Analisi di Reti
(modulo II)

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Algorithmic Game Theory

Algorithmic Issues in Non-cooperative (i.e., strategic) Distributed Systems

Two Research Traditions

- Theory of Algorithms: computational issues
  - What can be feasibly computed?
  - How much does it take to compute a solution?
  - Which is the quality of a computed solution?
- Game Theory: interaction between self-interested individuals
  - What is the outcome of the interaction?
  - Which social goals are compatible with selfishness?

one of the foremost mathematicians of the 20th century

John von Neumann
(1903-1957)

Games and Economic Behavior
(1944, with O. Morgenstern)

Different Assumptions

- Theory of Algorithms (in distributed systems):
  - Processors are *obedient*, *faulty*, or *adversarial*
  - Large systems, limited computational resources

- Game Theory:
  - Players are *strategic* (selfish)
  - Small systems, unlimited computational resources

The World of Modern Networks

- Agents often autonomous (users)
  - Users have their own individual goals
  - Network components owned by providers
- Often involve large networks
  - Massive systems
  - Limited communication/computational resources
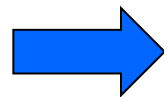 ⇨ *Both strategic and computational issues!*

Algorithmic Game Theory = *Theory of Algorithms* + *Game Theory*

- **Game Theory** provides a bunch of tools useful for addressing computational problems in non-cooperative scenarios

  - networks/systems used by self-interested users

- **Theory of algorithms** sheds light on results of **Game Theory**

  - for several results on the existence of equilibria/mechanisms we have that such an equilibium/mechanism cannot be found/implemented efficiently

Basics of Game Theory:
games & equilibria

A game

- A game consists of:
  - A set of players
  - A set of rules of encounter: Who should act when, and what are the possible actions (strategies)
  - A specification of payoffs for each combination of strategies
- **Game Theory** attempts to predict the outcome of the game (solution) by taking into account the individual behavior of the players
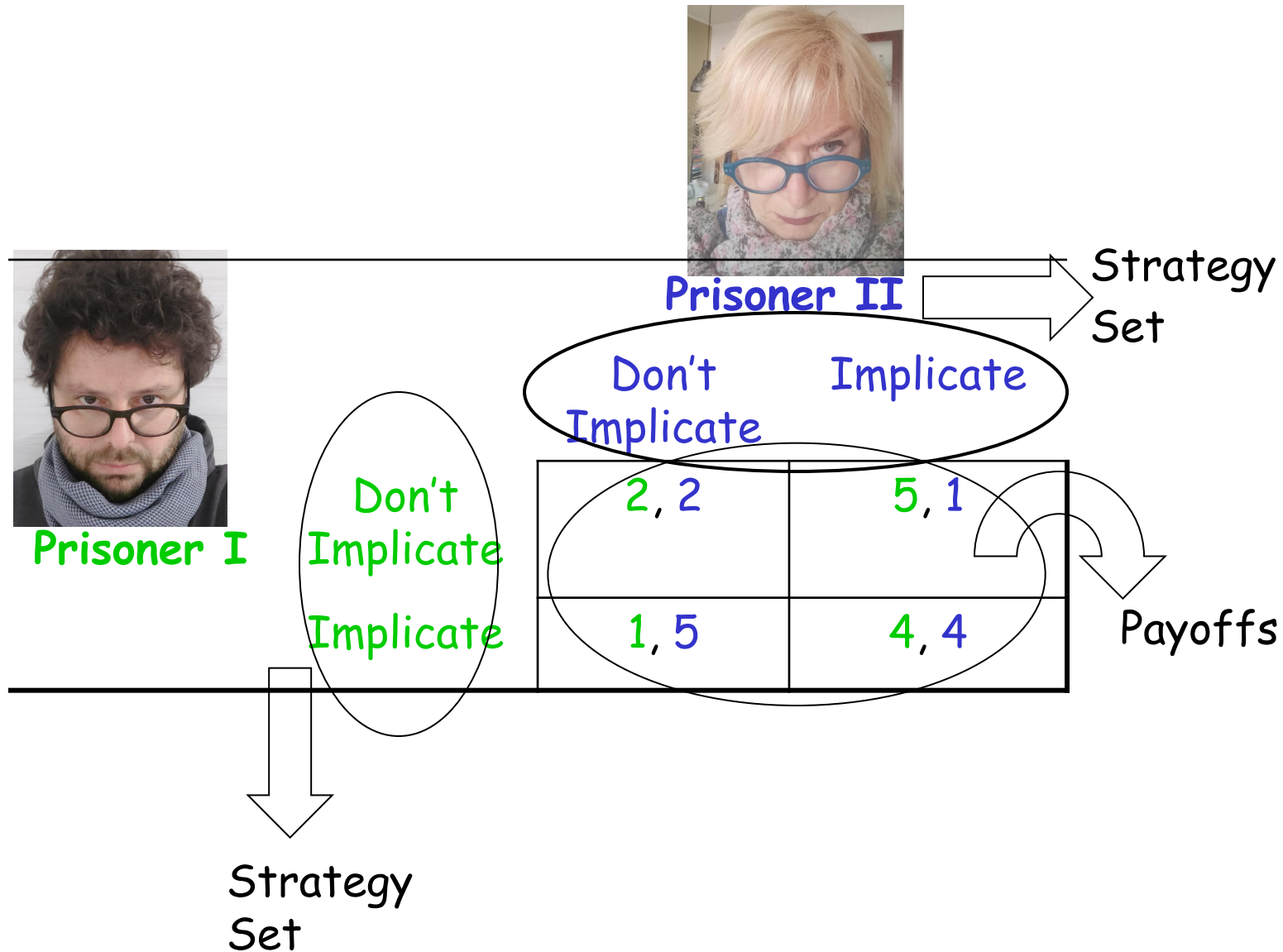
an equilibrium!

A famous one-shot game: the Prisoner's Dilemma

...the story of two strange and dangerous fellows...

A famous one-shot game: the Prisoner's Dilemma

Prisoner II

Strategy Set

Don't Implicate   Implicate

Prisoner I

Don't Implicate

Implicate

Strategy Set

	Don't Implicate	Implicate
Don't Implicate	2, 2	5, 1
Implicate	1, 5	4, 4

Payoffs

Prisoner I's decision

Prisoner II

	Don't Implicate	Implicate
Don't Implicate	2, 2	5, 1
Implicate	1, 5	4, 4

Prisoner I

- Prisoner I's decision:
  - If II chooses Don't Implicate then it is best to Implicate
  - If II chooses Implicate then it is best to Implicate
  - It is best to Implicate for I, **regardless** of what II does: *Dominant Strategy*

Prisoner II's decision

Prisoner II

	Don't Implicate	Implicate
Don't Implicate	2, 2	5, 1
Implicate	1, 5	4, 4

Prisoner I

- Prisoner II's decision:
  - If I chooses Don't Implicate then it is best to Implicate
  - If I chooses Implicate then it is best to Implicate
  - It is best to Implicate for II, **regardless** of what I does: *Dominant Strategy*

Hence...

	Prisoner II	
	Don't Implicate	Implicate
Prisoner I Don't Implicate	2, 2	5, 1
Implicate	1, 5	4, 4

- It is best for both to implicate **regardless** of what the other one does
- **Implicate** is a Dominant Strategy for both
- (Implicate, Implicate) becomes the **Dominant Strategy Equilibrium**
- Note: If they might collude, then it's beneficial for both to Not Implicate, but it's not an equilibrium as both have incentive to deviate

A network game

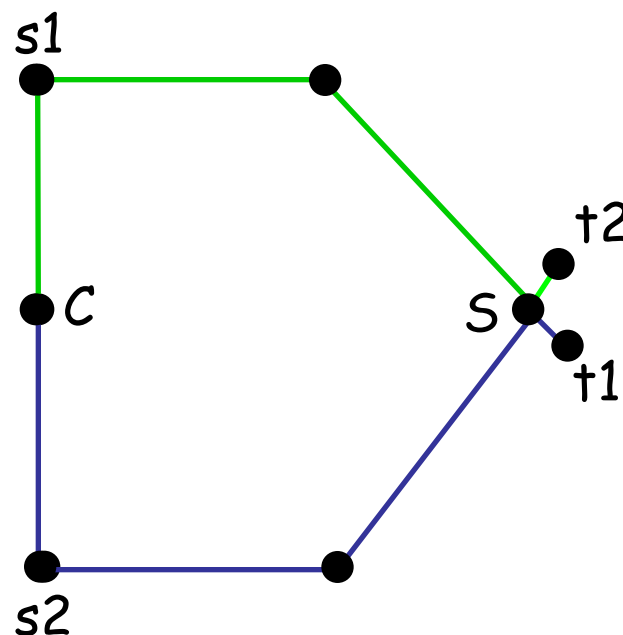C, S: peering points

two Internet Service Providers (ISP):
ISP1 e ISP2

ISP1 wants to send traffic from s1 to t1

ISP2 wants to send traffic from s2 to t2

(long) links have cost 1
(for ISP owning the link)

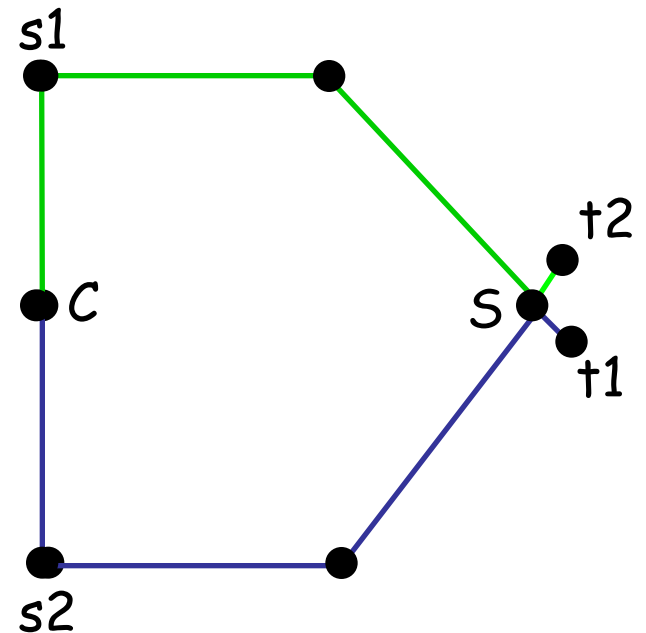Each ISPi can use two paths: the one passing through C o
the one passing through S

A network game

C, S: peering points

Cost Matrix

		ISP2	
		through S	through C
ISP1	through S	2, 2	5, 1
	through C	1, 5	4, 4

Formal representation of a game: Normal Form

- N **rational** players
- S_i = Strategy set of player i
- The strategy combination $(s_1, s_2, ..., s_N)$ gives payoff $(p_1, p_2, ..., p_N)$ to the N players
  $$\Rightarrow \quad S_1 \times S_2 \times ... \times S_N \text{ payoff matrix}$$

Dominant Strategy Equilibrium

- Dominant Strategy Equilibrium: is a strategy combination $s^* = (s_1^*, s_2^*, ..., s_N^*)$, such that s_i^* is a dominant strategy for each i, namely, for any possible alternative strategy profile $s = (s_1, s_2, ..., s_i, ..., s_N)$:
    - if p_i is a utility, then $p_i(s_1, s_2, ..., s_i^*, ..., s_N) \geq p_i(s_1, s_2, ..., s_i, ..., s_N)$
    - if p_i is a cost, then $p_i(s_1, s_2, ..., s_i^*, ..., s_N) \leq p_i(s_1, s_2, ..., s_i, ..., s_N)$

- Dominant Strategy is the best response to *any* strategy of other players
- If a game has a DSE, then players will immediately converge to it
- Of course, not all games (only *very few* in the practice!) have a dominant strategy equilibrium

A more relaxed solution concept: Nash Equilibrium [1951]

- **Nash Equilibrium**: is a strategy combination $s^* = (s_1^*, s_2^*, ..., s_N^*)$ such that for each i, s_i^* is a best response to $(s_1^*, ..., s_{i-1}^*, s_{i+1}^*, ..., s_N^*)$, namely,

  for any possible alternative strategy s_i of player i
  - if p_i is a utility, then $p_i(s_1^*, s_2^*, ..., s_i^*, ..., s_N^*) \geq p_i(s_1^*, s_2^*, ..., s_i, ..., s_N^*)$
  - if p_i is a cost, then $p_i(s_1^*, s_2^*, ..., s_i^*, ..., s_N^*) \leq p_i(s_1^*, s_2^*, ..., s_i, ..., s_N^*)$

Nash Equilibrium: The Battle of the Sexes (coordination game)

		Woman	
		Stadium	Cinema
Man	Stadium	6, 5	2, 2
	Cinema	1, 1	5, 6

- (Stadium, Stadium) is a NE: Best responses to each other
- (Cinema, Cinema) is a NE: Best responses to each other

☹ but they are not Dominant Strategy Equilibria … are we really sure they will eventually go out together????
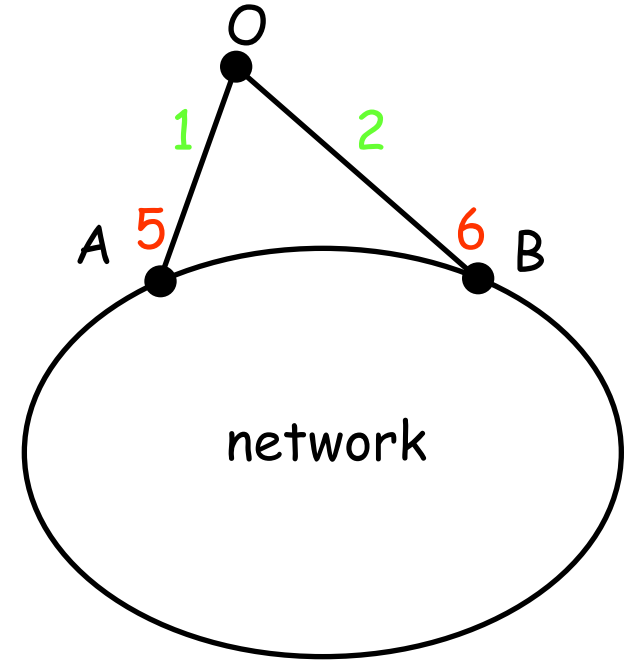
A similar game: routing congestion game

two traffic streams originated at node O need to be routed to the rest of the network

Costs without congestion:
 c(O,A)=1   c(O,B)=2
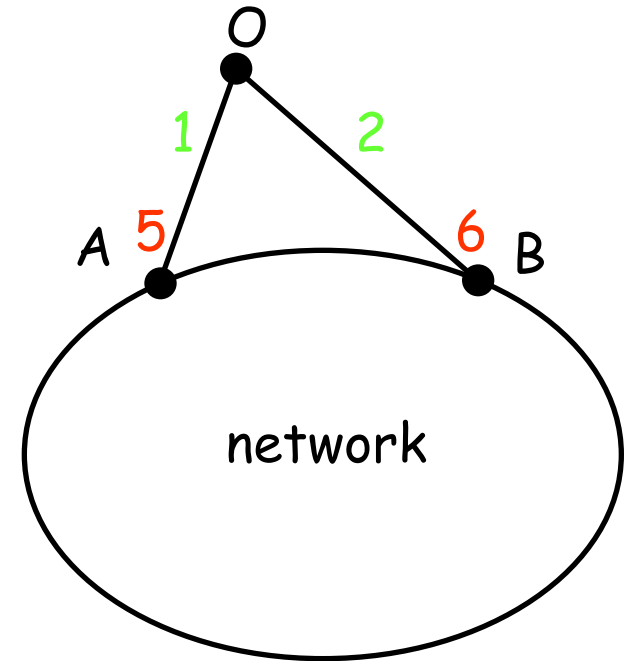
Costs with congestion:
 c(O,A)=5   c(O,B)=6

Each stream can use two paths: the one passing through A o the one passing through B

A similar game: routing congestion game

Cost Matrix

		stream 2	
		throungh A	through B
stream 1	throungh A	5, 5	1, 2
	through B	2, 1	6, 6

Nash Equilibrium

- In a NE no agent can unilaterally deviate from its strategy given others' strategies as fixed

- Agent has to deliberate about the strategies of the other agents

- Dominant Strategy Equilibrium $\Rightarrow$ Nash Equilibrium (but the converse is not true)

- If the game is played repeatedly and players converge to a solution, then it has to be a NE

A big game theoretic issue: the existence of a NE

- Unfortunately, for pure strategies games (as those seen so far), it is easy to see that we cannot have a general result of existence

- In other words, there may be no, one, or many NE, depending on the game

A conflictual game: Matching pennies

		Player II	
		Head	Tail
Player I	Head	1,-1	-1,1
	Tail	-1,1	1,-1

⇒ In any configuration, one of the players prefers to change his strategy
⇒ no NE!

On the existence of a NE (2)

- However, when a player can select his strategy randomly by using a probability distribution over his set of possible strategies (mixed strategy), then the following general result holds:

- Theorem (Nash, 1951): Any game with a finite set of players and a finite set of strategies has a NE of mixed strategies (i.e., the expected payoff cannot be improved by changing unilaterally the selected probability distribution).

- Matching pennies: if each player sets p(Head)=p(Tail)=1/2, then the expected payoff of each player is 0, and this is a NE, since no player can improve on this by choosing a different probability distribution!

Analysis of a game: some typical issues

1. Establish whether the game always has a NE
2. Find a NE, once it does exist
3. In a repeated game, establish whether and in how many steps the system will eventually converge to a NE
4. Establish the quality of a NE

On the quality of a NE

- How inefficient is a NE in comparison to an ideal situation in which the players would strive to collaborate with the common goal of choosing the best outcome? Best outcome w.r.t. what?

- we need a social-choice function C mapping strategy profiles into real numbers
  - C measures the overall quality of an outcome s
  - e.g. $C(s)$: sum of all players' costs/utilities

A worst-case perspective: the Price of Anarchy (PoA)

- Definition (Koutsopias & Papadimitriou, 1999): Given a game G and a social-choice function C, let S be the set of all NE. If the payoff represents a cost (resp., a utility) for a player, let OPT be the outcome of G minimizing (resp., maximizing) C. Then, the *Price of Anarchy (PoA)* of G w.r.t. C is

$$\text{PoA}_G(C) = \sup_{s \in S} \frac{C(s)}{C(\text{OPT})} \left(\text{resp.}, \inf_{s \in S} \frac{C(s)}{C(\text{OPT})} \right)$$

The price of stability (PoS)

- **Definition** (Schulz & Moses, 2003)
  Given a game G and a social-choice function C, let S be the set of all NE. If the payoff represents a cost (resp., a utility) for a player, let OPT be the outcome of G minimizing (resp., maximizing) C. Then, the *Price of Stability (PoS)* of G w.r.t. C is:

$$PoS_G(C) = \inf_{s \in S} \frac{C(s)}{C(\text{OPT})} \left(\text{resp.}, \sup_{s \in S} \frac{C(s)}{C(\text{OPT})} \right)$$
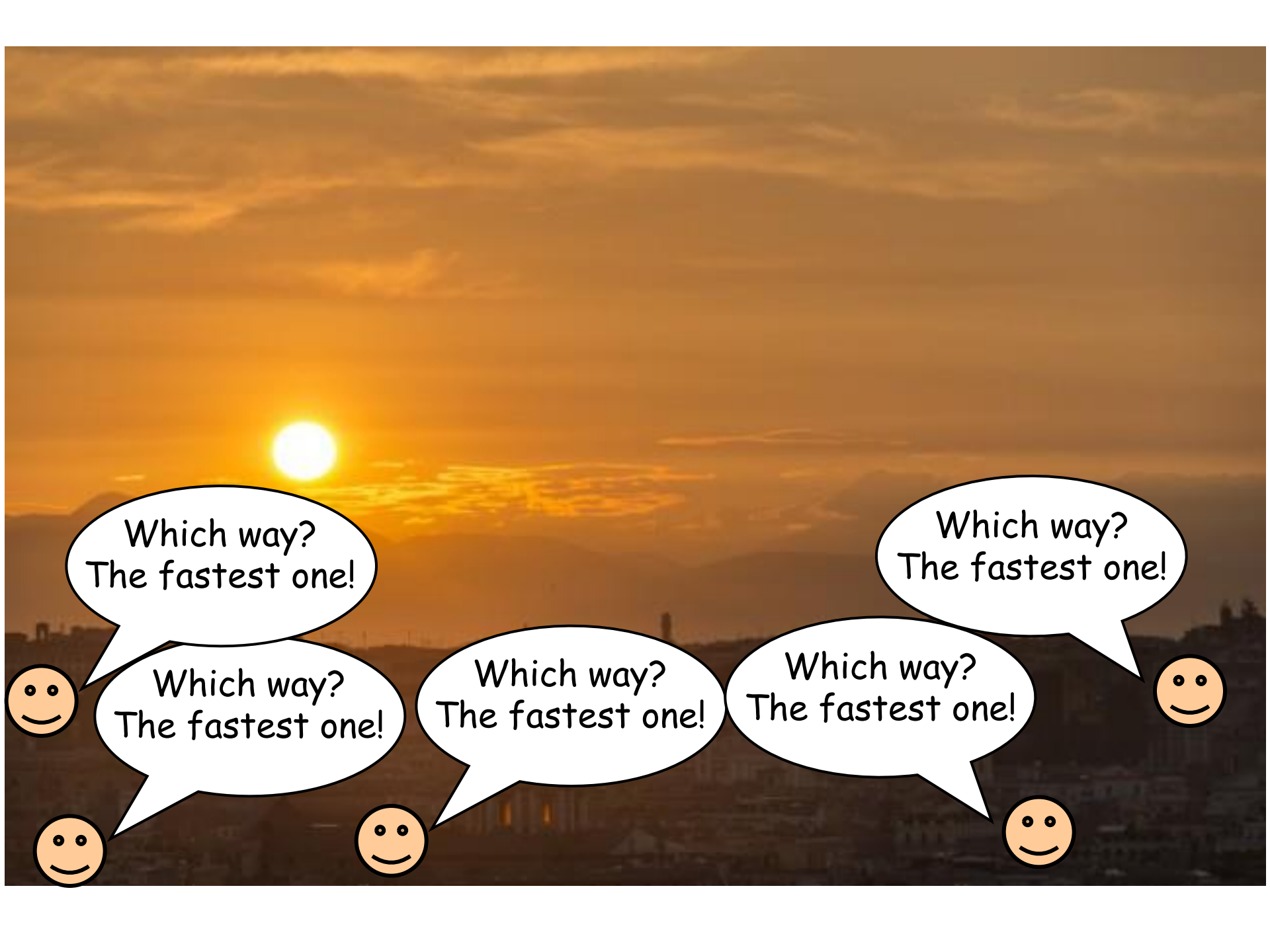
Some remarks

- PoA and PoS are
  - ≥ 1 for minimization problems
  - ≤ 1 for maximization problems
- PoA and PoS are small when they are close to 1
- PoS is at least as close to 1 than PoA
- In a game with a unique equilibrium PoA=PoS
- PoA is similar to the concept of approximation ratio of a heuristic
- a bound on the PoS provides a significantly weaker guarantee than a bound on the PoA
- Why to study the PoS?
  - sometimes a nontrivial bound is possible only for PoS
  - PoS quantifies the necessary degradation in quality under the game-theoretic constraint of stability

An example: Selfih Routing

selfish routing

A large network can be modelled by using game theory
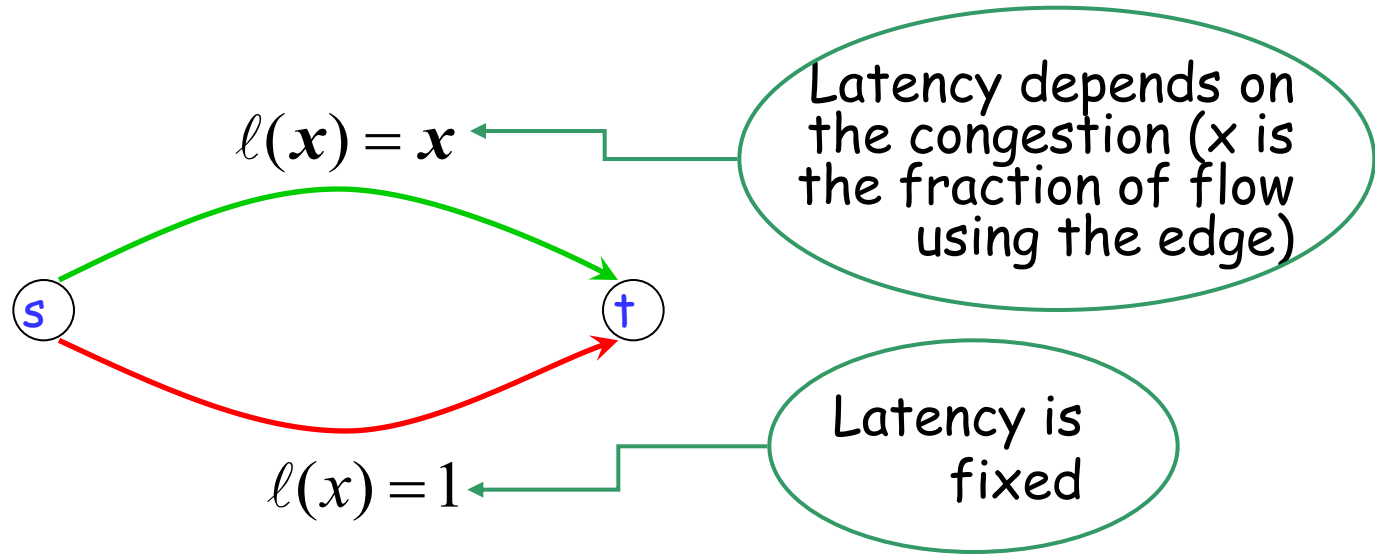
players $\longrightarrow$ users

strategies $\longrightarrow$ paths over which users can route their traffic

Non-atomic Selfish Routing:

- there is a large number of (selfish) users
- every user controls a tiny fraction of the traffic
- each edge has a cost function measuring the travel time as function of amount of traffic on the edge
- every user tries to minimize his travel time
- social-choice function (to minimize): average travel time incurred by players

Example: Pigou's game [1920]

One unit
of traffic

$$\ell(x) = x$$

Latency depends on the congestion (x is the fraction of flow using the edge)

s — t

$$\ell(x) = 1$$

Latency is fixed

- What is the NE of this game?
- Trivial: all the fraction of flow tends to travel on the upper edge $\Rightarrow$ the cost of the flow is 1·1 +0·1 =1
- How bad is this NE?
- The optimal solution is the minimum of C(x)=x·x +(1-x)·1 $\Rightarrow$ C '(x)=2x-1 $\Rightarrow$ OPT=1/2 $\Rightarrow$ C(OPT)=1/2·1/2+(1-1/2)·1=0.75
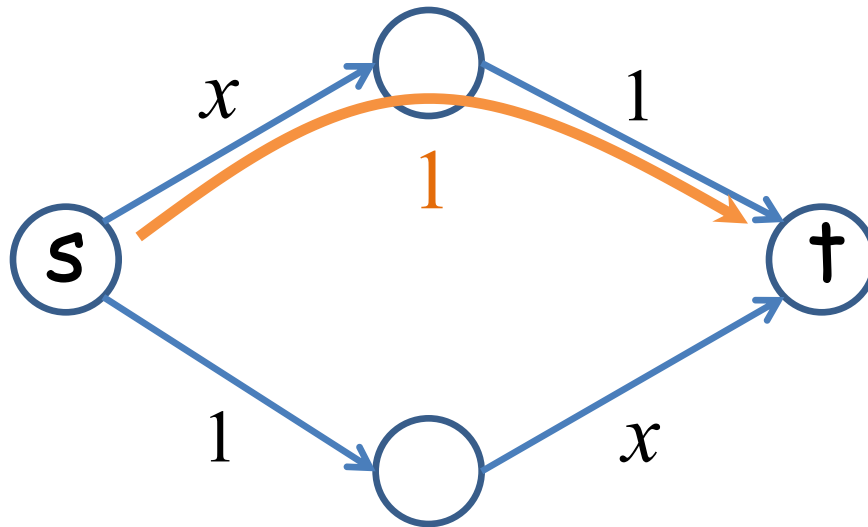
ratio between the two costs (NE vs Opt) = 1/0.75 = **4/3**

Do we have to take into account selfish behaviour of the users when we design a network?

The Braess's paradox
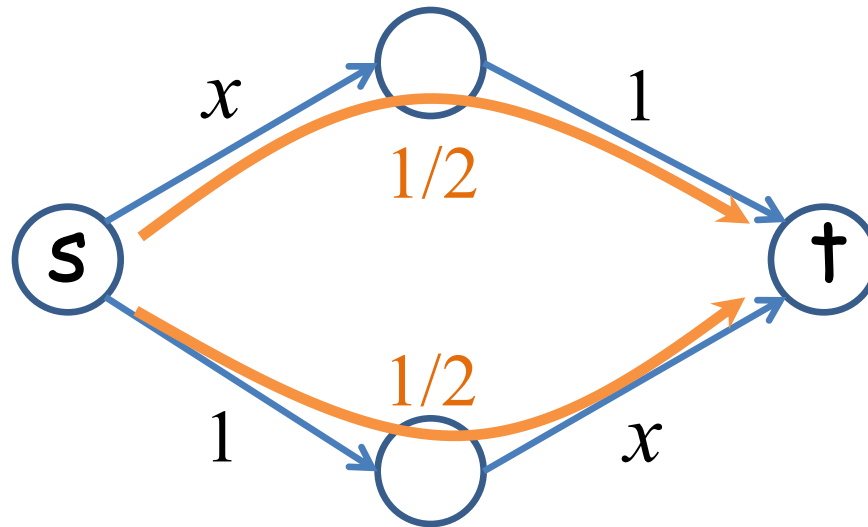
One unit
of traffic

average
travel time $= 2$

x

1

1

s

t

1

x

is it a NE?

...no!

The Braess's paradox

One unit
of traffic

average
travel time $= 1.5$

the only NE

Notice: this
is also the
optimal
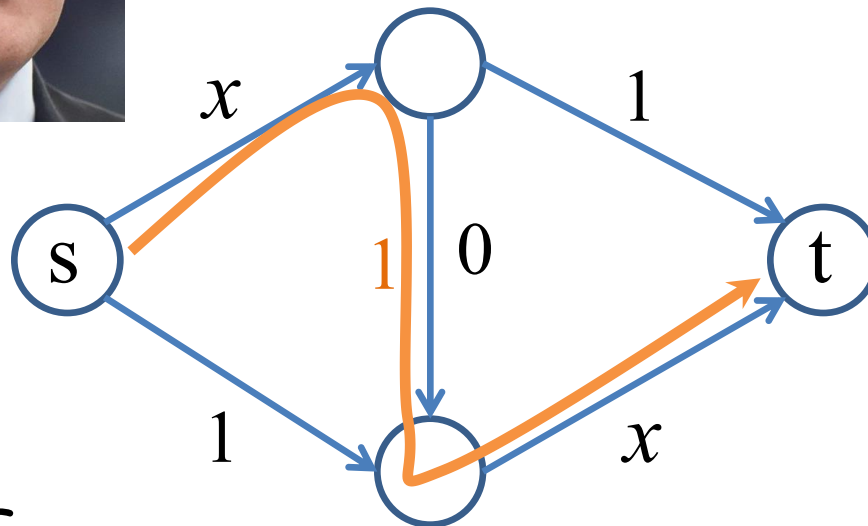outcome.

The Braess's paradox

One unit of traffic

the only NE

The Braess's paradox

One unit of traffic

the only NE

average travel time $= 2$

Notice:
- the optimal outcome as before:
- ½ up & ½ down
- average travel time of 1.5

4/3 as in the Pigou's example

ratio between the two costs (NE vs Opt) $= 2/1.5 = $ **4/3**

Theorem (Roughgarden&Tardos 2000)

The Price of Anarchy of the Selfish Routing Game with linear latency function is at most 4/3

There are *n* countries. Each country faces the choice of either passing legislation to control pollution or not. Assume that pollution control has a cost of 3 for the country, but each country that pollutes adds 1 of all countries (in term of added health costs).
The cost of controlling pollution is 3.

...notice that the cost of controlling pollution
is considerably larger than the cost a country
pays for being socially irresponsible...

can we bound the PoA?
And the PoS?

Tragedy of commons

There are *n* players. Each player wants to send information along a shared channel of known maximum capacity 1. Player i's strategy is to send x_i units of flow along the channel, for some $x_i \in [0,1]$.
Each player would like to have a large fraction of the bandwidth but the quality of the channel deteriorates as the total assigned bandwidth increases. More precisely, the value of a player i is $x_i(1- \Sigma_j x_j)$.

can we bound the PoA?
And the PoS?