Specialization report

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1 Introduction

This report has three main goals. First we shall establish the transition between pictures in quantum mechanics and quantum field theory. We go over the more common ones, Schrödinger, Heisenberg and Interaction picture, to derive and justify the two In and Out Pictures. By working out a few handy techniques and methods on the way, we will proof the Gell-Mann Low formula.

2 Pictures in Quantum Mechanics

2.1 Schrödinger picture and Heisenberg picture

In the Schrödinger the operators are time-independent but the wavefunctions are time dependent. The time evolution of the state vector is controlled by the Schrödinger equation. Let $|\Psi(t)\rangle$ denote the state vector at time t. It satisfies

$$i\hbar \frac{\partial}{\partial t} |\Psi_S(t)\rangle = \hat{H} |\Psi_S(t)\rangle, \tag{1}$$

where $\hat{H}$ is the Hamiltonian of the system. When assuming it time independent, the solution of eq. (1) can be formally written as

$$|\Psi_S(t)\rangle = \hat{U} |\Psi_S(t_0)\rangle \tag{2}$$

with $\hat{U} = e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}$ denoting the time evolution operator. Here $|\Psi_S(t_0)\rangle$ is a ket of $t = t_0$. We shall generally take $t_0 = 0$ and write

$$|\Psi_S(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t} |\Psi_H\rangle. \tag{3}$$

The state one the right-hand side now has no more time dependence. This turns it in contrast to Eq.(2) into a state in a new representation called the Heisenberg picture.

These two pictures differ by the way of storing the time dependence of the total system. In Schrödinger the states carried it, leaving for the Heisenberg picture only operators as a possibility. To perform this transition of dependence we look at matrix elements of operators are given by

$$\langle \Psi_S'(t)| \hat{A}_S |\Psi_S(t)\rangle = \langle \Psi_H'| e^{\frac{i}{\hbar}t\hat{H}} \hat{A}_S e^{-\frac{i}{\hbar}t\hat{H}} |\Psi_H\rangle. \tag{4}$$

$$\hat{A}_H(t) = e^{\frac{i}{\hbar}t\hat{H}}\hat{A}_S e^{-\frac{i}{\hbar}t\hat{H}} = \hat{U}^{-1}\hat{A}_S\hat{U}. \tag{5}$$

This new operator $\hat{A}_H$ in combination with the state Ψ_H build the set to work in the Heisenberg picture. The choice of a picture always requires the transition of states as well as operators. Now its time evolution is dictated by an equation that follows from differentiating the equation above and reads

$$\frac{d}{dt}\hat{A}_H(t) = \frac{i}{\hbar}\hat{H}e^{\frac{i}{\hbar}t\hat{H}}\hat{A}_S e^{-\frac{i}{\hbar}t\hat{H}} + e^{\frac{i}{\hbar}t\hat{H}}\hat{A}_S\left(-\frac{i}{\hbar}\hat{H}\right)e^{-\frac{i}{\hbar}t\hat{H}}. \tag{6}$$

Here we used the derivative of the unitary operator itself:

$$\frac{i}{\hbar}\partial_t\hat{U} = \hat{H}\hat{U}. \tag{7}$$

$$\frac{d}{dt}\hat{A}_H(t) = \frac{i}{\hbar}\hat{U}^{-1}\hat{H}\hat{A}_S\hat{U} - \frac{i}{\hbar}\hat{U}^{-1}\hat{A}_S\hat{H}\hat{U} \tag{8a}$$

$$= \frac{i}{\hbar}\hat{U}^{-1}\hat{H}\hat{U}\hat{U}^{-1}\hat{A}_S\hat{U} - \frac{i}{\hbar}\hat{U}^{-1}\hat{A}_S\hat{U}\hat{U}^{-1}\hat{H}\hat{U}. \tag{8b}$$

The inserted 1 allows us to transform the operators from Schrödinger picture to Heisenberg, including The Hamiltonian.

$$\frac{d}{dt}\hat{A}_H(t) = \frac{i}{\hbar}\hat{H}(t)_H\hat{A}(t)_H - \frac{i}{\hbar}\hat{A}(t)_H\hat{H}(t)_H. \tag{9}$$

Giving us:

$$\frac{d}{dt}\hat{A}_H(t) = \frac{-i}{\hbar}\left[\hat{A}_H(t), \hat{H}_H(t)\right]. \tag{10}$$

2.2 Interaction picture

The third picture is called the Interaction picture (sometimes called the Dirac picture). In it the state and any operator has a time dependence.
Suppose the Hamiltonian from the Schrödinger picture $\hat{H} = \hat{H}_0 + V$. Normally the $\hat{H}_0$ represents the free motion of a particle. V represents in general an external influence or new term complicating a basic problem. Although it often used in a perturbative approach the Interaction picture does not require V to be small or of any special form.
Inserting this new $\hat{H}$ in the unitary operator introduced above:

$$\hat{U} = e^{-\frac{i}{\hbar}t\hat{H}} = e^{-\frac{i}{\hbar}t\hat{H}_0 + V} = e^{-\frac{i}{\hbar}t\hat{H}_0}\hat{\Omega}(t) \tag{11}$$

This expression leads us to the formula of transformation to the Schrödinger picture, by again looking at a matrix element.

$$\langle \Psi'_S(t)| \, \hat{A}_S \, |\Psi_S(t)\rangle = \langle \Psi'_H| \, (e^{-\frac{i}{\hbar}t\hat{H}_0}\Omega(t))^\dagger \hat{A}_S e^{-\frac{i}{\hbar}t\hat{H}_0}\Omega(t) \, |\Psi_H\rangle \tag{12a}$$

$$= \langle \Psi'_H| \, \Omega(t)^{-1} e^{+\frac{i}{\hbar}t\hat{H}_0} \hat{A}_S e^{-\frac{i}{\hbar}t\hat{H}_0}\Omega(t) \, |\Psi_H\rangle \tag{12b}$$

$$\underset{\text{def.picture}}{=} \langle \Psi'_H| \, \Omega(t)^{-1} \hat{A}_I \Omega(t) \, |\Psi_H\rangle \tag{12c}$$

$$= \langle \Psi'_I(t)| \, \hat{A}_I(t) \, |\Psi_I(t)\rangle \, . \tag{12d}$$

Here we defined a new operator:

$$\hat{A}_I(t) = e^{+\frac{i}{\hbar}t\hat{H}_0} \hat{A}_S e^{-\frac{i}{\hbar}t\hat{H}_0}. \tag{13}$$

And new state :

$$|\Psi_I(t)\rangle = \Omega(t) \, |\Psi_H\rangle \, . \tag{14}$$

At the level of operators, the connection between Interaction and Heisenberg be established by substituting $\hat{A}_S$ in Eq.(5) into Eq.(13)

$$\hat{A}_I(t) = e^{+\frac{i}{\hbar}t\hat{H}_0} \hat{U} \hat{A}_H(t) \hat{U}^{-1} e^{-\frac{i}{\hbar}t\hat{H}_0}. \tag{15a}$$

$$= e^{+\frac{i}{\hbar}t\hat{H}_0} e^{-\frac{i}{\hbar}t\hat{H}} \hat{A}_H(t) e^{\frac{i}{\hbar}t\hat{H}} e^{-\frac{i}{\hbar}t\hat{H}_0}. \tag{15b}$$

Resulting in:

$$\hat{A}_I(t) = \hat{\Omega}(t) \hat{A}_H(t) \hat{\Omega}(t)^{-1} \tag{16}$$

Eq.(13) leads on the same way as before to:

$$i\hbar \frac{\partial}{\partial t} \hat{A}_I = \left[\hat{A}_I, \hat{H}_0 \right]. \tag{17}$$

Observe that $\hat{\Omega}(t)$, obeys the equation

$$\hat{\Omega}(t) = e^{\frac{i}{\hbar}t\hat{H}_0} \hat{U} \tag{18a}$$

$$i\hbar \partial_t \hat{\Omega}(t) = e^{\frac{i}{\hbar}t\hat{H}_0} \left(i\hbar \partial_t \hat{U} \right) - \hat{H}_0 e^{\frac{i}{\hbar}t\hat{H}_0} \hat{U} \tag{18b}$$

$$= \hat{H} e^{\frac{i}{\hbar}t\hat{H}_0} \hat{U} - \hat{H}_0 e^{\frac{i}{\hbar}t\hat{H}_0} \hat{U}, \tag{18c}$$

resulting in

$$i\hbar \frac{\partial}{\partial t} \hat{\Omega}(t) = \hat{V}_I(t) \hat{\Omega}(t). \tag{19}$$

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3 The In and Out picture

As a motivation for this section, consider the set-up of most experiments especially in elementary particle and nuclear physics. Several particles approach each other from a macroscopic scale, interact in a microscopical section relative to the de Broglie wavelength of the particles. On this scale vacuum fluctuations can accrue, which influence the incoming particle and even make it impossible to distinguish between fluctuation and particle in itself. Afterwards, the products of the interaction spread up to a macroscopic level, again been clearly distinguishable from each other and surrounding space. Therefore, a good description can be derived by taking the states of the system at the beginning and end as direct products of single particle effectively non-interacting states.

To bring this concept on into our formulation let's consider $V = j(t)\hat{q}(t)$ with $\hat{q}(t)$ as the operator of position and j an external source. This will resulting in an unstable vacuum.

3.1 External currents and the unstable vacuum

In a system with an external current the vacuum state can evolve over time into a state with particles. Starting with the action of a scalar field with mass m

$$I = \int d^4x \left(\frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2}m^2\Phi^2 - \Phi j \right) \tag{20}$$

, by taking the functional derivative in respect to Φ and setting it to zero, we obtain the equation of motion

$$\left(\partial^2 + m^2 \right) \Phi = j. \tag{21}$$

We will confine the field in a box and impose periodic boundary conditions on it. This is called quantisation in a box. It allows us to write the field in terms of modes.

$$\Phi(x) = \Phi(\underline{x}, t) = \sum_k q_k(t)u_k(\underline{x}). \tag{22}$$

In this separated time and space dependency, we choose the Fourier basis for u_k.

$$u_k(\underline{x}) = \frac{1}{\sqrt{V}}e^{ik\underline{x}} \tag{23}$$

The normalisation over the volume V comes form the box we impose for the field. In addition u_k fulfills the completeness relation.

$$\int d^3\underline{x}\; u_{k'}^*(\underline{x})u_k(\underline{x}) = \delta_{kk'} \tag{24}$$

With these we can rewrite the equation of motion:

$$\left(\partial^2 + m^2 \right) \sum_k q_k(t)u_k(\underline{x}) = j(\underline{x}, t) \tag{25a}$$

$$\sum_k \left[\left(\frac{\partial}{\partial t^2} - \nabla^2 + m^2 \right) q_k(t)u_k(\underline{x}) \right] = j(\underline{x}, t). \tag{25b}$$

The basis for u_k will enable us to take the spacial derivative.

$$\sum_k \left[\ddot{q}_k(t)u_k(\underline{x}) - (i)^2k^2q_k(t)u_k(\underline{x}) + m^2q_k(t)u_k(\underline{x}) \right] = j(\underline{x}, t). \tag{26}$$

To get an equation for q_k alone we need to get rid of u_k and remove the space dependency in the current. Multiplying with $u_{k'}^*$ and integrating over

the whole space will form the completeness relation on the left-hand side and the Fourier transformation for the current.

$$\int d^3\underline{x}\, u_{k'}^* \cdot | \sum_k \left[\ddot{q}_k(t)u_k(\underline{x}) - (i)^2 k^2 q_k(t)u_k(\underline{x}) + m^2 q_k(t)u_k(\underline{x}) \right] = \int d^3\underline{x}\, j(\underline{x},t)\frac{1}{\sqrt{V}}e^{ik\underline{x}}$$

$$\text{(27)}$$

$$\sum_k \left[\int d^3\underline{x}\, u_{k'}^* u_k \left(\ddot{q}_k(t) + \left(k^2 + m^2 \right) q_k(t) \right) \right] = \tilde{j}(k',t). \qquad \text{(28)}$$

Using the completeness relation

$$\sum_k \delta_{kk'} \left[\ddot{q}_k(t) + \left(k^2 + m^2 \right) q_k(t) \right] = \tilde{j}(k',t). \qquad \text{(29)}$$

The Kronecker-delta reduces the sum to one term. With change of variable we set choose k and we define $(k^2 + m^2) = \omega_k^2$.

$$\ddot{q}_k(t) + \omega_k^2 q_k(t) = \tilde{j}(k',t). \qquad \text{(30)}$$

We now make the assumption that the current vanishes outside a finite time interval. This gives us two cases. For early time it would go to the homogeneous solution. We call it $q_k(t) \to q_{k,In}(t)$.

$$\ddot{q}_{k,In}(t) + \omega_k^2 q_{k,In}(t) = 0, \qquad \text{(31)}$$

with the general solution in terms of creation $a_{k,In}$ and annihilation operators $a_{k,In}^\dagger$,

$$q_{k,In}(t) = \frac{1}{2\omega_k} \left(a_{k,In}e^{-i\omega_k t} + a_{k,In}^\dagger e^{i\omega_k t} \right). \qquad \text{(32)}$$

At late times it would also reduce the differential equation to a homogeneous type. This will be $q_k(t) \to q_{k,Out}(t)$.

$$q_{k,Out}(t) = \frac{1}{2\omega_k} \left(a_{k,Out}e^{-i\omega_k t} + a_{k,Out}^\dagger e^{i\omega_k t} \right). \qquad \text{(33)}$$

The full solution for $q_k(t)$ would then consist of homogeneous solution plus a term for the dependence of the current.

$$q_k(t) = q_{k,In}(t) + \frac{1}{\omega_k} \int_{-\infty}^t dt'\, sin\left(\omega_k(t - t') \right) \tilde{j}(k',t), \qquad \text{(34)}$$

for late times we again go over to the different homogeneous solution $q_{k,Out}(t)$ and take the integral to $+\infty$.

$$q_{k,Out}(t) = q_{k,In}(t) + \frac{1}{\omega_k} \int_{-\infty}^{\infty} dt' \, sin\left(\omega_k(t - t')\right) \tilde{j}(k', t). \tag{35}$$

Now we will apply a second Fourier transformation in respect to t. The commutation between k, x, t, E ensures no problem to do these transformations separate and we move the k dependence to a index notation.

$$\bar{j}_k(E) = \int_{-\infty}^{\infty} dt \tilde{j}_k(t) e^{iEt}. \tag{36}$$

After splitting sinus we write:

$$q_{k,Out}(t) = q_{k,In}(t) - \frac{i}{2\omega_k} e^{i\omega_k t} \, \bar{j}_k(-\omega_k) + \frac{i}{2\omega_k} e^{-i\omega_k t} \, \bar{j}_k(\omega_k) \tag{37}$$

From this equation we can obtain a very important information. The connection between creation and annihilation operators.

$$a_{k,Out} = a_{k,In} + i\bar{j}_k(\omega_k) \tag{38a}$$

$$a^{\dagger}_{k,Out} = a^{\dagger}_{k,In} - i\bar{j}_k(-\omega_k) \tag{38b}$$

This shows the operators are not the same in the present of a current. Therefore we need to differ between the eigenstates to these operators. Especially, it has to be stated that vacua also differ in this scenario. The concept of early and later times to fully solve the full equations will be discussed further in the later chapters.

Still, we need a way to transition from one operator to another and from eigenstate to eigenstate. This link is called the scattering operator S.

$$|Out\rangle = S^{-1} |In\rangle \tag{39a}$$

$$|In\rangle = S |Out\rangle . \tag{39b}$$

The scalar product of the mentioned two vacua can be written as :

$$\langle 0_{Out}|0_{In}\rangle = \langle 0_{Out}|S|0_{Out}\rangle \tag{40}$$

For now we choose S in a more specific form , where we used $\bar{j}_k(-\omega_k) = \bar{j}_k^*(\omega_k)$. The general one will be obtained later:

$$S = e^{\frac{i}{2\omega_k} \bar{j}_k(\omega_k) a^{\dagger}_{k,In}} e^{\frac{i}{2\omega_k} \bar{j}_k(-\omega_k) a_{k,In}} e^{-\frac{1}{4\omega_k} |\bar{j}_k(-\omega_k)|^2} \tag{41}$$

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For the operators we can write:

$$S^{-1}a_{k,In}S = a_{k,Out} = a_{k,In} + i \ \bar{j}_k(\omega_k) \tag{42}$$

Using this, we obtain first of all the probability for staying in the ground state for one mode:

$$p_{0,k} = |\langle 0_{Out}|0_{In}\rangle|^2 = |\langle 0_{In}|S|0_{In}\rangle|^2 \tag{43a}$$

$$= \left| \langle 0_{In}| e^{\frac{i}{2\omega_k}\bar{j}_k(\omega_k)a^\dagger_{k,In}} e^{\frac{i}{2\omega_k}\bar{j}_k(-\omega_k)a_{k,In}} e^{-\frac{1}{4\omega_k}|\bar{j}_k(-\omega_k)|^2} |0_{In}\rangle \right|^2. \tag{43b}$$

The annihilation operator will return a 0 in the exponent. Therefore only one factor remain important. The probabilty while taking to account any mode transition just involves the integral over all k:

$$p_0 = exp\left\{ -\int \frac{d^3k}{(2\pi)^3} \left| \frac{\bar{j}_k(\omega_k)}{\sqrt{2\omega_k}} \right|^2 \right\}. \tag{44}$$

This confirms our statement about an unstable vacuum. The negative sign in the exponent translates to smaller probability at higher external currents. So the vacuum can change it's state if a current is > 0.

Taking these new results into account, we need to impose for the case $V = j(t)\hat{q}(t)$ from the beginning a similar approach. On the realm of pictures this means asymptotic conditions on $j(t)$. We again assume it vanishes outside of finite intervals.

$$j(t) \to 0 \text{ for } t \to \pm\infty. \tag{45}$$

The effect of these conditions on Eq.(19) leads to

$$\Omega \to 1 \tag{46}$$

The early time condition defines the In picture in reminiscence to the first homogeneous equation and it writes :

$$i\hbar \frac{\partial}{\partial t}\hat{\Omega}_{in}(t) = \hat{V}_{in}(t)\hat{\Omega}_{in}(t). \tag{47}$$

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Fulfilling:
$$\hat{\Omega}_{in}(-\infty) = 1. \tag{48}$$

The so called *Out* picture will fulfill the condition for late times:

$$i\hbar\frac{\partial}{\partial t}\hat{\Omega}_{out}(t) = \hat{V}_{out}(t)\hat{\Omega}_{out}(t). \tag{49}$$

Stating:
$$\hat{\Omega}_{out}(+\infty) = 1. \tag{50}$$

The Interaction picture as introduced above does not require V to be of any special form but can still be applied for external source terms. To distinguish between the other two pictures above from the condition is chosen :

$$\hat{\Omega}_I(0) = 1. \tag{51}$$

The connecting operator between the pictures is called scattering operator or S-Matrix. It is defined as :[1]

$$S = \Omega_{in}(t)\Omega_{out}^{-1}(t) = \Omega_I(\infty). \tag{52}$$

4 Scattering operator

4.1 Solutions for the Interaction picture

The verification of statement (52) will make use of explicit expressions of all Ω.

Starting with the Interaction picture depended on t' and integrate the left-hand side of (19) :

$$\int_{t_0}^{t}\mathrm{d}t'i\hbar\frac{\partial}{\partial t'}\Omega_I(t') = i\hbar\left[\Omega_I(t) - \Omega_I(t_0)\right] \tag{53a}$$

,here we state $t_0 = 0$ as it is the condition for this picture and for now $t > 0$.

$$\int_{0}^{t}\mathrm{d}t'i\hbar\frac{\partial}{\partial t'}\Omega_I(t') = i\hbar\left[\Omega_I(t) - 1\right]. \tag{53b}$$

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[1]We will now go to natural units $c = \hbar = 1$ and drop the operator hat for convenience sake

Using this formula and the integral over the right-hand side of (19), we find an expression for $\Omega_I(t)$.

$$\Omega_I(t) = 1 - \frac{i}{\hbar} \int_0^t dt' V_I(t') \Omega_I(t') \tag{54}$$

,since the expression has an Ω_I on the other side we will go on by an iterative approach.

$$\Omega_I(t) = 1 - \frac{i}{\hbar} \int_0^t dt' V_I(t') \cdot \left(1 - \frac{i}{\hbar} \int_0^{t'} dt'' V_I(t'') \Omega_I(t'')\right) \tag{55a}$$

$$= 1 - \frac{i}{\hbar} \int_0^t dt' V_I(t') + (\frac{i}{\hbar})^2 \int_0^t dt' \int_0^{t'} dt'' V_I(t'') \Omega_I(t''). \tag{55b}$$

The iteration will lead us by single digit incrementation in the power and number of integrals. An problematic aspect of this series are the different integral borders. Each term introduces a new t and keeps the previous as an integral boundary which forces us to solve them in a strict order. To keep the different t as variables over the same integral $\int_0^t$, we use the chronological time ordering.

$$T(V(t_1), V(t_2)) = V(t_1) \, V(t_2) \, \theta(t_1 - t_2) \; + \; V(t_2) \, V(t_1) \, \theta(t_2 - t_1). \tag{56}$$

In chronological time ordering the Heaviside-Step-function is used to set the earlier times to the right and later to the left by checking the difference. The Step-function is 0 for negative values and 1 for positive.[2]
For convince sake we drop $\hbar$ again, t will be labelled with numbers instead of primes and time dependency will be indicated by index : $V(t_1) \to V_1$.
The solution to $\Omega_I(t)$ written as an infinite sum and time ordered :

$$\Omega_I(t) = \sum_{n=0}^{\infty} (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n V_I(t_1) \cdot \ldots \cdot V_I(t_n) \tag{57a}$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_0^t dt_1 \int_0^t dt_2 \ldots \int_0^t dt_n T \{V_I(t_1), \ldots, V_I(t_n)\} \tag{57b}$$

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[2]Proofs and elaborations for time ordering are to be found in the Appendices

The factorial sum over n converges to exponential function as seen in calculus and this non perturbativ expression reads:

$$\Omega_I(t) = T\left(e^{-i\int_0^t dt' V_I(t')}\right), \text{for } t > 0. \tag{58}$$

We needed to split the solution of $\Omega_I(t)$, since the condition for the picture is not on the limits of the time span. Assuming $t < 0$ changes the first integral to :

$$\int_t^0 dt' i\hbar \frac{\partial}{\partial t'} \Omega_I(t') = i\hbar \left[1 - \Omega_I(t)\right] \tag{59}$$

The sign difference to the expression for $t > 0$ can be fixed by switching the integral borders. This means our infinite sum expression still holds:

$$\Omega_I(t) = \sum_{n=0}^{\infty} (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n V_I(t_1) \cdot \ldots \cdot V_I(t_n). \tag{60}$$

The key difference now stands in the negativity of all t and a logical order for them would prefer later times to the right, coming closer to 0. This requires the anti-chronological time ordering :

$$\bar{T}(V(t_1), V(t_2)) = V(t_2)\, V(t_1)\, \theta(t_1 - t_2)\ +\ V(t_1)\, V(t_2)\, \theta(t_2 - t_1). \tag{61}$$

Using it similar as before:

$$\Omega_I(t) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_0^t dt_1 \int_0^t dt_2 \ldots \int_0^t dt_n \bar{T}\left\{V_I(t_1), \ldots, V_I(t_n)\right\}. \tag{62}$$

Converging to the second expression:

$$\Omega_I(t) = \bar{T}\left(e^{-i\int_0^t dt' V_I(t')}\right), \text{for } t < 0. \tag{63}$$

These two cases make the use of Ω_I safe in a sense of not having to watch out for sign flip in the integral while t runs. Second, the picture condition is clearly stated and not possible to hit while performing integration over t from negative to positive. But to perform the iterative solution ones only needs to state Ω_I being finite in the region of integration. Lowering the integral boundary to $-\infty$ instead of 0. Losing the advantages from above but gaining the single expression:

$$\Omega_I(t) = T\left(e^{-i\int_{-\infty}^t dt' V_I(t')}\right). \tag{64}$$

4.2 Solutions for the In and Out picture

Beginning with the In picture and eq.(47) we reach:

$$\int_{t_0}^{t} dt' i\hbar \frac{\partial}{\partial_{t'}} \Omega_{in}(t') = i\hbar \left[\Omega_I(in) - \Omega_{in}(t_0) \right] \qquad (65a)$$

,here we state $t_0 = -\infty$

$$\int_{-\infty}^{t} dt' i\hbar \frac{\partial}{\partial_{t'}} \Omega_{in}(t') = i\hbar \left[\Omega_{in}(t) - 1 \right]. \qquad (65b)$$

The condition of the In picture being a limit for the time span spares us any splitting. Following the first solution of the Interaction picture:

$$\Omega_{in}(t) = \sum_{n=0}^{\infty} (-i)^n \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \ldots \int_{-\infty}^{t_{n-1}} dt_n V_{in}(t_1) \cdot \ldots \cdot V_{in}(t_n) \qquad (66a)$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t} dt_2 \ldots \int_{-\infty}^{t} dt_n T \left\{ V_{in}(t_1), \ldots, V_{in}(t_n) \right\}. \qquad (66b)$$

Converging to :

$$\Omega_{in}(t) = T \left(e^{-i \int_{-\infty}^{t} dt' V_{in}(t')} \right). \qquad (67)$$

The Out picture on the other hand follows the second solution. Here we argue t being smaller then ∞ needs one change of sign like before and anti-chronological ordering,since t only coming closer to the limit as it runs.

$$\Omega_{out}(t) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{\infty}^{t} dt_1 \int_{\infty}^{t} dt_2 \ldots \int_{\infty}^{t} dt_n \bar{T} \left\{ V_{out}(t_1), \ldots, V_{out}(t_n) \right\}. \qquad (68)$$

Concluding :

$$\Omega_{out}(t) = \bar{T} \left(e^{-i \int_{\infty}^{t} dt' V_{in}(t')} \right). \qquad (69)$$

4.3 Connections

Using this set of solution, the first set of relations to be proven is:

$$S = \Omega_{in}(t)\Omega_{out}(t)^{-1} \qquad (70a)$$

$$= \Omega_{in}(\infty) \qquad (70b)$$

$$= \Omega_{out}(-\infty)^{-1}. \qquad (70c)$$

Writing the first equality out,

$$S = T\left(e^{-i\int_{-\infty}^{t} dt' V_{in}(t')}\right) \cdot \bar{T}\left(e^{-i\int_{\infty}^{t} dt'' V_{out}(t'')}\right) \tag{71a}$$

$$= \sum_n (-i)^n \int_{-\infty}^{t} dt_1 V_{in}(t_1) \ldots \int_{-\infty}^{t_{n-1}} dt_n V_{in}(t_n) \tag{71b}$$

$$\cdot \left(\sum_n (-i)^n \int_{\infty}^{t} dt'_1 V_{out}(t'_1) \ldots \int_{\infty}^{t'_{n-1}} dt'_n V_{out}(t'_n)\right)^{-1} \tag{71c}$$

As an unitar operator $\Omega_{out}(t)^{-1} = \Omega_{out}(t)^\dagger$

$$= T\left(e^{-i\int_{-\infty}^{t} dt' V_{in}(t')}\right) \cdot T\left(e^{+i\int_{\infty}^{t} dt'' V_{out}(t'')}\right) \tag{71d}$$

The lack of overlap in the integral limits allows us to fuse T

$$= T\left(e^{-i\int_{-\infty}^{t} dt' V_{in}(t')} \cdot e^{+i\int_{\infty}^{t} dt'' V_{out}(t'')}\right) \tag{71e}$$

Commutation in T allow ease use ofthe Baker-Campbell-Hausdorff-formula

$$= T\left(e^{-i\int_{-\infty}^{t} dt' V_{in}(t') - i\int_{t}^{\infty} dt'' V_{out}(t'')}\right). \tag{71f}$$

In addition the overlap makes S time-independent. Therefore t could be ∞:

$$S = T\left(e^{-i\int_{-\infty}^{\infty} dt' V_{in}(t') - i\int_{\infty}^{\infty} dt'' V_{out}(t'')}\right) = T\left(e^{-i\int_{-\infty}^{\infty} dt' V_{in}(t')}\right) = \Omega_{in}(\infty). \tag{72}$$

Or we choose $-\infty$:

$$S = T\left(e^{-i\int_{-\infty}^{-\infty} dt' V_{in}(t') - i\int_{\infty}^{-\infty} dt'' V_{out}(t'')}\right) = T\left(e^{-i\int_{-\infty}^{\infty} dt' V_{out}(t')}\right) = \Omega_{out}(-\infty)^{-1}. \tag{73}$$

Eq. (72) is very similar to the definition in eq.(52) for Ω_I. This comes from V_I and V_{in} both being picture transformations of V_H in the Heisenberg picture. We discussed the main difference lies in position of the condition on the time span. The time independence and integral over all t makes them equal.

$$S = T\left(e^{-i\int_{-\infty}^{\infty} dt V_I(t)}\right). \tag{74}$$

The next set of relations is :

$$\Omega_{in}(t)\Omega_I(t)^{-1} = S \ \Omega_I(\infty)^{-1} \tag{75a}$$
$$= \Omega_{in}(0). \tag{75b}$$

The first one is just 1, since the definition was $S = \Omega_I(\infty)$ was Using the exponential formulae:

$$\Omega_{in}(t)\Omega_I(t)^{-1} = T\left(e^{-i\int_{-\infty}^{t} dt' V_{in}(t')}\right)\left(\bar{T}\left(e^{-i\int_0^t dt' V_I(t')}\right)\right)^{-1} \tag{76a}$$

$$\underset{T\to\bar{T}}{=} T\left(e^{-i\int_{-\infty}^{t} dt' V_{in}(t')}\right) T\left(e^{+i\int_0^t dt' V_I(t')}\right) \tag{76b}$$

$$= T\left(e^{-i\int_{-\infty}^{t} dt' V_{in}(t')} e^{+i\int_0^t dt' V_I(t')}\right) \tag{76c}$$

$$\underset{\text{CBH}}{=} T\left(e^{-i\int_{-\infty}^{t} dt' V_{in}(t') - i\int_t^0 dt' V_I(t')}\right) \tag{76d}$$

$$= \Omega_{in}(0)\Omega_I(0) \underset{\Omega_I(0)=1}{=} \Omega_{in}(0) \tag{76e}$$

This last expression is time independent and is also 1 by looking at the differential equation for Ω_{in}.

The last relation is :

$$\Omega_{in}(t) = \bar{T}\left(e^{i\int_t^{\infty} dt' V_{in}(t')}\right) S. \tag{77}$$

First it must satisfy the differential equation for Ω_{in}:

$$i\partial_t \Omega_{in}(t) = i\partial_t \left(\sum_n i^n \int_t^{\infty} dt_1 V_{in}(t_1) \dots \int_{t_{n-1}}^{\infty} dt_n V_{in}(t_n)\right) S \tag{78a}$$

making an integration by part,

$$= i\left(\sum_n i^n \partial_t \left[\bar{V}_{in}(\infty) - \bar{V}_{in}(t)\right] \int_{t_1}^{\infty} dt_2 V_{in}(t_2) \dots \int_{t_{n-1}}^{\infty} dt_n V_{in}(t_n)\right) S \tag{78b}$$

partial differentiate in respect to t,

$$= i\left(\sum_n i^n \left[0 - V_{in}(t)\right] \int_{t_1}^{\infty} dt_2 V_{in}(t_2) \dots \int_{t_{n-1}}^{\infty} dt_n V_{in}(t_n)\right) S \tag{78c}$$

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multiplying and convergence,

$$= V_{in}(t) \sum_n \frac{i^{n-1}}{(n-1)!} \int_{t_1}^{\infty} \mathrm{d}t_2 \ldots \int_{t_{n-1}}^{\infty} \mathrm{d}t_n \bar{T} \left(V_{in}(t_2) \ldots V_{in}(t_n) \right) S \qquad (78\mathrm{d})$$

$$= \bar{T} \left(V_{in}(t) e^{i \int_t^{\infty} \mathrm{d}t' V_{in}(t')} \right) S \qquad (78\mathrm{e})$$

$$= V_{in}(t) \bar{T} \left(e^{i \int_t^{\infty} \mathrm{d}t' V_{in}(t')} \right) S \qquad (78\mathrm{f})$$

The last step was based on t being the earliest time in the integral and anti-chronological time ordering.
One also has to provide $\Omega_{in}(-\infty) = 1$:

$$\Omega_{in}(t) = \bar{T} \left(e^{i \int_{-\infty}^{\infty} \mathrm{d}t' V_{in}(t')} \right) S \qquad (79\mathrm{a})$$

$$= (\Omega_{in}(\infty))^{\dagger} S \qquad (79\mathrm{b})$$

$$= S^{\dagger} S = S^{-1} S = 1 \qquad (79\mathrm{c})$$

4.4   Unitarity of S

This last step required S to be unitar. Beginning with rewriting it.

$$S = T \left(e^{-i \int_{-\infty}^{\infty} \mathrm{d}t V_I(t)} \right) \qquad (80\mathrm{a})$$

$$= \sum_n \frac{(-i)^n}{(n)!} \int_{-\infty}^{\infty} \mathrm{d}t_1 \ldots \int_{-\infty}^{\infty} \mathrm{d}t_n T \left(V_{in}(t_1) \ldots V_{in}(t_n) \right) \qquad (80\mathrm{b})$$

$$= \sum_n (-i)^n \int_{-\infty}^{\infty} \mathrm{d}t_1 \ldots \int_{-\infty}^{\infty} \mathrm{d}t_n V_{in}(t_1) \ldots V_{in}(t_n). \qquad (80\mathrm{c})$$

Without variables in the limits of integration the order is arbitrary

$$= \sum_n \frac{(-i)^n}{(n)!} \int_{-\infty}^{\infty} \mathrm{d}t_1 \ldots \int_{-\infty}^{\infty} \mathrm{d}t_n \bar{T} \left(V_{in}(t_1) \ldots V_{in}(t_n) \right) \qquad (80\mathrm{d})$$

$$= \bar{T} \left(e^{-i \int_{-\infty}^{\infty} \mathrm{d}t V_I(t)} \right). \qquad (80\mathrm{e})$$

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Keeping this in mind we express $1 = 1$ as :

$$1 = T\left(e^0\right) \tag{81a}$$

$$= T\left(e^{i\int_{-\infty}^{\infty} dt V_I(t) - i\int_{-\infty}^{\infty} dt V_I(t)}\right) \tag{81b}$$

$$\underset{\text{CBH}}{=} T\left(e^{i\int_{-\infty}^{\infty} dt V_I(t)} e^{-i\int_{-\infty}^{\infty} dt V_I(t)} e^{\frac{1}{2}\left[i\int_{-\infty}^{\infty} dt V_I(t), -i\int_{-\infty}^{\infty} dt V_I(t)\right]}\right) \tag{81c}$$

$$\underset{e^{\frac{1}{2}\cdot 0}}{=} T\left(e^{i\int_{-\infty}^{\infty} dt V_I(t)} e^{-i\int_{-\infty}^{\infty} dt V_I(t)}\right) \tag{81d}$$

$$= T\left(e^{i\int_{-\infty}^{\infty} dt V_I(t)}\right) T\left(e^{-i\int_{-\infty}^{\infty} dt V_I(t)}\right) \tag{81e}$$

$$= T\left(e^{i\int_{-\infty}^{\infty} dt V_I(t)}\right) \bar{T}\left(e^{-i\int_{-\infty}^{\infty} dt V_I(t)}\right) \tag{81f}$$

$$= S \cdot S^\dagger \tag{81g}$$

$$= S \cdot S^{-1} \tag{81h}$$

$$= 1 \tag{81i}$$

5 Gell-Mann Low formula

To motivate Gell-Mann Low formula as the important asset, a common way of application and requirement will be laid out.

First, the formula allows us to transform a polynomial , chronological (or anti-chronological) ordered set of operators in the Heisenberg picture to the three pictures connected to the scattering operator.

This strikes as a fundamental step for dealing with the pictures. Furthermore this transition will be made very early when working on many topics of quantum field theory. As most of the time, one would begin with classical mechanics to get to QFT. Starting with the action of your problem in term of classical fields and then apply second quantisation to promote them to operators in the Heisenberg picture. This would be the point to transition and one needs the Gell-Mann Low formula.

Recalling Eq.(16) we can write:

$$Q_H(t) = \Omega_I(t)^{-1} Q_I(t) \, \Omega_I(t). \tag{82}$$

Since the proof didn't require any specifications on Ω as it only remains as the difference to the Heisenberg operator U. Allowing us to write:

$$Q_H(t) = \Omega_{in}(t)^{-1} Q_{in}(t) \, \Omega_{in}(t). \tag{83}$$

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Expressing Ω_{in} by S as seen above:

$$Q_H(t) = \left(\bar{T} \left(e^{i \int_t^\infty \mathrm{d}t' V_{in}(t')} \right) S \right)^{-1} Q_{in}(t) T \left(e^{-i \int_{-\infty}^t \mathrm{d}t' V_{in}(t')} \right) \tag{84}$$

being a unitary operator, we replace -1 by $\dagger$ and apply hermitian conjugation

$$Q_H(t) = \left(\bar{T} \left(e^{i \int_t^\infty \mathrm{d}t' V_{in}(t')} \right) S \right)^{\dagger} Q_{in}(t) T \left(e^{-i \int_{-\infty}^t \mathrm{d}t' V_{in}(t')} \right) \tag{85a}$$

$$= S^{\dagger} \left(\bar{T} \left(e^{i \int_t^\infty \mathrm{d}t' V_{in}(t')} \right) \right)^{\dagger} Q_{in}(t) T \left(e^{-i \int_{-\infty}^t \mathrm{d}t' V_{in}(t')} \right) \tag{85b}$$

$$= S^{-1} T \left(e^{-i \int_t^\infty \mathrm{d}t' V_{in}(t')} \right) Q_{in}(t) T \left(e^{-i \int_{-\infty}^t \mathrm{d}t' V_{in}(t')} \right). \tag{85c}$$

This step used the unitarity of S itself and the appendix for transition to chronological time ordering. Following that it is allowed to merge T due to zero overlap in the boundaries and correct order

$$Q_H(t) = S^{-1} T \left(e^{-i \int_t^\infty \mathrm{d}t' V_{in}(t')} Q_{in}(t) e^{-i \int_{-\infty}^t \mathrm{d}t' V_{in}(t')} \right). \tag{86}$$

The position of Q_{in} is still not ideal. Changing it will require commutations. Using the infinite series for e and removing time ordering will reduce the problem to commutations between Q_{in} and V_{in}.

$$Q_H(t) = S^{-1} \left(\sum_n (-i)^n \int_{-\infty}^t \mathrm{d}t_1' \int_t^\infty \mathrm{d}t_1' \dots \int_{-\infty}^{t_{n-1}} \mathrm{d}t_n' \int_{t_{n-1}}^\infty \mathrm{d}t_n' V_{in}(t_1') \dots V_{in}(t_n') Q_{in}(t) \right). \tag{87}$$

We see Q_{in} and V_{in} can commute without any problem since Q_{in} depends on t and $t \neq t'$. Moving it to the left and reapply T as well as e:

$$Q_H(t) = S^{-1} \left(\sum_n (-i)^n Q_{in}(t) \int_{-\infty}^t \mathrm{d}t_1' \int_t^\infty \mathrm{d}t_1' \dots \int_{-\infty}^{t_{n-1}} \mathrm{d}t_n' \int_{t_{n-1}}^\infty \mathrm{d}t_n' V_{in}(t_1') \dots V_{in}(t_n') \right) \tag{88a}$$

$$Q_H(t) = S^{-1} T \left(Q_{in}(t) e^{-i \int_{-\infty}^\infty \mathrm{d}t' V_{in}(t')} \right). \tag{88b}$$

Identifying the exponential function as S, the result is:

$$Q_H(t) = S^{-1} T \left(Q_{in}(t) S \right). \tag{89}$$

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Next will be for more than one operator. Starting with the left side in a non trivial time ordering and applying the transformation for each operator:

$$T\left(Q_H(t_1)Q_H(t_2)\dots\right) = T\left(\prod_j \Omega_{in}(t_j)^{-1}Q_{in}(t_j)\Omega_{in}(t_j)\right) \qquad (90a)$$

,inserting the expression with S for a Ω depending on t_j ,

$$= T\left(\prod_j S^{-1}T\left(e^{-i\int_{t_j}^{\infty} dt'_j V_{in}(t'_j)}\right)Q_{in}(t_j)T\left(e^{-i\int_{-\infty}^{t_j} dt'_j V_{in}(t'_j)}\right)\right) \qquad (90b)$$

,the series and commutation follows the same argumentation as for a single Q,

$$T\left(Q_H(t_1)Q_H(t_2)\dots\right) = T\left(\prod_j S^{-1}Q_{in}(t_j)S\right). \qquad (91)$$

Moving S outside of T gives the final form of the Gell-Mann Low formula:

$$T\left(Q_H(t_1)Q_H(t_2)\dots\right) = S^{-1}T\left(Q_{in}(t_1)Q_{in}(t_2)\dots S\right). \qquad (92)$$

To proof S can be outside of chronological time ordering we look at $T(S^{-1})$:

$$T(S^{-1}) = T(S^{\dagger}) \qquad (93a)$$

,using the expression of S in term of the In-picture and apply hermitian conjugation,

$$T(S^{-1}) = T\left(\bar{T}\left(e^{i\int_{-\infty}^{\infty} dt' V_{in}(t')}\right)\right) \qquad (93b)$$

,e^x as a series,

$$= T\left(\sum_n \frac{(i)^n}{n!}\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \dots \int_{-\infty}^{\infty} dt_n \bar{T}\left(V_{in}(t_1)\dots V_{in}(t_n)\right)\right) \qquad (93c)$$

and removing the anti-chronological time ordering,

$$= T\left(\sum_n \frac{(i)^n}{n!}i^n \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \dots \int_{-\infty}^{\infty} dt_n V_{in}(t_1)\dots V_{in}(t_n)\right). \qquad (93d)$$

The boundaries make every integral independent from each other so rearranging t_1 to t_n wouldn't change the result

$$\rightarrow T(S^{-1}\dots) = S^{-1}T(\dots)$$

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5.1 Vacuum expectation value

The Gell-Mann Low formula eq.(92) plays a central role in many perturbative or non-perturbative calculations. In particular by evaluating it sandwiched between eigenstates to rising and lowering operators in the In- and Out-picture. If they are in the groundstate of the theory aka. the vacuum state, we obtain the vacuum expectation value.

$$\langle 0_{out}|\, T\left(Q_H(t_1)Q_H(t_2)\dots\right)|0_{in}\rangle = \tag{94a}$$

$$= \langle 0_{out}|\, S^{-1}T\left(Q_{in}(t_1)Q_{in}(t_2)\dots S\right)|0_{in}\rangle \tag{94b}$$

$$= \langle 0_{out}|\, S^{-1}T\left(Q_{in}(t_1)Q_{in}(t_2)\dots e^{-i\int_{-\infty}^{\infty}dt'V_{in}(t')}\right)|0_{in}\rangle \tag{94c}$$

This average (expectation value) of a product of n operators is called the n-point correlation function. For example, using Wick's theorem and expansions in powers of $V_{in}(Q_{in}(t),t)$, we can gain knowledge about processes that involves in total n excitations of the ground state (or vacuum) in both asymptotic states.

At this point one can be in two situations. The ground state can be either stable or unstable. Over time it could spontaneously go into a state of excitation. Mathematically speaking instability leads to :

$$\langle 0_{out}|0_{in}\rangle \neq 1. \tag{95}$$

Here of course already normalized states. This possibility for excitations stem from external sources and therefore external currents. A stable state would lead to:

$$\langle 0_{out}|\, S^{-1} = \langle 0_{in}| \tag{96}$$

An asymptotic state would go over time into a asymptotic state of same excitation.

6   Conclusion

In this report we have presented a set of fundamental concepts and methods. These are essential for starting to study Quantum field theory. We began with well-known pictures from quantum mechanics and introduced the In and Out picture as consequences of working with asymptotic states. They lead us to the important connecting element between the pictures, the scattering operator. The connections allowed us to derive the Gell-Mann Low formula, which plays a major part in the second quantisation process one would do in quantum field theories.

Appendices

A   Chronological time ordering

Let's begin with at a regular function in form of:

$$I(t) = \int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 V(t_1)V(t_2) \tag{97}$$

We see that although the the right hand side has functions of t_2 and t_1 the left hand side doesn't. The integrals allow us to perform a change of variable and rewrite I as $I(t) = \dfrac{1}{2}I(t) + \dfrac{1}{2}I(t)$:

$$I(t) = \frac{1}{2} \int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 V(t_1)V(t_2) + \frac{1}{2} \int_{-\infty}^{t} \mathrm{d}t_2 \int_{-\infty}^{t_2} \mathrm{d}t_1 V(t_2)V(t_1). \tag{98}$$

We define the **chronological time ordering**[3] at this point to equalize the boundaries.
Note $V(t_1) \rightarrow V_1$:

$$T(V_1, V_2) = V_1\ V_2\ \theta(t_1 - t_2)\ +\ V_2\ V_1\ \theta(t_2 - t_1). \tag{99}$$

Being based on the Heaviside step-function it allows us to write the Eq.(97) as:

$$I(t) = \frac{1}{2} \int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t} \mathrm{d}t_2 V_1\ V_2\ \theta(t_1 - t_2) + \frac{1}{2} \int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t} \mathrm{d}t_2\ V_2\ V_1\ \theta(t_2 - t_1) \tag{100}$$

$$I(t) = \frac{1}{2!} \int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t} \mathrm{d}t_2 T(V_1, V_2) \tag{101}$$

The Heaviside function allows us to set terms to zero for negative arguments of θ and 1 for positive. By subtracting t_1 and t_2 we will be able to switch between the later and earlier points in time.
for $t_1 > t_2 \rightarrow \theta(t_1 - t_2)$ and for $t_2 > t_1 \rightarrow \theta(t_2 - t_1)$
To expand our concept to cases of more then two t we start by advancing T. In general it consists of a summation of all permutations P of a given set multiplied by Heaviside functions. These Heaviside functions have arguments

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[3]earlier times to the right and later times to the left

with the negative summation of the t in the same permutation P.
Definition:

$$T(V_1, V_2, \ldots, V_n) = \sum_{j=1}^{n!} P_j [V_1, V_2, \ldots, V_n] \cdot \theta \left(P_j \left[t_j - \sum_{i \neq j}^{n} t_i \right] \right) . \quad (102)$$

But this summation over j would not add up to the same value for $I(t)$ since it started in the order of one element in the sum. In this case we need a normalization in addition to T. From statistics we know a set of n different elements can be linear arranged in $n!$ ways. Coming in as a factor of $\dfrac{1}{n!}$ in the expressions later.

For a test we use $n = 3$

$$T(V_1, V_2, V_3) = \sum_{j=1}^{3!} P_j [V_1, V_2, V_3] \cdot \theta \left(P_j \left[t_j - \sum_{i \neq j}^{3} t_i \right] \right) \quad (103a)$$

$$= V_1 V_2 V_3 \cdot \theta(t_1 - t_2 - t_3) \quad (103b)$$
$$+ V_1 V_3 V_2 \cdot \theta(t_1 - t_3 - t_2) \quad (103c)$$
$$+ V_2 V_1 V_3 \cdot \theta(t_2 - t_1 - t_3) \quad (103d)$$
$$+ V_2 V_3 V_1 \cdot \theta(t_2 - t_3 - t_1) \quad (103e)$$
$$+ V_3 V_1 V_2 \cdot \theta(t_3 - t_1 - t_2) \quad (103f)$$
$$+ V_3 V_2 V_1 \cdot \theta(t_3 - t_2 - t_1) \quad (103g)$$

Now we can apply it to new I and do a proof by Induction based on the number of V. The Induction start is the case of two V. In the Induction step we say in a mathematicians way that it works for at least one unspecified higher order. Let's call it k : (*Note: $t_0 = t$*)

$$I_k(t) = \prod_{a=1}^{k} \int_{-\infty}^{t_{a-1}} dt_a \ V_a = \frac{1}{k!} (\prod_{a=1}^{k} \int_{-\infty}^{t} dt_a) T(V_1, \ldots, V_k). \quad (104)$$

Moving one increment higher in our 'chain' $k + 1$,

$$I_{k+1}(t) = \prod_{a=1}^{k+1} \int_{-\infty}^{t_{a-1}} dt_a \ V_a = \prod_{a=1}^{k} \int_{-\infty}^{t_{a-1}} dt_a \ V_a \cdot \int_{-\infty}^{t_k} dt_{k+1} \ V_{k+1} \quad (105)$$

Using the Induction Step and general definition for T:

$$I_{k+1}(t) = \frac{1}{k!}(\prod_{a=1}^{k} \int_{-\infty}^{t} \mathrm{d}t_a\ )T(V_1, \ldots, V_k) \cdot \int_{-\infty}^{t_k} \mathrm{d}t_{k+1}\ V_{k+1} \tag{106}$$

$$I_{k+1}(t) = I_k(t) \cdot \int_{-\infty}^{t_k} \mathrm{d}t_{k+1}\ V_{k+1} \tag{107}$$

This shows that the incrementation of k reduces to a multiplication with one more different element for the set. This increases the possible permutations by a factor of $k+1$ resulting in $(k+1)!$ in total. Giving us:

$$I_{k+1}(t) = \frac{1}{(k+1)!}(\prod_{a=1}^{k+1} \int_{-\infty}^{t} \mathrm{d}t_a\ )T(V_1, \ldots, V_{k+1}) \tag{108}$$

Time ordering is a very grounded concept in field theory, it appears very naturally for expressing propagators in term of fields. The reason apart from pure mathematics is comprehensible. The different functions, functionals or fields are best organized for summarizing a scattering or interaction event if they are time-like sorted.
Furthermore it comes with a great advantage. All products in $T(\ldots)$ do commute. Proof for two elements:

$$T(V_1, V_2) = V_1\ V_2\ \theta(t_1 - t_2)\ +\ V_2\ V_1\ \theta(t_2 - t_1) \tag{109a}$$
$$T(V_2, V_1) = V_2\ V_1\ \theta(t_2 - t_1)\ +\ V_1\ V_2\ \theta(t_1 - t_2) \tag{109b}$$

,since terms in sums always commutes,

$$T(V_1, V_2) = T(V_2, V_1) \tag{110}$$

In other words, the commutation relations say whether the subtraction of permutations of elements is zero or not. But in time ordering all permutations appear, we can rearrange the terms so subtraction of equal permutations happens. Therefore commutation in T holds.

B   Anti-chronological time ordering

On a close inspection of the introduction of the Heaviside-function and its' insertion into the integral, we actually skipped a choice. If we just would

have wanted to the overall boundaries the order of t_1 and t_2 in relation to V_1 and V_2 the insertion of θ could have been switched. This secretly led us to the definition of time-ordering or chronological time ordering as it is called more precisely. The other path would have resulted in anti-chronological time ordering:[4]

$$\bar{T}(V_1, V_2) = V_1\, V_2\, \theta(t_2 - t_1)\ +\ V_2\, V_1\, \theta(t_1 - t_2) \tag{111}$$

This section is not just to satisfy the observed readers but to proof a connection between both involing hermitian conjugation of the integrals, which can easily appear while doing picture transitions.

We choose the same starting point as for T:

$$I(t) = \frac{1}{2}\int_{-\infty}^{t}\mathrm{d}t_1\int_{-\infty}^{t_1}\mathrm{d}t_2 V_1 V_2 + \frac{1}{2}\int_{-\infty}^{t}\mathrm{d}t_2\int_{-\infty}^{t_2}\mathrm{d}t_1 V_2 V_1$$

$$= \frac{1}{2!}\int_{-\infty}^{t}\mathrm{d}t_1\int_{-\infty}^{t}\mathrm{d}t_2 T(V_1, V_2)$$

$$I(t)^\dagger = \left(\frac{1}{2}\int_{-\infty}^{t}\mathrm{d}t_1\int_{-\infty}^{t_1}\mathrm{d}t_2 V_1 V_2 + \frac{1}{2}\int_{-\infty}^{t}\mathrm{d}t_2\int_{-\infty}^{t_2}\mathrm{d}t_1 V_2 V_1\right)^\dagger \tag{112a}$$

$$= \frac{1}{2}\int_{-\infty}^{t}\mathrm{d}t_1\int_{-\infty}^{t_1}\mathrm{d}t_2\, (V_1 V_2)^\dagger + \frac{1}{2}\int_{-\infty}^{t}\mathrm{d}t_2\int_{-\infty}^{t_2}\mathrm{d}t_1\, (V_2 V_1)^\dagger \tag{112b}$$

$$= \frac{1}{2}\int_{-\infty}^{t}\mathrm{d}t_1\int_{-\infty}^{t_1}\mathrm{d}t_2 V_2^\dagger V_1^\dagger + \frac{1}{2}\int_{-\infty}^{t}\mathrm{d}t_2\int_{-\infty}^{t_2}\mathrm{d}t_1 V_1^\dagger V_2^\dagger \tag{112c}$$

, we observe that the switch of positions due to hermitian conjugation allows to use Eq.(111)

$$I(t)^\dagger = \frac{1}{2}\int_{-\infty}^{t}\mathrm{d}t_1\int_{-\infty}^{t}\mathrm{d}t_2 V_2^\dagger V_1^\dagger \theta(t_1 - t_2) + \frac{1}{2}\int_{-\infty}^{t}\mathrm{d}t_1\int_{-\infty}^{t}\mathrm{d}t_2 V_1^\dagger V_2^\dagger \theta(t_2 - t_1)$$

$$\tag{112d}$$

$$\Rightarrow \frac{1}{2}\int_{-\infty}^{t}\mathrm{d}t_1\int_{-\infty}^{t}\mathrm{d}t_2 \bar{T}(V_1^\dagger, V_2^\dagger) \tag{112e}$$

Now we assume our total Hamiltonian $\hat{H} = \hat{H}_0 + \hat{V}$ is hermitian and $\hat{V}$ won't lead to non fixed ground state energy $\rightarrow$ vacuum instability

$$I(t)^\dagger = \frac{1}{2}\int_{-\infty}^{t}\mathrm{d}t_1\int_{-\infty}^{t}\mathrm{d}t_2 \bar{T}(V_1, V_2) \tag{112f}$$

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[4]earlier times to the left and later times to the right

Literatur

[1] S.Randjbar-Daemi. *Course on Quantum Electrodynamics: Introduction to Quantum Field Theory*. The Abdus Salam International Centre for Theoretical Physics , 2007-2008.

[2] M.Peskin; D.Schroeder. *Quantum field theory*. Perseus Books Publishing, 1995.

[3] W.Greiner;J.Reinhart. *Quantum electrodynamics*. Verlag Harri Deutsch Thun und Frankfurt am Main, 1984.

[4] W.Greiner;J.Reinhart. *Feldquantisierung*. Verlag Harri Deutsch Thun und Frankfurt am Main, 1993.