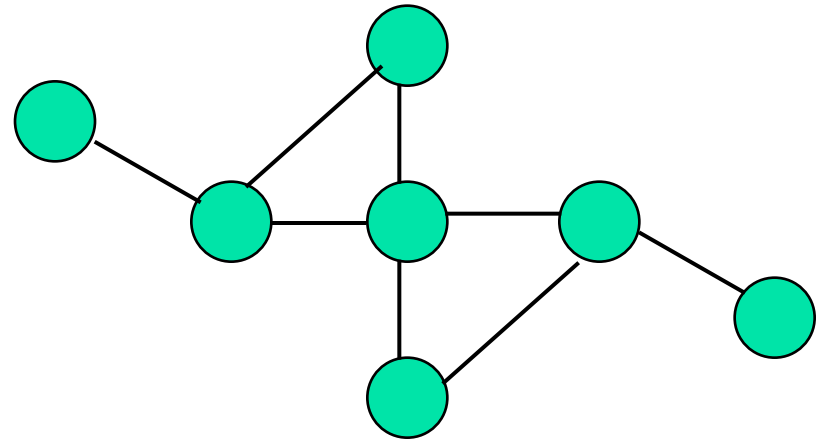
diam=4

Some more notation

An edge e is a **cut edge** of a graph $G=(V,E)$ if $G-e$ is disconnected

$$G-e=(V,E\setminus\{e\})$$

A simple property:
Any graph has at most $n-1$ cut edges



Theorem

The PoA is at most $O(\sqrt{\alpha})$.

proof

It follows from the following lemmas:

Lemma 1

The diameter of any stable network is at most $2\sqrt{\alpha} + 1$.

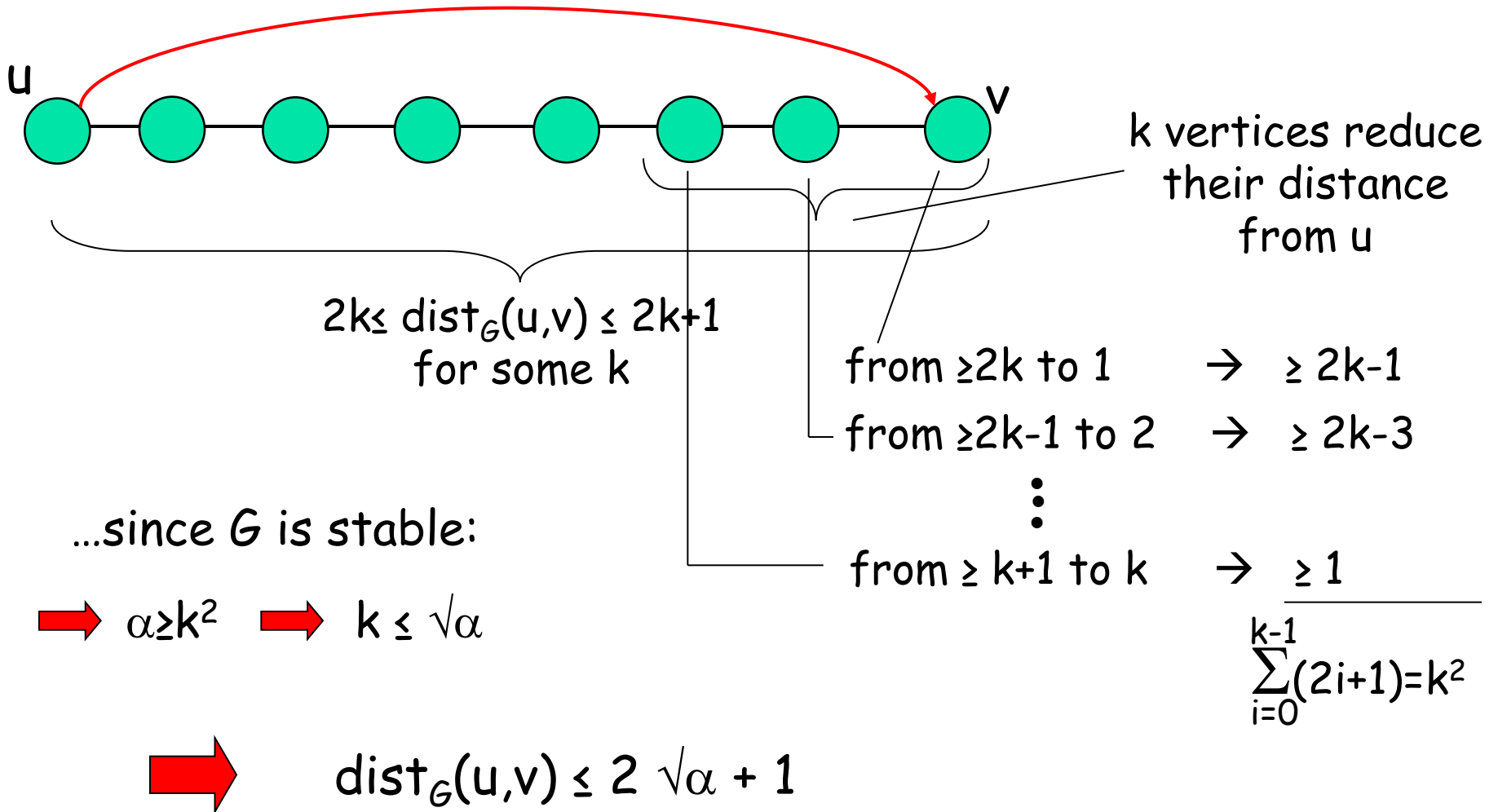
Lemma 2

The SC of any stable network with diameter d is at most $O(d)$ times the optimum SC.

proof of Lemma 1

G : stable network

Consider a shortest path in G between two nodes u and v



Lemma 2

The SC of any stable network $G=(V,E)$ with diameter d is at most $O(d)$ times the optimum SC.

idea of the proof (we'll formally prove it later)

$$\text{OPT} \geq \alpha (n-1) + n(n-1) \quad \rightarrow \quad \begin{aligned} \text{OPT} &\geq \alpha (n-1) \\ \text{OPT} &= \Omega(n^2) \end{aligned}$$

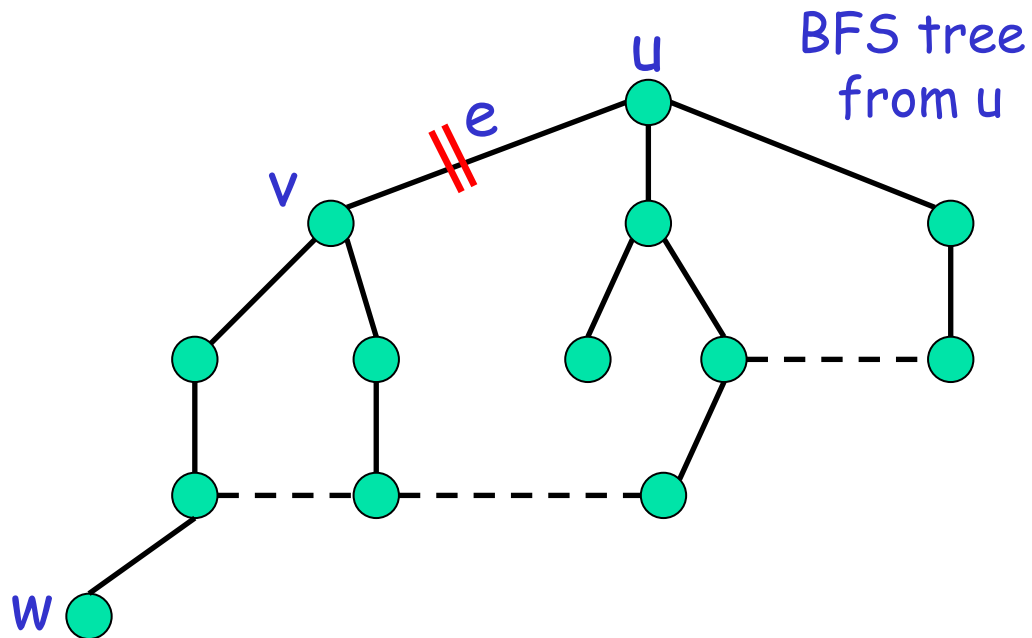
$$\text{SC}(G) = \sum_{u,v} d_G(u,v) + \alpha |E|$$

$$\begin{aligned} &= \underbrace{\sum_{u,v} d_G(u,v)}_{O(d n^2) = O(d) \text{ OPT}} + \underbrace{\alpha |E_{\text{cut}}|}_{\substack{\leq \text{OPT} \\ \leq (n-1)}} + \underbrace{\alpha |E_{\text{non-cut}}|}_{\substack{O(d) \text{ OPT} \\ O(n^2 d / \alpha)}} = O(d) \text{ OPT} \\ &\quad \text{that's the tricky bound} \end{aligned}$$

Proposition 1

Let G be a network with diameter d , and let $e=(u,v)$ be a non-cut edge. Then in $G-e$, every node w increases its distance from u by at most $2d$

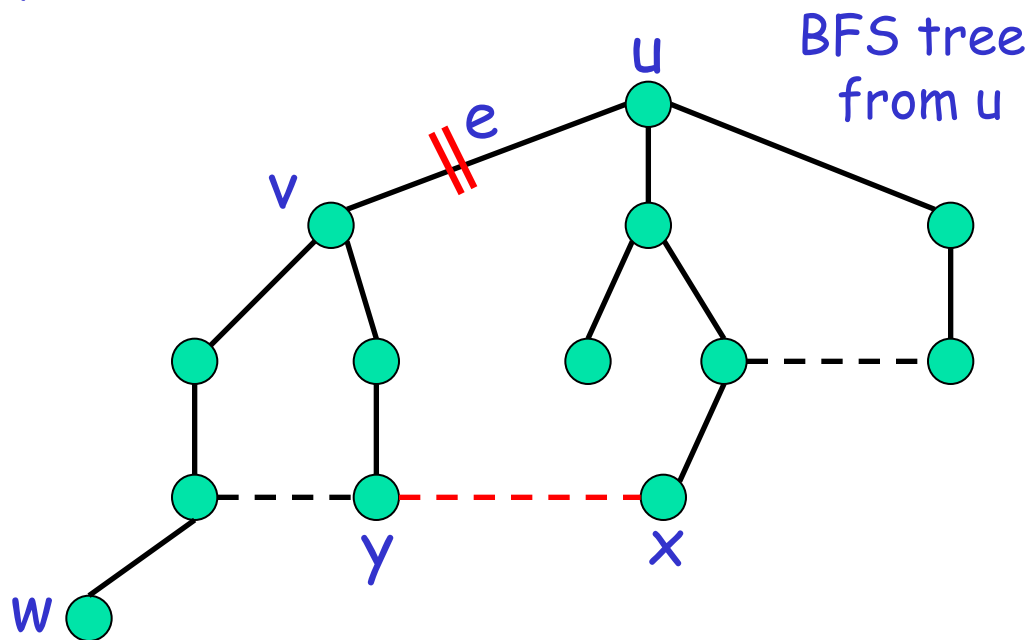
proof



Proposition 1

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proof

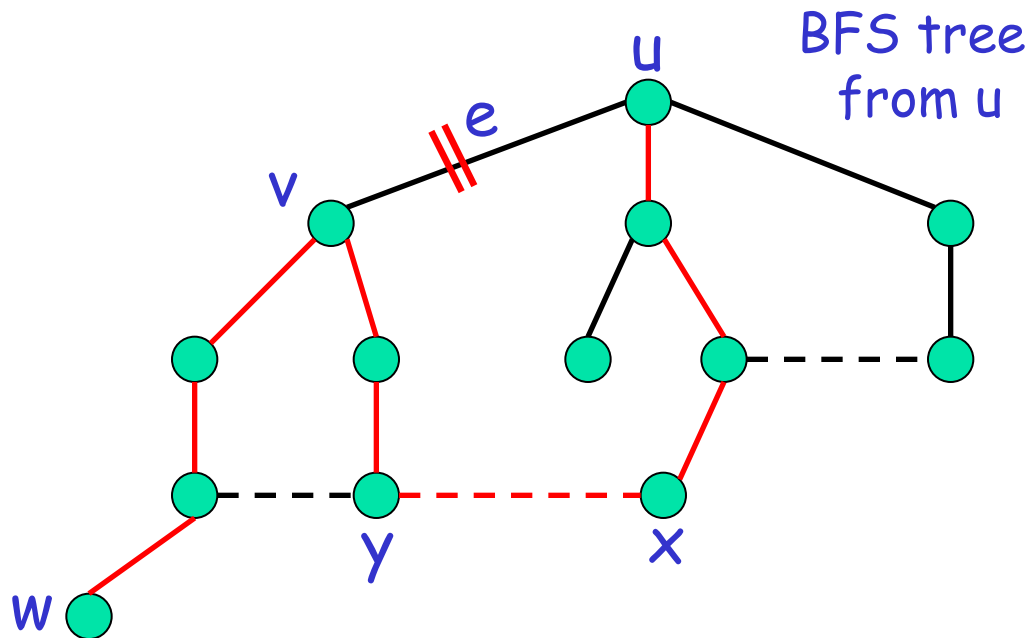


(x,y) :
any edge crossing
the cut induced
by the removal of e

Proposition 1

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proof



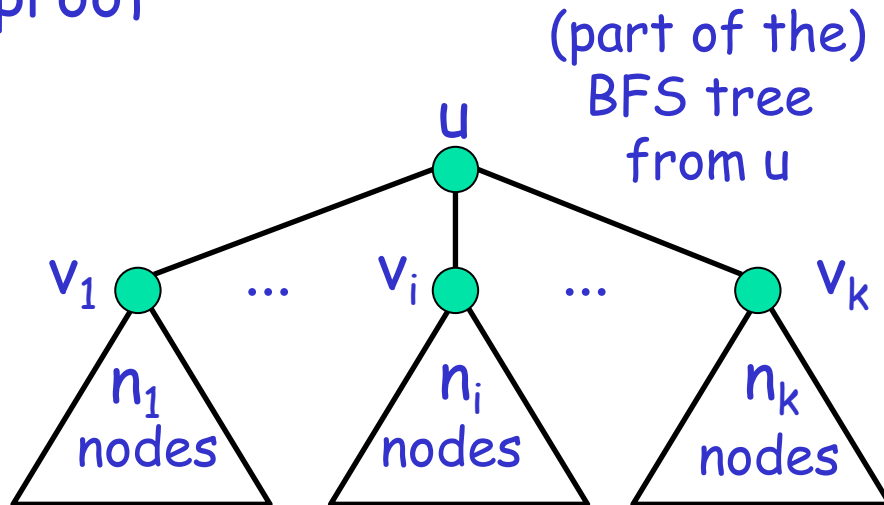
$$d_{G-e}(u,w) \leq \underbrace{d_G(u,x)}_{\leq d} + 1 + \underbrace{d_G(y,v)}_{\leq d} + \underbrace{d_G(v,w)}_{= d_G(u,w)-1} \leq d_G(u,w) + 2d$$



Proposition 2

Let G be a stable network, and let F be the set of Non-cut edges paid for by a node u . Then $|F| \leq (n-1)2d/\alpha$

proof



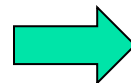
$$k = |F|$$

if u removes (u, v_i) saves α and its distance cost increases by at most $2d n_i$ (Prop. 1)

since G is stable:
 $\alpha \leq 2d n_i$

by summing up for all i

$$k \alpha \leq 2d \sum_{i=1}^k n_i \leq 2d (n-1)$$



$$k \leq (n-1) 2d/\alpha$$



Lemma 2

The SC of any stable network $G=(V,E)$ with diameter d is at most $O(d)$ times the optimum SC.

proof

$$OPT \geq \alpha (n-1) + n(n-1)$$

$$SC(G) = \underbrace{\sum_{u,v} d_G(u,v)}_{\leq dn(n-1) \leq d OPT} + \alpha |E| \leq d OPT + 2d OPT = 3d OPT$$

$$\alpha |E| = \alpha \underbrace{|E_{cut}|}_{\leq (n-1)} + \alpha \underbrace{|E_{non-cut}|}_{\leq n(n-1)2d/\alpha \text{ Prop. 2}} \leq \alpha(n-1) + n(n-1)2d \leq 2d OPT$$



Theorem

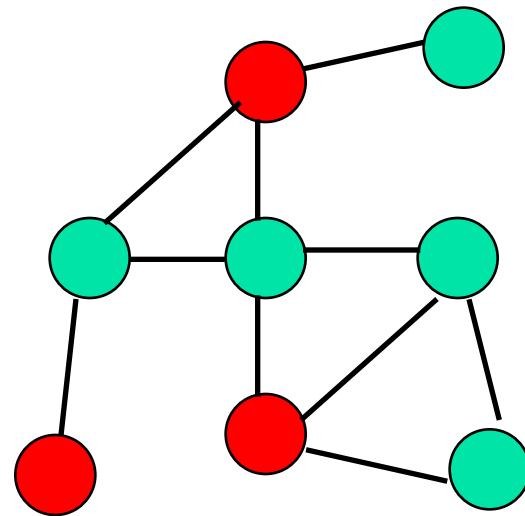
It is NP-hard, given the strategies of the other agents, to compute the best response of a given player.

proof

Reduction from dominating set problem

Dominating Set (DS) problem

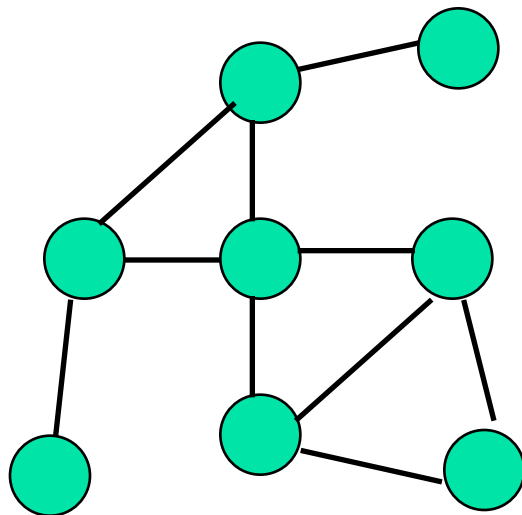
- Input:
 - a graph $G=(V,E)$
- Solution:
 - $U \subseteq V$, such that for every $v \in V - U$, there is $u \in U$ with $(u,v) \in E$
- Measure:
 - Cardinality of U



$1 < \alpha < 2$

player i 

the reduction

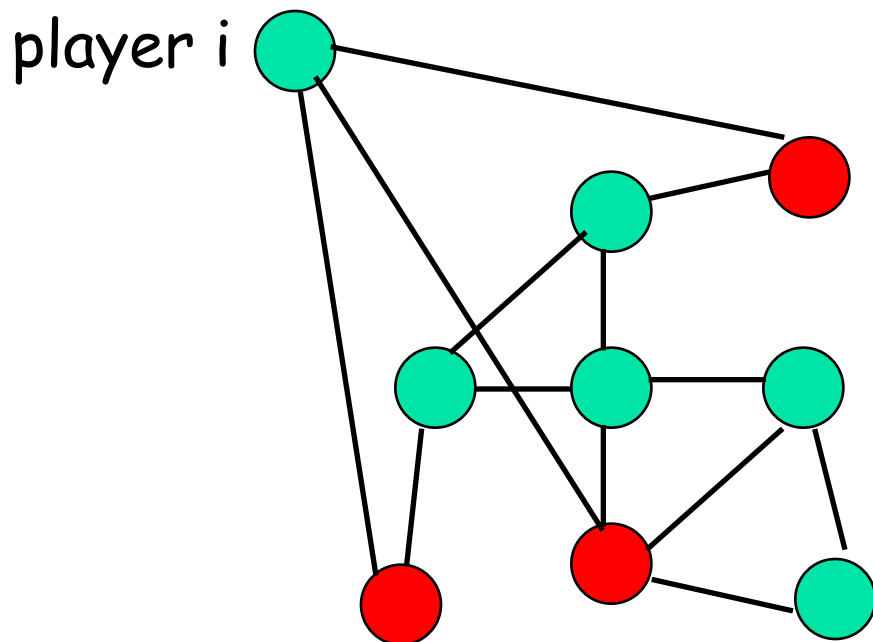


$$G=(V,E) = G(S_{-i})$$

Player i has a strategy yielding a cost $\leq \alpha k + 2n - k$ **if and only if** there is a DS of size $\leq k$

$$1 < \alpha < 2$$

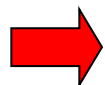
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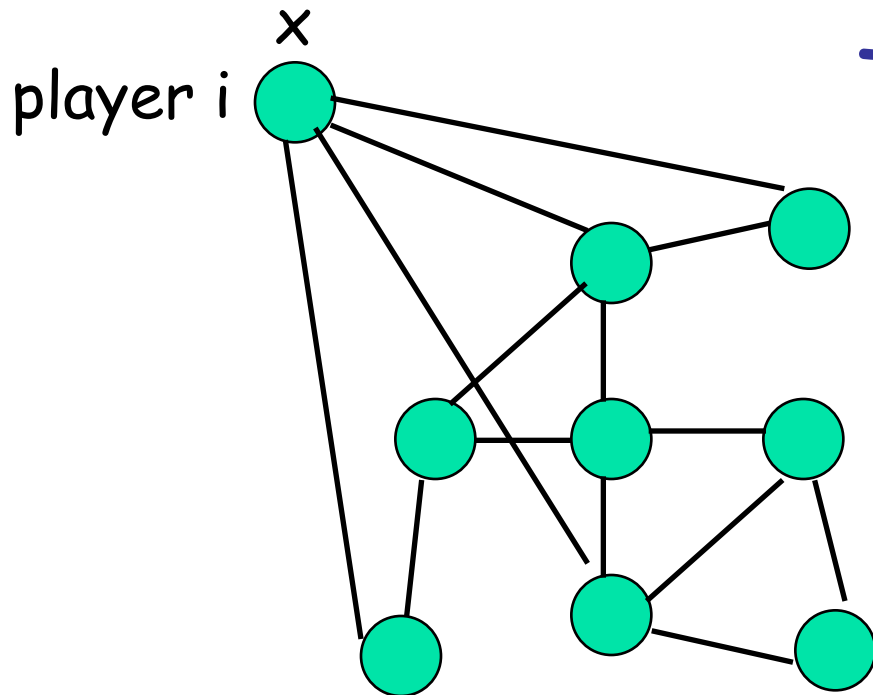
(\Leftarrow)

easy: given a dominating set U of size k , player i buys edges incident to the nodes in U



Cost for i is $\alpha k + 2(n-k) + k = \alpha k + 2n - k$

$$1 < \alpha < 2$$



the reduction

$$G=(V,E) = G(S_{-i})$$

(\Rightarrow)

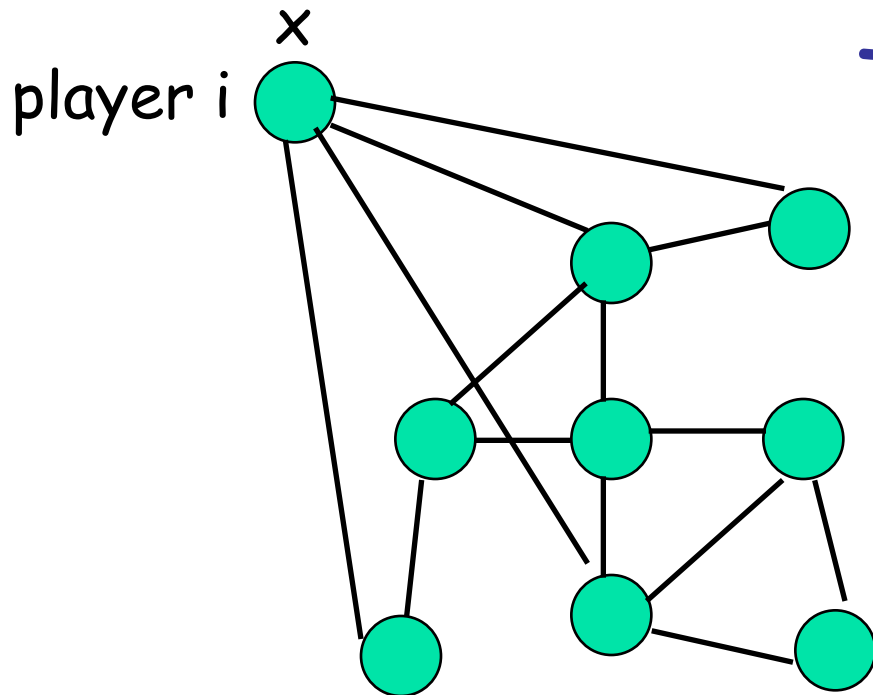
Let S_i be a strategy giving a cost $\leq \alpha k + 2n - k$

Modify S_i as follows:

repeat:

if there is a node v with distance ≥ 3 from x in $G(S)$,
then add edge (x,v) to S_i (this decreases the
cost)

$$1 < \alpha < 2$$



the reduction

$$G=(V,E) = G(S_{-i})$$

(\Rightarrow)

Let S_i be a strategy giving a cost $\leq \alpha k + 2n - k$

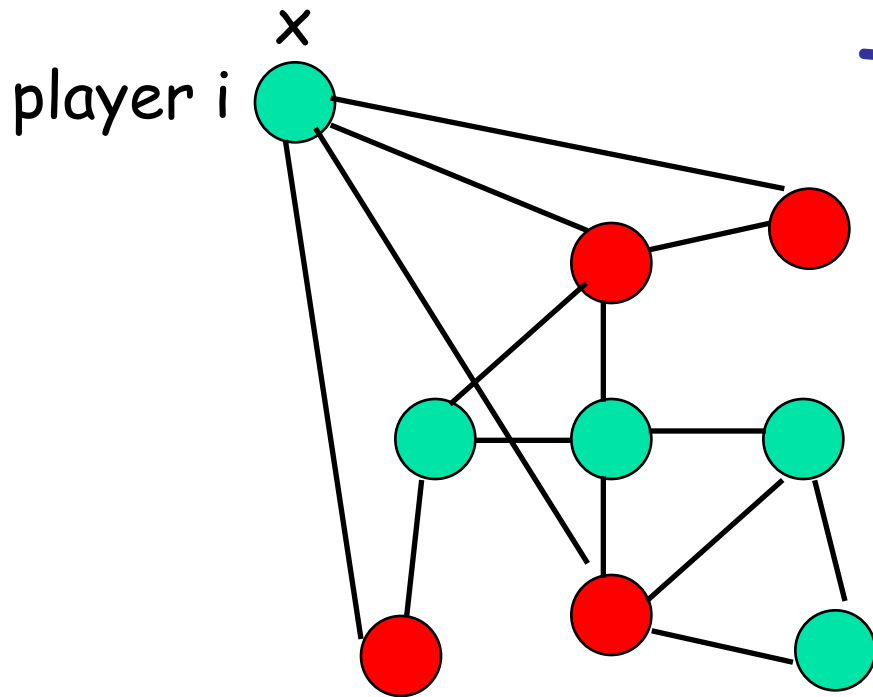
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Finally, every node has distance either 1 or 2 from x

$$1 < \alpha < 2$$



the reduction

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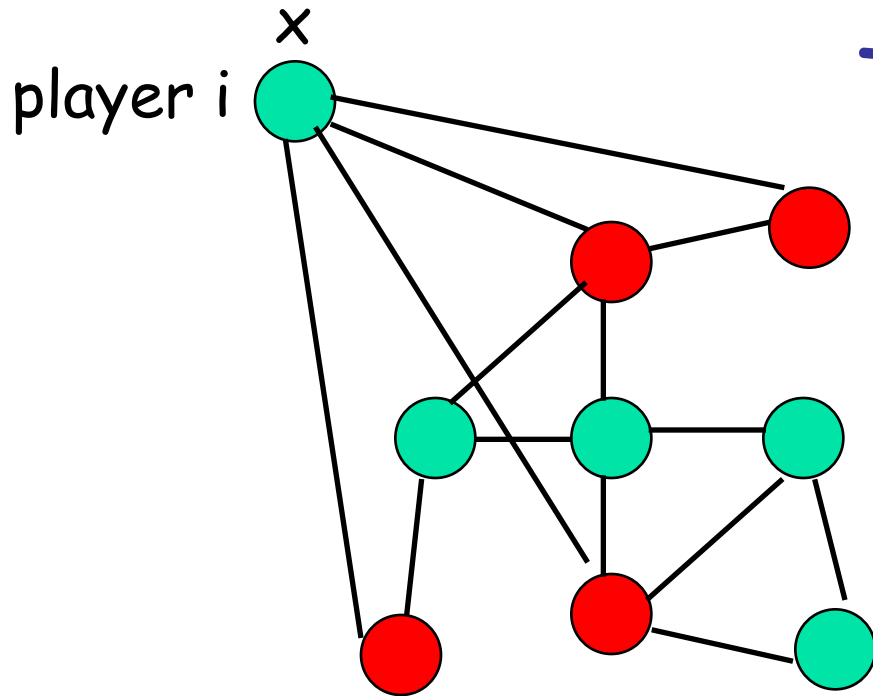
repeat:

if there is a node v with distance ≥ 3 from x in $G(S)$,
then add edge (x,v) to S_i (this decreases the
cost)

Finally, every node has distance either 1 or 2 from x

Let U be the set of nodes at distance 1 from x ...

$$1 < \alpha < 2$$



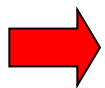
the reduction

$$G=(V,E) = G(S_{-i})$$

(\Rightarrow)

... U is a dominating set of the original graph G

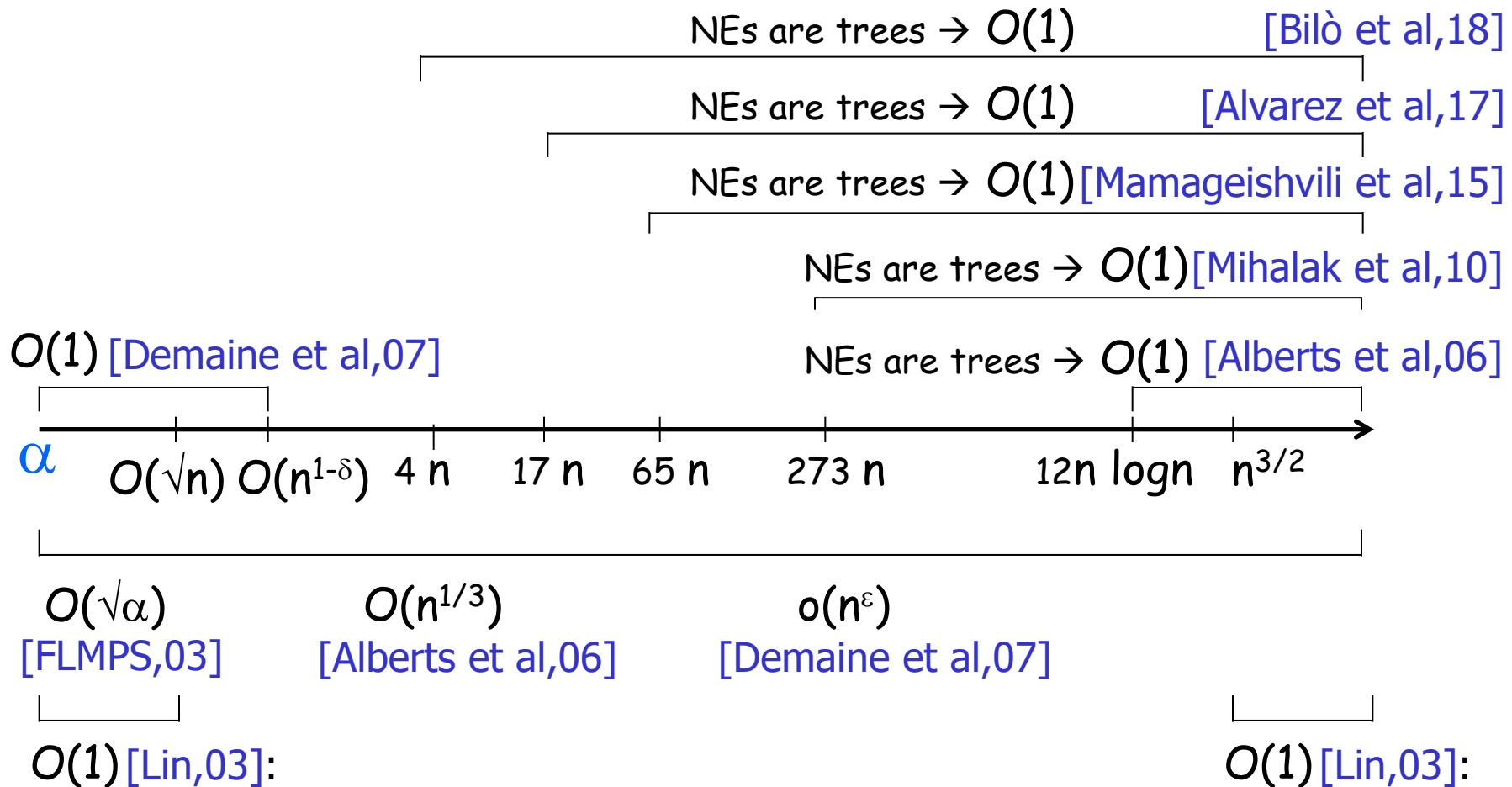
$$\text{We have } \text{cost}_i(S) = \alpha|U| + 2n - |U| \leq \alpha k + 2n - k$$



$$|U| \leq k$$



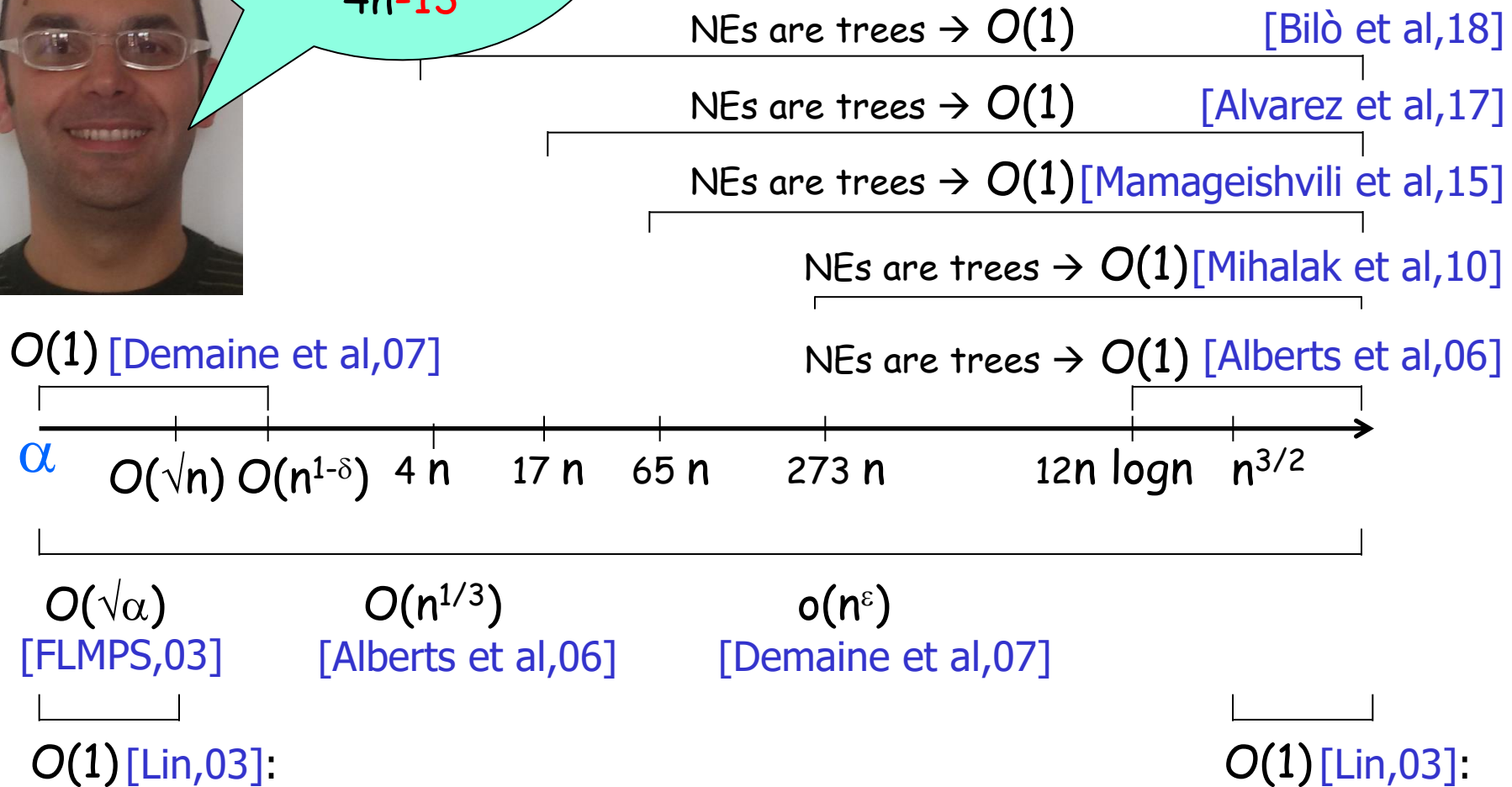
PoA as function of α : state of the art





Colleague,
remember to
mention that the
right bound is
 $4n-13$

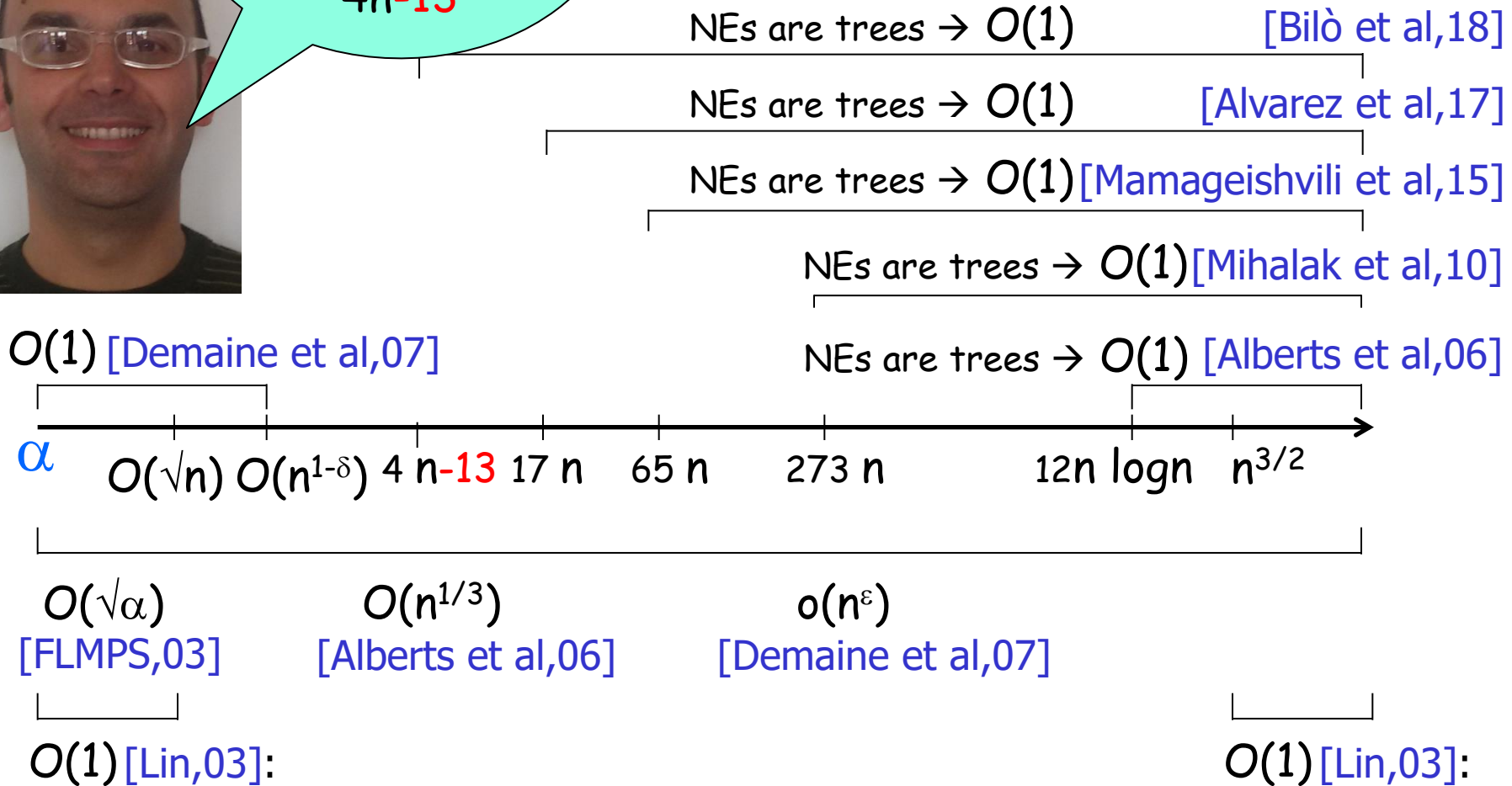
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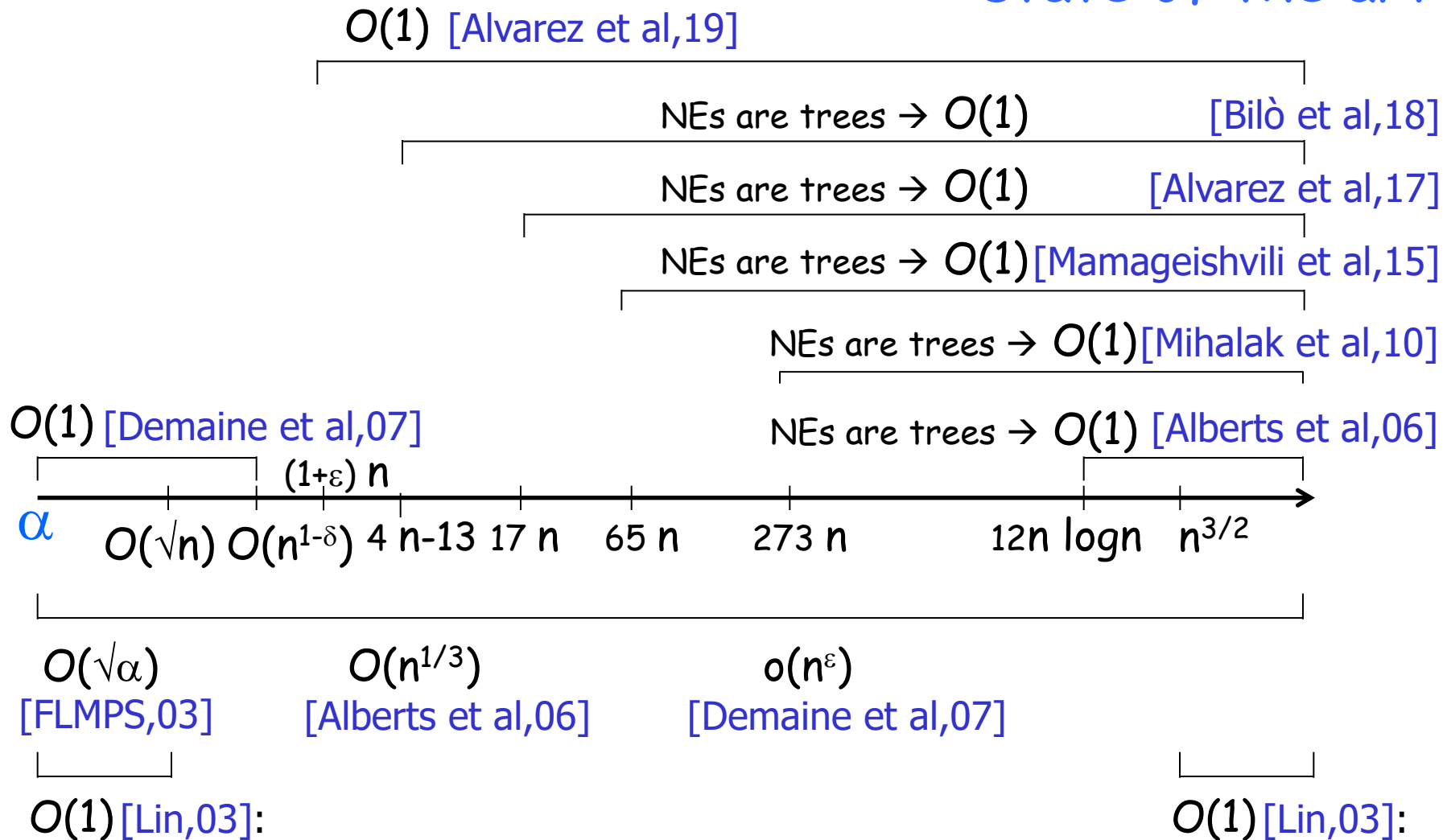
PoA as function of α : state of the art



[Bilò et al,18]:

D. Bilò, P. Lenzner, On the Tree Conjecture for the Network Creation Game, STACS'18

PoA as function of α : state of the art

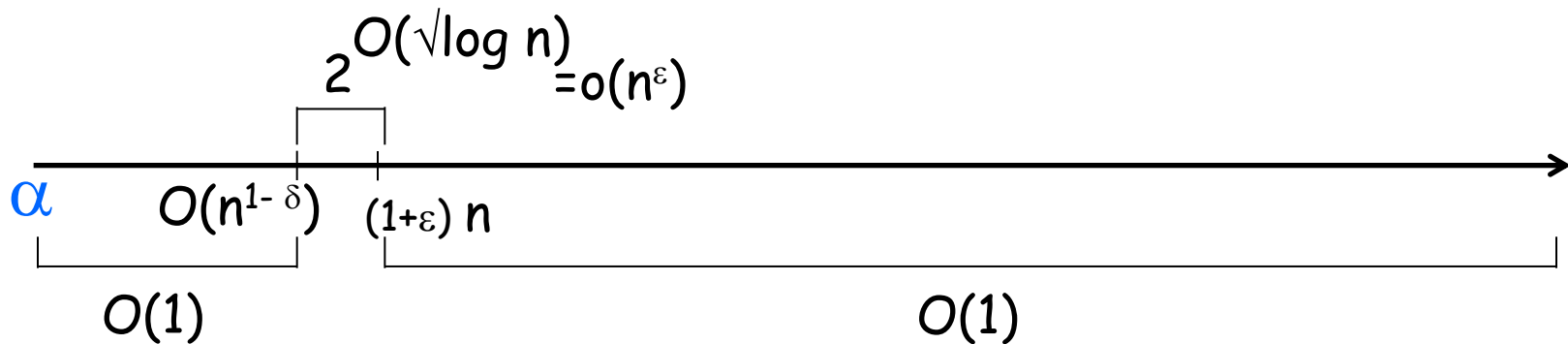


PoA as function of α :

state of the art



PoA as function of α : state of the art



Open: is POA always constant?