Final Assignment Random Signal Processing

September 22, 2020

1 Instructions

Please put your answers to the questions in a short document, and upload it in the Assignments in Brightspace. This assignment is done individually.

The report will be checked and the result of the assignment will be either PASS or FAIL. The grade for the written/online exam will only count in case you pass this assignment.

2 Random Signal Processing on GPS

In navigation systems, the position of an object, vehicle, person, or animal can be calculated from information provided by satellites. For instance, the TomTom and Garmin are well-known car navigation systems that measure the position of a car based on satellite (GPS) signals, and from these position measurements, speed and future positions can be predicted.

In this exercise we consider a (simple) position estimation problem, namely the position along a straight road. On that road, an object (say animal) is moving around. The position of the object is measured at regular time instances, and from these measurements the future position needs to be estimated. This problem is known as the position prediction problem.

Let us consider the time instance n. At that time instance, we have available the actual position measurements of the object at time instances n, $n-1$, $n-2$, etc, denoted by $X(n)$, $X(n-1)$, $X(n-2)$, etc. We model $X(n)$ as a WSS stochastic signal with expected value 0 and autocorrelation function $R_X(k)$.

Based on the position measurements $X(n)$, the object position at the time instance $n+1$ will be predicted (in a overly simple way) as follows:

$$\hat{X}(n+1) = X(n) + \alpha(X(n) - X(n-1)) \tag{1}$$

Here α is a tuning parameter that needs to be selected or calculated optimally. At time instance $n+1$ the actual position can be measured, and the difference between the predicted and actual position, called the position estimation error, $\varepsilon(n+1)$ can be evaluated:

$$\varepsilon(n+1) = X(n+1) - \hat{X}(n+1) \tag{2}$$

The position prediction system functions optimally if the average quadratic difference between the predicted and actual position (i.e. the mean squared position estimation error is minimized) is as small as possible. The quality of the position estimation is therefore expressed by the variance of the position estimation error, i.e.

$$\sigma_\varepsilon^2 = E[\varepsilon(n+1)^2] = E[(X(n+1) - \hat{X}(n+1))^2] \tag{3}$$

The value of the tuning parameter α is to be selected such that the variance (3) is minimized.

The file `positions1.mat` contains a series of $N = 1000$ measured object positions.

a. Calculate and plot the position estimation error $\varepsilon(n+1)$ for $\alpha = 0.1$ as a function of n ($n = 1, ..., N-1$). In other words, use the pair of measurements $(X(n), X(n-1))$ (for all N) to estimate the position at time instance $n+1$ using eq. (1) with $\alpha = 0.1$, and subtract the result from the actual (and given) position measurement $X(n+1)$.

b. Estimate the variance of position estimation error $\varepsilon(n+1)$ from the given measured car positions for the values of the tuning parameter $\alpha = -1.00, -0.99, -0.98, -0.97, \ldots 0.97, 0.98, 0.99, 1.00$. Plot the resulting relation between α and the estimated σ_ε^2. Determine a suitable value of α from this figure.

c. In (b.) you have calculated and plotted the variance of the estimation error directly from the data. The variance of the estimation error can also be expressed in terms of the autocorrelation function of the process $X(n)$. Show that the variance can be expressed as:

$$\sigma^2 = E[\varepsilon(n+1)^2] = (2 + 2\alpha + 2\alpha^2)R_X(0) + (-2 - 4\alpha - 2\alpha^2)R_X(1) + 2\alpha R_X(2) \tag{4}$$

d. Find an analytical expression for the *optimal* value of α by minimizing Equation (4) (taking derivative with respect to α).

e. Assume that the autocorrelation function of the random process $X(n)$ can be modeled as follows:

$$R_X(k) = \sigma_X^2 \rho^{|k|}, \qquad |\rho| < 1 \tag{5}$$

Use this expression to calculate and plot the optimal value of α as a function of ρ.

f. Estimate the value of ρ from the data $X(n)$, and find the optimal value of α.

g. Estimate the autocorrelation function $R_X(k)$ of the position measurements $X(n)$ in the file `positions1.mat`. Plot the resulting autocorrelation function.

h. Use the result under part (g.) and Equation (4) to plot the variance of position estimation error $\varepsilon(n+1)$ as a function of α. Determine a suitable value of α from this figure.

i. Use the result under parts (d.) and (g.) to find the optimal value of α.

j. Explain the differences (if they exist) between the calculated optimal/suitable values of α in parts (b.), (f.), and (i.).