Homework 3 - december 20, 2019

Deadline and delivery: january 10, 2020, 12:00-14:00 in C210

- **The homework can be solved by teams of at most two students.**
- **For LaTeX written solutions a bonus of 1 point is reserved.**
- **Identical or copied solutions will receive no points.**
- **The solutions must be written (printed) on papers, signed by its author(s), and physically delivered not by e-mail.**
- **Don't write again the texts of the problems! Each problem needs at most $1-2$ pages.**
- **ANY COPIED SOLUTION FOR A PROBLEM FROM BELOW WILL BE PENALISED WITH 2 POINTS.**

1. Let $G=(V,E)$ be a digraph, $c: E \to \mathbb{R}_+$, and $X, Y \subseteq V$ be two disjoint non-empty subsets of vertices in G. Suppose that we have two functions $\sigma : X \to \mathbb{R}_+$ (the supply) and $\theta : Y \to \mathbb{R}_+$ (the demand). $R=(G,X,Y,c)$ is an **e-network**; a function $x : E \to \mathbb{R}$ is called a **feasible e-flow** in the **e-network** $R=(G,X,Y,c)$ if

$$0 \leqslant x_{ij} \leqslant c_{ij}, \forall ij \in E,$$
$$\sum_k x_{ik} = \sum_k x_{ki}, \forall i \in V \setminus (X \cup Y),$$
$$\sum_k x_{ik} - \sum_k x_{ki} \leqslant \sigma_i, \forall i \in X$$
$$\sum_k x_{kj} - \sum_k x_{jk} \geqslant \theta_j, \forall j \in Y$$

Suppose that the total demand is at most the total supply, i. e., $\sigma = \sum_{i \in X} \sigma_i \geqslant \sum_{j \in Y} \theta_j = \theta$. Prove that there exists a feasible e-flow in R if and only if for every $S, T \subseteq V$ such that $S \cup T = V$ and $S \cap T = \varnothing$, we have

$$\sum_{i \in S, j \in T} c_{ij} \geqslant \sum_{j \in Y \cap T} \theta_j - \sum_{i \in X \cap T} \sigma_i.$$

(4 points)

2. At the CS department there are p students ($\mathcal{S} = \{S_1, S_2, \ldots, S_p\}$) who want to graduate and k professors ($\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$). For the final graduate examination (also called exit examination) teams of r professors will judge the students final projects.

For a given project each professor either has the competences to judge it or not, i. e., we know the set, $\mathcal{P}_i \subsetneq \mathcal{P}$ ($\mathcal{P}_i \neq \varnothing$), of professors specialized on the project of student S_i. Each professor P_l can participate to at most n_l teams.

Each student must present this project to a team of $r(\leqslant k)$ professors, $a(\leqslant r)$ of them being specialized on this project and the remaining $(r-a)$ are not.

(a) Devise a network flow model to organize the judging teams (which professor will attend which project presentation).

(b) Give a characterization of the existence of a solution to this problem in terms of maximum flow in the above network.

(c) What is the time complexity for deciding if a solution exists?

(1 + 1 + 1 = 3 points)

3. Consider the following decision problem:
3AN
Instance: $G = (V, E)$ a graph with $\Delta(G) \leqslant 3$ and $k \in \mathbb{N}^*$.
Question: Is there $U \subseteq V$, $|U| \leqslant k$ s. t. $\{u, v\} \cap U \neq \varnothing$, $\forall uv \in E$?

We consider also an instance of **3SAT** problem: $X = \{x_1, x_2, \ldots, x_n\}$ a set of boolean variables, $\mathcal{C} = C_1 \wedge C_2 \ldots \wedge C_m$ a set of disjunctive clauses over X, each clause containing exactly three literals: $C_j = v_{j1} \vee v_{j2} \vee v_{j3}$, $\forall j = \overline{1, m}$.

Let k_i to be the number of occurrences of x_i (as pozitive or negative literal) in $\mathcal{C}$ (we index these occurrences: the 1st, the 2nd etc). Define the following disjoint graphs and sets of edges:

(1) a $2k_i$ length cycle, $G_i = (V_i, E_i)$, for each boolean variable x_i, where $V_i = \{a_{i,1}, f_{i,1}, a_{i,2}, f_{i,2}, \ldots, a_{i,k_i}, f_{i,k_i}\}$ and $E_i = \{a_{i,h}f_{i,h}, f_{i,h}a_{i,h+1} \ : \ 1 \leqslant h \leqslant k_i\}$ (notation modulo k_i);

(2) a graph $H_j = (W_j, E(H_j)) \cong K_3$, for each clause C_j, where $W_j = \{w_{j,1}, w_{j,2}, w_{j,3}\}$;

(3) $A = \{a_{i,l}w_{j,k} \ : \ \text{if } v_{jk} = x_i \text{ is the } l^{th} \text{ occurrence of } x_i \text{ in } C_j\}$;

(4) $F = \{f_{i,l}w_{j,k} \ : \ \text{if } v_{jk} = \overline{x}_i \text{ is the } l^{th} \text{ occurrence of } x_i \text{ in } C_j\}$;

In the end define the following graph $G = (V, E)$:

$$V = \left(\bigcup_{i=1}^{n} V_i\right) \cup \left(\bigcup_{j=1}^{m} W_j\right), E = \left(\bigcup_{i=1}^{n} E_i\right) \cup \left(\bigcup_{j=1}^{m} E(H_j)\right) \cup A \cup F.$$

Prove that **3SAT** can be polinomially reduced to **3AN** (with the instance G from above and $k = 5m$) by showing that

(a) There is only two minimum cardinality vertex subsets that covers all the edges of G_i and their cardinalities are k_i, namely: $\{a_{i,1}, a_{i,2}, \ldots, a_{i,k_i}\}$ and $\{f_{i,1}, f_{i,2}, \ldots, f_{i,k_i}\}$.

(b) If $U \subset V(G)$ covers the edges of G and $|U| = 5m$, then

   (b1) $|U \cap W_j| \geqslant 2$, $\forall j = \overline{1, m}$;

   (b2) $\left| U \cap \left(\bigcup_{i=1}^{n} V_i\right) \right| \geqslant 3m$;

   (b3) $|U \cap W_j| = 2$, $\forall j = \overline{1, m}$, and $|U \cap V_i| = k_i$, $\forall i = \overline{1, n}$.

   (b4) the following truth assignment satisfies all the clauses from $\mathcal{C}$: for all i, $t(x_i) = true$ if and only if $U \cap V_i = \{a_{i,1}, a_{i,2}, \ldots, a_{i,k_i}\}$.

(c) Suppose $t : X \to \{true, false\}$ is a truth assignment that satisfies all the clauses from $\mathcal{C}$. Construct U as follows:

   - for each boolean variable x_i with $t(x_i) = true$ add to U the set $\{a_{i,1}, a_{i,2}, \ldots, a_{i,k_i}\}$;
   - for each boolean variable x_i with $t(x_i) = false$ add to U the set $\{f_{i,1}, f_{i,2}, \ldots, f_{i,k_i}\}$;
   - for each clause C_j add to U all the nodes in W_j except one which corresponds to a true literal in C_j.

   Prove that U has the required property and $|U| = 5m$.

(1 + (1 + 1 + 1 + 1) + 1 = 6 points)

4.

(a) Every graph G has a $\chi(G)$-coloring in which at least one of its coloring classes is a maximal stable set.

(b) Let $G = (V, E)$ be a graph and $x, y \in V$ such that $xy \notin E$. Prove that

$$\chi(G) = \min\{\chi(G + xy), \chi(G|xy)\},$$

where $G|xy$ denotes the contraction of the pair (x, y) in G.

(1 + 1 = 2 points)