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Logic and Foundation with Haskell

Exercise sheet 4

Exercise 1. Prove the following sequents involving conjunction:
(i) $\{\varphi\} \vdash (\varphi \wedge \varphi)$,
(ii) $\{\varphi, \psi, \chi\} \vdash (\varphi \wedge (\psi \wedge \chi))$.

Exercise 2. Show that $\{\varphi_1, \varphi_2\} \vdash \psi$ if and only if $\{\varphi_1 \wedge \varphi_2\} \vdash \psi$. Hence, we can view the set of assumptions as being a big conjunction.

Exercise 3. Show the following sequents involving implication:
(i) $\vdash (\varphi \to (\psi \to \psi))$,
(ii) $\vdash ((\varphi \to \varphi) \wedge (\psi \to \psi))$,
(iii) $\{(\varphi \to \psi), (\varphi \to \chi)\} \vdash (\varphi \to (\psi \wedge \chi))$.

Exercise 4. Show that $\Gamma \cup \{\varphi\} \vdash \psi$ if and only if $\Gamma \vdash (\varphi \to \psi)$. Hence implication 'internalizes' $\vdash$.

Exercise 5. Write down sequent rules ($\leftrightarrow$ I) and ($\leftrightarrow$ E) for equivalence.

Exercise 6. Prove the following sequents involving equivalence:
(i) $\{\varphi, (\varphi \leftrightarrow \psi)\} \vdash \psi$,
(ii) $\vdash (\varphi \leftrightarrow \varphi)$,
(iii) $\{\varphi \leftrightarrow (\psi \leftrightarrow \psi)\} \vdash \varphi$.

Exercise 7. Show that the relation $\varphi \sim \psi$ defined by $\vdash (\varphi \leftrightarrow \psi)$ is an equivalence relation.

Exercise 8. Prove the following sequents without (RAA):
(i) $\vdash (\neg(\varphi \wedge (\neg\varphi)))$,
(ii) $\vdash ((\neg(\varphi \to \psi)) \to (\neg\psi))$,
(iii) $\{(\varphi \to \psi)\} \vdash ((\neg\psi) \to (\neg\varphi))$,
(iv) $\{(\varphi \to \psi)\} \vdash (\neg(\varphi \wedge (\neg\psi)))$.

Exercise 9. Show using (RAA) that $\{((\neg\psi) \to (\neg\varphi))\} \vdash (\varphi \to \psi)$.

Exercise 10. Prove the following sequents without ($\vee$E):
(i) $\vdash (\varphi \to (\varphi \vee \psi))$,
(ii) $\{(\neg(\varphi \vee \psi))\} \vdash ((\neg\varphi) \wedge (\neg\psi))$,
(iii) $\vdash ((\varphi \to \psi) \to ((\neg\varphi) \vee \psi))$.

Exercise 11. Prove the following sequents with ($\vee$E):
(i) $\{(\varphi \vee \psi)\} \vdash (\psi \vee \varphi)$,
(ii) $\{(\varphi \vee \psi), (\varphi \to \chi), (\psi \to \chi)\} \vdash \chi$,
(iii) $\{(\varphi \vee \psi), (\neg\varphi)\} \vdash \psi$,
(iv) $\{((\neg\varphi) \wedge (\neg\psi))\} \vdash (\neg(\varphi \vee \psi))$.