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Logic and Foundation with Haskell

Exercise sheet 4

Exercise 1. Prove the following sequents involving conjunction:
(i) $\{\varphi\} \vdash (\varphi \wedge \varphi)$,
(ii) $\{\varphi, \psi, \chi\} \vdash (\varphi \wedge (\psi \wedge \chi))$.

Solution: (i): $\dfrac{\varphi \qquad \varphi}{(\varphi \wedge \varphi)}\,(\wedge\mathrm{I})$ (ii): $\dfrac{\varphi \qquad \dfrac{\psi \qquad \chi}{(\psi \wedge \chi)}\,(\wedge\mathrm{I})}{(\varphi \wedge (\psi \wedge \chi))}\,(\wedge\mathrm{I})$

Exercise 2. Show that $\{\varphi_1, \varphi_2\} \vdash \psi$ if and only if $\{(\varphi_1 \wedge \varphi_2)\} \vdash \psi$. Hence, we can view the set of assumptions as being a big conjunction.

Solution: Suppose that $\{\varphi_1, \varphi_2\} \vdash \psi$. Then there is a natural deduction derivation D with premises included in $\{\varphi_1, \varphi_2\}$, and conclusion ψ. By ($\wedge$E), we have derivations

$$\frac{(\varphi_1 \wedge \varphi_2)}{\varphi_1} \qquad\qquad \frac{(\varphi_1 \wedge \varphi_2)}{\varphi_2}$$

which we can append at the beginning of D to get a derivation witnessing $\{(\varphi_1 \wedge \varphi_2)\} \vdash \psi$.

Conversely, suppose that $\{(\varphi_1 \wedge \varphi_2)\} \vdash \psi$. Then there is a derivation D' that proves ψ from premises in $\{(\varphi_1 \wedge \varphi_2)\}$. We can append the introduction rule ($\wedge$I)

$$\frac{\varphi \qquad \psi}{(\varphi \wedge \psi)}$$

to the beginning of D' to get a derivation witnessing $\{\varphi_1, \varphi_2\} \vdash \psi$.

Exercise 3. Show the following sequents involving implication:
(i) $\vdash (\varphi \to (\psi \to \psi))$,
(ii) $\vdash ((\varphi \to \varphi) \wedge (\psi \to \psi))$,
(iii) $\{(\varphi \to \psi), (\varphi \to \chi)\} \vdash (\varphi \to (\psi \wedge \chi))$.

Solution: Note that $\dfrac{\cancel{\psi}}{(\psi \to \psi)}$ is a valid application of the ($\to$I) rule. Hence $\vdash (\psi \to \psi)$ for any formula ψ. Moreover, whenever $\vdash \psi$, then also $\vdash (\varphi \to \psi)$ for any φ by

$$\dfrac{\dfrac{\cdots}{\psi}}{(\varphi \to \psi)}\,(\to\mathrm{I})$$

where we use the version that does not cancel any premises.

(i): By the previous arguments $\vdash (\psi \to \psi)$, and hence also $\vdash (\varphi \to (\psi \to \psi))$. Explicitly, one can also write down the following derivation

$$\dfrac{\dfrac{\cancel{\psi}}{(\psi \to \psi)}}{(\varphi \to (\psi \to \psi))}$$

(ii): Since both $\vdash (\varphi \to \varphi)$ and $\vdash (\psi \to \psi)$ hold, we conclude $\vdash ((\varphi \to \varphi) \wedge (\psi \to \psi))$ by the sequent rule for $(\wedge\mathrm{I})$.

(iii): The sequent is proved by the following derivation

$$\dfrac{\dfrac{{\scriptstyle(\to\mathrm{E})}\ \dfrac{\cancel{\varphi} \qquad (\varphi \to \psi)}{\psi} \qquad \dfrac{\cancel{\varphi} \qquad (\varphi \to \chi)}{\chi}\ {\scriptstyle(\to\mathrm{E})}}{(\psi \wedge \chi)}\ {\scriptstyle(\wedge\mathrm{I})}}{(\varphi \to (\psi \wedge \chi))}\ {\scriptstyle(\to\mathrm{I})}$$

Exercise 4. Show that $\Gamma \cup \{\varphi\} \vdash \psi$ if and only if $\Gamma \vdash (\varphi \to \psi)$. Hence implication 'internalizes' $\vdash$.

Solution: Suppose that $\Gamma \cup \{\varphi\} \vdash \psi$. Then there is a derivation D of ψ from premises in $\Gamma \cup \{\varphi\}$. We can apply the $(\to\mathrm{I})$ rule to the end of D, while canceling φ to get a derivation of $(\varphi \to \psi)$ from premises in Γ. Conversely, suppose that $\Gamma \vdash (\varphi \to \psi)$. Then there is a derivation D' of $(\varphi \to \psi)$ from premises in Γ. Then the following derivation witnesses $\Gamma \cup \{\varphi\} \vdash \psi$:

$$\dfrac{\varphi \qquad \dfrac{\dfrac{\Gamma}{D'}}{(\varphi \to \psi)}}{\psi}\ {\scriptstyle(\to\mathrm{E})}$$

Exercise 5. Write down sequent rules $(\leftrightarrow \mathrm{I})$ and $(\leftrightarrow \mathrm{E})$ for equivalence.

Solution:

$(\leftrightarrow\mathrm{I})$: If $\Gamma \vdash (\varphi \to \psi)$ and $\Delta \vdash (\psi \to \varphi)$, then $\Gamma \cup \Delta \vdash (\varphi \leftrightarrow \psi)$.

$(\leftrightarrow\mathrm{E})$: If $\Gamma \vdash (\varphi \leftrightarrow \psi)$, then $\Gamma \vdash (\varphi \to \psi)$ and $\Gamma \vdash (\psi \to \varphi)$.

Exercise 6. Prove the following sequents involving equivalence:

(i) $\{\varphi, (\varphi \leftrightarrow \psi)\} \vdash \psi$,

(ii) $\vdash (\varphi \leftrightarrow \varphi)$,

(iii) $\{(\varphi \leftrightarrow (\psi \leftrightarrow \psi))\} \vdash \varphi$.

Solution: (i):

$$\dfrac{\varphi \qquad \dfrac{(\varphi \leftrightarrow \psi)}{\varphi \to \psi}\ {\scriptstyle(\leftrightarrow\mathrm{E})}}{\psi}\ {\scriptstyle(\to\mathrm{E})}$$

(ii):

$$\dfrac{\dfrac{\cancel{\psi}}{(\psi \to \psi)} \qquad \dfrac{\cancel{\psi}}{(\psi \to \psi)}}{(\psi \leftrightarrow \psi)}\ {\scriptstyle(\leftrightarrow\mathrm{I})}$$

(iii): $$\dfrac{\dfrac{D}{(\psi \leftrightarrow \psi)} \qquad \dfrac{(\varphi \leftrightarrow (\psi \leftrightarrow \psi))}{((\psi \leftrightarrow \psi) \to \varphi)}\,(\leftrightarrow\mathrm{E})}{\varphi}\,(\to\mathrm{E})$$ where D is the derivation from (ii).

Exercise 7. Show that the relation $\varphi \sim \psi$ defined by $\vdash (\varphi \leftrightarrow \psi)$ is an equivalence relation.

Solution: We need to show that $\sim$ is reflexive, symmetric and transitive. Reflexivity requires $\vdash (\varphi \leftrightarrow \varphi)$. This holds by Exercise 6(ii). Symmetry requires that $\vdash (\varphi \leftrightarrow \psi)$ implies $\vdash (\psi \leftrightarrow \varphi)$. This holds because $\vdash (\varphi \leftrightarrow \psi)$ implies both $\vdash (\psi \to \varphi)$ and $\vdash (\varphi \to \psi)$ by the sequent rule for ($\leftrightarrow$E), and hence $\vdash (\psi \leftrightarrow \varphi)$ by the sequent rule for ($\leftrightarrow$I).

Finally, transitivity requires that $\vdash (\varphi \leftrightarrow \psi)$ and $\vdash (\psi \leftrightarrow \chi)$ imply $\vdash (\varphi \leftrightarrow \chi)$. Using the ($\leftrightarrow$E) sequent rule, we conclude $\vdash (\varphi \to \psi)$ and $\vdash (\psi \to \chi)$. By Exercise 4, $\varphi \vdash \psi$ and $\psi \vdash \chi$. Applying the transitivity sequent rule (composing the witnessing derivations together) shows that $\varphi \vdash \chi$, and hence also $\vdash (\varphi \to \chi)$. A symmetric argument shows $\vdash (\chi \to \varphi)$. Thus $\vdash (\varphi \leftrightarrow \chi)$ by ($\leftrightarrow$I).

Exercise 8. Prove the following sequents without (RAA):

(i) $\vdash (\neg(\varphi \wedge (\neg\varphi)))$,
(ii) $\vdash ((\neg(\varphi \to \psi)) \to (\neg\psi))$,
(iii) $\{(\varphi \to \psi)\} \vdash ((\neg\psi) \to (\neg\varphi))$,
(iv) $\{(\varphi \to \psi)\} \vdash (\neg(\varphi \wedge (\neg\psi)))$.

Solution:

(i):

$$\dfrac{\dfrac{(\wedge\mathrm{E})\,\dfrac{\cancel{(\varphi \wedge (\neg\varphi))}}{\varphi} \qquad \dfrac{\cancel{(\varphi \wedge (\neg\varphi))}}{(\neg\varphi)}\,(\wedge\mathrm{E})}{\bot}\,(\neg\mathrm{E})}{(\neg(\varphi \wedge (\neg\varphi)))}\,(\neg\mathrm{I})$$

(ii):

$$\dfrac{\dfrac{\dfrac{(\to\mathrm{I})\,\dfrac{\cancel{\psi}}{(\varphi \to \psi)} \qquad \cancel{(\neg(\varphi \to \psi))}}{\bot}\,(\neg\mathrm{E})}{(\neg\psi)}\,(\neg\mathrm{I})}{((\neg(\varphi \to \psi)) \to (\neg\psi))}\,(\to\mathrm{I})$$

(iii):

$$\dfrac{\dfrac{\dfrac{(\neg\mathrm{E})\,\dfrac{\cancel{\varphi} \qquad (\varphi \to \psi)}{\psi}\,(\to\mathrm{E}) \qquad \cancel{(\neg\psi)}}{\bot}}{(\neg\varphi)}\,(\neg\mathrm{I})}{((\neg\psi) \to (\neg\varphi))}\,(\to\mathrm{I})$$

(iv:)

$$
\begin{prooftree}
\AxiomC{$\cancel{(\varphi \wedge (\neg\psi))}$}
\LeftLabel{($\wedge$E)}
\UnaryInfC{φ}
\AxiomC{$(\varphi \to \psi)$}
\LeftLabel{($\to$E)}
\BinaryInfC{ψ}
\AxiomC{$\cancel{(\varphi \wedge (\neg\psi))}$}
\RightLabel{($\wedge$E)}
\UnaryInfC{$(\neg\psi)$}
\RightLabel{($\neg$E)}
\BinaryInfC{$\bot$}
\RightLabel{($\neg$I)}
\UnaryInfC{$(\neg(\varphi \wedge (\neg\psi)))$}
\end{prooftree}
$$

Exercise 9. Show using (RAA) that $\{((\neg\psi) \to (\neg\varphi))\} \vdash (\varphi \to \psi)$.

Solution:

$$
\begin{prooftree}
\AxiomC{$\cancel{(\neg\psi)}$}
\AxiomC{$((\neg\psi) \to (\neg\varphi))$}
\LeftLabel{($\to$E)}
\BinaryInfC{$(\neg\varphi)$}
\AxiomC{$\cancel{\varphi}$}
\RightLabel{($\neg$E)}
\BinaryInfC{$\bot$}
\RightLabel{(RAA)}
\UnaryInfC{ψ}
\RightLabel{($\to$I)}
\UnaryInfC{$(\varphi \to \psi)$}
\end{prooftree}
$$

Exercise 10. Prove the following sequents without ($\vee$E):
(i) $\vdash (\varphi \to (\varphi \vee \psi))$,
(ii) $\{(\neg(\varphi \vee \psi))\} \vdash ((\neg\varphi) \wedge (\neg\psi))$,
(iii) $\vdash ((\varphi \to \psi) \to ((\neg\varphi) \vee \psi))$.

Solution:
(i):

$$
\begin{prooftree}
\AxiomC{φ}
\RightLabel{($\vee$I)}
\UnaryInfC{$(\varphi \vee \psi)$}
\RightLabel{($\to$I)}
\UnaryInfC{$(\varphi \to (\varphi \vee \psi))$}
\end{prooftree}
$$

(ii):

$$
\begin{prooftree}
\AxiomC{$(\neg(\varphi \vee \psi))$}
\AxiomC{φ}
\RightLabel{($\vee$I)}
\UnaryInfC{$(\varphi \vee \psi)$}
\LeftLabel{($\neg$E)}
\BinaryInfC{$\bot$}
\LeftLabel{($\neg$I)}
\UnaryInfC{$(\neg\varphi)$}
\AxiomC{$(\neg(\varphi \vee \psi))$}
\AxiomC{ψ}
\RightLabel{($\vee$I)}
\UnaryInfC{$(\varphi \vee \psi)$}
\LeftLabel{($\neg$E)}
\BinaryInfC{$\bot$}
\LeftLabel{($\neg$I)}
\UnaryInfC{$(\neg\psi)$}
\RightLabel{($\wedge$I)}
\BinaryInfC{$((\neg\varphi) \wedge (\neg\psi))$}
\end{prooftree}
$$

(iii): Using Exercise 4, we show the equivalent $\{(\varphi \to \psi)\} \vdash ((\neg\varphi) \vee \psi))$:

$$
\begin{prooftree}
\AxiomC{$\cancel{(\neg\varphi)}$}
\LeftLabel{($\vee$I)}
\UnaryInfC{$((\neg\varphi) \vee \psi)$}
\AxiomC{$\cancel{(\neg((\neg\varphi) \vee \psi))}$}
\LeftLabel{($\neg$E)}
\BinaryInfC{$\bot$}
\LeftLabel{(RAA)}
\UnaryInfC{φ}
\AxiomC{$\cancel{\varphi}$}
\AxiomC{$(\varphi \to \psi)$}
\RightLabel{($\to$E)}
\BinaryInfC{ψ}
\RightLabel{($\vee$I)}
\UnaryInfC{$((\neg\varphi) \vee \psi)$}
\AxiomC{$\cancel{(\neg((\neg\varphi) \vee \psi))}$}
\RightLabel{($\neg$E)}
\BinaryInfC{$\bot$}
\RightLabel{($\neg$I)}
\UnaryInfC{$(\neg\varphi)$}
\RightLabel{($\neg$E)}
\BinaryInfC{$\bot$}
\RightLabel{(RAA)}
\UnaryInfC{$((\neg\varphi) \wedge \psi)$}
\end{prooftree}
$$

Exercise 11. Prove the following sequents with $(\vee\mathrm{E})$:

(i) $\{(\varphi \vee \psi)\} \vdash (\psi \vee \varphi)$,
(ii) $\{(\varphi \vee \psi), (\varphi \to \chi), (\psi \to \chi)\} \vdash \chi$,
(iii) $\{(\varphi \vee \psi), (\neg\varphi)\} \vdash \psi$,
(iv) $\{((\neg\varphi) \wedge (\neg\psi))\} \vdash (\neg(\varphi \vee \psi))$.

Solution:

(i):

$$\dfrac{(\vee\mathrm{I})\,\dfrac{\cancel{\varphi}}{(\psi \vee \varphi)} \qquad \dfrac{\cancel{\psi}}{(\psi \vee \varphi)}\,(\vee\mathrm{I}) \qquad (\varphi \vee \psi)}{(\psi \vee \varphi)}\,(\vee\mathrm{E})$$

(ii):

$$\dfrac{(\to\mathrm{E})\,\dfrac{\cancel{\varphi} \qquad (\varphi \to \chi)}{\chi} \qquad \dfrac{\cancel{\psi} \qquad (\psi \to \chi)}{\chi}\,(\to\mathrm{E}) \qquad (\varphi \vee \psi)}{\chi}\,(\vee\mathrm{E})$$

(iii):

$$\dfrac{\dfrac{\dfrac{\cancel{\varphi} \qquad (\neg\varphi)}{\bot}\,(\neg\mathrm{E})}{\psi}\,(\mathrm{RAA}) \qquad \cancel{\psi} \qquad (\varphi \vee \psi)}{\psi}\,(\vee\mathrm{E})$$

(iv): Using Exercise 2, we show the equivalent $\{(\neg\varphi), (\neg\psi)\} \vdash (\neg(\varphi \vee \psi))$.

$$\dfrac{\dfrac{(\neg\mathrm{E})\,\dfrac{\cancel{\varphi} \qquad (\neg\varphi)}{\bot} \qquad \dfrac{\cancel{\psi} \qquad (\neg\psi)}{\bot}\,(\neg\mathrm{E}) \qquad \cancel{(\varphi \vee \psi)}}{\bot}\,(\vee\mathrm{E})}{(\neg(\varphi \vee \psi))}\,(\neg\mathrm{I})$$