Exercise

(1)

Let $B = (B_{ij})$ be an $N \times N$ real symmetric matrix (where $i, j = 1, \cdots, N$) . The inverse matrix of B is denoted by B^{-1}.

(a) Consider a system with canonical coordinates $q = (q_i)$ $(i = 1, \cdots, N)$ whose Lagrangian is given by

$$L(q, \dot{q}) = \sum_{ij} \frac{1}{2} B_{ij} \dot{q}_i \dot{q}_j - V(q), \tag{1}$$

where $\dot{q}_i$ is the time derivative of q_i, and $V(q)$ is a function of q. Find the canonical momenta p^i conjugate to q_i, and write down the canonical commutation relations.

(b) Write down the Hamiltonian of the above system as a function of q_i and p^i by using the inverse matrix B^{-1} of B.

(2)

(a) Let A be a 2×2 matrix given by

$$A = (A_{ij}) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \tag{2}$$

Find the eigenvalues of A.

(b) Consider a quantum mechanical system with two particles in one-dimension. The position and momentum of the particles are denoted by (x_i, p_i), where $i = 1, 2$ is the label that specifies the two particles. Suppose that the Hamiltonian is given by

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}\omega^2 \sum_{i,j} A_{ij} x_i x_j. \tag{3}$$

where ω is a parameter, and $A = (A_{ij})$ is the matrix given above. Find the energy spectrum of this Hamiltonian.