Basic quantum field theory

Kazuya Yonekura

Department of physics, graduate school of Science, Tohoku University

Abstract: Lecture notes for the course (Version: 2024/10/4)

Contents

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0 About this course

- Self-introduction: Kazuya Yonekura (high energy theory)
- Google classroom (zu7iym4)
- Evaluation: by exam at the end of the semester 2025/1/28 Tuesday, (January 28th) See the calendar of the graduate school of science.
- No class on 2024/10/25 (October 25th, University Festival),
  2024/11/1 (November 1st)
- Assumed knowledge :
  analytical mechanics, quantum mechanics (QM), electromagnetism (EM), special relativity

This course: Quantum Field Theory (QFT)

Abbreviation:
EM = electromagnetism
QM = quantum mechanics
QFT = quantum field theory

What is QFT?

- High energy physics: the fundamental framework for the laws of physics, including the standard model of particle physics
- Condensed matter physics: effective description of various many body physics phenomena (e.g. superconductivity)

Examples of fields

- All elementary particles are described by some fields.
  e.g. EM field, EM waves $\to$ photon
- Collective motions in many body systems
  e.g. sound waves $\to$ phonon

Remark
The language of relativistic QFT is used.
Natural units:

$$c: \text{ speed of light} \tag{0.1}$$
$$\hbar: \text{ Planck constant divided by } 2\pi \tag{0.2}$$
$$c = 1, \qquad \hbar = 1 \tag{0.3}$$

I will use natural units later.
For cond-mat., $c \to v$ (an appropriate speed of some field)

1 Non-relativistic QM

1.1 Notation, convention

$$t: \text{ time coordinate} \tag{1.1}$$
$$\vec{x} = (x^i) \quad (i = 1, 2, 3) \quad : \text{space coordinates} \tag{1.2}$$
$$\vec{p} = (p^i): \text{ momentum} \tag{1.3}$$
$$\tag{1.4}$$

A dot : time derivative

$$\dot{\vec{x}} = \frac{d\vec{x}}{dt} \qquad \ddot{\vec{x}} = \frac{d^2\vec{x}}{dt^2} \tag{1.5}$$

Einstein summation notation

$$A_i B_i = \sum_{i=1,2,3} A_i B_i = \vec{A} \cdot \vec{B} \tag{1.6}$$

Upper and lower indices mean the same thing: $A_i = A^i$.

Partial derivative:

$$\partial_i = \frac{\partial}{\partial x^i}, \qquad \partial_t = \frac{\partial}{\partial t} \tag{1.7}$$

Potentials for EM fields

$$\phi : \text{electric potential} \qquad \vec{A} = (A_i) : \text{vector potential} \tag{1.8}$$

EM fields

$$\vec{E} = -\vec{\partial}\phi - \frac{\partial \vec{A}}{\partial t} = -\partial_i \phi - \partial_t A_i \tag{1.9}$$
$$\vec{B} = \vec{\partial} \times \vec{A} = \frac{1}{2}\epsilon_{ijk}(\partial_i A_j - \partial_j A_i) \tag{1.10}$$

ϵ_{ijk} : totally antisymmetric tensor,

$$\epsilon_{ikj} = -\epsilon_{ijk}, \quad \epsilon_{kji} = -\epsilon_{ijk}, \quad \epsilon_{jik} = -\epsilon_{ijk}, \quad \epsilon_{123} = 1. \tag{1.11}$$

Explicitly

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1, \qquad \epsilon_{213} = \epsilon_{132} = \epsilon_{321} = -1, \qquad \text{others } = 0 \tag{1.12}$$

The exterior product

$$(\vec{A} \times \vec{B})_i = \epsilon_{ijk} A_j B_k. \quad \text{(Einstein's summation notation for } j, k) \tag{1.13}$$
$$= \sum_{j,k} \epsilon_{ijk} A_j B_k \tag{1.14}$$

1.2 Analytical mechanics

The action for a particle interacting with EM field:

$$S = \int dt\,(\frac{1}{2}m\dot{\vec{x}}^2 - e\phi) + e\int A_i \frac{dx^i}{dt}dt \tag{1.15}$$

$$m\text{: mass、 } e\text{: charge} \tag{1.16}$$

Equations of motion (EOM) is obtained by the principle of least action:
Under $\vec{x} \to \vec{x} + \delta\vec{x}$ ($\delta\vec{x}$: infinitesimal), the action is stationary, $\delta S = 0$.

$$\delta S = \int dt(m\delta\dot{x}^i\dot{x}^i - e\delta x^i\partial_i\phi) + e\int dt(\delta x^j\partial_j A_i\dot{x}^i + A_i\dot{\delta x^i}) \tag{1.17}$$

$$= \int dt(-m\ddot{x}^i - e\partial_i\phi)\delta x^i + e\int dt\,\delta x^i(\partial_i A_j\dot{x}^j - \partial_j A_i\dot{x}^j - \partial_t A_i) \tag{1.18}$$

where

$$\frac{d}{dt}A_i = \partial_j A_i\dot{x}^j + \partial_t A_i \tag{1.19}$$

is used.

$$\delta S = \int dt\left(-m\ddot{x}^i - e(\partial_i\phi + \partial_t A_i) + e(\partial_i A_j - \partial_j A_i)\dot{x}^j\right)\delta x^i = 0 \tag{1.20}$$

EOM

$$m\ddot{x}^i = -e(\partial_i\phi + \partial_t A_i) + e(\partial_i A_j - \partial_j A_i)\dot{x}^j \tag{1.21}$$

$$= eE^i + e\epsilon_{ijk}B^k\dot{x}^j \qquad (\partial_i A_j - \partial_j A_i = \epsilon_{ijk}B^k) \tag{1.22}$$

$$= e(\vec{E} + \dot{\vec{x}} \times \vec{B}) \text{ : EOM} \tag{1.23}$$

Lagrangian L ($S = \int dtL$)

$$L = \frac{1}{2}m\dot{\vec{x}}^2 - e\phi + eA_i\dot{x}^i \tag{1.24}$$

Canonical momentum

$$p_i = \frac{\partial L}{\partial\dot{x}^i} = m\dot{x}^i + eA^i \tag{1.25}$$

Hamiltonian

$$H = p_i\dot{x}^i - L \tag{1.26}$$

$$= \frac{1}{2}m\dot{\vec{x}}^2 + e\phi \tag{1.27}$$

$$= \frac{1}{2m}(\vec{p} - e\vec{A})^2 + e\phi. \tag{1.28}$$

1.3 Canonical quantization

So far classical mechanics.
Now quantum.

The rule for quantization:

$$x^i, p_j \quad : \text{operators on the Hilbert space of physical states} \tag{1.29}$$

Impose canonical commutation relations to x^i and p_j ,

$$[x^i, p_j] = i\hbar\delta^i_j \tag{1.30}$$

δ^i_j : the Kronecker delta

$$\delta^i_j = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases} \tag{1.31}$$

More explicitly, p_j is realized on wavefunctions $\Psi(\vec{x})$ by

$$p_j\Psi(\vec{x}) = -i\hbar\partial_j\Psi(\vec{x}) \tag{1.32}$$

Hamiltonian operator

$$H\Psi = \left[\frac{1}{2m}(\vec{p} - e\vec{A})^2 + e\phi\right]\Psi \tag{1.33}$$

$$= \left[\frac{1}{2m}(-i\hbar\partial_i - eA_i)^2 + e\phi\right]\Psi \tag{1.34}$$

(Time dependent) Schrödinger equation

$$i\hbar\partial_t\Psi = H\Psi \tag{1.35}$$

Here, the meaning of $(-i\hbar\partial_i - eA_i)^2$ is to act $(-i\hbar\partial_i - eA_i)$ twice and take some over i.
More explicitly, definite

$$(-i\hbar\partial_i - eA_i)^2\Psi = \sum_i(-i\hbar\partial_i - eA_i)[(-i\hbar\partial_i - eA_i)\Psi]. \tag{1.36}$$

1.4 Covariant derivatives and gauge transformations

Covariant derivative will be an important concept in QFT.

For Ψ in QM,

$$D_i = \partial_i - i\frac{e}{\hbar}A_i, \qquad D_t = \partial_t + i\frac{e}{\hbar}\phi \tag{1.37}$$

Schrödinger equation is now

$$i\hbar D_t\Psi = -\frac{\hbar^2}{2m}\vec{D}^2\Psi \tag{1.38}$$

$\vec{E}, \vec{B}$ are invariant under gauge transformations (exercise)

$$A_i \to A'_i = A_i + \partial_i \alpha, \qquad \phi \to \phi' = \phi - \partial_t \alpha. \tag{1.39}$$

α : an arbitrary function.

The transformation of Ψ is defined as

$$\Psi \to \Psi' = \exp(i\frac{e}{\hbar}\alpha)\Psi \tag{1.40}$$

Then

$$D'_i \Psi' = \left(\partial_i - i\frac{e}{\hbar}(A_i + \partial_i \alpha)\right)\left(\exp(i\frac{e}{\hbar}\alpha)\Psi\right) \tag{1.41}$$

$$= \exp(i\frac{e}{\hbar}\alpha)\left(\partial_i - i\frac{e}{\hbar}A_i\right)\Psi \tag{1.42}$$

$$= \exp(i\frac{e}{\hbar}\alpha) D_i \Psi \tag{1.43}$$

Similarly

$$(D'_i)^2 \Psi' = \exp(i\frac{e}{\hbar}\alpha)(D_i)^2 \Psi \tag{1.44}$$

$$D'_t \Psi' = \exp(i\frac{e}{\hbar}\alpha) D_t \Psi \tag{1.45}$$

Then

$$i\hbar D_t \Psi = -\frac{\hbar^2}{2m}\vec{D}^2 \Psi \quad \Longleftrightarrow \quad i\hbar D'_t \Psi' = -\frac{\hbar^2}{2m}\vec{D}'^2 \Psi' \tag{1.46}$$

Principle: physics is invariant under gauge transformations.

1.5 Harmonic oscillator

Harmonic oscillators will play very important roles in QFT.

In QM (one-dimension),

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2. \tag{1.47}$$

ω : parameter.

Define creation and annihilation operators $a, a^\dagger$:

$$a = \sqrt{\frac{m\omega}{2\hbar}}x + i\sqrt{\frac{1}{2\hbar\omega m}}p, \tag{1.48}$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}x - i\sqrt{\frac{1}{2\hbar\omega m}}p \tag{1.49}$$

Commutation relations

$$[a, a^\dagger] = \frac{1}{2\hbar}(-i[x,p] + i[p,x]) \tag{1.50}$$

$$= 1 \qquad ([x,p] = i\hbar) \tag{1.51}$$

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger), \qquad p = -i\sqrt{\frac{\hbar\omega m}{2}}(a - a^\dagger). \tag{1.52}$$

$$H = -\frac{\hbar\omega}{4}(a - a^\dagger)^2 + \frac{\hbar\omega}{4}(a + a^\dagger)^2 \tag{1.53}$$

$$= \frac{\hbar\omega}{2}(aa^\dagger + a^\dagger a) \tag{1.54}$$

$$= \hbar\omega(a^\dagger a + \frac{1}{2}). \tag{1.55}$$

Define state vectors $|n\rangle$ $(n = 0, 1, 2, \cdots)$ by

$$|0\rangle \ : \ a|0\rangle = 0, \tag{1.56}$$

$$|n+1\rangle = \frac{1}{\sqrt{n+1}}a^\dagger|n\rangle \qquad (n = 0, 1, 2, \cdots). \tag{1.57}$$

Number operator

$$N = a^\dagger a \tag{1.58}$$

Remark: for any operators A, B, C

$$[AB, C] = ABC - CAB = A(BC - CB) + (AC - CA)B = A[B,C] + [A,C]B \tag{1.59}$$

Then

$$[N, a^\dagger] = [a^\dagger a, a^\dagger] = a^\dagger[a, a^\dagger] + [a^\dagger, a^\dagger]a = a^\dagger \tag{1.60}$$

$$[N, a^\dagger] = a^\dagger, \qquad [N, a] = -a. \tag{1.61}$$

One can show

$$N|n\rangle = n|n\rangle. \tag{1.62}$$

Proof by induction:

1. For $n = 0$, $a^\dagger a|0\rangle = 0$ by definition of $|0\rangle$.

2. If $N|n\rangle = n|n\rangle$, then

$$N|n+1\rangle = \frac{1}{\sqrt{n+1}}Na^\dagger|n\rangle \tag{1.63}$$

$$= \frac{1}{\sqrt{n+1}}([N, a^\dagger] + a^\dagger N)|n\rangle \tag{1.64}$$

$$= \frac{1}{\sqrt{n+1}}(a^\dagger + a^\dagger N)|n\rangle \tag{1.65}$$

$$= \frac{1}{\sqrt{n+1}}a^\dagger(n+1)|n\rangle \tag{1.66}$$

$$= (n+1)|n+1\rangle. \tag{1.67}$$

Proof completed.

Hamiltonian

$$H = \hbar\omega(N + \frac{1}{2}) \tag{1.68}$$

$$[H, a^\dagger] = \hbar\omega a^\dagger, \qquad [H, a] = -\hbar\omega a. \tag{1.69}$$

The energy E_n of the state $|n\rangle$:

$$H|n\rangle = E_n|n\rangle \tag{1.70}$$

$$E_n = \hbar\omega(n + \frac{1}{2}). \tag{1.71}$$

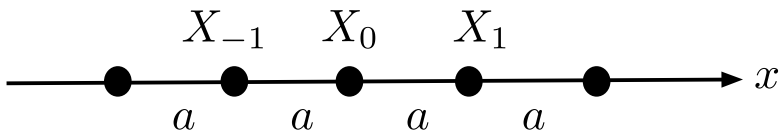
If you are not familiar with QM and analytical mechanics, learn basic things by yourself.

2 A condensed matter example of QFT : phonon

2.1 A lattice model

A simple model: a chain of harmonic oscillators in one-dimension.

$$(X_n, P_n) \quad (n = \cdots, -1, 0, 1, \cdots \quad) \tag{2.1}$$



Canonical commutation relations

$$[X_n, P_m] = i\hbar\delta_{nm} \tag{2.2}$$
$$[X_n, X_m] = 0, \qquad [P_n, P_m] = 0 \tag{2.3}$$

Hamiltonian

$$H = \sum_n \left(\frac{P_n^2}{2M} + \frac{1}{2}K(X_{n+1} - X_n - a)^2 \right) \tag{2.4}$$

$$a\,:\,\text{lattice spacing}, \qquad M\,:\,\text{mass of each nucleus}\;, \qquad K\,:\,\text{spring constant} \tag{2.5}$$

Classically, the energy is minimized for

$$X_n = na + C \qquad (C\,:\,\text{constant}) \tag{2.6}$$

We set $C = 0$ for simplicity.

For later convenience, define new variables (ϕ_n, π_n)

$$X_n = na + \sqrt{\frac{a}{M}}\phi_n, \qquad P_n = \sqrt{aM}\pi_n \tag{2.7}$$

Then,

$$[\phi_n, \pi_m] = \frac{1}{a}[X_n, P_m] = \frac{i\hbar\delta_{mn}}{a} \tag{2.8}$$

Also

$$\frac{P_n^2}{2M} = \frac{1}{2}a\pi_n^2 \tag{2.9}$$
$$\frac{1}{2}K(X_{n+1} - X_n - a)^2 = \frac{aK}{2M}(\phi_{n+1} - \phi_n)^2 = \frac{a}{2}v^2\left(\frac{\phi_{n+1} - \phi_n}{a}\right)^2 \tag{2.10}$$

where

$$v^2 = \frac{a^2 K}{M}. \tag{2.11}$$

Then

$$H = a \sum_n \left(\frac{1}{2}\pi_n^2 + \frac{1}{2}v^2 \left(\frac{\phi_{n+1} - \phi_n}{a} \right)^2 \right) \tag{2.12}$$

$$[\phi_n, \pi_m] = \frac{i\hbar\delta_{mn}}{a} \tag{2.13}$$

2.2 Continuum limit

Suppose we are interested in collective motions whose wavelength λ is large enough,

$$\lambda \gg a, \qquad \text{or} \qquad ka \ll 1, \quad k = \frac{2\pi}{\lambda} : \text{ wavenumber}. \tag{2.14}$$

So, we take the limit

$$a \to 0. \tag{2.15}$$

Define

$$x = na : \text{ the positions of nuclei (at the static positions)} \tag{2.16}$$

(ϕ_n, π_n) are functions of x,

$$\phi_n \to \phi(x), \qquad \pi_n \to \pi(x). \tag{2.17}$$

From the definition of derivatives and integrals,

$$\frac{\phi_{n+1} - \phi_n}{a} \to \frac{\partial \phi}{\partial x}(x) = \partial_x \phi(x), \tag{2.18}$$

$$a \sum_n \to \int dx. \tag{2.19}$$

Hamiltonian

$$H = a \sum_n \left(\frac{1}{2}\pi_n^2 + \frac{1}{2}v^2 \left(\frac{\phi_{n+1} - \phi_n}{a} \right)^2 \right) \tag{2.20}$$

$$\to \int dx \left(\frac{1}{2}\pi(x)^2 + \frac{1}{2}v^2 \left(\partial_x \phi(x)\right)^2 \right) \tag{2.21}$$

Define Dirac delta function:

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases} \tag{2.22}$$

such that

$$\int_{-\infty}^{\infty} dx \delta(x) = 1 \tag{2.23}$$

Then

$$\frac{\delta_{nm}}{a} \to \delta(x-y), \qquad x = na, \quad y = ma \tag{2.24}$$

because

$$\frac{\delta_{nm}}{a} = \begin{cases} \frac{1}{a} \to \infty & n-m=0 \\ 0 & n-m \neq 0 \end{cases} \tag{2.25}$$

and

$$a \sum_n \frac{\delta_{nm}}{a} = 1. \tag{2.26}$$

Then

$$[\phi_n, \pi_m] = \frac{i\hbar\delta_{mn}}{a} \tag{2.27}$$

$$\to \ [\phi(x), \pi(y)] = i\hbar\delta(x-y). \tag{2.28}$$

Summary:

$$H = \int dx \left(\frac{1}{2}\pi^2 + \frac{1}{2}v^2 \left(\partial_x\phi\right)^2 \right) \tag{2.29}$$

$$[\phi(x), \pi(y)] = i\hbar\delta(x-y). \tag{2.30}$$

An example of QFT, called phonon.

2.3 Lagrangian formulation

Let's return to classical theory.

The original Hamiltonian

$$H = \sum_n \left(\frac{P_n^2}{2M} + \frac{1}{2}K(X_{n+1} - X_n - a)^2 \right) \tag{2.31}$$

Lagrangian

$$L = \sum_n \left(\frac{M}{2}(\dot{X}_n)^2 - \frac{1}{2}K(X_{n+1} - X_n - a)^2 \right) \tag{2.32}$$

$$X_n = na + \sqrt{\frac{a}{M}}\phi_n \tag{2.33}$$

$$L = a\sum_n \left(\frac{1}{2}(\dot{\phi}_n)^2 - \frac{1}{2}v^2 \left(\frac{\phi_{n+1} - \phi_n}{a} \right)^2 \right) \tag{2.34}$$

$$\to \int dx \left(\frac{1}{2}(\partial_t\phi) - \frac{1}{2}v^2 \left(\partial_x\phi\right)^2 \right) \tag{2.35}$$

$$= \int dx \mathcal{L} \tag{2.36}$$

Here

$$\mathcal{L} = \frac{1}{2}(\partial_t \phi) - \frac{1}{2}v^2 \left(\partial_x \phi\right)^2 \quad : \text{Lagrangian density} \tag{2.37}$$

The canonical momentum

$$\pi_n = \frac{P_n}{\sqrt{aM}} = \sqrt{\frac{M}{a}}\dot{X}_n = \dot{\phi}_n. \tag{2.38}$$

hence

$$\pi(x) = \partial_t \phi(x). \tag{2.39}$$

This can be written as

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} \tag{2.40}$$

The action

$$S = \int dt dx \mathcal{L} \tag{2.41}$$

2.4 Classical dynamics

The principles of least action:

$$S = \int dt dx \left(\frac{1}{2}(\partial_t \phi) - \frac{1}{2}v^2 \left(\partial_x \phi\right)^2\right) \tag{2.42}$$

$$\phi \to \phi + \delta\phi \tag{2.43}$$

$$\begin{aligned} 0 = \delta S &= \int dt dx \left(\partial_t \phi \partial_t \delta\phi - v^2 \partial_x \phi \partial_x \delta\phi\right) && (2.44) \\ &= \int dt dx \left(-\partial_t^2 \phi + v^2 \partial_x^2 \phi\right) \delta\phi && (2.45) \end{aligned}$$

$$(-\partial_t^2 + v^2 \partial_x^2)\phi = 0 \tag{2.46}$$

If we consider

$$\phi = \cos(kx - \omega t), \tag{2.47}$$

$$\omega : \text{angular frequence}, \qquad k : \text{wavenumber} \tag{2.48}$$

then

$$\begin{aligned} &\omega^2 = v^2 k^2 \quad \Longrightarrow \quad \phi = \cos k(x \mp vt) && (2.49) \\ &v : \text{the speed of sound} && (2.50) \end{aligned}$$

3 Free scalar field theory

Aim: systematic quantization of a scalar field.

From now on,

- Consider relativistic QFT. (Sometimes good approximation for condensed matter systems if the speed of light c is replaced by an appropriate velocity v.)
- Use natural units

  $$\hbar = 1, \qquad c = 1. \tag{3.1}$$

  They can be recovered by dimensional analysis.
- Consider three spatial dimensions. (The previous example was one dimension. It is straightforward to generalize to other dimensions.)

3.1 Special relativity

Coordinates of spacetime

$$x^\mu = (ct, x^i) = (ct, x^1, x^2, x^3) = (x^0, \vec{x}) \tag{3.2}$$

In natural units $c = 1$, $x^0 = t$.
space indices : Latin letters $i, j, \cdots = 1, 2, 3$,
spacetime indices : Greek letters $\mu, \nu, \cdots = 0, 1, 2, 3$.

Spacetime metric tensor

$$g_{\mu\nu} = \begin{pmatrix} - & & & \\ & + & & \\ & & + & \\ & & & + \end{pmatrix} \tag{3.3}$$

$$ds^2 = -d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu = -(dx^0)^2 + (d\vec{x})^2. \tag{3.4}$$

It is the distance between two spacetime points x^μ and $x^\mu + dx^\mu$:

- When $ds^2 > 0$, spatial distance ds
- When $ds^2 < 0$, proper time difference $d\tau$

4-vector

$$A^\mu, \qquad A_\mu = g_{\mu\nu} A^\nu \tag{3.5}$$

Einstein summation notation

$$A^\mu B_\mu = \sum_{\mu=0}^{3} A^\mu B_\mu \tag{3.6}$$

$$A^2 = A^\mu A_\mu = -(A^0)^2 + (\vec{A})^2 \tag{3.7}$$

4-momentum of a particle

$$p^\mu = m\frac{dx^\mu}{d\tau} = m(\frac{1}{\sqrt{1-\vec{v}^2}}, \frac{\vec{v}}{\sqrt{1-\vec{v}^2}}) \tag{3.8}$$

$$\vec{v} = \frac{dx^i}{dt} \tag{3.9}$$

$$p^2 = p^\mu p_\mu = -(p^0)^2 + \vec{p}^2 = -m^2 \tag{3.10}$$

$$\Longrightarrow p^0 = \sqrt{\vec{p}^2 + m^2} := E_p \tag{3.11}$$

Partial derivative

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \tag{3.12}$$

EM potential

$$A^\mu = (+\phi, \vec{A}), \qquad A_\mu = (-\phi, \vec{A}). \tag{3.13}$$

EM tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.14}$$

$$= \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & B^3 & -B^2 \\ E^2 & -B^3 & 0 & B_1 \\ E^3 & B^2 & -B^1 & 0 \end{pmatrix} \tag{3.15}$$

$$F_{i0} = -F^{i0} = E_i \tag{3.16}$$

$$F_{ij} = \epsilon_{ijk} B^k \tag{3.17}$$

If you are not familiar with special relativity, learn basic things by yourself.

3.2 Real scalar field

Real scalar field ϕ
Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \tag{3.18}$$

$$S = \int d^4x \mathcal{L} \tag{3.19}$$

Equations of motion: $\phi \to \phi + \delta\phi$,

$$0 = \delta S = \int d^4x \delta\phi(\partial^2\phi - m^2). \tag{3.20}$$

$$(\partial^2 - m^2)\phi = 0 : \text{ Klein-Gordon eq.} \tag{3.21}$$

If we set

$$\phi = \cos(p \cdot x), \qquad p = (p^0, \vec{p}) : \text{ constant 4-vector} \tag{3.22}$$

then

$$(p^0)^2 - \vec{p}^2 = m^2 \tag{3.23}$$

$$\Longrightarrow p^0 = \pm E_p, \qquad E_p = \sqrt{\vec{p}^2 + m^2}. \tag{3.24}$$

$\vec{p}$ is later interpreted as momentum of a particle.

3.3 Quantization

Lagrangian

$$\mathcal{L} = \frac{1}{2}\left((\dot{\phi})^2 - (\vec{\partial}\phi)^2 - m^2\phi^2\right), \qquad \dot{\phi} = \partial_t\phi. \tag{3.25}$$

The canonical momentum

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi} \tag{3.26}$$

Hamiltonian

$$H = \int d^3x(\pi\dot{\phi} - \mathcal{L}) \tag{3.27}$$

$$= \int d^3x \frac{1}{2}\left(\pi^2 + (\vec{\partial}\phi)^2 + m^2\phi^2\right) \tag{3.28}$$

Equal time commutation relations

$$[\phi(t,\vec{x}), \pi(t,\vec{y})] = i\delta^3(\vec{x} - \vec{y}). \tag{3.29}$$

For technical concreteness, impose periodic boundary condition:

$$x^i \sim x^i + L \quad (i = 1, 2, 3) \tag{3.30}$$

This means

$$\phi(x^1, x^2, x^3) = \phi(x^1 + L, x^2, x^3) = \phi(x^1, x^2 + L, x^3) = \phi(x^1, x^2, x^3 + L) \tag{3.31}$$

Fourier modes

$$\exp(i\vec{p} \cdot \vec{x}) \tag{3.32}$$

$$\vec{p} = \frac{2\pi}{L}\vec{n}, \quad \vec{n} = (n_1, n_2, n_3) \text{ ; integers} \tag{3.33}$$

Fourier mode expansion

$$\phi = \frac{1}{L^{3/2}} \sum_{\vec{p}} Q(\vec{p}) \exp(i\vec{p}\cdot\vec{x}) \tag{3.34}$$

$$\pi = \frac{1}{L^{3/2}} \sum_{\vec{p}} P(\vec{p}) \exp(i\vec{p}\cdot\vec{x}) \tag{3.35}$$

The volume of space

$$\int d^3x = L^3 \tag{3.36}$$

Convenient formula

$$\frac{1}{L^3}\int d^3x \exp(i\vec{p}\cdot\vec{x}) = \begin{cases} 1 & \vec{p}=0 \\ 0 & \vec{p}=\frac{2\pi}{L}\vec{n}\neq 0 \end{cases} \tag{3.37}$$

Inverse Fourier transformation

$$Q(\vec{p}) = \frac{1}{L^{3/2}}\int d^3x \phi(\vec{x}) \exp(-i\vec{p}\cdot\vec{x}) \tag{3.38}$$

$$P(\vec{p}) = \frac{1}{L^{3/2}}\int d^3x \pi(\vec{x}) \exp(-i\vec{p}\cdot\vec{x}) \tag{3.39}$$

For a real scalar, $\phi = \phi^\dagger, \pi = \pi^\dagger$

$$Q(\vec{p})^\dagger = Q(-\vec{p}), \qquad P(\vec{p})^\dagger = P(-\vec{p}) \tag{3.40}$$

Commutations relations at a fixed time (say $t = 0$)

$$\begin{aligned}
[Q(\vec{p}), P(\vec{q})] &= \frac{1}{L^3}\int d^3x d^3y [\phi(\vec{x}), \pi(\vec{y})] \exp(-i\vec{p}\cdot\vec{x} - i\vec{q}\cdot\vec{y}) && (3.41)\\
&= \frac{1}{L^3}\int d^3x d^3y i\delta^3(\vec{x}-\vec{y}) \exp(-i\vec{p}\cdot\vec{x} - i\vec{q}\cdot\vec{y}) && (3.42)\\
&= \frac{i}{L^3}\int d^3x \exp(-i(\vec{p}+\vec{q})\cdot\vec{x}) && (3.43)\\
&= i\delta_{\vec{p},-\vec{q}} \qquad \text{(Kronecker delta)} && (3.44)
\end{aligned}$$

Hamiltonian

$$\begin{aligned}
H &= \frac{1}{2}\int d^3x \left(\pi^2 + (\vec{\partial}\phi)^2 + m^2\phi^2\right) && (3.45)\\
&= \frac{1}{2L^3}\int d^3x \sum_{\vec{p}}\sum_{\vec{q}} \left(P(\vec{p})P(\vec{q}) + (-\vec{p}\cdot\vec{q} + m^2)Q(\vec{p})Q(\vec{q})\right) \exp(-i(\vec{p}+\vec{q})\cdot\vec{x}) && (3.46)\\
&= \frac{1}{2}\sum_{\vec{p}}\sum_{\vec{q}} \left(P(\vec{p})P(\vec{q}) + (-\vec{p}\cdot\vec{q} + m^2)Q(\vec{p})Q(\vec{q})\right) \delta_{\vec{p},-\vec{q}} && (3.47)\\
&= \frac{1}{2}\sum_{\vec{p}} \left(|P(\vec{p})|^2 + (\vec{p}^2 + m^2)|Q(\vec{p})|^2\right) && (3.48)\\
&= \frac{1}{2}\sum_{\vec{p}} \left(|P(\vec{p})|^2 + E_p^2|Q(\vec{p})|^2\right) \qquad (E_p = \sqrt{\vec{p}^2 + m^2}) && (3.49)
\end{aligned}$$

Define creation and annihilation operators

$$A_{\vec{p}} = \frac{1}{\sqrt{2E_p}}(E_p Q(\vec{p}) + iP(\vec{p})) \tag{3.50}$$

$$A_{\vec{p}}^\dagger = \frac{1}{\sqrt{2E_p}}(E_p Q(-\vec{p}) - iP(-\vec{p})) \tag{3.51}$$

Commutation relations

$$[A_{\vec{p}}, A_{\vec{q}}^\dagger] = \frac{1}{2E_p}(-iE_p[Q(\vec{p}), P(-\vec{q})] + iE_p[P(\vec{p}), Q(-\vec{q})]) = \delta_{\vec{p},\vec{q}} \tag{3.52}$$

In this way,

$$[A_{\vec{p}}, A_{\vec{q}}^\dagger] = \delta_{\vec{p},\vec{q}}, \qquad [A_{\vec{p}}, A_{\vec{q}}] = [A_{\vec{p}}^\dagger, A_{\vec{q}}^\dagger] = 0 \tag{3.53}$$

Hamiltonian

$$H = \frac{1}{2}\sum_{\vec{p}} E_p \left(A_{\vec{p}}^\dagger A_{\vec{p}} + A_{\vec{p}}^\dagger A_{\vec{p}}\right) \tag{3.54}$$

$$= \sum_{\vec{p}} E_p \left(A_{\vec{p}}^\dagger A_{\vec{p}} + \frac{1}{2}\right) \tag{3.55}$$

For each Fourier mode $\vec{p}$, there is one harmonic oscillator with creation/annihilation operators $A_{\vec{p}}, A_{\vec{p}}^\dagger$.

$$[H, A_{\vec{p}}^\dagger] = E_p A_{\vec{p}}^\dagger, \qquad [H, A_{\vec{p}}] = -E_p A_{\vec{p}}. \tag{3.56}$$

Define vacuum energy

$$E_{\text{vac}} = \sum_{\vec{p}} \frac{1}{2} E_p \tag{3.57}$$

This is the zero point energy of harmonic oscillators.

$$H = \sum_{\vec{p}} E_p A_{\vec{p}}^\dagger A_{\vec{p}} + E_{\text{vac}}. \tag{3.58}$$

3.4 Hilbert space: Fock space

The Hilbert space of states is the same as a collection of harmonic oscillators.

Define the vacuum state $|\Omega\rangle$ by

$$A_{\vec{p}}|\Omega\rangle = 0 \qquad \text{(For all } \vec{p}) \tag{3.59}$$

The ground state of harmonic oscillators.

$$H|\Omega\rangle = E_{\text{vac}}|\Omega\rangle \tag{3.60}$$

E_{vac} is related to cosmological constant. I do not discuss it, and simply set

$$E_{\text{vac}} \to 0. \tag{3.61}$$

A state

$$A^\dagger_{\vec{p}}|\Omega\rangle \tag{3.62}$$

Its energy

$$HA^\dagger_{\vec{p}}|\Omega\rangle = ([H, A^\dagger_{\vec{p}}] + A^\dagger_{\vec{p}}H)|\Omega\rangle = E_p A^\dagger_{\vec{p}}|\Omega\rangle \qquad (E_{\text{vac}} = 0) \tag{3.63}$$

$$E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2} \tag{3.64}$$

Interpretation: A single particle with 4-momentum $p = (E_{\vec{p}}, \vec{p})$.
A particle appears from a field!

More general states:

$$|n\rangle = \left(\prod_{\vec{p}} (A^\dagger_{\vec{p}})^{n_{\vec{p}}}\right)|\Omega\rangle \tag{3.65}$$

$$n = (n_{\vec{p}}) : \text{ A nonnegative integer } n_{\vec{p}} \text{ for each } \vec{p} \tag{3.66}$$

$$(A^\dagger_{\vec{p}}A_{\vec{p}})|n\rangle = n_{\vec{p}}|n\rangle. \tag{3.67}$$

$$E = \sum_{\vec{p}} n_{\vec{p}} E_p \tag{3.68}$$

Interpretation :

- $n_{\vec{p}}$ particles with $p = (E_{\vec{p}}, \vec{p})$ for each $\vec{p}$.
- Identical particles are not distinguished.
- Any number $n_{\vec{p}}$ of particles is possible in a single one-particle state $\vec{p}$: Bose-Einstein statictics.

This kind of Hilbert space : Fock space

3.5 Infinite volume

L was artificial. We are going to take $L \to \infty$.

$$\phi = \frac{1}{L^{3/2}} \sum_{\vec{p}} Q(\vec{p}) \exp(i\vec{p}\cdot\vec{x}) \tag{3.69}$$

$$A_{\vec{p}} = \frac{1}{\sqrt{2E_p}}(E_p Q(\vec{p}) + iP(\vec{p})) \tag{3.70}$$

$$A^\dagger_{\vec{p}} = \frac{1}{\sqrt{2E_p}}(E_p Q(-\vec{p}) - iP(-\vec{p})) \tag{3.71}$$

$$\tag{3.72}$$

From them

$$Q(\vec{p}) = \frac{1}{\sqrt{2E_p}}(A_{\vec{p}} + A^\dagger_{-\vec{p}}), \quad P(\vec{p}) = -i\sqrt{\frac{E_p}{2}}(A_{\vec{p}} - A^\dagger_{-\vec{p}}) \tag{3.73}$$

$$\phi = \frac{1}{L^{3/2}} \sum_{\vec{p}} \frac{1}{\sqrt{2E_{\vec{p}}}} \Big(A_{\vec{p}} \exp(i\vec{p}\cdot\vec{x}) + A^\dagger_{\vec{p}} \exp(-i\vec{p}\cdot\vec{x}) \Big) \tag{3.74}$$

Recall

$$\vec{p} = \frac{2\pi}{L}\vec{n}, \qquad \vec{n} = (n_1, n_2, n_3). \tag{3.75}$$

When $L \to \infty$, $\vec{p}$ continuous,

$$\left(\frac{2\pi}{L}\right)^3 \sum_{\vec{n}} \to \int d^3p \tag{3.76}$$

Delta function

$$\frac{L^3}{(2\pi)^3}\delta_{\vec{p},\vec{q}} \to \delta^3(\vec{p})\text{: Dirac delta function.} \tag{3.77}$$

Check:

$$\int d^3p\, \delta^3(\vec{p}-\vec{q}) = \left(\frac{2\pi}{L}\right)^3 \sum_{\vec{p}} \frac{L^3}{(2\pi)^3}\delta_{\vec{p},\vec{q}} = 1. \tag{3.78}$$

Define

$$a_{\vec{p}} = L^{3/2} A_{\vec{p}}. \tag{3.79}$$

$$[a_{\vec{p}}, a^\dagger_{\vec{q}}] = L^3 \delta_{\vec{p},\vec{q}} \tag{3.80}$$

$$\to (2\pi)^3 \delta^3(\vec{p}-\vec{q}). \tag{3.81}$$

$$\phi = \frac{1}{L^3} \sum_{\vec{p}} \frac{1}{\sqrt{2E_{\vec{p}}}} \Big(a_{\vec{p}} \exp(i\vec{p}\cdot\vec{x}) + a^\dagger_{\vec{p}} \exp(-i\vec{p}\cdot\vec{x}) \Big) \tag{3.82}$$

$$\to \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \Big(a_{\vec{p}} \exp(i\vec{p}\cdot\vec{x}) + a^\dagger_{\vec{p}} \exp(-i\vec{p}\cdot\vec{x}) \Big). \tag{3.83}$$

$$H = \sum_{\vec{p}} E_p A^\dagger_{\vec{p}} A_{\vec{p}} = \frac{1}{L^3} \sum_{\vec{p}} E_p a^\dagger_{\vec{p}} a_{\vec{p}} \tag{3.84}$$

$$\to \int \frac{d^3p}{(2\pi)^3} E_p a^\dagger_{\vec{p}} a_{\vec{p}} \tag{3.85}$$

One particle state

$$|\vec{p}\rangle = \sqrt{2E_{\vec{p}}}a^\dagger_{\vec{p}}|\Omega\rangle. \tag{3.86}$$

($\sqrt{2E_{\vec{p}}}$ is just convention, often used in high energy physics.)
Normalization

$$\begin{aligned}
\langle\vec{p}|\vec{q}\rangle &= \sqrt{2E_{\vec{p}}}\sqrt{2E_{\vec{q}}}\langle\Omega|a_{\vec{p}}a^\dagger_{\vec{q}}|\Omega\rangle && (3.87)\\
&= \sqrt{2E_{\vec{p}}}\sqrt{2E_{\vec{q}}}\langle\Omega|\left((2\pi)^3\delta^3(\vec{p}-\vec{q}) + a^\dagger_{\vec{q}}a_{\vec{p}}\right)|\Omega\rangle && (3.88)\\
&= 2E_{\vec{p}}(2\pi)^3\delta^3(\vec{p}-\vec{q}). && (3.89)
\end{aligned}$$

4 More on a real scalar field

- Heisenberg picture
- Non-relativistic limit
- Second quantization

4.1 Heisenberg picture in QM

One particle quantum mechanics

$$[x,p]=i \quad (\ \hbar=1\ ), \qquad H=\frac{p^2}{2m}+V(x) \tag{4.1}$$

Schrödinger eq. for time-dependent states $|\Psi,t\rangle$:

$$i\frac{\partial}{\partial t}|\Psi,t\rangle = H|\Psi,t\rangle \tag{4.2}$$

Formal solution

$$|\Psi,t\rangle = e^{-itH}|\Psi\rangle \qquad (|\Psi\rangle = |\Psi,t=0\rangle) \tag{4.3}$$

It is called Schrödinger picture.
Here

$$e^A=\sum_{n=0}^{\infty}\frac{1}{n!}A^n \qquad \text{for any operator } A. \tag{4.4}$$

Heisenberg picture:
Define time dependent operators

$$x(t)=e^{itH}xe^{-itH}, \qquad p(t)=e^{itH}pe^{-itH} \tag{4.5}$$

For any operator O,

$$O(t)=e^{itH}Oe^{-itH} \tag{4.6}$$

The reason for the definition:
Expectation value of O at t is

$$\langle\Psi,t|O|\Psi,t\rangle = \langle\Psi|e^{itH}Oe^{-itH}|\Psi\rangle = \langle\Psi|O(t)|\Psi\rangle \tag{4.7}$$

- Schrödinger picture: states depend on t, operators do not.
- Heisenberg picture: operators depend on t, states do not.

They are physically equivalent. Just a matter of interpretation.

Note

$$\frac{\partial}{\partial t} e^{itH} = iHe^{itH} = e^{itH} iH \tag{4.8}$$

Heisenberg equations:

$$\frac{\partial}{\partial t} O(t) = \left(\frac{\partial}{\partial t} e^{itH} \right) O e^{-itH} + e^{itH} O \left(\frac{\partial}{\partial t} e^{-itH} \right) \tag{4.9}$$

$$= i[H, O(t)]. \tag{4.10}$$

Computations for $x(t)$, $p(t)$:

$$[p^2, x] = p[p, x] + [p, x]p = -2ip, \tag{4.11}$$

$$[V(x), p] = [V(x), -i\partial_x] = i\partial_x V(x) \tag{4.12}$$

$$\frac{d}{dt} x(t) = ie^{itH}[H, x]e^{-itH} = ie^{itH}[\frac{p^2}{2m}, x]e^{-itH} = e^{itH} \frac{p}{m} e^{-itH} = \frac{p(t)}{m} \tag{4.13}$$

$$\frac{d}{dt} p(t) = ie^{itH}[H, p]e^{-itH} = -e^{itH} \partial_x V(x) e^{-itH} = -\partial_x V(x(t)) \tag{4.14}$$

$$p(t) = m\dot{x}(t) \tag{4.15}$$

$$m\ddot{x}(t) = -\partial_x V(x(t)) \tag{4.16}$$

The same form as classical equations, but $x(t), p(t)$ are operators.

4.2 Heisenberg picture in QFT

In QFT, operators $\phi(\vec{x})$ depend on $\vec{x}$.
Relativity requires to treat $(t, \vec{x})$ on the same footing.
Heisenberg picture is more convenient.

$$\phi = \phi(t, \vec{x}) = \phi(x), \qquad x = (x^\mu) = (t, \vec{x}). \tag{4.17}$$

$$\phi(t = 0, \vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}} \exp(i\vec{p} \cdot \vec{x}) + a^\dagger_{\vec{p}} \exp(-i\vec{p} \cdot \vec{x}) \right). \tag{4.18}$$

In Heisenberg picture,

$$\phi(t, \vec{x}) = e^{itH} \phi(0, \vec{x}) e^{-itH} \tag{4.19}$$

Recall

$$[H, a_{\vec{p}}] = -E_p a_{\vec{p}}, \quad [H, a^\dagger_{\vec{p}}] = +E_p a^\dagger_{\vec{p}} \tag{4.20}$$

Then

$$e^{itH}a_{\vec{p}}e^{-itH} = e^{-itE_p}a_{\vec{p}}, \qquad e^{itH}a^\dagger_{\vec{p}}e^{-itH} = e^{itE_p}a^\dagger_{\vec{p}} \tag{4.21}$$

Proof: If $H|\Psi\rangle = E|\Psi\rangle$, then

$$Ha^\dagger_{\vec{p}}|\Psi\rangle = (E+E_p)a^\dagger_{\vec{p}}|\Psi\rangle \tag{4.22}$$

Then

$$e^{itH}a_{\vec{p}}{}^\dagger e^{-itH}|\Psi\rangle = e^{itH}a_{\vec{p}}{}^\dagger e^{-itE}|\Psi\rangle = e^{it(E+E_p)}e^{-itE}a_{\vec{p}}{}^\dagger|\Psi\rangle \tag{4.23}$$

$$= e^{itE_p}a^\dagger_{\vec{p}}|\Psi\rangle \tag{4.24}$$

$|\Psi\rangle$ was an arbitrary energy eigenstate, so

$$e^{itH}a_{\vec{p}}{}^\dagger e^{-itH} = e^{itE_p}a^\dagger_{\vec{p}}. \tag{4.25}$$

$e^{itH}a_{\vec{p}}e^{-itH} = e^{-itE_p}a_{\vec{p}}$ is the same.

$$\phi(t,\vec{x}) = e^{itH}\phi(0,\vec{x})e^{-itH} \tag{4.26}$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(e^{itH}a_{\vec{p}}e^{-itH}\exp(i\vec{p}\cdot\vec{x}) + e^{itH}a^\dagger_{\vec{p}}e^{-itH}\exp(-i\vec{p}\cdot\vec{x})\right) \tag{4.27}$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(a_{\vec{p}}e^{ip\cdot x} + a^\dagger_{\vec{p}}e^{-ip\cdot x}\right) \tag{4.28}$$

where

$$p\cdot x = -p^0 t + \vec{p}\cdot\vec{x}, \qquad p^0 = E_p. \tag{4.29}$$

Klein-Gordon equation

$$(\partial^2 - m^2)\phi = 0 \tag{4.30}$$

because

$$(\partial^2 - m^2)\exp(\pm ip\cdot x) = (-p^2 - m^2)\exp(\pm ip\cdot x) = 0. \tag{4.31}$$

4.3 Non-relativistic limit

Non-relativistic limit

$$|\vec{p}| \ll m \tag{4.32}$$

In this limit,

$$E_p \simeq m + \frac{\vec{p}^2}{2m} \tag{4.33}$$

m : Einstein's mass energy
Define

$$K_p = \frac{\vec{p}^2}{2m}. \tag{4.34}$$

$$\phi(x) = \frac{1}{\sqrt{2m}} e^{-\mathrm{i}mt} \varphi + \frac{1}{\sqrt{2m}} (e^{-\mathrm{i}mt} \varphi)^* \tag{4.35}$$

$$\varphi(x) = \int \frac{d^3 p}{(2\pi)^3} a_{\vec{p}}\, e^{-iK_p t + i\vec{p}\cdot\vec{x}}. \tag{4.36}$$

From

$$(\mathrm{i}\frac{\partial}{\partial t} + \frac{1}{2m}\vec{\partial}^2) e^{-iK_p t + i\vec{p}\cdot\vec{x}} = (K_p - \frac{\vec{p}^2}{2m}) e^{-iK_p t + i\vec{p}\cdot\vec{x}} = 0. \tag{4.37}$$

Equations of motion

$$\mathrm{i}\frac{\partial}{\partial t}\varphi = -\frac{1}{2m}\vec{\partial}^2 \varphi \tag{4.38}$$

The same form as Schrödinger eq. with zero potential (and $\hbar = 1$).

Suppose: $|\Psi\rangle$ is a one-particle state: superposition of $a^\dagger_{\vec{p}}|\Omega\rangle$.
Define

$$\Psi(t, \vec{x}) = \langle\Omega|\varphi|\Psi\rangle. \tag{4.39}$$

Then

$$\begin{aligned} \mathrm{i}\frac{\partial}{\partial t}\Psi(t,\vec{x}) &= \langle\Omega|\mathrm{i}\frac{\partial}{\partial t}\varphi|\Psi\rangle && (4.40)\\ &= \langle\Omega| -\frac{1}{2m}\vec{\partial}^2\varphi|\Psi\rangle && (4.41)\\ &= -\frac{1}{2m}\vec{\partial}^2\Psi(t,\vec{x}). && (4.42) \end{aligned}$$

Schrödinger eq.

Example:

$$|\Psi\rangle = a^\dagger_{\vec{p}}|\Omega\rangle. \tag{4.43}$$

Then

$$\begin{aligned} \Psi(t,\vec{x}) &= \langle\Omega| \left(\int \frac{d^3 q}{(2\pi)^3} a_{\vec{q}}\, e^{-iK_q t + i\vec{q}\cdot\vec{x}}\right) a^\dagger_{\vec{p}}|\Omega\rangle && (4.44)\\ &= \langle\Omega| \int \frac{d^3 q}{(2\pi)^3} e^{-iK_q t + i\vec{q}\cdot\vec{x}} (2\pi)^3 \delta^3(\vec{p}-\vec{q})|\Omega\rangle \quad (\because [a_{\vec{p}}, a^\dagger_{\vec{q}}] = (2\pi)^3\delta^3(\vec{p}-\vec{q})) && (4.45)\\ &= e^{-iK_p t + i\vec{p}\cdot\vec{x}} && (4.46) \end{aligned}$$

This is the wavefunction for the state with momentum $\vec{p}$.

4.4 Second quantization

Consider a QFT with action

$$S = \int dt d^3x \left(\frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\vec{\partial}\phi)^2 - \frac{1}{2}m^2\phi^2 - mV(\vec{x})\phi^2 \right) \tag{4.47}$$

ϕ : field
$V(\vec{x})$: some function.

EOM

$$-\partial_t^2 \phi = \left(-\vec{\partial}^{\,2} + m^2 + 2mV(\vec{x}) \right) \phi. \tag{4.48}$$

Take

$$\phi(x) = \frac{1}{\sqrt{2m}} e^{-\mathrm{i}mt} \varphi + \frac{1}{\sqrt{2m}} (e^{-\mathrm{i}mt} \varphi)^* \tag{4.49}$$

In the limit $m \to \infty$, EOM becomes

$$i\partial_t \phi = \left(-\frac{1}{2m} \vec{\partial}^{\,2} + V(\vec{x}) \right) \phi. \tag{4.50}$$

Terms oscillating rapidly as e^{2imt} are neglected.
The same form as Schrödinger eq.

Find eigenfunctions $\Psi_n(\vec{x})$,

$$H\Psi_n(\vec{x}) = E_n \Psi_n(\vec{x}) \qquad H = -\frac{1}{2m}\vec{\partial}^{\,2} + V(\vec{x}). \tag{4.51}$$

(For notational simplicity, assume discrete states.)
Expand the field φ,

$$\varphi(t, \vec{x}) = \sum_n \Psi_n(\vec{x}) e^{-iE_n t} A_n, \tag{4.52}$$

A_n : operators.
Quantization of the field ϕ turns out to give (exercise)

$$[\varphi(\vec{x}), \varphi(\vec{y})^\dagger] = \delta^3(\vec{x} - \vec{y}), \quad [\varphi(\vec{x}), \varphi(\vec{y})] = 0, \quad [\varphi(\vec{x})^\dagger, \varphi(\vec{y})^\dagger] \tag{4.53}$$

This gives (exercise)

$$[A_n, A_m^\dagger] = \delta_{nm}, \quad [A_n, A_m] = 0, \quad [A_n^\dagger, A_m^\dagger] = 0. \tag{4.54}$$

Interpretation:
$A_n, A_n^\dagger$ are creation and annihilation operators for the state corresponding to Ψ_n.

Schrödinger eq. is satisfied not by states but by the field φ :
Sometimes called second quantization.

5 Spinors and Dirac equations

5.1 Spins in QM

Spin degrees of freedom $\vec{S} = (S_i)$:
Commutation relations

$$[S_i, S_j] = i\epsilon_{ijk} S_k \qquad (\hbar = 1). \tag{5.1}$$

Electron (or quark) : spin $\frac{1}{2}$, two states

$$|\uparrow\rangle : \text{spin up} \quad S_3|\uparrow\rangle = +\frac{1}{2}|\uparrow\rangle \tag{5.2}$$

$$|\downarrow\rangle : \text{spin down} \quad S_3|\downarrow\rangle = -\frac{1}{2}|\downarrow\rangle. \tag{5.3}$$

State vectors (neglecting positions)

$$c_1|\uparrow\rangle + c_2|\downarrow\rangle \tag{5.4}$$

With basis vectors, $(|\uparrow\rangle, |\downarrow\rangle)$, state vectors are two-dimensional vector space

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad \text{(2 dim. vector)} \tag{5.5}$$

Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5.6}$$

Properties:

$$\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k \tag{5.7}$$

More explicitly,

$$(\sigma^1)^2 = 1, \qquad \sigma^1\sigma^2 = i\sigma^3, \quad \text{etc..} \tag{5.8}$$

Spin angular momentum for an electron is

$$S^i = \frac{1}{2}\sigma^i \tag{5.9}$$

Eigenstates for S_3

$$S^3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = +\frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad S^3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{5.10}$$

So far we have neglected position $\vec{x}$. Let's include $\vec{x}$.
Wave functions

$$\Psi(\vec{x}, t) = \begin{pmatrix} \Psi_1(\vec{x}, t) \\ \Psi_2(\vec{x}, t) \end{pmatrix} \tag{5.11}$$

Interpretation:
$|\Psi_1(\vec{x})|^2$ is the probability density for spin up states. Similarly for $|\Psi_2(\vec{x})|^2$.
Total probability =1

$$\int d^3x |\Psi(\vec{x})|^2 = 1, \qquad |\Psi(\vec{x})|^2 = |\Psi_1(\vec{x})|^2 + |\Psi_2(\vec{x})|^2 \tag{5.12}$$

Experimental fact :
Spin $\vec{S}$ and magnetic field $\vec{B}$ interact, with the Hamiltonian

$$H = H_0 + H_1 \tag{5.13}$$

H_0 does not involve spin,

$$H_0 = \frac{1}{2m}(-i\partial_i - eA_i)^2 + e\phi \tag{5.14}$$

H_1 includes spin,

$$H_1 = -g\frac{e}{2m}\vec{S}\cdot\vec{B} \tag{5.15}$$

g : dimensionless parameter

$$\frac{g-2}{2} = \begin{cases} 0.001\ 159\ 652\ 181\ 64(76) \text{ (theoretical calculation by QFT)} \\ 0.001\ 159\ 652\ 180\ 73(28) \text{ (experimental value)} \end{cases} \tag{5.16}$$

Extremely good agreement.

5.2 Gamma matrices and Clifford algebra

Dirac equation

$$(\gamma^\mu \partial_\mu + m)\psi = 0. \tag{5.17}$$

γ^μ $(\mu = 0, 1, 2, 3)$: gamma matrices, to be discussed.

Historical motivation: relativistic version of Schrödinger equation.
This motivation is not valid today. It is an EOM for fields.

- Klein-Gordon eq. : spin 0 fields
- Dirac eq. : spin $\frac{1}{2}$ fields
- Maxwell (or Yang-Mills) eq. : spin 1 fields

Properties of γ^μ: For plane waves $\Psi \propto \exp(ip_\mu x^\mu)$

$$(\gamma^\mu \partial_\mu + m)\Psi = (i\gamma_\mu p^\mu + m)\Psi = 0 \tag{5.18}$$

$$m\Psi = (-i\gamma_\mu p^\mu)\Psi \tag{5.19}$$

Using it twice,

$$m^2\Psi = -(\gamma_\mu p^\mu)^2\Psi \tag{5.20}$$

Requirement: there is a solution with $m^2 = -p^2$.

$$p^2 = (\gamma_\mu p^\mu)^2 = \gamma_\mu\gamma_\nu p^\mu p^\nu = \frac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu)p^\mu p^\nu. \tag{5.21}$$

In general, define anticommutator

$$\{A, B\} = AB + BA \tag{5.22}$$

If

$$\{\gamma_\mu, \gamma_\nu\} = \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu} \tag{5.23}$$

Then $p^2 = -m^2$. This anticommutation relation is called Clifford algebra.

γ_μ need to be matrices.
In three (not four) dimensions, Pauli matrices σ_i satisfy the desired relations:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5.24}$$

$$\{\sigma^i, \sigma^j\} = (\sigma^i\sigma^j + \sigma^j\sigma^i) = 2\delta_{ij} \tag{5.25}$$

Pauli matrices: 2×2 . In four dimensions, $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$ requires 4×4 matrices.

Explicit examples of γ^μ :

$$\gamma^i = \begin{pmatrix} 0 & -i\sigma^i \\ i\sigma^i & 0 \end{pmatrix} \tag{5.26}$$

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \tag{5.27}$$

Each block is 2×2. In total, 4×4
Check of Clifford algebra:

$$\gamma^i\gamma^j = \begin{pmatrix} 0 & -i\sigma^i \\ i\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i\sigma^j \\ i\sigma^j & 0 \end{pmatrix} = \begin{pmatrix} \sigma^i\sigma^j & 0 \\ 0 & \sigma^i\sigma^j \end{pmatrix} \tag{5.28}$$

$$\gamma^i\gamma^j + \gamma^j\gamma^i = \begin{pmatrix} 2\delta^{ij} & 0 \\ 0 & 2\delta^{ij} \end{pmatrix} = 2\delta^{ij} \tag{5.29}$$

$$\gamma^i\gamma^0 = \begin{pmatrix} 0 & -i\sigma^i \\ i\sigma^i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} -\sigma^j & 0 \\ 0 & \sigma^j \end{pmatrix} \tag{5.30}$$

$$\gamma^0\gamma^i = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i\sigma^i \\ i\sigma^i & 0 \end{pmatrix} = \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \tag{5.31}$$

$$\gamma^i\gamma^0 + \gamma^0\gamma^i = 0 \tag{5.32}$$

$$\{\gamma^0, \gamma^0\} = 2(\gamma^0)^2 = -2 \tag{5.33}$$

Combining them,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{5.34}$$

By using tensor products in linear algebra,

$$\gamma^i = \sigma^2 \otimes \sigma^i \tag{5.35}$$

$$\gamma^0 = -i\sigma^1 \otimes 1 \tag{5.36}$$

γ_μ need not be the above explicit form.
But we require

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \qquad (\gamma_0)^\dagger = -\gamma_0, \quad (\gamma_i)^\dagger = \gamma_i. \tag{5.37}$$

5.3 Lorentz invariance

Lorentz transformation

$$y^\mu = \Lambda^\mu{}_\nu x^\nu \tag{5.38}$$

ds^2 must be invariant:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{\mu\nu}dy^\mu dy^\nu \tag{5.39}$$

$$\Longrightarrow \quad g_{\rho\sigma}\Lambda^\rho{}_\mu\Lambda^\sigma{}_\nu = g_{\mu\nu} \tag{5.40}$$

Transformation of Dirac equation

$$(\gamma^\mu \frac{\partial}{\partial x^\mu} + m)\psi = (\gamma^\mu \frac{\partial y^\nu}{\partial x^\mu}\frac{\partial}{\partial y^\nu} + m)\psi \tag{5.41}$$

$$= (\gamma^\mu \Lambda^\nu{}_\mu \frac{\partial}{\partial y^\nu} + m)\psi \tag{5.42}$$

Assume ψ also transforms:

$$\psi'(y) = S(\Lambda)\psi(x) \tag{5.43}$$

$S(\Lambda)$: some matrix given in terms of γ^μ, to be determined.

Comparison: a vector field $A^\mu(x)$ transforms as

$$A'^\mu(y) = \Lambda^\mu{}_\nu A^\nu(x). \tag{5.44}$$

ψ is neither scalar nor vector. It is called spinor.

$$(\gamma^\mu \Lambda^\nu{}_\mu \frac{\partial}{\partial y^\nu} + m)\psi = \left(\gamma^\mu \Lambda^\nu{}_\mu \frac{\partial}{\partial y^\nu} + m\right) S(\Lambda)^{-1}\psi' \tag{5.45}$$

$$= S(\Lambda)^{-1} \left(\Lambda^\nu{}_\mu S(\Lambda)\gamma^\mu S(\Lambda)^{-1} \frac{\partial}{\partial y^\nu} + m\right)\psi' \tag{5.46}$$

If

$$\Lambda^\nu{}_\mu S(\Lambda)\gamma^\mu S(\Lambda)^{-1} = \gamma^\nu \tag{5.47}$$

$$\Longleftrightarrow \; S(\Lambda)^{-1}\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu \tag{5.48}$$

then

$$\left(\gamma^\mu \frac{\partial}{\partial x^\mu} + m\right)\psi(x) = S(\Lambda)^{-1}\left(\gamma^\mu \frac{\partial}{\partial y^\mu} + m\right)\psi'(y) \tag{5.49}$$

Then Dirac equation is Lorentz invariant.

We need to find $S(\Lambda)$.

Infinitesimal Lorentz transformation

Consider

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu, \qquad \omega : \text{ infinitesimal} \tag{5.50}$$

$$g_{\mu\nu} = g_{\rho\sigma}\Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu \tag{5.51}$$

$$= g_{\rho\sigma}(\delta^\rho{}_\mu + \omega^\rho{}_\mu)(\delta^\sigma{}_\nu + \omega^\sigma{}_\nu) \tag{5.52}$$

$$= g_{\mu\nu} + \omega_{\nu\mu} + \omega_{\mu\nu} \tag{5.53}$$

Here indices are raised and lowered by $g_{\mu\nu}$:

$$\omega_{\mu\nu} = g_{\mu\rho}\omega^\rho{}_\nu \tag{5.54}$$

We get

$$\omega_{\nu\mu} + \omega_{\mu\nu} = 0 \quad : \text{antisymmetric} \tag{5.55}$$

$S(\Lambda)$ for $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}$.
The answer that works :

$$S(\Lambda) = 1 + \frac{1}{4}\omega_{\rho\sigma}\gamma^{\rho}\gamma^{\sigma} \tag{5.56}$$

Not so many possibilities if it is given in terms of γ^{μ} and $\omega_{\mu\nu}$.

Check of $S(\Lambda)^{-1}\gamma^{\mu}S(\Lambda) = \Lambda^{\mu}{}_{\nu}\gamma^{\nu}$:
For infinitesimal $\omega_{\mu\nu}$,

$$S(\Lambda)^{-1} = 1 - \frac{1}{4}\omega_{\rho\sigma}\gamma^{\rho}\gamma^{\sigma} \tag{5.57}$$

Then

$$\begin{aligned}
&S(\Lambda)^{-1}\gamma^{\mu}S(\Lambda) && (5.58)\\
=&(1 - \frac{1}{4}\omega_{\rho\sigma}\gamma^{\rho}\gamma^{\sigma})\gamma^{\mu}(1 + \frac{1}{4}\omega_{\rho\sigma}\gamma^{\rho}\gamma^{\sigma}) && (5.59)\\
=&\gamma^{\mu} - \frac{1}{4}\omega_{\rho\sigma}(\gamma^{\rho}\gamma^{\sigma}\gamma^{\mu} - \gamma^{\mu}\gamma^{\rho}\gamma^{\sigma}) && (5.60)\\
=&\gamma^{\mu} - \frac{1}{4}\omega_{\rho\sigma}[\gamma^{\rho}(\gamma^{\sigma}\gamma^{\mu} + \gamma^{\mu}\gamma^{\sigma}) - (\gamma^{\rho}\gamma^{\mu} + \gamma^{\mu}\gamma^{\rho})\gamma^{\sigma}] && (5.61)\\
=&\gamma^{\mu} - \frac{1}{4}\omega_{\rho\sigma}[2\gamma^{\rho}g^{\mu\sigma} - 2g^{\rho\mu}\gamma^{\sigma}] && (5.62)\\
=&\gamma^{\mu} + \omega^{\mu}{}_{\rho}\gamma^{\rho} && (5.63)\\
=&\Lambda^{\mu}{}_{\rho}\gamma^{\rho} && (5.64)
\end{aligned}$$

This confirms the desired property of $S(\Lambda)$.

Finite Lorentz transformation

For finite ω, suppose

$$\Lambda = \exp(\omega) \tag{5.65}$$

In general, for a matrix $\omega = (\omega^{\mu}{}_{\nu})$,

$$\exp(\omega) = \sum_{k=0}^{\infty} \frac{1}{k!}\omega^{k}. \tag{5.66}$$

Another formula:

$$\exp(\omega) = \lim_{N\to\infty}\left(1 + \frac{1}{N}\omega\right)^{N} \tag{5.67}$$

$\exp(\omega)$ is given by doing an infinitesimal $(1 + \frac{1}{N}\omega)$ many times.
Then

$$S(\Lambda) = \exp(\frac{1}{4}\omega_{\rho\sigma}\gamma^\rho\gamma^\sigma) \tag{5.68}$$

$$= \lim_{N\to\infty}\left(1 + \frac{1}{4}\frac{\omega_{\mu\nu}}{N}\gamma^\mu\gamma^\nu\right)^N \tag{5.69}$$

Transformation law

$$\psi'(y) = S(\Lambda)\psi(x), \qquad (y = \Lambda x) \tag{5.70}$$

A field transforming like this: spinor field.
ψ has 4 components

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \tag{5.71}$$

$S(\Lambda)$ is a 4×4 matrix.

Summary: Under Lorentz transformation $y^\mu = \Lambda^\mu{}_\nu x^\nu$,

- Scalar : $\phi'(y) = \phi(x)$
- Vector : $A'^\mu(y) = \Lambda^\mu{}_\nu A^\nu(x)$
- Spinor : $\psi'(y) = S(\Lambda)\psi(x)$

5.4 More on Lorentz transformation

Summary of previous discussions:

$$y^\mu = \Lambda^\mu{}_\nu x^\nu \tag{5.72}$$

$$\psi'(y) = S(\Lambda)\psi(x) \tag{5.73}$$

Here

$$\Lambda = \exp(\omega) \qquad S(\Lambda) = \exp(\frac{1}{4}\gamma^\mu\gamma^\nu\omega_{\mu\nu}) \tag{5.74}$$

$$S(\Lambda)^{-1}\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu. \tag{5.75}$$

$S(\Lambda)$ is not a unitary matrix.
For later purposes, study $S(\Lambda)^\dagger$.

From $\gamma^i\gamma^0 = -\gamma^0\gamma^i$ and $(\gamma^i)^\dagger = \gamma^i$,

$$(\gamma^i)^\dagger = -(\gamma^0)\gamma^i(\gamma^0)^{-1} \tag{5.76}$$

From $\gamma^0\gamma^0 = \gamma^0\gamma^0$ and $(\gamma^0)^\dagger = -\gamma^0$,

$$(\gamma^0)^\dagger = -(\gamma^0)\gamma^0(\gamma^0)^{-1} \tag{5.77}$$

Summarizing,

$$(\gamma^\mu)^\dagger = -(\gamma^0)\gamma^\mu(\gamma^0)^{-1} \tag{5.78}$$

Then

$$(\gamma^\mu\gamma^\nu\omega_{\mu\nu})^\dagger = (\gamma^\nu)^\dagger(\gamma^\mu)^\dagger\omega_{\mu\nu} \tag{5.79}$$
$$=(\gamma^0)\gamma^\nu(\gamma^0)^{-1}(\gamma^0)\gamma^\mu(\gamma^0)^{-1}\omega_{\mu\nu} \tag{5.80}$$
$$=(\gamma^0)(\gamma^\nu\gamma^\mu\omega_{\mu\nu})(\gamma^0)^{-1} \tag{5.81}$$
$$=-(\gamma^0)(\gamma^\mu\gamma^\nu\omega_{\mu\nu})(\gamma^0)^{-1} \qquad (\because \omega_{\mu\nu} = -\omega_{\nu\mu}). \tag{5.82}$$

Recall the definition of exp of matrices,

$$\exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!}A^k \tag{5.83}$$

From it,

$$\exp(A)^\dagger = \exp(A^\dagger), \qquad \exp(BAB^{-1}) = B\exp(A)B^{-1}, \qquad \exp(A)^{-1} = \exp(-A) \tag{5.84}$$

Then

$$S(\Lambda)^\dagger = \exp(\frac{1}{4}(\gamma^\mu\gamma^\nu\omega_{\mu\nu})^\dagger) \tag{5.85}$$
$$= \exp(-\frac{1}{4}(\gamma^0)(\gamma^\mu\gamma^\nu\omega_{\mu\nu})(\gamma^0)^{-1}) \tag{5.86}$$
$$= (\gamma^0)\exp(-\frac{1}{4}(\gamma^\mu\gamma^\nu\omega_{\mu\nu}))(\gamma^0)^{-1} \tag{5.87}$$
$$= (\gamma^0)S(\Lambda)^{-1}(\gamma^0)^{-1} \tag{5.88}$$

$S(\Lambda)^{-1}$ is the inverse matrix of $S(\Lambda)$.
Summary:

$$S(\Lambda)^{-1} = (\gamma^0)^{-1}S(\Lambda)^\dagger(\gamma^0). \tag{5.89}$$

Notation :

$$\overline{\psi} = i\psi^\dagger\gamma^0 \tag{5.90}$$

Lorentz transformation

$$\overline{\psi}'(y) = i\psi^\dagger(x)S(\Lambda)^\dagger\gamma^0 \tag{5.91}$$
$$= i\psi^\dagger(x)\gamma^0(\gamma^0)^{-1}S(\Lambda)^\dagger\gamma^0 \tag{5.92}$$
$$= \overline{\psi}S(\Lambda)^{-1} \tag{5.93}$$

This is useful for constructing Lorentz covariant quantities.

Some examples:

1. A scalar

$$\overline{\psi}'\psi' = \overline{\psi}S(\Lambda)^{-1}S(\Lambda)\psi = \overline{\psi}\psi. \tag{5.94}$$

2. A vector

$$\overline{\psi}'\gamma^\mu\psi' = \overline{\psi}S(\Lambda)^{-1}\gamma^\mu S(\Lambda)\psi = \Lambda^\mu{}_\nu\overline{\psi}\gamma^\nu\psi. \tag{5.95}$$

   More general tensors

$$\overline{\psi}'\gamma^{\mu_1}\cdots\gamma^{\mu_k}\psi' = \Lambda^{\mu_1}_{\nu_1}\cdots\Lambda^{\mu_k}_{\nu_k}\overline{\psi}\gamma^{\nu_1}\cdots\gamma^{\nu_k}\psi \tag{5.96}$$

3. Another scalar

$$\overline{\psi}'\gamma^\mu\frac{\partial}{\partial y^\mu}\psi' = \overline{\psi}\gamma^\mu\frac{\partial}{\partial x^\mu}\psi \tag{5.97}$$

and so on.

5.5 Spin

Generally,

$$\text{symmetry} \Longrightarrow \text{conserved quantity} \tag{5.98}$$

In particular,

$$\text{rotational symmetry} \Longrightarrow \text{angular momentum} \tag{5.99}$$

The angular momentum is found by infinitesimal rotation

$$y^\mu = \Lambda^\mu{}_\nu x^\mu \tag{5.100}$$

$$\psi'(y) = S(\Lambda)\psi(x) \tag{5.101}$$

Change notation $y \to x$

$$\psi'(x) = S(\Lambda)\psi(\Lambda^{-1}x). \tag{5.102}$$

Rotation : $\omega_{\mu\nu}$ has only spatial components

$$\omega_{ij} \neq 0, \qquad \omega_{ij} = -\omega_{ji} \tag{5.103}$$

$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$.
Introduce ω^i

$$\omega_{ij} = \epsilon_{ikj}\omega^k, \quad \omega_{0i} = 0 \tag{5.104}$$

Upper and lower indices are not distinguished for $i = 1, 2, 3$.

$$\Lambda^i{}_j x^j = x^i + \epsilon^{ikj}\omega_k x_j = \vec{x} + \vec{\omega}\times\vec{x} \tag{5.105}$$

By previous results,

$$S(\Lambda) = 1 + \frac{1}{4}\omega_{ij}\gamma^i\gamma^j = 1 - \frac{1}{4}\epsilon_{ijk}\omega^i\gamma^j\gamma^k \tag{5.106}$$

$$\begin{aligned}
\psi'(\vec{x}) &= S(\Lambda)\psi(\Lambda^{-1}\vec{x}) && (5.107)\\
&= (1 - \frac{1}{4}\epsilon^{ijk}\omega_i\gamma_j\gamma_k)\psi(\vec{x} - \omega \times \vec{x}) && (5.108)\\
&= \psi(\vec{x}) - \left(\frac{1}{4}\epsilon_{ijk}\omega^i\gamma^j\gamma^k + \epsilon^{ijk}\omega_j x_k \partial_i\right)\psi(\vec{x}) && (5.109)\\
&= \psi(\vec{x}) - i\omega^i J_i \psi(\vec{x}) && (5.110)
\end{aligned}$$

where

$$J_i = -\frac{1}{4}i\epsilon_{ijk}\gamma^j\gamma^k - i\epsilon_{ijk}x^j\partial^k = S_i + L_i \tag{5.111}$$

The generator for infinitesimal rotation.

$$\begin{aligned}
\text{orbital angular momentum}: L_i &= -i\epsilon_{ijk}x^j\partial^k = \vec{x} \times \vec{p} \qquad (p_i = -i\partial_i) && (5.112)\\
\text{spin angular momentum}: S_i &= -\frac{1}{4}i\epsilon_{ijk}\gamma^j\gamma^k && (5.113)\\
\text{total angular momentum}: J_i &= L_i + S_i && (5.114)
\end{aligned}$$

Computation of S_i : using

$$\gamma^i = \begin{pmatrix} 0 & -i\sigma^i \\ i\sigma^i & 0 \end{pmatrix} \tag{5.115}$$

then

$$\begin{aligned}
S_i &= -\frac{1}{4}i\epsilon_{ijk}\gamma^j\gamma^k && (5.116)\\
&= -\frac{1}{4}i\epsilon_{ijk}\begin{pmatrix} \sigma^i\sigma^j & 0 \\ 0 & \sigma^i\sigma^j \end{pmatrix} && (5.117)\\
&= \frac{1}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} && (5.118)
\end{aligned}$$

$\frac{1}{2}\sigma^i$: spin operator in QM.
In particular,

$$S_z = S_3 = \frac{1}{2}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \tag{5.119}$$

$$\text{eigenvalue}: \quad \pm\frac{1}{2} \quad : \quad \text{spin } \frac{1}{2} \tag{5.120}$$

Remark:
The relation between QM wavefunction Ψ and the operator ψ :

$$\Psi(x) = \langle \Omega | \psi(x) | \Psi \rangle, \qquad | \Psi \rangle \ : \text{a single particle state in QFT} \tag{5.121}$$

S_i is the spin operator for QM wavefunction Ψ.

5.6 Coupling to EM fields

Recall

$$A_\mu = (-\phi, A_i) \quad : \text{EM potential} \tag{5.122}$$

Gauge transformation

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \alpha, \qquad \alpha \ : \text{arbitrary function} \tag{5.123}$$

Under it,

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu} \tag{5.124}$$

EM tensor $F_{\mu\nu}$ is invariant.

For wavefunction Ψ,

$$\Psi \to \Psi' = \exp(ie\alpha)\Psi. \tag{5.125}$$

Covariant derivative

$$D_\mu \Psi = (\partial_\mu - ieA_\mu)\Psi \tag{5.126}$$

For $D'_\mu = \partial_\mu - ieA'_\mu$

$$D'_\mu \Psi' = \exp(ie\alpha) D_\mu \Psi. \tag{5.127}$$

In the same way, for the operator ψ,

$$\psi \to \psi' = \exp(ie\alpha)\psi \tag{5.128}$$
$$D'_\mu \psi' = \exp(ie\alpha) D_\mu \psi. \tag{5.129}$$

More generally, if some $\Phi(x)$ transforms as

$$\Phi(x) \to \Phi'(x) = \exp(iq\alpha), \tag{5.130}$$

define

$$D_\mu \Phi(x) = (\partial_\mu - iqA_\mu)\Phi. \tag{5.131}$$

q : charge of the operator

For example,

$$\overline{\psi} = i\psi^\dagger\gamma^0 \tag{5.132}$$

$$\overline{\psi} \to \overline{\psi}' = \exp(-ie\alpha)\overline{\psi} \tag{5.133}$$

Then

$$D_\mu\overline{\psi} = (\partial_\mu + ieA_\mu)\overline{\psi}. \tag{5.134}$$

Integration by parts: for example,

$$\int d^4x\overline{\psi}\gamma^\mu(\partial_\mu - ieA_\mu)\psi = \int d^4x(-\partial_\mu\overline{\psi} - ieA_\mu\overline{\psi})\gamma^\mu\psi \tag{5.135}$$

hence

$$\int d^4x\overline{\psi}\gamma^\mu D_\mu\psi = -\int d^4x(D_\mu\overline{\psi})\gamma^\mu\psi. \tag{5.136}$$

Dirac eq. when $A_\mu \neq 0$: replace

$$(\gamma^\mu\partial_\mu + m)\psi = 0 \tag{5.137}$$

by

$$(\gamma^\mu D_\mu + m)\psi = 0 \tag{5.138}$$

Feynman slash notation : For any vector B^μ,

$$\not{B} = \gamma^\mu B_\mu. \tag{5.139}$$

Then

$$(\not{D} + m)\psi = (\gamma^\mu D_\mu + m)\psi = 0. \tag{5.140}$$

5.7 Magnetic moment

Purpose: to compute the interaction $\propto \vec{B}\cdot\vec{S}$ in QM.

Dirac eq.

$$(\gamma^\mu D_\mu + m)\psi = 0 \tag{5.141}$$

$D_\mu = \partial_\mu - ieA_\mu$

$$\begin{aligned}
m^2\psi &= (-\gamma^\mu D_\mu)^2\psi && (5.142)\\
&= \gamma^\mu D_\mu\gamma^\nu D_\nu\psi && (5.143)\\
&= \left(\frac{1}{2}\{\gamma^\mu,\gamma^\nu\} + \frac{1}{2}[\gamma^\mu,\gamma^\nu]\right) D_\mu D_\nu\psi && (5.144)\\
&= \left(g^{\mu\nu} + \frac{1}{2}[\gamma^\mu,\gamma^\nu]\right) D_\mu D_\nu\psi && (5.145)\\
&= \left(D^2 + \frac{1}{4}[\gamma^\mu,\gamma^\nu][D_\mu,D_\nu]\right)\psi && (5.146)\\
& && (5.147)
\end{aligned}$$

Commutation relation

$$[D_\mu, D_\nu] = [\partial_\mu - ieA_\mu, \partial_\nu - ieA_\nu] \tag{5.148}$$
$$= -ie(\partial_\mu A_\nu - \partial_\nu A_\mu) \tag{5.149}$$
$$= -ieF_{\mu\nu} \tag{5.150}$$

$$\left(D^2 - m^2 - ie\frac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}\right)\psi = 0 \tag{5.151}$$

Suppose

$$A_\mu = (0, A_i), \qquad \partial_t A_i = 0. \tag{5.152}$$

In particular, $\vec{B} \neq 0$, $\vec{E} = 0$.

$$F_{ij} = \epsilon_{ijk}B^k, \quad F_{0i} = 0. \tag{5.153}$$

Then

$$-ie\frac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu} \tag{5.154}$$
$$= -ie\frac{1}{2}(\gamma^i\gamma^j\epsilon_{ijk}B^k) \tag{5.155}$$
$$= 2eS_kB^k \qquad (S_k = -\frac{i}{4}\gamma^i\gamma^j\epsilon_{ijk} : \text{ spin }) \tag{5.156}$$

$$\left(-\partial_t^2 + \vec{D}^{\,2} - m^2 + 2eS_kB^k\right)\psi = 0 \tag{5.157}$$

Non-relativistic limit $m \to \infty$: define

$$\psi = e^{-imt}\widetilde{\psi} + e^{imt}\widetilde{\psi}' \tag{5.158}$$

$$-\partial_t^2(e^{-imt}\widetilde{\psi}) = e^{-imt}(m^2 + 2im\partial_t - \partial_t^2)\widetilde{\psi} \tag{5.159}$$

EOM

$$0 = e^{imt}\left(-\partial_t^2 + \vec{D}^{\,2} - m^2 + 2eS_kB^k\right)\psi \tag{5.160}$$
$$= \left(-\partial_t^2 + 2im\partial_t + \vec{D}^{\,2} + 2eS_kB^k\right)\widetilde{\psi} + e^{2imt}(\cdots)\widetilde{\psi}' \tag{5.161}$$

In the limit $m \to \infty$, neglect
(1) ∂_t^2 compared to $m\partial_t$
(2) rapidly oscillating e^{2imt}
Then

$$\left(2im\partial_t + \vec{D}^{\,2} + 2eS_kB^k\right)\widetilde{\psi} = 0 \tag{5.162}$$
$$\implies \quad i\partial_t\Psi = \left(-\frac{1}{2m}\vec{D}^2 - \frac{e}{m}\vec{S}\cdot\vec{B}\right)\widetilde{\psi} \tag{5.163}$$

In QM, it corresponds to the interaction

$$H_1 = -g\frac{e}{2m}\vec{S}\cdot\vec{B}, \tag{5.164}$$

$$g = 2 \tag{5.165}$$

This is a good approximation to the actual value

$$g = 2..002\cdots. \tag{5.166}$$

5.8 Solutions of Dirac equation

Purpose: to obtain solutions of Dirac eq. when $A_\mu = 0$:

$$(\partial\!\!\!/ + m)\psi = 0. \tag{5.167}$$

Modes with momentum p,

$$\psi = e^{\pm ip\cdot x}w, \qquad p^0 > 0, \quad w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} \text{ : constant.} \tag{5.168}$$

Then

$$(\partial\!\!\!/ + m)(e^{\pm ip\cdot x}w) = e^{\pm ip\cdot x}(\pm i p\!\!\!/ + m)w = 0. \tag{5.169}$$

$$\Longrightarrow (\pm i p\!\!\!/ + m)w = 0 \tag{5.170}$$

Computation of w :
Recall

$$\gamma^i = \begin{pmatrix} 0 & -i\sigma^i \\ i\sigma^i & 0 \end{pmatrix} \qquad \gamma^0 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \tag{5.171}$$

Write

$$w = \begin{pmatrix} \chi \\ \eta \end{pmatrix} \tag{5.172}$$

χ, η : 2-component

$$\begin{pmatrix} m & \mp(p^0 - \vec{\sigma}\vec{p}) \\ \mp(p^0 + \vec{\sigma}\vec{p}) & m \end{pmatrix} \begin{pmatrix} \chi \\ \eta \end{pmatrix} \tag{5.173}$$

Then

$$\pm(p^0 + \vec{\sigma}\vec{p})\chi = m\eta, \qquad \pm(p^0 - \vec{\sigma}\vec{p})\eta = m\chi. \tag{5.174}$$

From them,

$$m^2\eta = (p^0 + \vec{\sigma}\vec{p})(p^0 - \vec{\sigma}\vec{p})\eta \tag{5.175}$$
$$= [(p^0)^2 - (\vec{\sigma}\vec{p})^2]\eta \tag{5.176}$$
$$= [(p^0)^2 - \vec{p}^2]\eta \tag{5.177}$$
$$= -p^2\eta \tag{5.178}$$

Here

$$(\vec{\sigma}\vec{p})^2 = \frac{1}{2}\{\sigma_i, \sigma_j\}p^i p^j = \delta_{ij}p^i p^j. \tag{5.179}$$

$$m^2 = -p^2 \implies p^0 = E_p = \sqrt{\vec{p}^2 + m^2} \qquad : \text{on-shell} \tag{5.180}$$

$p^0 - \vec{\sigma}\vec{p}$ and $p^0 + \vec{\sigma}\vec{p}$ are
(i) hermitian matrices
(ii) with positive eigenvalues (exercise)
(iii) and commute with each other .
For any hermitian A with positive eigenvalues, $\sqrt{A}$ such that $(\sqrt{A})^2 = A$ can be defined.
Proof: If $A = UDU^\dagger$, with U unitary and D diagonal

$$D = \begin{pmatrix} D_1 & & \\ & D_2 & \\ & & \ddots \end{pmatrix} \tag{5.181}$$

Then $\sqrt{A} = U\sqrt{D}U^\dagger$ with

$$D = \begin{pmatrix} \sqrt{D_1} & & \\ & \sqrt{D_2} & \\ & & \ddots \end{pmatrix} \tag{5.182}$$

If $[A, B] = 0$, they are simultaneously diagonalizable and

$$\sqrt{A}\sqrt{B} = \sqrt{AB}. \tag{5.183}$$

We can define

$$\sqrt{p^0 - \vec{\sigma}\vec{p}}, \qquad \sqrt{p^0 + \vec{\sigma}\vec{p}} \tag{5.184}$$

and

$$\sqrt{p^0 - \vec{\sigma}\vec{p}}\sqrt{p^0 + \vec{\sigma}\vec{p}} = \sqrt{(p^0 - \vec{\sigma}\vec{p})(p^0 + \vec{\sigma}\vec{p})} = \sqrt{(p^0)^2 - \vec{p}^2} = m. \tag{5.185}$$

Solution for w :

$$w = \begin{pmatrix} \chi \\ \eta \end{pmatrix} = \begin{pmatrix} \sqrt{p^0 - \vec{\sigma}\vec{p}}\,\xi \\ \pm\sqrt{p^0 + \vec{\sigma}\vec{p}}\,\xi \end{pmatrix}, \qquad \xi : \text{arbitrary} \tag{5.186}$$

In fact,

$$\pm(p^0 + \vec{\sigma}\vec{p})\chi = \pm(p^0 + \vec{\sigma}\vec{p})\sqrt{p^0 - \vec{\sigma}\vec{p}}\,\xi = \pm m\sqrt{p^0 + \vec{\sigma}\vec{p}}\,\xi = m\eta. \tag{5.187}$$

$$\pm(p^0 - \vec{\sigma}\vec{p})\eta = (p^0 - \vec{\sigma}\vec{p})\sqrt{p^0 + \vec{\sigma}\vec{p}}\,\xi = m\sqrt{p^0 - \vec{\sigma}\vec{p}}\,\xi = m\chi. \tag{5.188}$$

Define

$$u_{\vec{p}}(\xi) = \begin{pmatrix} \sqrt{p^0 - \vec{\sigma}\vec{p}}\,\xi \\ \sqrt{p^0 + \vec{\sigma}\vec{p}}\,\xi \end{pmatrix} \tag{5.189}$$

$$v_{\vec{p}}(\xi) = \begin{pmatrix} \sqrt{p^0 - \vec{\sigma}\vec{p}}\,\xi \\ -\sqrt{p^0 + \vec{\sigma}\vec{p}}\,\xi \end{pmatrix} \tag{5.190}$$

ξ : arbitrary.
Then

$$(i\not{p} + m)u_{\vec{p}}(\xi) = 0, \qquad (-i\not{p} + m)v_{\vec{p}}(\xi) = 0. \tag{5.191}$$

Solutions of Dirac equation are

$$e^{ip\cdot x}u_{\vec{p}}(\xi), \quad e^{-ip\cdot x}v_{\vec{p}}(\xi) \qquad (p^0 = E_p) \tag{5.192}$$

Orthonormality properties:

$$u_{\vec{p}}(\xi)^\dagger u_{\vec{p}}(\xi') = \left(\xi^\dagger\sqrt{p^0 - \vec{\sigma}\vec{p}}\ \ \xi^\dagger\sqrt{p^0 + \vec{\sigma}\vec{p}}\right)\begin{pmatrix} \sqrt{p^0 - \vec{\sigma}\vec{p}}\,\xi' \\ \sqrt{p^0 + \vec{\sigma}\vec{p}}\,\xi' \end{pmatrix} \tag{5.193}$$

$$= \xi^\dagger\left((p^0 - \vec{\sigma}\vec{p}) + (p^0 + \vec{\sigma}\vec{p})\right)\xi' \tag{5.194}$$

$$= 2E_p\xi^\dagger\xi' \tag{5.195}$$

In the same way,

$$v_{\vec{p}}(\xi)^\dagger v_{\vec{p}}(\xi') = 2E_p\xi^\dagger\xi'. \tag{5.196}$$

Also,

$$u_{\vec{p}}(\xi)^\dagger v_{-\vec{p}}(\xi') = \left(\xi^\dagger\sqrt{p^0 - \vec{\sigma}\vec{p}}\ \ \xi^\dagger\sqrt{p^0 + \vec{\sigma}\vec{p}}\right)\begin{pmatrix} \sqrt{p^0 + \vec{\sigma}\vec{p}}\,\xi' \\ -\sqrt{p^0 - \vec{\sigma}\vec{p}}\,\xi' \end{pmatrix} \tag{5.197}$$

$$= \xi^\dagger(m - m)\xi' \tag{5.198}$$

$$= 0. \tag{5.199}$$

Setting

$$\xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \xi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{5.200}$$

and defining

$$u_{(\vec{p},s)} = u_{\vec{p}}(\xi_s), \qquad v_{(\vec{p},s)} = v_{\vec{p}}(\xi_s) \qquad (s = 1, 2) \tag{5.201}$$

Summary:
For a given $\vec{p}$, the following four vectors are orthogonal, with absolute value $2E_p$.

$$u_{(\vec{p},1)}, \quad u_{(\vec{p},2)}, \quad v_{(-\vec{p},1)}, \quad v_{(-\vec{p},2)} \tag{5.202}$$

6 Quantization of Dirac field

Recall: for Lorentz transformation $\Lambda = (\Lambda^\mu{}_\nu)$, $S(\Lambda)$ is a 4×4 matrix such that

$$S(\Lambda)^{-1}\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu \tag{6.1}$$

Dirac field ψ transforms

$$\psi'(x) = S(\Lambda)\psi(\Lambda^{-1}x) \tag{6.2}$$

ψ has 4 components

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \tag{6.3}$$

γ^μ and $S(\Lambda)$ are 4×4 matrices.

Feynman slash notation : For any vector B^μ,

$$\not{B} = \gamma^\mu B_\mu \tag{6.4}$$

In particular,

$$\not{\partial} = \gamma^\mu \partial_\mu \tag{6.5}$$

$$\not{D} = \gamma^\mu D_\mu \tag{6.6}$$

6.1 Analytical mechanics

The Lagrangian

$$\mathcal{L} = -\overline{\psi}(\not{D} + m)\psi \tag{6.7}$$

$$\overline{\psi} = \psi^\dagger(i\gamma^0) \tag{6.8}$$

This is Lorentz invariant.
Action

$$S = \int d^4x \mathcal{L} \tag{6.9}$$

Equations of motion : $\psi \to \psi + \delta\psi$,

$$\delta S = -\int d^4x \delta\overline{\psi}(\not{D} + m)\psi + \overline{\psi}(\not{D} + m)\delta\psi \tag{6.10}$$

$$= -\int d^4x \delta\overline{\psi}(\not{D} + m)\psi + (-D_\mu\overline{\psi}\gamma^\mu + m\overline{\psi})\delta\psi \tag{6.11}$$

where

$$D_\mu \overline{\psi} = (\partial_\mu + ieA_\mu)\overline{\psi}. \tag{6.12}$$

$$(\not{D} + m)\psi = 0 : \text{ Dirac eq.} \tag{6.13}$$

$$(-D_\mu \overline{\psi} \gamma^\mu + m\overline{\psi}) = 0. \tag{6.14}$$

The second equation follows from the first one:

$$(\gamma^\mu D_\mu + m)\psi = 0 \tag{6.15}$$

$$\Longrightarrow D_\mu \psi^\dagger (\gamma^\mu)^\dagger + m\psi^\dagger \qquad (\because \text{taking } \dagger) = 0. \tag{6.16}$$

Recall

$$(\gamma^\mu)^\dagger = -\gamma^0 \gamma^\mu (\gamma^0)^{-1}, \qquad \overline{\psi} = i\psi^\dagger \gamma^0. \tag{6.17}$$

Then

$$0 = D_\mu \psi^\dagger (\gamma^\mu)^\dagger + m\psi^\dagger = (-D_\mu \overline{\psi} \gamma^\mu + m\overline{\psi})(\gamma^0)^{-1}. \tag{6.18}$$

Canonical momentum : I do not give a systematic discussion.

From

$$\mathcal{L} = -\overline{\psi}\gamma^0 \partial_t \psi \cdots = i\psi^\dagger \partial_t \psi + \cdots \tag{6.19}$$

define canonical momentum

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\psi^\dagger \tag{6.20}$$

Regard ψ and $\psi^\dagger$ as independent canonical variables.

Hamiltonian

$$H = \int d^3x (\pi \partial_t \psi - \mathcal{L}) \tag{6.21}$$

6.2 Anti-commutation relation

Important point : quantization is different from the usual case.
For a harmonic oscillator,

$$\text{canonical commutation relation} \quad \Longrightarrow \quad \text{creation/annihilation operators } a,\ a^\dagger \tag{6.22}$$

$$[a, a^\dagger] = 1, \quad [a, a] = [a^\dagger, a^\dagger] = 0 \tag{6.23}$$

$$a|0\rangle = 0 \tag{6.24}$$

$$|n\rangle \propto (a^\dagger)^n |0\rangle, \qquad n = 0, 1, 2, 3, ... \tag{6.25}$$

$$H = \omega(a^\dagger a + \frac{1}{2}), \qquad H|n\rangle = \omega(n + \frac{1}{2})|n\rangle. \tag{6.26}$$

Applied to a scalar field, it has led to Bose-Einstein statistics.

This is not the case for fermions.

$$\text{Fact: Electrons obey Fermi-Dirac statistics.} \tag{6.27}$$
$$\text{For each single particle state, there is 0 or 1 particle.} \tag{6.28}$$

A general theorem:

$$\text{spin-statistics theorem}: \quad \begin{cases} \text{integer spin} = \text{Bose-Einstein statistics: boson} \\ \text{half-integer spin} = \text{Fermi-Dirac statistics: fermion} \end{cases} \tag{6.29}$$

A theoretical consequence of QFT, agreeing with experiment.

Fermi-Dirac requires anticommutation relation:
Creation and annihilation operators $a, a^\dagger$

$$\{a, a^\dagger\} = aa^\dagger + a^\dagger a = 1, \qquad \{a, a\} = \{a^\dagger, a^\dagger\} = 0 \tag{6.30}$$

In particular,

$$(a)^2 = \frac{1}{2}\{a, a\} = 0, \qquad (a^\dagger)^2 = \frac{1}{2}\{a^\dagger, a^\dagger\} = 0 \tag{6.31}$$

Only two states:

$$\begin{array}{ll} |0\rangle, & a|0\rangle = 0 \\ |1\rangle = a^\dagger|0\rangle, & a^\dagger|1\rangle = (a^\dagger)^2|0\rangle = 0 \end{array} \tag{6.32}$$

Interpretation: 0 or 1 particle.

Number operator

$$N = a^\dagger a \tag{6.33}$$

In fact,

$$[a^\dagger a, a] = (a^\dagger a)a - a(a^\dagger a) = a^\dagger a^2 - (aa^\dagger)a = 0 - (1 - a^\dagger a)a = -a \tag{6.34}$$
$$\implies [N, a] = -a. \tag{6.35}$$

In the same way,

$$[N, a^\dagger] = a^\dagger \tag{6.36}$$

$$N|0\rangle = a^\dagger a|0\rangle = 0, \qquad (a|0\rangle = 0). \tag{6.37}$$
$$N|1\rangle = Na^\dagger|0\rangle = (a^\dagger + a^\dagger N)|0\rangle = a^\dagger|0\rangle = |1\rangle. \tag{6.38}$$

If the Hamiltonian is

$$H = \omega a^\dagger a \tag{6.39}$$

Then

$$H|0\rangle = 0, \qquad H|1\rangle = \omega|1\rangle. \tag{6.40}$$

For a Dirac field ψ with canonical momentum $\pi = i\psi^\dagger$:
Define quantization by anti-commutation relations

$$\{\psi(t,\vec{x}), \pi(t,\vec{y})\} = i\delta^3(\vec{x}-\vec{y})I_4 \qquad \pi = i\psi^\dagger \tag{6.41}$$

I_4 : 4×4 unit matrix. More explicitly,

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}, \qquad \pi = (\pi_1, \pi_2, \pi_3, \pi_4) \tag{6.42}$$

Then

$$\{\psi_a(t,\vec{x}), \pi_b(t,\vec{y})\} = i\delta^3(\vec{x}-\vec{y})\delta_{ab} \qquad (a, b = 1, 2, 3, 4) \tag{6.43}$$

We will abbreviate it by omitting even I_4,

$$\{\psi(t,\vec{x}), \pi(t,\vec{y})\} = i\delta^3(\vec{x}-\vec{y}), \quad \text{both sides regarded as } 4\times 4 \text{ matrices} \tag{6.44}$$

Using $\pi = i\psi^\dagger$,

$$\{\psi(t,\vec{x}), \psi^\dagger(t,\vec{y})\} = \delta^3(\vec{x}-\vec{y}) \tag{6.45}$$

6.3 Creation and annihilation operators

Recall solutions of Dirac equations

$$e^{ip\cdot x}u_{(\vec{p},s)}, \quad e^{-ip\cdot x}v_{(\vec{p},s)}, \quad s = 1, 2, \quad p^0 = E_p \tag{6.46}$$

Properties :

$$(i\not{p} + m)u_{(\vec{p},s)} = 0, \qquad (-i\not{p} + m)v_{(\vec{p},s)} = 0, \tag{6.47}$$

Four vectors

$$u_{(\vec{p},s)}, \quad v_{(-\vec{p},s')} \quad (s = 1, 2) \tag{6.48}$$

are orthogonal.
Expand

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \Big(a_{(\vec{p},s)} u_{(\vec{p},s)} \exp(ip \cdot x) + b^\dagger_{(\vec{p},s)} v_{(\vec{p},s)} \exp(-ip \cdot x) \Big) \tag{6.49}$$

$a_{(\vec{p},s)}$, $b_{(\vec{p},s)}$: operators.
The adjoint on $b_{(\vec{p},s)}$ is for later convenience.

Commutation relations turn out to be

$$\{a_{(\vec{p},s)}, a^\dagger_{(\vec{q},s')}\} = \delta_{s,s'} (2\pi)^3 \delta(\vec{p} - \vec{q}) \qquad \{b_{(\vec{p},s)}, b^\dagger_{(\vec{q},s')}\} = \delta_{s,s'} (2\pi)^3 \delta(\vec{p} - \vec{q}) \tag{6.50}$$

$$\text{other anticommutatiors} = 0 \tag{6.51}$$

Derivation:
At $t = 0$,

$$\psi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \Big(a_{(\vec{p},s)} u_{(\vec{p},s)} + b^\dagger_{(-\vec{p},s)} v_{(-\vec{p},s)} \Big) \exp(i\vec{p} \cdot \vec{x}) \tag{6.52}$$

Recall

$$\{\psi(\vec{x}), \psi^\dagger(\vec{y})\} = \delta^3(\vec{x} - \vec{y}) \tag{6.53}$$

Using

$$\int d^3x \exp(i\vec{p} \cdot \vec{x}) = (2\pi)^3 \delta^3(\vec{p}) \tag{6.54}$$

and orthogonality

$$u^\dagger_{(\vec{p},s)} u_{(\vec{p},s')} = 2E_p \delta_{ss'}, \quad v^\dagger_{(-\vec{p},s)} v_{(-\vec{p},s')} = 2E_p \delta_{ss'}, \quad u^\dagger_{(\vec{p},s)} v_{(-\vec{p},s')} = 0. \tag{6.55}$$

then (exercise)

$$a_{(\vec{p},s)} = \frac{1}{\sqrt{2E_p}} \int d^3x \, u^\dagger_{(\vec{p},s)} \psi(\vec{x}) \exp(-i\vec{p} \cdot \vec{x}) \tag{6.56}$$

$$b^\dagger_{(-\vec{p},s)} = \frac{1}{\sqrt{2E_p}} \int d^3x \, v^\dagger_{(-\vec{p},s)} \psi(\vec{x}) \exp(-i\vec{p} \cdot \vec{x}) \tag{6.57}$$

From it,

$$\{a_{(\vec{p},s)}, a^\dagger_{(\vec{q},s')}\} \tag{6.58}$$

$$= \frac{1}{\sqrt{2E_p}\sqrt{2E_q}} \int d^3x d^3y \, u^\dagger_{(\vec{p},s)} \{\psi(\vec{x}), \psi(\vec{y})^\dagger\} u_{(\vec{q},s')} \exp(-i\vec{p} \cdot \vec{x} + i\vec{q} \cdot \vec{y}) \tag{6.59}$$

$$= \frac{1}{\sqrt{2E_p}\sqrt{2E_q}} \int d^3x d^3y \, u^\dagger_{(\vec{p},s)} \delta^3(\vec{x} - \vec{y}) u_{(\vec{q},s')} \exp(-i\vec{p} \cdot \vec{x} + i\vec{q} \cdot \vec{y}) \tag{6.60}$$

$$= \frac{1}{\sqrt{2E_p}\sqrt{2E_q}} \int d^3x \, u^\dagger_{(\vec{p},s)} u_{(\vec{q},s')} \exp(-i(\vec{p} - \vec{q}) \cdot \vec{x}) \tag{6.61}$$

$$= \frac{1}{2E_p} u^\dagger_{(\vec{p},s)} u_{(\vec{p},s')} (2\pi)^3 \delta^3(\vec{p} - \vec{q}) \tag{6.62}$$

$$= \delta_{s,s'} (2\pi)^3 \delta^3(\vec{p} - \vec{q}) \tag{6.63}$$

Other relations can be obtained similarly.

6.4 Hamiltonian and Fock space

Hamiltonian

$$H = \int d^3x \, (\pi \partial_t \psi - \mathcal{L}) \,, \qquad \pi = i\psi^\dagger \tag{6.64}$$

EOM $(\not{\partial} + m)\psi = 0$ gives

$$\mathcal{L} = -\overline{\psi}(\not{\partial} + m)\psi = 0. \tag{6.65}$$

Thus

$$H = \int d^3x \, i\psi^\dagger \partial_t \psi \tag{6.66}$$

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \Big(a_{(\vec{p},s)} u_{(\vec{p},s)} \exp(ip \cdot x) + b^\dagger_{(\vec{p},s)} v_{(\vec{p},s)} \exp(-ip \cdot x) \Big) \tag{6.67}$$

Using orthogonality of $u_{(\vec{p},s)}, v_{(-\vec{p},s)}$, one get (exercise)

$$\begin{aligned} H &= \int \frac{d^3p}{(2\pi)^3} \sum_s E_p \left(a^\dagger_{(\vec{p},s)} a_{(\vec{p},s)} - b_{(\vec{p},s)} b^\dagger_{(\vec{p},s)} \right) && (6.68) \\ &= \int \frac{d^3p}{(2\pi)^3} \sum_s E_p \left(a^\dagger_{(\vec{p},s)} a_{(\vec{p},s)} + b^\dagger_{(\vec{p},s)} b_{(\vec{p},s)} \right) + E_{\text{vac}} && (6.69) \end{aligned}$$

Here

$$E_{\text{vac}} = \int \frac{d^3p}{(2\pi)^3} \sum_s E_p \left(-2(2\pi)^3 \delta(0) \right) \tag{6.70}$$

We neglect it.

$a^\dagger_{(\vec{p},s)}, b^\dagger_{(\vec{p},s)}$ creat energies:

$$[H, a^\dagger_{(\vec{p},s)}] = E_p a^\dagger_{(\vec{p},s)}, \qquad [H, b^\dagger_{(\vec{p},s)}] = E_p b^\dagger_{(\vec{p},s)} \tag{6.71}$$

Define vacuum $|\Omega\rangle$

$$a_{(\vec{p},s)}|\Omega\rangle = 0, \qquad b_{(\vec{p},s)}|\Omega\rangle = 0 \tag{6.72}$$

One particle states

$$\begin{aligned} &a^\dagger_{(\vec{p},s)}|\Omega\rangle \quad : \text{particle} && (6.73) \\ &b^\dagger_{(\vec{p},s)}|\Omega\rangle \quad : \text{anti-particle} && (6.74) \end{aligned}$$

Particle and anti-particle has the same energy, opposite charge:
$\psi \sim a + b^\dagger \implies a, b^\dagger$ has the same charge, or $a^\dagger$ and $b^\dagger$ has different charge.

More general states: act each of $a^\dagger_{(\vec{p},s)}, b^\dagger_{(\vec{p},s)}$ on $|\Omega\rangle$ zero or one time.
From

$$(a^\dagger_{(\vec{p},s)})^2 = 0, \qquad (b^\dagger_{(\vec{p},s)})^2 = 0 \tag{6.75}$$

each of them cannot act twice.

Summary:

- $a^\dagger_{(\vec{p},s)}$ create a particle with momentum $\vec{p}$ and spin s
- $b^\dagger_{(\vec{p},s)}$ create an anti-particle with momentum $\vec{p}$ and spin s
- Each one-particle state has 0 or 1 particle: Fermi-Dirac statistics.

7 Quantization of electromagnetic field

EM field $A_\mu \to$ photon

Interaction of A_μ and ψ : quantum electrodynamics (QED).
But we only consider free A_μ.

7.1 Analytical mechanics

EM tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{7.1}$$

Gauge transformation

$$A_\mu \to A_\mu + \partial\alpha \qquad \alpha: \text{ arbitrary} \tag{7.2}$$

Physics invariant under it.

Lagrangian $\mathcal{L}$ and action S

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(\vec{E}^2 - \vec{B}^2) \tag{7.3}$$

$$S = \int d^4x\mathcal{L} \tag{7.4}$$

$F_{ij} = \epsilon_{ijk}B^k$, $F_{0i} = -E_i$

Equations of motion: from $\delta S = 0$,

$$\delta S = -\frac{1}{4}\int d^4x 2F^{\mu\nu}(\partial_\mu\delta A_\nu - \partial_\nu\delta A_\mu) \tag{7.5}$$

$$= -\int d^4x F^{\mu\nu}\partial_\mu\delta A_\nu \tag{7.6}$$

$$= \int d^4x(\partial_\mu F^{\mu\nu})\delta A_\nu \tag{7.7}$$

then

$$\partial_\mu F^{\mu\nu} = 0: \text{ Maxwell eq.} \tag{7.8}$$

Canonical momentum and quantization are subtle. We use ad hoc method.
Terms including time derivatives:

$$\mathcal{L} = +\frac{1}{2}(\partial_0 A_i - \partial_i A_0)^2 + \cdots \tag{7.9}$$

Canonical momenta for A_i $(i = 1, 2, 3)$:

$$\pi^i = \frac{\partial\mathcal{L}}{\partial\dot{A}_i} = \partial_0 A_i - \partial_i A_0 = F_{0i} = -E_i \tag{7.10}$$

A_0 does not have time derivative. No canonical momentum for it.
Equation of motion from δA_0

$$\partial_i F^{i0} = \partial_i \pi^i = 0 \tag{7.11}$$

Hamiltonian

$$H = \int d^3x (\pi^i \dot{A}_i - \mathcal{L}) \tag{7.12}$$
$$= \int d^3x (\pi^i(\pi_i + \partial_i A_0) - \mathcal{L}) \tag{7.13}$$
$$= \int d^3x \frac{1}{2}(\vec{\pi}^2 + \vec{B}^2 - A_0 \partial_i \pi^i) \tag{7.14}$$
$$= \int d^3x \frac{1}{2}(\vec{\pi}^2 + \vec{B}^2) \quad (\because \partial_i \pi^i = 0) \tag{7.15}$$

7.2 Gauge invariance of states

Naive canonical commutation relations:

$$[A_i(\vec{x}), \pi^j(\vec{y})] = i\delta_i^j \delta^3(\vec{x} - \vec{y}) \tag{7.16}$$

However,

$$\partial_i \pi^i = 0 \tag{7.17}$$

These two are inconsistent,

$$[A_i(\vec{x}), \partial_j \pi^j(\vec{y})] = i\frac{\partial}{\partial y^i} \delta^3(\vec{x} - \vec{y}) \neq 0 \tag{7.18}$$

Interpretation?

Conceptual (not practical) understanding: wavefunction of $\vec{A}$,

$$\Psi(\vec{A}) \tag{7.19}$$

This is a functional of the function $\vec{A}$. Canonical momentum

$$\pi^i = -i\frac{\delta}{\delta A_i} \qquad \text{functional derivative} \tag{7.20}$$

Interpret $\partial_i \pi_i = 0$ as follows:
$\partial_i \pi^i$ is nonzero as an operator.
But impose that physical states satisfy

$$(\partial_i \pi^i)\Psi(\vec{A}) = 0 \tag{7.21}$$

$\partial_i \pi_i = 0$ is not an equation for the operators π^i but a condition for physical states.

More consideration:
For infinitesimal $\alpha(\vec{x})$,

$$0 = \int d^3x \alpha(\vec{x}) \partial_i \pi^i(\vec{x}) \Psi(\vec{A}) \tag{7.22}$$

$$0 = -\int d^3x \partial_i \alpha(\vec{x}) \pi^i(\vec{x}) \Psi(\vec{A}) \tag{7.23}$$

$$= i \int d^3x (\partial_i \alpha) \frac{\delta \Psi(\vec{A})}{\delta A_i} \tag{7.24}$$

$$= i \left[\Psi(\vec{A} + \vec{\partial}\alpha) - \Psi(\vec{A}) \right] \tag{7.25}$$

then

$$\Psi(\vec{A} + \vec{\partial}\alpha) = \Psi(\vec{A}) \tag{7.26}$$

Interpretation: $\Psi(\vec{A})$ is invariant under gauge transformation

$$A_i \to A_i + \partial_i \alpha \tag{7.27}$$

Physical states are gauge invariant.

7.3 Quantization

Practically, it is convenient to fix the gauge.

Fourier modes

$$A_i(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \tilde{A}_i(\vec{p}) \exp(i\vec{p} \cdot \vec{x}) \tag{7.28}$$

Gauge transformation in momentum space:

$$\tilde{A}_i(\vec{p}) \to \tilde{A}_i(\vec{p}) + i p_i \tilde{\alpha}(\vec{p}) \tag{7.29}$$

$\tilde{A}_i$ in the direction $\vec{p}/|\vec{p}|$ is not physical degrees of freedom.

Gauge fixing:

$$p_i \tilde{A}^i = 0 \implies \partial_i A^i = 0 : \text{Coulomb gauge} \tag{7.30}$$

Then

$$\partial_i \pi^i = 0, \qquad \partial_i A^i = 0 \tag{7.31}$$

Among 3 components ofA_i and π^i, only 2 perpendicular to $\vec{p}$ are physical degrees of freedom.

Mode expansion:

$$\vec{A}_i(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} (\vec{\epsilon}_1 Q_1(\vec{p}) + \vec{\epsilon}_2 Q_2(\vec{p})) \exp(i\vec{p} \cdot \vec{x}) \tag{7.32}$$

$$\pi_i(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} (\vec{\epsilon}_1 P_1(\vec{p}) + \vec{\epsilon}_2 P_2(\vec{p})) \exp(i\vec{p} \cdot \vec{x}) \tag{7.33}$$

$\vec{\epsilon}_1$, $\vec{\epsilon}_2$: polarization vectors

$$\vec{\epsilon}_k \cdot \vec{\epsilon}_\ell = \delta_{k,\ell}, \qquad \vec{p} \cdot \vec{\epsilon}_k = 0 \qquad (k, \ell = 1, 2) \tag{7.34}$$

Hamiltonian

$$H = \int d^3x \frac{1}{2}(\vec{\pi}^2 + \vec{B}^2) \tag{7.35}$$

$$= \int d^3(\frac{1}{2}\vec{\pi}^2 + \frac{1}{4}(\partial_i A_j - \partial_j A_i)^2) \tag{7.36}$$

$$= \frac{1}{2}\int d^3x(\vec{\pi}^2 + (\partial_i A_j)^2 - \partial_i A_j \partial_j A_i) \tag{7.37}$$

$$= \frac{1}{2}\int d^3x(\vec{\pi}^2 + (\partial_i A_j)^2) \qquad \text{(integration by parts, } \partial_i A^i) = 0 \tag{7.38}$$

$$= \text{...similar computation to scalar fields} \tag{7.39}$$

$$= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}(|P_1(\vec{p})|^2 + |P_2(\vec{p})|^2 + \vec{p}^2(|Q_1(\vec{p})|^2 + |Q_2(\vec{p})|^2)) \tag{7.40}$$

Each pair (Q_1, P_1), (Q_2, P_2) behaves similarly to a real scalar field with $m = 0$.

The rest is similar to real scalar.
Results:
Commutation relations

$$[Q_k(\vec{p}), P_\ell(\vec{q})] = i\delta_{k,\ell}(2\pi)^3\delta(\vec{p} - \vec{q}) \tag{7.41}$$

Creation, annihilation operators

$$a_{(\vec{p},k)} = \frac{1}{\sqrt{2E_p}}(E_p Q_k(\vec{p}) + iP_k(\vec{p})) \tag{7.42}$$

$$a^\dagger_{(\vec{p},k)} = \frac{1}{\sqrt{2E_p}}(E_p Q_k(-\vec{p}) - iP_k(-\vec{p})) \tag{7.43}$$

$$[a_{(\vec{p},k)}, a^\dagger_{(\vec{q},\ell)}] = \delta_{k,\ell}(2\pi)^3\delta(\vec{p} - \vec{q}) \tag{7.44}$$

Hamiltonian

$$H = \sum_k \int \frac{d^3p}{(2\pi)^3} E_p \left(a^\dagger_{(\vec{p},k)} a_{(\vec{p},k)}\right) + E_{\text{vac}} \tag{7.45}$$

$$E_p = |\vec{p}|, \qquad E_{\text{vac}} = \sum_{\vec{p}} E_p \tag{7.46}$$

Interpretation:

- $a^\dagger_{(\vec{p},k)}$, $(k = 1, 2)$ create two polarization photons with $p^\mu = (|\vec{p}|, \vec{p})$.
- Bose-Einstein statistics

$$\vec{A}(t, \vec{x}) = \sum_k \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \vec{\epsilon}_k \left(a_{(\vec{p},k)} \exp(ip \cdot x) + a^\dagger_{(\vec{p},k)} \exp(-ip \cdot x)\right) \tag{7.47}$$