Tutorium: Signals and Systems

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1 Signals and Systems in Time Domain

1.1 Basic Continuous Signals

Basic signals are simple signals which are used to describe more complex signals.

1.1.1 Step Function

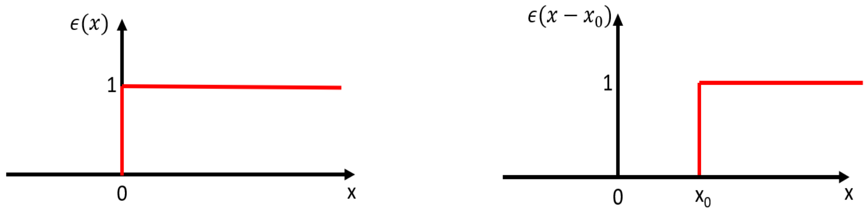
The unit step function $\epsilon(x)$, also known as the Heaviside unit function, is defined as

$$\epsilon(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases} \tag{1}$$

The shifted step function $\epsilon(x - x_0)$ is defined as

$$\epsilon(x - x_0) = \begin{cases} 1 & \text{for } x \geq x_0 \\ 0 & \text{for } x < x_0 \end{cases} \tag{2}$$

Fig. 1(a) and 1(b) show the step function and its shifted version.



(a) Step function (b) Shifted version

Figure 1: Step function

1.1.2 Rectangular Function

The rectangular function $rect(\frac{x}{T})$ is defined as

$$rect(\frac{x}{T}) = \begin{cases} 1 & \text{for } |x| \leq \frac{T}{2} \\ 0 & \text{for } |x| > \frac{T}{2} \end{cases} \tag{3}$$

The function can also be describe using the step function.

$$rect(\frac{x}{T}) = \epsilon(x + \frac{T}{2}) - \epsilon(x - \frac{T}{2}) \tag{4}$$

Figure 2 shows the general form of the rectangular function.

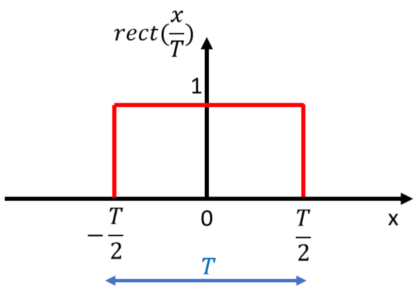


Figure 2: Rectangular function

1.1.3 Unit Impulse Function (Dirac delta function)

The dirac delta function $\delta(x)$ plays a significant role in system theory. It is defined as

$$\delta(x) = \begin{cases} 0 & \text{for } x \neq 0 \\ \infty & \text{for } x = 0 \end{cases} \tag{5}$$

The dirac delta function is drawn as an "arrow" which height is equal to the constant multiplied to the function: $A * \delta(x)$ (Fig. 3)

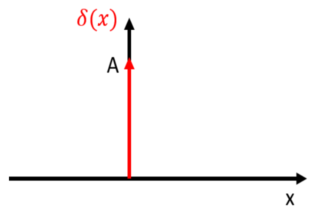


Figure 3: Dirac delta function

The dirac delta function can be considered as the rectangular function which is squeezed to an infinitesimal small width whereby the area of the function remains 1 (Fig. 4). So, the height of the function is "∞", the width is 0 and therefore the integral is 1.

Since in the Riemann integral sense a ordinary function, which is unequal 0 at just one point, must have to integral 0, the dirac delta function cannot be an ordinary function. Mathematically, the dirac delta function is actually not a function but a distribution. However, the dirac delta function can still be handled as a function and thus distributions do not have to be considered in this course.

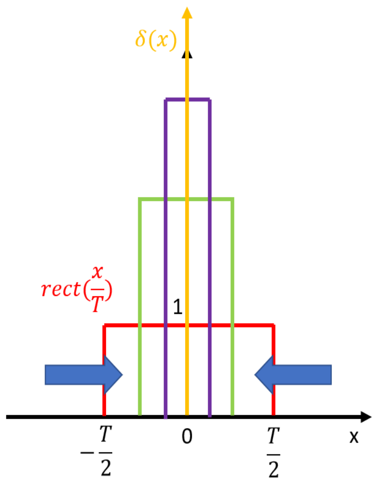


Figure 4: From rectangular to delta function

Important Properties

a) Main definition:

$$\int_{-\infty}^{\infty} \delta(x)dx = 1 \tag{6}$$

$$\int_{-\infty}^{\infty} a * \delta(x)dx = a \int_{-\infty}^{\infty} \delta(x)dx = a$$

b) Scaling:

$$\delta(ax) = \frac{1}{|a|}\delta(x) \tag{7}$$

c) The dirac delta function is an even function:

$$\delta(-x) = \delta(x) \rightarrow \delta(x - x_0) = \delta(x_0 - x) \tag{8}$$

d) The dirac delta function is the first derivative of the step function:

$$\frac{d\epsilon(x)}{dx} = \delta(x) \tag{9}$$
$$\int_{-\infty}^{x} \delta(x)dx = \epsilon(x)$$

e) "Hide property":

$$f(x)\delta(x - x_0) = f(x_0)\delta(x - x_0) \tag{10}$$

So the value of the function $f(x)$ at the point x_0 becomes the weight of the delta function $\delta(x - x_0)$ located at x_0.
Thus it follows

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)dx = \int_{-\infty}^{\infty} f(x)\delta(x_0 - x)dx = f(x_0) \tag{11}$$

1.1.4 Complex Exponential Signals

Exponential signals, especially complex exponential signals, are extremely important for describing signals in the frequency domain using the Fourier transform and the Laplace transform. According to Euler's formula a complex exponential signal is defined as

$$x(t) = e^{j\omega t} = cos(\omega t) + jsin(\omega t) \tag{12}$$

So, the complex exponential function consists of a cosine wave as real part and a sine wave as imaginary part. More general, extending the definition as

$$x(t) = e^{st} = e^{(\sigma + j\omega)t} = e^{\sigma t} * e^{j\omega t} = e^{\sigma t}[cos(\omega t) + jsin(\omega t)] \tag{13}$$

a complex exponential signal is received with an time-dependent amplitude.
If $\Re(s) = \sigma$ is smaller (greater) than 0, the amplitude of the complex exponential function (respectively the cosine or sine wave) decreases (increases) with time. If $\Re(s) = \sigma$ is equal to 0, Eq. 12 is received.
A sine wave with a time-dependent amplitude is depicted in Fig. 5.
The complex exponential signal should be kept in mind for the Fourier and Laplace transform considered in later sections.

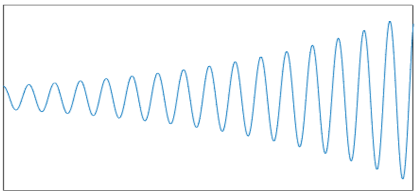


Figure 5: Real part of complex exponential signal

1.2 LTI-Systems in Time Domain

1.2.1 Describing LTI-Systems with Differential Equations

A common way to describe LTI-Systems in general and therefore electrical circuits are differential equations. The general expression is as follows:

$$\sum_{m=0}^{M} b_m \frac{d^m x}{dt^m} = \sum_{n=0}^{N} a_n \frac{d^n y}{dt^n} \tag{14}$$
$$y = \frac{1}{a_0}\left(\sum_{m=0}^{M} b_m \frac{d^m x}{dt^m} - \sum_{n=1}^{N} a_n \frac{d^n y}{dt^n}\right)$$

In this case, $x(t)$ is the input signal of the system and $y(t)$ the output signal. So, solving the differential equation delivers the output signal of the system. This concept has already been discussed in previous lectures and is additionally a quite complex way of describing systems. Consequently, this approach is not going to be considered further. However, this concept should always be kept in mind since this approach is related to describing systems with the Fourier and Laplace transform.

1.2.2 Describing LTI-Systems with Convolution

Impulse Response

The impulse response of a system is the response of the system to a dirac delta function as input signal: $x(t) = \delta(t)$. The impulse response is generally assigned to $h(t)$.

$$h(t) = S\{\delta(t)\} \tag{15}$$

Step Response

The step response of a system is the response of the system to a step function as input signal: $x(t) = \epsilon(t)$. The impulse response is generally assigned to $g(t)$.

$$g(t) = S\{\epsilon(t)\} \tag{16}$$

Relationship of Impulse and Step Response

Equally to the relationship of the dirac delta impulse and the step function, the impulse and step response are related to each other as

$$\begin{aligned} h(t) &= \frac{dg(t)}{dt} \\ g(t) &= \int_{-\infty}^{t} h(\tau)d\tau \end{aligned} \tag{17}$$

Describing The System

According to Eq. 11, each arbitrary signal can be described as a superposition of weighted and shifted dirac delta functions.

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t) \circledast \delta(t) \tag{18}$$

Due to the characteristic of LTI-systems (linear, time-invariant) it is applied for the output signal of the system:

$$\begin{aligned} y(t) = S\{x(t)\} = S\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\} = &\int_{-\infty}^{\infty} x(\tau)S\{\delta(t-\tau)\}d\tau \\ = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\tau = x(t) \circledast h(t) & \end{aligned} \tag{19}$$

Conclusion: If the impulse response of a LTI-system is known, the system can be fully described.

Causality and Stability

Considering the impulse response of a LTI-system it can be found out if the LTI-system is causal and stable.

a) Causality
A LTI-system is causal, if the impulse response starts earliest at $t = 0$.

$$h(t) = 0 \quad for \quad t < 0 \tag{20}$$

b) Stability
A LTI-system is stable, if the impulse response is absolutely integrable.

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty \tag{21}$$

2 Signals and Systems in Frequency Domain

In the previous chapter, signals and LTI-systems are considered in the time domain. However, describing signals and LTI-systems in the time domain can become complicated. In the following chapter, it is going to be discussed how signals and LTI-systems can be considered differently, namely in the frequency domain.

2.1 Harmonic Signals

2.1.1 Describing Signals

As already learnt in previous lectures, harmonic signals, i.e. sine and cosine signals, can be easily described by using Euler's formula (Eq. 12) which is also known as complex AC-calculation. The main idea beyond this approach is to algebraise the differential equation, which describes the in- and output behaviour of the system, by transferring it into the frequency domain. Therefore, the following applies:

$$\frac{dx^n(t)}{dt^n} \xrightarrow{\mathcal{F}} (j\omega)^n \underline{X}(\omega) \tag{22}$$

$$\int x(t)dt \xrightarrow{\mathcal{F}} \frac{1}{j\omega}\underline{X}(\omega) \tag{23}$$

By doing this, the harmonic input signal $x(t) = Xsin(\omega t+\phi_x)$ is considered as a complex exponential signal $\underline{x}(t) = Xe^{j(\omega+\phi_x)} = Xcos(\omega t+\phi_x)+jXsin(\omega t+\phi_x)$ although it also carries a cosine signal as real part.

$$x(t) = Xsin(\omega t + \phi_x) \rightarrow \underline{x}(t) = Xe^{j(\omega t+\phi_x)} \tag{24}$$

Since only linear systems are considered, the frequency of the input and output signal always remains the same. So, the time-dependency does not need to be considered anymore. Only the complex amplitude $\underline{X}$ consisting of the signal's amplitude and phase is used. The complex amplitude represents the corresponding time-signal $x(t)$ in the frequency domain.

$$x(t) = Xsin(\omega t + \phi_x) \rightarrow \underline{X} = Xe^{j\phi_x} \tag{25}$$

By doing the calculations in the frequency domain by using the complex amplitude $\underline{X}$, the complex amplitude $\underline{Y} = Ye^{j\phi_y}$ of the output signal $y(t)$ is received. Since the output signal has the same frequency as the input signal, the complex amplitude $\underline{Y}$ can be extended to the corresponding complex exponential signal $\underline{y}(t)$.

$$\underline{Y} = Ye^{j\phi_y} \rightarrow \underline{y}(t) = Ye^{j(\omega t + \phi_y)} \tag{26}$$

The complex signal delivers the real time-signal by just taking the imaginary part.

$$\underline{y}(t) = Ye^{j(\omega t + \phi_y)} \rightarrow y(t) = \Im\{\underline{y}(t)\} = Ysin(\omega t + \phi_y) \tag{27}$$

The whole cycle is shown in Figure 6.

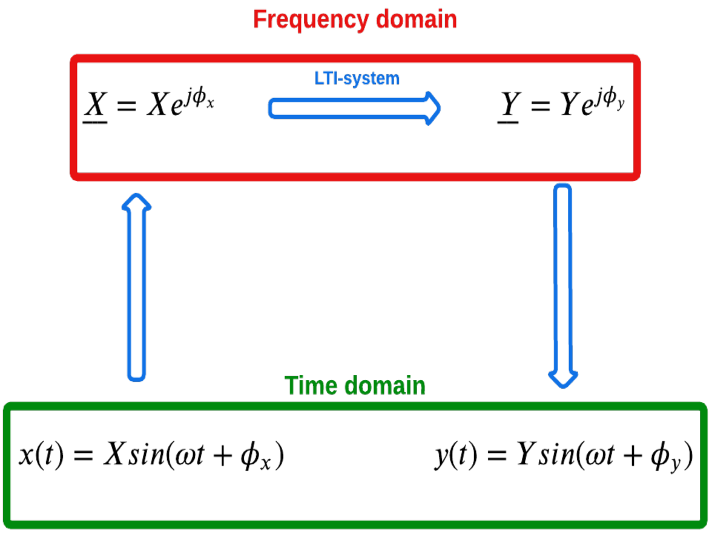


Figure 6: Cycle time and frequency domain

Conclusion: The complex amplitude of a harmonic signal represents the time-signal in the frequency domain. By using the complex AC-calculation, differential equations, which describe the behaviour of a system, can be transferred into the frequency domain which leads to algebraic expressions. In the frequency domain, the complex amplitude fully represents the harmonic signal. Linear systems never change the frequency of the output signal so the frequency of the input and output sine wave is always the same.

Example 1: Assuming an electrical circuit is given which is described in the time domain by the following differential equation.

$$y(t) = LC\ddot{x}(t) + RC\dot{x}(t) - LC\ddot{y}(t) - RC\dot{y}(t)$$

Solving this differential equation might become pretty complex. However, considering this expression in the frequency domain delivers the following algebraic expression.

$$\underline{Y} = -\omega^2 LC\underline{X} + j\omega RC\underline{X} + \omega^2 LC\underline{Y} - j\omega RC\underline{Y}$$
$$\underline{Y} = \underline{X}\frac{-\omega^2 LC + j\omega RC}{-\omega^2 LC + j\omega RC + 1}$$

Assuming $L = 1H$, $C = 100\mu F$, $R = 10\Omega$ and $x(t) = 2V * sin(5\frac{1}{s}t - 1.4)$ and describing the input signal as complex exponential equation $x(t) = 2Ve^{j(5\frac{1}{s}t-1.4)}$ the sought signal $y(t)$ is

$$\underline{Y} = 2Ve^{-j1.4} * \frac{-(5\frac{1}{s})^2 LC + j5\frac{1}{s}RC}{-(5\frac{1}{s})^2 LC + j5\frac{1}{s}RC + 1} = 0.01Ve^{j0.63}$$
$$y(t) = 0.01V * sin(5\frac{1}{s}t + 0.63)$$

2.1.2 Describing LTI-Systems

2.1.2.1 Frequency Response

As already mentioned, LTI-systems are described by differential equations in the time domain which is often a pretty complex approach. Using the complex AC-calculation, the differential equation can be considered as algebraic expression in the frequency domain. Considering a system as a black-box, it is just important to know what output signal the system generates depending on a specific input signal. Since the frequency of the output signal is the same as the input signal, it is sufficient to know how the system affects the amplitude and phase of the harmonic input signal for a specific frequency. This information is provided by determining the quotient of the complex amplitude of the output and the input signal. In order to get a description of the system for all frequencies, the real amplitude and phase of the complex amplitude and thus the complex amplitude itself remains frequency-dependent.

$$\underline{H}(\omega) = \frac{\underline{Y}(\omega)}{\underline{X}(\omega)} = \frac{Y(\omega)}{X(\omega)}e^{j(\phi_y(\omega)-\phi_x(\omega))} = H(\omega)e^{j\phi_H(\omega)} \tag{28}$$

Here, $\underline{H}(\omega)$ is called the frequency response of the system and provides for each frequency ω how the system affects the complex amplitude, i.e. the real amplitude and phase, of the input signal. The effect on the real amplitude is described by the real amplitude of the frequency response (amplitude response), the effect on the phase by the phase of the frequency response (phase response).

Amplitude Response

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \sqrt{\Re\{\underline{H}(\omega)\}^2 + \Im\{\underline{H}(\omega)\}^2} \tag{29}$$

For $H(\omega) > 1$: The input signal is amplified
For $H(\omega) < 1$: The input signal is attenuated
For $H(\omega) = 1$: Input and output signal are equal in amplitude

Phase Response

$$\angle\underline{H}(\omega) = \phi_y(\omega) - \phi_x(\omega) = arctan(\frac{\Im\{\underline{H}(\omega)\}}{\Re\{\underline{H}(\omega)\}}) \tag{30}$$

For $\phi_H(\omega) > 0$: Output signal rushes ahead of input signal
For $\phi_H(\omega) < 0$: Output signal hurries after input signal
For $\phi_H(\omega) = 0$: Output and input signal are equal in phase

So, if the frequency response of a system is known, the complex output signal is always defined as

$$\underline{Y}(\omega) = \underline{H}(\omega)\underline{X}(\omega) = X(\omega)H(\omega)e^{j(\phi_x(\omega)+\phi_H(\omega))} = Y(\omega)e^{j\phi_y(\omega)} \tag{31}$$

And therefore the corresponding time-signal as

$$y(t) = X(\omega)H(\omega)sin(\omega t + \phi_x(\omega) + \phi_H(\omega)) = Y(\omega)sin(\omega t + \phi_y(\omega)) \tag{32}$$

Considering example 1, the frequency response of the system would look like this:

$$\begin{gathered}
\underline{Y}(\omega) = -\omega^2 LC\underline{X}(\omega) + j\omega RC\underline{X}(\omega) + \omega^2 LC\underline{Y}(\omega) - j\omega RC\underline{Y}(\omega) \\
\underline{Y}(\omega) = \underline{X}(\omega)\frac{-\omega^2 LC + j\omega RC}{-\omega^2 LC + j\omega RC + 1} \\
\underline{H}(\omega) = \frac{\underline{Y}(\omega)}{\underline{X}(\omega)} = \frac{-\omega^2 LC + j\omega RC}{-\omega^2 LC + j\omega RC + 1}
\end{gathered}$$

For the particular frequency $\omega = 5\frac{1}{s}$ the effect of the system to the input signal can be described as

$$\underline{H}(\omega = 5\frac{1}{s}) = 0.006e^{j2.03}$$

So, the system attenuates the amplitude of the input signal by the factor of 0.006 and changes the phase by 2.03 rad.
Consequently, the complex amplitude follows as

$$\underline{Y} = 0.006e * 2V * e^{j(-1.4+2.03)} = 0.01Ve^{j0.63}$$

and the real time-signal is

$$y(t) = \Im\{\underline{Y}\} = 0.006 * 2V * sin(5\frac{1}{s}t - 1.4 + 2.03) = 0.01Vsin(5\frac{1}{s}t + 0.63)$$

Conclusion: The frequency response of a system describes how the system affects the complex amplitude and thus the real amplitude and phase of the harmonic input signal. The frequency response is received by applying the complex AC-calculation and determining the quotient of the complex amplitudes of the output and input signal.

2.1.2.2 Transfer Function

In case the input signal is a harmonic signal but with a time-dependent real amplitude, the approach above has to be extended.
This is done by extending the time-signal and thus the argument of the complex exponential signal by the parameter σ. This process is also known as the extended complex AC-calculation.

$$x(t) = Xe^{\sigma t}sin(\omega t + \phi_x) \tag{33}$$

$$\underline{x}(t) = Xe^{\sigma t}e^{j(\omega t+\phi_x)} = Xe^{\sigma t}e^{j\omega t}e^{j\phi_x} = Xe^{(\sigma+j\omega)t}e^{j\phi_x} = Xe^{st}e^{j\phi_x} \tag{34}$$

The parameter $s = \sigma + j\omega$ is a complex number and therefore called the complex frequency. The attenuation constant of the signal is received by $\Re\{s\} = \sigma$, the actual frequency by $\Im\{s\} = \omega$. It applies:
For $\sigma < 0$: Harmonic signal with exponentially decreasing amplitude
For $\sigma > 0$: Harmonic signal with exponentially increasing amplitude
For $\sigma < 0$: Harmonic signal with constant amplitude

Thus, the complex amplitude of the signal follows by removing the time-dependency of the signal is

$$\underline{X} = Xe^{j\phi_x} \tag{35}$$

Since the complex amplitude is still the same as for the complex AC-calculation, adding the parameter σ does not have an impact on the main approach. However, using this extended approach additionally delivers the opportunity describing systems with harmonic input signals which have a time-dependent amplitude.
By using this extended approach, the frequency response is now called transfer function. So, the transfer function is defined as

$$\underline{H}(s) = \frac{\underline{Y}(s)}{\underline{X}(s)} = \frac{Y(s)}{X(s)} e^{j(\phi_y(s)-\phi_x(s))} = H(s)e^{j\phi_H(s)} \tag{36}$$

Analogous to the frequency response of a system, the amplitude of the transfer function describes how the system affects the amplitude of the harmonic input signal. Its phase describes how the phase of the input signal is affected.

Amplitude

$$H(s) = \frac{Y(s)}{X(s)} = \sqrt{\Re\{\underline{H}(s)\}^2 + \Im\{\underline{H}(s)\}^2} \tag{37}$$

For $H(s) > 1$: The input signal is amplified
For $H(s) < 1$: The input signal is attenuated
For $H(s) = 1$: Input and output signal are equal in amplitude

Phase

$$\angle\underline{H}(s) = \phi_y(s) - \phi_x(s) = arctan(\frac{\Im\{\underline{H}(s)\}}{\Re\{\underline{H}(s)\}}) \tag{38}$$

For $\phi_H(s) > 0$: Output signal rushes ahead of input signal
For $\phi_H(s) < 0$: Output signal hurries after input signal
For $\phi_H(s) = 0$: Output and input signal are equal in phase

Conclusion: The transfer function of a system provides the exact same information as the frequency response but for harmonic signals with a time-dependent real amplitude. The transfer function follows from the frequency response by just substituting the frequency $j\omega$ with the complex frequency s.

Pole-Zero Plot

The pole-zero plot is a graphical way of describing the transfer function of a system in the complex s-plane. It follows from the factorised form of the transfer function.

$$\underline{H}(s) = \frac{b_M \prod_{l=1}^{M}(s - z_l)}{a_N \prod_{k=1}^{N}(s - p_k)} \tag{39}$$

- z_l: Zeroes
- p_k: Poles
- $\frac{b_M}{a_N}$: Constant K

Note:

- Each transfer function is except for the constant K unambiguously defined by its pole-zero plot.
- In the plot, poles are depicted as crosses, zeroes as circles.
- For real signals, the poles and zeroes are either real or pairwise conjugated-complex. So, they lay either on or mirror-symmetrically to the real axis (σ-axis).

Example 2: Using the extended complex AC-calculation for example 1 leads to the following transfer function.

$$\underline{H}(s) = \frac{\underline{Y}(s)}{\underline{X}(s)} = \frac{s^2LC + sRC}{s^2LC + sRC + 1} = \frac{s(s+10)}{(s+5-j99.87)(s+5+j99.87)}$$

The associated pz-plot is shown in Figure 7.

Causality and Stability

a) *Causality*
A LTI-system is causal if the number of poles is at least equal to the number of zeroes.

b) *Stability*
A LTI-system is stable if all poles lay on the left-hand side of the imaginary axis (ω-axis).

So, the system from example 2 is stable and causal.

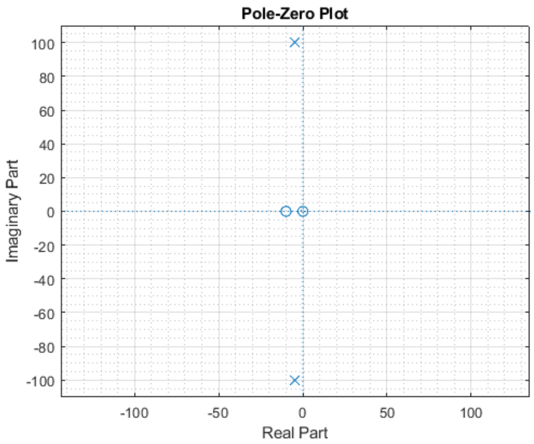


Figure 7: Pole-Zero Plot

2.1.2.3 Bode Diagram

The bode diagram is a graphical depiction of the frequency response. Amplitude and phase response are depicted separately with respect to ω. In both graphics, the frequency is depicted logarithmically. Additionally, the amplitude is depicted in decibel (dB). For power magnitudes, the applied pre-factor is 10

$$H(\omega)|_{dB} = 10lg(\frac{P_2}{P_1})dB \tag{40}$$

for field magnitudes (e.g., voltage, current) it is 20.

$$H(\omega)|_{dB} = 20lg(\frac{U_2}{U_1})dB \tag{41}$$

Generally, it applies:
For $H(\omega)|_{dB} > 0 \rightarrow H(\omega) > 1$: The input signal is amplified
For $H(\omega)|_{dB} < 0 \rightarrow H(\omega) < 1$: The input signal is attenuated
For $H(\omega)|_{dB} > 0 \rightarrow H(\omega) = 1$: Input and output signal are equal in amplitude

In the following sections, field magnitudes are assumed.

Bode Diagrams of Basic LTI-Systems

1) <u>P-Element</u>

- Frequency response

$$\underline{H}(\omega) = K \tag{42}$$

- Amplitude response

$$H(\omega)|_{dB} = 20lg(K) = const. \tag{43}$$

- Phase response

$$\angle H(\omega) = arctan(\frac{0}{K}) = 0 \tag{44}$$

- Bode diagram: Fig. 8

2) <u>D-Element</u>

- Frequency response

$$\underline{H}(\omega) = j\omega K \tag{45}$$

- Amplitude response

$$H(\omega)|_{dB} = 20lg(K) + 20lg(\omega) \tag{46}$$

- Phase response

$$\angle H(\omega) = arctan(\frac{\omega K}{0}) = \frac{\pi}{2} \tag{47}$$

- Bode diagram: Fig. 9

3) <u>I-Element</u>

- Frequency response

$$\underline{H}(\omega) = \frac{K}{j\omega} \tag{48}$$

- Amplitude response

$$H(\omega)|_{dB} = 20lg(K) - 20lg(\omega) \tag{49}$$

- Phase response

$$\angle H(\omega) = arctan(-\frac{\frac{K}{\omega}}{0}) = -\frac{\pi}{2} \tag{50}$$

- Bode diagram: Fig. 10

4) $\underline{PT_1\text{-Element}}$

- Frequency response

$$\underline{H}(\omega) = \frac{K}{j\omega T + 1} = \frac{K\omega_g}{j\omega + \omega_g} \tag{51}$$

- Amplitude response

$$H(\omega)|_{dB} = \begin{cases} 20lg(K) & \text{for } \omega << \omega_g \\ 20lg(\frac{K}{T}) - 20lg(\omega) & \text{for } \omega >> \omega_g \end{cases} \tag{52}$$

- Phase response

$$\angle H(\omega) = \begin{cases} arctan(\frac{0}{K}) = 0 & \text{for } \omega << \omega_g \\ arctan(-\frac{\frac{K}{\omega T}}{0}) = -\frac{\pi}{2} & \text{for } \omega >> \omega_g \end{cases} \tag{53}$$

- Bode diagram: Fig. 11

Note: The parameter T is a time constant and is the reciprocal of the cutoff frequency ω_g. The cutoff frequency is the point where the bode diagram of the PT1-Element decreases by $3dB$ (or linearly by the factor of $\sqrt{2}$). The gradient of a PT_1-Element is $-20\frac{dB}{dec}$.

5) $\underline{PT_2\text{-Element}}$

- Frequency response

$$\underline{H}(\omega) = \frac{K}{(j\omega)^2T^2 + 2\theta j\omega T + 1} = \frac{K\omega_g^2}{(j\omega)^2 + 2\theta j\omega\omega_g + \omega_g^2} \tag{54}$$

- Amplitude response

$$H(\omega)|_{dB} = \begin{cases} 20lg(K) & \text{for } \omega << \omega_g \\ 20lg(\frac{K}{T^2}) - 40lg(\omega) & \text{for } \omega >> \omega_g \end{cases} \tag{55}$$

- Phase response

$$\angle H(\omega) = \begin{cases} arctan(\frac{0}{K}) = 0 & \text{for } \omega << \omega_g \\ arctan(\frac{0}{-\frac{K}{\omega^2}T^2}) - \pi = -\pi & \text{for } \omega >> \omega_g \end{cases} \tag{56}$$

- Bode diagram: Fig. 12

Note: The PT2-Element can actually be considered as a serial connection of two PT1-Elements. The parameter θ is defined as decay factor. If θ is less than 1, the frequency response overshoots at some point. The gradient of a PT_2-Element is $-40\frac{dB}{dec}$.

6) $\underline{PT_n\text{-Element}}$

- Frequency response

$$\underline{H}(\omega) = \frac{K}{a_n(j\omega)^2 + ... + a_2(j\omega)^2 + a_1 j\omega + 1} \tag{57}$$

- Amplitude response

$$H(\omega)|_{dB} = \begin{cases} 20lg(K) & \text{for } \omega << \omega_g \\ 20lg(\frac{K}{a_n}) - n * 20lg(\omega) & \text{for } \omega >> \omega_g,\ n = 1, 2, ... \end{cases} \tag{58}$$

- Phase response

$$\angle H(\omega) = \begin{cases} 0 & \text{for } \omega << \omega_g \\ n * (-\frac{\pi}{2}) & \text{for } \omega >> \omega_g,\ n = 1, 2, ... \end{cases} \tag{59}$$

Note: The gradient of a PT_n-Element is $-n * 20\frac{dB}{dec}$.

7) $\underline{DT_1\text{-Element}}$

- Frequency response

$$\underline{H}(\omega) = \frac{j\omega K}{j\omega T + 1} = j\omega K * \frac{1}{j\omega T + 1} \tag{60}$$

→ The DT_1-Element consists of a D-Element and a PT_1-Element.

- Amplitude response

$$H(\omega)|_{dB} = \begin{cases} 20lg(K) + 20lg(\omega) & \text{for } \omega << \omega_g \\ 20lg(\frac{K}{T}) - 40lg(\omega) & \text{for } \omega >> \omega_g \end{cases} \tag{61}$$

- Phase response

$$\angle H(\omega) = \begin{cases} arctan(\frac{j\omega K}{0}) = \frac{\pi}{2} & \text{for } \omega << \omega_g \\ arctan(\frac{0}{\frac{K}{T}}) = 0 & \text{for } \omega >> \omega_g \end{cases} \tag{62}$$

- Bode diagram: Fig. 13

Note: Analogous to the DT_1-Element, DT_n-Elements consist of n D-Elements and one PT_n-Element.

7) Inverse LTI-systems

Inverse LTI-systems are reciprocals of basic LTI-systems. Thus it generally applies:

a) Amplitude response

$$H(\omega)_{inv}|_{dB} = \frac{1}{H(\omega)}|_{dB} = H(\omega)^{-1}|_{dB} = -H(\omega)|_{dB} \tag{63}$$

b) Phase response

$$\angle\underline{H}(\omega)_{inv} = -\angle\underline{H(\omega)} \tag{64}$$

So, bode diagrams of inverse LTI-systems can be determined by mirroring the bode diagrams of basic LTI-systems with respect to the x-axis.

Note: All bode diagrams were created with the following parameter values:

- $K = 2dB$
- $T = 0.01s \rightarrow \omega_g = \frac{1}{T} = 100\frac{1}{s}$
- $\theta = 0.2$

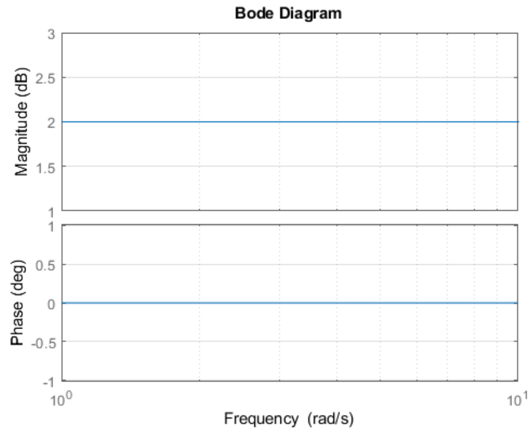


Figure 8: P-Element

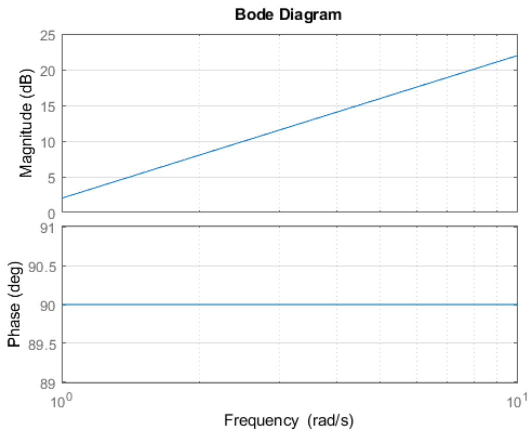


Figure 9: D-Element

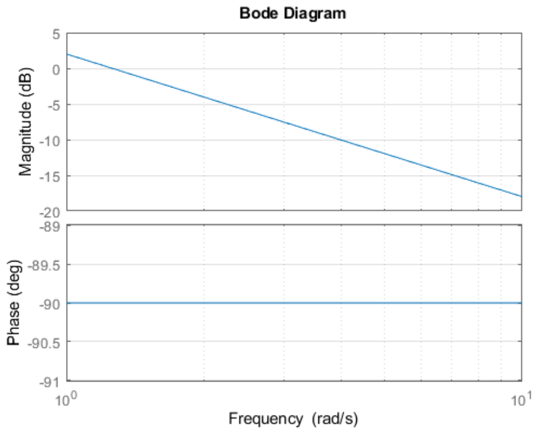


Figure 10: I-Element

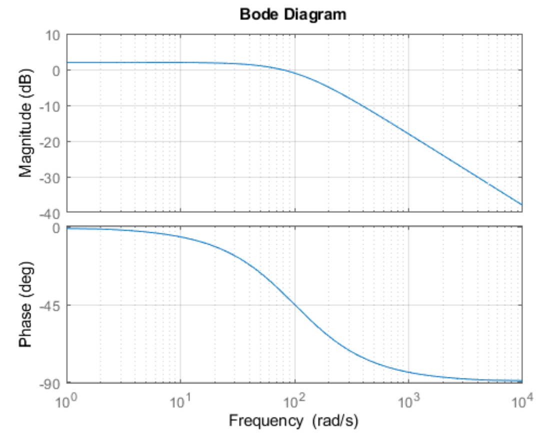


Figure 11: PT_1-Element

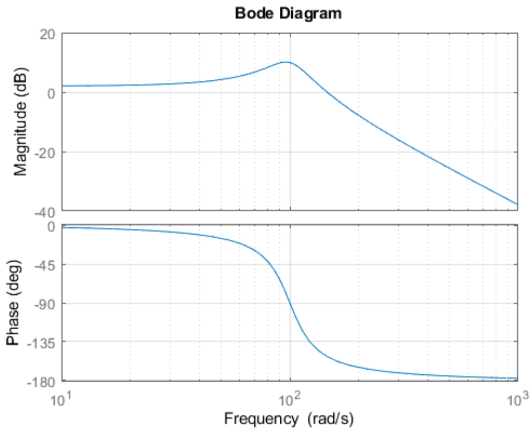


Figure 12: PT_2-Element

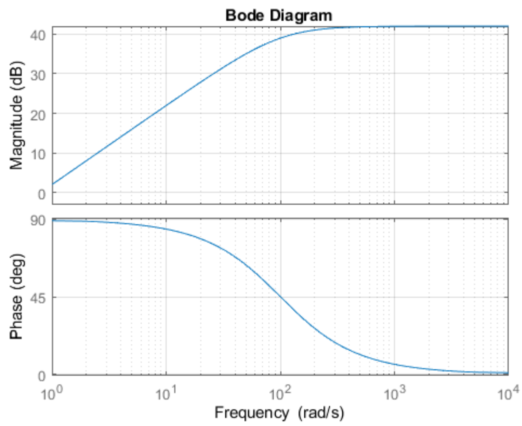


Figure 13: DT_1-Element

Common Filters

a) **Low pass**
Frequencies below the cutoff frequency can pass, higher frequencies are attenuated.
Example: PT_1-Element

b) **High pass**
Frequencies above the cutoff frequency can pass, lower frequencies are attenuated.
Example: DT_1-Element

c) **Band gap**
All frequencies out of a specific interval are attenuated.
Example: Serial connection of PT_1-Element and DT_1-Element.

d) **Band stop**
All frequencies within a specific interval are attenuated.
Example: Serial connection of 2 PT_2-Elements.

Determining Bode Diagrams

The bode diagram of a system can be derived from the corresponding frequency response or transfer function. In the following, two major approaches are explained.

a) Deriving from basic LTI-systems

1. Decompose the given system in several basic systems
2. Determine the start and end point of the bode diagram of each basic system as $\lim_{\omega\to 0} \underline{H}(\omega)$ and $\lim_{\omega\to\infty} \underline{H}(\omega)$
3. Determine the cutoff frequency of each basic system
4. Determine the final bode diagram by adding the bode diagrams of all basic systems (in dB)

b) Deriving from poles and zeroes of the transfer function

1. Determine the poles and zeroes of the transfer function
2. The absolute value of each pole delivers a cutoff frequency after which the bode diagram falls by $-20\frac{dB}{Dec}$

3. The absolute value of each zero delivers a cutoff frequency after which the bode diagram rises by $+20\frac{dB}{Dec}$

Example 3: The following frequency response is given:

$$\underline{H}(\omega) = \frac{j\omega K}{(j\omega)^2 T + j\omega T + 1}$$

Approach a):
The system can be decomposed into a PT_1-Element and a DT_1-Element.

$$\underline{H}(\omega) = \frac{j\omega K}{j\omega T_1 + 1} * \frac{1}{j\omega T_2 + 1} = \underline{H}_{DT_1}(\omega) * \underline{H}_{PT_1}(\omega)$$

The start and end points of both basic systems are

$$\lim_{\omega\to 0} H_{DT_1}(\omega)|_{dB} \to -\infty$$
$$\lim_{\omega\to \infty} H_{DT_1}(\omega)|_{dB} = 20lg(K)$$

$$\lim_{\omega\to 0} H_{PT_1}(\omega)|_{dB} = 20lg(K)$$
$$\lim_{\omega\to \infty} H_{PT_1}(\omega)|_{dB} \to -\infty$$

The cutoff frequencies are

$$\omega_{g_{DT_1}} = \frac{1}{T_1}$$
$$\omega_{g_{PT_1}} = \frac{1}{T_2}$$

This leads to the bode diagrams of the basic systems in Figure 14.
By adding both bode diagrams together, the bode diagram of the given system is received (Fig. 15) by assuming $T_1 = 0.01s$ and $T_2 = 10^{-3}s$.

Approach b):
The poles and zeroes of the system are

$$p_1 = \frac{1}{T_1} = \omega_1$$
$$p_2 = \frac{1}{T_2} = \omega_2$$
$$z_1 = 0$$

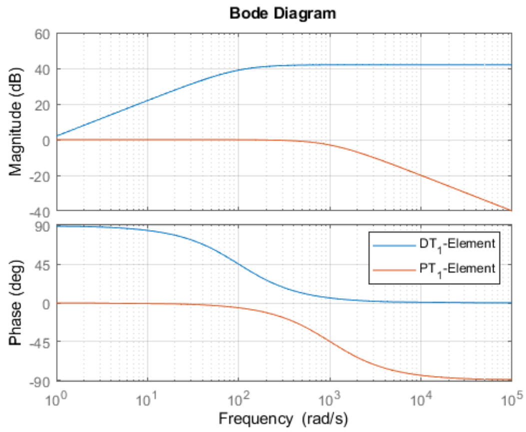


Figure 14: Bode diagrams basic systems

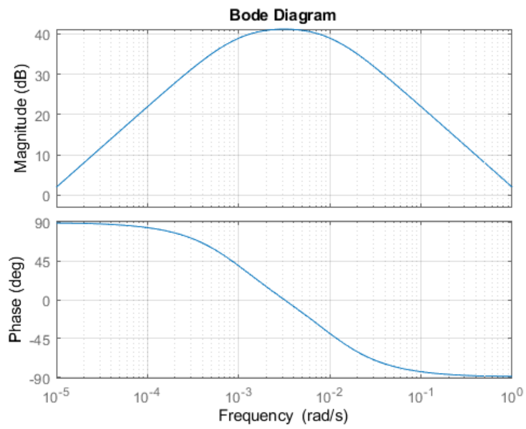


Figure 15: Bode diagram given system

So from $\omega = 0$, the bode diagram rises with 20 $\frac{dB}{dec}$. From $\omega = \omega_1$, the bode diagram changes its gradient by -20 $\frac{dB}{dec}$ and thus has a gradient of 0. From $\omega = \omega_2$, the bode diagram changes its gradient by -20 $\frac{dB}{dec}$ again so from this point the bode diagram falls by -20 $\frac{dB}{dec}$.

2.2 Non-Harmonic Periodic Signals

It can mathematically be shown that (almost) each non-harmonic periodic signal can be described as superposition of an infinitive number of cosine and sine waves, the so-called Fourier Series. The frequencies of the cosine and sine waves are integer multiple of the frequency ω_1 of the periodic signal. There exist 3 possible formats to describe the periodic signal.

1) Trigonometric Format

$$x(t) = a_0 + \sum_{n=1}^{\infty}[a_n cos(n\omega_1 t) + b_n sin(n\omega_1 t)] \tag{65}$$

with:

$$a_0 = \frac{1}{T}\int_{\tau}^{\tau+T} x(t)dt \tag{66}$$

$$a_n = \frac{2}{T}\int_{\tau}^{\tau+T} x(t)cos(n\omega_1 t)dt \tag{67}$$

$$b_n = \frac{2}{T}\int_{\tau}^{\tau+T} x(t)sin(n\omega_1 t)dt \tag{68}$$

So, the integration must always be done for one period.

Symmetrics
a) Even function $x(t)$
The Fourier Series only contains cosine components. For calculating the coefficients the integral limits can be halved and the integral itself duplicated.

$$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)dt = \frac{2}{T}\int_{0}^{\frac{T}{2}} x(t)dt \tag{69}$$

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)cos(n\omega_1 t)dt = \frac{4}{T}\int_{0}^{\frac{T}{2}} x(t)cos(n\omega_1 t)dt \tag{70}$$

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)sin(n\omega_1 t)dt = 0 \tag{71}$$

b) Odd function $x(t)$

The Fourier Series only contains sine components and no steady component. For calculating the coefficients the integral limits can be halved and the integral itself duplicated.

$$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)dt = 0 \tag{72}$$

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)cos(n\omega_1 t)dt = 0 \tag{73}$$

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)sin(n\omega_1 t)dt = \frac{4}{T}\int_{0}^{\frac{T}{2}} x(t)sin(n\omega_1 t)dt \tag{74}$$

2) <u>Harmonic Format</u>

$$x(t) = A_0 + \sum_{n=1}^{\infty} A_n sin(n\omega_1 t + \alpha_n) = A_0 + \sum_{n=1}^{\infty} A_n cos(n\omega_1 t - \beta_n) \tag{75}$$

The amplitudes A_0 and A_n and the phases α_n and β_n can be determined from the Fourier coefficients a_n and b_n and vice versa.

$$A_0 = a_0 \tag{76}$$

$$A_n = \sqrt{a_n^2 + b_n^2} \tag{77}$$

$$\alpha_n = arctan(\frac{a_n}{b_n}) = \frac{\pi}{2} - \beta_n \tag{78}$$

$$\beta_n = arctan(\frac{b_n}{a_n}) = \frac{\pi}{2} - \alpha_n \tag{79}$$

$$a_n = A_n sin(\alpha_n) = A_n cos(\beta_n) \tag{80}$$

$$b_n = A_n cos(\alpha_n) = A_n sin(\beta_n) \tag{81}$$

3) <u>Complex Format</u>

Since cosine and sine waves can be described by complex exponential signals (Eq. 12), the real Fourier Series can be formatted into the complex Fourier Series.

$$x(t) = \sum_{n=-\infty}^{\infty} \underline{c}_n e^{jn\omega_1 t} \tag{82}$$

with:

$$\underline{c}_n = \frac{1}{T}\int_{\tau}^{\tau+T} x(t)e^{-jn\omega_1 t}dt = |\underline{c}_n|e^{j\phi_n} \tag{83}$$

The parameter $\underline{c}_n$ is called the (discrete) spectrum of the signal $x(t)$. It is a (discrete) complex function with respect to the variable n.

Symmetrics
Generally, the complex spectrum $\underline{c}_n$ can be described as:

$$\begin{aligned} &\underline{c}_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_1 t}dt = \\ &\frac{1}{T}\int_0^T x(t)[cos(n\omega_1 t) - jsin(n\omega_1 t)]dt = \\ &\frac{1}{T}[\int_0^T x(t)cos(n\omega_1 t)dt - j\int_0^T x(t)sin(n\omega_1 t)dt] \end{aligned} \tag{84}$$

Thus, for the symmetrical functions applies:
a) Even function $x(t)$

$$\underline{c}_n = \frac{2}{T}\int_0^{\frac{T}{2}} x(t)cos(n\omega_1 t)dt = \Re\{\underline{c}_n\} \tag{85}$$

b) Odd function $x(t)$

$$\underline{c}_n = -j\frac{2}{T}\int_0^{\frac{T}{2}} x(t)sin(n\omega_1 t)dt = \Im\{\underline{c}_n\} \tag{86}$$

And therefore specific properties apply for the spectrum.

Properties

1. Real part is even: $\Re\{\underline{c}_n\} = \Re\{\underline{c}_{-n}\}$
2. Imaginary part is odd: $\Im\{\underline{c}_n\} = -\Im\{\underline{c}_{-n}\}$
3. Amplitude spectrum is even: $|\underline{c}_n| = |\underline{c}_{-n}|$
4. Phase spectrum is odd: $\angle\underline{c}_n = -\angle\underline{c}_{-n}$

Interpretation
In the previous chapter it was explained that harmonic signals are fully represented in the frequency domain by their corresponding complex amplitude which holds their real amplitude and phase. The spectrum $\underline{c}_n$ actually does the same but for all sine and cosine waves included in the non-harmonic periodic time-signal. For each value of n, the spectrum describes the sine\cosine wave with the frequency $n\omega_1$ included in the time-signal. The absolute value (phase) of $\underline{c}_n$ describes the real amplitude (phase) of this wave. So, the spectrum $\underline{c}_n$ can be seen as a function which includes all complex amplitudes of the sine\cosine waves which are included in the given time-signal. This means, the amplitude spectrum $|\underline{c}_n|$ indicates with which amplitude (i.e. how significantly) each sine\cosine wave appears in the related time-signal. The phase spectrum ϕ_n indicates which phase each sine\cosine wave presents.

Analysing Networks
Since the spectrum describes all complex amplitudes for all harmonic waves included in the given time-signal, determining the output signal of a system for non-harmonic periodic signals is exactly the same as for harmonic signals. The only difference is that the input signal is not described by one complex amplitude but by the Fourier Series of the signal and the relating spectrum $\underline{c}_n$. This means, in order to obtain the output signal of the system, the given time-signal is considered in the frequency domain by its spectrum $\underline{c}_n$. The output signal follows from multiplying the spectrum of the input signal with the frequency response of the system. Thus, for each harmonic wave included in the signal, it is determined how the system affects its amplitude and wave. The frequency of each harmonic signal remains the same since the system is linear. Additionally, due to the linearity, the output signal is just the superposition of all harmonic waves sent through the system (Fig. 16). In the following, it is shown how the output signal of a LTI-system can be described mathematically by using its Fourier Series.

According to Figure **??**, the input signal is described in the frequency domain by its complex Fourier Series:

$$x(t) = \sum_{-\infty}^{\infty} \underline{c}_{x_n} e^{jn\omega_1 t} \rightarrow \underline{c}_{x_n} = |\underline{c}_{x_n}| e^{j\phi_{x_n}} \tag{87}$$

Hence, for the output signal, which is also described by its complex Fourier Series, applies:

$$\begin{aligned} &\underline{c}_{y_n} = \underline{c}_{x_n}\underline{H}(\omega = n\omega_1) = \\ &|\underline{c}_{x_n}|e^{j\phi_{x_n}}\underline{H}(\omega = n\omega_1)e^{j\phi_H(\omega=n\omega_1)} = \\ &|\underline{c}_{x_n}|H(\omega = n\omega_1)e^{j\phi_{x_n}+\phi_H(\omega=n\omega_1)} = |\underline{c}_{y_n}|e^{j\phi_{y_n}} \end{aligned} \tag{88}$$

Therefore, it follows for the time-signal (i.e. trigonometric or harmonic format):

$$\begin{aligned} &y(t) = a_0H(\omega = 0) + \sum_{n=1}^{\infty}[a_nH(\omega = n\omega_1)cos(n\omega_1 t + \phi_H(\omega = n\omega_1)) + \\ &b_nH(\omega = n\omega_1)sin(n\omega_1 t + \phi_H(\omega = n\omega_1))] = \\ &A_0H(\omega = 0) + \sum_{n=1}^{\infty} A_nH(\omega = n\omega_1)sin(n\omega_1 t + \alpha_n + \phi_H(\omega = n\omega_1)) = \\ &A_0H(\omega = 0) + \sum_{n=1}^{\infty} A_nH(\omega = n\omega_1)cos(n\omega_1 t - \beta_n + \phi_H(\omega = n\omega_1)) \end{aligned} \tag{89}$$

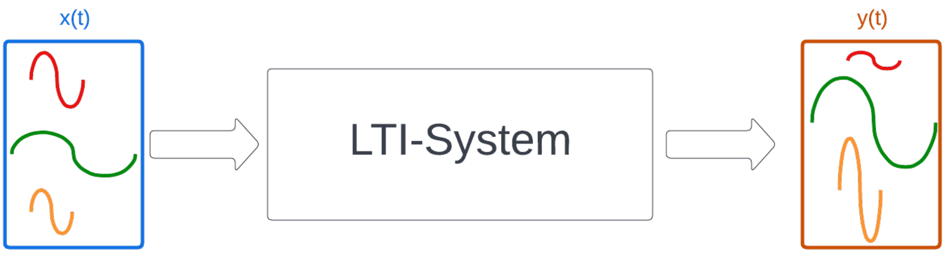


Figure 16: Fourier Series and LTI-system

2.3 Non-Harmonic Non-Periodic Signals

2.3.1 Fourier Transform

2.3.1.1 Describing Signals

So far, it has been shown how harmonic signals (i.e., sine and cosine waves) and non-harmonic but periodic signals can be described. Since some signals may also be non-periodic, it has to be explained how to describe these signals.

Basically, the idea is exactly the same as for harmonic or periodic signals. For harmonic signals, the complex amplitude is used as an representative of the signal in the frequency domain. For periodic signals, the spectrum $\underline{c}_n$ is used, which can be seen as a function of the frequency ω which represents the complex amplitudes of all harmonic signals (i.e., sine and cosine waves) included in the periodic signal. By applying the frequency response (respectively transfer function) of a system to the complex amplitude or the spectrum, the output signal of the system is obtained.
The same concept is applied for non-periodic signals. Therefore, the idea of the Fourier Series must be extended.

Equation 84 shows that the spectrum $\underline{c}_n$ of a periodic signal can be obtained by considering one period of the signal (see integral limits). This is possible since the signal is periodic in the time domain and therefore one period contains the whole information of the signal. To receive the whole information of a non-periodic signal, the whole signal in the time domain must be considered. This means in regards to the Fourier Series that the period duration must go to infinity. Thus, the frequency $\omega = \frac{2\pi}{T}$ is replaced by $d\omega$ and the summation is replaced by an integral. Since the new periodic duration is not T anymore but ∞, the integral limits are $-\infty$ and ∞. Additionally, by replacing $\underline{c}_n$ with $\frac{1}{T}\int_{\tau}^{\tau+T} x(t)e^{-jn\omega_1 t}dt$ (see Eq. 83), whereby $\frac{1}{T} = \frac{\omega_1}{2\pi}$, the complex Fourier Integral is obtained:

$$\begin{aligned} x(t) &= \sum_{n=-\infty}^{\infty} \underline{c}_n e^{jn\omega_1 t} = \sum_{n=-\infty}^{\infty} \frac{1}{T}\int_{\tau}^{\tau+T} x(t)e^{-jn\omega_1 t}dt e^{jn\omega_1 t} \\ \rightarrow x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underline{X}(\omega)e^{j\omega t}d\omega \end{aligned} \tag{90}$$

Consequently, a non-periodic time signal can be described as a summation of an infinitive number of sine and cosine waves but compared to the Fourier Series there is no fundamental frequency ω_1 anymore which means that the frequencies of the sine and cosine waves are not longer integer multiples of a specific frequency.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underline{X}(\omega)e^{j\omega t}d\omega \tag{91}$$

Same as for the Fourier Series, $\underline{X}(\omega)$ is called the spectrum of the signal $x(t)$ and has the exact same meaning as $\underline{c}_n$. It is a function of ω and represents the complex amplitudes of all harmonic signals which are contained in the non-periodic signal $x(t)$. However, while $\underline{c}_n$ is a discrete function of ω, respectively spectrum, $\underline{X}(\omega)$ is a continuous function of ω, respectively spectrum.

By applying the frequency response of a system to the Fourier-Transform of the input signal, each single harmonic wave, included in the signal, is affected in terms of its real amplitude and phase. The output signal follows from the summation of all modified harmonic waves since the considered systems are always linear.

$$\underline{X}(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \tag{92}$$

Existence of the Fourier Transform
The Fourier Transform of a signal basically exists if the signal $x(t)$ itself is absolutely integrable.

$$\int_{-\infty}^{\infty} |x(t)| dt < 0 \tag{93}$$

However, by extending the functions to distributions and therefore using the dirac delta function, it is also possible to determine the Fourier Transforms of not-absolutely integrable signals (e.g., sine and cosine).

Symmetric
The symmetric properties of the Fourier Transform is similar to them of the Fourier Series.
a) Even function $x(t)$

$$\underline{X}(\omega) = \int_{-\infty}^{\infty} x(t) cos(\omega t) dt = 2 \int_{0}^{\infty} x(t) cos(\omega t) dt = \Re\{\underline{X}(\omega)\} \tag{94}$$

b) Odd function $x(t)$

$$\underline{X}(\omega) = -j \int_{-\infty}^{\infty} x(t) sin(\omega t) dt = -2j \int_{0}^{\infty} x(t) sin(\omega t) dt = -j \Im\{\underline{X}(\omega)\} \tag{95}$$

Properties
The properties of the Fourier Transform are the same as for the Fourier Series.

1. Real part is even: $\Re\{\underline{X}(\omega)\} = \Re\{\underline{X}(-\omega)\}$
2. Imaginary part is odd: $\Im\{\underline{X}(\omega)\} = -\Im\{\underline{X}(-\omega)\}$
3. Amplitude spectrum is even: $|\underline{X}(\omega)| = |\underline{X}(-\omega)|$

4. Phase spectrum is odd: $\angle\underline{X}(\omega) = -\angle\underline{X}(-\omega)$

Time-Bandwith-Law
The product of the signal duration T_D and the bandwidth ω_B is always constant.

$$T_D * \omega_B = const \tag{96}$$

This means that a limited signal in the time domain is always infinite in the frequency domain and vice versa.

2.3.1.2 Describing LTI-Systems

The convolution theorem shows that the frequency response of a system is the Fourier Transform of its impulse response.

$$\underline{H}(\omega) = \mathcal{F}\{h(t)\} \tag{97}$$

Of course, the frequency response can also be determined by applying the complex AC-calculation method.

2.3.1.3 Analysing Networks

The concept is exactly the same as for the Complex AC-Calculation or the Fourier Series.
To avoid differential equations in the time domain, the non-periodic signals are transformed into the frequency domain by using the Fourier Transform. The Fourier Transform describes the complex amplitudes of all harmonic signals which are contained in the non-periodic signal $x(t)$, analogous to the discrete spectrum $\underline{c}_n$ of a periodic signal. Therefore, by applying (multiplying) the frequency response (respectively transfer function) of a system to the Fourier Transform of the input signal, the Fourier Transform of the output signal is received. By re-transforming the Fourier-Transform of the output signal into the time domain, the output signal in the time domain can be determined.

According to Figure 6, the input signal is described in the frequency domain by its Fourier Transform:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underline{X}(\omega)e^{j\omega t}d\omega \rightarrow \underline{X}(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{98}$$

Hence, for the output signal, which is also described by its Fourier Transform, applies:

$$\underline{Y}(\omega) = \underline{H}(\omega) * \underline{X}(\omega) \tag{99}$$

Therefore, it follows for the time-signal:

$$y(t) = \mathcal{F}^{-1}\{\underline{Y}(\omega)\} \tag{100}$$

Summary Example

The system of example 1 (see Describing Harmonic Signals) is given:

$$\underline{H}(\omega) = \frac{-\omega^2 LC + j\omega RC}{-\omega^2 LC + j\omega RC + 1}$$

The following non-periodic signal is fed into the system (Fig. 17):

$$x(t) = rect(\frac{t-2}{4})$$

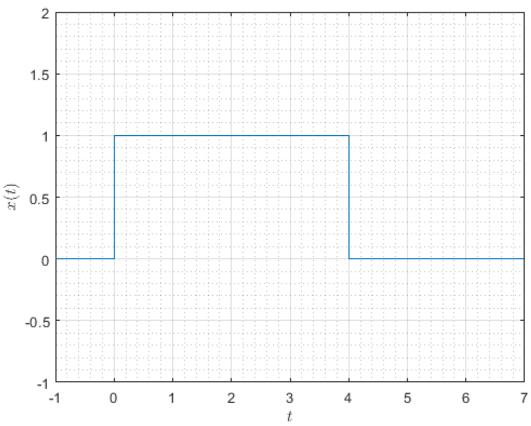


Figure 17: Input Signal

The associated Fourier Transform is (Fig. 18):

$$\underline{X}(\omega) = 4 * si(2\omega) * e^{-2j\omega}$$

The spectrum of the output signal follows as (Fig. 19):

$$\underline{Y}(\omega) = \underline{H}(\omega) * \underline{X}(\omega) = \frac{-\omega^2 LC + j\omega RC}{-\omega^2 LC + j\omega RC + 1} * 4si(2\omega)$$

The re-transformation of the spectrum in the time domain is pretty complicated and therefore neglected.

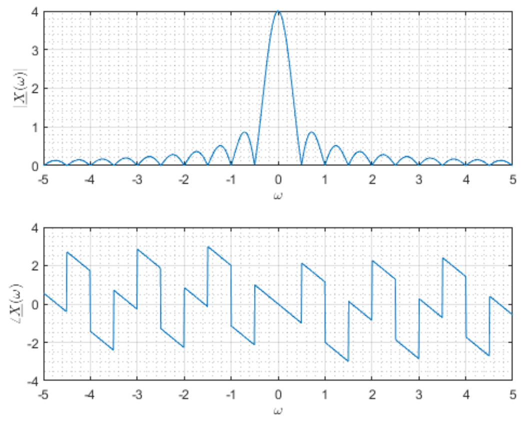


Figure 18: FT Input Signal

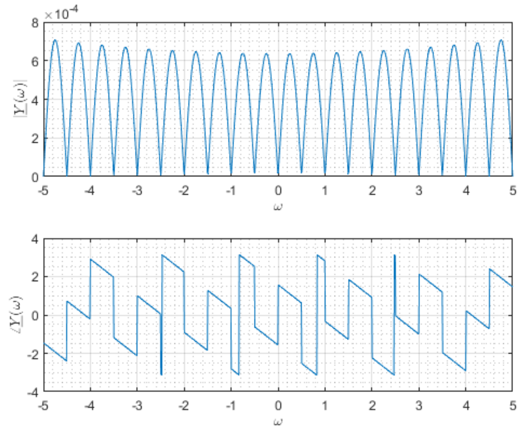


Figure 19: FT Output Signal

2.3.2 Laplace Transform

Determinations

There are two types of the Laplace transform, the one-sided and the two-sided Laplace transform. In this tutorium, only the one-sided approach is considered. This means that all signals in the time domain are only known for $t \geq 0$. Thus, all signals in the time domain must always be multiplied by the step function $\epsilon(t)$.

2.3.2.1 Describing Signals

The Laplace-Transform is an extension or generalization of the Fourier-Transform. Some signals, which a Fourier-Transform does not exist for, often have got a corresponding Laplace-Transform. Whereas the Fourier-Transform is used to investigate the spectrum of a signal, the Laplace-Transform is better suitable to calculate differential equations and to analyse networks, especially transient processes.
Also, whereas the Fourier-Transform can be considered as an extension of the complex AC-calculation, the Laplace-Transform can be considered as an extension of the extended AC-calculation.

Formulas
Usually, the Laplace-Transform of a signal is determined using specific tables. However, for the sake of completeness, the formulas are depicted below.

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\omega}^{\sigma + j\omega} \underline{X}(s) e^{st} ds \text{ for } t \geq 0 \tag{101}$$

$$\underline{X}(s) = \int_{0}^{\infty} x(t) e^{-st} dt \tag{102}$$

with: $s = \sigma + j\omega$

Interpretation
The Laplace-Transform can be interpreted as the Fourier-Transform with the addition that the not-periodic signal does not only contain harmonic waves with constant amplitudes but their amplitudes can vary with time. This follows from the fact that the Laplace-Transform uses the complex frequency s, where the parameter σ represents the time-dependence of the amplitude, instead of the circular frequency ω. This relationship is explained in chapter 2.1.2.2.
Therefore, each Laplace-Transform contains a real amplitude and phase which mean the same as the real amplitude and phase of a Fourier-Transform.

2.3.2.2 Describing LTI-Systems

The convolution theorem shows that the transfer function of a system is the Laplace transform of its impulse response.

$$\underline{H}(s) = \mathcal{L}\{h(t)\} \tag{103}$$

Of course, the transfer function can also be determined by applying the extended complex AC-calculation method.

Pole-Zero Plot

Each Laplace transform is clearly represented by its poles and zeros apart from a constant factor. Therefore, a Laplace transform is often depicted as a pole-zero plot. The pole-zero plot is normally used to describe LTI-systems.

$$\underline{X}(s) = \frac{b_M}{a_N} \frac{(s - z_1)(s - z_2)...(s - z_M)}{(s - p_1)(s - p_2)...(s - p_N)} \tag{104}$$

For real systems, the poles and zeros are always real or complex-conjugated.

Stability and Causality

a) Causality: The corresponding LTI-system is causal if a transfer function exists. In this case the condition $h(t) = 0$ for $t > 0$ is automatically fulfilled.

b) Stability: Since in the time domain the condition $\int_{-\infty}^{\infty} |h(t)|dt < \infty$ must always be fulfilled, all poles of the transfer function of a stable system must be left-handed to the imaginary axis which means they have got a negative real part.

$$\Re\{p_n\} < 0 \ \ \forall n \in \mathbb{N} \tag{105}$$

2.3.2.3 Convergence Area

The convergence area shows for which values for σ the Laplace-Transform exists.
Basically, similar to the Fourier-Transform, the Laplace-Transform exists if $x(t)e^{-\sigma t}$ is absolutely integrable.

$$\int_0^{\infty} |x(t)e^{-\sigma t}| < \infty \tag{106}$$

Since this expression can only reach ∞ if $x(t)$ is an exponential function, a closer look has to be taken on this scenario.
It applies: $x(t) = e^{at}$

For the time domain follows:

$$x(t)e^{-\sigma t} = e^{at}e^{-\sigma t} = e^{t(a-\sigma)} \tag{107}$$

The expression is absolutely integrable if $a < \sigma$. This shows that in the time domain σ must always be higher than a.

For the image domain follows:

$$x(t) = e^{at} \xrightarrow{\mathcal{L}} \underline{X}(s) = \frac{1}{s-a} \tag{108}$$

This shows that the highest pole of the Laplace-Transform sets the beginning of the convergence area.

Relationship between the FT and the LT
Since $s = \sigma + j\omega$, the Laplace-Transform is equal to the Fourier-Transform if $\sigma = 0$ respectively $s = j\omega$.

$$\mathcal{F}\{x(t)\} = \underline{X}(\omega) = \underline{X}(s)|_{s=j\omega} \tag{109}$$

So, the Fourier-Transform can be determined from the Laplace-Transform by just replacing s by ω but only if the Fourier-Transform exists for $\sigma = 0$ which is the case if the imaginary-axis is part of the convergence area or rather if all poles are left-handed to the imaginary-axis. This means, all poles must provide a negative real part.

2.3.2.4 Methods of the inverse Laplace Transform

Since determining the Laplace transform and re-Transform of a signal by using the formula is mostly pretty laborious, transform tables are usually used. However, in some cases, the corresponding signal of a Laplace transform is not contained in the table. Consequently, the given Laplace transform must be modified to obtain an expression which can be re-transformed by using the transform table. Since the considered Laplace transforms are normally rational functions, the partial fraction decomposition is a commonly used technique.
There are 3 different cases:

a) Laplace transform with single real poles

$$\underline{X}(s) = \frac{A_1}{s-p_1} + \frac{A_2}{s-p_2} + ... + \frac{A_n}{s-p_n} \tag{110}$$

with the correspondence:

$$\frac{A_n}{s-p_n} \xrightarrow{\mathcal{L}^{-1}} A_n e^{p_n t}$$

b) Laplace transform with multiple real poles
If p_0 is a pole with multiplicity k, the following decomposition applies:

$$\underline{X}(s) = \frac{\underline{Z}(s)}{(s-p_0)^k} = \frac{A_1}{s-p_0} + \frac{A_2}{(s-p_0)^2} + ... + \frac{A_n}{(s-p_0)^n} \tag{111}$$

with the correspondence:

$$\frac{A_n}{(s-p_0)} \xrightarrow{\mathcal{L}^{-1}} A_n e^{p_0 t} \frac{t^{n-1}}{(n-1)!}$$

c) Laplace transform with single complex poles

$$\underline{X}(s) = \frac{\underline{Z}(s)}{\underline{N}(s)} = \frac{\underline{Z}(s)}{(s-p_1)(s-p_1^*)} = \frac{C_1 s + C_2}{s^2 - 2as + a^2 + b^2} \tag{112}$$

with the correspondence:

$$\frac{C_1 s + C_2}{s^2 - 2as + a^2 + b^2} \xrightarrow{\mathcal{L}} e^{at}[C_1 cos(bt) + \frac{C_2 + aC_1}{b} sin(bt)]$$

Note: Is it often also helpful to consider the transformation theorems of the Laplace transform.

2.3.2.5 Analysing Networks

The concept is exactly the same as for the Fourier transform.
To avoid differential equations in the time domain, the non-periodic signals are transformed into the image domain by using the Laplace transform. The Laplace transform describes the complex amplitudes of all harmonic signals with a time-dependent real amplitude which are contained in the non-periodic signal $x(t)$. Therefore, by applying (multiplying) the transfer function of a system to the Laplace transform of the input signal, the Laplace transform of the output signal is received. By re-transforming the Laplace transform of the output signal into the time domain, the output signal in the time domain can be determined.

The input signal is described in the image domain by its Laplace transform:

$$x(t) \rightarrow \underline{X}(s) \tag{113}$$

Hence, for the output signal, which is also described by its Fourier Transform, applies:

$$\underline{Y}(s) = \underline{H}(s) * \underline{X}(s) \tag{114}$$

Therefore, it follows for the time-signal:

$$y(t) = \mathcal{L}^{-1}\{\underline{Y}(s)\} \tag{115}$$

Summary Example

An electrical circuit is given which delivers the following differential equation in the time domain:

$$RC\frac{du_a(t)}{dt} + u_a(t) = u_e(t)$$

The output signal $u_a(t)$ wants to be determined. The signal $u_e(t)$ is the input signal of the system. Since solving a differential equation can sometimes be very laborious, the differential equation is avoided by transferring the equation into the image domain by using the Laplace transforms of the signals.

$$s\underline{U_a}(s) - U_a(0) + \frac{1}{RC}\underline{U}_a(s) = \frac{1}{RC}\underline{U}_e(s)$$

Now the equation can be modified:

$$\underline{U}_a(s) = (\underline{U}_e(s) + RC * U_a(0)) * \frac{1}{sRC+1}$$

Here, the factor $\frac{1}{sRC+1}$ represents the transfer function of the system. This transfer function can also be determined by just applying the extended complex AC-calculation method.
In order to get the output signal, the Laplace transform of the input signal must be determined. Assuming the input signal is as follows (Fig. 20):

$$u_e(t) = t * \epsilon(t)$$

For the corresponding Laplace transform follows:

$$\underline{U}_e(s) = \frac{1}{s^2}$$

Therefore, in case $U_a(0) = 0$, the Laplace transform of the output signal is:

$$\underline{U}_a(s) = \frac{1}{s^2} * \frac{1}{sRC+1} = \frac{1}{s^2(sRC+1)} = \frac{\frac{1}{RC}}{s^2(s+\frac{1}{RC})}$$

The inverse Laplace transform delivers the output signal in the time domain (Fig. **??** with $R = C = 1$):

$$u_a(t) = RC[\frac{1}{RC}t - 1 + e^{-\frac{t}{RC}}]$$

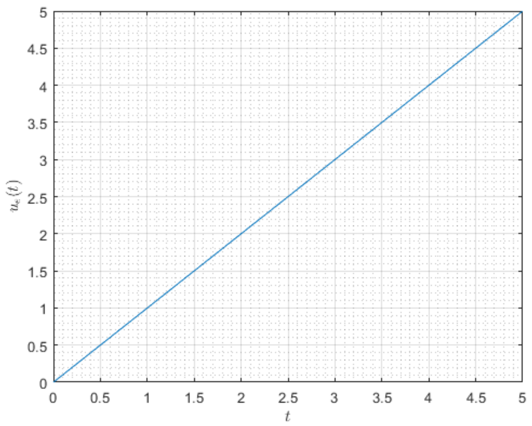


Figure 20: Input Signal

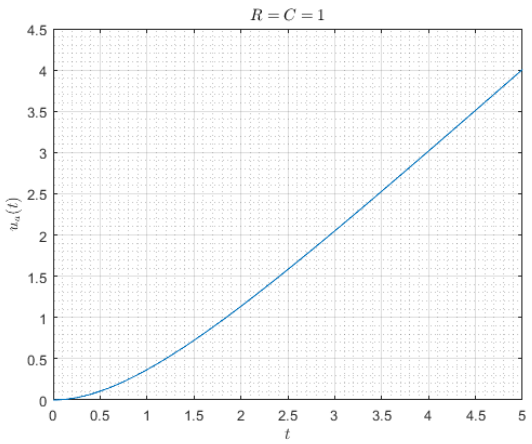


Figure 21: Output Signal

3 Switching Processes

By using the Laplace transformation, switching processes can be described more easily because in this case solving differential equations in the time domain can be avoided. When using this approach, the initial conditions of each inductance and capacity must be considered by creating an equivalent circuit. The output signal can then be determined by using this equivalent circuit.

3.1 Equivalent Circuits

a) Capacities
It applies in the time domain:

$$u_c(t=0-) = u_c(t=0+) \neq 0 \tag{116}$$

$$i_c(t) = C\frac{du_c(t)}{dt} \tag{117}$$

This means the initial condition for the voltage of the capacity must also be considered in the image domain. From the Laplace transformation follows:

$$\underline{I}_c(s) = C(s\underline{U}_c(s) - u_c(0)) = sC\underline{U}_c(s) - Cu_c(0) = \underline{I}_{c_1}(s) - \underline{I}_{c_2}(s) \tag{118}$$

Transforming this equation delivers also:

$$\underline{U}_c(s) = \frac{\underline{I}_c(s)}{sC} + \frac{u_c(0)}{s} = \underline{U}_{c_1}(s) + \underline{U}_{c_2}(s) \tag{119}$$

Thus, the capacity can be described in two different ways. In equation 118, $\underline{I}_{c_1}(s)$ describes Ohm's law in the image domain as usual. $\underline{I}_{c_2}(s)$ considers the initial condition as an additional parallel current source. In equation 119, $\underline{U}_{c_1}(s)$ describes Ohm's law in the image domain as usual whereas $\underline{U}_{c_2}(s)$ considers the initial condition as an additional serial voltage source.
Graphically, it looks like in Fig. 22.

b) Inductances
It applies in the time domain:

$$i_L(t=0-) = i_L(t=0+) \neq 0 \tag{120}$$

$$u_L(t) = L\frac{di_L(t)}{dt} \tag{121}$$

This means the initial condition for the current of the inductance must also be considered in the image domain. From the Laplace transformation follows:

$$\underline{U}_L(s) = L(s\underline{I}_L(s) - i_L(0)) = sL\underline{I}_L(s) - Li_L(0) = \underline{U}_{L_1}(s) - \underline{U}_{L_2}(s) \tag{122}$$

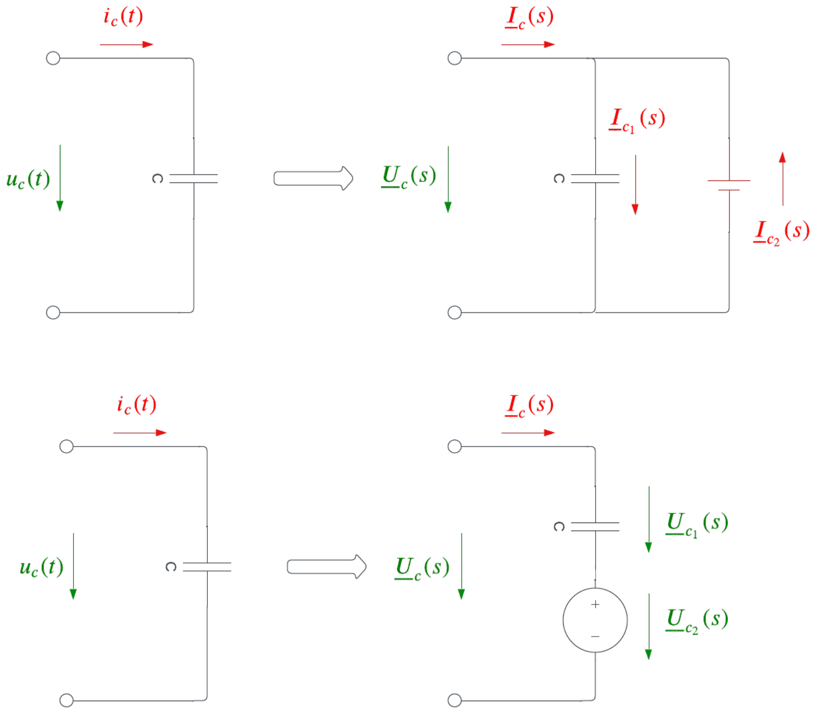


Figure 22: Equivalent Circuits Capacity

Transforming this equation delivers also:

$$\underline{I}_L(s) = \frac{\underline{U}_L(s)}{sL} + \frac{i_L(0)}{s} = \underline{I}_{L_1}(s) + \underline{I}_{L_2}(s) \tag{123}$$

Thus, the inductance can be described in two different ways. In equation 122, $\underline{U}_{L_1}(s)$ describes Ohm's law in the image domain as usual. $\underline{U}_{L_2}(s)$ considers the initial condition as an additional serial voltage source. In equation 123, $\underline{I}_{L_1}(s)$ describes Ohm's law in the image domain as usual whereas $\underline{I}_{L_2}(s)$ considers the initial condition as an additional parallel current source.
Graphically, it looks like in Fig. 23.

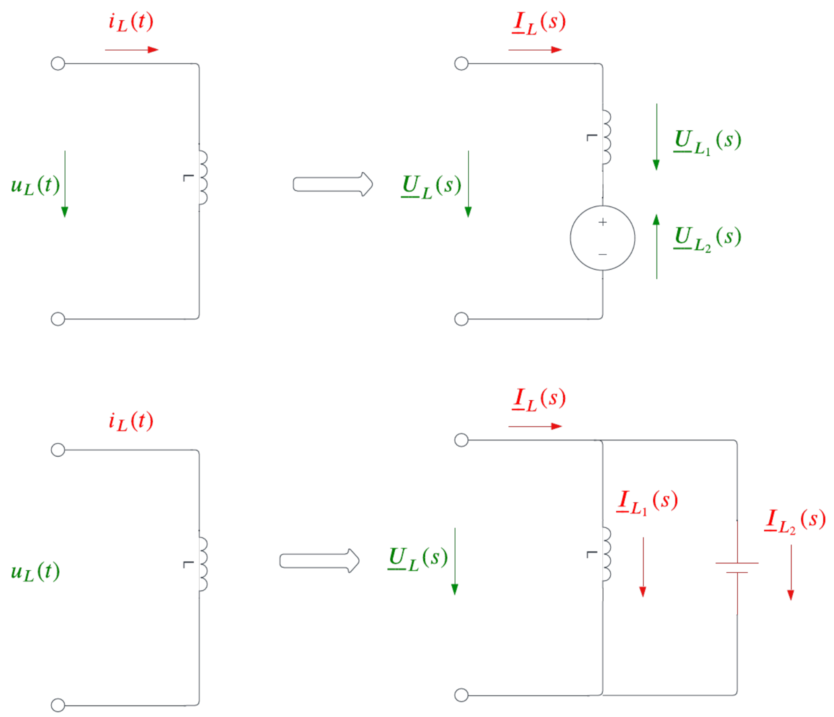


Figure 23: Equivalent Circuits Inductance

3.2 Interpretation

Since determining the output signal by using the Laplace transform in the image domain is equivalent to solving the differential equation of the system in the time domain, the output signal, which is the solution of the differential equation, always consists of a homogeneous solution and a particulate solution. The particulate solution represents the settled state of the output signal. This term never includes a decaying e-function and therefore does not disappear for $t \to \infty$. The homogeneous solution describes the transient process of the output signal and always includes a decaying e-function. Consequently, this term always disappears for $t \to \infty$.

$$y(t \to \infty) = y_{hom}(t \to \infty) + y_{part}(t \to \infty) = 0 + y_{part}(t \to \infty) \tag{124}$$

Example 4:

The electrical circuit in Fig. 24 is given.

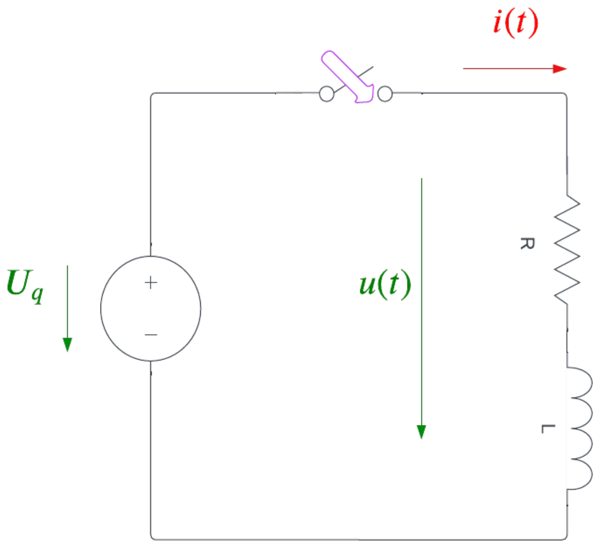


Figure 24: Electrical Circuit Example 4

The current $i(t)$ wants to be determined when the switch is flipped. The voltage U_q is the input signal. Since the electrical circuit is interrupted for $t = 0-$, it applies $i(t = 0-) = 0$ and thus no initial conditions must be considered.
First the output signal of the system in the image domain is determined:

$$\underline{U}_q(s) = \underline{I}(s)(R + sL)$$
$$\underline{I}(s) = \frac{\underline{U}(s)}{sL + R}$$

Since $u(t)$ is just a step function $\epsilon(t)$ the Laplace transform of the input voltage is

$$u(t) = U_q\epsilon(t) \xrightarrow{\mathcal{L}} \underline{U}(s) = \frac{U_q}{s}$$

So, for the current follows:

$$\underline{I}(s) = \frac{U_q}{s(sL + R)} = \frac{1}{L}\frac{U_q}{s(s + \frac{R}{L})}$$

Transferring this expression into the time domain delivers:

$$i(t) = \frac{U_q}{L} * \frac{L}{R}(1 - e^{-\frac{R}{L}t}) = \frac{U_q}{R}(1 - e^{-\frac{R}{L}t}) = \frac{U_q}{R} - \frac{U_q}{R}e^{-\frac{R}{L}t} = i_{part}(t) + i_{hom}(t)$$

For $t \to \infty$ it applies:

$$i(t \to \infty) = i_{part}(t \to \infty) + i_{hom}(t \to \infty) = i_{part}(t \to \infty) + 0 = \frac{U_q}{R}$$

Signal $i(t)$ and its single components are depicted in Fig. 25. It can be observed that the homogeneous solution is responsible for the transient process whereas the particulate solution is responsible for the end state of the output signal.

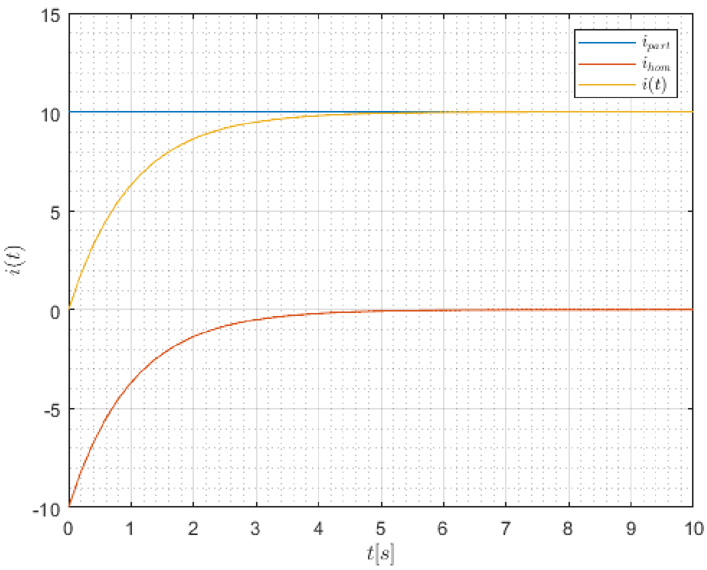


Figure 25: Plotting signal i(t) and its components

4 Time-Discrete Signals and Systems

4.1 Time-Discrete Signals in Time Domain

A time-discrete signal $x(n)$ results from sampling a continuous signal in the time domain with a sampling time T respectively a sampling frequency $f_s = \frac{1}{T}$. This means, the continuous signal $x_c(t)$ is only considered at the particular moments $t = nT$ where n is an integer.

$$x(n) = x_c(nT) = x_c(t)|_{t=nT} \tag{125}$$

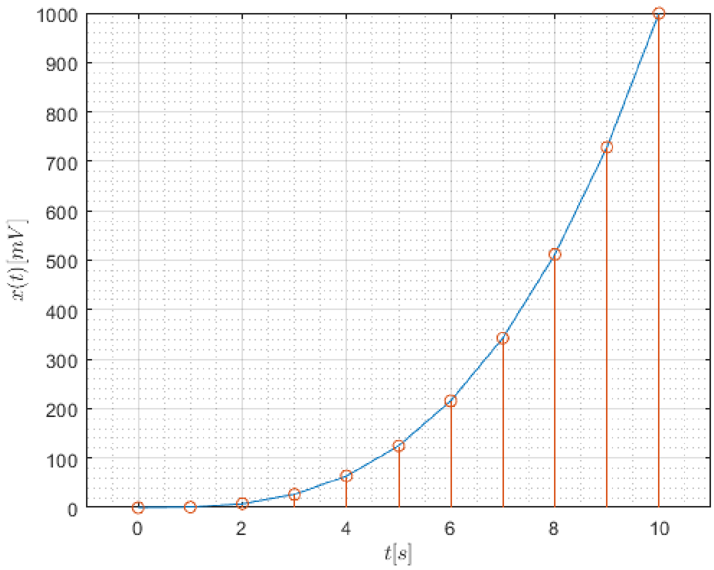


Figure 26: Sampling continuous signal

Figure 26 shows an example of sampling a continuous signal. In this case, the sampling time is $T = 1s$.

4.1.1 Basic Time-Discrete Signals

The basic time-discrete signals just follow from the continuous ones by sampling them.

c) Time-Discrete Unit Impulse Function

$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases} = \epsilon(n) - \epsilon(n-1) \tag{126}$$

Figure 27 depicts the function.

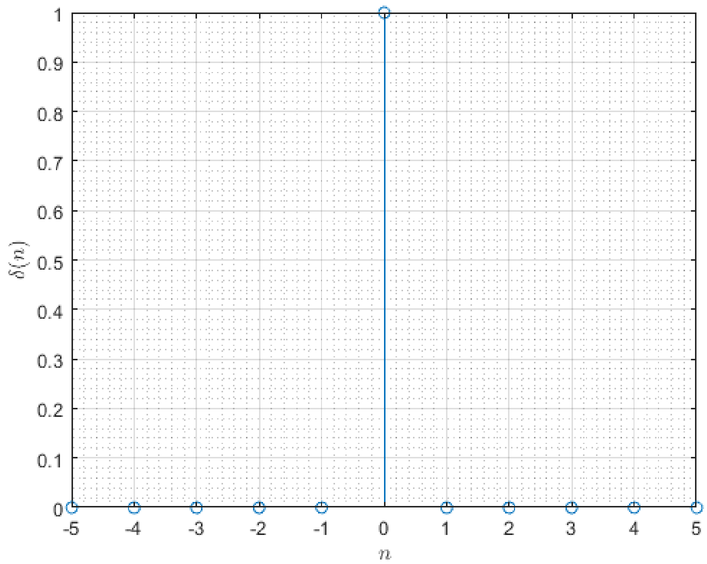


Figure 27: Time discrete unit impulse function

Properties

The properties of the time-discrete delta function are more or less the same as for the continuous one.

a) Main definition:

$$\sum_{n=-\infty}^{\infty} \delta(n) = 1 \tag{127}$$

b) The time-discrete delta function is an even function:

$$\delta(-n) = \delta(n) \rightarrow \delta(n-k) = \delta(k-n) \tag{128}$$

c) "Hide Property":

$$x(n)\delta(n-k) = x(k)\delta(n-k) \tag{129}$$

Thus it follows:

$$x(k) = \sum_{n=-\infty}^{\infty} x(n)\delta(n-k) \tag{130}$$

d) Each time-discrete signal can be expressed as a sequence of weighted time-discrete delta functions:

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k) \tag{131}$$

b) Time-Discrete Unit Step

$$\epsilon(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases} = \sum_{k=-\infty}^{n} \delta(k) \tag{132}$$

Figure 28 depicts the function.

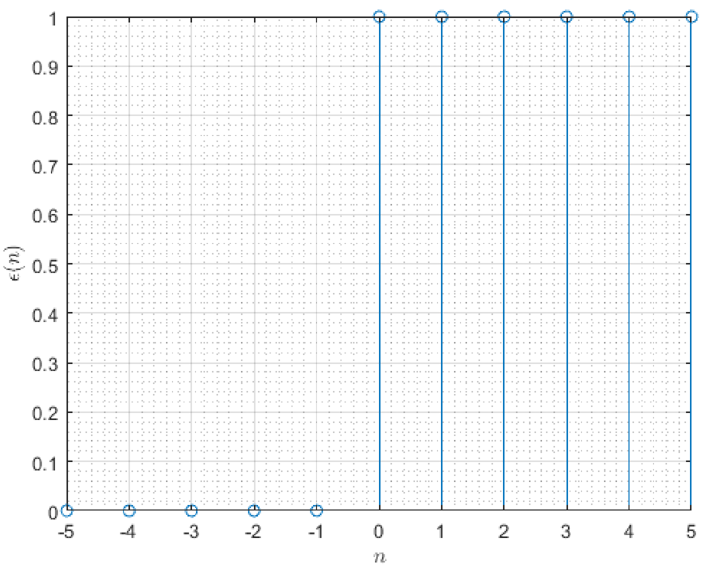


Figure 28: Time discrete unit step function

c) Time-Discrete Rectangular Sequence

$$rect_N(n) = \begin{cases} 1 & \text{for } 0 \leq n \leq N-1 \\ 0 & \text{for else} \end{cases} = \epsilon(n) - \epsilon(n-N) = \sum_{k=0}^{N-1} \delta(n-k) \tag{133}$$

Figure 29 depicts an example of the sequence.

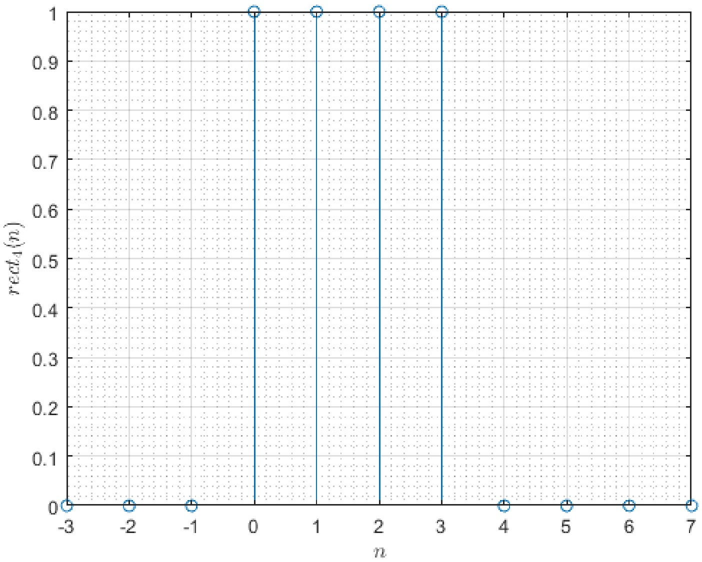


Figure 29: Time discrete rectangular sequence for N = 4

d) Time-Discrete Complex Exponential Signal

- For constant real amplitudes:

$$x(n) = \underline{A}e^{j\Omega n} \tag{134}$$

- For time-dependent real amplitudes:

$$\begin{aligned} x(n) = \underline{A}e^{st}|_{t=nT} = \underline{A}e^{snT} = \underline{A}e^{(\sigma+j\omega)nT} = \\ \underline{A}e^{(\sigma T+j\omega T)n} = \underline{A}e^{(\Sigma+j\Omega)n} = \underline{A}e^{Sn} \end{aligned} \tag{135}$$

This means:

$$S = sT = (\sigma + j\omega)T = \sigma T + j\omega T = \Sigma + j\Omega \tag{136}$$
$$\Sigma = \sigma T \tag{137}$$
$$\Omega = \omega T = 2\pi f T = 2\pi \frac{f}{f_s} \tag{138}$$

The parameter Σ has the same meaning for time-discrete signals as the parameter σ for time-continuous signals. It determines the decay or growth of the real time-dependent amplitude. The parameter Ω is called standardize frequency and has got the following value range:

$$\Omega \in [0, 2\pi) \; or \; [-\pi.\pi)$$

e) Time-Discrete Sinusoidal Signal

$$x(n) = A * sin(\Omega n + \phi) \tag{139}$$

with:

$$\Omega = \omega T = 2\pi f T = 2\pi \frac{f}{f_s} \tag{140}$$

4.2 Time-Discrete LTI-Systems in Time Domain

There are two ways of describing time-discrete LTI-systems in the time domain, difference equations and the convolution.

4.2.1 Difference Equations

The difference equations for time-discrete systems are correspond to the differential equation of time-continuous systems.

$$\sum_{k=1}^{N} a_k * y(n-k) = \sum_{l=0}^{M} b_l * x(n-l) \tag{141}$$

or:

$$y(n) = \frac{1}{a_0}(\sum_{l=0}^{M} b_l * x(n-l) - \sum_{k=1}^{N} a_k * y(n-k)) \tag{142}$$

Each transfer function of a system can directly be derived from its difference equation and vice versa.
A graphical depiction of the difference equation of a system is its corresponding signal-flow-graph. Figure 30 depicts the universal form.

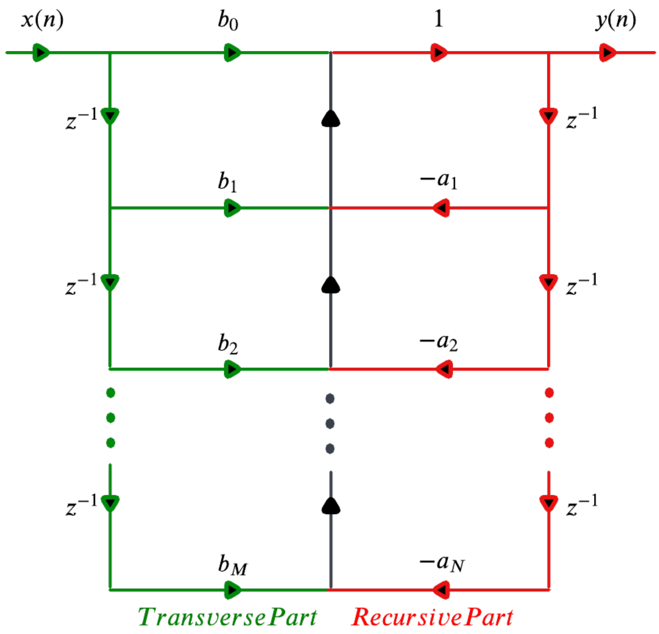


Figure 30: Signal-flow-graph

The left part of the graph is called transverse part, the right one is called recursive part.

4.2.2 Convolution

Impulse Response

$$h(n) = S\{\delta(n)\} \tag{143}$$

Step Response

$$g(n) = S\{\epsilon(n)\} \tag{144}$$

Relationship between Impulse and Step Response

$$h(n) = g(n) - g(n-1) \tag{145}$$

$$g(n) = \sum_{k=-\infty}^{n} h(k) \tag{146}$$

Describing the System

The concept is the same as for time-continuous LTI-systems.
As already mentioned, each signal $x(n)$ can be described as a sum of weighted time-discrete delta functions.

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k) \tag{147}$$

Due to the superposition principle it applies for the output signal:

$$\begin{aligned} &y(n) = S\{x(n)\} = S\{\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\} = \\ &\sum_{k=-\infty}^{\infty} x(k)S\{\delta(n-k)\} = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = x(n) \circledast h(n) \end{aligned} \tag{148}$$

Length of the Output Signal

The output signal always has the following length:

$$L_y = L_h + L_x - 1 \tag{149}$$

Causality and Stability

a) Causality
A time-discrete LTI-system is causal if its impulse response is right-sided.

$$h(n) = 0 \ for \ t < 0 \tag{150}$$

b) Stability
A time-discrete LTI-system is stable if its impulse response is absolutely summable.

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty \tag{151}$$

4.3 Time-Discrete Signals in Frequency Domain

4.3.1 Discrete-Time Fourier Transform (DTFT)

The DTFT is the same as the FT but for time-discrete signals. The concept is basically the same but instead of the continuous frequency ω the standardized frequency Ω is used.

Transformation and Re-Transformation

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \underline{X}(\Omega) e^{j\Omega n} d\Omega \tag{152}$$

This formula can exactly be interpreted as for the time-continuous signals. The signal $x(n)$ can be considered as a superposition of an infinity number of time-discrete sine waves with frequency Ω. In contrast to the re-transformation for continuous signals, the integration is not from $-\infty$ to ∞ but from $-\pi$ to π. This is possible due to 2π-periodicity of the spectrum of time-discrete signals, one period already contains the information of the complete spectrum.

$$\underline{X}(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n} \tag{153}$$

Same as for the spectrum $\underline{X}(\omega)$ of time-continuous signals, the spectrum $\underline{X}(\Omega)$ of time-discrete signals represents the complex amplitudes of all time-discrete sine waves which are included in the time signal $x(n)$. Since the summation is not over the continuous parameter t but over the discrete parameter n, the integral becomes a summation.

Existence

The DTFT of a signal exists if the following is fulfilled:

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty \tag{154}$$

Properties

a) *Dualities*

- periodic $\leftrightarrow$ discrete
  In the chapter about the Fourier Series it was seen that if the time signal is

periodic, the spectrum is discrete. Same applies vice versa. If the time signal is discrete, the spectrum of this signal is always periodic.

- aperiodic $\leftrightarrow$ continuous
  In the chapter about the Fourier transform it has also been seen that aperiodic time signals always own a continuous spectrum. Vice Versa, an aperiodic spectrum indicates that the corresponding time signal is continuous.

b) *Spectrum*

- Amplitude Spectrum
  - For real signals, it is always even, i.e., mirror-symmetric to the y-axis
  - Due to the 2π-periodicity, it is additionally mirror-symmetric to all $k\pi$ with $k \in \mathbb{Z}$
  - Because of the previous two points, for time-discrete signals and systems it is sufficient to indicate the amplitude spectrum for the interval $[0, \pi]$. This interval contains the whole information of the amplitude.
- Phase Spectrum
  - For real signals, it is always odd, i.e., point-symmetric to the origin
  - Due to the 2π-periodicity, it is additionally point-symmetric to all $k\pi$ with $k \in \mathbb{Z}$
  - Because of the previous two points, for real time-discrete signals and systems it is sufficient to indicate the phase spectrum for the interval $[0, \pi]$. This interval contains the whole information of the phase spectrum.

Figure 31 shows a sample spectrum of a time-discrete signal.

4.3.2 Z-Transformation

The z-transformation is basically the Laplace transformation for time-discrete signals. In other words, the z-transformation is related to the Laplace transformation as the DTFT is related to the Fourier transform and therefore the z-transformation is basically an extension of the DTFT. In this tutorium, only the one-sided z-transformation is considered. This means, she is only valid for one-sided time signals.

It has already been discussed that the Laplace transform follows from the Fourier transform by introducing the parameter σ which describes the time-dependence of the real

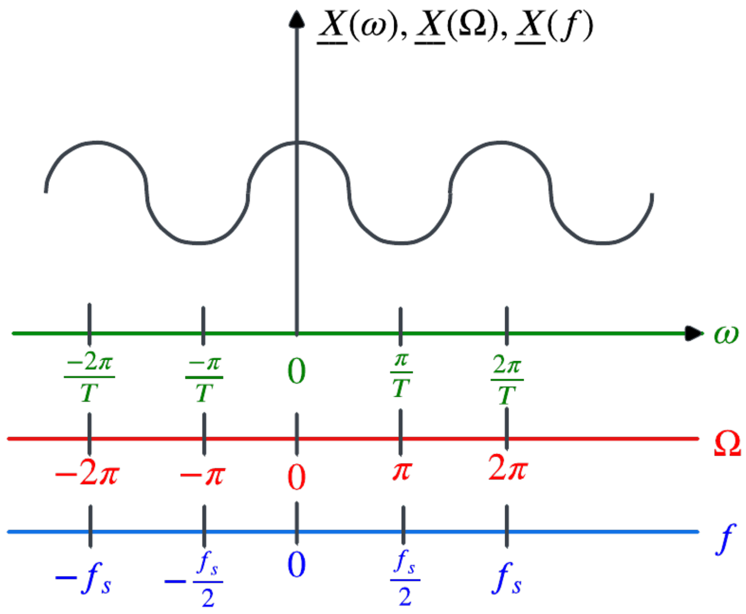


Figure 31: Sample spectrum time-discrete signal

amplitude.

$$e^{j\omega t} \to e^{(\sigma+j\omega)t} = e^{st}$$

The same concept applies for the z-transformation. The z-transformation follows from the DTFT by introducing the parameter Σ which also describes the time-dependence of the real amplitude. Additionally, the expression e^S is substituted by the parameter z which has mathematical reasons. It would also be possible not to apply this substitution.

$$e^{j\Omega n} \to e^{(\Sigma+j\Omega)n} = e^{Sn}$$

Interpretation

- Laplace transformation
  As already discussed, the function $\underline{X}(s)$ represents the complex amplitudes of all sine waves which are included in the considered time signal $x(t)$. In this case, the complex frequency $s = \sigma + j\omega$ describes the time-dependency by the parameter σ as well as the frequency by the parameter ω. This means, each value of s stands

for a sine wave with a specific decay/growth of the real amplitude and a specific frequency ω. The real part $\Re\{s\} = \sigma$ of s represents the decay/growth whereas the imaginary part $\Im\{s\} = \omega$ represents the frequency.

- Z-transformation
  The z-transformation works according to the same principle. Basically, the function $\underline{X}(z)$ represents the complex amplitudes of all time-discrete sine waves included in the considered time-discrete signal $x(n)$. However, in this case the decay/growth of the real amplitude is described by the parameter Σ and the frequency by the parameter Ω. This means, since $z = e^{\Sigma + j\Omega} = e^{\Sigma} * e^{j\Omega} = r * e^{j\Omega}$, not the real and imaginary parts of z but its amplitude $r = e^{\Sigma} = |z|$ and phase $\Omega = arg(z)$ describe the decay/growth respectively the frequency of a specific time-discrete sine wave.

Transformation and Re-transformation

$$x(n) = \frac{1}{2\pi j} \oint \underline{X}(z) z^{n-1} dz \tag{155}$$

The derivation of this formula is quite complex and not really important since the re-transformation is usually done by using a transformation table, same as for the Laplace transformation.

$$\underline{X}(z) = \sum_{n=0}^{\infty} x(n) z^{-n} \tag{156}$$

This expression directly follows from equation 102 by replacing e^{sT} with z.

4.3.2.1 Convergence Area

S-Plane → Z-Plane

Due to the substitution $z = e^{sT} = e^{\Sigma + j\Omega} = e^{\Sigma} * (cos(\Omega) + jsin(\Omega)) = r * (cos(\Omega) + jsin(\Omega))$, the complex s-plane is non-linearly mapped onto the complex z-plane whereby the imaginary axis of the s-plane becomes a unit circle in the z-plane. The expression $r * [cos(x) + jsin(x)]$ always describes a circle in the complex plane with radius r. This means, each value of the frequency $\Omega = arg(z)$ represents the angle of this circle whereas the parameter $r = e^{\Sigma} = |z|$ determines the radius of this circle. Thus, the parameter $z = r * e^{j\Omega} = r * [cos(\Omega) + jsin(\Omega)]$ represents all points of the complex z-plane, each point is unambiguously determined by an angle Ω and a radius r. Figure 32 depicts the transition graphically.

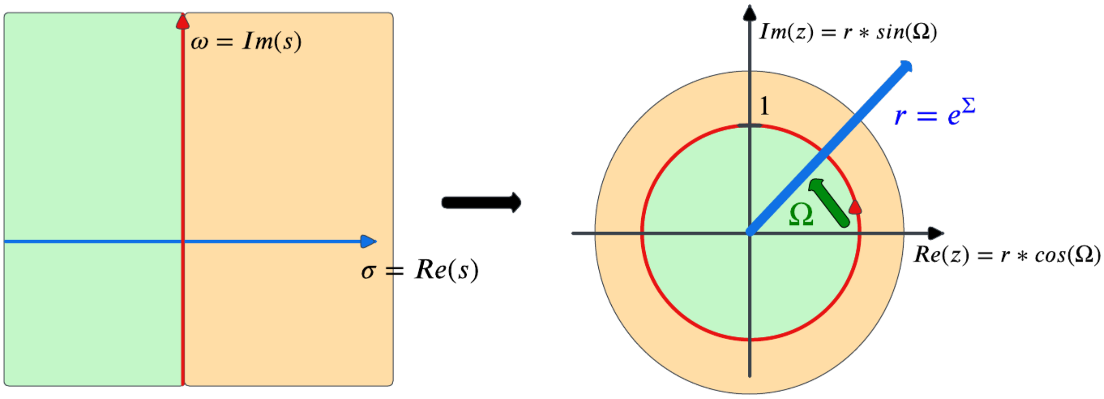


Figure 32: Transition from s-plane to z-plane

Convergence

Basically, the z-transform of a signal $x(n)$ exists if the following criteria is fulfilled:

$$\sum_{n=0}^{\infty} |x(n)e^{-\Sigma n}| = \sum_{n=0}^{\infty} |x(n)r^{-n}| < \infty \tag{157}$$

This means, the existence of the z-transform depends on the value of the parameter $r = e^{\Sigma}$ and thus of the value of the parameter Σ. Consequently, the parameter Σ, or rather r, determines, same as the parameter σ for the Laplace transformation, the convergence area of the signal.
For the Laplace transformation, the case $x(t) = e^{at}$ was considered more detailed since this is the only case where divergence could appear (see equation 107). The same can be done for the z-transformation to figure out the convergence area. However, since e^S is replaced by z, also e^A has to be replaced by a parameter, which is a.

$$x(n) = e^{An} = a^n \tag{158}$$

Then, for the time domain follows:

$$x(n)r^{-n} = a^n r^{-n} = (\frac{a}{r})^n \tag{159}$$

Since $n \in \mathbb{N}$ because only right-sided signals are considered, the expression above can only converge if $|r| > |a|$ which finally determines the convergence area. This also shows that the convergence area of right-sided signals is always represented by the area outside the circle with the radius r.

Nevertheless, there is also the option to determine the convergence area by considering the z-transform of the signal in the image domain.
Considering the same scenario where $x(n) = a^n$, the following applies in the image domain:

$$x(n) = a^n \xrightarrow{\mathcal{Z}} \underline{X}(z) = \frac{z}{z-a} \tag{160}$$

Due to this relationship, it is possible to just determine the convergence area by determining the pole with the highest absolute value. The absolute value $|p_{highest}|$ of this pole is equal to the radius r of the circle which determines the convergence area. Figure 33 shows an example where the absolute value of the pole is 3.

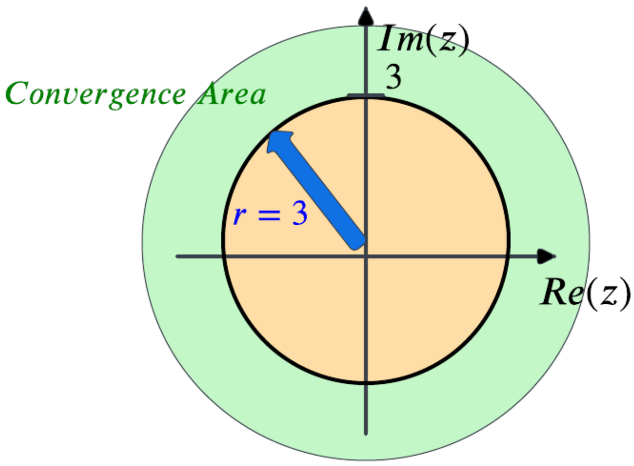


Figure 33: Convergence area for r = 3 respectively $|p_{highest}| = 3$

Relationship between the DTFT and ZT

The relationship is the same as for the FT and the LT (see equation 109). If the parameter Σ equals 0 or rather the parameter r equals 1, the ZT is equal to the DTFT. This means that the DTFT of a signal can be derived from its z-transform by replacing

z with $e^{j\Omega}$ if $r \leq 1$ because then equation **??** is definitely fulfilled.

$$\underline{X}(\Omega) = \underline{X}(z)|_{z=e^{j\Omega}} \tag{161}$$

In other words, the DTFT of a signal can be derived from its z-transform if one of the following conditions applies:

- The unit circle is part of the convergence area
- The absolute values of all poles of the z-transform are smaller than 1

4.3.2.2 Inverse Z-Transformation

Usually a transformation table is used to determine the signal in the time domain. However, sometimes the z-transform does not have the right form in order to do an inverse transform. In this case, it might be helpful to conduct a partial fraction decomposition but with dividing the z-transform by z first.

1) Before PFD divide z-transform by z
2) Conduct PFD as usual
3) Multiply expression by z
4) Conduct inverse transformation by using transformation table

Example 6

$$\underline{X}(z) = \frac{z^2+1}{z^2-0.81} = \frac{z^2+1}{(z-0.9)(z+0.9)}$$

$$\frac{\underline{X}(z)}{z} = \frac{z^2+1}{z(z-0.9)(z+0.9)} = \frac{-1.23}{z} + \frac{1.12}{z-0.9} + \frac{1.12}{z+0.9}$$

$$\underline{X}(z) = \frac{-1.23z}{z} + \frac{1.12z}{z-0.9} + \frac{1.12z}{z+0.9} = -1.23 + 1.12 + 1.12$$

$$x(n) = -1.23\delta(n) + 1.12 * 0.9^n\epsilon(n) + 1.12(-0.9)^n\epsilon(n)$$

4.4 Time-Discrete LTI-Systems in Frequency Domain

Time-discrete LTI-systems exactly described the same way as time-continuous LTI-systems by using the DTFT instead of the FT or the zT instead of the LT.

4.4.1 Frequency Domain

$$\underline{Y}(\Omega) = \underline{X}(\Omega) * \underline{H}(\Omega) \tag{162}$$

with the frequency response of the system:

$$\underline{H}(\Omega) = \frac{\underline{Y}(\Omega)}{\underline{X}(\Omega)} = \mathcal{F}_*\{h(n)\} = |\underline{H}(\Omega)|e^{j\phi_H(\Omega)} \tag{163}$$

So, the frequency response of a time-discrete LTI-system is the DTFT of its impulse response.

4.4.2 Image Domain

$$\underline{Y}(z) = \underline{X}(z) = \underline{H}(z) \tag{164}$$

with the transfer function of the system:

$$\underline{H}(z) = \frac{\underline{Y}(z)}{\underline{X}(z)} = \mathcal{Z}\{h(n)\} \tag{165}$$

So, the transfer function of a time-discrete LTI-system is the z-transform of its impulse response.

Causality and Stability

a) *Causality*
A time-discrete LTI-system is causal if the number of poles is higher than or equal the number of zeros of the transfer function.

b) *Stability*
A time-discrete LTI-system is stable if one of the following conditions is fulfilled:
- The unit circle belongs to the convergence area
- The absolute value of all poles of the transfer function is less than 1, i.e., all poles lay inside the unit circle
- The DTFT can be determined from the z-transform by replacing $z = e^{j\Omega}$

5 Two-Ports

Two-ports are special quadripoles which fulfill the port-condition. This means that the current, which flows into the system from one terminal, always has to flow out at the other terminal of the same port. If the two-port is used as transfer element between two one-ports, the port-condition is always fulfilled.
In this section, only linear two-ports are discussed whereby the symmetric reference-arrow system is used (Fig. 17).

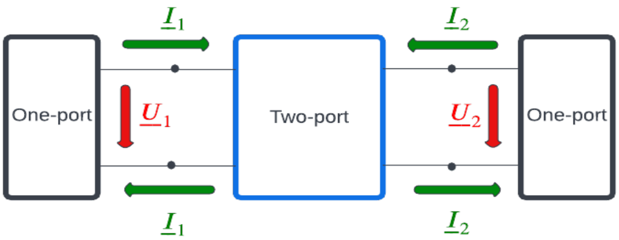


Figure 34: Two-port

5.1 Describing Two-Ports

Two-ports are described by the so-called two-port equations or two-port matrices. The two-port matrices follow from the two-port equations and vice versa. The two-port equations describe the relationship between $\underline{U}_1$, $\underline{I}_1$, $\underline{U}_2$ and $\underline{I}_2$. It is generally assumed that two magnitudes are known so the other two have to be determined. This leads to a linear equation system with two unknown variables. There are 6 possible structures of the two-port equations.

5.1.1 Two-Port Equations

1. Impedance Form and Impedance Matrix

$$\begin{aligned} \underline{U}_1 &= \underline{Z}_{11}\underline{I}_1 + \underline{Z}_{12}\underline{I}_2 \\ \underline{U}_2 &= \underline{Z}_{21}\underline{I}_1 + \underline{Z}_{22}\underline{I}_2 \end{aligned} \tag{166}$$

$$\begin{bmatrix} \underline{U}_1 \\ \underline{U}_2 \end{bmatrix} = \begin{bmatrix} \underline{Z}_{11} & \underline{Z}_{12} \\ \underline{Z}_{21} & \underline{Z}_{22} \end{bmatrix} \begin{bmatrix} \underline{I}_1 \\ \underline{I}_2 \end{bmatrix}$$

2. Admittance Form and Admittance Matrix

$$\underline{I}_1 = \underline{Y}_{11}\underline{U}_1 + \underline{Y}_{12}\underline{U}_2 \qquad (167)$$
$$\underline{I}_2 = \underline{Y}_{21}\underline{U}_1 + \underline{Y}_{22}\underline{U}_2$$

$$\begin{bmatrix}\underline{I}_1\\ \underline{I}_2\end{bmatrix} = \begin{bmatrix}\underline{Y}_{11} & \underline{Y}_{12}\\ \underline{Y}_{21} & \underline{Y}_{22}\end{bmatrix}\begin{bmatrix}\underline{U}_1\\ \underline{U}_2\end{bmatrix}$$

3. Serial Form and Serial Matrix

$$\underline{U}_1 = \underline{A}_{11}\underline{U}_2 + \underline{A}_{12}(\underline{-I}_2) \qquad (168)$$
$$\underline{I}_1 = \underline{A}_{21}\underline{U}_2 + \underline{A}_{22}(\underline{-I}_2)$$

$$\begin{bmatrix}\underline{U}_1\\ \underline{I}_1\end{bmatrix} = \begin{bmatrix}\underline{A}_{11} & \underline{A}_{12}\\ \underline{A}_{21} & \underline{A}_{22}\end{bmatrix}\begin{bmatrix}\underline{-I}_2\\ \underline{-I}_2\end{bmatrix}$$

4. Hybrid Form 1 and Hybrid Matrix 1

$$\underline{U}_1 = \underline{H}_{11}\underline{I}_1 + \underline{H}_{12}(\underline{U}_2) \qquad (169)$$
$$\underline{I}_2 = \underline{H}_{21}\underline{I}_1 + \underline{H}_{22}(\underline{U}_2)$$

$$\begin{bmatrix}\underline{U}_1\\ \underline{I}_2\end{bmatrix} = \begin{bmatrix}\underline{H}_{11} & \underline{H}_{12}\\ \underline{H}_{21} & \underline{H}_{22}\end{bmatrix}\begin{bmatrix}\underline{I}_1\\ \underline{U}_2\end{bmatrix}$$

5. Hybrid Form 2 and Hybrid Matrix 2

$$\underline{I}_1 = \underline{C}_{11}\underline{U}_1 + \underline{C}_{12}(\underline{I}_2) \qquad (170)$$
$$\underline{U}_2 = \underline{C}_{21}\underline{U}_1 + \underline{C}_{22}(\underline{I}_2)$$

$$\begin{bmatrix}\underline{I}_1\\ \underline{U}_2\end{bmatrix} = \begin{bmatrix}\underline{C}_{11} & \underline{C}_{12}\\ \underline{C}_{21} & \underline{C}_{22}\end{bmatrix}\begin{bmatrix}\underline{U}_1\\ \underline{I}_2\end{bmatrix}$$

6. Serial Form and Serial Matrix backwards

$$\underline{U}_2 = \underline{B}_{11}\underline{U}_1 + \underline{B}_{12}(\underline{-I}_1) \qquad (171)$$
$$\underline{I}_2 = \underline{B}_{21}\underline{U}_1 + \underline{B}_{22}(\underline{-I}_1)$$

$$\begin{bmatrix}\underline{U}_2\\ \underline{I}_2\end{bmatrix} = \begin{bmatrix}\underline{B}_{11} & \underline{B}_{12}\\ \underline{B}_{21} & \underline{B}_{22}\end{bmatrix}\begin{bmatrix}\underline{U}_1\\ \underline{-I}_1\end{bmatrix}$$

Determining the Unknown Variables

There are two possible approaches.

1) Set one parameter value to 0
   One unknown parameter is set to 0. This means the two-port is either operated in idle (I=0) or in short mode (U=0).

2) Using a table
   For many two-ports, a table already exists which provides formulas to determine the missing parameter values. Sometimes, the given two-port is composed from several elementary two-ports. In this case, the parameter values of the given two-port can be determined by determining the parameter values of each elementary two-port and finally calculating the values according to the formulas below.

5.1.2 Interconnecting Two-Ports

1) Row-Row-Wiring

$$[\underline{Z}_{ges}] = [\underline{Z}_1] + [\underline{Z}_2] \tag{172}$$

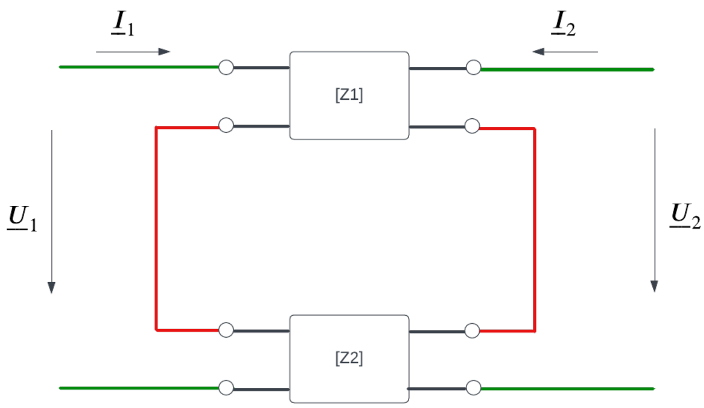


Figure 35: Row-Row-Wiring

2) Parallel-Parallel-Wiring

$$[\underline{Y}_{ges}] = [\underline{Y}_1] + [\underline{Y}_2] \tag{173}$$

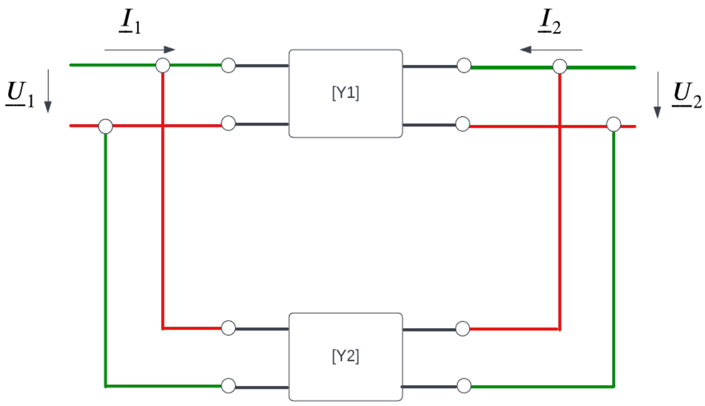


Figure 36: Parallel-Parallel-Wiring

3) Row-Parallel-Wiring

$$[\underline{H}_{ges}] = [\underline{H}_1] + [\underline{H}_2] \tag{174}$$

4) Parallel-Row-Wiring

$$[\underline{C}_{ges}] = [\underline{C}_1] + [\underline{C}_2] \tag{175}$$

5) Chain-Wiring

$$[\underline{A}_{ges}] = [\underline{A}_1] * [\underline{A}_2] \tag{176}$$

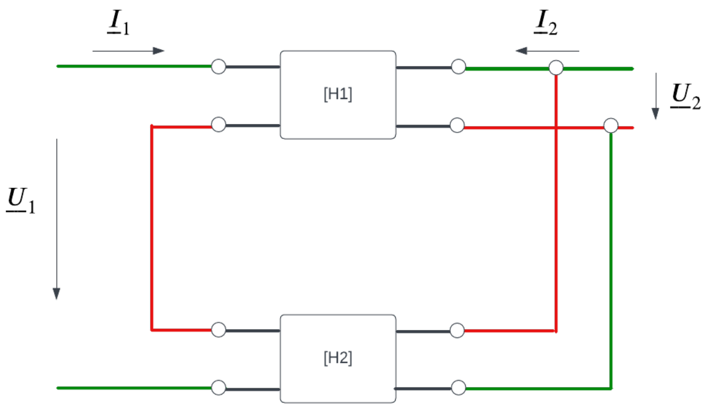


Figure 37: Row-Parallel-Wiring

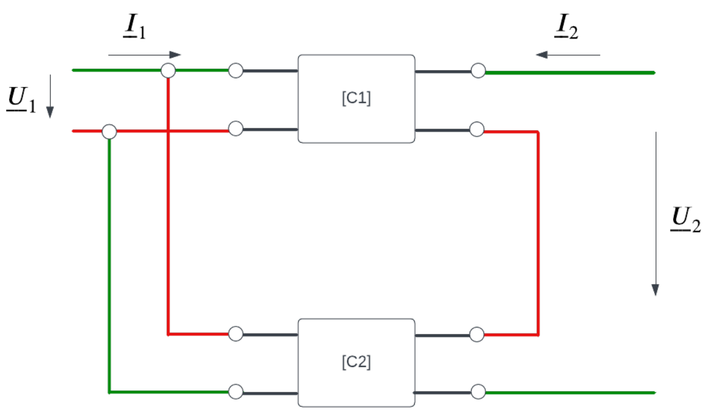


Figure 38: Parallel-Row-Wiring

Note: In order to calculate the parameter values of a given two-port, which is composed of several basic two-ports, both the main two-port and the basic two-ports must fulfill the port-condition.

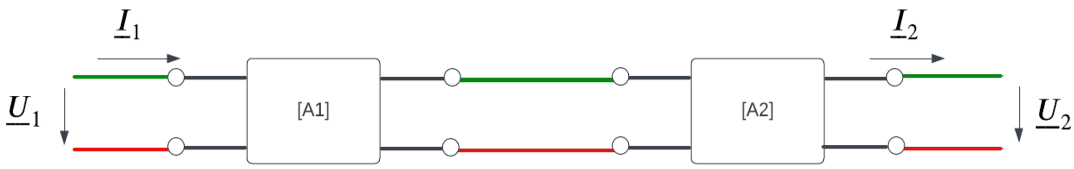


Figure 39: Chain-Wiring

5.1.3 Transforming the Two-Port Parameters

Sometimes, some two-port parameters are already known but the values of other parameters must be found. In this case, the table from Figure 40 can be used with the determinants from Figure 41.

5.1.4 Equivalent Circuits

For simplifying calculations, complex two-ports can also be described as simpler circuits if specific two-port parameters are known.

π-Circuit

Fig. 42 shows the circuit and how its parameters can be determined by using the Y-Matrix of the given two-port.

T-Circuit

Fig. 43 shows the circuit and how its parameters can be determined by using the Y-Matrix of the given two-port.

5.1.5 Connected Two-Ports

Input Impedance

If an impedance is connected to a two-port, the two-port becomes a one-port which can be described as one input impedance $\underline{Z}_E$. Fig. 44 shows the concept for operating the two-port from side 1, Fig. 45 for side 2.
Fig. 46 shows how the input impedance can be determined if one two-port matrix is given.

(Y)	$\underline{Y}_{11}$	$\underline{Y}_{12}$	$\frac{\underline{Z}_{22}}{\det\underline{Z}}$	$\frac{-\underline{Z}_{12}}{\det\underline{Z}}$	$\frac{1}{\underline{H}_{11}}$	$\frac{-\underline{H}_{12}}{\underline{H}_{11}}$	$\frac{\det\underline{C}}{\underline{C}_{22}}$	$\frac{\underline{C}_{12}}{\underline{C}_{22}}$	$\frac{\underline{A}_{22}}{\underline{A}_{12}}$	$\frac{-\det\underline{A}}{\underline{A}_{12}}$
	$\underline{Y}_{21}$	$\underline{Y}_{22}$	$\frac{-\underline{Z}_{21}}{\det\underline{Z}}$	$\frac{\underline{Z}_{11}}{\det\underline{Z}}$	$\frac{\underline{H}_{21}}{\underline{H}_{11}}$	$\frac{\det\underline{H}}{\underline{H}_{11}}$	$\frac{-\underline{C}_{21}}{\underline{C}_{22}}$	$\frac{1}{\underline{C}_{22}}$	$\frac{-1}{\underline{A}_{12}}$	$\frac{\underline{A}_{11}}{\underline{A}_{12}}$
(Z)	$\frac{\underline{Y}_{22}}{\det\underline{Y}}$	$\frac{-\underline{Y}_{12}}{\det\underline{Y}}$	$\underline{Z}_{11}$	$\underline{Z}_{12}$	$\frac{\det\underline{H}}{\underline{H}_{22}}$	$\frac{\underline{H}_{12}}{\underline{H}_{22}}$	$\frac{1}{\underline{C}_{11}}$	$\frac{-\underline{C}_{12}}{\underline{C}_{11}}$	$\frac{\underline{A}_{11}}{\underline{A}_{21}}$	$\frac{\det\underline{A}}{\underline{A}_{21}}$
	$\frac{-\underline{Y}_{21}}{\det\underline{Y}}$	$\frac{\underline{Y}_{11}}{\det\underline{Y}}$	$\underline{Z}_{21}$	$\underline{Z}_{22}$	$\frac{-\underline{H}_{21}}{\underline{H}_{22}}$	$\frac{1}{\underline{H}_{22}}$	$\frac{\underline{C}_{21}}{\underline{C}_{11}}$	$\frac{\det\underline{C}}{\underline{C}_{11}}$	$\frac{1}{\underline{A}_{21}}$	$\frac{\underline{A}_{22}}{\underline{A}_{21}}$
(H)	$\frac{1}{\underline{Y}_{11}}$	$\frac{-\underline{Y}_{12}}{\underline{Y}_{11}}$	$\frac{\det\underline{Z}}{\underline{Z}_{22}}$	$\frac{\underline{Z}_{12}}{\underline{Z}_{22}}$	$\underline{H}_{11}$	$\underline{H}_{12}$	$\frac{\underline{C}_{22}}{\det\underline{C}}$	$\frac{-\underline{C}_{12}}{\det\underline{C}}$	$\frac{\underline{A}_{12}}{\underline{A}_{22}}$	$\frac{\det\underline{A}}{\underline{A}_{22}}$
	$\frac{\underline{Y}_{21}}{\underline{Y}_{11}}$	$\frac{\det\underline{Y}}{\underline{Y}_{11}}$	$\frac{-\underline{Z}_{21}}{\underline{Z}_{22}}$	$\frac{1}{\underline{Z}_{22}}$	$\underline{H}_{21}$	$\underline{H}_{22}$	$\frac{-\underline{C}_{21}}{\det\underline{C}}$	$\frac{\underline{C}_{11}}{\det\underline{C}}$	$\frac{-1}{\underline{A}_{22}}$	$\frac{\underline{A}_{21}}{\underline{A}_{22}}$
(C)	$\frac{\det\underline{Y}}{\underline{Y}_{22}}$	$\frac{\underline{Y}_{12}}{\underline{Y}_{22}}$	$\frac{1}{\underline{Z}_{11}}$	$\frac{-\underline{Z}_{12}}{\underline{Z}_{11}}$	$\frac{\underline{H}_{22}}{\det\underline{H}}$	$\frac{-\underline{H}_{12}}{\det\underline{H}}$	$\underline{C}_{11}$	$\underline{C}_{12}$	$\frac{\underline{A}_{21}}{\underline{A}_{11}}$	$\frac{-\det\underline{A}}{\underline{A}_{11}}$
	$\frac{-\underline{Y}_{21}}{\underline{Y}_{22}}$	$\frac{1}{\underline{Y}_{22}}$	$\frac{\underline{Z}_{21}}{\underline{Z}_{11}}$	$\frac{\det\underline{Z}}{\underline{Z}_{11}}$	$\frac{-\underline{H}_{21}}{\det\underline{H}}$	$\frac{\underline{H}_{11}}{\det\underline{H}}$	$\underline{C}_{21}$	$\underline{C}_{22}$	$\frac{1}{\underline{A}_{11}}$	$\frac{\underline{A}_{12}}{\underline{A}_{11}}$
(A)	$\frac{-\underline{Y}_{22}}{\underline{Y}_{21}}$	$\frac{-1}{\underline{Y}_{21}}$	$\frac{\underline{Z}_{11}}{\underline{Z}_{21}}$	$\frac{\det\underline{Z}}{\underline{Z}_{21}}$	$\frac{-\det\underline{H}}{\underline{H}_{21}}$	$\frac{-\underline{H}_{11}}{\underline{H}_{21}}$	$\frac{1}{\underline{C}_{21}}$	$\frac{\underline{C}_{22}}{\underline{C}_{21}}$	$\underline{A}_{11}$	$\underline{A}_{12}$
	$\frac{-\det\underline{Y}}{\underline{Y}_{21}}$	$\frac{-\underline{Y}_{11}}{\underline{Y}_{21}}$	$\frac{1}{\underline{Z}_{21}}$	$\frac{\underline{Z}_{22}}{\underline{Z}_{21}}$	$\frac{-\underline{H}_{22}}{\underline{H}_{21}}$	$\frac{-1}{\underline{H}_{21}}$	$\frac{\underline{C}_{11}}{\underline{C}_{21}}$	$\frac{\det\underline{C}}{\underline{C}_{21}}$	$\underline{A}_{21}$	$\underline{A}_{22}$

Figure 40: Transformation Parameters [1, p. 181]

Output Impedance and Equivalent Voltage Source

If a real voltage source is connected to the two-port, the two-port becomes a active one-port which can be described as a equivalent real voltage source $\underline{U}_{qe}$ with the output impedance $\underline{Z}_A$ as inner impedance. Fig. 47 shows the concept for operating the two-port from side 1, Fig. 48 for side 2.

Fig. 49 shows how the output impedance and the equivalent voltage source can be determined if one two-port matrix is given. Fig. 50 shows it for the equivalent voltage source.

$$\det \underline{Y} = \underline{Y}_{11}\underline{Y}_{22} - \underline{Y}_{12}\underline{Y}_{21} = \frac{1}{\det \underline{Z}} = \frac{\underline{H}_{22}}{\underline{H}_{11}} = \frac{\underline{C}_{11}}{\underline{C}_{22}} = \frac{\underline{A}_{21}}{\underline{A}_{12}}$$

$$\det \underline{Z} = \frac{1}{\det \underline{Y}} = \underline{Z}_{11}\underline{Z}_{22} - \underline{Z}_{12}\underline{Z}_{21} = \frac{\underline{H}_{11}}{\underline{H}_{22}} = \frac{\underline{C}_{22}}{\underline{C}_{11}} = \frac{\underline{A}_{12}}{\underline{A}_{21}}$$

$$\det \underline{H} = \frac{\underline{Y}_{22}}{\underline{Y}_{11}} = \frac{\underline{Z}_{11}}{\underline{Z}_{22}} = \underline{H}_{11}\underline{H}_{22} - \underline{H}_{12}\underline{H}_{21} = \frac{1}{\det \underline{C}} = \frac{\underline{A}_{11}}{\underline{A}_{22}}$$

$$\det \underline{C} = \frac{\underline{Y}_{11}}{\underline{Y}_{22}} = \frac{\underline{Z}_{22}}{\underline{Z}_{11}} = \frac{1}{\det \underline{H}} = \underline{C}_{11}\underline{C}_{22} - \underline{C}_{12}\underline{C}_{21} = \frac{\underline{A}_{22}}{\underline{A}_{11}}$$

$$\det \underline{A} = \frac{\underline{Y}_{12}}{\underline{Y}_{21}} = \frac{\underline{Z}_{12}}{\underline{Z}_{21}} = -\frac{\underline{H}_{12}}{\underline{H}_{21}} = -\frac{\underline{C}_{12}}{\underline{C}_{21}} = \underline{A}_{11}\underline{A}_{22} - \underline{A}_{12}\underline{A}_{21}$$

Figure 41: Transformation Parameters Determinants [1, p. 182]

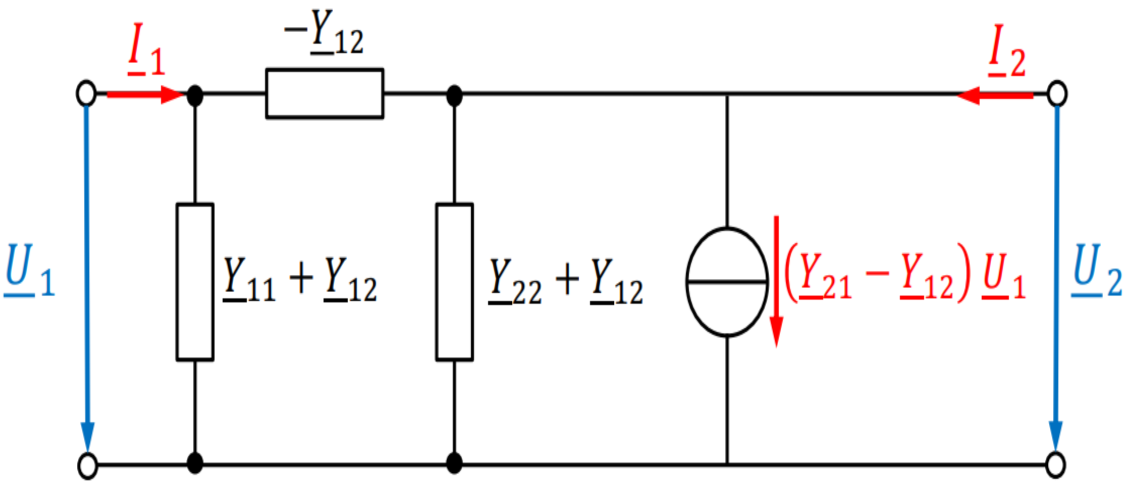


Figure 42: π-Circuit, SUS-Vorlesung, Dr. Armin Sehr

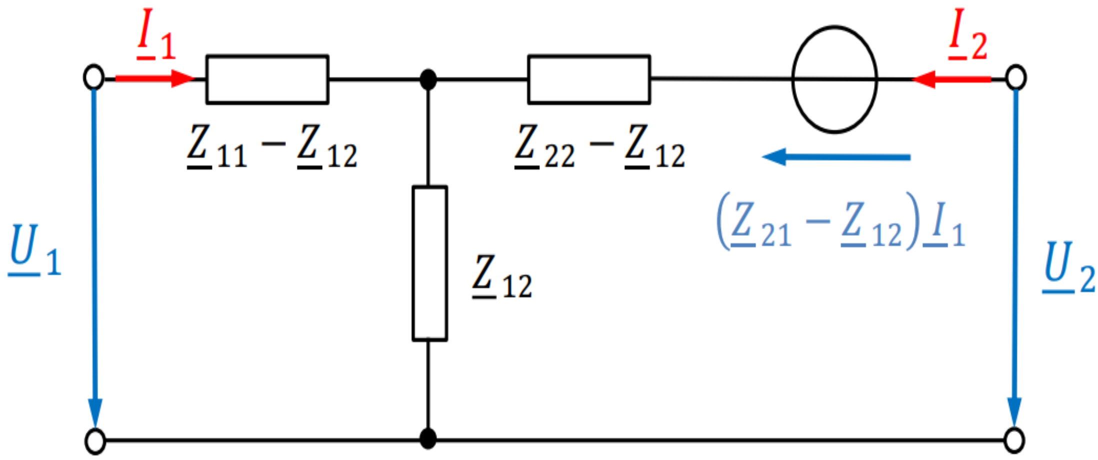


Figure 43: T-Circuit, SUS-Vorlesung, Dr. Armin Sehr

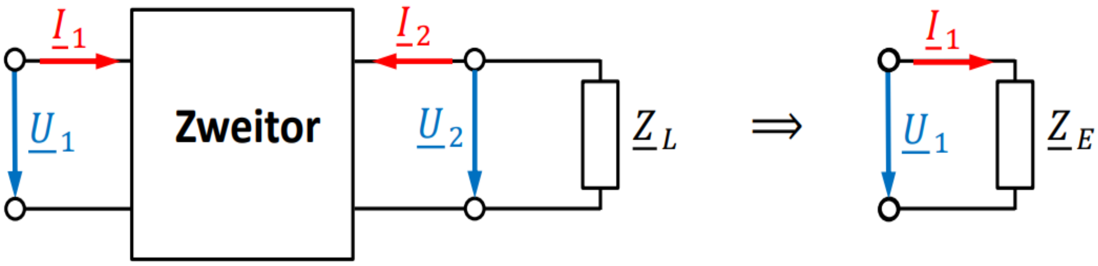


Figure 44: Input impedance if operating from side 1, SUS-Vorlesung, Dr. Armin Sehr

5.1.6 Symmetric Properties

Reciprocal Two-Ports

A two-port is reciprocal if it delivers the same output currents when operated by the same voltage source in forward and backwards direction (Fig. 51).
It must apply:

- $\underline{U}_{1a} = \underline{U}_{2b} = \underline{U}_q$

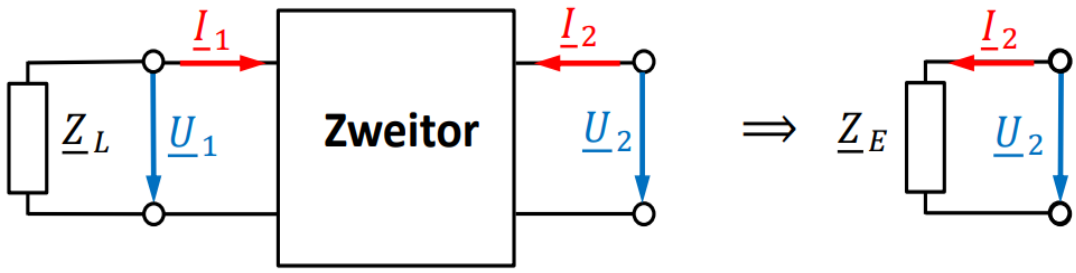


Figure 45: Input impedance if operating from side 2, SUS-Vorlesung, Dr. Armin Sehr

Parameter	Betrieb von Seite 1	Betrieb von Seite 2
A	$\underline{Z}_E = \dfrac{\underline{A}_{11}\underline{Z}_L + \underline{A}_{12}}{\underline{A}_{21}\underline{Z}_L + \underline{A}_{22}}$	$\underline{Z}_E = \dfrac{\underline{A}_{22}\underline{Z}_L + \underline{A}_{12}}{\underline{A}_{21}\underline{Z}_L + \underline{A}_{11}}$
Z	$\underline{Z}_E = \underline{Z}_{11} - \dfrac{\underline{Z}_{12}\,\underline{Z}_{21}}{\underline{Z}_{22} + \underline{Z}_L}$	$\underline{Z}_E = \underline{Z}_{22} - \dfrac{\underline{Z}_{12}\,\underline{Z}_{21}}{\underline{Z}_{11} + \underline{Z}_L}$
Y	$\underline{Y}_E = \underline{Y}_{11} - \dfrac{\underline{Y}_{12}\,\underline{Y}_{21}}{\underline{Y}_{22} + \underline{Y}_L}$	$\underline{Y}_E = \underline{Y}_{22} - \dfrac{\underline{Y}_{12}\,\underline{Y}_{21}}{\underline{Y}_{11} + \underline{Y}_L}$
H	$\underline{Z}_E = \underline{H}_{11} - \dfrac{\underline{H}_{12}\,\underline{H}_{21}}{\underline{H}_{22} + \underline{Y}_L}$	$\underline{Y}_E = \underline{H}_{22} - \dfrac{\underline{H}_{12}\,\underline{H}_{21}}{\underline{H}_{11} + \underline{Z}_L}$

Figure 46: Calculating input impedance, SUS-Vorlesung, Dr. Armin Sehr

- $\underline{I}_{2a} = \underline{I}_{1b}$

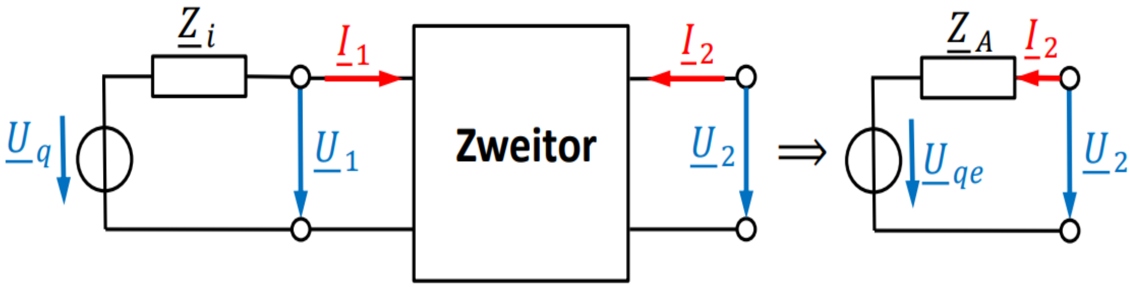


Figure 47: Output impedance if operating from side 1, SUS-Vorlesung, Dr. Armin Sehr

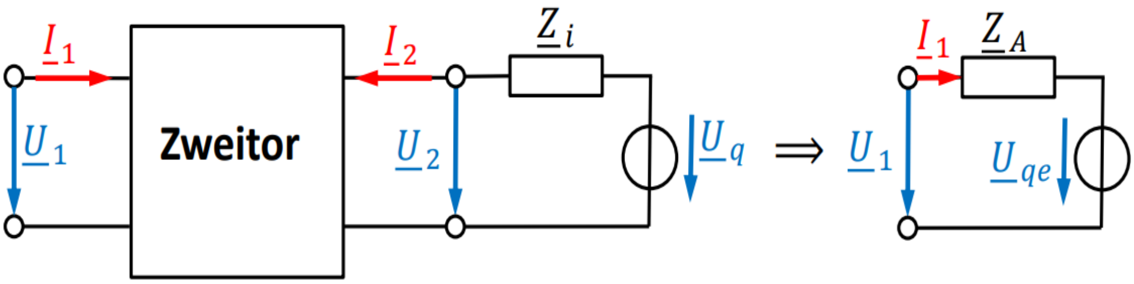


Figure 48: Output impedance if operating from side 2, SUS-Vorlesung, Dr. Armin Sehr

It basically applies:

- All passive two-ports are reciprocal
- Reciprocal two-ports can always be described by just three parameters. If three parameters are known, the fourth is also known.
- The following relations between the parameters exist:

  $\underline{Z}_{12} = \underline{Z}_{21}$

  $\underline{Y}_{12} = \underline{Y}_{21}$

  $\underline{H}_{12} = -\underline{H}_{21}$

  $\underline{C}_{12} = -\underline{C}_{21}$

  $det\underline{A} = 1$

Parameter	Betrieb von Seite 1	Betrieb von Seite 2
A	$\underline{Z}_A = \frac{\underline{A}_{22}\underline{Z}_i + \underline{A}_{12}}{\underline{A}_{21}\underline{Z}_i + \underline{A}_{11}}$	$\underline{Z}_A = \frac{\underline{A}_{11}\underline{Z}_i + \underline{A}_{12}}{\underline{A}_{21}\underline{Z}_i + \underline{A}_{22}}$
Z	$\underline{Z}_A = \underline{Z}_{22} - \frac{\underline{Z}_{12}\,\underline{Z}_{21}}{\underline{Z}_{11} + \underline{Z}_i}$	$\underline{Z}_A = \underline{Z}_{11} - \frac{\underline{Z}_{12}\,\underline{Z}_{21}}{\underline{Z}_{22} + \underline{Z}_i}$
Y	$\underline{Y}_A = \underline{Y}_{22} - \frac{\underline{Y}_{12}\,\underline{Y}_{21}}{\underline{Y}_{11} + \underline{Y}_i}$	$\underline{Y}_A = \underline{Y}_{11} - \frac{\underline{Y}_{12}\,\underline{Y}_{21}}{\underline{Y}_{22} + \underline{Y}_i}$
H	$\underline{Y}_A = \underline{H}_{22} - \frac{\underline{H}_{12}\,\underline{H}_{21}}{\underline{H}_{11} + \underline{Z}_i}$	$\underline{Z}_A = \underline{H}_{11} - \frac{\underline{H}_{12}\,\underline{H}_{21}}{\underline{H}_{22} + \underline{Y}_i}$

Figure 49: Calculating output impedance, SUS-Vorlesung, Dr. Armin Sehr

Symmetric Two-Ports

Symmetric two ports always have the same input impedance $\underline{Z}_E$ if they are connected to the same load impedance regardless of at which side the load impedance is located (Fig. 52).
It must apply:

- $\underline{Z}_{E1} = \underline{Z}_{E2}$

It basically applies:

- Symmetric two-ports can always be described by just three parameters. If three parameters are known, the fourth is also known.
- The following relations between the parameters exist:

Parameter	Betrieb von Seite 1	Betrieb von Seite 2
A	$\underline{U}_{qe} = \dfrac{1}{\underline{A}_{21}\underline{Z}_i + \underline{A}_{11}} \underline{U}_q$	$\underline{U}_{qe} = \dfrac{\det(\underline{A})}{\underline{A}_{21}\underline{Z}_i + \underline{A}_{22}} \underline{U}_q$
Z	$\underline{U}_{qe} = \dfrac{\underline{Z}_{21}}{\underline{Z}_{11} + \underline{Z}_i} \underline{U}_q$	$\underline{U}_{qe} = \dfrac{\underline{Z}_{12}}{\underline{Z}_{22} + \underline{Z}_i} \underline{U}_q$
Y	$\underline{I}_{qe} = -\dfrac{\underline{Y}_{21}}{\underline{Y}_{11} + \underline{Y}_i} \underline{I}_q$	$\underline{I}_{qe} = -\dfrac{\underline{Y}_{12}}{\underline{Y}_{22} + \underline{Y}_i} \underline{I}_q$
H	$\underline{I}_{qe} = -\dfrac{\underline{H}_{21}}{\underline{H}_{11} + \underline{Z}_i} \underline{U}_q$	$\underline{U}_{qe} = \dfrac{\underline{H}_{12}}{\underline{H}_{22} + \underline{Y}_i} \underline{I}_q$

Figure 50: Calculating output impedance, SUS-Vorlesung, Dr. Armin Sehr

$$\underline{Z}_{11} = \underline{Z}_{22}$$
$$\underline{Y}_{11} = \underline{Y}_{22}$$
$$det\underline{H} = 1$$
$$det\underline{C} = 1$$
$$\underline{A}_{11} = \underline{A}_{22}$$

5.1.7 Longitudinal Symmetric Two-Ports

All two-ports, which are reciprocal as well as symmetric, are longitudinal symmetric. Since these two-ports fulfill both the properties of reciprocal and symmetric two-ports, these two-ports are fully described by only two parameters. The following relations between the parameters exist:

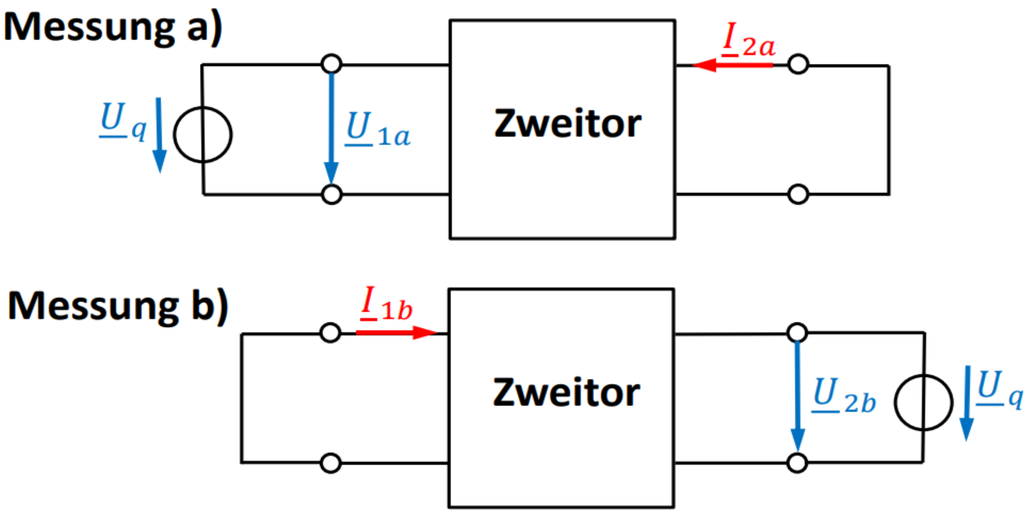


Figure 51: Reciprocal two-port, SUS-Vorlesung, Dr. Armin Sehr

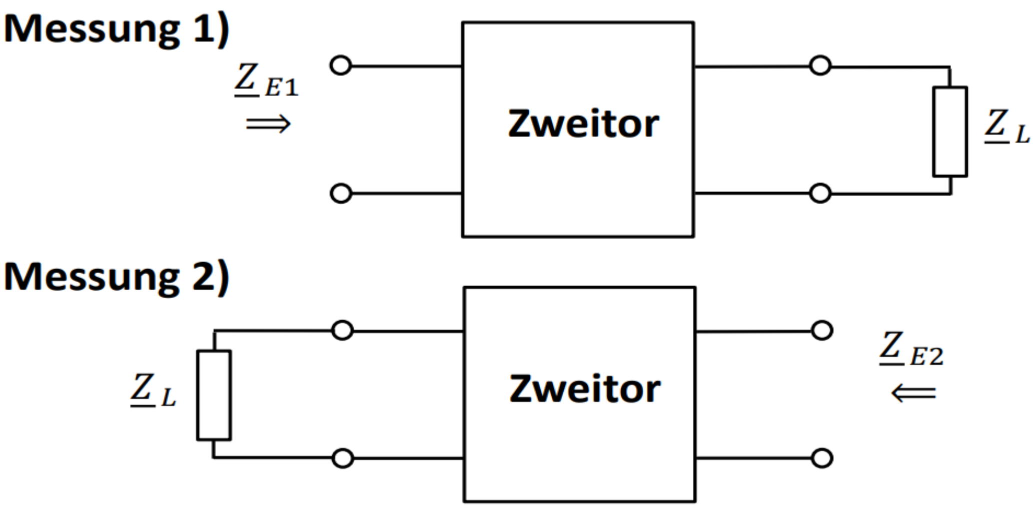


Figure 52: Symmetric two-port, SUS-Vorlesung, Dr. Armin Sehr

$\underline{Z}_{12} = \underline{Z}_{21}$ *and* $\underline{Z}_{11} = \underline{Z}_{22}$

$\underline{Y}_{12} = \underline{Y}_{21}$ *and* $\underline{Y}_{11} = \underline{Y}_{22}$

$\underline{H}_{12} = -\underline{H}_{21}$ *and* $det\underline{H} = 1$

$\underline{C}_{12} = -\underline{C}_{21}$ *and* $det\underline{C} = 1$

$det\underline{A} = 1$ *and* $\underline{A}_{11} = \underline{A}_{22}$

5.1.8 Wave Impedance

If a two-port is connected to its wave impedance $\underline{Z}_{w2}$ at side 2, the two-port has got the input impedance $Z_E = Z_{w1}$ from side 1. The same applies vice versa if the wave impedance Z_{w1} is connected to the two-port at side 1 (Fig. 53 and Fig. 54).

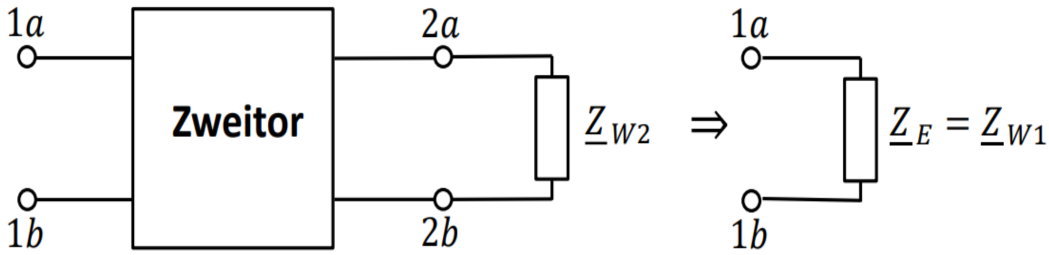


Figure 53: Wave impedance connected to side 2, SUS-Vorlesung, Dr. Armin Sehr

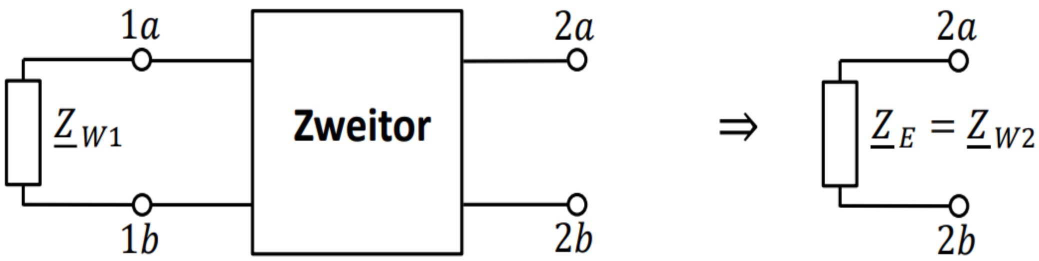


Figure 54: Wave impedance connected to side 1, SUS-Vorlesung, Dr. Armin Sehr

Calculation

Fig. 55 shows how the wave impedances can be determined.

Parameter	An Tor 1	An Tor 2
A	$\underline{Z}_{W1} = \sqrt{\frac{\underline{A}_{11}\,\underline{A}_{12}}{\underline{A}_{21}\,\underline{A}_{22}}}$	$\underline{Z}_{W2} = \sqrt{\frac{\underline{A}_{22}\,\underline{A}_{12}}{\underline{A}_{21}\,\underline{A}_{11}}}$
Z	$\underline{Z}_{W1} = \sqrt{\frac{\underline{Z}_{11}\det \underline{Z}}{\underline{Z}_{22}}}$	$\underline{Z}_{W2} = \sqrt{\frac{\underline{Z}_{22}\det \underline{Z}}{\underline{Z}_{11}}}$
Y	$\underline{Z}_{W1} = \sqrt{\frac{\underline{Y}_{22}}{\underline{Y}_{11}\det \underline{Y}}}$	$\underline{Z}_{W2} = \sqrt{\frac{\underline{Y}_{11}}{\underline{Y}_{22}\det \underline{Y}}}$
H	$\underline{Z}_{W1} = \sqrt{\frac{\underline{H}_{11}\det \underline{H}}{\underline{H}_{22}}}$	$\underline{Z}_{W2} = \sqrt{\frac{\underline{H}_{11}}{\underline{H}_{22}\det \underline{H}}}$

Figure 55: Calculation wave impedance, SUS-Vorlesung, Dr. Armin Sehr

Wave Impedance for Symmetric Two-Ports

Since for symmetric two-ports the input impedance is the same from both sides, symmetric two-ports only have one wave impedance.

$$\underline{Z}_{E1} = \underline{Z}_{E2} = \underline{Z}_{w1} = \underline{Z}_{w2} = \underline{Z}_{w}$$

5.1.9 Apparent Power Matching

Basically, apparent power matching is present if the following condition is fulfilled (Fig. 56):

$$\underline{Z}_i = \underline{Z}_L \tag{177}$$

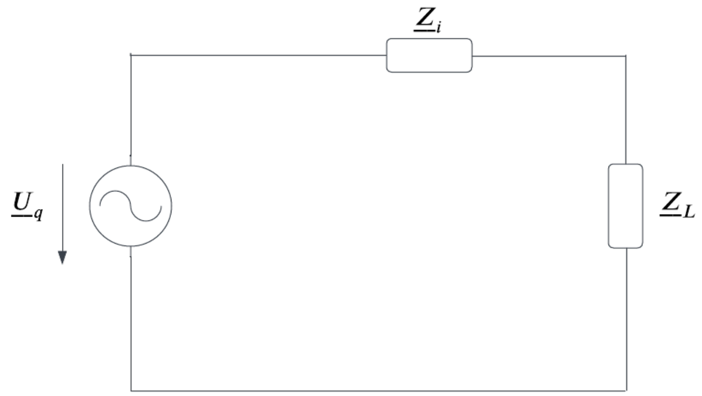


Figure 56: Apparent power matching

Therefore, for two-ports apparent power matching exists if a two-port is connected to its wave impedance at one side (e.g., $\underline{Z}_{w2}$) whereas on the other side the inner impedance of the voltage source is the corresponding wave impedance (e.g., $\underline{Z}_{w1}$) (Fig. 57). Because in this case the input impedance $\underline{Z}_E$ of the two-port is equal to $\underline{Z}_{w1}$ and therefore to the inner impedance of the voltage source. So it basically must apply:

$$\underline{Z}_i = \underline{Z}_{w1} \; and \; \underline{Z}_L = \underline{Z}_{w2} \tag{178}$$

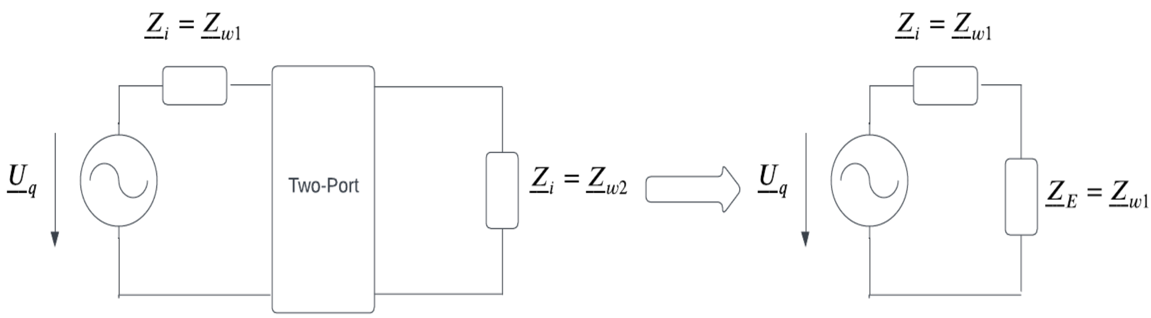


Figure 57: Apparent power matching two-port

References

[1] Wilfried Weißgerber. *10 Vierpoltheorie*, pages 171–263. Springer Fachmedien Wiesbaden, Wiesbaden, 2015.