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Logic and Foundation with Haskell

Exercise sheet 11

In lecture, we defined the natural numbers ω in ZFC as consisting of all sets that are contained in every inductive set.

Definition 1. A set A is *inductive*, if it contains $\emptyset$, and if $a \in A$ implies $a^+ := a \cup \{a\} \in A$.

Definition 2. The *natural numbers* ω consist of sets n that are contained in every inductive set.

Furthermore, recall that a *Peano system* $\langle N, s, e\rangle$ consists of a set N, a function $s : N \to N$, and a designated element $e \in N$ that satisfies the Peano axioms:

(i) $e \notin \mathrm{ran}(s)$
(ii) s is injective,
(iii) any subset $A \subseteq N$ that contains e and is closed under s is equal to N.

In this sheet, we will show that the naturals numbers ω are a Peano system. For this, it is helpful to introduce the following definition:

Definition 3. A set A is called *transitive* if it satisfies the following condition:

$$\forall a \in A\, \forall x \in a\, (x \in a \in A \;\to\; x \in A).$$

Exercise 1. Show that the following conditions are equivalent to being a transitive set:

(i) $\bigcup A \subseteq A$,
(ii) $\forall a \in A\, (a \in A \;\to\; a \subseteq A)$,
(iii) $A \subseteq \mathcal{P}(A)$.

Exercise 2. Show that for any transitive set $\bigcup(a^+) = a$.

Exercise 3. Show that every natural number $n \in \omega$ is a transitive set.

Exercise 4 (Bonus)**.** Show that the natural numbers ω form a transitive set.

Exercise 5. Prove that $\langle \omega, (-)^+, \emptyset\rangle$ form a Peano system.
Hint: Use Exercises 2 and 3 to prove injectivity of the successor operation $(-)^+$.