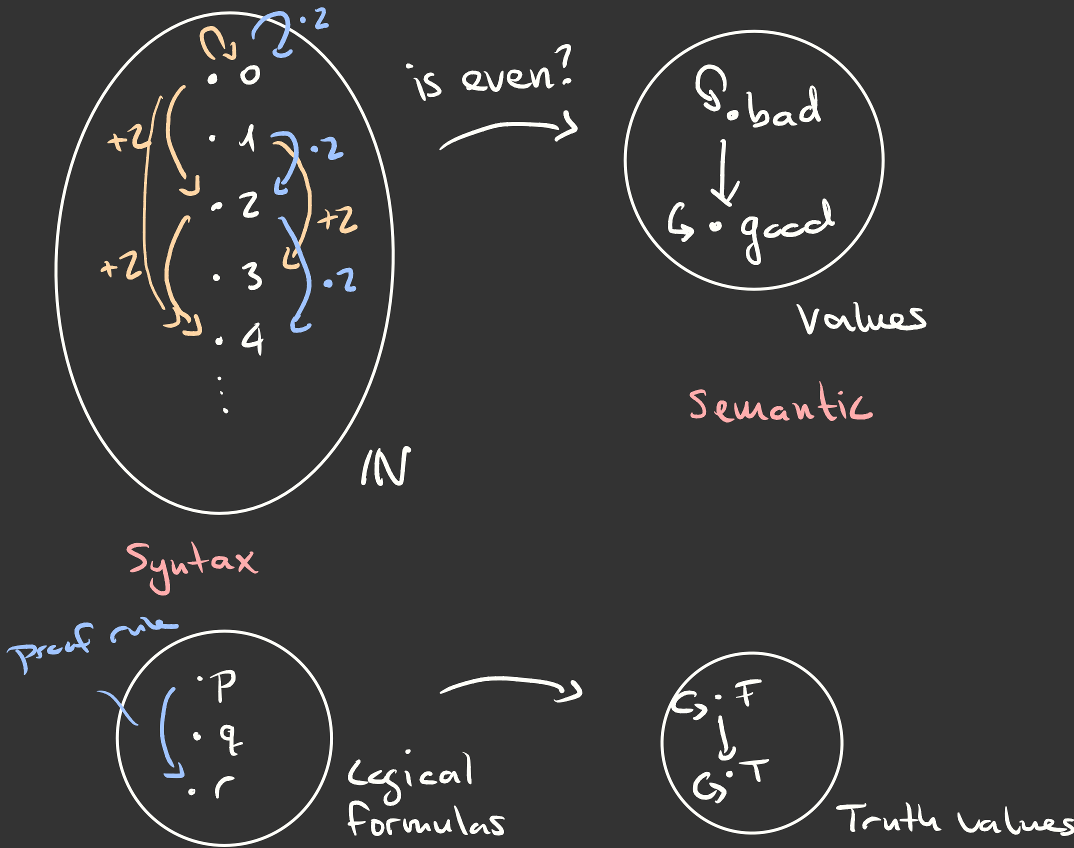
What is mathematical logic?



Qs:
- Are rules sound?
- Are rules complete?
- Are rules minimal?
- Decision procedure? with what complexity?

What will we be studying?

- Propositional logic
  1. Naive
  4. Prop. logic done right

propositions : $p, q, r, \ldots$

connectives : $\neg, \wedge, \vee, \to, \leftrightarrow$

$$(\underbrace{r}_{\text{rain}} \wedge \neg \underbrace{i}_{\text{indoors}}) \to \underbrace{c}_{\text{clouds}}$$

- First order logic
  2. Naive
  3. Naive proof theory.
  5. First order logic done right.

$$\underbrace{\forall x \forall y}_{\text{quantifier}} (\underbrace{\underbrace{f(x) = f(y)}_{\text{function symbol}}}_{P(x,y)} \xrightarrow{\text{implication}} \underbrace{\underbrace{x = y}_{\text{equality}}}_{q(x,y)})$$

variables

— injectivity of f.

Naive Propositional Logic

Idea: Combine statements using logical connectives.

$$\underbrace{\text{not}}_{\neg}\ (\underbrace{\text{"it's raining"}}_{(P_1}\ \underbrace{\text{or}}_{\vee}\ \underbrace{\text{"}\pi\text{ is irrational"}}_{P_2)})\ \Big|\ \underbrace{\text{not}}_{\neg}\ \underbrace{\text{"i'm hungry"}}_{P_3}$$

$$\equiv$$

$$(\text{not "it's raining"})\text{ and }(\text{not "}\pi\text{ is irrational"})$$

$$(\neg\ P_1)\quad\wedge\quad(\neg P_2)$$

Syntax:
- atomic propositions: $P_1, P_2, P_3, \dots$
- logical connectives:
  - $\neg$ negation (not)
  - $\wedge$ conjunction (and)
  - $\vee$ disjunction (or)
  - $\rightarrow$ implication
  - $\leftrightarrow$ equivalence (if and only if)

wff: $(P_1 \wedge P_2) \vee P_3$ $(\neg P_1) \rightarrow P_2$

not wff: $\neg P_1 \rightarrow P_2$ $P_1 P_2 \neg$ $P_1 \neg P_2$ $P_1 \rightarrow (\rightarrow P_3)$

Semantics: 1) Assing truth values to atomic Propositions $v: \{p_i : i \in \mathbb{N}_{\geq 0}\} \to \{1,0\}$

2) Calculate the truth value of a formula using rules for logical connectives.

Φ	$\neg\Phi$
1	0
0	1

Φ	Ψ	$\Phi \wedge \Psi$	$\Phi \vee \Psi$	$\Phi \to \Psi$	$\Phi \leftrightarrow \Psi$
1	1	1	1	1	1
1	0	0	1	0	0
0	1	0	1	1	0
0	0	0	0	1	1

Ex:

$$(\neg p_1) \to (p_2 \vee p_3)$$

$$\dfrac{\dfrac{\dfrac{0}{1} \qquad \dfrac{1 \quad 0}{1}}{}}{1}$$

$$(p_1 \leftrightarrow p_2) \wedge p_3$$

$$\dfrac{\dfrac{1 \quad 0}{0} \qquad 1}{0}$$

Does this always work?

Def: A formula Φ is valid / tautology iff Φ is true for any assignment of truth values to the atomic propositions occurring in it.

Ex $p_1 \vee (\neg p_1)$

p_1	$\neg p_1$	$\boxed{p_1 \vee (\neg p_1)}$
1	0	1
0	1	1

$p_1 \to (p_2 \to p_1)$

p_1	p_2	$p_2 \to p_1$	$\boxed{p_1 \to (p_2 \to p_1)}$
1	1	1	1
1	0	1	1
0	1	0	1
0	0	1	1

2^n

How can one come up w/ tautologies?

Def: Two formulas Φ and Ψ are logically equivalent iff they both have the same truth value for any assignment of truth values to the atomic propositions occurring in them.

Ex $\neg(p_1 \vee p_2) \equiv (\neg p_1) \wedge (\neg p_2)$

p_1	p_2	$p_1 \vee p_2$	$\neg(p_1 \vee p_2)$
1	1	1	0
1	0	1	0
0	1	1	0
0	0	0	1

p_1	p_2	$\neg p_1$	$\neg p_2$	$(\neg p_1) \wedge (\neg p_2)$
1	1	0	0	0
1	0	0	1	0
0	1	1	0	0
0	0	1	1	1

Lem: $\Phi \equiv \Psi$ iff $\Phi \leftrightarrow \Psi$ is tautology.

1. $P \equiv \neg(\neg P)$

2. $P \wedge P \equiv P \qquad P \vee P \equiv P$

$(\neg R) \to S \equiv (\neg(\neg R)) \vee S$

3. $P \to Q \equiv (\neg P) \vee Q$

4. $P \to Q \equiv (\neg Q) \to (\neg P)$

5. $P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P)$

6. $P \wedge Q \equiv Q \wedge P \qquad P \vee Q \equiv Q \vee P$

7. $\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q) \qquad \neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$

8. $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R \qquad P \vee (Q \vee R) \equiv (P \vee Q) \vee R$

9. $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R) \qquad P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

<u>Ex</u>: $\neg((\neg R) \to S) \overset{\checkmark}{\equiv} (\neg R) \wedge (\neg S)$

$(3.) \equiv$

$\neg((\neg(\neg R)) \vee S)$

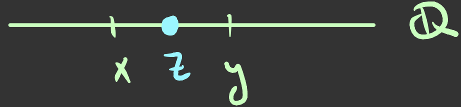
$(1.) \equiv$

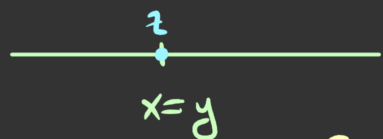
$\neg(R \vee S) \underset{(7.)}{\equiv} (\neg R) \wedge (\neg S)$

Naive First Order Logic

Between every two $\underbrace{\text{rational numbers}}_{p/q}$ there is a third one.

$p, q \in \mathbb{Z}$, $q \neq 0$





$\forall$ if   then $\exists$

For all $\overbrace{\text{rational numbers}}^{\mathbb{Q}}$ x, y where $x<y$ there is a rational number z satisfying $x<z$ and $z<y$.

$$\forall x \in \mathbb{Q}, \forall y \in \mathbb{Q} \, (x<y \to \exists z \in \mathbb{Q} \, (x<z \wedge z<y))$$

$$\forall x\in\mathbb{Q},\ \forall y\in\mathbb{Q}\,(x<y \to \exists z\in\mathbb{Q}\,(x<z \wedge z<y))$$

$\forall x\in\mathbb{Q}$

$$\forall y\in\mathbb{Q}\,(x<y \to \exists z\in\mathbb{Q}\,(x<z \wedge z<y))$$

$\forall y\in\mathbb{Q}$

$$(x<y \to \exists z\in\mathbb{Q}\,(x<z \wedge z<y))$$

$\to$

$\forall x\,\forall y$

$\exists x\,\exists y\ P(x,y)$ $\quad x<y \quad\quad \exists z\in\mathbb{Q}\,(x<z \wedge z<y)$

$<$ $\quad\quad \exists z\in\mathbb{Q}$

$x \quad y \quad\quad (x<z \wedge z<y)$

$\wedge$

$q(x,z)\ \ x<z \quad\quad z<y\ \ r(z,y)$

$< \quad\quad <$

$x \quad z \quad\quad z \quad y$

Restricted quantifyers: $\forall x \in \mathbb{Q}$ (restricted) $\quad \forall x$ (unrestricted)

$$\forall x \in \mathbb{R}\,(x>0) \rightsquigarrow \forall x\,(x \in \mathbb{R} \overset{\wedge}{\longrightarrow} x>0)$$

$$\exists x \in \mathbb{R}\,(x>0) \rightsquigarrow \exists x\,(x \in \mathbb{R} \underset{\longrightarrow}{\wedge} x>0)$$

$$\exists x \in \mathbb{R}\,(x>x) \qquad \exists x\,(x \in \mathbb{R} \wedge x>x)$$

$$\forall x \in \mathbb{Q}\ \exists n,m \in \mathbb{Z}\,(n \neq 0 \wedge x = m/n)$$

$$\forall x \in \mathbb{Q}\ \exists n \in \mathbb{Z}\ \exists m \in \mathbb{Z}\,(n \neq 0 \wedge x = m/n)$$

$$\forall x \in \mathbb{Q}\ \exists n \in \mathbb{Z}\ \exists m\,(m \in \mathbb{Z} \wedge (n \neq 0 \wedge x = m/n))$$

$$\forall x \in \mathbb{Q}\ \exists n\ \exists m\,(n \in \mathbb{Z} \wedge (m \in \mathbb{Z} \wedge (n \neq 0 \wedge x = m/n)))$$

$$\forall x\ \exists n \exists m\,(x \in \mathbb{Q} \to (n \in \mathbb{Z} \wedge (m \in \mathbb{Z} \wedge (n \neq 0 \wedge x = m/n))))$$

Ex i) The equation $x^2+1=0$ has a solution.

There exists an x such that $x^2+1=0$.

$\exists x\,(x^2+1=0)$ ⟵ To assign a truth value, must be interpreted in a structure

$\exists x$

$x^2+1=0$ $\quad p(x)$

$=$

$x^2+1 \qquad 0$

$+$

$x^2 \qquad 1$

$\cdot^2$

x

e.g. $(\mathbb{R}, \cdot^2, +, 1, 0)$

$(\mathbb{C}, \cdot^2, +, 1, 0)$

$p_1 \wedge p_2$ $\qquad \Phi \equiv \Psi$

$\forall x\,(x=x)$

$\forall x \forall y\, \Phi(x,y) \equiv \forall y \forall x\, \Phi(x,y)$

$\neg(\forall x\, \Phi(x)) \equiv \exists x\,(\neg \Phi(x))$

Ex ii) There are infinitely many primes.

For every natural number n there exists a prime number p such that $p \geq n$.

$p > 1$ and (if $d \mid p$ then ($d = 1$ or $d = p$.))

$$\forall n \in \mathbb{N}\, \exists p \in \mathbb{N}\, (\underbrace{\text{prime}(p)} \wedge p \geq n)$$

$$\text{prime}(p) := p > 1 \wedge \forall d \in \mathbb{N}\, (d \mid p \rightarrow (d = 1 \vee d = p))$$

$$d \mid p := \exists q \in \mathbb{N}\, (q \cdot d = p)$$

Ex iii) There is only one prime number.

There is a prime number p and for all prime numbers q we have $p=q$.

$$\underbrace{\exists p\in\mathbb{N}\,(\text{prime}(p)\wedge \underbrace{\forall q\in\mathbb{N}\,(\text{prime}(q)\to p=q)}_{\forall q\in\mathbb{P}\,(p=q)})}$$

$$\exists p\in\mathbb{P}\;\forall q\in\mathbb{P}\,(p=q)$$

Informal Proof Theory

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Style

1/ write correct English, express yourself clearly.

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1/ Write correct English, express yourself clearly.

2/ Tell the reader what you are doing.

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1/ write correct English, express yourself clearly.

2/ Tell the reader what you are doing.

Let $x := y - z$

3/ Say what you mean when introducing variables.

Informal Proof Theory

Style

1/ Write correct English, express yourself clearly.

2/ Tell the reader what you are doing.

3/ Say what you mean when introducing variables.

4/ Don't start sentences with symbols. Don't write formulas only.

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Style

1/ Write correct English, express yourself clearly.

2/ Tell the reader what you are doing.

3/ Say what you mean when introducing variables.

4/ Don't start sentences with symbols. Don't write formulas only.

5/ Use thus, therefore, hence, it follows to link up parts.

$\Longrightarrow \quad \forall \quad \exists$

Tips

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6/ Use schema:

Given: ______

To be proved: ______

Proof: ______

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7/ Use layout to identify subproofs.

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Tips

6/ Use schema: Given: ______
To be proved: ______
Proof: ______

7/ Use layout to identify subproofs.

8/ Look up definitions to rewrite givens

9/ Make a neat final version.

10/ Don't expect to get it on the first try.

11/ Sleep on it

Velleman "How to prove it".

Proof rules

Implication ($\rightarrow$)

Intro: To prove $\Phi \rightarrow \psi$, then we assume Φ and prove ψ.

↳ deduction rule

Given: ___

To prove: $\Phi \rightarrow \psi$

Proof: Suppose Φ.

To prove: ψ

Proof: ___

Therefore $\Phi \rightarrow \psi$.

Given: $P \rightarrow Q$, $Q \rightarrow R$

To prove: $P \rightarrow R$.

Proof: Suppose P.

To prove: R.

Proof: Since P is true and $P \rightarrow Q$, we have Q. Since Q is true and $Q \rightarrow R$, we have R.

Therefore $P \rightarrow R$.

Proof rules

Implication ($\to$)

Elimination: Give $\Phi \to \psi$ and Φ, conclude ψ.

← modus ponens.

Given: $\Phi \to \psi$, Φ

Thus ψ.

Given: $P \to Q$, $P \to (Q \to R)$

To prove: $P \to R$.

Proof: Suppose P.

To prove: R.

Proof: Since P and $P \to Q$, we know that Q.

Since P and $P \to (Q \to R)$, we know that $Q \to R$.

Since Q and $Q \to R$, we conclude R.

Therefore $P \to R$.

Conjunction ($\wedge$)

Intro: To prove $\Phi \wedge \Psi$, then prove Φ and Ψ.

Givens: Φ, Ψ

Thus $\Phi \wedge \Psi$

Elimination: Given $\Phi \wedge \Psi$, conclude both Φ and Ψ.

Given: $\Phi \wedge \Psi$

Thus Φ

Given: $\Phi \wedge \Psi$

Thus Ψ.

Let $n, m \in \mathbb{N}$

Prove: $((n \text{ even}) \wedge (m \text{ even})) \to n+m$ even.

Proof: Assume [$((n \text{ even}) \wedge (m \text{ even}))$. Hence] n even and m even.

Then $n = 2 \cdot p$ and $m = 2 \cdot q$ for $p, q \in \mathbb{N}$.

Hence $n+m = 2p+2q = 2(p+q)$ so $n+m$ is even.

Equivalence ($\leftrightarrow$)

Note: $\Phi \leftrightarrow \psi$ is logically eq. to $(\Phi \to \psi) \wedge (\psi \to \Phi)$.

Intro: To prove $\Phi \leftrightarrow \psi$, then prove $\Phi \to \psi$ and $\psi \to \Phi$.

Given: ___

To prove: $\Phi \leftrightarrow \psi$.

- Suppose Φ.
  To prove: ψ
  Hence $\Phi \to \psi$
- Suppose ψ
  To prove: Φ
  Hence $\psi \to \Phi$

Hence $\Phi \leftrightarrow \psi$.

$\leftrightarrow$ if and only if

iff

Elimination:

- Given $\Phi \leftrightarrow \psi$, Φ
  Thus ψ
- Given $\Phi \leftrightarrow \psi$, ψ
  Thus Φ.

Negation ($\neg$)

Advice: Use log. equiv. to transform negative statements into positive ones whenever possible.

$$\neg \forall x\,(x<a \vee b \geq x) \equiv \exists x\, \neg (x<a \vee b \geq x)$$
$$\equiv \exists x\, (\neg (x<a) \wedge \neg (b \leq x))$$
$$\equiv \exists x\, (x \geq a \wedge b > x)$$

Intro To prove $\neg \Phi$, assume Φ and derive a contradiction.

Given: ___

To prove: $\neg \Phi$

Proof: Suppose Φ

To prove: $\perp$ $\quad P \wedge \neg P$

Proof: ...

Hence $\neg \Phi$.

Intro To prove $\neg\Phi$, assume Φ and derive a contradiction.

Given: ___

To prove: $\neg\Phi$

Proof: Suppose Φ

To prove: $\perp$

Proof: ...

Hence $\neg\Phi$.

Proof by contradiction.

Given: ___

To prove: Φ

Proof: Suppose $\neg\Phi$

To prove: $\perp$

Hence $\neg(\neg\Phi) \equiv \Phi$

Given: $P \to Q$

To prove: $\neg Q \to \neg P$.

Proof: Suppose $\neg Q$.

To prove: $\neg P$.

Proof: Suppose P. Since $P \to Q$, we know Q. But $\neg Q$, contradiction. Hence $\neg P$.

Hence $\neg Q \to \neg P$.

Elimination:

Given: Φ, $\neg\Phi$

Hence ψ.

Disjunction ($\vee$)

Intro To prove $\Phi \vee \Psi$, prove Φ or Ψ.

Given: Φ | Given: Ψ
Hence $\Phi \vee \Psi$ | Hence $\Phi \vee \Psi$.

Elimination: To use $\Phi \vee \Psi$ to prove Γ, then we must show that Γ follows from Φ and Γ follows from Ψ.

$$\left\{\begin{array}{l}\Gamma \text{ follows from } \Phi \\ \Gamma \text{ follows from } \Psi.\end{array}\right.$$

Given: $\Phi \vee \Psi$
To prove: Γ

- Suppose Φ
  To prove: Γ
- Suppose: Ψ
  To prove: Γ

Hence Γ.

Given: $P \vee Q$, $\neg P$
To prove: Q
Proof: [Suppose P. But $\neg P$, so Q.
[Suppose Q. Hence Q.
Hence Q.

Given: $\neg(P \vee Q)$

To prove: $\neg P \wedge \neg Q$.

Proof: Assume P. Then $P \vee Q$. ($\vee$-intro)

This contradicts $\neg(P \vee Q)$.

Hence $\neg P$. $\leftarrow$ $\neg$-intro

Assume Q. Then $P \vee Q$

This contradicts $\neg(P \vee Q)$

Hence $\neg Q$.

Hence $\neg P \wedge \neg Q$. $\leftarrow$ $\wedge$-intro

Given: $\neg P \wedge \neg Q$

To prove: $\neg(P \vee Q)$.

Proof by cases

Givens: ____

To prove: Γ

$\rightarrow$

Givens: ____, $\Phi \vee \neg\Phi$

To prove: Γ

Proof: $\begin{cases} \text{Suppose } \Phi \\ \text{To prove: } \Gamma \end{cases}$

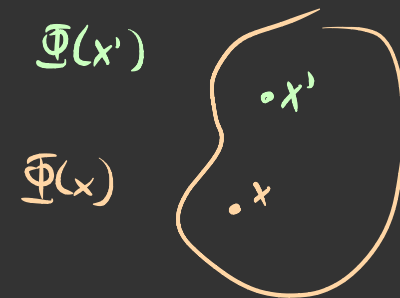
$\begin{cases} \text{Suppose: } \neg\Phi \\ \text{To prove: } \Gamma. \end{cases}$ Hence Γ.

Ex: Let $n \in \mathbb{N}$. Then $n^2 - n$ is even.

Universal Quantifier ($\forall$)

$\forall x \in A \; \Phi(x)$ $\qquad\qquad$ $x \in A$

Intro To prove $\forall x \, \Phi(x)$, then take an <u>arbitrary</u> element x and prove $\Phi(x)$ for this choice of x.



Given: ___

To prove: $\forall x \, \Phi(x)$

Proof: Let x be arbitrary.

To prove: $\Phi(x)$

Proof: ...

Hence $\forall x \, \Phi(x)$

Elimination: For $\forall x \, \Phi(x)$ we can conclude $\Phi(x_0)$ holds for any x_0.

Given: $\forall x \, \Phi(x)$

Hence $\Phi(x_0)$

Given: $\forall x \in A \; \Phi(x)$

Hence $\Phi(x_0)$, where $x_0 \in A$.

Existential Quantifier ($\exists$)

$\exists x \in A \; \Phi(x)$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $x_0 \in A$

Intro To prove $\exists x \, \Phi(x)$, we must specify an x_0 with $\Phi(x_0)$.

Given: $\Phi(x_0)$

Hence $\exists x \, \Phi(x)$

Elimination: Given $\exists x \, \Phi(x)$, we may choose an x_0 satifying $\Phi(x_0)$.

$\exists x \in A \; \Phi(x)$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $x_0 \in A$

Given: $\exists x \, \Phi(x)$

To prove: Γ

Proof: Let x_0 satisfy $\Phi(x_0)$

To prove: Γ.

Hence Γ.

Useful logical equivalences with quantifiers:

1. $\forall x \forall y\ \Phi(x,y) \overset{\longleftrightarrow}{\equiv} \forall y \forall x\ \Phi(x,y)$

   $\exists x \exists y\ \Phi(x,y) \equiv \exists y \exists x\ \Phi(x,y)$

2. $\neg \forall x\ \Phi(x) \equiv \exists x\ \neg \Phi(x)$

   $\neg \exists x\ \Phi(x) \equiv \forall x\ \neg \Phi(x)$

3. $\forall x\ (\Phi(x) \wedge \Psi(x)) \equiv (\forall x\ \Phi(x)) \wedge (\forall x\ \Psi(x))$

   $\exists x\ (\Phi(x) \vee \Psi(x)) \overset{\longleftrightarrow}{\equiv} (\exists x\ \Phi(x)) \vee (\exists x\ \Psi(x))$

($\longrightarrow$) Suppose $\exists x\ (\Phi(x) \vee \Psi(x))$.
Let x_0 satisfy $\Phi(x_0) \vee \Psi(x_0)$.
Suppose $\Phi(x_0)$. Then $\exists x\ \Phi(x)$, so also $\exists x\ \Phi(x) \vee \exists x\ \Psi(x)$.
Suppose $\Psi(x_0)$. Then $\exists x\ \Psi(x)$ so also $\exists x\ \Phi(x) \vee \exists x\ \Psi(x)$.
Hence $\exists x\ \Phi(x) \vee \exists x\ \Psi(x)$.

($\longleftarrow$) Suppose $\exists x\ \Phi(x) \vee \exists x\ \Psi(x)$.
In case $\exists x\ \Phi(x)$, we can choose x_0 satisfying $\Phi(x_0)$.
Then $\Phi(x_0) \vee \Psi(x_0)$, so $\exists x\ (\Phi(x) \vee \Psi(x))$.
In case $\exists x\ \Psi(x)$, we have some x_0 with $\Psi(x_0)$.
Hence $\Phi(x_0) \vee \Psi(x_0)$. Therefore $\exists x\ (\Phi(x) \vee \Psi(x))$.

Useful logical equivalences with quantifiers:

1. $\forall x \forall y\, \Phi(x,y) \equiv \forall y \forall x\, \Phi(x,y)$

   $\exists x \exists y\, \Phi(x,y) \equiv \exists y \exists x\, \Phi(x,y)$

2. $\neg \forall x\, \Phi(x) \equiv \exists x\, \neg \Phi(x)$

   $\neg \exists x\, \Phi(x) \equiv \forall x\, \neg \Phi(x)$

3. $\forall x\, (\Phi(x) \wedge \Psi(x)) \equiv (\forall x\, \Phi(x)) \wedge (\forall x\, \Psi(x))$

   $\exists x\, (\Phi(x) \vee \Psi(x)) \equiv (\exists x\, \Phi(x)) \vee (\exists x\, \Psi(x))$

Natural Deduction

Syntax	Semantics
• Language	• Logical validity
• Proof	
$\boxed{\Gamma \vdash \varphi}$	$\Gamma \models \varphi$

$$\text{Syntax} \xrightarrow{\text{sound}} \text{Semantics}$$

$$\text{Syntax} \xleftarrow{\text{complete}} \text{Semantics}$$

Natural Deduction

Def A derivation of a statement φ is formal proof in the form of a tree whose

- leaves contain its assumptions
- root is φ

and each node follows a natural deduction rule.

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Ex:

$$\dfrac{\dfrac{\varphi \qquad \psi}{(\varphi \wedge \psi)}\,(\wedge I) \qquad \chi}{((\varphi \wedge \psi) \wedge \chi)}\,(\wedge I)$$

assumptions

conclusion

Def: A sequent is an expression

$$(\Gamma \vdash \psi)$$

where ψ is a statement (conclusion)

Γ is a set of statements (assumptions)

It is correct if "There is a derivation of ψ whose assumptions are all contained in Γ"

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Ex:

$$\dfrac{\dfrac{\varphi \qquad \psi}{(\varphi \wedge \psi)}(\wedge I) \qquad \chi}{((\varphi \wedge \psi) \wedge \chi)}(\wedge I) \qquad \underbrace{\{\sigma, \varphi, \psi, \chi\}}_{\Gamma} \vdash \underbrace{((\varphi \wedge \psi) \wedge \chi)}_{\text{conclusion}}$$

ND rule: (Axiom)

For any statement φ, we have a derivation

$$\varphi$$

with conclusion φ and assumption φ.

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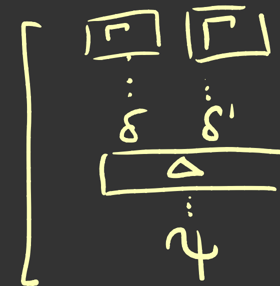

Sequent rule: (Transitivity)

If $(\Delta \vdash \psi)$ and $(\Gamma \vdash \delta)$ for every $\delta \in \Delta$, then also $(\Gamma \vdash \psi)$.

ND rule: $(\wedge I)$ Given derivations $\begin{array}{c}\Gamma\\ \mathcal{D}\\ \varphi\end{array}$ and $\begin{array}{c}\Delta\\ \mathcal{D}'\\ \psi\end{array}$,

there is a derivation

$$\dfrac{\begin{array}{c}\Gamma\\ \mathcal{D}\\ \varphi\end{array} \qquad \begin{array}{c}\Delta\\ \mathcal{D}'\\ \psi\end{array}}{(\varphi \wedge \psi)}\,(\wedge I)$$

whose assumptions are those of $\mathcal{D}$ and $\mathcal{D}'$ combined.

ND rule: $(\wedge I)$ Given derivations $\begin{matrix}\Gamma\\ \mathcal{D}\\ \varphi\end{matrix}$ and $\begin{matrix}\Delta\\ \mathcal{D}'\\ \psi\end{matrix}$,

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whose assumptions are those of $\mathcal{D}$ and $\mathcal{D}'$ combined.

Sequent rule $(\wedge I)$

If $(\Gamma \vdash \varphi)$ and $(\Delta \vdash \psi)$ then $(\Gamma \cup \Delta \vdash (\varphi \wedge \psi))$

Exercise:

(a) $\{\varphi, \psi, \chi\} \vdash (\varphi \wedge (\psi \wedge \chi))$

(b) $\{\varphi\} \vdash ((\varphi \wedge \varphi) \wedge \varphi)$

(c) $\{\varphi \wedge \psi\} \vdash ((\varphi \wedge \psi) \wedge (\varphi \wedge \psi))$

(b)' $\{\varphi\} \vdash (\varphi \wedge \varphi)$

$$\frac{\varphi \qquad \varphi}{(\varphi \wedge \varphi)}\ (\wedge I)$$

$$\frac{\begin{array}{c} \{\varphi\} - \Gamma \\ D \\ \varphi \end{array} \qquad \begin{array}{c} \Delta - \{\varphi\} \\ D' \\ \psi - \varphi \end{array}}{(\varphi \wedge \underset{\varphi}{\psi})}\ (\wedge I)$$

$(\wedge E)$ If $\begin{array}{c}\Gamma\\ \mathcal{D}\\ (\varphi\wedge\psi)\end{array}$ is a derivation,

then so are
$$\dfrac{\begin{array}{c}\Gamma\\ \mathcal{D}\\ (\varphi\wedge\psi)\end{array}}{\varphi}\,(\wedge E) \quad \text{and} \quad \dfrac{\begin{array}{c}\Gamma\\ \mathcal{D}\\ (\varphi\wedge\psi)\end{array}}{\psi}\,(\wedge E)$$

($\wedge$E) If $\begin{matrix}\Gamma\\ \mathcal{D}\\ (\varphi\wedge\psi)\end{matrix}$ is a derivation,

then so are $\dfrac{\begin{matrix}\Gamma\\ \mathcal{D}\\ (\varphi\wedge\psi)\end{matrix}}{\varphi}\,(\wedge E)$ and $\dfrac{\begin{matrix}\Gamma\\ \mathcal{D}\\ (\varphi\wedge\psi)\end{matrix}}{\psi}\,(\wedge E)$

($\wedge$E) If $(\Gamma\vdash(\varphi\wedge\psi))$, then $(\Gamma\vdash\varphi)$ and $(\Gamma\vdash\psi)$.

($\wedge$E) If $\begin{matrix}\Gamma \\ \mathcal{D} \\ (\varphi \wedge \psi)\end{matrix}$ is a derivation,

then so are $\dfrac{\begin{matrix}\Gamma \\ \mathcal{D} \\ (\varphi \wedge \psi)\end{matrix}}{\varphi}\,(\wedge E)$ and $\dfrac{\begin{matrix}\Gamma \\ \mathcal{D} \\ (\varphi \wedge \psi)\end{matrix}}{\psi}\,(\wedge E)$

($\wedge$E) If $(\Gamma \vdash (\varphi \wedge \psi))$, then $(\Gamma \vdash \varphi)$ and $(\Gamma \vdash \psi)$.

Ex: Show $\{(\varphi \wedge \psi)\} \vdash (\psi \wedge \varphi)$.

$$\dfrac{\dfrac{(\varphi \wedge \psi)}{\psi}\,(\wedge E) \qquad \dfrac{(\varphi \wedge \psi)}{\varphi}\,(\wedge E)}{(\psi \wedge \varphi)}\,(\wedge I)$$

- $\{(\varphi \wedge \psi)\} \vdash (\varphi \wedge \psi)$ (Axiom)
- $\{(\varphi \wedge \psi)\} \vdash \varphi$ and $\{(\varphi \wedge \psi)\} \vdash \psi$. ($\wedge$E)
- $\{(\varphi \wedge \psi)\} \vdash (\psi \wedge \varphi)$ ($\wedge$I)

Exercises:

1/ (a) $\{(\varphi \wedge \psi)\} \vdash (\varphi \wedge \varphi)$

(b) $\{((\varphi \wedge \psi) \wedge \chi)\} \vdash (\varphi \wedge (\psi \wedge \chi))$

(c) $\{\varphi, (\psi \wedge \chi)\} \vdash (\chi \wedge \varphi)$

2/ Show that

$\{\varphi_1, \varphi_2\} \vdash \psi$ if and only if $\{(\varphi_1 \wedge \varphi_2)\} \vdash \psi$.

$(\to I)$ If $\begin{array}{c}\Gamma \\ D \\ \psi\end{array}$ is a derivation, then so are

$$\dfrac{\begin{array}{c}\cancel{\varphi} \\ D \\ \psi\end{array}}{(\varphi \to \psi)}\,(\to I)$$

$$\dfrac{\begin{array}{c}\Gamma \setminus \{\varphi\} \\ D \\ \psi\end{array}}{(\varphi \to \psi)}\,(\to I) \quad \text{and} \quad \dfrac{\begin{array}{c}\Gamma \\ D \\ \psi\end{array}}{(\varphi \to \psi)}\,(\to I)$$

$(\to I)$ If $\begin{matrix} \Gamma \\ \mathcal{D} \\ \psi \end{matrix}$ is a derivation, then so are

$$\dfrac{\begin{matrix} \Gamma \setminus \{\varphi\} \\ \mathcal{D} \\ \psi \end{matrix}}{\varphi \to \psi}(\to I) \quad \text{and} \quad \dfrac{\begin{matrix} \Gamma \\ \mathcal{D} \\ \psi \end{matrix}}{\varphi \to \psi}(\to I)$$

$(\to I)$ If $(\Gamma \cup \{\varphi\} \vdash \psi)$, then $(\Gamma \vdash (\varphi \to \psi))$.

$(\to I)$ If $\begin{matrix}\Gamma\\ D\\ \psi\end{matrix}$ is a derivation, then so are

$$\dfrac{\begin{matrix}\Gamma\setminus\{\varphi\}\\ D\\ \psi\end{matrix}}{\varphi\to\psi}\,(\to I) \qquad \text{and} \qquad \dfrac{\begin{matrix}\Gamma\\ D\\ \psi\end{matrix}}{\varphi\to\psi}\,(\to I)$$

$(\to I)$ If $(\Gamma\cup\{\varphi\}\vdash\psi)$, then $(\Gamma\vdash(\varphi\to\psi))$.

Ex: Show $\vdash(\varphi\to(\psi\to(\varphi\wedge\psi)))$

$$\dfrac{\dfrac{\dfrac{2\ \cancel{\varphi} \qquad \cancel{\psi}\ 1}{(\varphi\wedge\psi)}\,(\wedge I)}{(\psi\to(\varphi\wedge\psi))}\,(\to I)\ 1}{(\varphi\to(\psi\to(\varphi\wedge\psi)))}\,(\to I)\ 2$$

$$\dfrac{\dfrac{\dfrac{\{\varphi\}\vdash\varphi \qquad \{\psi\}\vdash\psi}{\{\varphi,\psi\}\vdash\varphi\wedge\psi}}{\{\varphi\}\vdash(\psi\to(\varphi\wedge\psi))}}{\vdash(\varphi\to(\psi\to(\varphi\wedge\psi)))}$$

$(\to E)$ If $\begin{array}{c}\Gamma \\ \mathcal{D} \\ \varphi\end{array}$ and $\begin{array}{c}\Delta \\ \mathcal{D}' \\ (\varphi \to \psi)\end{array}$ are derivations, then so is

$$\dfrac{\begin{array}{cc}\Gamma & \Delta \\ \mathcal{D} & \mathcal{D}' \\ \varphi & (\varphi \to \psi)\end{array}}{\psi}\ (\to E)$$

$(\to E)$ If $\begin{array}{c}\Gamma\\ \mathcal{D}\\ \varphi\end{array}$ and $\begin{array}{c}\Delta\\ \mathcal{D}'\\ (\varphi\to\psi)\end{array}$ are derivations, then so is

$$\dfrac{\begin{array}{c}\Gamma\\ \mathcal{D}\\ \varphi\end{array} \qquad \begin{array}{c}\Delta\\ \mathcal{D}'\\ (\varphi\to\psi)\end{array}}{\psi}\,(\to E)$$

$(\to E)$ If $(\Gamma \vdash \varphi)$ and $(\Delta \vdash (\varphi \to \psi))$, then $(\Gamma \cup \Delta \vdash \psi)$.

$(\to E)$ If $\begin{array}{c}\Gamma\\ D\\ \varphi\end{array}$ and $\begin{array}{c}\Delta\\ D'\\ (\varphi \to \psi)\end{array}$ are derivations, then so is

$$\dfrac{\begin{array}{c}\Gamma\\ D\\ \varphi\end{array} \qquad \begin{array}{c}\Delta\\ D'\\ (\varphi \to \psi)\end{array}}{\psi}\ (\to E)$$

$(\to E)$ If $(\Gamma \vdash \varphi)$ and $(\Delta \vdash (\varphi \to \psi))$, then $(\Gamma \cup \Delta \vdash \psi)$.

Ex: Show $\{(\varphi \to \psi), (\psi \to \chi)\} \vdash (\varphi \to \chi)$

$$\dfrac{\dfrac{\dfrac{① \ \cancel{\varphi} \qquad (\varphi \to \psi)}{\psi}\ (\to E) \qquad (\psi \to \chi)}{\chi}\ (\to E)}{(\varphi \to \chi)}\ (\to I)\ ①$$

Exercises:

1/ Show

(a) $\vdash (\varphi \to (\psi \to \psi))$

(b) $\vdash ((\varphi \to \varphi) \wedge (\psi \to \psi))$

(c) $\{(\varphi \to \psi), (\varphi \to \chi)\} \vdash (\varphi \to (\psi \wedge \chi))$

2/ Show that $\{\varphi\} \vdash \psi$ if and only if $\vdash (\varphi \to \psi)$

$(\leftrightarrow I)$ If $\begin{matrix}\Gamma \\ \mathcal{D} \\ (\varphi \to \psi)\end{matrix}$ and $\begin{matrix}\Delta \\ \mathcal{D}' \\ (\psi \to \varphi)\end{matrix}$ are derivations

then so is

$$\dfrac{\begin{matrix}\Gamma \\ \mathcal{D} \\ (\varphi \to \psi)\end{matrix} \qquad \begin{matrix}\Delta \\ \mathcal{D}' \\ (\psi \to \varphi)\end{matrix}}{(\varphi \leftrightarrow \psi)} \ (\leftrightarrow I)$$

$(\varphi \to \psi) \land (\psi \to \varphi)$

$(\leftrightarrow I)$ If $\begin{array}{c}\Gamma\\ \mathcal{D}\\ (\varphi \to \psi)\end{array}$ and $\begin{array}{c}\Delta\\ \mathcal{D}'\\ (\psi \to \varphi)\end{array}$ are derivations

then so is

$$\dfrac{\begin{array}{c}\Gamma\\ \mathcal{D}\\ (\varphi \to \psi)\end{array} \qquad \begin{array}{c}\Delta\\ \mathcal{D}'\\ (\psi \to \varphi)\end{array}}{(\varphi \leftrightarrow \psi)}\,(\leftrightarrow I)$$

$(\leftrightarrow E)$ If $\begin{array}{c}\Delta\\ \mathcal{D}\\ (\varphi \leftrightarrow \psi)\end{array}$ is a derivation, then so are

$$\dfrac{(\varphi \to \psi) \land (\psi \to \varphi)}{(\varphi \to \psi)}\,(\land E) \quad \dfrac{\begin{array}{c}\Delta\\ \mathcal{D}\\ (\varphi \leftrightarrow \psi)\end{array}}{(\varphi \to \psi)}\,(\leftrightarrow E) \quad \text{and} \quad \dfrac{\begin{array}{c}\Delta\\ \mathcal{D}\\ (\varphi \leftrightarrow \psi)\end{array}}{(\psi \to \varphi)}\,(\leftrightarrow E)$$

Exercises:

1/ Write sequent rules for $(\leftrightarrow I)$ and $(\leftrightarrow E)$.

2/ (a) $\{\varphi, (\varphi \leftrightarrow \psi)\} \vdash \psi$

(b) $\vdash (\varphi \leftrightarrow \varphi)$

(c) $\{(\varphi \leftrightarrow (\psi \leftrightarrow \psi))\} \vdash \varphi$

3/ Show that the relation $\varphi \sim \psi :\Longleftrightarrow \vdash (\varphi \leftrightarrow \psi)$

is an equivalence relation:

- $\varphi \sim \varphi$ for all φ. $\vdash (\varphi \leftrightarrow \varphi)$
- $\varphi \sim \psi$ implies $\psi \sim \varphi$.
- $\varphi \sim \psi$ and $\psi \sim \chi$ imply $\varphi \sim \chi$.

$(\neg E)$ If $\begin{array}{c}\Gamma\\ D\\ \varphi\end{array}$ and $\begin{array}{c}\Delta\\ D'\\ (\neg\varphi)\end{array}$ then

$$\dfrac{\begin{array}{cc}\Gamma & \Delta\\ D & D'\\ \varphi & (\neg\varphi)\end{array}}{\bot}\,(\neg E)$$

$(\neg E)$ If $\begin{array}{c}\Gamma\\ \mathcal{D}\\ \varphi\end{array}$ and $\begin{array}{c}\Delta\\ \mathcal{D}'\\ (\neg\varphi)\end{array}$ then $\dfrac{\begin{array}{cc}\Gamma & \Delta\\ \mathcal{D} & \mathcal{D}'\\ \varphi & (\neg\varphi)\end{array}}{\bot}\,(\neg E)$

$(\neg I)$ If $\begin{array}{c}\Gamma\\ \mathcal{D}\\ \bot\end{array}$ is a derivation, then so are

$$\dfrac{\begin{array}{c}\Gamma\setminus\{\varphi\}\\ \mathcal{D}\\ \bot\end{array}}{(\neg\varphi)}\,(\neg I) \quad \text{and} \quad \dfrac{\begin{array}{c}\Gamma\\ \mathcal{D}\\ \bot\end{array}}{(\neg\varphi)}\,(\neg I)$$

$\neg\varphi \rightsquigarrow (\varphi \to \bot)$

$$\dfrac{\begin{array}{cc}\Gamma & \Delta\\ \mathcal{D} & \mathcal{D}'\\ \varphi & (\varphi\to\bot)\end{array}}{\bot}\,(\to E) \qquad \dfrac{\begin{array}{c}\Gamma\setminus\{\varphi\}\\ \mathcal{D}\\ \bot\end{array}}{(\varphi\to\bot)}\,(\to I)$$

(¬E) If $\begin{matrix}\Gamma\\ \mathcal{D}\\ \varphi\end{matrix}$ and $\begin{matrix}\Delta\\ \mathcal{D}'\\ (\neg\varphi)\end{matrix}$ then

$$\dfrac{\begin{matrix}\Gamma\\ \mathcal{D}\\ \varphi\end{matrix} \qquad \begin{matrix}\Delta\\ \mathcal{D}'\\ (\neg\varphi)\end{matrix}}{\bot}\,(\neg E)$$

(¬I) If $\begin{matrix}\Gamma\\ \mathcal{D}\\ \bot\end{matrix}$ is a derivation, then so are

$$\dfrac{\begin{matrix}\Gamma\setminus\{\varphi\}\\ \mathcal{D}\\ \bot\end{matrix}}{(\neg\varphi)}\,(\neg I) \quad \text{and} \quad \dfrac{\begin{matrix}\Gamma\\ \mathcal{D}\\ \bot\end{matrix}}{(\neg\varphi)}\,(\neg I)$$

Ex $\vdash (\varphi\to(\neg(\neg\varphi)))$

$$\dfrac{\dfrac{\dfrac{\overset{2}{\cancel{\varphi}} \qquad \overset{1}{\cancel{(\neg\varphi)}}}{\bot}\,(\neg E)}{(\neg(\neg(\varphi)))}\,(\neg I)\ 1}{(\varphi\to(\neg(\neg\varphi)))}\,(\to I)\ 2$$

(RAA) If $\begin{array}{c}\Gamma\\ \mathcal{D}\\ \bot\end{array}$ is a derivation, then so are

$$\dfrac{\begin{array}{c}\Gamma\setminus\{(\neg\varphi)\}\\ \mathcal{D}\\ \bot\end{array}}{\varphi}\,(\mathrm{RAA}) \qquad \text{and} \qquad \dfrac{\begin{array}{c}\Gamma\\ \mathcal{D}\\ \bot\end{array}}{\varphi}\,(\mathrm{RAA})$$

$\neg\varphi \rightsquigarrow (\varphi \to \bot)$

$$\dfrac{\begin{array}{c}\Gamma\setminus\{(\neg\varphi)\}\\ \mathcal{D}\\ \bot\end{array}}{((\neg\varphi)\to\bot)}\,(\to I) \quad \rightsquigarrow \quad (\varphi\to\bot)\to\bot$$

$\vdash (\varphi \to (\neg(\neg\varphi)))$ ← w/o (RAA)

$\vdash ((\neg(\neg\varphi)) \to \varphi)$ ← w/ (RAA)

(RAA) If $\begin{matrix}\Gamma\\ \mathcal{D}\\ \bot\end{matrix}$ is a derivation, then so are

$$\dfrac{\begin{matrix}\Gamma \setminus \{(\neg\varphi)\}\\ \mathcal{D}\\ \bot\end{matrix}}{\varphi}\,(RAA) \quad \text{and} \quad \dfrac{\begin{matrix}\Gamma\\ \mathcal{D}\\ \bot\end{matrix}}{\varphi}\,(RAA)$$

Ex: $\vdash ((\neg(\neg\varphi)) \to \varphi)$

$$\dfrac{\dfrac{\dfrac{\overset{①}{\cancel{(\neg\varphi)}} \qquad \overset{②}{\cancel{(\neg(\neg\varphi))}}}{\bot}\,(\neg E)}{\varphi}\,(RAA)\ ①}{((\neg(\neg\varphi)) \to \varphi)}\,(\to I)\ ②$$

$(\neg E)$ If $(\Gamma \vdash \varphi)$ and $(\Delta \vdash (\neg \varphi))$, then $(\Gamma \cup \Delta \vdash \bot)$.

($\neg$E) If $(\Gamma \vdash \varphi)$ and $(\Delta \vdash (\neg \varphi))$, then $(\Gamma \cup \Delta \vdash \bot)$.

($\neg$I) If $(\Gamma \cup \{\varphi\} \vdash \bot)$, then $(\Gamma \vdash (\neg \varphi))$.

($\neg$E) If $(\Gamma \vdash \varphi)$ and $(\Delta \vdash (\neg \varphi))$, then $(\Gamma \cup \Delta \vdash \bot)$.

($\neg$I) If $(\Gamma \cup \{\varphi\} \vdash \bot)$, then $(\Gamma \vdash (\neg \varphi))$.

(RAA) If $(\Gamma \cup \{(\neg \varphi)\} \vdash \bot)$, then $(\Gamma \vdash \varphi)$.

($\neg$E) If $(\Gamma \vdash \varphi)$ and $(\Delta \vdash (\neg\varphi))$, then $(\Gamma \cup \Delta \vdash \bot)$.

($\neg$I) If $(\Gamma \cup \{\varphi\} \vdash \bot)$, then $(\Gamma \vdash (\neg\varphi))$.

(RAA) If $(\Gamma \cup \{(\neg\varphi)\} \vdash \bot)$, then $(\Gamma \vdash \varphi)$.

Exercises:

1/ Show without RAA

(a) $\vdash (\neg(\varphi \wedge (\neg\varphi)))$

(b) $\vdash ((\neg(\varphi \to \psi)) \to (\neg\psi))$

(c) $\{(\varphi \to \psi)\} \vdash ((\neg\psi) \to (\neg\varphi))$

(d) $\{(\varphi \to \psi)\} \vdash (\neg(\varphi \wedge (\neg\psi)))$

2/ Show with RAA $\{((\neg\psi) \to (\neg\varphi))\} \vdash (\varphi \to \psi)$

(∨I) If $\begin{array}{c}\Gamma\\ \mathcal{D}\\ \varphi\end{array}$ then $\dfrac{\begin{array}{c}\Gamma\\ \mathcal{D}\\ \varphi\end{array}}{(\varphi \vee \psi)}(\vee I)$ and $\dfrac{\begin{array}{c}\Gamma\\ \mathcal{D}\\ \varphi\end{array}}{(\psi \vee \varphi)}(\vee I)$

($\vee$I) If $\begin{matrix}\Gamma \\ \mathcal{D} \\ \varphi\end{matrix}$ then $\dfrac{\begin{matrix}\Gamma \\ \mathcal{D} \\ \varphi\end{matrix}}{(\varphi \vee \psi)}(\vee I)$ and $\dfrac{\begin{matrix}\Gamma \\ \mathcal{D} \\ \varphi\end{matrix}}{(\psi \vee \varphi)}(\vee I)$

($\vee$I) If at least one of $(\Gamma \vdash \varphi)$ and $(\Gamma \vdash \psi)$ is correct, then $(\Gamma \vdash (\varphi \vee \psi))$.

(∨I) If $\begin{array}{c}\Gamma\\ D\\ \varphi\end{array}$ then $\dfrac{\begin{array}{c}\Gamma\\ D\\ \varphi\end{array}}{(\varphi \vee \psi)}(\vee I)$ and $\dfrac{\begin{array}{c}\Gamma\\ D\\ \varphi\end{array}}{(\psi \vee \varphi)}(\vee I)$

(∨I) If at least one of $(\Gamma \vdash \varphi)$ and $(\Gamma \vdash \psi)$ is correct, then $(\Gamma \vdash (\varphi \vee \psi))$.

Ex: $\vdash (\varphi \vee (\neg \varphi))$

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\cancel{(\neg\varphi)}^{①}}{(\varphi\vee(\neg\varphi))}(\vee I) \quad \neg\cancel{(\varphi\vee(\neg\varphi))}^{③}}{\bot}(\neg E)}{\varphi}(RAA)\ ①
\qquad
\dfrac{\dfrac{\dfrac{\cancel{\varphi}^{②}}{(\varphi\vee(\neg\varphi))}(\vee I) \quad \neg\cancel{(\varphi\vee(\neg\varphi))}^{③}}{\bot}(\neg E)}{(\neg\varphi)}(\neg I)\ ②
}{
\dfrac{\bot}{(\varphi\vee(\neg\varphi))}(RAA)\ ③
}(\neg E)
$$

($\vee$E) Given $\begin{array}{c}\Gamma\\ \mathcal{D}\\ (\varphi\vee\psi)\end{array}$, $\begin{array}{c}\Delta\\ \mathcal{D}'\\ \chi\end{array}$ and $\begin{array}{c}\Phi\\ \mathcal{D}''\\ \chi\end{array}$, we have a derivation

$$\dfrac{\begin{array}{c}\Gamma\\ \mathcal{D}\\ (\varphi\vee\psi)\end{array}\quad\begin{array}{c}\cancel{\varphi}\\ \mathcal{D}'\\ \chi\end{array}\quad\begin{array}{c}\cancel{\psi}\\ \mathcal{D}''\\ \chi\end{array}}{\chi}$$

$$\dfrac{\begin{array}{c}\Gamma\\ \mathcal{D}\\ (\varphi\vee\psi)\end{array}\quad\begin{array}{c}\Delta\underbrace{\setminus\{\varphi\}}\\ \mathcal{D}'\\ \chi\end{array}\quad\begin{array}{c}\Phi\underbrace{\setminus\{\psi\}}_{}\\ \mathcal{D}''\\ \chi\end{array}}{\chi}\,(\vee E)$$

(optional)

(∨E) Given $\begin{array}{c}\Gamma\\ \mathcal{D}\\ (\varphi\vee\psi)\end{array}$, $\begin{array}{c}\Delta\\ \mathcal{D}'\\ \chi\end{array}$ and $\begin{array}{c}\Phi\\ \mathcal{D}''\\ \chi\end{array}$, we have a derivation

$$\dfrac{\begin{array}{c}\Gamma\\ \mathcal{D}\\ (\varphi\vee\psi)\end{array}\qquad\begin{array}{c}\Delta\underbrace{\setminus\{\varphi\}}\\ \mathcal{D}'\\ \chi\end{array}\qquad\begin{array}{c}\Phi\underbrace{\setminus\{\psi\}}\\ \mathcal{D}''\\ \chi\end{array}}{\chi}\,(\vee E)$$

(optional)

(∨E) If $(\Gamma\cup\{\varphi\}\vdash\chi)$ and $(\Delta\cup\{\psi\}\vdash\chi)$
then $(\Gamma\cup\Delta\cup\{(\varphi\vee\psi)\}\vdash\chi)$

Exercises:

1/ Show without $(\vee E)$

(a) $\vdash (\varphi \to (\varphi \vee \psi))$

(b) $\{(\neg(\varphi \vee \psi))\} \vdash ((\neg \varphi) \wedge (\neg \psi))$

(c) $\vdash ((\varphi \to \psi) \to ((\neg \varphi) \vee \psi))$

2/ Show with $(\vee E)$

(a) $\{(\varphi \vee \psi)\} \vdash (\psi \vee \varphi)$

(b) $\{(\varphi \vee \psi), (\varphi \to \chi), (\psi \to \chi)\} \vdash \chi$

(c) $\{(\varphi \vee \psi), (\neg \varphi)\} \vdash \psi$

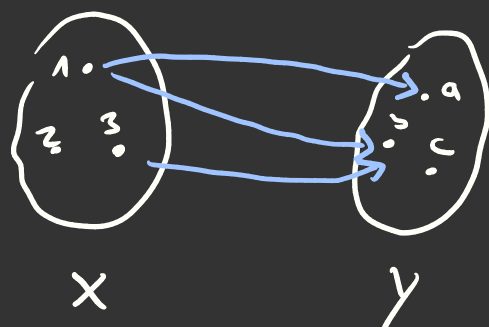
(d) $\{((\neg \varphi) \wedge (\neg \psi))\} \vdash (\neg(\varphi \vee \psi))$

Relations

Def Given set X, Y, a **relation** $R: X \nrightarrow Y$ from X to Y is a subset $R \subseteq X \times Y$.

$\{(x,y) \mid x \in X, y \in Y\}$

Ex:



$\{(1,a), (1,b), (3,c)\} = R$

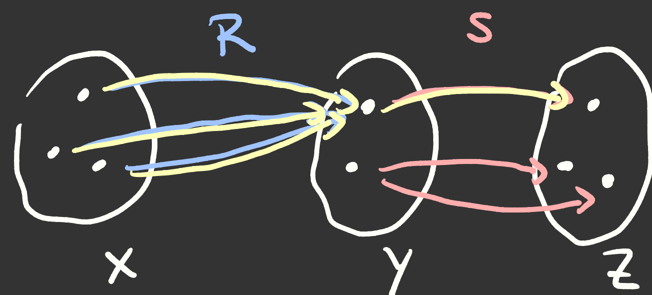
$(1,a) \in R$

$(2,a) \notin R$

Def Given $R: X \nrightarrow Y$ and $S: Y \nrightarrow Z$ define the **composite relation** $S \circ R: X \nrightarrow Z$ by

$$(x,z) \in S \circ R \iff \exists y \in Y: (x,y) \in R \wedge (y,z) \in S.$$

Ex

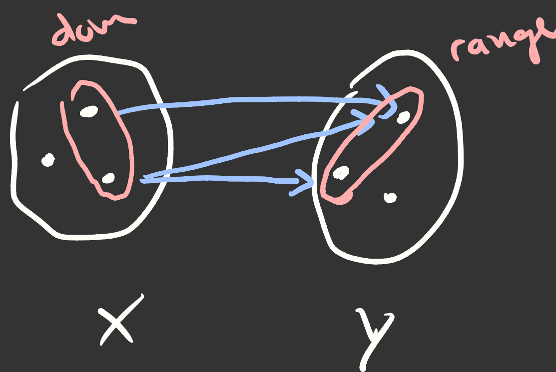


Def: Given a relation $R: X \nrightarrow Y$ we define its

domain to be $\{x \mid (x,y) \in R\}$

range to be $\{y \mid (x,y) \in R\}$.

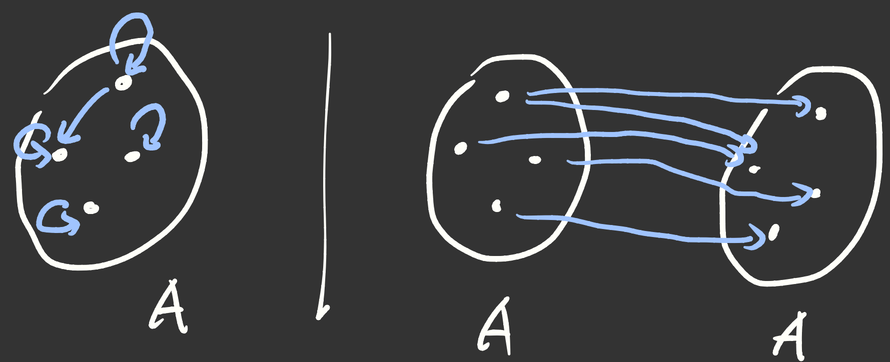
Ex:



From now on consider $R: A \nrightarrow A$.

Def: R is called

Reflexive if $(x,x) \in R$ for all $x \in A$



From now on consider $R: A \nrightarrow A$.

Def: R is called

Reflexive if $(x,x) \in R$ for all $x \in A$

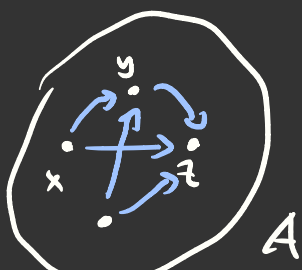
Symmetric if $(x,y) \in R$ implies $(y,x) \in R$
for all $x,y \in A$

From now on consider $R: A \to A$.

Def: R is called

equivalence relation:

- **Reflexive** if $(x,x) \in R$ for all $x \in A$
- **Symmetric** if $(x,y) \in R$ implies $(y,x) \in R$ for all $x,y \in A$
- **Transitive** if $(x,y) \in R$ and $(y,z) \in R$ implies $(x,z) \in R$ for all $x,y,z \in A$.



Language of Propositions

Def: A signature σ is a countable set of symbols not containing any of the symbols

$$\neg,\ \wedge,\ \vee,\ \rightarrow,\ \leftrightarrow,\ \perp,\ (,\ ).$$

Language of Propositions

Def: A signature σ is a countable set of symbols not containing any of the symbols

$$\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \bot, (,).$$

Ex: $\sigma = \{p_0, p_1, \dots\} = \{p_i \mid i \in \mathbb{N}\}$

Language of Propositions

Def: A *signature* σ is a countable set of symbols not containing any of the symbols

$$\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \bot, (,).$$

Ex: $\sigma = \{p_0, p_1, \dots\} = \{p_i \mid i \in \mathbb{N}\}$

Def: We write $LP(\sigma)$ for the language of propositions with signature σ.

Def: Formulas of $LP(\sigma)$ are defined inductively:

i) Every symbol of $\sigma \cup \{\perp\}$ is a formula.

ii) If φ is a formula, then so is $(\neg\varphi)$.

iii) If φ and ψ are formulas, then so are $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \to \psi)$, $(\varphi \leftrightarrow \psi)$.

iv) Nothing else is a formula.

Def: Formulas of $LP(\sigma)$ are defined inductively:

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Def: The connective of ii) / iii) is the head of the formula.

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- i) Every symbol of $\sigma \cup \{\perp\}$ is a formula.
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iv) Nothing else is a formula.

Def: The connective of ii) / iii) is the head of the formula.

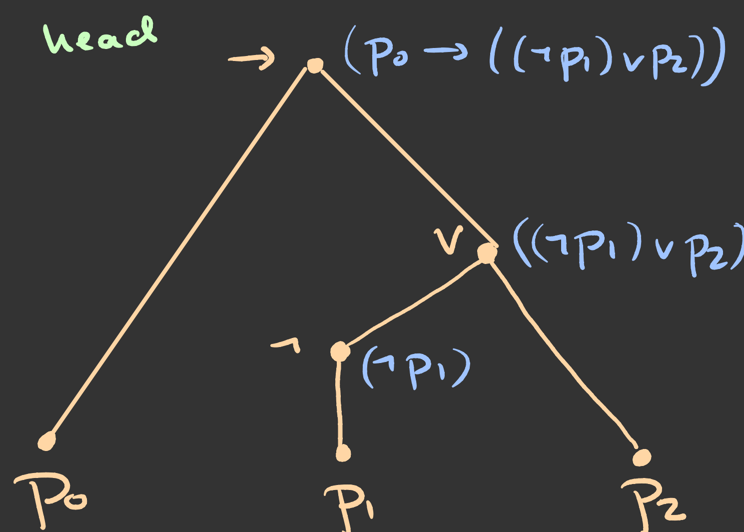
head

Ex: $(p_1 \to (\perp \wedge (\neg p_0)))$ $\qquad (\varphi \to \psi)$

__Ex:__ Every formula has a parsing tree.

~~$p_0 \to p_1 \wedge p_2$~~

$$(p_0 \to ((\neg p_1) \vee p_2))$$



Def: The complexity of a formula φ is number of connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$ occurring in φ.

Formulas of complexity 0 are called atomic.

Def: The complexity of a formula φ is number of connectives $\neg, \wedge, \vee, \to, \leftrightarrow$ occurring in φ.

Formulas of complexity 0 are called atomic.

Ex: $(p_0 \to ((\neg p_1) \vee p_2))$ $\quad 3$

$\bot \quad 0$

$p_0 \quad 0$

Def: Consider a string $\varepsilon := a_1 \cdots a_n$ of n symbols.

A *segment* of ε is a substring $a_i \cdots a_j$ $1 \leq i \leq j \leq n$.

It is *initial* if $i=1$. It is *proper* if its length $< n$.

For each initial segment s, its *depth* is the number of "(" minus the number of ")" in s.

Def: Consider a string $\varepsilon := a_1 \cdots a_n$ of n symbols.

A **segment** of ε is a substring $a_i \cdots a_j$ $\quad 1 \le i \le j \le n$.

It is **initial** if $i = 1$. It is **proper** if its length $< n$.

For each initial segment s, its **depth** is the number of "(" minus the number of ")" in s.

Ex:

$$\overbrace{(p_0 \overset{1}{\to} ((\overset{3}{\neg} p_1)}^{3-1=2} \overset{2}{\vee} p_2))$$

$a_1 \; a_2 \; a_3 \; \dots \qquad a_{12}$

$\to ((\neg p_1) \vee$

$($	1
$(p_0$	1
$(p_0 \to$	1
$(p_0 \to ($	2
$(p_0 \to (($	3
$\vdots$	

Lemma Let χ be a formula of $LP(\sigma)$. Then

(a) χ has depth 0.

Lemma Let χ be a formula of $LP(\sigma)$. Then

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Lemma Let χ be a formula of $LP(\sigma)$. Then

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(b) Every proper initial segment of χ has depth > 0.

(c) If χ has complexity >0, there is exactly one connective at depth 1 which is the head of χ.

Proof by induction on complexity:

Case complexity $\chi = k > 0$: χ has at least one of the forms $(\neg\varphi)$, $(\varphi\wedge\psi)$, $(\varphi\vee\psi)$, $(\varphi\to\psi)$, $(\varphi\leftrightarrow\psi)$.

Suppose $\chi = (\varphi\wedge\psi)$. Then φ and ψ have compl. $< k$.

Hence φ, ψ satisfy (a)–(c). The initial segments of χ are

$$\begin{array}{lcccccccc} & (, & (s \text{ - prop. init. of } \varphi, & (\varphi , & (\varphi\wedge , & (\varphi\wedge t \text{ - prop. init. } \psi, & (\varphi\wedge\psi , & (\varphi\wedge\psi) = \chi \\ d: & 1 & >1 & 1 & 1 & >1 & 1 & 0 \end{array}$$

Hence (a)–(c) hold for χ. $\square$

Thm: Any formula X of $LP(\sigma)$ has exactly one of the following forms:

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(c) χ is exactly one of $(\varphi\wedge\psi)$, $(\varphi\vee\psi)$, $(\varphi\to\psi)$, $(\varphi\leftrightarrow\psi)$, where φ and ψ are formulas.

Moreover φ, ψ are uniquely determined.

Proof: Inspection shows if χ is atomic. Suppose not. The first symbol of χ is "(". If the second is "$\neg$", then we are in case (b), otherwise (c). Uniqueness of the head of χ makes all forms in (c) exclusive. $\square$

Functions

Def A *function* $f: X \to Y$ is a relation $f \subseteq X \times Y$ where for all $x \in X$, there exist a unique $y \in Y$ with $(x, y) \in f$.

We write $f(x) = y$ whenever $(x, y) \in f$.

Functions

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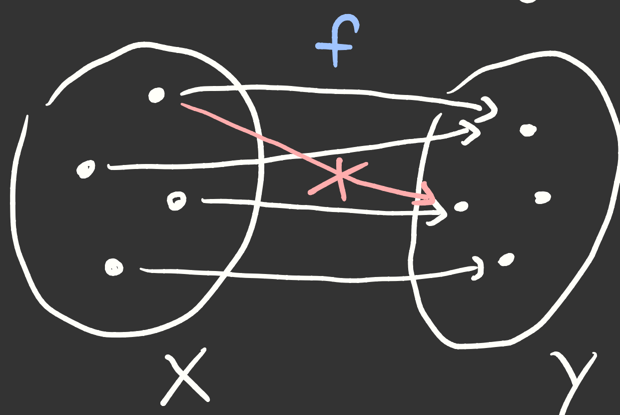
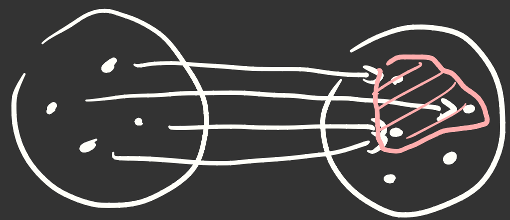
Ex



Def • The *image* of a function $f : X \to Y$ is

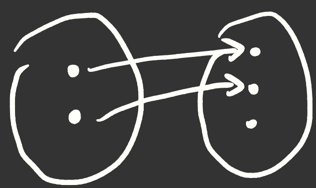
$$\mathrm{Im}(f) := \{\, y \mid (x,y) \in f \,\}$$



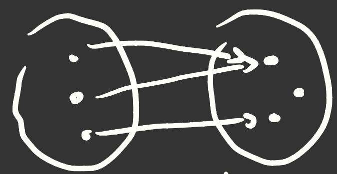
Def • The image of a function $f: X \to Y$ is

$$\mathrm{Im}(f) := \{ y \mid (x,y) \in f \}$$

• We call f injective if for every $y \in Y$ there is at most one $x \in X$ with $f(x) = y$.



injective



not injective

Def • The *image* of a function $f: X \to Y$ is

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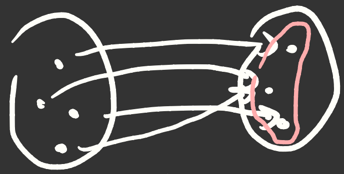
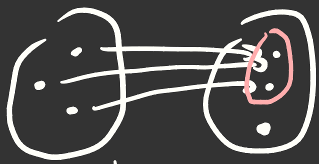
• We call f *injective* if for every $y \in Y$ there is at most one $x \in X$ with $f(x) = y$.

• We call f *surjective* if $\mathrm{Im}(f) = Y$.



surjective



not surjective.

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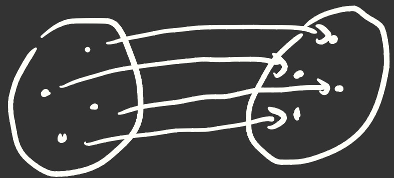
- We call f *injective* if for every $y \in Y$ there is at most one $x \in X$ with $f(x) = y$.
- We call f *surjective* if $\mathrm{Im}(f) = Y$.
- We call f *bijective* if it is injective and surjective.



Semantics for LP

Def: Let σ be a signature. A σ- structure is a function

$$A: \sigma \to \{0,1\}$$

which assigns each symbol $p \in \sigma$ a truth value $A(p)$.

Def: Every σ-structure A can be extended to A^* which assigns every formula χ of $LP(\sigma)$ a truth value:

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(a) If $\chi = p$ for $p \in \sigma$, then $\mathcal{A}^*(\chi) = \mathcal{A}(p)$,

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(c) $\mathcal{A}^*((\neg\varphi)) = 1$ if and only if $\mathcal{A}^*(\varphi) = 0$,

(d) $\mathcal{A}^*((\varphi \wedge \psi)) = 1$ if and only if $\mathcal{A}^*(\varphi) = \mathcal{A}^*(\psi) = 1$,

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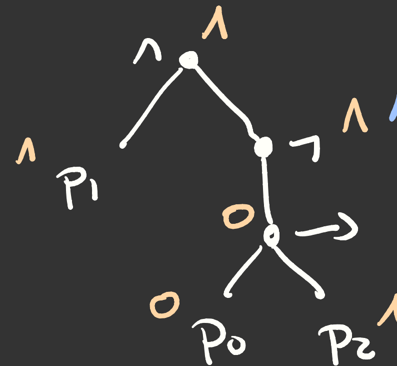
(g) $\mathcal{A}^*((\varphi \leftrightarrow \psi)) = 1$ if and only if $\mathcal{A}^*(\varphi) = \mathcal{A}^*(\psi)$.

Ex: $(p_1 \wedge (\neg(p_0 \to p_2)))$

$A :=$

p_0	p_1	p_2
0	1	1



$A^*(\neg(p_0 \to p_1))$

$A^*(p_0 \to p_1)$

$A^*(p_2) = A(p_2) = 1$

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iv) We say φ is a contradiction / incosistent if it has no models.

Ex: $(\varphi \vee (\neg\varphi))$

φ	$\neg\varphi$	$(\varphi \vee \neg\varphi)$	$\neg(\varphi \vee \neg\varphi)$
1	0	1	0
0	1	1	0

Def: Two $LP(\sigma)$ formulas φ, ψ are logically equivalent if any of the following conditions hold:

i) For every σ-structure A, $A^*(\varphi) = A^*(\psi)$

ii) The formula $(\varphi \leftrightarrow \psi)$ is a tautology.

Soundness of Natural Deduction for LP

Def: Let σ be a signature, Γ a set of LP(σ) formulas, and ψ an LP(σ) formula

i) We say a σ-structure $\mathcal{A}$ is a model for Γ if $\mathcal{A}$ is a model for every $\varphi \in \Gamma$.

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i) We say a σ-structure A is a model for Γ if A is a model for every $\varphi \in \Gamma$.

$\Gamma \vdash \psi$

ii) We write $\Gamma \models \psi$ if every model of Γ is also a model of ψ. This is called a semantic sequent.

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ii) We write $\Gamma \models \psi$ if every model of Γ is also a model of ψ. This is called a semantic sequent.

iii) We write $\Gamma \not\models \psi$, if $\Gamma \models \psi$ does not hold.

Thm (Soundness) If $\underbrace{\Gamma \vdash \psi}_{\text{There is a ND proof of } \psi \text{ based only on assumptions in } \Gamma.}$ then $\underbrace{\Gamma \models \psi}_{\text{Every model of } \Gamma \text{ is a model of } \psi.}$.

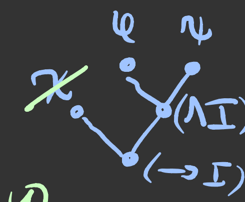
There is a ND proof of ψ based only on assumptions in Γ.

Every model of Γ is a model of ψ.

Proof: By induction on the height of a derivation.

Length of longest path from root to a leaf.

$$\dfrac{\overset{1}{\cancel{\chi}} \qquad \dfrac{\varphi \qquad \psi}{(\varphi \wedge \psi)}\,(\wedge I)}{(\chi \to (\varphi \wedge \psi))}\,(\to I)\ 1$$



Base case ✓ height $= 0$. Then our derivation has the form

$$D := \quad \bullet\, \psi$$

If D proves that $\Gamma \vdash \psi$, then $\psi \in \Gamma$.
Let A be a model of Γ. Then A is a model for every $\varphi \in \Gamma$, in particular A is a model of ψ.

__Case__: The derivation D that proves $\Gamma \vdash \psi$ has height $k > 0$.

By induction we may assume that any derivation of height $< k$ satifies the theorem.

Let R be the last rule applied in D.

__$R = (\to I)$__ ✓ In this case D has the form

$$\dfrac{\begin{array}{c}\Gamma \cup \{\cancel{\varphi}\} \; ① \\ D' \\ \chi\end{array}}{(\varphi \to \chi)}\,(\to I)\; ①$$

$\Gamma \vdash \psi$

$\Downarrow$

$\Gamma \vDash \psi$

Then D' has height $< k$. and it's assumptions lie in $\Gamma \cup \{\varphi\}$

Let A be a model for Γ.

Suppose that A is not a model for $(\varphi \to \chi)$, i.e. $A^*((\varphi \to \chi)) = 0$. Then $A^*(\varphi) = 1$ and $A^*(\chi) = 0$.

Hence A is a model for $\Gamma \cup \{\varphi\}$. Since D' proves that $\Gamma \cup \{\varphi\} \vdash \chi$, we conclude $\Gamma \cup \{\varphi\} \vDash \chi$, hence $A^*(\chi) = 1$. contradiction.

$R = (\to E)$ ✓ Then D has the form

$$\dfrac{\dfrac{\begin{array}{c}\Delta_1\\ D_1\end{array}}{\varphi} \qquad \dfrac{\begin{array}{c}\Delta_2\\ D_2\end{array}}{(\varphi \to \psi)}}{\psi}\,(\to E)$$

$\Gamma \vdash \psi$

$\Downarrow$

$\Gamma \models \psi$

D_1, D_2 have height $< k$ and $\Delta_1, \Delta_2 \subseteq \Gamma$.

Let A be a model of Γ. Since $\Delta_1, \Delta_2 \subseteq \Gamma$, A is a model for both Δ_1 and Δ_2.

Hence by induction A is a model of both φ and $(\varphi \to \psi)$.

From truth tables we conclude that A is a model for ψ.

φ	ψ	$(\varphi \to \psi)$
1	1	1
1	0	0
0	1	1
0	0	1

$R = (\mathrm{RAA})$ ✓ Then D has the form

$$\cfrac{\begin{matrix} \Gamma \cup (\cancel{\neg\psi})\ ① \\ D' \\ \bot \end{matrix}}{\psi}\ (\mathrm{RAA})\ ①$$

$\Gamma \vdash \psi$

$\Downarrow$

$\Gamma \models \psi$

we know D' has height $< k$ and has assumption in $\Gamma \cup \{\neg\psi\}$.

Let $\mathcal{A}$ be a model of Γ.

By induction we know that every model $\mathcal{B}$ of $\Gamma \cup \{(\neg\psi)\}$ is a model of $\bot$. Since $\mathcal{B}^*(\bot) = 0$, this is a contradiction. Hence $\Gamma \cup \{\neg(\psi)\}$ has no models.

Suppose that $\mathcal{A}$ is not a model for ψ.

Then $\mathcal{A}^*(\psi) = 0$ so $\mathcal{A}^*((\neg\psi)) = 1$.

This would mean that $\mathcal{A}$ is a model for $\Gamma \cup \{(\neg\psi)\}$, contradiction. Therefore $\mathcal{A}$ must be a model of ψ.

$\square$

Completeness of Natural Deduction for LP

Thm: (Completeness) If $\Gamma \models \psi$ then $\Gamma \vdash \psi$.

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We will show this for $\underbrace{\text{reduced LP}}_{\text{rLP}}$ with connectives $\wedge, \neg, \bot$.

Completeness of Natural Deduction for LP

Thm: (Completeness) If $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$.

We will show this for $\underbrace{\text{reduced LP}}_{\text{rLP}}$ with connectives $\wedge, \neg, \bot$.

Def: A set of rLP(σ) formulas is syntactically consistent if $\Gamma \nvdash \bot$.

$$\Gamma \vdash \bot \implies \Gamma \cup \{\neg\varphi\} \vdash \bot \overset{\text{RAA}}{\implies} \Gamma \vdash \varphi$$

Inconsistent sets prove any formula.

Lemma: To prove completeness it suffices to show that every syntactically consistent set Γ has a model.

Proof: Suppose that every syntactically consistent set Γ has a model.

Suppose that $\Gamma \models \psi$. $(*)$

Claim $\Gamma \cup \{\neg\psi\}$ has no models.

By $(*)$ we know that every model of Γ is also a model of ψ, hence is not a model of $(\neg\psi)$.

Since $\Gamma \cup \{\neg\psi\}$ has models it can't be syntactically consistent.

Hence $\Gamma \cup \{\neg\psi\} \vdash \bot$. Therefore by RAA $\Gamma \vdash \psi$. $\square$

Def: A set of $rLP(\sigma)$ formulas Γ is a Hintikka set if

i) If $(\varphi \wedge \psi) \in \Gamma$, then both $\varphi \in \Gamma$ and $\psi \in \Gamma$,

Def: A set of $\mathrm{LP}(\sigma)$ formulas Γ is a Hintikka set if

i) If $(\varphi \wedge \psi) \in \Gamma$, then both $\varphi \in \Gamma$ and $\psi \in \Gamma$,

ii) If $(\neg(\varphi \wedge \psi)) \in \Gamma$, then $(\neg \varphi) \in \Gamma$ or $(\neg \psi) \in \Gamma$,

$$\underset{(\neg \varphi) \vee (\neg \psi)}{\equiv}$$

Def: A set of $\mathrm{rLP}(\sigma)$ formulas Γ is a **Hintikka set** if

i) If $(\varphi \wedge \psi) \in \Gamma$, then both $\varphi \in \Gamma$ and $\psi \in \Gamma$,

ii) If $(\neg(\varphi \wedge \psi)) \in \Gamma$, then $(\neg\varphi) \in \Gamma$ or $(\neg\psi) \in \Gamma$,

iii) If $(\neg(\neg\varphi)) \in \Gamma$, then $\varphi \in \Gamma$,

Def: A set of $\neg LP(\sigma)$ formulas Γ is a Hintikka set if

i) If $(\varphi \wedge \psi) \in \Gamma$, then both $\varphi \in \Gamma$ and $\psi \in \Gamma$,

ii) If $(\neg(\varphi \wedge \psi)) \in \Gamma$, then $(\neg\varphi) \in \Gamma$ or $(\neg\psi) \in \Gamma$,

iii) If $(\neg(\neg\varphi)) \in \Gamma$, then $\varphi \in \Gamma$,

iv) $\bot \notin \Gamma$

Def: A set of $\mathrm{rLP}(\sigma)$ formulas Γ is a **Hintikka set** if

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iii) If $(\neg(\neg\varphi)) \in \Gamma$, then $\varphi \in \Gamma$,

iv) $\bot \notin \Gamma$

v) There is no propositional symbol p such that p and $(\neg p)$ are both in Γ.

Lemma: Every Hintikka set has a model.

Proof: Let Γ be a Hintikka set and we define σ-structure

$$A(p) = \begin{cases} 1 & \text{if } p \in \Gamma \\ 0 & \text{if } p \notin \Gamma. \end{cases} \qquad (*)$$

Claim: For every $\mathrm{rLP}(\sigma)$ formula φ, we have:

(a) If $\varphi \in \Gamma$ then $A^*(\varphi) = 1$. ($\Rightarrow$ A is a model for Γ)

(b) If $(\neg\varphi) \in \Gamma$ then $A^*(\varphi) = 0$.

Proof of Claim: By induction on complexity of φ.

(Case 0): $\varphi = p \in \sigma$. Then $A^*(p) = A(p)$.

(a) If $\varphi = p \in \Gamma$, then by $(*)$ $A(p) = 1$.

(b) If $(\neg\varphi) = (\neg p) \in \Gamma$, then (V) implies that $p \notin \Gamma$.

By $(*)$ $A(p) = 0$ and hence $A^*(p) = 0$.

__Claim:__ For every rLP(σ) formula φ, we have:

(a) If $\varphi \in \Gamma$ then $A^*(\varphi) = 1$.

(b) If $(\neg\varphi) \in \Gamma$ then $A^*(\varphi) = 0$.

(case 0'): $\varphi = \bot$. By (iv), $\bot \notin \Gamma$. Hence (a) holds.

By def. $A^*(\bot) = 0$. Hence (b) holds.

(case 1): $\varphi = (\neg\psi)$. By induction ψ satisfies (a) and (b).

(a) If $\varphi = (\neg\psi) \in \Gamma$. By (b) $A^*(\psi) = 0$. Hence $A^*(\neg\psi) = 1$.

(b) If $(\neg\varphi) = (\neg(\neg\psi)) \in \Gamma$, then by (iii) $\psi \in \Gamma$. By (a), $A^*(\psi) = 1$.

Hence, $A^*(\underbrace{\neg\psi}_{\varphi}) = 0$.

Claim: For every $\mathrm{rLP}(\sigma)$ formula φ, we have:

(a) If $\varphi \in \Gamma$ then $A^*(\varphi) = 1$. $\Longrightarrow$ A is a model for Γ.

(b) If $(\neg\varphi) \in \Gamma$ then $A^*(\varphi) = 0$.

(case 1): $\varphi = (\neg \psi)$. By induction ψ satisfies (a) and (b).

(a) If $\varphi = (\neg\psi) \in \Gamma$. By (b) $A^*(\psi) = 0$. Hence $A^*(\neg\psi) = 1$.

(b) If $(\neg\varphi) = (\neg(\neg\psi)) \in \Gamma$, then by (iii) $\psi \in \Gamma$. By (a), $A^*(\psi) = 1$.
Hence, $A^*(\underbrace{\neg\psi}_{\varphi}) = 0$.

(Case 2): $\varphi = (\psi \wedge \chi)$. By induction (a) and (b) hold for ψ and χ.

(a) If $\varphi = (\psi \wedge \chi) \in \Gamma$. Then by (i) $\psi \in \Gamma$ and $\chi \in \Gamma$.
Hence $A^*(\psi) = A^*(\chi) = 1$. By def. $A^*(\psi \wedge \chi) = 1$.

(b) If $(\neg\varphi) = (\neg(\psi \wedge \chi)) \in \Gamma$. Then (ii) $(\neg\psi) \in \Gamma$ or $(\neg\chi) \in \Gamma$.
Suppose $(\neg\psi) \in \Gamma$, then by (b) $A^*(\psi) = 0$. Hence $A^*(\underbrace{\psi \wedge \chi}_{\varphi}) = 0$
Suppose $(\neg\chi) \in \Gamma$, ______ " ______

$\square$

Lemma: If Γ is a syntactically consistent set of $rLP(\sigma)$ formulas, it can be extended to a Hintikka set $\Delta \supseteq \Gamma$.

Proof: **Claim:** There is an enumeration of $rLP(\sigma)$ formulas, where each formula occurs infinitely often.

Pf: Since σ is countable we conclude that the set Φ of $rLP(\sigma)$ formulas is countable. (Gödel numbers)

$$\left(\Phi = \bigcup_{n \in \mathbb{N}} \Phi_n \quad , \quad \text{where } \Phi_n = \{ \text{formulas of complexity } n \} \right)$$

$$|\Phi_n| \leq |\underbrace{\sigma \times \cdots \times \sigma}_{n\text{-times}}|$$

Hence there is an enumeration $\psi_0, \psi_1, \psi_2, \ldots$ of Φ.

The list $\psi_0,\ \psi_0, \psi_1,\ \psi_0, \psi_1, \psi_2, \ldots$

has the desired property. $\square$

Let $(\varphi_i)_{i=0}^{\infty}$ be enumeration from the claim.

We build a sequence $\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \dots$ by

$$\Gamma_0 := \Gamma$$

(α) If $\varphi_i = (\chi_1 \wedge \chi_2)$ and $\varphi_i \in \Gamma_i$, then $\Gamma_{i+1} := \Gamma_i \cup \{\chi_1, \chi_2\}$

(β) If $\varphi_i = (\neg(\chi_1 \wedge \chi_2))$ and $\varphi_i \in \Gamma_i$, then

$$\Gamma_{i+1} := \begin{cases} \Gamma_i \cup \{(\neg \chi_1)\} & \text{if } \Gamma_i \cup \{(\neg \chi_1)\} \text{ syntactically consistent,} \\ \Gamma_i \cup \{(\neg \chi_2)\} & \text{otherwise.} \end{cases}$$

(γ) If $\varphi_i = (\neg(\neg \chi))$ and $\varphi_i \in \Gamma_i$, then $\Gamma_{i+1} := \Gamma_i \cup \{\chi\}$.

(δ) Otherwise set $\Gamma_{i+1} := \Gamma_i$.

Let $\Delta := \bigcup_{i \in \mathbb{N}} \Gamma_i$ <u>Claim</u>: Δ is syntactically consistent.

Let $\Delta := \bigcup_{i \in \mathbb{N}} \Gamma_i$ <u>Claim:</u> Δ is syntactically consistent.

<u>Pf:</u> We will show that each Γ_i is syntactically consistent by induction:

(case $i=0$) We know that $\Gamma_0 = \Gamma$ is s.c. by assumption.

(case $i=k+1$) We may assume that Γ_k is s.c.

Suppose that Γ_{k+1} was obtained by rule (α).

(α) If $\varphi_i = (\chi_1 \wedge \chi_2)$ and $\varphi_i \in \Gamma_i$, then $\Gamma_{i+1} := \Gamma_i \cup \{\chi_1, \chi_2\}$

Suppose that Γ_{k+1} is not s.c. Then there is a derivation D of $\perp$ from assumptions in Γ_{k+1}. Since Γ_k is s.c., we must have χ_1 or χ_2 as assumptions in D.

Now replace the assumption χ_1 by $\dfrac{(\chi_1 \wedge \chi_2)}{\chi_1}(\wedge E)$ and χ_2 by $\dfrac{(\chi_1 \wedge \chi_2)}{\chi_2}(\wedge E)$. This produces a derivation D' using only assumptions in Γ_k. ↯.

Suppose that Γ_{k+1} was obtained from rule (β).

(β) If $\varphi_i = (\neg(\chi_1 \wedge \chi_2))$ and $\varphi_i \in \Gamma_i$, then

$$\Gamma_{i+1} := \begin{cases} \Gamma_i \cup \{(\neg \chi_1)\} & \text{if } \Gamma_i \cup \{(\neg \chi_1)\} \text{ syntactically consistent,} \\ \Gamma_i \cup \{(\neg \chi_2)\} & \text{otherwise.} \end{cases}$$

Suppose that Γ_{k+1} is not s.c. Hence $\Gamma_{k+1} = \Gamma_k \cup \{(\neg \chi_2)\}$ and $\Gamma_k \cup \{(\neg \chi_1)\}$ is not s.c. Then

$$\dfrac{\dfrac{\text{(RAA)}\ \dfrac{\begin{matrix}\Gamma_k \cup \{(\neg \cancel{\chi_1})\} \\ D_1 \\ \bot\end{matrix}}{\chi_1} \qquad \dfrac{\begin{matrix}\Gamma_k \cup \{(\neg \cancel{\chi_2})\} \\ D_2 \\ \bot\end{matrix}}{\chi_2}\ \text{(RAA)}}{(\chi_1 \wedge \chi_2)}\ (\wedge I) \qquad (\neg(\chi_1 \wedge \chi_2))}{\bot}\ (\neg I)$$

is a derivation of $\bot$ from assumptions in Γ_k. ↯

Suppose Γ_{k+1} was obtained by

(γ) If $\varphi_i = (\neg(\neg\chi))$ and $\varphi_i \in \Gamma_i$, then $\Gamma_{i+1} := \Gamma_i \cup \{\chi\}$.

Suppose Γ_{k+1} is not s.c. Then there is a derivation of $\perp$ from assumptions in Γ_k plus χ. Replace every assumption χ with the derivation

$$\dfrac{\dfrac{\cancel{(\neg\chi)} \qquad (\neg(\neg\chi))}{\perp}(\neg I)}{\chi}(RAA)$$

The result is a derivation of $\perp$ from Γ_k. $\lightning$.

Finally suppose Γ_{k+1} was obtained using (δ). Then $\Gamma_{k+1} = \Gamma_k$ which is s.c by assumption. $\square$

Suppose $\Delta = \bigcup_{k\in\mathbb{N}} \Gamma_k$ is not s.c. Then $\Phi \vdash \perp$ for some finite subset $\Phi \subseteq \Delta$. Each $\varphi_i \in \Phi$ lies in some Γ_{k_i}, for $k_i \in \mathbb{N}$. Since the Γ_{k_i} are nested, all φ_i lie in $\Gamma_{\max\{k_i\}}$. $\lightning$ $\square$

Claim: Δ is a Hintikka set.

(i) Suppose that $(\varphi \wedge \psi) \in \Delta = \bigcup \Gamma_k$, we conclude that $(\varphi \wedge \psi) \in \Gamma_i$ for $i \in \mathbb{N}$. Since $(\varphi \wedge \psi)$ occurs infinitely often in the enumeration $(\varphi_i)_{i=0}^{\infty}$ we know $\varphi_j = (\varphi \wedge \psi)$ for some $j \geq i$. Since $\Gamma_i \subseteq \Gamma_j$, the conditions for $(\wedge)$ are satisfied for defining Γ_{j+1}. Hence Γ_{j+1} was obtained by rule (α) and $\Gamma_{j+1} = \Gamma_j \cup \{\varphi, \psi\}$. Hence $\varphi, \psi \in \Delta$.

(ii) Suppose that $(\neg(\varphi \wedge \psi)) \in \Delta$. By an analogous argument we find $\Gamma_j \ni (\neg(\varphi \wedge \psi))$ with $\varphi_j = (\neg(\varphi \wedge \psi))$, so

$$\Gamma_{j+1} = \begin{cases} \Gamma_j \cup \{(\neg\varphi)\} \\ \Gamma_j \cup \{(\neg\psi)\} \end{cases}$$

by rule (β). Hence $(\neg\varphi)$ or $(\neg\psi) \in \Delta$.

(iii) Suppose that $(\neg(\neg\varphi)) \in \Delta$. $\xrightarrow{\text{similar}}$ $\varphi \in \Delta$.

Claim: Δ is a Hintikka set.

(iv) Since Δ is s.c. we know $\bot \notin \Delta$.

(v) Moreover for any prop. symbol p, we can't have p and $(\neg p)$ are both elements of Δ, otherwise we could derive $\bot$ by

$$\dfrac{p \qquad (\neg p)}{\bot}\,(\neg I)$$

Since $\Gamma_0 = \Gamma$ and $\Delta = \bigcup \Gamma_k$, we have $\Gamma \subseteq \Delta$. $\square$

Thm: (Completeness) If $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$.

Pf: By lemma 1 it is sufficient to show that every syntactically consistent set Γ has a model.

By lemma 3 every s.c. Γ can be extended to a Hintikka set $\Delta \supseteq \Gamma$.

By lemma 2, every Hintikka set has a model, so Δ has a model. Since $\Gamma \subseteq \Delta$, that same model is also a model for Γ. $\square$

$$\Gamma \models \varphi \iff \Gamma \vdash \varphi.$$

Naive Set Theory

Def: A set is a collection of things regarded as a single object. If A is a set, we write

$x \in A$ if x is an element of A,

$x \notin A$ otherwise.

Two sets A, B are equal if the have the same elements: $\forall x : x \in A \iff x \in B$.

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Two sets A, B are equal if the have the same elements: $\forall x : x \in A \iff x \in B$.

Ex: $A := \{2, 3, 5\}$ $\quad B := \{ \text{Solutions to } x^3 - 10x^2 + 31x - 30 = 0 \}$

Ex:

- Order is unimportant $\{x, y\} = \{y, x\}$
- Multiples are unimportant $\{x, x\} = \{x\}$

$\forall x: \ x \in A \iff x \in B.$

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Ex:
- For every A, $\emptyset \subseteq A$. $C := \{1,2\} \subseteq \{1,2,3\} =: D$
- $\{\emptyset\} \in \{\{\emptyset\}\}$ but $\{\emptyset\} \nsubseteq \{\{\emptyset\}\}$.

Def: Let $\Phi(x)$ be a property. We denote by

$$\{x : \Phi(x)\}$$

the collection of elements x which satisfy $\Phi(x)$.

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Ex: $A := \{1,2,3\}$, $B := \{1,4,9\}$

$A \cup B = \{1,2,3,4,9\}$

$A \cap B = \{1\}$

$A \setminus B = \{2,3\}$.

Def: The powerset of A is the set of its subsets

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Ex: $A := \{1, 2\}$ $\mathcal{P}(A) := \{ \emptyset, \{1\}, \{2\}, \{1,2\} \}$

$2^{|A|}$

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Ex: $A := \{1,2\}$

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$\langle x,y \rangle = \langle x',y' \rangle$ if and only if $x=x'$ and $y=y'$.

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Ex: $A := \{1,2\}$, $B := \{x,y\}$.

$$A\times B = \{\langle 1,x\rangle, \langle 1,y\rangle, \langle 2,x\rangle, \langle 2,y\rangle\}$$

Prop: (commutativity)

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

$$A \cup B = \{x : \underbrace{x \in A \text{ or } x \in B}_{\Phi(x)}\}$$

$$= \{x : \underbrace{x \in B \text{ or } x \in A}_{\Psi(x)}\}$$

$$= B \cup A.$$

Prop:

(commutativity)

$A \cup B = B \cup A$

$A \cap B = B \cap A$

(associativity)

$(A \cup B) \cup C = A \cup (B \cup C)$

$(A \cap B) \cap C = A \cap (B \cap C)$

Prop:

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(associativity)

$$(A \cup B) \cup C = A \cup (B \cup C)$$

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(empty set)

$$A \cup \emptyset = A$$

$$A \cap \emptyset = \emptyset$$

(de Morgan)

$$C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$$

$$C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B)$$

"How to prove it"
Velleman.

Problems: • Consider

$\Phi(x)$

???

$y \cancel{\notin} \in \{ x : \cancel{\text{x a natural number that can be described in 11 words}} \}$

$y :=$ The smallest natural number that can't be described in 11 words.

Problems: • Consider

$$\left\{ x : \begin{array}{l} x \text{ a natural number that can} \\ \text{be described in 11 words} \end{array} \right\}$$

$y :=$ The smallest natural number that can't be described in 11 words.

• (Russel's paradox) Consider

$$A := \{ x : x \notin x \}$$

not a set.

Is $A \overset{???}{\in} A$? If $A \in A$. Then it is not the case that $A \notin A$. Hence $A \notin A$. ↯

If $A \notin A$, then $A \in A$ ↯.

$\{ x \in S : \Phi(x) \}$

Axiomatic Set Theory

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$$X = \{A_1, A_2, A_3\}$$

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7. **Replacement** If a class F is a function and X a set, then $Y = F(X) = \{F(x) : x \in X\}$ is a set.

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7. **Replacement** If a class F is a function and X a set, then $Y = F(X) = \{F(x) : x \in X\}$ is a set.
8. **Regularity** Every nonempty set has an $\in$-minimal element.
9. **Choice** Every family of nonempty sets has a choice function.

The language of sets

We use first order logic with relation symbols $=$, $\in$.

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We write $\varphi(u_1, \ldots, u_n)$ when all free variables of φ occur among $u_1, \ldots, u_n$.

Classes

If $\varphi(x, p_1, \ldots, p_n)$ is a formula, we call

$$C = \{ x : \varphi(x, p_1, \ldots, p_n) \}$$

a class. It's members satify

$$x \in C \iff \varphi(x, p_1, \ldots, p_n).$$

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a class. It's members satify

$$x \in C \iff \varphi(x, p_1, \dots, p_n).$$

Two classes are considered equal if they have the same elements. This is equivalent to their defining formulas being logically equivalent.

The universal class is $V = \{x : x = x\}$.

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We define inclusion $C \subseteq D$ if $\forall x : x \in C \rightarrow x \in D$.

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We define inclusion $C \subseteq D$ if $\forall x : x \in C \to x \in D$.

We define operations on classes C, D:

$$C \cap D := \{x : x \in C \text{ and } x \in D\}$$

$$C \cup D := \{x : x \in C \text{ or } x \in D\}$$

$$C \setminus D := \{x : x \in C \text{ and } x \notin D\}$$

$$\bigcup C := \{x : x \in S \text{ for some } S \in C\}$$

$$\bigcup\{S_1, S_2, \ldots\} = S_1 \cup S_2 \cup \cdots$$

The **universal class** is $V = \{x : x = x\}$.

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$$\bigcup C := \{x : x \in S \text{ for some } S \in C\}$$

Every set S is a class $\{x : x \in S\}$.

A class that is not a set is called **proper class**.

1. Extensionality

If X and Y have the same elements, then $X = Y$.

$$\forall u\,(u \in X \leftrightarrow u \in Y) \rightarrow X = Y$$

The converse also holds by logic.

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If X and Y have the same elements, then $X=Y$.

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2. Pairing

For any sets a and b there is a set $\{a,b\}$ containing exactly a and b.

$$\forall a \forall b \exists c \forall x\,(x \in c \leftrightarrow (x=a \vee x=b))$$

By extensionality the pairing $\{a, b\}$ is unique.

The singleton $\{a\}$ is defined as $\{a\} := \{a, a\}$.

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We can extend ordered pairs to n-tuples by

$$\langle a,b,c \rangle := \langle \langle a,b \rangle, c \rangle$$
$$\langle a,b,c,d \rangle := \langle \langle a,b,c \rangle, d \rangle$$
$$\vdots$$

3. Separation

Let $\varphi(u,p)$ be a formula. For any sets X and p, there is a set $Y = \{u \in X : \varphi(u,p)\}$.

$$\forall X \, \forall p \, \exists Y \, \forall u \, (u \in Y \leftrightarrow (u \in X \wedge \varphi(u,p)))$$

For each formula $\varphi(u,p)$, the above is a axiom.

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For each formula $\varphi(u,p)$, the above is a axiom.

Using n-tuples, the axiom implies separation for formulas $\varphi(u, p_1, \dots, p_n)$ by setting

$$\varphi(u,p) = \exists p_1 \cdots \exists p_n \, (p = \langle p_1, \dots, p_n \rangle \wedge \varphi(u, p_1, \dots, p_n))$$

Let $C = \{u : \varphi(u, p_1, \ldots, p_n)\}$ be a class. Then for any set X, $C \cap X = \{u \in X : \varphi(u, p_1, \ldots, p_n)\}$ is a set. Hence subclasses of sets are sets.

$$\underset{\text{class}}{C} \subseteq \underset{\text{set}}{X} \rightsquigarrow \underbrace{C \cap X}_{\text{set}} = \underset{\text{set}}{C}$$

Let $C = \{u : \varphi(u, p_1, \ldots, p_n\}$ be a class. Then for any set X, $C \cap X = \{u \in X : \varphi(u, p_1, \ldots, p_n)\}$ is a set. Hence subclasses of sets are sets.

Since $X \cap Y \subseteq Y$ and $X - Y \subseteq X$, the operations work on sets.

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Since $X \cap Y \subseteq Y$ and $X - Y \subseteq X$, the operations work on sets.

Provided that at least one set X exists,

$$\emptyset := \{u \in X : u \neq u\}$$

defines the empty set.

Two sets X, Y are called disjoint if $X \cap Y = \emptyset$.

If C is a nonempty class of sets, we define

$$\bigcap C := \bigcap \{X : X \in C\} = \{u : u \in X \text{ for each } X \in C\}.$$

4. Union

For any set X, there is a set $Y = \bigcup X$.

$$\forall X \exists Y \forall u (u \in Y \leftrightarrow \exists z (z \in X \wedge u \in z))$$

4. Union

$\bigcup\{S_1, S_2, \dots\}$

For any set X, there is a set $Y = \bigcup X$.

$$\forall X \exists Y \forall u (u \in Y \leftrightarrow \exists z (z \in X \wedge u \in z))$$

Define $\quad X \cup Y := \bigcup\{X, Y\}$

and $\quad \{a_1, \dots, a_n\} := \{a_1\} \cup \dots \cup \{a_n\}$.

5. Power set

For any set X there is a set $Y = P(X)$.

$$\forall X \, \exists Y \, \forall u \, (u \in Y \leftrightarrow u \subseteq X)$$

where $u \subseteq X \; :\Longleftrightarrow \; \forall z \, (z \in u \rightarrow z \in X)$

The set $P(X) = \{ u : u \subseteq X \}$ is called power set.

We can now define products of sets X, Y by

$$X \times Y = \{ \langle x, y \rangle : x \in X \text{ and } y \in Y \}$$

$$= \{ u : \exists x \exists y \, (u = \langle x, y \rangle \wedge x \in X \wedge y \in Y) \}$$

Since $X \times Y \subseteq PP(X \cup Y)$ this defines a set.

$$\begin{array}{c} \rotatebox{90}{$\in$} \\ \langle x, y \rangle \\ \rotatebox{90}{$=$} \end{array}$$

$$PP(X \cup Y) \ni \{ \underbrace{\{x\}}_{\in P(X) \subseteq P(X \cup Y)}, \underbrace{\{x, y\}}_{\in P(X \cup Y)} \} \subseteq P(X \cup Y)$$

We can now define products of sets X, Y by

$$X \times Y = \{ \langle x, y \rangle : x \in X \text{ and } y \in Y \}$$
$$= \{ u : \exists x \exists y \, (u = \langle x, y \rangle \wedge x \in X \wedge y \in Y) \}$$

Since $X \times Y \subseteq PP(X \cup Y)$ this defines a set.

We extend by $X_1 \times \dots \times X_{n+1} := (X_1 \times \dots \times X_n) \times X_{n+1}$

and write $X^n := \underbrace{X \times \dots \times X}_{n\text{-times}}$.

An *n-ary relation* R is a set of n-tuples.

If R is a binary relation, we write

$$xRy \;:\Longleftrightarrow\; \langle x, y \rangle \in R.$$

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Define

$$\text{(domain)} \qquad \mathrm{dom}(R) := \{u : \exists v : uRv\}$$

$$\text{(range)} \qquad \mathrm{ran}(R) := \{v : \exists u : uRv\}.$$

These are sets since both are subsets of $\bigcup\bigcup R$.

$$R = \{\underset{\{\{x\},\{x,y\}\}}{\underset{\shortparallel}{\langle x, y\rangle}}, \overset{\langle u,v\rangle}{\text{—}}\}$$

$$\bigcup R = \{\{x\}, \{x,y\}, \overset{\{u\},\{u,v\}}{\ldots}\}$$

$$\bigcup\bigcup R = \{x, y, \overset{u,v}{\ldots}\}$$

A binary relation f is a function if

$$\forall x \forall y \forall z \, (x f y \wedge x f z \rightarrow y = z)$$

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All functions $f : X \rightarrow Y$ form a set $Y^X \subseteq P(X \times Y)$.

6. Infinity

There exists an ~~infinite~~ inductive set.

$$\exists S\,(\underbrace{\phi}_{\{u\in S\,:\,u\neq u\}} \in S \wedge (\forall x \in S)\; x \cup \{x\} \in S).$$

$$S = \{\phi, \{\phi\}, \{\phi, \{\phi\}\}, \ldots\}$$

$$\phi \cup \{\phi\} = \{\phi\}$$

$$\{\phi\} \cup \{\{\phi\}\} = \{\phi, \{\phi\}\}$$

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7. Replacement

If a class F is a function and X a set, then the image $F(X)$ is a set. $F_p = \{ \langle x, y \rangle : \varphi(x, y, p) \}$

$$\forall p \left[\begin{array}{l} \forall x \forall y \forall z \, (\varphi(x, y, p) \wedge \varphi(x, z, p) \rightarrow y = z) \\ \quad \longrightarrow \forall X \exists Y \forall y \, (y \in Y \leftrightarrow (\exists x \in X) \, \varphi(x, y, p)) \end{array} \right.$$

8. Regularity

Every nonempty set has an $\in$-minimal element.

$$\forall S \left(S \neq \emptyset \rightarrow (\exists x \in S) \; S \cap x = \emptyset \right)$$

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$$\forall S \left(S \neq \emptyset \to (\exists x \in S)\ S \cap x = \emptyset \right)$$

Considering $S = \{x_0, x_1, \ldots\}$ there is no infinite sequence

$$x_0 \ni x_1 \ni x_2 \ni \cdots$$

(Otherwise $\forall x \in S\ \exists y \in S : y \in x$ implies $\forall x \in S : S \cap x \neq \emptyset$.)

$x_n \in S$

$$\left.\begin{matrix} x_{n+1} \in x_n \\ x_{n+1} \in S \end{matrix}\right\} S \cap x_n \neq \emptyset.$$

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$x \ni x \ni x \ni \cdots$

In particular there are no sets $X \in X$

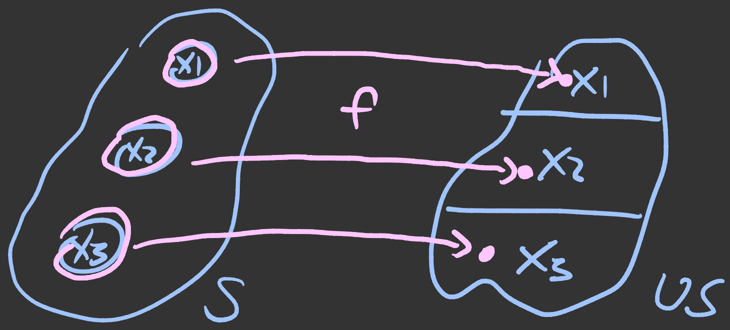
and no cycles $x_0 \in x_1 \in \cdots \in x_n \in x_0$.

9. Choice

Every family of nonempty sets has a choice function.

If S is a family of sets and $\emptyset \notin S$, then a choice function for S, is a function

$$f: S \to \bigcup S \text{ where } f(X) \in X \text{ for all } X \in S.$$



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The axiom of choice is equivalent to

Every cartesian product of nonempty sets is nonempty.

Zorn's Lemma. $\overset{0}{\bullet} \leq \overset{1}{\bullet} \leq \overset{2}{\bullet} \leq \overset{3}{\bullet} \cdots$

Every set can be well-ordered. $\mathbb{R}$