MAT605 Exam

Logic and Foundations with Haskell

June 1st, 2023

Full Name : ______________________________

Student No.: ______________________________

Instructions

(i) Fill in your info on the lines provided above and on each new page.

(ii) Clearly mark your answers by crossing the corresponding checkboxes.

(iii) Each question may have multiple correct answers (except those marked *single choice*).

(iv) Points are awarded for each correctly checked / unchecked box.

(v) If a question seems unclear, you can write a justification for your answers.

Question:	1	2	3	4	5	6	7	8	9
Points:	1	1	1	1	2	2	1	1½	1
Score:									

Question:	10	11	12	13	14	15	16	17	18
Points:	1	1	1	1	1	1	1	2	1
Score:									

Question:	19	20	21	22	23	24	25	26	27
Points:	2	1	½	1	½	1	1	1	1
Score:									

Question:	28	29	30	31	32	33	34	35	Total
Points:	1	2	4	1½	2	2	1	2	46
Score:									

Essential theory questions

Propositional Logic Let ϕ and ψ be formulas of propositional logic.

1. (1 point) Suppose $p_0 = \texttt{True}$, $p_1 = \texttt{False}$ and $p_2 = \texttt{False}$. What is the truth value of the following propositional logic formulas (single choice):

   (a) $((\neg p_0) \vee (p_1 \rightarrow p_2))$ **A.** `True` B. `False`

   (b) $(p_0 \wedge ((\neg p_1) \leftrightarrow p_2))$ A. `True` **B.** `False`

2. (1 point) Write down the parsing tree for $(p_0 \wedge (\neg p_1))$:

   Solution: Start with p_0 and p_1 as leaves and apply $\neg$ to p_1. Then combine p_0 and $(\neg p_1)$ using $\wedge$.

3. (1 point) Which of the following statements are correct?

   - ☐ A formula is a *tautology / logical validity* if it is true for some assignment of truth values to atomic propositions.
   - ■ **A formula is a *tautology / logical validity* if it is true for any assignment of truth values to atomic propositions.**
   - ☐ If ϕ is a tautology, then $(\phi \rightarrow \psi)$ is a tautology.
   - ■ **If ϕ is a tautology, then $(\psi \rightarrow \phi)$ is a tautology.**

4. (1 point) Which of the following statements are correct?

   - ■ **ϕ and ψ are *logically equivalent* if $(\phi \leftrightarrow \psi)$ is a tautology.**
   - ☐ ϕ and ψ are *logically equivalent* if they contain the same atomic propositions.
   - ☐ ϕ and ψ are *logically equivalent* if whenever ϕ is true, then ψ is true.
   - ■ **ϕ and ψ are *logically equivalent* if the have the same truth value for any assignment of truth values to atomic propositions.**

First Order Logic

5. (2 points) Which of the following first order logic formulas translate the following statements? We allow for restricted quantification.

   (a) There is a smallest natural number n.

   - ☐ $\exists n \in \mathbb{N}\, \exists m \in \mathbb{N} : n \leq m$
   - ■ $\exists n \in \mathbb{N}\, \forall m \in \mathbb{N} : n \leq m$
   - ☐ $\forall m \in \mathbb{N}\, \exists n \in \mathbb{N} : n \leq m$
   - ☐ $\forall m \in \mathbb{N}\, \forall n \in \mathbb{N} : n \leq m$

   (b) For each natural number n there is a natural number m which is bigger than n.

   - ■ $\forall n\, \exists m : (n \in \mathbb{N} \to (m \in \mathbb{N} \land n < m))$
   - ■ $\forall n \in \mathbb{N}\, \exists m \in \mathbb{N} : n < m$
   - ☐ $\forall n\, \exists m : (n \in \mathbb{N} \land (m \in \mathbb{N} \land n < m))$
   - ☐ $\forall n\, \exists m : (n \in \mathbb{N} \to (m \in \mathbb{N} \to n < m))$

Informal Proof Theory

6. (2 points) Which of the following are valid proof rules for statements Φ and Ψ:

   - ☐ To prove $\Phi \lor \Psi$, it is necessary to prove both Φ and $\neg\Psi$.
   - ■ **To prove $\Phi \land \Psi$, it is necessary to prove both Φ and Ψ.**
   - ■ **To prove $\Phi \to \Psi$, it is sufficient to assume Φ and prove Ψ.**
   - ☐ Given $\neg\Phi$, one can conclude Ψ.
   - ☐ To prove $\forall x \Phi(x)$, prove $\Phi(x)$ for some x of your choice.
   - ■ **Given $\neg\Phi$ and Φ, one can conclude Ψ.**
   - ■ **Given $\Phi \leftrightarrow \Psi$ and Ψ, one can conclude Φ**
   - ☐ Given $\exists\Phi(x)$ and any x, one can conclude $\Phi(x)$.

Natural Deduction

7. (1 point) Which of the following statements are correct?

   - ■ **A *derivation* is a formal proof of a conclusion ϕ that uses the natural deduction proof rules.**
   - ☐ A *derivation* of ϕ is a sequence of sequents that proves a formula ϕ.
   - ■ **A *sequent* is a formal expression $\Gamma \vdash \phi$ saying that there is a derivation of ϕ from assumptions in Γ.**
   - ☐ A *sequent* is a formal proof $\Gamma \vdash \phi$ of a conclusion ϕ from assumptions in Γ that uses the natural deduction sequent rules.

8. (1 ½ points) Which of the following are correct natural deduction proofs?

☐ $\dfrac{\phi \qquad \cancel{\psi}}{(\phi \to \psi)}$ ■ $\dfrac{\phi \qquad \psi}{(\phi \wedge \psi)}$ ■ $\dfrac{\psi}{(\phi \vee \psi)}$ ■ $\dfrac{\dfrac{\cancel{(\neg\phi)} \qquad \phi}{\bot}}{\neg(\neg\phi)}$ ☐ $\dfrac{(\phi \vee \psi) \qquad (\neg\psi)}{\phi}$

9. (1 point) Consider the introduction rule for $\leftrightarrow$:

$$\dfrac{\dfrac{\dfrac{\Gamma}{\vdots}}{(\phi \to \psi)} \qquad \dfrac{\dfrac{\Delta}{\vdots}}{(\psi \to \phi)}}{(\phi \leftrightarrow \psi)}$$

Which of the following sequent rules corresponds to the above (single choice)?

A. If $\Gamma \vdash (\phi \to \psi)$ and $\Delta \vdash (\psi \to \phi)$, then $\Gamma \vdash (\phi \leftrightarrow \psi)$.

B. If $\Gamma \vdash (\phi \to \psi)$ and $\Delta \vdash (\psi \to \phi)$, then $\Gamma \cup \Delta \vdash (\phi \leftrightarrow \psi)$.

C. If $\Gamma \cup \Delta \vdash (\phi \leftrightarrow \psi)$, then $\Gamma \vdash (\phi \to \psi)$ and $\Delta \vdash (\psi \to \phi)$.

D. If $\Gamma \vdash (\phi \to \psi)$ and $\Delta \vdash (\psi \to \phi)$, then $\Gamma \vdash (\phi \leftrightarrow \psi)$ or $\Delta \vdash (\phi \leftrightarrow \psi)$

Formal Propositional Logic

10. (1 point) Which of the following are formulas of $\mathsf{LP}(\sigma)$ for $\sigma := \{p_i : i \in \mathbb{N}\}$?

■ $((p_0 \wedge \bot) \to p_{55})$ ☐ $(q_1 \wedge p_1)$ ■ $((\neg p_{2023}) \leftrightarrow (p_1 \vee p_{40}))$ ☐ $\exists p_0 : (p_0 \wedge p_1)$

11. (1 point) What does the unique parsing theorem say (single choice)?

A. Any formula ϕ can be expressed as $\bot$, p, $(\neg\psi)$, or $(\psi \diamondsuit \chi)$, where ψ and χ are formulas, $p \in \sigma$ and propositional symbol, and $\diamondsuit$ is a binary logical connective.

B. Every formula ϕ given by a parsing tree is uniquely determined.

C. Any formula ϕ has exactly one of the following forms: $\bot$, p, $(\neg\psi)$, or $(\psi \diamondsuit \chi)$, where ψ and χ are formulas, $p \in \sigma$ and propositional symbol, and $\diamondsuit$ is a binary logical connective.

D. Every formula ϕ can be parsed into a parsing tree.

12. (1 point) Which of the following statements are correct?

■ **A *σ-structure* is a function $\sigma \to \{0, 1\}$.**

☐ A *σ-structure* is a set of propositional symbols that generate all true $\mathsf{LP}(\sigma)$ formulas.

■ **Any *σ-structure* A can be extended to a function A^* that assigns each $\mathsf{LP}(\sigma)$ formula a truth value in $\{0, 1\}$.**

☐ Any *σ-structure* A can be extended to a function A^* that assigns each $\mathsf{LP}(\sigma)$ formula the value 1.

13. (1 point) Let A be a σ-structure. Which of the following statements are true?
    - ■ **If $A^*(\neg\phi) = 1$, then $A^*(\phi) = 0$.**
    - ☐ If $A^*(\phi \vee \psi) = 1$, then $A^*(\phi) = 1$.
    - ■ **If $A^*(\phi \to \psi) = 0$, then $A^*(\phi) = 1$.**
    - ■ **If $A^*(\phi \to \psi) = 0$, then $A^*(\psi) = 0$.**

14. (1 point) Which of the following statements are correct?
    - ■ **A σ-structure A is a model of ϕ if $A^*(\phi) = 1$.**
    - ■ **$\Gamma \models \phi$ if every model of Γ is a model of ϕ.**
    - ☐ A σ-structure A is a model of ϕ if $A(p) = 1$ for each atomic proposition p occurring in ϕ.
    - ☐ $\Gamma \models \phi$ if each formula $\gamma \in \Gamma$ has a model A which is also a model for ϕ.

15. (1 point) Which of the following statements are correct?
    - ☐ *Soundness* means that $\Gamma \models \phi$ implies $\Gamma \vdash \phi$.
    - ■ ***Soundness* means that $\Gamma \vdash \phi$ implies $\Gamma \models \phi$.**
    - ■ ***Completeness* means that $\Gamma \models \phi$ implies $\Gamma \vdash \phi$.**
    - ☐ *Completeness* means that $\Gamma \vdash \phi$ implies $\Gamma \models \phi$.

Set Theory

16. (1 point) Which of the following are valid set-theoretic identities?
    - ☐ $A \setminus B = B \setminus A$
    - ■ $A \cup (B \cap \emptyset) = A$
    - ☐ $A \cap (B \cup C) = (A \cup B) \cap (A \cup C)$
    - ■ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

17. (2 points) Which of the following are axioms of ZFC?
    - ☐ Every nonempty family of sets has an $\subseteq$-minimal element.
    - ☐ Every nonempty family of sets has a choice function.
    - ■ **For any set X, there is a set $\bigcup X$.**
    - ☐ For any set X, there is a set $\bigcap X$.
    - ☐ For any set p and any formula $\Phi(x, p)$, there is a set $\{x : \Phi(x, p)\}$.
    - ■ **For any sets X, Y, there is a set $\{X, Y\}$.**
    - ☐ If a class F is a function and X is another class, then $F(X)$ is a set.
    - ■ **For any sets X, p and any formula $\Phi(x, p)$, there is a set $\{x \in X : \Phi(x, p)\}$.**

18. (1 point) Which of the following are consequences of the regularity axiom?

    ☐ There is no infinite sequence of sets $X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots$.

    ■ **There is no set X satisfying $X \in X$.**

    ■ **There is no infinite sequence of sets $X_0 \ni X_1 \ni X_2 \ni \ldots$.**

    ☐ There is no infinite sequence of sets $X_0 \in X_1 \in X_2 \in \ldots$.

Essential Haskell questions

19. (2 points) Which of the following commands produce the list `[0,1,2,3]`?

    ☐ `[0]:[1,2,3]`

    ■ `0:[1,2,3]`

    ■ `[0..3]`

    ☐ `[x | x in N, x < 4]`

    ☐ `[0,1,2] ++ 3`

    ■ `[x | x <- [0..100], x < 4]`

    ■ `take 4 [0..]`

    ☐ `[0,1,2] ++ [2,3]`

20. (1 point) Which of the following commands return `4`?

    ■ `head [x^2 | x <- [2,4,6]]`

    ☐ `(1 :: Int) + (2 :: Integer)`

    ■ `(+) 1 3`

    ☐ `1 (+) 3`

21. (½ point) What is the correct way to define a pair (single choice)?

    A. `pair 1 2`

    B. `[1,2]`

    C. `(1,2)`

    D. `{1,2}`

22. (1 point) What is the type signature of `max` (single choice)?

    A. `max :: Eq a => a -> a -> a`

    B. `max :: Ord a => a -> a -> a`

    C. `max :: Ord a => (a -> a) -> a`

    D. `max :: (Ord a, Ord b) => a -> b -> a`

23. (½ point) What typeclass allows you to convert its members to strings (single choice)?

    A. `String` B. `Print` C. `Display` **D.** `Show`

24. (1 point) Which of the following are correct signatures for `f x y = x + y` ?

    ☐ `f :: Int -> Integer -> Int`

    ■ `f :: Num a => a -> a -> a`

    ☐ `f :: (Num a, Num b) => a -> b -> a`

    ■ `f :: Int -> Int -> Int`

25. (1 point) Which of the following are correct signatures for `f (x,y) = x + y` ?

    ☐ `f :: Num a => (a -> a) -> a`

    ■ `f :: Num a => (a,a) -> a`

    ☐ `f :: Num a => a -> a -> a`

    ☐ `f :: Num (a,a) => a -> a`

26. (1 point) Which of theses patterns will match every list with at least two elements?
    ☐ `x:[y,xs]` ☐ `[x,y] ++ xs` ■ `(x:y:xs)` ☐ `[x,y,xs]`

27. (1 point) Which of the following functions are syntactically correct?

☐
```
f x y | x = y = True
      | otherwise = False
```

☐
```
f x y = | x = y = True
        | otherwise = False
```

■
```
f x y | x == y = True
      | otherwise = False
```

☐
```
f x y = | x == y = True
        | otherwise = False
```

28. (1 point) Which of the following expressions return `[1,4,9]`?

    ■ `map (^2) [1,2,3]`

    ☐ `all (^2) [1,2,3]`

    ☐ ``filter (\x -> x `in` [1,4,9]) [1..10]``

    ■ ``filter (\x -> x `elem` [1,4,9]) [1..10]``

29. (2 points) Which of the following expressions return `True`?

    ☐ `if False then True else False`

    ■ `if True then True else False`

    ■ `and [True, True]`

    ☐ `or []`

    ☐ `any (<3) [4,5]`

    ■ `and []`

    ■ `all (<3) [0,1]`

    ☐ `filter (== True) [True, False]`

Advanced questions

30. (4 points) Prove the following using natural deduction. Indicate what rules you are using in each step.

    (a) $\vdash (\phi \to (\phi \vee \psi))$

    Solution:

$$\dfrac{\dfrac{\cancel{\phi}}{(\phi \vee \psi)}\,(\vee I)}{(\phi \to (\phi \vee \psi))}\,(\to I)$$

    (b) $\{(\neg(\phi \to \psi))\} \vdash (\neg\psi)$

    Solution:

$$\dfrac{\dfrac{(\to I)\;\dfrac{\cancel{\psi}}{(\phi \vee \psi)} \qquad (\neg(\phi \to \psi))}{\perp}\,(\neg I)}{(\neg\psi)}\,(\neg E)$$

31. (1 ½ points) Consider the following (fictitious) natural deduction rule:

$$\dfrac{\dfrac{\dfrac{\Gamma}{\vdots}}{(\phi \to \psi)} \qquad \dfrac{\dfrac{\Delta}{\vdots}}{(\psi \to \chi)}}{(\phi \to \chi)}$$

Write down the corresponding sequent rule:

Solution: If $\Gamma \vdash (\phi \to \psi)$ and $\Delta \vdash (\psi \to \chi)$ then $\Gamma \cup \Delta \vdash (\phi \to \chi)$.

32. (2 points) Argue from scratch using the ZFC axioms that the symmetric difference $X \triangle Y := (X \setminus Y) \cup (Y \setminus X)$ of two sets X, Y is a set:

Solution: Let X, Y be sets. By separation $X \setminus Y := \{x \in X : x \notin Y\}$ is a set. Similarly, $Y \setminus X$ is a set. By pairing $\{(X \setminus Y), (Y \setminus X)\}$ is a set. Finally, by union $\bigcup\{(X \setminus Y), (Y \setminus X)\} = (X \setminus X) \cup (Y \setminus X)$ is a set.

33. (2 points) Let $R \subseteq X \times X$ be relation on X. Which of the following are true?

- ☐ If R is reflexive, then $R \circ R \subseteq R$.
- ■ **If R is reflexive, then $R \subseteq R \circ R$.**
- ■ **If R is transitive, then $R \circ R \subseteq R$.**
- ☐ If R is transitive, then $R \subseteq R \circ R$.

34. (1 point) Which of the following expressions returns 3 (single choice)?

A. `foldr (\x acc -> acc) 0 [1,2,3]`

B. `foldr (\x acc -> x) 0 [1,2,3]`

C. `foldl (\acc x -> x) 0 [1,2,3]`

D. `foldl (\acc x -> acc) 0 [1,2,3]`

35. (2 points) Write a safe division function using `Maybe` (without type signature).

Solution:

```
safeDiv x 0 = Nothing
safeDiv x y = Just (x / y)
```