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Estimating value-at-risk using a multivariate copula-based volatility model: Evidence from European banks

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ABSTRACT

This paper proposes a multivariate copula-based volatility model for estimating Value-at-Risk (VaR) in the banking sector of selected European countries by combining dynamic conditional correlation (DCC) multivariate GARCH (M-GARCH) volatility model and copula functions. Non-normality in multivariate models is associated with the joint probability of the univariate models' marginal probabilities –the joint probability of large market movements, referred to as *tail dependence*. In this paper, we use copula functions to model the tail dependence of large market movements and test the validity of our results by performing back-testing techniques. The results show that the copula-based approach provides better estimates than the common methods currently used and captures VaR reasonably well based on the differences in the numbers of exceptions produced during different observation periods at the same confidence level.

1. Introduction

Value-at-Risk (VaR) is a standard risk measure in financial risk management and is widely used in the banking sector and other financial institutions. It summarises the worst possible loss of a portfolio of financial assets at a given confidence level over a given time period. European banks began adopting VaR in the early 1990s (Holton, 2002). International bank regulators also influenced the development and use of VaR when the Basel Committee on Banking Supervision (BCBS) chose VaR as the international standard method for evaluating the market risk of a portfolio of financial assets for regulatory purposes (Goodhart, 2011).

There are many methods to estimate VaR (e.g., Holton (2003), Jorion (2007), Malz (2011) and the references therein), and the most common methods used by banks include the variance-covariance method (developed by J.P. Morgan using its RiskMetrics in 1993), historical simulation, and the Monte Carlo simulation. Due to their simplicity, the variance-covariance and historical simulation methods have been prevalent in the banking sector and financial institutions for calculating VaR (Pérignon and Smith, 2010). These methods are based on the assumption that asset returns are independently and identically normally distributed. This assumption contradicts empirical evidence by Sheikh and Qiao (2010), which shows that in many cases, financial asset returns are not independent and normally distributed and are, in fact, leptokurtic and fat-tailed, leading to underestimation or overestimation of VaR; this is because extremely large positive and negative asset returns are more common in practice than normally distributed models.

Another drawback of these methods is in the estimation of the conditional volatility of financial returns. Most financial asset returns exhibit heavy tails with respect to conditional volatility over time. Berkowitz et al. (2011) also showed evidence of changing

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volatility and non-normality using desk-level data from a large international commercial bank. VaR models are highly dependent on the type of volatility model used. A good volatility model should be able to capture the behaviour of the tail distribution of asset returns, be easily implemented for a wide range of asset returns, and be easily extensible to portfolios with many risk factors of different kinds (Malz, 2011). For multivariate volatility models of VaR, we must focus on the tail dependence, which is the principal factor associated with non-normality.

Many volatility models have been proposed, for example, the generalised autoregressive conditional heteroskedasticity (GARCH) models and its extensions have been used to capture the effects of volatility clustering and asymmetry in VaR estimation. Many studies have applied a variety of univariate GARCH models in VaR estimation; see So and Philip (2006), Berkowitz and ÓBrien (2002), and McNeil and Frey (2000). In addition, Kuester et al. (2006) provides an extensive review of VaR estimation methods with a focus on univariate GARCH models. The results of all these studies suggest that GARCH models provide more accurate VaR estimates than traditional methods. Because financial applications typically deal with a portfolio of assets with several risk factors (as considered in this study), a multivariate GARCH (M-GARCH) model would be very useful for VaR estimation. Univariate VaR models focus on an individual portfolio, whereas the multivariate approach explicitly model the correlation structure of the covariance or volatility matrix of multiple asset returns over time. Bauwens and Laurent (2012) provides a comprehensive review of univariate volatility models and their applications.

Numerous M-GARCH models have since been developed, for example, Tsay (2013); Fengler and Herwartz (2008); Engle and Kroner (1995); Bollerslev et al. (1994) and the references therein. Bauwens et al. (2006) divide M-GARCH models into three categories: (1) direct generalisation of univariate GARCH models (e.g., exponentially-weighted moving average (EWMA), vector error correction (VEC), BEKK, etc.), (2) linear combinations of univariate GARCH models (e.g., generalised orthogonal GARCH (GO-GARCH), principal component GARCH (PGARCH), etc.), and (3) nonlinear combinations of univariate GARCH models (e.g., dynamic conditional correlation (DCC) and constant conditional correlation (CCC) models). The article by Silvennoinen and Teräsvirta (2009) gives a concise review of most common M-GARCH models; parametric and semi-parametric models and their properties. See also Ghalanos (2015) for more details on M-GARCH models.

Most volatility models fail to satisfy the positive definite conditions of the covariance matrix of asset returns. In this study, we employed the M-GARCH DCC volatility model by Engle (2002) because of conditions (as seen later) that will guarantee the conditional volatility matrix to be positive-definite almost surely. Furthermore, the DCC model allows the correlation matrix of asset returns to be time varying and thus reflects the current market conditions.

Non-normality in multivariate models is associated with the joint probability of the univariate models' marginal probabilities, that is, the joint probability of large market movements referred to as *tail dependence*. The VaR estimation for a portfolio of assets can become very difficult due to the complexity of joint multivariate distributions. To overcome these problems, we use the copula theory, which enables us to construct a flexible multivariate distribution with different margins and different dependence structures; this allows the joint distribution of a portfolio to be free from assumptions of normality and linear correlation. Additionally, copulas can easily capture extreme dependencies such as tail dependence, while the normal distribution assumes no extreme dependencies.

The copula theory was first developed by Sklar (1959) and later introduced to the finance literature by Embrechts et al. (2002), Frey and McNeil (2003), and Li (2000). Consequently, Embrechts et al. (2002) introduced the application of copula theory to financial asset returns, and Patton (2004) expanded the framework of the copula theory with respect to the time-varying nature of financial dependence schemes. The copula theory has also been used in risk management to measure the VaR of portfolios, including both unconditional (Cherubini and Luciano (2001), Embrechts et al. (2001), and Cherubini et al. (2004)) and, recently, conditional distributions (Silva Filho et al. (2014), Huang et al. (2009) and Fantazzini (2008)).

This paper thus presents an approach of estimating VaR using an M-GARCH DCC volatility model and copulas. To the best of our knowledge, this is the first paper to report the results for estimating VaR in banking sector of selected European countries, namely; Germany, United Kingdom (UK), Sweden, France, Italy, Spain, and Greece. Therefore, this work offers a new contribution to the literature in this study area.

The rest of the paper is structured as follows: Section 2 presents the methodology of the proposed approach, the DCC model and copula functions. Section 3 discusses the data used, empirical results and back-testing the VaR model, followed by the conclusion in Section 4.

2. Methodology

2.1. The dynamic conditional correlation (DCC) model

To properly understand the volatility matrix of the DCC model, it is important to first understand the mean and variance equations of the GARCH(1,1) model. The GARCH(1,1) model, first proposed by Bollerslev (1986), allows conditional variance to be dependent upon previous lags. The GARCH(1,1) model has the following form

$$r_{i,t} = \mu_i + a_{i,t}, \quad a_{i,t} = \sigma_{i,t}\eta_{i,t} \tag{1a}$$

$$\sigma^2_{i,t} = \alpha_i + \beta_i a^2_{i,t-1} + \gamma_i \sigma^2_{i,t-1}, \tag{1b}$$

$$\text{for} \quad i = 1, \dots, N, t = 1, \dots, T,$$

where $r_{i,t}$ are the log return series of daily stock prices, μ_i are the conditional means of the log returns, $a_{i,t}$ are the residuals of the mean equation (Eqn. (1a), $\eta_{i,t}$ represents white noise with zero mean and unit variance, $\sigma_{i,t}$ are the conditional volatility series from the variance equation (Eqn. (1b)), N represents the total number of stocks under investigation, T is the sample size, and α, β and γ are the GARCH parameters.

In the DCC model, the covariance matrix at time t is given by

$$\Sigma_t = D_t \rho_t D_t, \tag{2}$$

and the conditional correlation matrix is then

$$\rho_t = D_t^{-1} \Sigma_t D_t^{-1}, \tag{3}$$

where D_t is the diagonal matrix of the N conditional volatilities of the stock returns; that is, $D_t = \text{diag}\left\{\sqrt{\sigma_{11,t}}, \ldots, \sqrt{\sigma_{NN,t}}\right\}$, and $\sigma_{ij,t}$ is the (i,j)th element of the volatility matrix (Tsay, 2005).

Two types of DCC models have been proposed in the literature. The first type, proposed by Tse and Tsui (2002) is given by

$$\rho_t = (1 - \theta_1 - \theta_2)\overline{\rho} + \theta_1 \rho_{t-1} + \theta_2 \psi_{t-1}, \tag{4}$$

where $\overline{\rho}$ is the unconditional correlation matrix of η_t, θ_1 and θ_2 are non-negative real numbers satisfying $0 \le \theta_1 + \theta_2 < 1$. ψ_{t-1} is a correlation matrix of the most recent returns that depends on $\{\eta_{t-1} \ldots \eta_{t-m}\}$ for some integer m and is defined as

$$\psi_{ij,t-1} = \frac{\sum_{i=1}^{m} \eta_{i,t-i}\eta_{j,t-i}}{\sqrt{(\sum_{i=1}^{m} \eta_{i,t-i}^2)(\sum_{i=1}^{m} \eta_{j,t-i}^2)}}. \tag{5}$$

The choice of the parameter m can be regarded as a smoothing parameter. The larger the m is, the smoother the resulting correlations. Thus, a choice of $m > k$ ensures ψ_{t-1} and hence ρ_t are guaranteed to be positive-definite.

The second type of DCC models is proposed by Engle (2002); the correlation matrix as in Eq. (3) is defined as

$$\rho_t = (1 - \theta_1 - \theta_2)\overline{\rho_t} + \theta_1 \rho_{t-1} + \theta_2 \eta_{t-1}\eta_{t-1}', \tag{6}$$

where θ_1 and θ_2 are non-negative real numbers and $0 < \theta_1 + \theta_2 < 1$; ρ_t is a positive-definite matrix. Unlike the DCC model of Tse and Tsui (2002) that requires the choice of m in practice, Eq. (6) uses η_{t-1} only so that the correlation matrix re-normalised at each time $t-1$ (for details, see Tsay (2013)). In this study, we use the DCC model of Engle (2002).

2.2. Copulas

In multivariate settings, we use the following version of Sklar's theorem as given in Cherubini et al. (2004) for the purpose of VaR estimation.

Sklar's theorem: Consider an n-dimensional joint distributional function $F(x)$, with uniform margins $F_1(x_1), \ldots, F_n(x_n)$; $x = (x_1, \ldots, x_n)$, with $-\infty \le x_i \le \infty$, then there exists a copula $C : [0,1]^n \to [0,1]$ such that

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)), \tag{7}$$

determined under absolute continuous margins as

$$C(u_1, \ldots, u_n) = F(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n)), \tag{8}$$

otherwise, C is uniquely determined on the range $R(F_1) \times \cdots \times R(F_n)$. Equally, if C is a copula and $F_1, \ldots, F_n$ are univariate distribution functions, then Eq. (7) is a joint distribution function with margins $F_1, \ldots, F_n$ (Tsay, 2013).

The copula $C(u_1, \ldots, u_n)$ has density $c(u_1, \ldots, u_n)$ associated to it, which is defined as

$$c(u_1, \ldots, u_n) = \frac{\partial_n C(u_1, \ldots, u_n)}{\partial u_1, \ldots, \partial u_n}, \tag{9}$$

and is related to the density function F for continuous random variables denoted as f, by the canonical copula representation

$$f(x_1,\dots,x_n)=c(F_1(x_1),\dots,F_n(x_n))\prod_{i=1}^{n}f_i(x_i), \tag{10}$$

where f_i are the marginal densities that can be different from each other (Ghalanos, 2015; Tsay, 2013; Huang et al., 2009; Cherubini et al., 2004).

Cherubini et al. (2011) discuss two commonly used families of copulas in financial applications: the elliptical and the Archimedean copulas.

Elliptical copulas are derived from the elliptical distribution by applying Sklar's theorem. The most common are the Gaussian and the Student's-t copulas, which are symmetric (Cherubini et al., 2011). Their dependence structure is determined by a standardised correlation or dispersion matrix because of the invariant property of copulas. Consider a symmetric positive definite matrix ρ with $\mathrm{diag}(\rho)=(1,1,\dots,1)^{\mathrm{tr}}$; we can represent the multivariate Gaussian copula (MGC) as

$$C_{\rho}^{G_a}=\Phi_{\rho}(\Phi^{-1}(u_1),\dots,\Phi^{-1}(u_n)), \tag{11}$$

where Φ_ρ is the standardised multivariate normal distribution and Φ_{ρ}^{-1} is the inverse standard univariate normal distribution function of u with correlation matrix ρ. If the margins are normal, then the Gaussian copula will generate the standard Gaussian joint distribution function with density function

$$c_{\rho}^{G_a}(u_1,u_2,\dots,u_n)=\frac{1}{|\rho|^{\frac{1}{2}}}\exp\left(-\frac{1}{2}\varsigma'(\rho^{-1}-\mathbf{I})\varsigma\right), \tag{12}$$

where $\varsigma=(\Phi^{-1}(u_1),\dots,\Phi^{-1}(u_n))'$ and $\mathbf{I}$ is the identity matrix.

On the other hand, the multivariate Student's-t copula (MTC) can be represented as

$$T_{\rho,v}(u_1,\dots,u_n)=t_{\rho,v}(t_v^{-1}(u_1),\dots,t_v^{-1}(u_n)) \tag{13}$$

with density function

$$c_{\rho,v}(u_1,\dots,u_n)=|\rho|^{-\frac{1}{2}}\frac{\Gamma(\frac{v+n}{2})}{\Gamma(\frac{v}{2})}\left(\frac{\Gamma(\frac{v}{2})}{\Gamma(\frac{v+1}{2})}\right)^{n}\frac{(1+\frac{1}{v}\varsigma'\rho^{-1}\varsigma)^{-\frac{v+n}{2}}}{\prod_{j=1}^{n}\left(1+\frac{\varsigma_j^2}{v}\right)^{-\frac{v+1}{2}}}, \tag{14}$$

where $t_{\rho,v}$ is the standardised Student's-t distribution with correlation matrix ρ and v degrees of freedom.

Archimedean copulas are useful in risk management analysis because they capture an asymmetric tail dependence between financial asset returns. The most commonly used Archimedean copulas in financial applications are the Gumbel (1960), Clayton (1978) and Frank (1979) copulas, which are built via a generator as

$$C(u_1,\dots,u_n)=\varphi^{-1}(\varphi(u_1)+\cdots+\varphi(u_n)), \tag{15}$$

with density function

$$c(u_1,\dots,u_n)=\varphi^{-1}(\varphi(u_1)+\cdots+\varphi(u_n))\prod_{i=1}^{n}\varphi'(u_i), \tag{16}$$

where φ is the copula generator and φ^{-1} is completely monotonic on $[0,\infty]$. That is, φ must be infinitely differentiable with derivatives of ascending order and alternative sign such that $\varphi^{-1}(0)=1$ and $\lim_{x\to+\infty}\varphi(x)=0$ (Cherubini et al., 2011; Yan et al., 2007). Thus, $\varphi'(u)<0$ (i.e., φ is strictly decreasing) and $\varphi''(u)>0$ (i.e., φ is strictly convex).

The Gumbel copula captures upper tail dependence, is limited to positive dependence, and has generator function $\varphi(u)=(-\ln(u))^{\alpha}$ and generator inverse $\varphi^{-1}(x)=\exp(-x^{\frac{1}{\alpha}})$. This will generate a Gumbel n-copula represented by

$$C(u_1,\dots,u_n)=\exp\left\{-\left[\sum_{i=1}^{n}(-\ln u_i)^{\alpha}\right]^{\frac{1}{\alpha}}\right\}\quad \alpha>1. \tag{17}$$

The generator function for the Clayton copula is given by $\varphi(u)=u^{-\alpha}-1$ and generator inverse $\varphi^{-1}(x)=(x+1)^{-\frac{1}{\alpha}}$, which yields a Clayton n-copula represented by

$$C(u_1,\ldots,u_n) = \left[\sum_{i=1}^{n} u_i^{-\alpha} - n + 1\right]^{-\frac{1}{\alpha}} \quad \alpha > 0. \tag{18}$$

The Frank copula has generator function $\varphi(u) = \ln\left(\frac{\exp(-\alpha u)-1}{\exp(-\alpha)-1}\right)$ and generator inverse $\varphi^{-1}(x) = -\frac{1}{\alpha}\ln(1 + e^{x}(e^{-\alpha} - 1))$, which will result in a Frank n-copula represented by

$$C(u_1,\ldots,u_n) = -\frac{1}{\alpha}\ln\left\{1 + \frac{\prod_{i=1}^{n}(e^{-\alpha u_i} - 1)}{(e^{-\alpha} - 1)^{n-1}}\right\} \quad \alpha > 0. \tag{19}$$

We follow Breymann et al. (2003) and employ Gaussian, Student's-t, Gumbel, Clayton and Frank copulas in this study.

2.3. Measuring dependence

The traditional way to measure the relationship between markets and risk factors is by looking at their linear correlations, which depend both on the marginal and joint distributions of the risk factors. If there is no linear relationship –in the case of non-normality –the results might be misleading (Cherubini et al., 2011). In this situation, non-parametric invariant measures that are not dependent on marginal probability distributions are more appropriate. Copulas measure a form of dependence between pairs of risk factors (i.e., asset returns) known as concordance using invariant measures. Two observations (x_i, y_i) and (x_j, y_j) from a vector (X, Y) of continuous random variables are concordant if $(x_i - x_j)(y_i - y_j) > 0$ and discordant if $(x_i - x_j)(y_i - y_j) < 0$. Large values of X are paired with large values of Y and small values of X are paired with small values of Y as the proportion of concordant pairs in the sample increases. On the other hand, the proportion of concordant pairs decreases as large values of X are paired with small values of Y and small values of X are paired with large values of Y (Alexander, 2008).

The most commonly used invariant measures are the Kendall's τ and Spearman's ρ. Consider n paired continuous observations (x_i, y_i) ranked from smallest to largest, with the smallest ranked 1, the second smallest ranked 2, and so on. Then, Kendall's τ is defined as the sum of the number of concordant pairs minus the sum of the number of discordant pairs divided by the total number of pairs, i.e., the probability of concordance minus the probability of discordance:

$$\tau_{X,Y} = P[(x_i - x_j)(y_i - y_j) > 0] - P[(x_i - x_j)(y_i - y_j) < 0] = \frac{\mathbf{C} - \mathbf{D}}{\mathbf{C} + \mathbf{D}}, \tag{20}$$

where $\mathbf{C}$ is the number of concordant pairs below a particular rank that are larger in value than that particular rank, and D is the number of discordant pairs below a particular rank that are smaller in value than that particular rank.

Spearman's ρ, on the other hand, is defined as the probability of concordance minus the probability of discordance of the pair of vectors (x_1, y_1) and (x_2, y_3) with the same margins. That is,

$$\rho_{X,Y} = 3(P[(x_1 - x_2)(y_1 - y_3) > 0] - P[(x_1 - x_2)(y_1 - y_3)] < 0).$$

The joint distribution function of (x_1, y_1) is $H(x, y)$, while the joint distribution function of (x_2, y_3) is $F(x), G(y)$ because x_2 and y_3 are independent (Nelsen, 2007). Alternatively,

$$\rho_{X,Y} = 1 - \frac{6\sum_{i=1}^{n} d_i^2}{n(n^2 - 1)},$$

where d is the difference between the ranked samples.

Nelsen (2007) has shown that Kendall's τ and Spearman's ρ depend on the vectors (x_1, y_1), (x_2, y_2) and (x_1, y_1), (x_2, y_3), respectively, through their copulas C, and that the following relationship holds:

$$\tau_{X,Y} = 4\int_0^1\int_0^1 C(u,v)dC(u,v) - 1$$

$$\rho_{X,Y} = 12\int_0^1\int_0^1 C(u,v)dudv - 3.$$

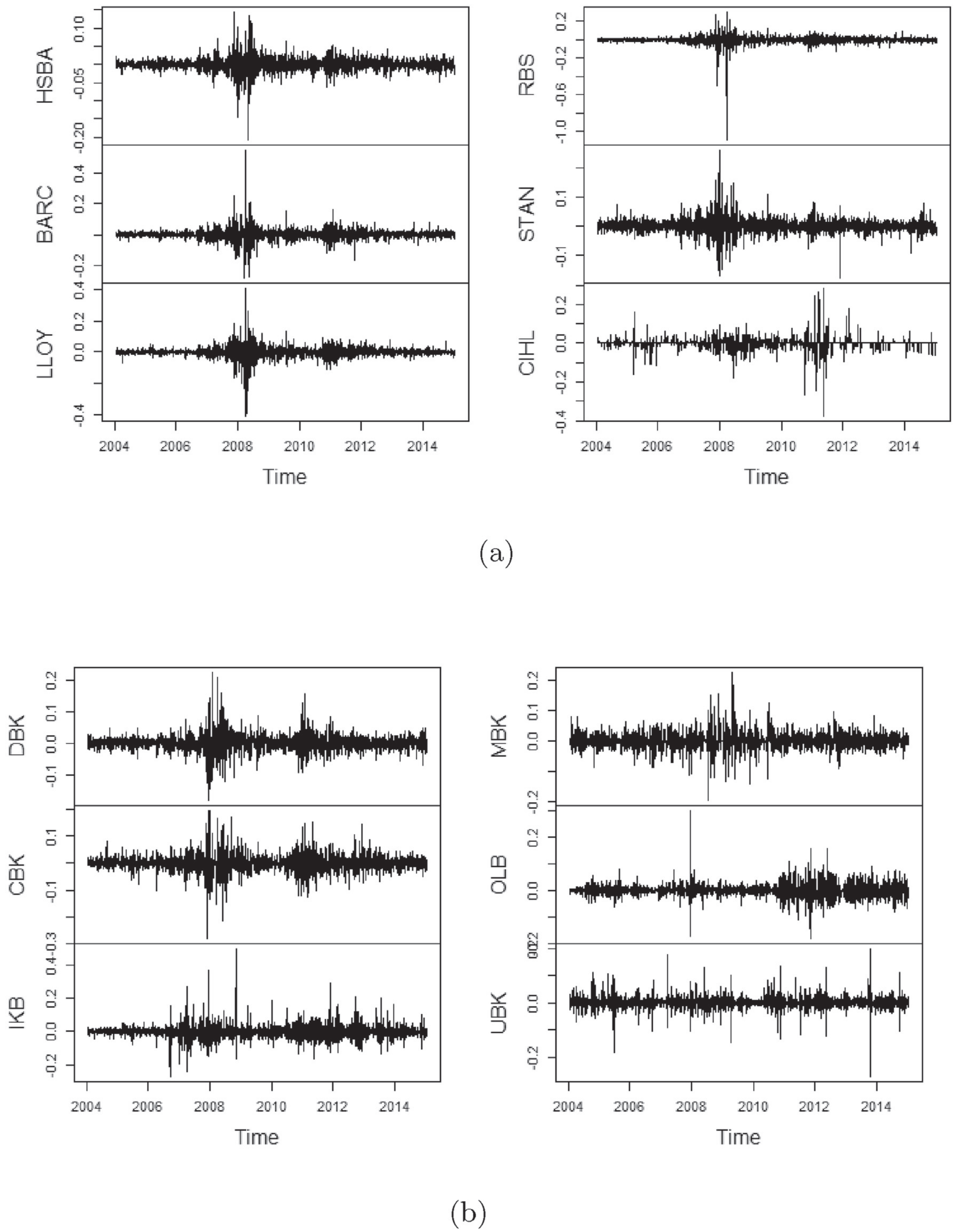
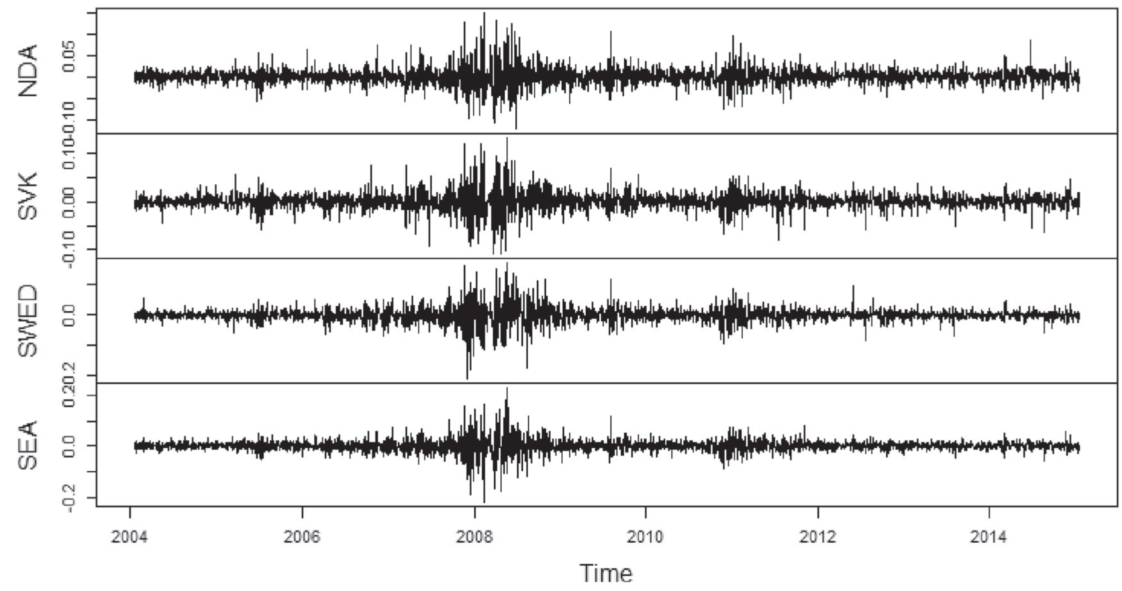
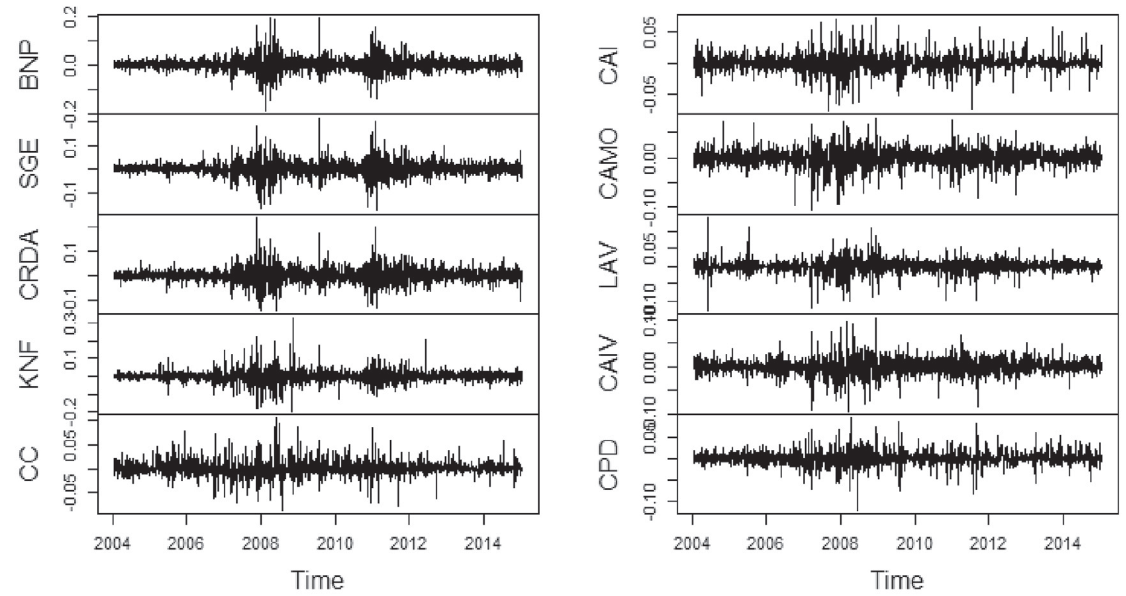


Fig. 1. Time plots of daily log return series for UK stocks (panel (a)), and German stocks (panel (b)), for the period from 31 December 2004 to 31 December 2015.

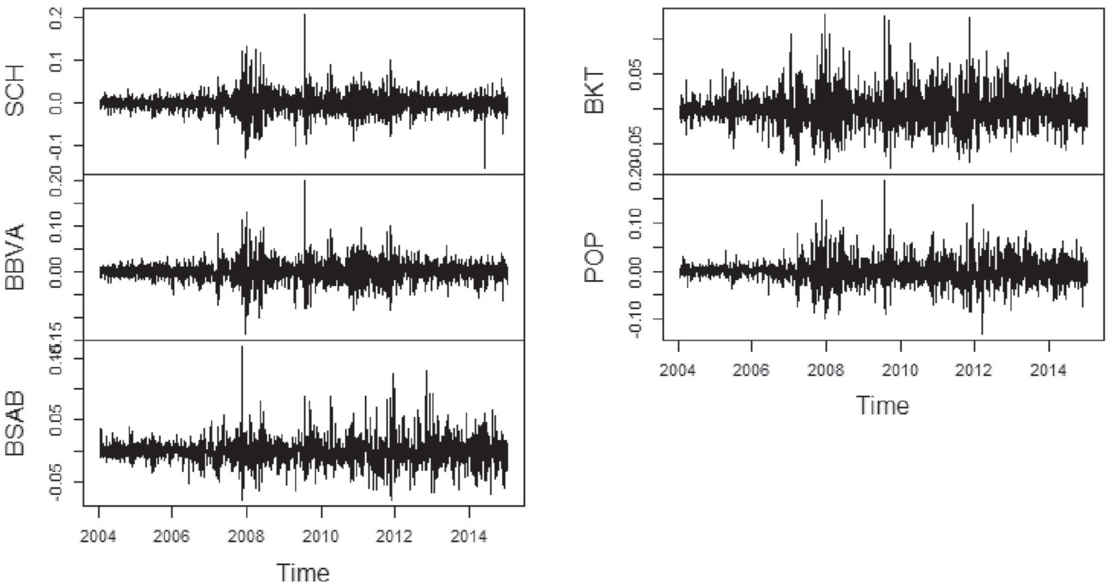


(a)

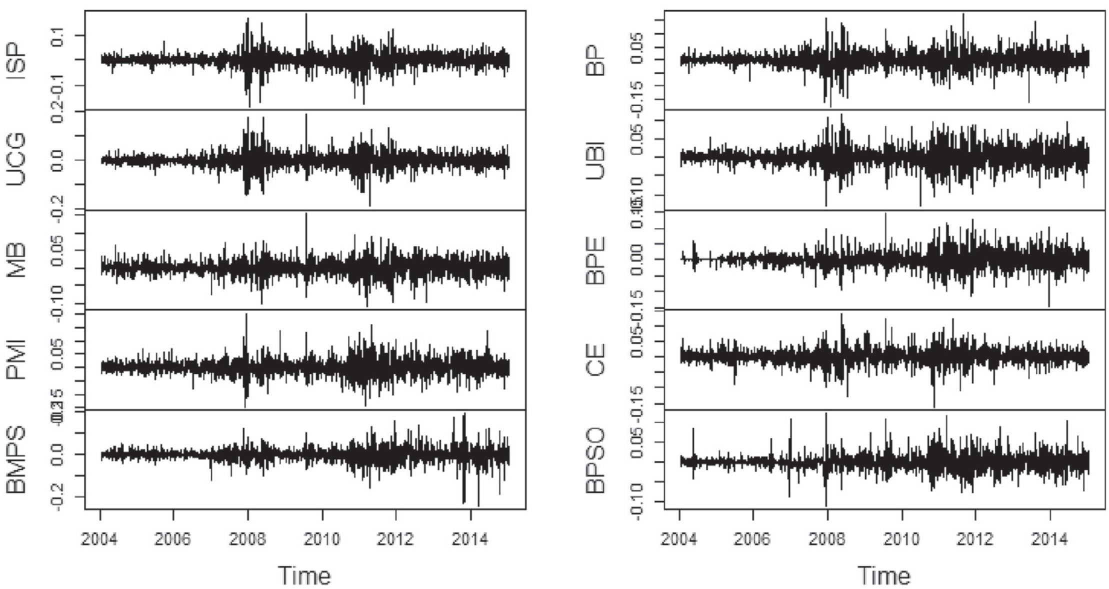


(b)

Fig. 2. Time plots of daily log return series for Swedish stocks (panel (a)), and French stocks (panel (b)), for the period from 31 December 2004 to 31 December 2015.



(a)



(b)

Fig. 3. Time plots of daily log return series for Spanish stocks (panel (a)), and Italian stocks (panel (b)), for the period from 31 December 2004 to 31 December 2015.

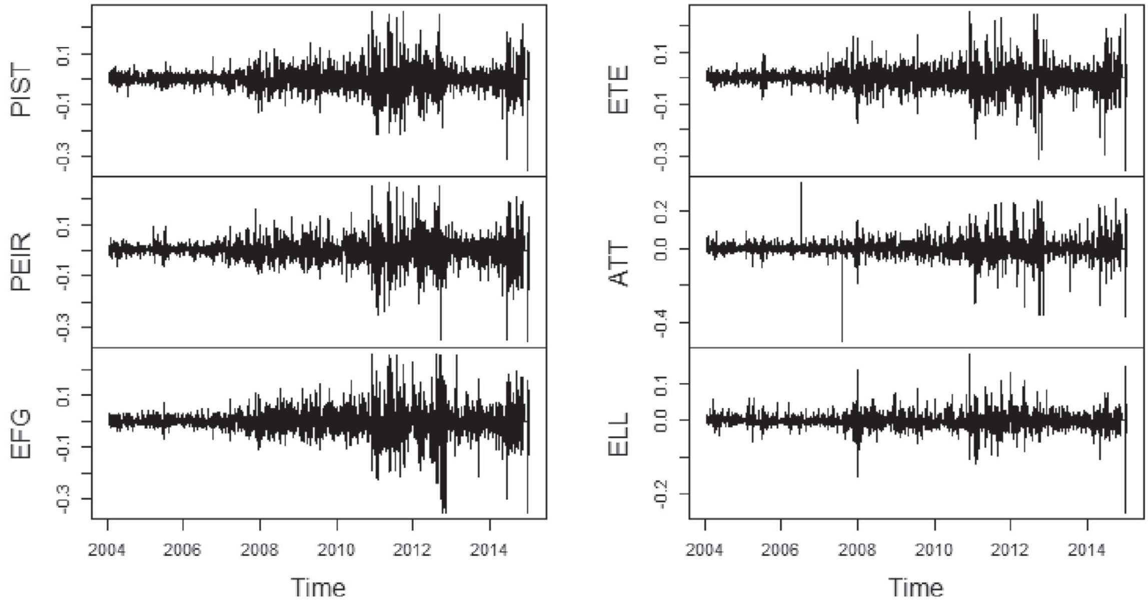


Fig. 4. Time plots of daily log return series using stocks from Greece for the period from 31 December 2004 to 31 December 2015. The plots show the effects of the 2008 global financial crisis and 2011 European financial crisis, and the presence of volatility clustering.

3. Data and empirical results

3.1. Data and description

The data employed in this study consist of daily closing prices of 47 stocks from the banking sector of seven selected European countries, namely, Germany, UK, Sweden, France, Italy, Spain, and Greece. These stocks are within the top ten banks for each country. We selected our data starting from 31 December 2004 to 31 December 2015 to include the 2008 global financial crisis and 2011 European financial crisis period. All data are from DataStream and consist of 2870 observations.

The daily log return series of the stocks are calculated via

$$r_t = \left[\log\left(\frac{S_{1,t+\tau}}{S_{1,t}}\right), \ldots, \log\left(\frac{S_{N,t+\tau}}{S_{N,t}}\right)\right] = (r_{1t}, \ldots, r_{Nt}), \tag{21}$$

where N represents the number of stocks in the sample. Figs. 1–4 show time series plots of daily log returns series for the different countries. From the plots, we can observe the presence of volatility clustering. That is, small changes in volatility tend to be followed by small changes for a prolonged period of time and large changes in volatility tend to be followed by large changes for a prolonged period of time.

Basic statistics of the stock returns are reported in Table 1. We see from the table that the stock returns are far from being normally distributed, as indicated by their high excess kurtosis and skewness. We confirm this by running a multivariate autoregressive conditional heteroscedasticity (ARCH) test based on Ljung-Box test statistics $Q_k(m)$ and its modification known as robust $Q_k^r(m)$ test on the log returns at 5% significance, where m is the number of lags of cross-correlation matrices used in the tests. The modification involves discarding those observations from the return series whose corresponding standardised residuals exceed 95-th quantile in order to reduce the effect of heavy tails. The motivation for the $Q_k^r(m)$ test is because $Q_k(m)$ may fare poorly in finite samples when the residuals of the time series, $a_{i,t} = \sigma_{i,t}\eta_{i,t}$, have heavy tails (Tsay, 2013). The test statistic is given as follows:

$$Q_k(m) = T^2 \sum_{i=1}^{m} \frac{1}{T-i} \mathbf{b}_i'(\hat{\rho}_0^{-1} \otimes \hat{\rho}_0^{-1})\mathbf{b}_i \approx \chi^2_{k^2}(m), \tag{22}$$

which is asymptotically equivalent to the multivariate Lagrange multiplier (LM) test for conditional heteroscedasticity by Engle (1982); here, k is the dimension of $a_{i,t}$, T is the sample size, $\mathbf{b}_i = vec(\hat{\rho}_i')$ with $\hat{\rho}_j$ being the lag-j cross-correlation matrix of $a_{i,t}^2$. The results shown in Table 2 indicate the presence of conditional heteroscedasticity.

Table 1
Summary statistics of daily log-returns (in percentages). High excess kurtosis and skewness suggest stock returns are not normally distributed.

Country		Stocks from various banks									
France		F.BNP	F.SGE	F.CRDA	F.KNF	F.CC	F.CAI	F.CAMO	F.LAV	F.CAIV	F.CPD
	Mean	0.0006	−0.0151	−0.0215	−0.0019	0.0006	−0.0108	−0.0102	−0.0011	−0.0103	−0.0166
	Variance	0.0647	0.0807	0.0764	0.0959	0.0197	0.0145	0.0262	0.0267	0.0225	0.0202
	Std. deviation	2.5427	2.8399	2.7645	3.0969	1.4027	1.2029	1.6182	1.6340	1.5003	1.4212
	Skewness	34.4305	6.9738	27.7890	60.0710	62.1017	7.6754	−43.2826	−46.9617	5.3467	−1.5654
	Excess kurtosis	867.8928	680.0429	624.0289	1209.1365	821.6399	678.3979	476.7556	1107.4525	601.0420	702.8488
Italy		I.ISP	I.UCG	I.MB	I.PMI	I.BMPS	I.BP	I.UBI	I.BPE	I.CE	I.BPSO
	Mean	−0.0023	−0.0519	−0.0078	−0.0250	−0.1260	−0.0635	−0.0278	−0.0129	−0.0022	−0.0132
	Variance	0.0671	0.0829	0.0429	0.0773	0.0870	0.0814	0.0566	0.0574	0.0517	0.0335
	Std. deviation	2.5905	2.8797	2.0722	2.7800	2.9488	2.8539	2.3799	2.3966	2.2734	1.8305
	Skewness	−25.1383	−16.0663	3.8658	29.6811	−24.7777	2.9842	−4.4575	24.7941	1.4148	58.9718
	Excess kurtosis	656.5973	589.0746	320.6962	383.9322	891.5712	461.0507	273.2804	392.4517	408.2981	553.0105
UK		HSBA	BARC	LLOY	RBS	STAN	CIHL				
	Mean	−0.0124	−0.0306	−0.0407	−0.0971	−0.0112	−0.1171				
	Variance	0.0293	0.1030	0.1077	0.1504	0.0594	0.0762				
	Std. deviation	1.7122	3.2098	3.2823	3.8785	2.4376	2.7600				
	Skewness	−33.6697	143.8658	−105.4936	−840.1325	31.6077	9.0031				
	Excess kurtosis	1690.7965	4021.7880	3727.5413	23552.6326	1308.5010	4940.1441				
Germany		D.DBK	D.CBK	D.IKB	D.MBK	D.OLB	D.UBK				
	Mean	−0.0320	−0.0782	−0.1287	0.0008	−0.0457	0.0879				
	Variance	0.0615	0.0904	0.1529	0.0550	0.0478	0.0466				
	Std. deviation	2.4803	3.0066	3.9100	2.3445	2.1863	2.1586				
	Skewness	32.5034	−13.9034	151.9830	31.5160	83.5634	−31.7385				
	Excess kurtosis	1022.5506	873.5286	1914.4094	1145.6160	2008.7958	1984.3534				
Greece		G.PIST	G.PEIR	G.EFG	G.ETE	G.ATT	G.ELL				
	Mean	−0.1603	−0.3371	−0.3377	−0.2872	−0.2045	−0.0642				
	Variance	0.2287	0.2876	0.3162	0.2487	0.2818	0.0448				
	Std. deviation	4.7826	5.3624	5.6234	4.9866	5.3087	2.1157				
	Skewness	−12.5182	−105.8321	−61.0233	−103.6301	−59.4886	−7.5496				
	Excess kurtosis	827.1241	1063.2666	865.8341	1044.0274	1317.2709	1519.9341				
Spain		E.SCH	E.BBVA	E.BSAB	E.BKT	E.POP					
	Mean	−0.0070	−0.0154	−0.0199	0.0117	−0.0702					
	Variance	0.0463	0.0444	0.0359	0.0512	0.0529					
	Std. deviation	2.1519	2.1078	1.8940	2.2617	2.2991					
	Skewness	20.4736	32.7311	71.6545	49.2691	43.5654					
	Excess kurtosis	828.4460	668.4718	623.2511	304.9592	494.7029					
Sweden		W.NDA	W.SVK	W.SWED	W.SEA						
	Mean	0.0202	0.0234	0.0109	0.0104						
	Variance	0.0419	0.0347	0.0631	0.0643						
	Std. deviation	2.0471	1.8633	2.5116	2.5361						
	Skewness	52.6835	12.2476	−20.9301	5.3782						
	Excess kurtosis	657.9097	690.7580	906.9637	1278.0779						

Table 2
Multivariate-ARCH test on the standardised residuals. The null hypothesis of no ARCH effect and no serial correlation is rejected at 95% significance level.

	Multivariate-ARCH Test						
	France	Italy	UK	Germany	Greece	Spain	Sweden
$Q_k(10)$	3686	4008	1244	1187	3586	808	5747
p-value	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$Q_k^r(10)$	5053	3769	3195	3518	3180	1546	1931
p-value	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 3
Copula parameter values and model selection based on MLE values and AIC criteria. The copula that best fits the data (in bold) is selected based on the highest MLE value, and the smallest AIC value.

Country stocks		Archimedean copula			Elliptical copula	
		Gumbel	Clayton	Frank	Gaussian	Student's-t
UK	Copula parameter	1.30	0.47	2.33	ρ_n	ρ_t
	MLE	1691	**1813**	1810	3834	**4047**
	AIC	−3380	**−3624**	−3618	−7638	**−8062**
Germany	Copula parameter	1.05	0.10	0.43	ρ_n	ρ_t
	MLE	81.47	**152.5**	105.2	986.6	**1018**
	AIC	−161	**−303**	−208	−1943	**−2004**
Greece	Copula parameter	1.55	0.81	3.86	ρ_n	ρ_t
	MLE	**3861**	3609	3825	4717	**5092**
	AIC	**−7720**	−7216	−7648	−9404	**−10152**
Spain	Copula parameter	1.88	1.34	5.73	ρ_n	ρ_t
	MLE	4817	4616	**4952**	5936	**6362**
	AIC	−9631	−9230	**−9902**	−11852	**−12702**
France	Copula parameter	1.16	0.24	1.32	ρ_n	ρ_t
	MLE	1352	**1615**	1298	4789	**5059**
	AIC	−2702	**−3228**	−2594	−9488	**−10026**
Italy	Copula parameter	1.45	0.70	3.55	ρ_n	ρ_t
	MLE	6959	6816	**7024**	8846	**9493**
	AIC	−13916	−13630	**−14046**	−17602	**−18894**
Sweden	Copula parameter	1.95	1.32	6.00	ρ_n	ρ_t
	MLE	**3747**	3142	3715	**4057**	4047
	AIC	**−7492**	−6282	−7028	**−8102**	−8080

3.2. Modeling results

From the analysis, we obtained the volatility matrix, which consists of the marginal standardised residuals $\{\eta_{i,t}\}_{t=1}^{T}$, by applying the M-GARCH DCC model to the log return series and setting the conditional distribution of the standardised residuals to the Student's-*t* distribution to account for the heavy tails.

Copula parameters are estimated by the canonical maximum likelihood (CML) method (Cherubini et al., 2004). That is, we use pseudo-observations of the standardised residuals from the fitted DCC model to estimate the marginals and then estimate the copula parameters by means of maximum likelihood estimation (MLE):

$$\widehat{\Theta}_2 = ArgMax_{\Theta_2} \sum_{t=1}^{T} \ln c(\widehat{F}_1(x_{1t}), \ldots, \widehat{F}_n(x_{nt}); \Theta_2). \tag{23}$$

The best copula to be used for our VaR modeling is selected by comparing their MLE values. The copula that best fits the data should be the one with the highest MLE value. Chollete et al. (2009) argued that by selecting the best model based on the MLE method or a related criterion such as the Akaike information criterion (AIC) or Bayesian information criterion (BIC), the restriction will not matter because copulas that allow for negative dependence will still be chosen if the data set contains periods with negative dependence of data. As seen in Table 3, the same copula types have been selected based on the AIC and the MLE method, i.e., the copula with the smallest AIC value. We select two models, one from each copula family. Table 3 shows the MLE and copula parameter values for both elliptical and Archimedean copulas.

Next, we specify the desired marginal distributions, which we set to a Student's-*t* distribution. Student's-*t* distribution is selected for the margins because the multivariate ARCH test on the standardised residuals, $\{\eta_{i,t}\}_{t=1}^{T}$, i.e., before fitting the DCC models and the residuals after fitting the DCC models, fails to reject the null hypothesis of no conditional heteroscedasticity. This is a weakness of the DCC models because it is hard to justify that all correlations evolve in the same manner regardless of the assets involved (Tsay, 2013).

Finally, the estimated copula parameters are then used to generate a new matrix of $T \times N$ simulations,

$$\hat{\Sigma} = \{\zeta_{i,t}\}, i = 1, \ldots, N, t = 1, \ldots, T \tag{24}$$

with margins that are completely free from assumptions of normality or linear correlations. N represents the stocks in each country and T represents the length or the sample size of original data. We then reintroduce the GARCH(1,1) model, Eqns. (1a) and (1b), and convert the daily simulated data with t-margins to daily risk factor returns. That is,

$$\mathbf{r}_{i,t} = \mu_i + \xi_{i,t}, \xi_{i,t} = \hat{\sigma}_{i,t}\zeta_{i,t}, \tag{25}$$

where $\zeta_{i,t}$ are the daily simulated observations from the copulas with t-margins (Eqn. (24)).

3.3. Estimating VaRs

For each country, we apply the risk factor mappings to construct a simulated portfolio of returns consisting of all represented stocks. The portfolio return on day t is a weighted average of the return on individual stocks. Let **Inv** be the total amount invested in the portfolio, x_i be the fraction of the total investment in stock i, and $\mathbf{r}_{i,t}$ be the return of stock i at time t; then, the weight applied to $\mathbf{r}_{i,t}$ is the fraction of the portfolio invested in stock i calculated as $w_i = \frac{x_i}{\mathbf{Inv}}$. We assume equal weights, and therefore, the return on the portfolio at time t is given by

$$\overline{R}_{p,t} = E\left(R_{p,t}\right) = \sum_{i=1}^{N} w_i E\left(\mathbf{r}_{i,t}\right), \sum_{i=1}^{N} w_i = 1. \tag{26}$$

The one day VaR at time t with $q\%$ confidence level is calculated as

$$VaR_{q,t} = F_q^{-1}(\overline{R}_p), \tag{27}$$

where F_q^{-1} represents the q-th quantile of the distribution of $\overline{R}_p$ below which lies $(1-q)\%$ of the observations and above which lies $q\%$ of the observations. Thus, we are $q\%$ confident that in the worst case scenario, the losses on the portfolio will not exceed the $1-q$ quantile. Table 4 shows VaR estimates with confidence levels $q = 99\%, 95\%$, and 90%, based on the selected elliptical and Archimedean copulas for the constructed portfolios.

3.4. Back-testing

To check that the model does not overestimate or underestimate risk, we do back-testing on the model. This involves comparing the estimated VaRs for a given number of days to the subsequent portfolio returns. The number of days X in which the loss on the portfolio exceeds VaR is recorded as the number of exceptions or failures. Too many exceptions implies the VaR model underestimates the level of risk on the portfolio, and too few exceptions implies the model overestimates risk. The number of exceptions should be reasonably close to $T(1-q)\%$, where q is confidence level (CL); depends on the choice of q and follows a binomial distribution

$$f(X) = \binom{T}{X} p^X q^{T-X}, \tag{28}$$

with mean $= pT$ and variance $= pqT$; $p = 1-q$ (Best, 2000).

The most common back-testing methods include the standard normal hypothesis (or failure rate) test, Basel's *traffic light* test, Kupiec's proportion of failures (POF) test by Kupiec (1995), and Christoffersen's test (Christoffersen, 1998). We employ the standard normal hypothesis test, Basel's *traffic light* test, and Kupiec's POF test to check the reliability of the VaR model on a window of 250 and 500 observation periods and record the number of exceptions produced.

From the central limit theorem and with sufficiently large T, Eq. (28) can be approximated by the normal distribution

$$z = \frac{X - pT}{\sqrt{pqT}} \approx \Phi(0,1), \tag{29}$$

which is also the test statistic for a standard normal hypothesis test to assess the reliability of the VaR model (Jorion, 2007). The VaR model is rejected if $z <$ -$z_{p/2}$ or $z > z_{p/2}$ for a two-tailed test and if $z > z_p$ for a one-tailed test. $z_{p/2}$ and z_p are the cutoff values for the inverse standard normal cumulative distribution of p and $p/2$, respectively. Tables 5 and 6 show back-testing results based on the standard normal hypothesis test.

Table 4
VaR estimates based on the selected Archimedean and elliptical copulas for the constructed portfolios.

Portfolio	Confidence level	Copula family	
		Archimedean	Elliptical
UK		Clayton	Student's-*t*
	99%	−7.52	−7.85
	95%	−3.41	−3.76
	90%	−2.19	−2.39
Germany		Clayton	Student's-*t*
	99%	−6.24	−5.31
	95%	−3.52	−2.90
	90%	−2.53	−2.13
Greece		Gumbel	Student's-*t*
	99%	−14.82	−18.88
	95%	−7.23	−8.68
	90%	−5.04	−5.71
Spain		Frank	Student's-*t*
	99%	−6.12	−7.02
	95%	−3.75	−3.78
	90%	−2.79	−2.65
France		Clayton	Student's-*t*
	99%	−5.11	−4.54
	95%	−2.49	−2.52
	90%	−1.81	−1.77
Italy		Frank	Student's-*t*
	99%	−5.76	−7.35
	95%	−3.77	−4.07
	90%	−2.87	−2.78
Sweden		Gumbel	Gaussian-*t*
	99%	−6.44	−8.11
	95%	−3.53	−3.37
	90%	−2.34	−2.47

Table 5
Testing the reliability of the VaR model based on the standard normal hypothesis test. The returns generated using the selected Archimedean copulas.

Portfolio	CL	Exceptions		VaR coverage (%)		z-score		One-tailed test		Two-tailed test	
		250	500	250	500	250	500	250	500	250	500
UK	99%	28	27	0.98	0.94	−0.13	−0.32	Accept	Accept	Accept	Accept
	95%	141	138	4.94	4.81	−0.21	−0.47	Accept	Accept	Accept	Accept
	90%	277	264	9.65	9.20	−0.62	−1.43	Accept	Accept	Accept	Accept
Germany	99%	27	26	0.94	0.91	−0.32	−0.50	Accept	Accept	Accept	Accept
	95%	137	132	4.78	4.60	−0.55	−0.98	Accept	Accept	Accept	Accept
	90%	266	259	9.27	9.03	−1.24	−1.66	Accept	Accept	Accept	Accept
Greece	99%	27	27	0.94	0.94	−0.32	−0.32	Accept	Accept	Accept	Accept
	95%	143	140	4.98	4.88	−0.04	−0.30	Accept	Accept	Accept	Accept
	90%	282	279	9.83	9.27	−0.30	−0.49	Accept	Accept	Accept	Accept
Spain	99%	27	27	0.94	0.94	−0.32	−0.32	Accept	Accept	Accept	Accept
	95%	142	142	4.95	4.95	−0.12	−0.12	Accept	Accept	Accept	Accept
	90%	283	283	9.86	9.86	−0.24	−0.24	Accept	Accept	Accept	Accept
France	99%	26	26	0.91	0.91	−0.50	−0.50	Accept	Accept	Accept	Accept
	95%	140	130	4.88	4.53	−0.30	−1.15	Accept	Accept	Accept	Accept
	90%	272	253	9.48	8.82	−0.93	−2.11	Accept	Accept	Accept	Accept
Italy	99%	27	27	0.94	0.94	−0.32	−0.32	Accept	Accept	Accept	Accept
	95%	143	143	4.98	4.98	−0.04	−0.04	Accept	Accept	Accept	Accept
	90%	283	281	9.86	9.79	−0.24	−0.37	Accept	Accept	Accept	Accept
Sweden	99%	27	25	0.94	0.87	−0.32	−0.69	Accept	Accept	Accept	Accept
	95%	141	137	4.91	4.78	−0.21	−0.55	Accept	Accept	Accept	Accept
	90%	280	269	9.76	9.38	−0.43	−1.11	Accept	Accept	Accept	Accept

The second back-testing approach is Basel's *traffic light*, that was originally proposed by BCBS in 1996. In the new accord, BCBS further came up with a set of requirements that the VaR model must satisfy for it to be considered a reliable risk measure (Resti, 2008). That is, (i) VaR must be calculated with 99% confidence, (ii) back-testing must be done using a minimum of a one year observation period and must be tested over at least 250 days, (iii) regulators should be 95% confident that they are not erroneously rejecting a valid VaR model, and (iv) Basel specifies a one-tailed test –it is only interested in the underestimation of risk. For an unbiased VaR model, we expect a maximum of $2.5 \approx 3$ exceptions over a period of 250 days at $p = 1\%$ confidence level. Depending on the number of exceptions, the bank is placed in a green, yellow, or red zone; see Table 7 that shows the acceptance region. In the

Table 6
Testing the reliability of the VaR model based on the standard normal hypothesis test. The returns generated using the selected elliptical copulas.

Portfolio	CL	Exceptions		VaR coverage (%)		z-score		One-tailed test		Two-tailed test	
		250	500	250	500	250	500	250	500	250	500
UK	99%	27	27	0.94	0.94	−0.32	−0.32	Accept	Accept	Accept	Accept
	95%	142	141	4.95	4.91	−0.12	−0.21	Accept	Accept	Accept	Accept
	90%	281	273	9.79	9.52	−0.37	−0.87	Accept	Accept	Accept	Accept
Germany	99%	27	27	0.94	0.94	−0.32	−0.32	Accept	Accept	Accept	Accept
	95%	140	132	4.88	4.60	−0.30	−0.98	Accept	Accept	Accept	Accept
	90%	279	264	9.72	9.20	−0.47	−1.36	Accept	Accept	Accept	Accept
Greece	99%	23	23	0.80	0.80	−1.07	−1.07	Accept	Accept	Accept	Accept
	95%	122	121	4.25	4.22	−1.84	−1.92	Accept	Accept	Accept	Accept
	90%	253	245	8.82	8.54	−2.11	−2.61	Accept	Accept	Accept	Accept
Spain	99%	27	27	0.94	0.94	−0.32	−0.32	Accept	Accept	Accept	Accept
	95%	142	137	4.95	7.78	−0.12	−0.55	Accept	Accept	Accept	Accept
	90%	279	271	9.72	9.45	−0.49	−0.99	Accept	Accept	Accept	Accept
France	99%	27	27	0.94	0.94	−0.32	−0.32	Accept	Accept	Accept	Accept
	95%	139	135	4.84	4.71	−0.38	−0.72	Accept	Accept	Accept	Accept
	90%	280	286	9.76	9.34	−0.43	−1.18	Accept	Accept	Accept	Accept
Italy	99%	27	27	0.94	0.94	−0.32	−0.32	Accept	Accept	Accept	Accept
	95%	140	140	4.88	4.88	−0.30	−0.30	Accept	Accept	Accept	Accept
	90%	282	275	9.83	9.59	−0.74	−0.30	Accept	Accept	Accept	Accept
Sweden	99%	27	25	0.94	0.87	−0.32	−0.69	Accept	Accept	Accept	Accept
	95%	141	134	4.91	4.67	−0.21	−0.81	Accept	Accept	Accept	Accept
	90%	278	263	9.69	9.17	−0.55	−1.49	Accept	Accept	Accept	Accept

red zone, the VaR model underestimates risk and is out-rightly rejected. Test results based on this test are shown in Table 8.

Table 7
Acceptance region for Basel's *traffic light* approach for back-testing VaR models.

Zone	Number of exceptions (X)	Cumulative probability
Green	≤ 4	89.22%
Yellow	5	95.88%
	6	98.63%
	7	99.60%
	8	99.89%
	9	99.97%
Red	≥ 10	99.99%

In Kupiec's POF test, Kupiec (1995) developed a 95% confidence region for unconditional coverage (UC) whereby the number of exceptions produced by the model must be within this interval for it to be considered a reliable risk measure. The test is based on the likelihood ratio

$$LR_{POF} = -2\ln \frac{q^{T-X}p^{X}}{\left(1-\frac{X}{T}\right)^{T-X}\left(\frac{X}{T}\right)^{X}}, \tag{30}$$

asymptotically distributed χ_1^2. We consider an indicator function on the exceptions as

$$\mathbb{I}_t(p) = \mathbb{I}_{\{L_t > VaR_t(q)\}} = \begin{cases} 1, & \text{if } L_t > VaR_t(q) \\ 0, & \text{otherwise.} \end{cases} \tag{31}$$

Under the UC, the null hypothesis for LR_{POF} is $H_0 : \mathrm{E}[\mathbb{I}_t(p)] = \frac{X}{T} = p$ against $H_a : \mathrm{E}[\mathbb{I}_t(p)] = \frac{X}{T} \neq p$. The VaR model is rejected if $LR_{POF} > \chi_1^2 = 3.841$. Hence by equating Eqn. (30) to 3.841 and solving for X, we obtain two values of X; the rejection region is $[X_1, X_2]$. The VaR model is rejected if $X \notin [X_1, X_2]$ and accepted if $X \in [X_1, X_2]$ (Holton, 2002). Results based on this test are shown in Tables 9 and 10.

The results show that as the observation period increases from 250 to 500 days at a 99% confidence level, the number of exceptions produced is unchanged using elliptical copulas with the exception of Sweden, which exhibits a difference of 2. With Archimedean copulas, there is a difference of 1, 1, and 2 for the UK, Germany and Sweden, respectively, at the 99% confidence level. The number of exceptions produced is also very close to the expectation $T(1-q)\%$, i.e., 29, with the exception of Greece for

Table 8
Testing the reliability of the VaR model based on BCBS requirements.

Portfolio	Copula family	Exceptions		C.Pr (one-tailed test) %		C.Pr (two-tailed test) %		Zone (one-tailed test)		Zone (two-tailed test)		One-tailed test		Two-tailed test	
		250	500	250	500	250	500	250	500	250	500	250	500	250	500
UK	Archimedean	28	27	49.81	42.32	99.96	99.91	Green	Green	Yellow	Yellow	Accept	Accept	Accept	Accept
	Elliptical	27	27	42.32	42.32	99.91	99.91	Green	Green	Yellow	Yellow	Accept	Accept	Accept	Accept
Germany	Archimedean	27	26	42.32	35.01	99.91	98.82	Green	Green	Yellow	Yellow	Accept	Accept	Accept	Accept
	Elliptical	27	27	42.32	42.32	99.91	99.91	Green	Green	Yellow	Yellow	Accept	Accept	Accept	Accept
Greece	Archimedean	27	27	42.32	42.32	99.91	99.91	Green	Green	Yellow	Yellow	Accept	Accept	Accept	Accept
	Elliptical	23	23	16.52	16.52	98.80	98.80	Green	Green	Yellow	Yellow	Accept	Accept	Accept	Accept
Spain	Archimedean	27	27	42.32	42.32	99.91	99.91	Green	Green	Yellow	Yellow	Accept	Accept	Accept	Accept
	Elliptical	27	27	42.32	42.32	99.91	99.91	Green	Green	Yellow	Yellow	Accept	Accept	Accept	Accept
France	Archimedean	26	26	35.00	35.00	99.82	99.82	Green	Green	Yellow	Yellow	Accept	Accept	Accept	Accept
	Elliptical	27	27	42.32	42.32	99.91	99.91	Green	Green	Yellow	Yellow	Accept	Accept	Accept	Accept
Italy	Archimedean	27	27	42.32	42.32	99.91	99.91	Green	Green	Yellow	Yellow	Accept	Accept	Accept	Accept
	Elliptical	27	27	42.32	42.32	99.91	99.91	Green	Green	Yellow	Yellow	Accept	Accept	Accept	Accept
Sweden	Archimedean	27	25	42.32	28.14	99.91	99.65	Green	Green	Yellow	Yellow	Accept	Accept	Accept	Accept
	Elliptical	27	25	42.32	28.14	99.91	99.65	Green	Green	Yellow	Yellow	Accept	Accept	Accept	Accept

Note: C.Pr represents the cumulative probability (see Table 7).

Table 9
Testing the reliability of the VaR model based on Kupiec's POF coverage test. The returns generated using the Archimedean copulas.

Portfolio	CL	Exceptions		Test statistic		Kupiec's POF test	
		250	500	250	500	250	500
UK	99%	28	27	0.02	0.10	Accept	Accept
	95%	141	138	0.04	0.22	Accept	Accept
	90%	277	264	NaN	NaN	Reject	Reject
Germany	99%	27	26	0.10	0.26	Accept	Accept
	95%	137	132	0.31	0.99	Accept	Accept
	90%	266	259	NaN	NaN	Reject	Reject
Greece	99%	27	27	0.1	0.1	Accept	Accept
	95%	143	140	0.00	0.09	Accept	Accept
	90%	282	279	NaN	NaN	Reject	Reject
Spain	99%	27	27	0.10	0.10	Accept	Accept
	95%	142	142	0.02	0.02	Accept	Accept
	90%	283	283	NaN	NaN	Reject	Reject
France	99%	26	26	0.26	0.26	Accept	Accept
	95%	140	130	0.09	1.37	Accept	Accept
	90%	272	253	NaN	NaN	Reject	Reject
Italy	99%	27	27	0.10	0.10	Accept	Accept
	95%	143	143	0.00	0.00	Accept	Accept
	90%	283	281	NaN	NaN	Reject	Reject
Sweden	99%	27	25	0.10	0.50	Accept	Accept
	95%	141	137	0.04	0.31	Accept	Accept
	90%	280	269	NaN	NaN	Reject	Reject

Table 10
Testing the reliability of the VaR model based on Kupiec's POF coverage test. The returns generated using the elliptical copulas.

Portfolio	CL	Exceptions		Test statistic		Kupiec's POF test	
		250	500	250	500	250	500
UK	99%	27	27	0.10	0.10	Accept	Accept
	95%	142	141	0.02	0.04	Accept	Accept
	90%	281	273	NaN	NaN	Reject	Reject
Germany	99%	27	27	0.10	0.10	Accept	Accept
	95%	140	132	0.09	0.99	Accept	Accept
	90%	279	264	NaN	NaN	Reject	Reject
Greece	99%	23	23	1.22	1.22	Accept	Accept
	95%	122	121	3.55	3.90	Accept	Reject
	90%	253	245	NaN	NaN	Reject	Reject
Spain	99%	27	27	0.10	0.10	Accept	Accept
	95%	142	137	0.02	0.31	Accept	Accept
	90%	279	271	NaN	NaN	Reject	Reject
France	99%	27	27	0.10	0.10	Accept	Accept
	95%	139	135	0.15	0.53	Accept	Accept
	90%	280	268	NaN	NaN	Reject	Reject
Italy	99%	27	27	0.10	0.10	Accept	Accept
	95%	140	140	0.00	0.00	Accept	Accept
	90%	282	275	NaN	NaN	Reject	Reject
Sweden	99%	27	25	0.10	0.50	Accept	Accept
	95%	141	134	0.04	0.67	Accept	Accept
	90%	278	263	NaN	NaN	Reject	Reject

the elliptical Student's-*t* copula. This suggests that the model at a 99% confidence level captures VaR extremely well because the difference in exceptions is very minimal or zero.

At the 95% confidence level, there is a significant difference between the number of exceptions produced in some of the countries when using 250- and 500-observation periods for both Archimedean and elliptical copulas. Although the number of exceptions is quite close to the expectation, i.e., 143, the significant difference indicates that there is greater room for error in estimating VaR at the 95% compared with the 99% confidence level. However, back-testing results indicate that the VaR model performed reasonably well at the 95% confidence level except for the models of Greece in the 500-observation period.

At a 90% confidence level, the difference in the number of exceptions is quite high for both copula families. The tail dependence structure of the portfolio returns is not quite accounted for and hence, the model fails to capture extreme events.

Back-testing results from Kupiec's POF test confirm the above analysis. The model is accepted at the 99% and 95% confidence levels for both copula families except for Greece, where the elliptical Student's-*t* copula rejects the model at a 95% confidence level with a 500-day observation period. At a 90% confidence level, Kupiec's POF test rejects the model for both copula families.

For the standard normal hypothesis test and Basel's *traffic light* test, we perform one- and two-tailed tests. Although Basel is only concerned with the underestimation of risk, we performed a two-tailed test to make sure that the model does not overestimate risk and thereby result in excess capital being provided (Best, 2000). The model is accepted in all cases for both one-tailed and two-tailed tests. However, for both Archimedean and elliptical copulas, the Basel's *traffic light* tests place the VaR model in the yellow zone for a two-tailed test when using 250- and 500-observation periods, suggesting that there might be some instances of overestimation of risk since for a one-tailed test the model falls in the green zone in all instances.

It is important to note that the standard normal hypothesis test and the Basel's *traffic light* test do not specify whether the number of exceptions produced is too small. That is, the model will not be rejected if the number of exceptions is too small, which would lead to overestimation of VaR. Kupiec's POF test rejects the model if the number of exceptions produced is too large or too small. This is why the standard normal hypothesis test and the Basel's *traffic light* test fail to reject the model at the 90% confidence level. Thus, from the banks perspective, Kupiec's POF test is preferable and superior because it accounts for both underestimation and overestimation of risk.

4. Conclusion

Because VaR models attempt to capture the behaviour of asset returns in the left tail, it is important that the model is constructed such that it does not underestimate the proportion of outliers and hence the true VaR. The normality assumption of asset returns might severely underestimate the true VaR because extreme values are assumed to be very unlikely to occur. For a reliable VaR model, it is important to take into account the dependent structure of the stock returns, and the type of volatility models being used. We construct our VaR model for a time horizon of one day using M-GARCH DCC model as the underlying volatility model and copula functions to model dependence among the stock returns.

The back-testing results cover the period between 2008 and 2011, and the VaR estimates presented in Table 4 clearly show that during these periods, which includes the 2008 global financial crisis and 2011 European financial crisis, the risk of collapse of the banking system of Greece was extremely high. This was especially because capital was insufficient to provide a cushion capable of withstanding sudden losses in periods of financial distress. This was almost double compared to the other countries, which should have been a red flag for the investors.

Back-testing results based on the standard normal hypothesis test, Basel's *traffic light* test and Kupiec's POF test suggest that the model is reliable as a risk measure at 99% and 95% confidence levels. The results of back-testing also suggest that the type of copula family used (i.e., elliptical or Archimedean) to model the dependence structure does not have a significant effect when dealing with quantile VaRs. The results are quite similar based on the number of exceptions produced.

This study also suggests a challenging yet possible development in the world of risk management, which is to design a back-testing method based on the Basel requirements that detects when the number of exceptions produced by a VaR model are too small; this is because the current *traffic light* back-testing method does not address this issue.

In subsequent studies, we suggest employing the extreme value theory to model the tail behaviour of the risk factors to see if there will be any improvement at the 90% confidence level where the VaR model is rejected by Kupiec's POF test.

Appendix A. Supplementary data

Supplementary data related to this article can be found at https://doi.org/10.1016/j.inteco.2018.03.001.

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