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Specialization report

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Chapter 1

Introduction

Quantum field theory is a formalism that allow us to describe the interaction between elementary particles. Of great interest are observable quantities like cross-sections or decay rates, which can be established theoretically and contrasted with experimental results. Both observables can be determined from the amplitude of the corresponding process, which are calculated by using pertubation theory. The main goal of this report is the derivation of the Gell-Mann Low formula that enables us to apply this technique.

In chapter 2 we discuss the more common Schrödinger, Heisenberg and Interaction picture. Followed up by *in* and *out* pictures to which Gell-Mann Low formula allows easy transition. These pictures are based on boundary conditions, which are motivated in section 2.3, imposing free motion on particles along them. We derive them from experimental and physical points of view. After deriving concrete solutions for the unitary operators associated with these new pictures, we define the scattering operator $\hat{S}$ in chapter 3 and establish the Gell-Mann Low formula.

Chapter 2

Pictures in Quantum Mechanics

2.1 Schrödinger picture and Heisenberg picture

The choice of a picture always requires to establish the states but also the corresponding operators. In the Schrödinger picture the operators are time-independent but the wavefunctions are time dependent. The time evolution of a state vector is controlled by the Schrödinger equation. Let $|\Psi(t)\rangle$ denote a state vector at time t. It satisfies

$$i\hbar\frac{\partial}{\partial t}|\Psi_S(t)\rangle = \hat{H}|\Psi_S(t)\rangle\,, \tag{2.1}$$

where $\hat{H}$ is the Hamiltonian of the system. When assuming it time independent, the solution of Eq. (2.1) can be formally written as

$$|\Psi_S(t)\rangle = \hat{U}(t-t_0)|\Psi_S(t_0)\rangle \tag{2.2}$$

with $\hat{U}(t-t_0) = e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}$ the time evolution operator, which satisfies the differential equation

$$i\hbar\partial_t\hat{U}(t-t_0) = \hat{H}\hat{U}(t-t_0). \tag{2.3}$$

Under the general assumption of the Hamiltonian being hermitian $\hat{U}(t-t_0)$ is also an unitary operator, meaning:

$$\begin{aligned}&\hat{U}(t-t_0)\times\hat{U}^{\dagger}(t-t_0) = e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}e^{\frac{i}{\hbar}\hat{H}(t-t_0)} = 1\\&=\hat{U}(t-t_0)\times\hat{U}^{-1}(t-t_0)\end{aligned} \tag{2.4}$$

Going back to Eq. (2.2) we see $|\Psi_S(t_0)\rangle$ is a ket of $t=t_0$. We shall generally take $t_0 = 0$ and write

$$|\Psi_S(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|\Psi_H\rangle\,. \tag{2.5}$$

The state on the right-hand side has no longer time dependence. This defines the state in the Heisenberg picture.

The above two pictures differ between each other in the way of storing the time dependence. In the Schrödinger picture only the states carry such a dependence, whereas in the Heisenberg picture only operators has this possibility. To verify this statement we study the matrix element of an operator in the Schrödinger picture

$$\langle\Psi'_S(t)|\,\hat{A}^S\,|\Psi_S(t)\rangle = \langle\Psi'_H|\,e^{\frac{i}{\hbar}t\hat{H}}\hat{A}^S e^{-\frac{i}{\hbar}t\hat{H}}\,|\Psi_H\rangle\,, \tag{2.6}$$

where Eq. (2.5) has been used. As a consequence,

$$\hat{A}^H(t) = e^{\frac{i}{\hbar}t\hat{H}}\hat{A}^S e^{-\frac{i}{\hbar}t\hat{H}} = \hat{U}(t)^{-1}\hat{A}^S\hat{U}(t). \tag{2.7}$$

This new operator $\hat{A}^H(t)$ in combination with the state $|\Psi_H\rangle$ defines the Heisenberg picture. Observe that the time evolution of $\hat{A}^H(t)$ is dictated by an equation that follows from differentiating the equation above with respect to t:

$$\frac{d}{dt}\hat{A}^H(t) = \frac{i}{\hbar}\hat{H}e^{\frac{i}{\hbar}t\hat{H}}\hat{A}^S e^{-\frac{i}{\hbar}t\hat{H}} + e^{\frac{i}{\hbar}t\hat{H}}\hat{A}^S\left(-\frac{i}{\hbar}\hat{H}\right)e^{-\frac{i}{\hbar}t\hat{H}}. \tag{2.8}$$

Here we have used the time evolution equation (2.3). Hence,

$$\begin{aligned}\frac{d}{dt}\hat{A}^H(t) &= \frac{i}{\hbar}\hat{U}(t)^{-1}\hat{H}\hat{A}^S\hat{U}(t) - \frac{i}{\hbar}\hat{U}(t)^{-1}\hat{A}^S\hat{H}\hat{U}(t)\\ &= \frac{i}{\hbar}\hat{U}(t)^{-1}\hat{H}\underbrace{\hat{U}(t)\hat{U}(t)^{-1}}_{=1}\hat{A}^S\hat{U}(t)\\ &\qquad -\frac{i}{\hbar}\hat{U}(t)^{-1}\hat{A}^S\underbrace{\hat{U}(t)\hat{U}(t)^{-1}}_{=1}\hat{H}\hat{U}(t).\end{aligned} \tag{2.9}$$

The inserted 1 allows us to express Eq. (2.9) in term of operators in the Heisenberg picture.

$$\frac{d}{dt}\hat{A}^H(t) = \frac{i}{\hbar}\hat{H}^H(t)\hat{A}^H(t) - \frac{i}{\hbar}\hat{A}^H(t)\hat{H}^H(t), \tag{2.10}$$

where $\hat{H}^H(t)$ is the respective Hamiltonian in the Heisenberg picture. Therefore:

$$i\frac{d}{dt}\hat{A}^H(t) = \frac{1}{\hbar}\left[\hat{A}^H(t),\hat{H}^H(t)\right]. \tag{2.11}$$

2.2 Interaction picture

A third picture can be introduced: the Interaction picture (sometimes called the Dirac picture). We will see very shortly that, in the Interacting picture both the states and the respective operators are time dependent. Let us suppose that the Hamiltonian in the Schrödinger picture can be splitted as follows $\hat{H} = \hat{H}_0 + \hat{V}$. Normally $\hat{H}_0$ describe the free motion of a system, whereas $\hat{V}$ represents its interaction, which could be with an external source. Although it often used in a perturbative approach, the Interaction picture does not require $\hat{V}$ to be small as compared with $\hat{H}_0$. Inserting this decomposition of $\hat{H}$ in the unitary operator introduced below Eq. (2.2):

$$\hat{U}(t) = e^{-\frac{i}{\hbar}t\hat{H}} = e^{-\frac{i}{\hbar}t\left(\hat{H}_0+V\right)} = e^{-\frac{i}{\hbar}t\hat{H}_0}\hat{\Omega}_I(t) \tag{2.12}$$

This expression helps us to establish a formula from which operators and states in the interaction picture can be defined[1]. For this, consider a matrix element $\langle\Psi'_S(t)|\,\hat{A}^S\,|\Psi_S(t)\rangle$. Taking into account Eq. (2.5) and (2.12) we find

$$\langle\Psi'_S(t)|\,\hat{A}^S\,|\Psi_S(t)\rangle = \langle\Psi'_H|\,(e^{-\frac{i}{\hbar}t\hat{H}_0}\Omega_I(t))^\dagger \hat{A}^S e^{-\frac{i}{\hbar}t\hat{H}_0}\Omega_I(t)\,|\Psi_H\rangle \tag{2.13a}$$
$$= \langle\Psi'_H|\,\Omega_I(t)^{-1}\hat{A}^I(t)\Omega_I(t)\,|\Psi_H\rangle\,. \tag{2.13b}$$

Here the operator in the interaction picture reads

$$\hat{A}^I(t) = e^{+\frac{i}{\hbar}t\hat{H}_0}\hat{A}^S e^{-\frac{i}{\hbar}t\hat{H}_0}, \tag{2.14}$$

whereas a corresponding state in this picture is

$$|\Psi_I(t)\rangle = \Omega_I(t)\,|\Psi_H\rangle\,. \tag{2.15}$$

At the level of operators, the connection between the Interaction and the Heisenberg picture is established by inverting Eq. (2.7) and inserting the resulting $\hat{A}^S$ into Eq. (2.14). This leads to

$$\hat{A}^I(t) = e^{+\frac{i}{\hbar}t\hat{H}_0}\hat{U}(t)\hat{A}^H(t)\hat{U}(t)^{-1}e^{-\frac{i}{\hbar}t\hat{H}_0}. \tag{2.16a}$$
$$= e^{+\frac{i}{\hbar}t\hat{H}_0}e^{-\frac{i}{\hbar}t\hat{H}}\hat{A}^H(t)e^{\frac{i}{\hbar}t\hat{H}}e^{-\frac{i}{\hbar}t\hat{H}_0}, \tag{2.16b}$$

ending with

$$\hat{A}^I(t) = \hat{\Omega}_I(t)\hat{A}^H(t)\hat{\Omega}_I(t)^{-1}. \tag{2.17}$$

The time evolution equation for $\hat{A}^I(t)$ can be found as done for $\hat{A}^H(t)$ [see below Eq. (2.7)]:

$$i\hbar\frac{\partial}{\partial t}\hat{A}^I = \left[\hat{A}^I, \hat{H}_0\right]. \tag{2.18}$$

Furthermore, an equation for $\hat{\Omega}_I(t)$ can be determined. To this end we invert Eq. (2.12) and express $\hat{\Omega}_I(t) = e^{\frac{i}{\hbar}t\hat{H}_0}\hat{U}(t)$. Afterwards we differentiate with respect to times:

$$i\hbar\partial_t\hat{\Omega}_I(t) = e^{\frac{i}{\hbar}t\hat{H}_0}\left(i\hbar\partial_t\hat{U}(t)\right) - \hat{H}_0 e^{\frac{i}{\hbar}t\hat{H}_0}\hat{U}(t) \tag{2.19a}$$
$$= \hat{H}e^{\frac{i}{\hbar}t\hat{H}_0}\hat{U}(t) - \hat{H}_0 e^{\frac{i}{\hbar}t\hat{H}_0}\hat{U}(t), \tag{2.19b}$$

where Eq. (2.3) has been used. Using the definition of $\hat{\Omega}_I(t)$ we end up with

$$i\hbar\frac{\partial}{\partial t}\hat{\Omega}_I(t) = \hat{V}_I(t)\hat{\Omega}_I(t). \tag{2.20}$$

To find a well defined solution, an initial condition is needed. In Eq. (2.12) we see that at $t = 0$, the time evolution operator reduces to $\hat{U}(0) = 1$, and from this the following condition $\hat{\Omega}_I(0) = 1$ arises. We remark that $\hat{V}_I(t)$ in the Interaction picture as introduced above does not require $\hat{V}$ to be of any specific form, but can still be applied in presence of external sources.

[1]By substituting Eq. (2.12) in Eq. (2.4) one can see directly that $\Omega_I(t)$ is also unitary

2.3 The *in* and *out* picture: External currents

Consider the set-up of most experiments in elementary particle and nuclear physics. Several particles approach each other from a macroscopic scale and interact in a microscopic section comparable to the Compton wavelength of the incoming particles. On this scale vacuum fluctuations are no longer negligible for the dynamic of the involved particles and make them impossible to distinguish between each other. As a result, the products of the interaction spread up to a macroscopic distances and the distinguishability between outgoing particles is admitted. Therefore, at such asymptotically distances, the description of the incoming and outgoing multi-particle states can be approached by direct products of single-particle effectively non-interacting states.

To bring this concept into our formulation let's consider the action of a scalar field Φ with mass $m = m_0 c/\hbar$ coupled to an external source $j(\underline{x}, t)$[2]:

$$I = \int d^4x \, \mathcal{L}(\Phi, \dot{\Phi}, j) = \int d^4x \left(\frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} m^2 \Phi^2 - \Phi j \right). \tag{2.21}$$

Taking the functional derivative with respect to Φ and setting it to zero, we obtain the equation of motion

$$\left(\partial^2 + m^2 \right) \Phi = j. \tag{2.22}$$

To proceed, we quantize our field in a box of volume V and length L. The classical field and its canonical momentum $\Pi = \partial \mathcal{L} / \partial \dot{\Phi}(\underline{x}, t) = \dot{\Phi}(\underline{x}, t)$ are then promoted to operators $\hat{\Phi}(\underline{x}, t)$ and $\hat{\Pi}(\underline{x}, t)$ in the Heisenberg picture. Satisfying the equal-time commutator:

$$\left[\hat{\Phi}^H(\underline{x}, t), \hat{\Pi}^H(\underline{x}', t) \right] = i\delta^3(\underline{x} - \underline{x}'). \tag{2.23}$$

We then expand the field operator as follows:

$$\hat{\Phi}^H(\underline{x}, t) = \sum_{\underline{k}} \hat{q}^H_{\underline{k}}(t) u_{\underline{k}}(\underline{x}). \tag{2.24}$$

The 3 dim. wave vector $\underline{k}$ for the modes is represented by $\underline{k} = \frac{2\pi}{L}(n_x, n_y, n_z)$ with $n_i \in \mathbb{Z}$. In this separated time and space dependency, we choose the Fourier basis for $u_{\underline{k}}(\underline{x})$

$$u_{\underline{k}}(\underline{x}) = \frac{1}{L^{3/2}} e^{i\underline{k} \cdot \underline{x}}, \tag{2.25}$$

where the volume L^3 provides the required normalization. We remark that $u_{\underline{k}}(\underline{x})$ constitutes an orthonormalized basis in the Hilbert space

$$\int d^3x \, u^*_{\underline{k}'}(\underline{x}) u_{\underline{k}}(\underline{x}) = \delta_{\underline{k}, \underline{k}'} \tag{2.26}$$

[2]From now on we will work in natural units and set $c = \hbar = 1$

$$\sum_{\underline{k}} u^*_{\underline{k}}(\underline{x}) u_{\underline{k}}(\underline{x}') = \delta^3\left(\underline{x} - \underline{x}'\right). \tag{2.27}$$

We now substitute Eq. (2.25) into the equation of motion (2.22) . As a consequence

$$\sum_{\underline{k}} \left[\ddot{\hat{q}}^H_{\underline{k}}(t) u_{\underline{k}}(\underline{x}) + \underline{k}^2 \hat{q}^H_{\underline{k}}(t) u_{\underline{k}}(\underline{x}) + m^2 \hat{q}^H_{\underline{k}}(t) u_{\underline{k}}(\underline{x})\right] = j(\underline{x}, t). \tag{2.28}$$

To get an equation for $\hat{q}^H_{\underline{k}}(t)$ alone we need to get rid of $u_{\underline{k}}(\underline{x})$ and remove the space dependence in the current. Multiplying with $u^*_{\underline{k}'}(\underline{x})$ and integrating over the whole space we find,

$$\begin{aligned} &\sum_{\underline{k}} \left[\int d^3x \, u^*_{\underline{k}'} u_{\underline{k}} \left(\ddot{\hat{q}}^H_{\underline{k}}(t) + \left(\underline{k}^2 + m^2\right) \hat{q}^H_{\underline{k}}(t) \right) \right] \\ &\qquad = \underbrace{\int d^3x \, j(\underline{x}, t) \frac{1}{L^{3/2}} e^{i\underline{k}\cdot\underline{x}}}_{=\tilde{j}(\underline{k},t)}. \end{aligned} \tag{2.29}$$

After using the orthonormality relation (2.26) this expression reduces to

$$\ddot{\hat{q}}^H_{\underline{k}}(t) + \omega^2_{\underline{k}} \hat{q}^H_{\underline{k}}(t) = \tilde{j}(\underline{k}, t), \tag{2.30}$$

where $\omega^2_{\underline{k}} = (\underline{k}^2 + m^2)$ is the energy of the particle in mode $\underline{k}$.

We now make the assumption that the current vanishes outside a finite time interval,

$$j(\underline{k}, t) \to 0 \text{ for } t \to \pm\infty. \tag{2.31}$$

As a consequence one can distinguish between early and late times. For early time Eq. (2.30) approaches the homogeneous differential equation. We will call its asymptotic solution by $\hat{q}^H_{\underline{k}}(t) \to \hat{q}^{in}_k(t)$. Explicitly,

$$\hat{q}^{in}_{\underline{k}}(t) \approx \frac{1}{2\omega_{\underline{k}}} \left(\hat{a}^{in}_{\underline{k}} e^{-i\omega_{\underline{k}} t} + \hat{a}^{in\,\dagger}_{-\underline{k}} e^{i\omega_{\underline{k}} t} \right), \quad t \to -\infty, \tag{2.32}$$

where $\hat{a}_{\underline{k},in}$ denotes the annihilation operator, whereas $\hat{a}^\dagger_{\underline{k},in}$ is the corresponding creation operator. Their commutator is

$$\left[\hat{a}^{in}_{\underline{k}}, \hat{a}^{in\,\dagger}_{\underline{k}'}\right] = 2\omega_{\underline{k}} \delta_{\underline{k},\underline{k}'}. \tag{2.33}$$

At late times Eq. (2.30) also reduces to a homogeneous type. In this case the asymptotic solution $\hat{q}^H_{\underline{k}}(t) \to \hat{q}^{out}_k(t)$ reads

$$\hat{q}_{\underline{k},out}(t) \approx \frac{1}{2\omega_{\underline{k}}} \left(\hat{a}^{out}_{\underline{k}} e^{-i\omega_{\underline{k}} t} + \hat{a}^{out\,\dagger}_{-\underline{k}} e^{i\omega_{\underline{k}} t} \right), \quad t \to +\infty, \tag{2.34}$$

where the new operators fulfil a commutation relation similar to Eq. (2.33). The solution for $\hat{q}_{\underline{k}}(t)$, at times for which $j(\underline{x}, t)$ is active, would then consist of the homogeneous solution plus a term containing the current:

$$\hat{q}^{H}_{\underline{k}}(t) = \hat{q}^{in}_{\underline{k}}(t) + \frac{1}{\omega_{\underline{k}}} \int_{-\infty}^{t} dt' \sin\left[\omega_{\underline{k}}(t-t')\right] \tilde{j}(\underline{k}, t'), \tag{2.35}$$

where $\bar{j}_{\underline{k}}(\omega_{\underline{k}}) = \int_{-\infty}^{\infty} dt \tilde{j}(\underline{k}, t) e^{i\omega_{\underline{k}} t}$ is the temporal Fourier transform of the current. For late times $t \to +\infty$ the expression above approaches to

$$\hat{q}^{out}_{\underline{k}}(t) \approx \hat{q}^{in}_{\underline{k}}(t) + \frac{1}{\omega_{\underline{k}}} \int_{-\infty}^{\infty} dt' \sin\left[\omega_{\underline{k}}(t-t')\right] \tilde{j}(\underline{k}, t'). \tag{2.36}$$

After splitting the sinus function, we find

$$\hat{q}^{out}_{\underline{k}}(t) = \hat{q}^{in}_{\underline{k}}(t) - \frac{i}{2\omega_{\underline{k}}} e^{i\omega_{\underline{k}} t} \bar{j}_{\underline{k}}(-\omega_{\underline{k}}) + \frac{i}{2\omega_{\underline{k}}} e^{-i\omega_{\underline{k}} t} \bar{j}_{\underline{k}}(\omega_{\underline{k}}), \tag{2.37}$$

From this equation we can obtain the connection between creation and annihilation operators associated with the asymptotically far fields $t \to \pm\infty$. In compact notation

$$\hat{a}^{out}_{\underline{k}} = \hat{a}^{in}_{\underline{k}} + i\bar{j}_{\underline{k}}(\omega_{\underline{k}}), \tag{2.38a}$$

$$\hat{a}^{out\,\dagger}_{\underline{k}} = \hat{a}^{in\,\dagger}_{\underline{k}} - i\bar{j}_{\underline{k}}(-\omega_{\underline{k}}). \tag{2.38b}$$

This shows that, in the presence of an external current, the two sets of second quantization operators are not the same. Therefore we need to differ between the corresponding *in* and *out* eigenstates. Particularly, it has to be stated that the vacua also differ in this scenario.

It is important to stress, that the full solution $\hat{q}^{H}_{\underline{k}}(t)$ found in Eq. (2.35) has to be understood in the Heisenberg picture. From this we can proceed as shown in section **2.2**. We split the Hamiltonian as done there: $\hat{H} = \hat{H}_0 + \hat{V}_H$.

$$\hat{H}_0(\Phi, \Pi) = \int d^3x \left[\frac{1}{2}(\hat{\Pi}^H)^2 + \frac{1}{2}(\nabla\hat{\Phi}^H)^2 + \frac{1}{2}m^2(\hat{\Phi}^H)^2\right], \tag{2.39}$$

$$\hat{V}_H(\Phi) = \int d^3x\, j\hat{\Phi}^H. \tag{2.40}$$

Expressing both field operators in terms of the Fourier basis given in (2.24), and using the orthonormality relation Eq. (2.26), as well as the reality condition of the field for $\hat{q}_{-\underline{k}}(t) = \hat{q}^{\star}_{\underline{k}}(t)$ we can express the Hamiltonian as follows:

$$\hat{H}_0(q, \dot{q}) = \sum_{\underline{k}} \left\{\frac{1}{2}(\dot{\hat{q}}^{H}_{\underline{k}}(t))^2 + \frac{1}{2}\omega^2_{\underline{k}}(\hat{q}^{H}_{\underline{k}}(t))^2\right\}, \tag{2.41}$$

$$\hat{V}_H(q) = \sum_{\underline{k}} \tilde{j}(\underline{k}, t)\hat{q}^{H}_{\underline{k}}(t). \tag{2.42}$$

From this form we go to the Interaction picture. In the present context, the potential V_I appearing in Eq. (2.20) reads:

$$\hat{V}_I(q_I) = \sum_{\underline{k}} \tilde{j}(\underline{k}, t)\hat{q}^I_{\underline{k}}(t), \tag{2.43}$$

where we used Eq. (2.17) to transform $\hat{q}_{\underline{k}}(t)$ into the Interaction picture

$$\hat{q}^I_{\underline{k}}(t) = \hat{\Omega}_I(t)\hat{q}^H_{\underline{k}}(t)\hat{\Omega}_I^{-1}(t). \tag{2.44}$$

To have a well defined operator $\hat{\Omega}_I(t)$ we need conditions for any $\hat{\Omega}$ so that $\hat{\Omega} \to 1$ as stated for the Interaction picture in Eq. (2.20) which is at the moment mostly depended on the current j. The early time condition at $t = -\infty$ defines the *in* picture in reminiscence to the first asymptotic solution given in Eq (2.32) and it writes:

$$i\frac{\partial}{\partial t}\hat{\Omega}_{in}(t) = \hat{V}_{in}(t)\hat{\Omega}_{in}(t), \tag{2.45}$$

where the initial condition $\hat{\Omega}_{in}(-\infty) = 1$ has to be fulfilled. Contrary to the previous case the operator of the *out* picture will satisfy the differential equation:

$$i\frac{\partial}{\partial t}\hat{\Omega}_{out}(t) = \hat{V}_{out}(t)\hat{\Omega}_{out}(t), \tag{2.46}$$

with $\hat{\Omega}_{out}(+\infty) = 1$.

Chapter 3

Scattering operator

3.1 Solutions for the Interaction, *in* and *out* picture

In this section we solve the differential equations for the various pictures established in section 2.2 and 2.3. We start with the Interaction picture depended on t' and integrate both sides of Eq. (2.20). For $t > 0$ its left-hand side gives:

$$\int_0^t \mathrm{d}t' i \frac{\partial}{\partial_{t'}} \hat{\Omega}_I(t') = i\left[\hat{\Omega}_I(t) - 1\right], \tag{3.1}$$

where the initial condition $\hat{\Omega}_I(0) = 1$ has been used. With this formula and the integral over the right-hand side of (2.20), we find an expression for $\hat{\Omega}_I(t)$.

$$\hat{\Omega}_I(t) = 1 - i\int_0^t \mathrm{d}t' \hat{V}_I(t')\hat{\Omega}_I(t'), \tag{3.2}$$

since the expression has an $\hat{\Omega}_I(t)$ on the other side we will go on by an iterative approach.

$$\begin{aligned} \hat{\Omega}_I(t) &= 1 - i\int_0^t \mathrm{d}t' \hat{V}_I(t') \cdot \left(1 - i\int_0^{t'} \mathrm{d}t'' \hat{V}_I(t'')\hat{\Omega}_I(t'')\right) \\ &= 1 - i\int_0^t \mathrm{d}t' \hat{V}_I(t') + i^2 \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \hat{V}_I(t'')\hat{\Omega}_I(t''). \end{aligned} \tag{3.3}$$

The iteration increments the power of i and the number of integrals. By repeating the operation described above we can write

$$\hat{\Omega}_I(t) = \sum_{n=0}^{\infty} (-i)^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \ldots \int_0^{t_{n-1}} \mathrm{d}t_n \hat{V}_I(t_1) \cdot \ldots \cdot \hat{V}_I(t_n). \tag{3.4}$$

A problematic aspect of this series are the different integral limits. Each term introduces a new t_i and keeps the previous t_{i-1} as an integral variable which forces us to solve them in a strict order. To circumvent this formal aspect we will perform some additional operations. Let us consider the term from Eq. (3.4) containing the product of two interactions :

$$I(t) = \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \hat{V}_I(t_1)\hat{V}(t_2). \tag{3.5}$$

By developing the change of variable $t_2 \longleftrightarrow t_1$,[1] this integral can be written as

$$I(t) = \int_0^t \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_1 \hat{V}_I(t_2)\hat{V}_I(t_1). \tag{3.6}$$

We find an alternative representation of $I(t)$ by adding (3.5) and (3.6):

$$I(t) = \frac{1}{2}\int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \hat{V}_I(t_1)\hat{V}_I(t_2) + \frac{1}{2}\int_0^t \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_1 \hat{V}_I(t_2)\hat{V}_I(t_1). \tag{3.7}$$

In order to have a common integration limit t, we introduce the chronological time ordering.

$$T(\hat{V}_I(t_1)\ \hat{V}_I(t_2)) = \hat{V}_I(t_1)\ \hat{V}_I(t_2)\ \theta(t_1 - t_2)\ +\ \hat{V}_I(t_2)\ \hat{V}_I(t_1)\ \theta(t_2 - t_1). \tag{3.8}$$

The chronological time ordering sets operators depending of earlier times to the right and later to the left. The the Heaviside-Step-function is 0 for negative values of its argument and 1 when it becomes positive. By subtracting t_1 and t_2 in the argument of the step functions we are able to switch between the two terms in Eq. (3.7) and extending the integral limits to t, since it sets terms to zero for negative arguments. Therefore no change appears in the result of the integral by extending the limit. We used for $t_1 > t_2 \to \theta(t_1 - t_2)$ and for $t_2 > t_1 \to \theta(t_2 - t_1)$. By applying Eq. (3.8) at Eq. (3.7), we find the desired notation:

$$\begin{aligned} I(t) = \frac{1}{2}\int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_2 \hat{V}_I(t_1)\ \hat{V}_I(t_2)\ \theta(t_1 - t_2) \\ + \frac{1}{2}\int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_2\ \hat{V}_I(t_2)\ \hat{V}_I(t_1)\ \theta(t_2 - t_1), \end{aligned}$$

which can be compactly written in the following form.

$$I(t) = \frac{1}{2!}\int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_2 T(\hat{V}_I(t_1)\ \hat{V}_I(t_2)). \tag{3.9}$$

This case of two interactions is generalized to terms involving $\hat{V}(t)$ n-times in the appendix 4.A. Applying the T operator allows us to write the solution to $\hat{\Omega}_I(t)$ given in Eq. (3.4) in the time-ordered form:

$$\hat{\Omega}_I(t) = \frac{(-i)^n}{n!}\int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_2 \ldots \int_0^t \mathrm{d}t_n T\left\{\hat{V}_I(t_1)\ \ldots\ \hat{V}_I(t_n)\right\}. \tag{3.10}$$

Observe that this expression is a non-pertubative result, which can be written in a symbolically notation

$$\hat{\Omega}_I(t) = T\left(e^{-i\int_0^t \mathrm{d}t' \hat{V}_I(t')}\right), \text{ for } t \geq 0. \tag{3.11}$$

To not limit the Interaction picture only to positive t values, we need a complementary expression for negative t's. Assuming $t < 0$, the integral in Eq.

[1] The Jacobian of this change of variable is the unity

(3.1) is changed to:

$$\int_t^0 \mathrm{d}t' i \frac{\partial}{\partial_{t'}} \hat{\Omega}_I(t') = i \left[1 - \hat{\Omega}_I(t)\right]. \tag{3.12}$$

Alongside performing in the integral of the right-hand side of Eq. (2.20) in the new limits, we find:

$$\hat{\Omega}_I(t) = 1 + i \int_t^0 \mathrm{d}t' \hat{V}_I(t') \hat{\Omega}_I(t'). \tag{3.13}$$

From this, our infinite sum expression still holds up to a different sign:

$$\hat{\Omega}_I(t) = \sum_{n=0}^{\infty} i^n \int_t^0 \mathrm{d}t_1 \int_{t_1}^0 \mathrm{d}t_2 \ldots \int_{t_{n-1}}^0 \mathrm{d}t_n \hat{V}_I(t_1) \ \ldots \ \hat{V}_I(t_n). \tag{3.14}$$

The key difference now stands in the negativity of all t and a logical order for them would prefer later times to the right, coming closer to 0. This requires the anti-chronological time ordering:

$$\bar{T}(\hat{V}(t_1)\ \hat{V}(t_2)) = \hat{V}(t_2)\ \hat{V}(t_1)\ \theta(t_1 - t_2)\ +\ \hat{V}(t_1)\ \hat{V}(t_2)\ \theta(t_2 - t_1). \tag{3.15}$$

A generalized expression containing the product of several $\hat{V}(t)$'s is given in the appendix 4.A. Using it similar as before:

$$\hat{\Omega}_I(t) = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int_t^0 \mathrm{d}t_1 \int_t^0 \mathrm{d}t_2 \ldots \int_t^0 \mathrm{d}t_n \bar{T} \left\{ \hat{V}_I(t_1) \ \ldots \ \hat{V}_I(t_n) \right\}. \tag{3.16}$$

As a consequence, we obtain a second expression:

$$\hat{\Omega}_I(t) = \bar{T} \left(e^{i \int_t^0 \mathrm{d}t' \hat{V}_I(t')} \right), \ \text{ for } t < 0. \tag{3.17}$$

A notation for $\hat{\Omega}_I(t)$ without specifying the values of t can be derived by using Heaviside-Step-functions:

$$\hat{\Omega}_I(t) = T \left(e^{-i \int_0^t \mathrm{d}t' \hat{V}_I(t')} \right) \theta(t) + \bar{T} \left(e^{+i \int_t^0 \mathrm{d}t' \hat{V}_I(t')} \right) \theta(-t). \tag{3.18}$$

For the *in* picture we proceed in an almost identical fashion to the Interaction picture for $t > 0$. Only the lower boundary in the integral is changed to $-\infty$ as it is the asymptotic condition of this picture. This resolves the need for a two term solution. After resumation, we obtain:

$$\hat{\Omega}_{in}(t) = T \left(e^{-i \int_{-\infty}^t \mathrm{d}t' \hat{V}_{in}(t')} \right). \tag{3.19}$$

The *out* picture on the other hand follows the derivation of the expression for $t < 0$. We start at Eq. (3.12) with ∞ instate of 0. Here we argue t being smaller then ∞ needs one change of sign like before and anti-chronological ordering $\bar{T}$ introduced in Eq. (3.15), since t only coming closer to the limit as it runs.

$$\hat{\Omega}_{out}(t) = \sum_{n=0}^{\infty} \frac{(i)^n}{n!} \int_t^{\infty} \mathrm{d}t_1 \int_t^{\infty} \mathrm{d}t_2 \ldots \int_t^{\infty} \mathrm{d}t_n \bar{T} \left\{ \hat{V}_{out}(t_1) \; \ldots \; \hat{V}_{out}(t_n) \right\}. \quad (3.20)$$

Performing a flip in the integral limits we conclude:

$$\hat{\Omega}_{out}(t) = \bar{T} \left(e^{-i \int_{\infty}^{t} \mathrm{d}t' \hat{V}_{in}(t')} \right). \quad (3.21)$$

3.2 Representation of the Scattering operator

In section 2.3 we saw that the creation and annihilation operators associated with the *in* and *out* pictures are related between each other in Eq. (2.38) via the external current. We wish to establish how this connection manifests at the level of the corresponding scattering states. To this end, we particularize Eq. (2.15) to the case in which the initial condition is taken at $t \to \pm\infty$. If the initial condition is $t \to -\infty$, we have

$$|\Psi_{in}(t)\rangle = \hat{\Omega}_{in}(t) \, |\Psi_H\rangle \,. \quad (3.22)$$

Likewise for the *out* picture we write:

$$|\Psi_{out}(t)\rangle = \hat{\Omega}_{out}(t) \, |\Psi_H\rangle \,. \quad (3.23)$$

Combining both Eq. (3.22) and Eq. (3.23) we find

$$|\Psi_{in}(t)\rangle = \hat{\Omega}_{in}(t) \hat{\Omega}_{out}^{-1}(t) \, |\Psi_{out}(t)\rangle \,. \quad (3.24)$$

The product of $\hat{\Omega}$'s defines the scattering operator $\hat{S}$:

$$\hat{S} = \hat{\Omega}_{in}(t) \hat{\Omega}_{out}^{-1}(t). \quad (3.25)$$

We will verify explicitly that Eq. (3.25) is time independent. For this we take the partial derivative of $\hat{S}$ with respect to t. This operation leads to

$$\begin{aligned} i\partial_t \left(\hat{\Omega}_{in}(t) \hat{\Omega}_{out}^{-1}(t) \right) &= i\dot{\hat{\Omega}}_{in}(t) \hat{\Omega}_{out}^{-1}(t) + i\hat{\Omega}_{in}(t) \dot{\hat{\Omega}}_{out}^{-1}(t) \\ &= \hat{V}_{in}(t) \hat{\Omega}_{in}(t) \hat{\Omega}_{out}^{-1}(t) + i\hat{\Omega}_{in}(t) \dot{\hat{\Omega}}_{out}^{-1}(t). \end{aligned} \quad (3.26)$$

In the first term of the second line we have used Eq. (2.45). The evaluation of $\dot{\hat{\Omega}}_{out}^{-1}(t)$ requires some additional steps. We start by differentiating the identity $\hat{\Omega}_{out}(t) \hat{\Omega}_{out}^{-1}(t) = 1$. We then find the relation

$$i\hat{\Omega}_{out}(t) \dot{\hat{\Omega}}_{out}^{-1}(t) = -i\dot{\hat{\Omega}}_{out}(t) \hat{\Omega}_{out}^{-1}(t) \quad (3.27)$$

Afterwards, we multiply the left by $\hat{\Omega}_{out}(t)$ and use Eq. (2.46) in the right-hand side. As a consequence,

$$i\dot{\hat{\Omega}}_{out}^{-1}(t) = -\hat{\Omega}_{out}^{-1}(t)\hat{V}_{out}(t). \tag{3.28}$$

This results is inserted in Eq. (3.26):

$$i\partial_t \left(\hat{\Omega}_{in}(t)\hat{\Omega}_{out}^{-1}(t)\right) = \hat{V}_{in}(t)\hat{\Omega}_{in}(t)\hat{\Omega}_{out}^{-1}(t) - \hat{\Omega}_{in}(t)\hat{\Omega}_{out}^{-1}(t)\hat{V}_{out}(t) \tag{3.29}$$

A picture transformation like stated for the Interaction picture in Eq. (2.17) allows us to write both potentials in the Heisenberg representation.

$$i\partial_t \hat{S} = \hat{\Omega}_{in}(t)\hat{V}_H\hat{\Omega}_{out}^{-1}(t) + \hat{\Omega}_{in}(t)(-\hat{V}_H)\hat{\Omega}_{out}^{-1}(t) = 0. \tag{3.30}$$

This time independence of the product gives us the choice to set t to any value. In particular by setting it to $t = \infty$ we find that Eq. (3.25) reduces to

$$\hat{S} = \hat{\Omega}_{in}(\infty), \tag{3.31}$$

where the initial conditions, $\hat{\Omega}_{out}(\infty) = 1$ has been used. Similarly, if $t = \infty$ we use $\hat{\Omega}_{in}(-\infty) = 1$ and

$$\hat{S} = \hat{\Omega}_{out}^{-1}(-\infty), \tag{3.32}$$

Taking into account, that a lot of literature around quantum field theory relates the $\hat{S}$ operator in term of the Interaction picture, we shall verify a secondary set of relations. Starting in a state in the *in* picture, we go to the Heisenberg and then to the Interaction picture similar to what we have done in Eq. (3.25).

$$\begin{aligned} |\Psi_{in}(t)\rangle &= \hat{\Omega}_{in}(t)\,|\Psi_H\rangle \\ &= \hat{\Omega}_{in}(t)\hat{\Omega}_I^{-1}(t)\,|\Psi_I(t)\rangle\,. \end{aligned} \tag{3.33}$$

This new product of unitary operators is also time independent. This can be shown by taking the partial derivative as seen in Eq. (3.26). The derivative of the inverse operator $\left[\dot{\hat{\Omega}}_I^{-1}(t)\right]$ will also result in a $-\hat{V}_H\hat{\Omega}_I^{-1}(t)$. Thus,

$$i\partial_t\hat{\Omega}_{in}(t)\hat{\Omega}_I^{-1}(t) = \hat{\Omega}_{in}(t)\hat{V}_H\hat{\Omega}_I^{-1}(t) + \hat{\Omega}_{in}(t)(-\hat{V}_H)\hat{\Omega}_I^{-1}(t) = 0. \tag{3.34}$$

To bring the $\hat{S}$ operator into our equation, we substitute $\hat{\Omega}_{in}(t)$ using the definition in Eq. (3.25):

$$\hat{\Omega}_{in}(t)\hat{\Omega}_I^{-1}(t) = \hat{S}\hat{\Omega}_{out}(t)\hat{\Omega}_I^{-1}(t). \tag{3.35}$$

We can set t to the initial condition of a picture now including $\hat{\Omega}_I(0) = 1$, since we have verified time independence.

$$\begin{aligned}\hat{\Omega}_{in}(t)\hat{\Omega}_I^{-1}(t) &= \hat{S}\,\hat{\Omega}_I^{-1}(\infty) \\ &= \hat{\Omega}_{in}(0).\end{aligned} \tag{3.36}$$

Finally, we want to derive the following expression:

$$\hat{\Omega}_{in}(t) = \bar{T}\left(e^{i\int_t^\infty \mathrm{d}t'\hat{V}_{in}(t')}\right)\hat{S}. \tag{3.37}$$

This formula is essential in our way of establishing the Gell-Mann Low formula, which is carried out in the next section. First it must satisfy the differential equation for $\hat{\Omega}_{in}(t)$ [see Eq. (2.45)]. To reduce the number of integrals directly depending on t to one, we use the representation for the exponential function without anti-chronological time ordering similar to Eq. (3.14):

$$i\partial_t\hat{\Omega}_{in}(t) = i\partial_t\left(\sum_n i^n \int_t^\infty \mathrm{d}t_1\hat{V}_{in}(t_1)\ldots\int_{t_{n-1}}^\infty \mathrm{d}t_n\hat{V}_{in}(t_n)\right)\hat{S}. \tag{3.38a}$$

The Leibniz integral rule can be applied to obtain,

$$i\partial_t\hat{\Omega}_{in}(t) = i\left(\sum_n i^n\left(-\hat{V}_{in}(t)\right)\int_{t_1}^\infty \mathrm{d}t_2\hat{V}_{in}(t_2)\ldots\int_{t_{n-1}}^\infty \mathrm{d}t_n\hat{V}_{in}(t_n)\right)\hat{S}. \tag{3.38b}$$

An i taken out of the sum allows us to remove the negative sign and restores the correct power $n-1$. By using the definition of anti-chronological time ordering (see appendix 4.A) and changing the summation index $(n-1=\tilde{n})$,

$$i\partial_t\hat{\Omega}_{in}(t) = \hat{V}_{in}(t)\sum_{\tilde{n}=1}^{\infty}\frac{i^{\tilde{n}}}{\tilde{n}!}\int_{t_0'}^\infty \mathrm{d}t_1'\ldots\int_{t_0'}^\infty \mathrm{d}t_{\tilde{n}}'\bar{T}\left(\hat{V}_{in}(t_1')\ldots\hat{V}_{in}(t_{\tilde{n}}')\right)\hat{S}. \tag{3.39}$$

$$i\partial_t\hat{\Omega}_{in}(t) = \hat{V}_{in}(t)\underbrace{\bar{T}\left(e^{i\int_t^\infty \mathrm{d}t'\hat{V}_{in}(t')}\right)\hat{S}}_{=\hat{\Omega}_{in}(t)}. \tag{3.40}$$

Now we verify that the initial condition $\left[\hat{\Omega}_{in}(-\infty) = 1\right]$ still holds for the representation given in Eq. (3.37). For this we evaluate it as follows:

$$\hat{\Omega}_{in}(-\infty) = \bar{T}\left(e^{i\int_{-\infty}^\infty \mathrm{d}t'\hat{V}_{in}(t')}\right)\hat{S} \tag{3.41}$$

At this point we need to make use of an important property of the time ordered operators.

The hermitian conjugation of anti-chronological time ordering of a product of operators $\hat{V}(t)$ turns it into the chronological time ordering of same operators, as long as $\hat{V}(t)$ is hermitian. Note that this can be seen immediately by taking the hermitian conjugate of Eq. (3.15):

$$\left[\bar{T}(\hat{V}(t_1)\dots\hat{V}(t_n))\right]^\dagger = T(\hat{V}(t_1)\dots\hat{V}(t_n)). \tag{3.42}$$

By keeping this in mind, we can recognize in Eq. (3.41) the hermitian conjugate of Eq. (3.19) evaluated at $t=\infty$:

$$\begin{aligned}\hat{\Omega}_{in}(-\infty) &= (\hat{\Omega}_{in}(\infty))^\dagger\hat{S}\\ &\underbrace{=}_{\text{Eq. (3.31)}} \hat{S}^{-1}\hat{S} = 1,\end{aligned} \tag{3.43}$$

where the last step relies on the unitary of $\hat{S}$. This can be seen straight away by taking the hermitian conjugate of Eq. (3.25).

3.3 Gell-Mann Low formula

Let us consider Eq. (2.17) with the initial condition taken at $t\to-\infty$:

$$\hat{q}^H_{\underline{k}}(t) = \hat{\Omega}^{-1}_{in}(t)\hat{q}^{in}_{\underline{k}}(t)\,\hat{\Omega}_{in}(t). \tag{3.44}$$

Expressing $\hat{\Omega}^{-1}_{in}(t)$ using Eq. (3.37) and $\hat{\Omega}_{in}(t)$ as in Eq. (3.19). We find

$$\hat{q}^H_{\underline{k}}(t) = \left[\bar{T}\left(e^{i\int_t^\infty \mathrm{d}\tilde{t}\hat{V}_{in}(\tilde{t})}\right)\hat{S}\right]^{-1}\hat{q}^{in}_{\underline{k}}(t)T\left(e^{-i\int_{-\infty}^t \mathrm{d}t'\hat{V}_{in}(t')}\right). \tag{3.45}$$

Since $\hat{\Omega}_{in}(t)$ is a unitary operator we can replace its inverse by its hermitian conjugate:

$$\hat{q}^H_{\underline{k}}(t) = \hat{S}^{-1}\left[\bar{T}\left(e^{i\int_t^\infty \mathrm{d}\tilde{t}\hat{V}_{in}(\tilde{t})}\right)\right]^\dagger\hat{q}^{in}_{\underline{k}}(t)T\left(e^{-i\int_{-\infty}^t \mathrm{d}t'\hat{V}_{in}(t')}\right), \tag{3.46}$$

where the unitarity of $\hat{S}$ has been used. Taking Eq. (3.42) into account, this expression can be written as

$$\hat{q}^H_{\underline{k}}(t) = \hat{S}^{-1}T\left(e^{-i\int_t^\infty \mathrm{d}\tilde{t}\hat{V}_{in}(\tilde{t})}\right)\hat{q}^{in}_{\underline{k}}(t)T\left(e^{-i\int_{-\infty}^t \mathrm{d}t'\hat{V}_{in}(t')}\right). \tag{3.47}$$

Let us temporary denote

$$\hat{M}(t) = T\left(e^{-i\int_t^\infty \mathrm{d}\tilde{t}\hat{V}_{in}(\tilde{t})}\right)\hat{q}^{in}_{\underline{k}}(t). \tag{3.48}$$

Observe that the product

$$\hat{M}(t)T\left(e^{-i\int_{-\infty}^t \mathrm{d}t'\hat{V}_{in}(t')}\right) = T\left(\hat{M}(t)e^{-i\int_{-\infty}^t \mathrm{d}t'\hat{V}_{in}(t')}\right), \tag{3.49}$$

because $\hat{M}(t)$ is fixed at the latest time (see the integral limits). Therefore,

$$\hat{q}^H_{\underline{k}}(t) = \hat{S}^{-1} T \left[T \left(e^{-i\int_t^\infty \mathrm{d}\tilde{t} \hat{V}_{in}(\tilde{t})} \right) \hat{q}^{in}_{\underline{k}}(t) e^{-i\int_{-\infty}^t \mathrm{d}t' \hat{V}_{in}(t')} \right]. \tag{3.50}$$

We want to move the exponential function as well as the $\hat{q}^{in}_{\underline{k}}(t)$ under inner time ordering operator T. In order to achieve this, we first introduce $\hat{F}(t) = \hat{q}^{in}_{\underline{k}}(t) e^{-i\int_{-\infty}^t \mathrm{d}t' \hat{V}_{in}(t')}$ and consider the operation:

$$\begin{aligned} T &\left(e^{-i\int_t^\infty \mathrm{d}\tilde{t} \hat{V}_{in}(\tilde{t})} \right) \hat{F}(t) \\ &= \Big\{ 1 - i \int_t^\infty \mathrm{d}\tilde{t}_1 T \left[\hat{V}_{in}(\tilde{t}_1) \right] \\ &\qquad + \frac{(-i)^2}{2} \int_t^\infty \mathrm{d}\tilde{t}_1 \int_{\tilde{t}_1}^\infty \mathrm{d}\tilde{t}_2 T \left[\hat{V}_{in}(\tilde{t}_1) \hat{V}_{in}(\tilde{t}_2) \right] \Big\} \hat{F}(t). \end{aligned} \tag{3.51}$$

As $\hat{F}(t)$ is evaluated at the earliest time the expression above can be written as:

$$\begin{aligned} \hat{F}(t) - i &\int_t^\infty \mathrm{d}\tilde{t}_1 T \left[\hat{V}_{in}(\tilde{t}_1) \hat{F}(t) \right] \\ &+ \frac{(-i)^2}{2} \int_t^\infty \mathrm{d}t\tilde{t}_1 \int_{\tilde{t}_1}^\infty \mathrm{d}\tilde{t}_2 T \left[\hat{V}_{in}(\tilde{t}_1) \hat{V}_{in}(\tilde{t}_2) \hat{F}(t) \right] + \dots \\ &= T \left(e^{-i\int_t^\infty \mathrm{d}\tilde{t} \hat{V}_{in}(\tilde{t})} \hat{F}(t) \right). \end{aligned} \tag{3.52}$$

Inside of a T-product, the factors commute[2] and can be written in any order (see appendix 4.A). In particular,

$$\hat{q}^H_{\underline{k}}(t) = \hat{S}^{-1} T \left[T \left(\hat{q}^{in}_{\underline{k}}(t) e^{-i\int_t^\infty \mathrm{d}\tilde{t} \hat{V}_{in}(\tilde{t})} e^{-i\int_{-\infty}^t \mathrm{d}t' \hat{V}_{in}(t')} \right) \right]. \tag{3.53}$$

To merge both exponential functions, the BCH-formula must be applied. Note that, due to the chronological time ordering, we could set the involving commutator to $T[\hat{V}_{in}(\tilde{t}), \hat{V}_{in}(t')] = 0$. After dropping the redundant second time ordering operator, it reads:

$$\hat{q}^H_{\underline{k}}(t) = \hat{S}^{-1} T \Big[\hat{q}^{in}_{\underline{k}}(t) \underbrace{e^{-i\int_t^\infty \mathrm{d}\tilde{t} \hat{V}_{in}(\tilde{t}) - i\int_{-\infty}^t \mathrm{d}t' \hat{V}_{in}(t')}}_{=\exp(-i\int_{-\infty}^\infty \mathrm{d}t' \hat{V}_{in}(t'))} \Big]. \tag{3.54}$$

$$\hat{q}^H_{\underline{k}}(t) = \hat{S}^{-1} T \left[\hat{q}^{in}_{\underline{k}}(t) e^{-i\int_{-\infty}^\infty \mathrm{d}t' \hat{V}_{in}(t')} \right]. \tag{3.55}$$

This representation of the Gell-Mann Low formula is used, when actually performing perturbative calculations.

As an alternative, we want to establish the equivalent scattering operator notation. It is based on the identification of the exponential function with the

[2] In the case of fermions, one has to apply anti-commutation. Nevertheless, the result stays the same. The proof and the general form for fermionic external current can be seen in [6].

expression for the $\hat{S}$-operator given in the second line of Eq. (3.31). Going back to Eq. (3.50), we first commute the exponential function and the $\hat{q}^{in}_{\underline{k}}(t)$ operator and define $\hat{F}(t) = e^{-i\int_{-\infty}^{t} \mathrm{d}t' \hat{V}_{in}(t')}$. It reads:

$$\hat{q}^{H}_{\underline{k}}(t) = \hat{S}^{-1} T \left[\hat{q}^{in}_{\underline{k}}(t) T \left(e^{-i\int_{t}^{\infty} \mathrm{d}\tilde{t} \hat{V}_{in}(\tilde{t})} \right) \hat{F}(t) \right]. \tag{3.56}$$

Following the same steps involving our new $\hat{F}(t)$ as seen in Eq. (3.52), we derive:

$$\hat{q}^{H}_{\underline{k}}(t) = \hat{S}^{-1} T \left[\hat{q}^{in}_{\underline{k}}(t) T \left(e^{-i\int_{t}^{\infty} \mathrm{d}\tilde{t} \hat{V}_{in}(\tilde{t})} e^{-i\int_{-\infty}^{t} \mathrm{d}t' \hat{V}_{in}(t')} \right) \right]. \tag{3.57}$$

With the BCH-formula we can merge the exponential functions, while keeping the commutator $T[\hat{V}_{in}(\tilde{t}), \hat{V}_{in}(t')] = 0$, we can then directly apply Eq. (3.31).

$$\hat{q}^{H}_{\underline{k}}(t) = \hat{S}^{-1} T \left[\hat{q}^{in}_{\underline{k}}(t) T \left(e^{-i\int_{-\infty}^{\infty} \mathrm{d}t' \hat{V}_{in}(t')} \right) \right]. \tag{3.58}$$

$$\hat{q}^{H}_{\underline{k}}(t) = \hat{S}^{-1} T \left[\hat{q}^{in}_{\underline{k}}(t) \hat{S} \right]. \tag{3.59}$$

We remark that, even though the $\hat{S}$-operator is time independent, it has to remain inside of the time ordering. The reasons for this are commutators involved in the BCH-formula and repositioning $\hat{q}^{H}_{\underline{k}}(t)$. Only the overall time ordering ensures them to be zero. This must also be kept in mind for the exponential function in Eq. (3.55), since we have shown, that both expressions for the factor involving the interaction part $\hat{V}_{in}(t)$ are derived on the same way. Only difference was when the $\hat{q}^{in}_{\underline{k}}(t_1)$ is repositioned. In both cases the same commutators have to be evaluated.

Next, consider the case with more than one $\hat{q}^{H}_{\underline{k}}$-operator. We begin with two time-ordered operators in the Heisenberg-picture:

$$\begin{aligned} T\left[\hat{q}^{H}_{\underline{k}}(t_1) \hat{q}^{H}_{\underline{k}'}(t_2) \right] =& \hat{q}^{H}_{\underline{k}}(t_1) \hat{q}^{H}_{\underline{k}'}(t_2) \theta(t_1 - t_2) \\ &+ \hat{q}^{H}_{\underline{k}'}(t_2) \hat{q}^{H}_{\underline{k}}(t_1) \theta(t_2 - t_1). \end{aligned} \tag{3.60}$$

In this form the following steps can be visualized and extended to products combining more then two operators, easily. We first apply Eq. (3.47) to the later time in each term:

$$\begin{aligned} T\left[\hat{q}^{H}_{\underline{k}}(t_1) \hat{q}^{H}_{\underline{k}'}(t_2) \right] = \hat{S}^{-1} \Big\{ & T \left(e^{-i\int_{t_1}^{\infty} \mathrm{d}t_1' \hat{V}_{in}(t_1')} \right) \hat{q}^{in}_{\underline{k}}(t_1) \hat{\Omega}_{in}(t_1) \\ & \times \hat{q}^{H}_{\underline{k}'}(t_2) \theta(t_1 - t_2) \\ & + T \left(e^{-i\int_{t_2}^{\infty} \mathrm{d}t_2' \hat{V}_{in}(t_2')} \right) \hat{q}^{in}_{\underline{k}'}(t_2) \hat{\Omega}_{in}(t_2) \\ & \times \hat{q}^{H}_{\underline{k}}(t_1) \theta(t_2 - t_1) \Big\}. \end{aligned} \tag{3.61}$$

One can immediately see, that one $\hat{S}^{-1}$-operator can be moved out from both lines. Furthermore, the conditions of the cases ensures that the respective operators to the right can be moved inside of the inner chronological time ordering. This is similar to what was done between Eq. (3.51) and (3.54).

$$\begin{aligned} T\left[\hat{q}^H_{\underline{k}}(t_1)\hat{q}^H_{\underline{k}'}(t_2)\right] = \hat{S}^{-1}\Big\{ & T\left(\hat{q}^{in}_{\underline{k}}(t_1)e^{-i\int_{t_1}^{\infty}\mathrm{d}t_1'\hat{V}_{in}(t_1')}\hat{\Omega}_{in}(t_1)\hat{q}^H_{\underline{k}'}(t_2)\right) \\ & \times\theta(t_1-t_2) \\ & + T\Big(\hat{q}^{in}_{\underline{k}'}(t_2)\underbrace{e^{-i\int_{t_2}^{\infty}\mathrm{d}t_2'\hat{V}_{in}(t_2')}\hat{\Omega}_{in}(t_2)}_{=\hat{S}}\hat{q}^H_{\underline{k}}(t_1)\Big) \\ & \times\theta(t_2-t_1)\Big\}. \end{aligned} \tag{3.62}$$

Observe, that $\hat{S}$ can be derived from the product of the exponential function and the unitary operator $\hat{\Omega}_{in}$(see Eq. (3.52) and below). The $\hat{q}^H$ operator must be expressed using Eq. (3.47):

$$\begin{aligned} T\left[\hat{q}^H_{\underline{k}}(t_1)\hat{q}^H_{\underline{k}'}(t_2)\right] = \hat{S}^{-1}\Big\{ & T\Big[\hat{q}^{in}_{\underline{k}}(t_1)\hat{S}\hat{S}^{-1}\hat{q}^{in}_{\underline{k}'}(t_2)T\Big(e^{-i\int_{t_2}^{\infty}\mathrm{d}t_2'\hat{V}_{in}(t_2')}\Big) \\ & \times T\Big(e^{-i\int_{-\infty}^{t_2}\mathrm{d}t_2'\hat{V}_{in}(t_2')}\Big)\Big]\theta(t_1-t_2) \\ & + T\Big[\hat{q}^{in}_{\underline{k}'}(t_2)\underbrace{\hat{S}\hat{S}^{-1}}_{=1}\hat{q}^{in}_{\underline{k}}(t_1)T\Big(e^{-i\int_{t_1}^{\infty}\mathrm{d}t_1'\hat{V}_{in}(t_1')}\Big) \\ & \times T\Big(e^{-i\int_{-\infty}^{t_1}\mathrm{d}t_1'\hat{V}_{in}(t_1')}\Big)\Big]\theta(t_2-t_1)v\Big\}. \end{aligned} \tag{3.63}$$

After cancelling the scattering operator and its inverse, we now merging both exponential function regarding the earlier time:

$$\begin{aligned} T\left[\hat{q}^H_{\underline{k}}(t_1)\hat{q}^H_{\underline{k}'}(t_2)\right] = & \hat{S}^{-1}\Big\{T\left[\hat{q}^{in}_{\underline{k}}(t_1)\hat{q}^{in}_{\underline{k}'}(t_2)\hat{S}\right]\theta(t_1-t_2) \\ & + T\left[\hat{q}^{in}_{\underline{k}'}(t_2)\hat{q}^{in}_{\underline{k}}(t_1)\hat{S}\right]\theta(t_2-t_1).\Big\} \end{aligned} \tag{3.64}$$

Note, rewriting our notation using cases back to the chronological time operator T, results in $T(T[\ldots])$. After dropping the redundant second time ordering operator, it reads:

$$T\left(\hat{q}^H_{\underline{k}}(t_1)\hat{q}^H_{\underline{k}'}(t_2)\right) = \hat{S}^{-1}T\left(\hat{q}^{in}_{\underline{k}}(t_1)\hat{q}^{in}_{\underline{k}'}(t_2)\hat{S}\right). \tag{3.65}$$

Following these step multiple time, one can easily arrive at the final form of the Gell-Mann Low formula:

$$T\left(\hat{q}^H_{\underline{k}}(t_1)\hat{q}^H_{\underline{k}'}(t_2)\ldots\right) = \hat{S}^{-1}T\left(\hat{q}^{in}_{\underline{k}}(t_1)\hat{q}^{in}_{\underline{k}'}(t_2)\ldots\hat{S}\right). \tag{3.66}$$

Like Eq. (3.55), we can substitute $\hat{S}$ with the exponential function by the inner T-operator.

Its relevance to quantum field theory lies in performing perturbation theory. When sandwiching Eq. (3.66) between the asymptotic vacuum states of the *in* and *out* picture, it reads:

$$\begin{aligned}&\langle 0_{out}| \, T\left(\hat{q}^{H}_{\underline{k}}(t_1)\hat{q}^{H}_{\underline{k}'}(t_2)\dots\right)|0_{in}\rangle \\ &= \langle 0_{out}| \, \hat{S}^{-1}T\left(\hat{q}^{in}_{\underline{k}}(t_1)\hat{q}^{in}_{\underline{k}'}(t_2)\dots\hat{S}\right)|0_{in}\rangle \,.\end{aligned} \tag{3.67}$$

The inverse of the scattering operator allows to transform these vacuum states like Eq. (3.24): $\langle 0_{out}| \, \hat{S}^{-1} = \langle 0_{in}|$. One important aspect to consider at this point are disconnected graphs. These are terms containing vacuum fluctuations not linked to incoming and out-going particles [vacuum bubbles]. In order to avoid these highly divergent objects, it is convenient to work out the connected Green's functions. To derive these, we have to divide Eq. (3.67) by $\langle 0_{out}|0_{in}\rangle = \langle 0_{in}|\hat{S}|0_{in}\rangle$ (see [1]).

Chapter 4

Conclusion

In this report we have reviewed a set of fundamental concepts and methods, essential for starting a study of quantum field theory. We began with the well-known quantum mechanical pictures and introduced the *in* and *out* picture as consequences of working with asymptotic states. By performing transitions between these states, a formula involving the unitary operators was found, which we defined as the scattering operator $\hat{S}$. With this object, we could derive the Gell-Mann Low formula, which is the key to performing perturbation theory.

4.A Chronological and Anti-Chronological time ordering

To expand our concept of chronological time ordering to cases of more then two t_i we start by advancing the definition stated in Eq. (3.8). In general time ordering consists of a summation of all permutations P of a given set multiplied by Heaviside functions. The notation P_j refers to a specific arrangement, while $P_j(i)$ returns the index of the operator at this position i in the permutation j. We keep the number of brackets low by denoting $\hat{V}_I(t_1) \rightarrow \hat{V}_1$ Definition:

$$T(\hat{V}_1\hat{V}_2\ldots\hat{V}_n) = \sum_{j=1}^{n!} P_j \left[\prod_{l}^{n} \hat{V}_l\right] \left[\prod_{i}^{n-1} \theta\left(t_{P_j(i)} - t_{P_j(i+1)}\right)\right]. \tag{4.1}$$

The generalisation of anti-chronological time ordering given in Eq. (3.15) is based on the same principle. Definition:

$$\bar{T}(\hat{V}_1, \hat{V}_2, \ldots, \hat{V}_n) = \sum_{j=1}^{n!} P_j \left[\prod_{l}^{n} \hat{V}_l\right] \left[\prod_{i}^{n-1} \theta\left(t_{P_j(i+1)} - t_{P_j(i)}\right)\right]. \tag{4.2}$$

As a test, we set $n = 3$ in Eq. (4.1):

$$\begin{aligned} T(\hat{V}_1\hat{V}_2\hat{V}_3) = \sum_{j=1}^{3!} P_j \left[\hat{V}_1\hat{V}_2\hat{V}_3\right] &\left[\prod_{i}^{3-1} \theta\left(t_{P_j(i)} - t_{P_j(i+1)}\right)\right] \\ = \quad &\hat{V}_1\hat{V}_2\hat{V}_3\, \theta(t_1 - t_2)\theta(t_2 - t_3) \\ &+ \hat{V}_1\hat{V}_3\hat{V}_2\, \theta(t_1 - t_3)\theta(t_3 - t_2) \\ &+ \hat{V}_2\hat{V}_1\hat{V}_3\, \theta(t_2 - t_1)\theta(t_1 - t_3) \\ &+ \hat{V}_2\hat{V}_3\hat{V}_1\, \theta(t_2 - t_3)\theta(t_3 - t_1) \\ &+ \hat{V}_3\hat{V}_1\hat{V}_2\, \theta(t_3 - t_1)\theta(t_1 - t_2) \\ &+ \hat{V}_3\hat{V}_2\hat{V}_1\, \theta(t_3 - t_2)\theta(t_2 - t_1). \end{aligned} \tag{4.3}$$

As well as $n = 3$ in Eq. (4.2)

$$\begin{aligned} \bar{T}(\hat{V}_1\hat{V}_2\hat{V}_3) = \sum_{j=1}^{3!} P_j \left[\hat{V}_1, \hat{V}_2, \hat{V}_3\right] &\left[\prod_{i}^{3-1} \theta\left(t_{P_j(i+1)} - t_{P_j(i)}\right)\right] \\ = \quad &\hat{V}_1\hat{V}_2\hat{V}_3\, \theta(t_2 - t_1)\theta(t_3 - t_2) \\ &+ \hat{V}_1\hat{V}_3\hat{V}_2\, \theta(t_3 - t_1)\theta(t_2 - t_3) \\ &+ \hat{V}_2\hat{V}_1\hat{V}_3\, \theta(t_1 - t_2)\theta(t_3 - t_1) \\ &+ \hat{V}_2\hat{V}_3\hat{V}_1\, \theta(t_3 - t_2)\theta(t_1 - t_3) \\ &+ \hat{V}_3\hat{V}_1\hat{V}_2\, \theta(t_1 - t_3)\theta(t_2 - t_1) \\ &+ \hat{V}_3\hat{V}_2\hat{V}_1\, \theta(t_2 - t_3)\theta(t_1 - t_2). \end{aligned} \tag{4.4}$$

The following results and calculations can be performed in the same fashion for chronological and anti-chronological time ordering. We will show them explicitly for the case of chronological time ordering.

In Eq. (3.5) we defined an function $I(t)$ for two potentials $V_I(t)$. Expanding this function to the case of n requires a normalization factor alongside T. A set of n different elements can be linear arranged in $n!$ ways. Therefore a factor of $1/n!$ in the expressions for $I(t)$ is required, when chronological time ordering is applied.

We begin with Eq. (4.1) applied to a larger $I(t)$ and perform a proof by induction based on the number of $\hat{V}_o$. The induction start is the case of two $\hat{V}$. In the induction step we state that it works for at least one unspecified higher order k: (*Note:* $t_0 = t$)

$$I_k(t) = \prod_{a=1}^{k} \int_{-\infty}^{t_{a-1}} \mathrm{d}t_a \, \hat{V}_a = \frac{1}{k!} \left(\prod_{a=1}^{k} \int_{-\infty}^{t} \mathrm{d}t_a \right) T(\hat{V}_1, \ldots, \hat{V}_k). \tag{4.5}$$

By moving one increment higher in our 'chain' to $k+1$, we have to show:

$$I_{k+1}(t) = \prod_{a=1}^{k+1} \int_{-\infty}^{t_{a-1}} \mathrm{d}t_a \, \hat{V}_a = \frac{1}{(k+1)!} \left(\prod_{a=1}^{k+1} \int_{-\infty}^{t} \mathrm{d}t_a \right) T(\hat{V}_1 \ldots \hat{V}_{k+1}). \tag{4.6}$$

By separating the product on the left-hand side at the point of $k \times (k+1)$, the induction step allows us to apply the definition in Eq. (4.1):

$$\begin{aligned} I_{k+1}(t) &= \prod_{a=1}^{k} \int_{-\infty}^{t_{a-1}} \mathrm{d}t_a \, \hat{V}_a \int_{-\infty}^{t_k} \mathrm{d}t_{k+1} \, \hat{V}_{k+1} \\ &= \frac{1}{k!} \left(\prod_{a=1}^{k} \int_{-\infty}^{t} \mathrm{d}t_a \right) T(\hat{V}_1, \ldots, \hat{V}_k) \int_{-\infty}^{t_k} \mathrm{d}t_{k+1} \, \hat{V}_{k+1} \\ &= \frac{1}{k!} \left(\prod_{a=1}^{k} \int_{-\infty}^{t} \mathrm{d}t_a \right) \sum_{j=1}^{k!} P_j \left[\prod_{l}^{k} \hat{V}_l \right] \\ &\qquad \times \left[\prod_{i}^{n-1} \theta \left(t_{P_j(i)} - t_{P_j(i+1)} \right) \right] \int_{-\infty}^{t_k} \mathrm{d}t_{k+1} \, \hat{V}_{k+1}. \end{aligned} \tag{4.7}$$

Now we have to manually recreate the sum over all positions of $\hat{V}_{k+1}$ in each term of $\sum_{j=1}^{n!} P_j \left[\prod_l^n \hat{V}_l \right]$. This is similar to the splitting of $I(t)$ in all permutations in Eq. (3.7). Instead of a factor $1/2$, each term will have $1/(k+1)$.

$$\begin{aligned} I_{k+1}(t) = \frac{1}{k!} &\left(\prod_{a=1}^{k} \int_{-\infty}^{t} \mathrm{d}t_a \right) \int_{-\infty}^{t_k} \mathrm{d}t_{k+1} \\ &\times \frac{1}{(k+1)} \sum_{u=1}^{(k+1)!} P_u \left\{ \sum_{j=1}^{k!} P_j \left[\prod_{l}^{k} \hat{V}_l \right] \hat{V}_{k+1} \right\} \\ &\qquad \times \left[\prod_{i}^{k-1} \theta \left(t_{P_j(i)} - t_{P_j(i+1)} \right) \right]. \end{aligned} \tag{4.8}$$

Like in Eq. (3.8), we can equalise the last integration limit t_k to t by introducing one more Heaviside function θ in each term. In addition to that we write these two permutations in a single total P_f.

$$I_{k+1}(t) = \frac{1}{(k+1)!} \left(\prod_{a=1}^{k+1} \int_{-\infty}^{t} \mathrm{d}t_a \right) \sum_{u=1}^{(k+1)!} P_f \left\{ \prod_{l}^{k+1} \hat{V}_l \right\} \times \left[\prod_{i}^{k} \theta \left(t_{P_f(i)} - t_{P_f(i+1)} \right) \right]. \tag{4.9}$$

This is then chronological time ordered expression for a $k+1$ long product, which should be shown.

$$I_{k+1}(t) = \frac{1}{(k+1)!} \left(\prod_{a=1}^{k+1} \int_{-\infty}^{t} \mathrm{d}t_a \right) T(\hat{V}_1, \ldots, \hat{V}_{k+1}) \tag{4.10}$$

An very useful aspect of chronological as well as anti-chronological time ordering is all operators V_i in $T(\ldots)$ or $\bar{T}(\ldots)$ commute. For two elements:

$$T(\hat{V}_1, \hat{V}_2) = \hat{V}_1 \, \hat{V}_2 \, \theta(t_1 - t_2) \; + \; \hat{V}_2 \, \hat{V}_1 \, \theta(t_2 - t_1) \tag{4.11a}$$

$$T(\hat{V}_2, \hat{V}_1) = \hat{V}_2 \, \hat{V}_1 \, \theta(t_2 - t_1) \; + \; \hat{V}_1 \, \hat{V}_2 \, \theta(t_1 - t_2), \tag{4.11b}$$

after switching the terms,

$$T(\hat{V}_1, \hat{V}_2) = T(\hat{V}_2, \hat{V}_1) \tag{4.12}$$

In other words, the commutation relations reflect, whether the subtraction of permutations of elements is zero or not. But in time ordering all permutations occur, we can rearrange the terms so subtraction of equal permutations happens. Therefore commutation holds for more then two V_i.

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