



Specialization report

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1 Introduction

Quantum field theory is a set of methods and concepts that allow us to describe elementary particle processes. Of great interest are derived quantities like cross-sections σ or decay rates Γ as most common comparisons to experiments. From the side of theoretical physics, they can be extracted by calculating the amplitudes of the related correlation functions. Such calculations require perturbative approaches. However the starting points for setting up correlations functions like a Lagrangian, operators and states are described and formulated in the way of the Heisenberg picture without any parts that allow for perturbation theory. The main goal of this report is the derivation of the solution to this dilemma. The Gell-Mann Low formula. It allows transition to pictures inside the correlation functions without loss of information by including the scattering operator S . This S allows to be calculated perturbatively and therefore also the actual correlation functions. We will working out a few handy techniques and methods on the way to Gell-Mann Low formula. This includes discussing the more common Schrödinger, Heisenberg and Interaction picture. Followed up by *In* and *Out* pictures which Gell-Mann Low formula allows easy transition inside the corr. function. These are based on boundary conditions imposing free motion on particles along them. We derive them from experimental and physical point of views.

2 Pictures in Quantum Mechanics

2.1 Schrödinger picture and Heisenberg picture

The choice of a picture always requires to establish the states but also the corresponding operators. In the Schrödinger picture the operators are time-independent but the wavefunctions are time dependent. The time evolution of a state vector is controlled by the Schrödinger equation. Let $|\Psi(t)\rangle$ denote a state vector at time t . It satisfies

$$i\hbar \frac{\partial}{\partial t} |\Psi_S(t)\rangle = \hat{H} |\Psi_S(t)\rangle, \quad (1)$$

where \hat{H} is the Hamiltonian of the system. When assuming it time independent, the solution of Eq. (1) can be formally written as

$$|\Psi_S(t)\rangle = \hat{U} |\Psi_S(t_0)\rangle \quad (2)$$

with $\hat{U}(t - t_0) = e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}$ the time evolution operator, which satisfies the differential equation

$$i\hbar\partial_t\hat{U}(t - t_0) = \hat{H}\hat{U}(t - t_0). \quad (3)$$

Here $|\Psi_S(t_0)\rangle$ is a ket of $t = t_0$. We shall generally take $t_0 = 0$ and write

$$|\Psi_S(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t} |\Psi_H\rangle. \quad (4)$$

The state on the right-hand side has no longer time dependence. This defines the state in the Heisenberg picture.

The above two pictures differ between each other in the way of storing the time dependence. In the Schrödinger picture only the states carry such a dependence, whereas in the Heisenberg picture only operators has this possibility. To verify this statement we study the matrix element of an operator in the Schrödinger picture

$$\langle\Psi'_S(t)|\hat{A}_S|\Psi_S(t)\rangle = \langle\Psi'_H|e^{\frac{i}{\hbar}t\hat{H}}\hat{A}_Se^{-\frac{i}{\hbar}t\hat{H}}|\Psi_H\rangle, \quad (5)$$

where Eq. (4) has been used. As a consequence,

$$\hat{A}_H(t) = e^{\frac{i}{\hbar}t\hat{H}}\hat{A}_Se^{-\frac{i}{\hbar}t\hat{H}} = \hat{U}(t)^{-1}\hat{A}_S\hat{U}(t). \quad (6)$$

This new operator $\hat{A}_H(t)$ in combination with the state $|\Psi_H\rangle$ defines the Heisenberg picture. Observe that the time evolution of $\hat{A}_H(t)$ is dictated by an equation that follows from differentiating the equation above with respect to t :

$$\frac{d}{dt}\hat{A}_H(t) = \frac{i}{\hbar}\hat{H}e^{\frac{i}{\hbar}t\hat{H}}\hat{A}_Se^{-\frac{i}{\hbar}t\hat{H}} + e^{\frac{i}{\hbar}t\hat{H}}\hat{A}_S\left(-\frac{i}{\hbar}\hat{H}\right)e^{-\frac{i}{\hbar}t\hat{H}}. \quad (7)$$

Here we have used the time evolution equation (3). Hence,

$$\begin{aligned} \frac{d}{dt}\hat{A}_H(t) &= \frac{i}{\hbar}\hat{U}(t)^{-1}\hat{H}\hat{A}_S\hat{U}(t) - \frac{i}{\hbar}\hat{U}(t)^{-1}\hat{A}_S\hat{H}\hat{U}(t) \\ &= \frac{i}{\hbar}\hat{U}(t)^{-1}\hat{H}\underbrace{\hat{U}(t)\hat{U}(t)^{-1}}_{=1}\hat{A}_S\hat{U}(t) - \frac{i}{\hbar}\hat{U}(t)^{-1}\hat{A}_S\underbrace{\hat{U}(t)\hat{U}(t)^{-1}}_{=1}\hat{H}\hat{U}(t). \end{aligned} \quad (8)$$

The inserted 1 allows us to express Eq. (8) in term of operators in the Heisenberg picture.

$$\frac{d}{dt}\hat{A}_H(t) = \frac{i}{\hbar}\hat{H}_H(t)\hat{A}_H(t) - \frac{i}{\hbar}\hat{A}_H(t)\hat{H}_H(t), \quad (9)$$

where $\hat{H}_H(t)$ is the respective Hamiltonian in the Heisenberg picture. Therefore:

$$i\frac{d}{dt}\hat{A}_H(t) = \frac{1}{\hbar} [\hat{A}_H(t), \hat{H}_H(t)]. \quad (10)$$

2.2 Interaction picture

A third picture can be introduced: the Interaction picture (sometimes called the Dirac picture). We will see very shortly that, in the Interacting picture both the states and the respective operators are time dependent. Let us suppose that the Hamiltonian in the Schrödinger picture can be splitted as follows $\hat{H} = \hat{H}_0 + \hat{V}$. Normally \hat{H}_0 describe the free motion of a system, whereas \hat{V} represents its interaction, which could be with an external source. Although it often used in a perturbative approach, the Interaction picture does not require \hat{V} to be small as compared with \hat{H}_0 . Inserting this decomposition of \hat{H} in the unitary operator introduced below Eq. (2):

$$\hat{U}(t) = e^{-\frac{i}{\hbar}t\hat{H}} = e^{-\frac{i}{\hbar}t\hat{H}_0 + \hat{V}} = e^{-\frac{i}{\hbar}t\hat{H}_0}\hat{\Omega}(t) \quad (11)$$

This expression helps us to establish a formula from which operators and states in the interaction picture can be defined. For this, consider a matrix element $\langle \Psi'_S(t) | \hat{A}_S | \Psi_S(t) \rangle$. Taking into account Eq. (4) and (11) we find

$$\langle \Psi'_S(t) | \hat{A}_S | \Psi_S(t) \rangle = \langle \Psi'_H | (e^{-\frac{i}{\hbar}t\hat{H}_0}\Omega(t))^\dagger \hat{A}_S e^{-\frac{i}{\hbar}t\hat{H}_0}\Omega(t) | \Psi_H \rangle \quad (12a)$$

$$= \langle \Psi'_H | \Omega(t)^{-1} \hat{A}_I(t) \Omega(t) | \Psi_H \rangle. \quad (12b)$$

Here the operator in the interaction picture reads

$$\hat{A}_I(t) = e^{+\frac{i}{\hbar}t\hat{H}_0} \hat{A}_S e^{-\frac{i}{\hbar}t\hat{H}_0}, \quad (13)$$

whereas a corresponding state in this picture is

$$|\Psi_I(t)\rangle = \Omega(t) |\Psi_H\rangle. \quad (14)$$

At the level of operators, the connection between the Interaction and the Heisenberg picture is established by inverting Eq. (6) and inserting the resulting \hat{A}_S into Eq. (13). This leads to

$$\hat{A}_I(t) = e^{+\frac{i}{\hbar}t\hat{H}_0} \hat{U}(t) \hat{A}_H(t) \hat{U}(t)^{-1} e^{-\frac{i}{\hbar}t\hat{H}_0}. \quad (15a)$$

$$= e^{+\frac{i}{\hbar}t\hat{H}_0} e^{-\frac{i}{\hbar}t\hat{H}} \hat{A}_H(t) e^{\frac{i}{\hbar}t\hat{H}} e^{-\frac{i}{\hbar}t\hat{H}_0}, \quad (15b)$$

ending with

$$\hat{A}_I(t) = \hat{\Omega}(t)\hat{A}_H(t)\hat{\Omega}(t)^{-1}. \quad (16)$$

The time evolution equation for $\hat{A}_I(t)$ can be found as done for $\hat{A}_H(t)$ [see below Eq. (6)]:

$$i\hbar\frac{\partial}{\partial t}\hat{A}_I = [\hat{A}_I, \hat{H}_0]. \quad (17)$$

Furthermore, an equation for $\hat{\Omega}(t)$ can be determined. To this end we invert Eq. (11) and express $\hat{\Omega}(t) = e^{\frac{i}{\hbar}t\hat{H}_0}\hat{U}(t)$. Afterwards we differentiate with respect to times:

$$i\hbar\partial_t\hat{\Omega}(t) = e^{\frac{i}{\hbar}t\hat{H}_0} \left(i\hbar\partial_t\hat{U}(t) \right) - \hat{H}_0 e^{\frac{i}{\hbar}t\hat{H}_0}\hat{U}(t) \quad (18a)$$

$$= \hat{H} e^{\frac{i}{\hbar}t\hat{H}_0}\hat{U}(t) - \hat{H}_0 e^{\frac{i}{\hbar}t\hat{H}_0}\hat{U}(t), \quad (18b)$$

where Eq. (3) has been used. Using the definition of $\hat{\Omega}(t)$ we end up with

$$i\hbar\frac{\partial}{\partial t}\hat{\Omega}(t) = \hat{V}_I(t)\hat{\Omega}(t). \quad (19)$$

2.3 The In and Out picture: External currents

Consider the set-up of most experiments in elementary particle and nuclear physics. Several particles approach each other from a macroscopic scale and interact in a microscopic section comparable to the Compton wavelength of the incoming particles. On this scale vacuum fluctuations are no longer negligible for the dynamic of the involved particles and make them impossible to distinguish between each other. As a result, the products of the interaction spread up to a macroscopic distances and the distinguishability between outgoing particles is admitted. Therefore, at such asymptotically distances, the description of the incoming and outgoing multi-particle states can be approached by direct products of single-particle effectively non-interacting states.

To bring this concept into our formulation let's consider the action of a scalar field Φ with mass m coupled to an external source $j(\underline{x}, t)$:

$$I = \int d^4x \mathcal{L}(\Phi, \dot{\Phi}, j) = \int d^4x \left(\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}m^2\Phi^2 - \Phi j \right). \quad (20)$$

Taking the functional derivative with respect to Φ and setting it to zero, we obtain the equation of motion

$$(\partial^2 + m^2) \Phi = j. \quad (21)$$

To proceed, we quantize our field in a box of volume V and length L . The classical field and its canonical momentum $\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}(\underline{x}, t)} = \dot{\Phi}(\underline{x}, t)$ are then promoted to operators $\hat{\Phi}(\underline{x}, t)$ and $\hat{\Pi}(\underline{x}, t)$ in the Heisenberg picture. Satisfying the equal-time commutator:

$$[\hat{\Phi}(\underline{x}, t), \hat{\Pi}(\underline{x}', t)] = i\delta^3(\underline{x} - \underline{x}'). \quad (22)$$

We then expand the field operator as follows:

$$\hat{\Phi}(\underline{x}, t) = \sum_{\underline{k}} \hat{q}_{\underline{k}}(t) u_{\underline{k}}(\underline{x}). \quad (23)$$

The 3 dim. wave vector \underline{k} for the modes is defines as $\underline{k} = \frac{2\pi}{L}(n_x, n_y, n_z)$ with $n_i \in \mathbb{Z}$. In this separated time and space dependency, we choose the Fourier basis for $u_{\underline{k}}(\underline{x})$

$$u_{\underline{k}}(\underline{x}) = \frac{1}{\sqrt{V}} e^{i\underline{k} \cdot \underline{x}}, \quad (24)$$

where the volume V provides the required normalization. We remark that $u_{\underline{k}}(\underline{x})$ constitutes an orthonormalized basis in the Hilbert space

$$\int d^3x u_{\underline{k}'}^*(\underline{x}) u_{\underline{k}}(\underline{x}) = \delta_{\underline{k}, \underline{k}'} \quad (25)$$

$$\sum_{\underline{k}} u_{\underline{k}}^*(\underline{x}) u_{\underline{k}}(\underline{x}') = \delta^3(\underline{x} - \underline{x}'). \quad (26)$$

We now substitute Eq. (24) into the equation of motion (21). As a consequence

$$\sum_{\underline{k}} [\ddot{\hat{q}}_{\underline{k}}(t) u_{\underline{k}}(\underline{x}) + \underline{k}^2 \hat{q}_{\underline{k}}(t) u_{\underline{k}}(\underline{x}) + m^2 \hat{q}_{\underline{k}}(t) u_{\underline{k}}(\underline{x})] = j(\underline{x}, t). \quad (27)$$

To get an equation for $\hat{q}_{\underline{k}}(t)$ alone we need to get rid of $u_{\underline{k}}(\underline{x})$ and remove the space dependence in the current. Multiplying with $u_{\underline{k}'}^*(\underline{x})$ and integrating over the whole space we find,

$$\sum_{\underline{k}} \left[\int d^3x u_{\underline{k}'}^* u_{\underline{k}} \left(\ddot{\hat{q}}_{\underline{k}}(t) + (\underline{k}^2 + m^2) \hat{q}_{\underline{k}}(t) \right) \right] = \underbrace{\int d^3x j(\underline{x}, t) \frac{1}{\sqrt{V}} e^{i\underline{k} \cdot \underline{x}}}_{=\tilde{j}(\underline{k}, t)}. \quad (28)$$

After using the orthonormality relation (25) this expression reduces to

$$\ddot{\hat{q}}_{\underline{k}}(t) + \omega_{\underline{k}}^2 \hat{q}_{\underline{k}}(t) = \tilde{j}(\underline{k}, t), \quad (29)$$

where $(\underline{k}^2 + m^2) = \omega_{\underline{k}}^2$.

We now make the assumption that the current vanishes outside a finite time interval,

$$j(t) \rightarrow 0 \text{ for } t \rightarrow \pm\infty, \quad (30)$$

as a consequence for Eq. (29), this allows us to distinguish between early and late times. For early time Eq. (29) approaches a homogeneous differential equation. We will call its asymptotic solution by $\hat{q}_{\underline{k}}(t) \rightarrow \hat{q}_{\underline{k},in}(t)$. Explicitly,

$$\hat{q}_{\underline{k},in}(t) = \frac{1}{2\omega_{\underline{k}}} \left(\hat{a}_{\underline{k},in} e^{-i\omega_{\underline{k}} t} + \hat{a}_{\underline{k},in}^\dagger e^{i\omega_{\underline{k}} t} \right), \leftrightarrow t \rightarrow -\infty, \quad (31)$$

where $\hat{a}_{\underline{k},in}$ denotes the annihilation operator, whereas $\hat{a}_{\underline{k},in}^\dagger$ is the corresponding creation operator. At late times Eq. (29) also reduces to a homogeneous type. In this case asymptotic the solution $\hat{q}_{\underline{k}}(t) \rightarrow \hat{q}_{\underline{k},out}(t)$ reads

$$\hat{q}_{\underline{k},out}(t) = \frac{1}{2\omega_{\underline{k}}} \left(\hat{a}_{\underline{k},out} e^{-i\omega_{\underline{k}} t} + \hat{a}_{\underline{k},out}^\dagger e^{i\omega_{\underline{k}} t} \right), \leftrightarrow t \rightarrow +\infty. \quad (32)$$

The solution for $\hat{q}_{\underline{k}}(t)$, at times for which $j(\underline{x}, t)$ is active, would then consist of the homogeneous solution plus a term containing the current:

$$\hat{q}_{\underline{k}}(t) = \hat{q}_{\underline{k},in}(t) + \frac{1}{\omega_{\underline{k}}} \int_{-\infty}^t dt' \sin[\omega_{\underline{k}}(t - t')] \tilde{j}(\underline{k}, t'). \quad (33)$$

For late times $t \rightarrow +\infty$ the expression above approaches to

$$\hat{q}_{\underline{k},out}(t) \approx \hat{q}_{\underline{k},in}(t) + \frac{1}{\omega_{\underline{k}}} \int_{-\infty}^{\infty} dt' \sin[\omega_{\underline{k}}(t - t')] \tilde{j}(\underline{k}, t'). \quad (34)$$

After splitting the sinus function, we find that

$$\hat{q}_{\underline{k},out}(t) = \hat{q}_{\underline{k},in}(t) - \frac{i}{2\omega_{\underline{k}}} e^{i\omega_{\underline{k}}t} \omega_{\underline{k}} + \frac{i}{2\omega_{\underline{k}}} e^{-i\omega_{\underline{k}}t} \bar{j}_{\underline{k}}(\omega_{\underline{k}}), \quad (35)$$

where $\bar{j}_{\underline{k}}(\omega_{\underline{k}}) = \int_{-\infty}^{\infty} dt \tilde{j}_{\underline{k}}(t) e^{i\omega_{\underline{k}}t}$. From this equation we can obtain the connection between creation and annihilation operators associated with the asymptotically far fields $t \rightarrow \pm\infty$. In compact notation

$$\hat{a}_{\underline{k},out} = \hat{a}_{\underline{k},in} + i\bar{j}_{\underline{k}}(\omega_{\underline{k}}), \quad (36a)$$

$$\hat{a}_{\underline{k},out}^\dagger = \hat{a}_{\underline{k},in}^\dagger - i\bar{j}_{\underline{k}}(-\omega_{\underline{k}}). \quad (36b)$$

This shows that, in the presence of an external current, the two sets of second quantization operators are not the same. Therefore we need to differ between the corresponding *in* and *out* eigenstates. Particularly, it has to be stated that vacua also differ in this scenario.

These results were all derived and described in the Heisenberg picture. We kept using the full Lagrangian and found a general solution in Eq. (33). From this we can proceed as shown in section **2.2**. We split the Hamiltonian as done there: $\hat{H} = \hat{H}_0 + V$.

$$\hat{H}_0(\Phi, \Pi) = \frac{1}{2}\hat{\Pi}^2 + \frac{1}{2}(\nabla\hat{\Phi})^2 + \frac{1}{2}m^2\hat{\Phi}^2, \quad (37)$$

$$\hat{V}(\Phi) = j\hat{\Phi}. \quad (38)$$

Expressing both operators in terms of the Fourier basis given (23), and the orthonormality relations [Eq. (25),(26)], we can express the Hamiltonian as follows:

$$\hat{H}_0(q, \dot{q}) = \sum_{\underline{k}} \frac{1}{2} \dot{\hat{q}}_{\underline{k}}^2(t) + \frac{1}{2} \omega_{\underline{k}}^2 \hat{q}_{\underline{k}}^2(t), \quad (39)$$

$$\hat{V}(q) = \sum_{\underline{k}} \tilde{j}(\underline{k}, t) \hat{q}_{\underline{k}}(t). \quad (40)$$

From this form we go to the Interaction picture. The potential V_I appearing in Eq. (19) reads :

$$\hat{V}_I(q) = \sum_{\underline{k}} \tilde{j}(\underline{k}, t) \hat{q}_{\underline{k}_I}(t), \quad (41)$$

where we used Eq. (16) to transform $\hat{q}_{\underline{k}}(t)$:

$$\hat{q}_{\underline{k}_I}(t) = \hat{\Omega}_I(t)\hat{q}_{\underline{k}}(t)\hat{\Omega}_I^{-1}(t). \quad (42)$$

But to have a well defined operator $\hat{\Omega}(t)$ we need conditions for $\Omega \rightarrow 1$ which is mostly depended on the current j . There are 3 possibilities. Either at $t = \pm\infty$ or $t = 0$. This leads then to 3 different pictures. The early time condition defines the *In* picture in reminiscence to the first asymptotic solution Eq (31) and it writes :

$$i\hbar \frac{\partial}{\partial t} \hat{\Omega}_{in}(t) = \hat{V}_{in}(t)\hat{\Omega}_{in}(t). \quad (43)$$

Fulfilling:

$$\hat{\Omega}_{in}(-\infty) = 1. \quad (44)$$

The so called *Out* picture will fulfill the condition for late times:

$$i\hbar \frac{\partial}{\partial t} \hat{\Omega}_{out}(t) = \hat{V}_{out}(t)\hat{\Omega}_{out}(t). \quad (45)$$

Stating:

$$\hat{\Omega}_{out}(+\infty) = 1. \quad (46)$$

The Interaction picture as introduced above does not require V to be of any special form but can still be applied for external source terms. To distinguish between the other two pictures above from the condition is chosen :

$$\hat{\Omega}_I(0) = 1. \quad (47)$$

The connecting operator between the pictures is called scattering operator or S-Matrix. It is defined as :¹

$$S = \Omega_I(\infty). \quad (48)$$

3 Scattering operator

3.1 Solutions for the Interaction picture

The verification of statement (48) will make use of explicit expressions of all Ω .

¹We will now go to natural units $c = \hbar = 1$ and drop the operator hat for convenience sake

Starting with the Interaction picture depended on t' and integrate the left-hand side of (19) :

$$\int_{t_0}^t dt' i\hbar \frac{\partial}{\partial t'} \Omega_I(t') = i\hbar [\Omega_I(t) - \Omega_I(t_0)], \quad (49a)$$

here we state $t_0 = 0$ as it is the condition for this picture and for now $t > 0$.

$$\int_0^t dt' i\hbar \frac{\partial}{\partial t'} \Omega_I(t') = i\hbar [\Omega_I(t) - 1]. \quad (49b)$$

Using this formula and the integral over the right-hand side of (19), we find an expression for $\Omega_I(t)$.

$$\Omega_I(t) = 1 - \frac{i}{\hbar} \int_0^t dt' V_I(t') \Omega_I(t') \quad (50)$$

,since the expression has an Ω_I on the other side we will go on by an iterative approach.

$$\Omega_I(t) = 1 - \frac{i}{\hbar} \int_0^t dt' V_I(t') \cdot \left(1 - \frac{i}{\hbar} \int_0^{t'} dt'' V_I(t'') \Omega_I(t'') \right) \quad (51a)$$

$$= 1 - \frac{i}{\hbar} \int_0^t dt' V_I(t') + \left(\frac{i}{\hbar}\right)^2 \int_0^t dt' \int_0^{t'} dt'' V_I(t'') \Omega_I(t''). \quad (51b)$$

The iteration will lead us by single digit incrementation in the power and number of integrals. An problematic aspect of this series are the different integral borders. Each term introduces a new t and keeps the previous as an integral boundary which forces us to solve them in a strict order. To keep the different t as variables over the same integral \int_0^t , we use the chronological time ordering.

$$T(V(t_1), V(t_2)) = V(t_1) V(t_2) \theta(t_1 - t_2) + V(t_2) V(t_1) \theta(t_2 - t_1). \quad (52)$$

In chronological time ordering the Heaviside-Step-function is used to set the earlier times to the right and later to the left by checking the difference. The Step-function is 0 for negative values and 1 for positive.²

²Proofs and elaborations for time ordering are to be found in the Appendices

For convince sake we drop \hbar again, t will be labelled with numbers instead of primes and time dependency will be indicated by index : $V(t_1) \rightarrow V_1$.
The solution to $\Omega_I(t)$ written as an infinite sum and time ordered :

$$\Omega_I(t) = \sum_{n=0}^{\infty} (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n V_I(t_1) \cdot \dots \cdot V_I(t_n) \quad (53a)$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n T \{V_I(t_1), \dots, V_I(t_n)\} \quad (53b)$$

The factorial sum over n converges to exponential function as seen in calculus and this non perturbativ expression reads:

$$\Omega_I(t) = T \left(e^{-i \int_0^t dt' V_I(t')} \right), \text{ for } t > 0. \quad (54)$$

We needed to split the solution of $\Omega_I(t)$, since the condition for the picture is not on the limits of the time span. Assuming $t < 0$ changes the first integral to :

$$\int_t^0 dt' i\hbar \frac{\partial}{\partial t'} \Omega_I(t') = i\hbar [1 - \Omega_I(t)] \quad (55)$$

The sign difference to the expression for $t > 0$ can be fixed by switching the integral borders. This means our infinite sum expression still holds:

$$\Omega_I(t) = \sum_{n=0}^{\infty} (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n V_I(t_1) \cdot \dots \cdot V_I(t_n). \quad (56)$$

The key difference now stands in the negativity of all t and a logical order for them would prefer later times to the right, coming closer to 0. This requires the anti-chronological time ordering :

$$\bar{T}(V(t_1), V(t_2)) = V(t_2) V(t_1) \theta(t_1 - t_2) + V(t_1) V(t_2) \theta(t_2 - t_1). \quad (57)$$

Using it similar as before:

$$\Omega_I(t) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \bar{T} \{V_I(t_1), \dots, V_I(t_n)\}. \quad (58)$$

Converging to the second expression:

$$\Omega_I(t) = \bar{T} \left(e^{-i \int_0^t dt' V_I(t')} \right), \text{ for } t < 0. \quad (59)$$

These two cases make the use of Ω_I safe in a sense of not having to watch out for sign flip in the integral while t runs. Second, the picture condition is clearly stated and not possible to hit while performing integration over t from negative to positive. But to perform the iterative solution one only needs to state Ω_I being finite in the region of integration. Lowering the integral boundary to $-\infty$ instead of 0. Losing the advantages from above but gaining the single expression:

$$\Omega_I(t) = T \left(e^{-i \int_{-\infty}^t dt' V_I(t')} \right). \quad (60)$$

3.2 Solutions for the In and Out picture

Beginning with the In picture and Eq. (43) we reach:

$$\int_{t_0}^t dt' i\hbar \frac{\partial}{\partial t'} \Omega_{in}(t') = i\hbar [\Omega_I(in) - \Omega_{in}(t_0)] \quad (61a)$$

,here we state $t_0 = -\infty$

$$\int_{-\infty}^t dt' i\hbar \frac{\partial}{\partial t'} \Omega_{in}(t') = i\hbar [\Omega_{in}(t) - 1]. \quad (61b)$$

The condition of the In picture being a limit for the time span spares us any splitting. Following the first solution of the Interaction picture:

$$\Omega_{in}(t) = \sum_{n=0}^{\infty} (-i)^n \int_{-\infty}^t dt_1 \int_{-\infty}^{t_1} dt_2 \dots \int_{-\infty}^{t_{n-1}} dt_n V_{in}(t_1) \cdot \dots \cdot V_{in}(t_n) \quad (62a)$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^t dt_1 \int_{-\infty}^{t_1} dt_2 \dots \int_{-\infty}^{t_{n-1}} dt_n T \{V_{in}(t_1), \dots, V_{in}(t_n)\}. \quad (62b)$$

Converging to :

$$\Omega_{in}(t) = T \left(e^{-i \int_{-\infty}^t dt' V_{in}(t')} \right). \quad (63)$$

The Out picture on the other hand follows the second solution. Here we argue t being smaller than ∞ needs one change of sign like before and anti-chronological ordering, since t only coming closer to the limit as it runs.

$$\Omega_{out}(t) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{\infty}^t dt_1 \int_{\infty}^{t_1} dt_2 \dots \int_{\infty}^{t_{n-1}} dt_n \bar{T} \{V_{out}(t_1), \dots, V_{out}(t_n)\}. \quad (64)$$

Concluding :

$$\Omega_{out}(t) = \bar{T} \left(e^{-i \int_{-\infty}^t dt' V_{in}(t')} \right). \quad (65)$$

3.3 Connections

Using this set of solution, the first set of relations to be proven is:

$$S = \Omega_{in}(t) \Omega_{out}(t)^{-1} \quad (66a)$$

$$= \Omega_{in}(\infty) \quad (66b)$$

$$= \Omega_{out}(-\infty)^{-1}. \quad (66c)$$

Writing the first equality out,

$$S = T \left(e^{-i \int_{-\infty}^t dt' V_{in}(t')} \right) \cdot \bar{T} \left(e^{-i \int_{-\infty}^t dt'' V_{out}(t'')} \right) \quad (67a)$$

$$= \sum_n (-i)^n \int_{-\infty}^t dt_1 V_{in}(t_1) \dots \int_{-\infty}^{t_{n-1}} dt_n V_{in}(t_n) \quad (67b)$$

$$\cdot \left(\sum_n (-i)^n \int_{-\infty}^t dt'_1 V_{out}(t'_1) \dots \int_{-\infty}^{t'_{n-1}} dt'_n V_{out}(t'_n) \right)^{-1} \quad (67c)$$

As an unitar operator $\Omega_{out}(t)^{-1} = \Omega_{out}(t)^\dagger$

$$= T \left(e^{-i \int_{-\infty}^t dt' V_{in}(t')} \right) \cdot T \left(e^{+i \int_{-\infty}^t dt'' V_{out}(t'')} \right) \quad (67d)$$

The lack of overlap in the integral limits allows us to fuse T

$$= T \left(e^{-i \int_{-\infty}^t dt' V_{in}(t')} \cdot e^{+i \int_{-\infty}^t dt'' V_{out}(t'')} \right) \quad (67e)$$

Commutation in T allow ease use of the Baker-Campbell-Hausdorff-formula

$$= T \left(e^{-i \int_{-\infty}^t dt' V_{in}(t') - i \int_t^\infty dt'' V_{out}(t'')} \right). \quad (67f)$$

In addition the overlap makes S time-independent. Therefore t could be ∞ :

$$S = T \left(e^{-i \int_{-\infty}^\infty dt' V_{in}(t') - i \int_\infty^\infty dt'' V_{out}(t'')} \right) = T \left(e^{-i \int_{-\infty}^\infty dt' V_{in}(t')} \right) = \Omega_{in}(\infty). \quad (68)$$

Or we choose $-\infty$:

$$S = T \left(e^{-i \int_{-\infty}^{\infty} dt' V_{in}(t') - i \int_{-\infty}^{\infty} dt'' V_{out}(t'')} \right) = T \left(e^{-i \int_{-\infty}^{\infty} dt' V_{out}(t')} \right) = \Omega_{out}(-\infty)^{-1}. \quad (69)$$

Eq. (68) is very similar to the definition in Eq. (48) for Ω_I . This comes from V_I and V_{in} both being picture transformations of V_H in the Heisenberg picture. We discussed the main difference lies in position of the condition on the time span. The time independence and integral over all t makes them equal.

$$S = T \left(e^{-i \int_{-\infty}^{\infty} dt V_I(t)} \right). \quad (70)$$

The next set of relations is :

$$\Omega_{in}(t) \Omega_I(t)^{-1} = S \Omega_I(\infty)^{-1} \quad (71a)$$

$$= \Omega_{in}(0). \quad (71b)$$

The first one is just 1, since the definition was $S = \Omega_I(\infty)$ was Using the exponential formulae:

$$\Omega_{in}(t) \Omega_I(t)^{-1} = T \left(e^{-i \int_{-\infty}^t dt' V_{in}(t')} \right) \left(\bar{T} \left(e^{-i \int_0^t dt' V_I(t')} \right) \right)^{-1} \quad (72a)$$

$$\stackrel{T \rightarrow \bar{T}}{=} T \left(e^{-i \int_{-\infty}^t dt' V_{in}(t')} \right) T \left(e^{+i \int_0^t dt' V_I(t')} \right) \quad (72b)$$

$$= T \left(e^{-i \int_{-\infty}^t dt' V_{in}(t')} e^{+i \int_0^t dt' V_I(t')} \right) \quad (72c)$$

$$\stackrel{CBH}{=} T \left(e^{-i \int_{-\infty}^t dt' V_{in}(t') - i \int_t^0 dt' V_I(t')} \right) \quad (72d)$$

$$= \Omega_{in}(0) \Omega_I(0) \stackrel{\Omega_I(0)=1}{=} \Omega_{in}(0) \quad (72e)$$

This last expression is time independent and is also 1 by looking at the differential equation for Ω_{in} .

The last relation is :

$$\Omega_{in}(t) = \bar{T} \left(e^{i \int_t^{\infty} dt' V_{in}(t')} \right) S. \quad (73)$$

First it must satisfy the differential equation for Ω_{in} :

$$i \partial_t \Omega_{in}(t) = i \partial_t \left(\sum_n i^n \int_t^{\infty} dt_1 V_{in}(t_1) \dots \int_{t_{n-1}}^{\infty} dt_n V_{in}(t_n) \right) S \quad (74a)$$

making an integration by part,

$$= i \left(\sum_n i^n \partial_t [\bar{V}_{in}(\infty) - \bar{V}_{in}(t)] \int_{t_1}^{\infty} dt_2 V_{in}(t_2) \dots \int_{t_{n-1}}^{\infty} dt_n V_{in}(t_n) \right) S \quad (74b)$$

partial differentiate in respect to t ,

$$= i \left(\sum_n i^n [0 - V_{in}(t)] \int_{t_1}^{\infty} dt_2 V_{in}(t_2) \dots \int_{t_{n-1}}^{\infty} dt_n V_{in}(t_n) \right) S \quad (74c)$$

multiplying and convergence,

$$= V_{in}(t) \sum_n \frac{i^{n-1}}{(n-1)!} \int_{t_1}^{\infty} dt_2 \dots \int_{t_{n-1}}^{\infty} dt_n \bar{T}(V_{in}(t_2) \dots V_{in}(t_n)) S \quad (74d)$$

$$= \bar{T} \left(V_{in}(t) e^{i \int_t^{\infty} dt' V_{in}(t')} \right) S \quad (74e)$$

$$= V_{in}(t) \bar{T} \left(e^{i \int_t^{\infty} dt' V_{in}(t')} \right) S \quad (74f)$$

The last step was based on t being the earliest time in the integral and anti-chronological time ordering.

One also has to provide $\Omega_{in}(-\infty) = 1$:

$$\Omega_{in}(t) = \bar{T} \left(e^{i \int_{-\infty}^{\infty} dt' V_{in}(t')} \right) S \quad (75a)$$

$$= (\Omega_{in}(\infty))^{\dagger} S \quad (75b)$$

$$= S^{\dagger} S = S^{-1} S = 1 \quad (75c)$$

3.4 Unitarity of S

This last step required S to be unitar. Beginning with rewriting it.

$$S = T \left(e^{-i \int_{-\infty}^{\infty} dt V_I(t)} \right) \quad (76a)$$

$$= \sum_n \frac{(-i)^n}{(n)!} \int_{-\infty}^{\infty} dt_1 \dots \int_{-\infty}^{\infty} dt_n T(V_{in}(t_1) \dots V_{in}(t_n)) \quad (76b)$$

$$= \sum_n (-i)^n \int_{-\infty}^{\infty} dt_1 \dots \int_{-\infty}^{\infty} dt_n V_{in}(t_1) \dots V_{in}(t_n). \quad (76c)$$

Without variables in the limits of integration the order is arbitrary

$$= \sum_n \frac{(-i)^n}{(n)!} \int_{-\infty}^{\infty} dt_1 \dots \int_{-\infty}^{\infty} dt_n \bar{T}(V_{in}(t_1) \dots V_{in}(t_n)) \quad (76d)$$

$$= \bar{T}\left(e^{-i \int_{-\infty}^{\infty} dt V_I(t)}\right). \quad (76e)$$

Keeping this in mind we express $1 = 1$ as :

$$1 = T(e^0) \quad (77a)$$

$$= T\left(e^{i \int_{-\infty}^{\infty} dt V_I(t) - i \int_{-\infty}^{\infty} dt V_I(t)}\right) \quad (77b)$$

$$\stackrel{\text{CBH}}{=} T\left(e^{i \int_{-\infty}^{\infty} dt V_I(t)} e^{-i \int_{-\infty}^{\infty} dt V_I(t)} e^{\frac{1}{2} [i \int_{-\infty}^{\infty} dt V_I(t), -i \int_{-\infty}^{\infty} dt V_I(t)]}\right) \quad (77c)$$

$$\stackrel{e^{\frac{1}{2} \cdot 0}}{=} T\left(e^{i \int_{-\infty}^{\infty} dt V_I(t)} e^{-i \int_{-\infty}^{\infty} dt V_I(t)}\right) \quad (77d)$$

$$= T\left(e^{i \int_{-\infty}^{\infty} dt V_I(t)}\right) T\left(e^{-i \int_{-\infty}^{\infty} dt V_I(t)}\right) \quad (77e)$$

$$= T\left(e^{i \int_{-\infty}^{\infty} dt V_I(t)}\right) \bar{T}\left(e^{-i \int_{-\infty}^{\infty} dt V_I(t)}\right) \quad (77f)$$

$$= S \cdot S^\dagger \quad (77g)$$

$$= S \cdot S^{-1} \quad (77h)$$

$$= 1 \quad (77i)$$

4 Gell-Mann Low formula

To motivate Gell-Mann Low formula as the important asset, a common way of application and requirement will be laid out.

First, the formula allows us to transform a polynomial , chronological (or anti-chronological) ordered set of operators in the Heisenberg picture to the three pictures connected to the scattering operator.

This strikes as a fundamental step for dealing with the pictures. Furthermore this transition will be made very early when working on many topics of quantum field theory. As most of the time, one would begin with classical mechanics to get to QFT. Starting with the action of your problem in term of classical fields and then apply second quantisation to promote them to operators in the Heisenberg picture. This would be the point to transition and one needs the Gell-Mann Low formula.

Recalling Eq. (16) we can write:

$$Q_H(t) = \Omega_I(t)^{-1} Q_I(t) \Omega_I(t). \quad (78)$$

Since the proof didn't require any specifications on Ω as it only remains as the difference to the Heisenberg operator U . Allowing us to write:

$$Q_H(t) = \Omega_{in}(t)^{-1} Q_{in}(t) \Omega_{in}(t). \quad (79)$$

Expressing Ω_{in} by S as seen above:

$$Q_H(t) = \left(\bar{T} \left(e^{i \int_t^\infty dt' V_{in}(t')} \right) S \right)^{-1} Q_{in}(t) T \left(e^{-i \int_{-\infty}^t dt' V_{in}(t')} \right) \quad (80)$$

being a unitary operator, we replace -1 by \dagger and apply hermitian conjugation

$$Q_H(t) = \left(\bar{T} \left(e^{i \int_t^\infty dt' V_{in}(t')} \right) S \right)^\dagger Q_{in}(t) T \left(e^{-i \int_{-\infty}^t dt' V_{in}(t')} \right) \quad (81a)$$

$$= S^\dagger \left(\bar{T} \left(e^{i \int_t^\infty dt' V_{in}(t')} \right) \right)^\dagger Q_{in}(t) T \left(e^{-i \int_{-\infty}^t dt' V_{in}(t')} \right) \quad (81b)$$

$$= S^{-1} T \left(e^{-i \int_t^\infty dt' V_{in}(t')} \right) Q_{in}(t) T \left(e^{-i \int_{-\infty}^t dt' V_{in}(t')} \right). \quad (81c)$$

This step used the unitarity of S itself and the appendix for transition to chronological time ordering. Following that it is allowed to merge T due to zero overlap in the boundaries and correct order

$$Q_H(t) = S^{-1} T \left(e^{-i \int_t^\infty dt' V_{in}(t')} Q_{in}(t) e^{-i \int_{-\infty}^t dt' V_{in}(t')} \right). \quad (82)$$

The position of Q_{in} is still not ideal. Changing it will require commutations. Using the infinite series for e and removing time ordering will reduce the problem to commutations between Q_{in} and V_{in} .

$$Q_H(t) = S^{-1} \left(\sum_n (-i)^n \int_{-\infty}^t dt'_1 \int_t^\infty dt'_1 \dots \int_{-\infty}^{t_{n-1}} dt'_n \int_{t_{n-1}}^\infty dt'_n V_{in}(t'_1) \dots V_{in}(t'_n) Q_{in}(t) \right). \quad (83)$$

We see Q_{in} and V_{in} can commute without any problem since Q_{in} depends on t and $t \neq t'$. Moving it to the left and reapply T as well as e :

$$Q_H(t) = S^{-1} \left(\sum_n (-i)^n Q_{in}(t) \int_{-\infty}^t dt'_1 \int_t^\infty dt'_1 \dots \int_{-\infty}^{t_{n-1}} dt'_n \int_{t_{n-1}}^\infty dt'_n V_{in}(t'_1) \dots V_{in}(t'_n) \right) \quad (84a)$$

$$Q_H(t) = S^{-1} T \left(Q_{in}(t) e^{-i \int_{-\infty}^\infty dt' V_{in}(t')} \right). \quad (84b)$$

Identifying the exponential function as S , the result is:

$$Q_H(t) = S^{-1}T(Q_{in}(t)S). \quad (85)$$

Next will be for more than one operator. Starting with the left side in a non trivial time ordering and applying the transformation for each operator:

$$T(Q_H(t_1)Q_H(t_2)\dots) = T\left(\prod_j \Omega_{in}(t_j)^{-1}Q_{in}(t_j)\Omega_{in}(t_j)\right) \quad (86a)$$

,inserting the expression with S for a Ω depending on t_j ,

$$= T\left(\prod_j S^{-1}T\left(e^{-i\int_{t_j}^{\infty} dt'_j V_{in}(t'_j)}\right)Q_{in}(t_j)T\left(e^{-i\int_{-\infty}^{t_j} dt'_j V_{in}(t'_j)}\right)\right) \quad (86b)$$

,the series and commutation follows the same argumentation as for a single Q ,

$$T(Q_H(t_1)Q_H(t_2)\dots) = T\left(\prod_j S^{-1}Q_{in}(t_j)S\right). \quad (87)$$

Moving S outside of T gives the final form of the Gell-Mann Low formula:

$$T(Q_H(t_1)Q_H(t_2)\dots) = S^{-1}T(Q_{in}(t_1)Q_{in}(t_2)\dots S). \quad (88)$$

To proof S can be outside of chronological time ordering we look at $T(S^{-1})$:

$$T(S^{-1}) = T(S^\dagger) \quad (89a)$$

,using the expression of S in term of the In-picture and apply hermitian conjugation,

$$T(S^{-1}) = T\left(\bar{T}\left(e^{i\int_{-\infty}^{\infty} dt' V_{in}(t')}\right)\right) \quad (89b)$$

, e^x as a series,

$$= T\left(\sum_n \frac{(i)^n}{n!} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \dots \int_{-\infty}^{\infty} dt_n \bar{T}(V_{in}(t_1)\dots V_{in}(t_n))\right) \quad (89c)$$

and removing the anti-chronological time ordering,

$$= T\left(\sum_n \frac{(i)^n}{n!} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \dots \int_{-\infty}^{\infty} dt_n V_{in}(t_1)\dots V_{in}(t_n)\right). \quad (89d)$$

The boundaries make every integral independent from each other so rearranging t_1 to t_n wouldn't change the result

$$\rightarrow T(S^{-1}\dots) = S^{-1}T(\dots)$$

4.1 Vacuum expectation value

The Gell-Mann Low formula Eq. (88) plays a central role in many perturbative or non-perturbative calculations. In particular by evaluating it sandwiched between eigenstates to rising and lowering operators in the In- and Out-picture. If they are in the groundstate of the theory aka. the vacuum state, we obtain the vacuum expectation value.

$$\langle 0_{out} | T(Q_H(t_1)Q_H(t_2) \dots) | 0_{in} \rangle = \quad (90a)$$

$$= \langle 0_{out} | S^{-1} T(Q_{in}(t_1)Q_{in}(t_2) \dots S) | 0_{in} \rangle \quad (90b)$$

$$= \langle 0_{out} | S^{-1} T\left(Q_{in}(t_1)Q_{in}(t_2) \dots e^{-i \int_{-\infty}^{\infty} dt' V_{in}(t')}\right) | 0_{in} \rangle \quad (90c)$$

This average (expectation value) of a product of n operators is called the n -point correlation function. For example, using Wick's theorem and expansions in powers of $V_{in}(Q_{in}(t), t)$, we can gain knowledge about processes that involves in total n excitations of the ground state (or vacuum) in both asymptotic states.

At this point one can be in two situations. The ground state can be either stable or unstable. Over time it could spontaneously go into a state of excitation. Mathematically speaking instability leads to :

$$\langle 0_{out} | 0_{in} \rangle \neq 1. \quad (91)$$

Here of course already normalized states. This possibility for excitations stem from external sources and therefore external currents. A stable state would lead to:

$$\langle 0_{out} | S^{-1} = \langle 0_{in} | \quad (92)$$

An asymptotic state would go over time into a asymptotic state of same excitation.

5 Conclusion

In this report we have presented a set of fundamental concepts and methods. These are essential for starting to study Quantum field theory. We began with well-known pictures from quantum mechanics and introduced the In and Out picture as consequences of working with asymptotic states. They lead us to the important connecting element between the pictures, the scattering operator. The connections allowed us to derive the Gell-Mann Low formula, which plays a major part in the second quantisation process one would do in quantum field theories.

5.A Vacuum transition probability under external current

Still, we need a way to transition from one operator to another and from eigenstate to eigenstate. This link is called the scattering operator S .

$$|out\rangle = S^{-1} |in\rangle \quad (93a)$$

$$|in\rangle = S |out\rangle. \quad (93b)$$

The scalar product of the mentioned two vacua can be written as :

$$\langle 0_{out} | 0_{in} \rangle = \langle 0_{out} | S | 0_{out} \rangle \quad (94)$$

These applications for the operator require S to be of the form

$$\begin{aligned} S &= e^{i\Lambda} \\ S^\dagger &= e^{-i\Lambda^\dagger}, \end{aligned} \quad (95)$$

since $S = S^\dagger$ should hold $\rightarrow \Lambda = -\Lambda^\dagger$. We choose for Λ the chronological time-ordered sum $\sum_{\underline{k}} \int dt \bar{j}_{\underline{k}}(t) \hat{q}_{\underline{k},in}(t)$.

Explicitly,

$$S = T \left[\exp \left(i \sum_{\underline{k}} \int dt \bar{j}_{\underline{k}}(t) \hat{q}_{\underline{k},in}(t) \right) \right] \quad (96a)$$

$$\stackrel{\text{Eq.(31)}}{=} T \left[\exp \left(i \sum_{\underline{k}} \int dt \bar{j}_{\underline{k}}(t) \frac{1}{2\omega_{\underline{k}}} \left(\hat{a}_{\underline{k},in} e^{-i\omega_{\underline{k}}t} + \hat{a}_{\underline{k},in}^\dagger e^{i\omega_{\underline{k}}t} \right) \right) \right], \quad (96b)$$

using the the CBH-formula allows us to separate,

$$\begin{aligned} S = & T \left[\exp \left(i \sum_{\underline{k}} \int dt \bar{j}_{\underline{k}}(t) \frac{1}{2\omega_{\underline{k}}} \left(\hat{a}_{\underline{k},in} e^{-i\omega_{\underline{k}}t} \right) \right) \right. \\ & \cdot \exp \left(i \sum_{\underline{k}} \int dt \bar{j}_{\underline{k}}(t) \frac{1}{2\omega_{\underline{k}}} \left(\hat{a}_{\underline{k},in}^\dagger e^{i\omega_{\underline{k}}t} \right) \right) \\ & \cdot \exp \left(-i \sum_{\underline{k}} \int dt \bar{j}_{\underline{k}}(t) \frac{1}{2 \cdot 2\omega_{\underline{k}}} \left[\hat{a}_{\underline{k},in} e^{-i\omega_{\underline{k}}t}, \hat{a}_{\underline{k},in}^\dagger e^{i\omega_{\underline{k}}t} \right] \right) \left. \right]. \end{aligned} \quad (97)$$

The integral over t will Fourier transform $\bar{j}_{\underline{k}}(t)$ into our relativistic notation,

$$S = \left[\exp \left(i \sum_{\underline{k}} \frac{1}{2\omega_{\underline{k}}} j_{-k\mu} \hat{a}_{\underline{k},in} \right) \cdot \exp \left(i \sum_{\underline{k}} \frac{1}{2\omega_{\underline{k}}} j_{k\mu} \hat{a}_{\underline{k},in}^\dagger \right) \cdot \exp \left(-i \sum_{\underline{k}} \frac{1}{4\omega_{\underline{k}}} |j_{k\mu}|^2 \right) \right]. \quad (98)$$

We dropped the time ordering since the transformation made it time independent.

For now we choose S in a more specific form, where we used $\bar{j}_{\underline{k}}(-\omega_{\underline{k}}) = \bar{j}_{\underline{k}}^*(\omega_{\underline{k}})$. The general one will be obtained later:

$$S = e^{\frac{i}{2\omega_{\underline{k}}} \bar{j}_{\underline{k}}(\omega_{\underline{k}}) \hat{a}_{\underline{k},in}^\dagger} e^{\frac{i}{2\omega_{\underline{k}}} \bar{j}_{\underline{k}}(-\omega_{\underline{k}}) \hat{a}_{\underline{k},in}} e^{-\frac{1}{4\omega_{\underline{k}}} |\bar{j}_{\underline{k}}(-\omega_{\underline{k}})|^2} \quad (99)$$

For the operators we can write:

$$S^{-1} \hat{a}_{\underline{k},in} S = \hat{a}_{\underline{k},out} = \hat{a}_{\underline{k},in} + i \bar{j}_{\underline{k}}(\omega_{\underline{k}}) \quad (100)$$

Using this, we obtain first of all the probability for staying in the ground state for one mode:

$$p_{0,k} = |\langle 0_{out} | 0_{in} \rangle|^2 = |\langle 0_{in} | S | 0_{in} \rangle|^2 \quad (101a)$$

$$= \left| \langle 0_{in} | e^{\frac{i}{2\omega_{\underline{k}}} \bar{j}_{\underline{k}}(\omega_{\underline{k}}) \hat{a}_{\underline{k},in}^\dagger} e^{\frac{i}{2\omega_{\underline{k}}} \bar{j}_{\underline{k}}(-\omega_{\underline{k}}) \hat{a}_{\underline{k},in}} e^{-\frac{1}{4\omega_{\underline{k}}} |\bar{j}_{\underline{k}}(-\omega_{\underline{k}})|^2} | 0_{in} \rangle \right|^2. \quad (101b)$$

The annihilation operator will return a 0 in the exponent. Therefore only one factor remain important. The probability while taking to account any mode transition just involves the integral over all k :

$$p_0 = \exp \left\{ - \int \frac{d^3 k}{(2\pi)^3} \left| \frac{\bar{j}_{\underline{k}}(\omega_{\underline{k}})}{\sqrt{2\omega_{\underline{k}}}} \right|^2 \right\}. \quad (102)$$

This confirms our statement about an unstable vacuum. The negative sign in the exponent translates to smaller probability at higher external currents. So the vacuum can change it's state if a current is > 0 .

5.B Chronological time ordering

Let's begin with at a regular function in form of:

$$I(t) = \int_{-\infty}^t dt_1 \int_{-\infty}^{t_1} dt_2 V(t_1) V(t_2) \quad (103)$$

We see that although the the right hand side has functions of t_2 and t_1 the left hand side doesn't. The integrals allow us to perform a change of variable and rewrite I as $I(t) = \frac{1}{2}I(t) + \frac{1}{2}I(t)$:

$$I(t) = \frac{1}{2} \int_{-\infty}^t dt_1 \int_{-\infty}^{t_1} dt_2 V(t_1) V(t_2) + \frac{1}{2} \int_{-\infty}^t dt_2 \int_{-\infty}^{t_2} dt_1 V(t_2) V(t_1). \quad (104)$$

We define the **chronological time ordering**³ at this point to equalize the boundaries.

Note $V(t_1) \rightarrow V_1$:

$$T(V_1, V_2) = V_1 V_2 \theta(t_1 - t_2) + V_2 V_1 \theta(t_2 - t_1). \quad (105)$$

Being based on the Heaviside step-function it allows us to write the Eq. (103) as:

$$I(t) = \frac{1}{2} \int_{-\infty}^t dt_1 \int_{-\infty}^{t_1} dt_2 V_1 V_2 \theta(t_1 - t_2) + \frac{1}{2} \int_{-\infty}^t dt_1 \int_{-\infty}^{t_1} dt_2 V_2 V_1 \theta(t_2 - t_1) \quad (106)$$

$$I(t) = \frac{1}{2!} \int_{-\infty}^t dt_1 \int_{-\infty}^{t_1} dt_2 T(V_1, V_2) \quad (107)$$

The Heaviside function allows us to set terms to zero for negative arguments of θ and 1 for positive. By subtracting t_1 and t_2 we will be able to switch between the later and earlier points in time.

³earlier times to the right and later times to the left

for $t_1 > t_2 \rightarrow \theta(t_1 - t_2)$ and for $t_2 > t_1 \rightarrow \theta(t_2 - t_1)$

To expand our concept to cases of more then two t we start by advancing T . In general it consists of a summation of all permutations P of a given set multiplied by Heaviside functions. These Heaviside functions have arguments with the negative summation of the t in the same permutation P .

Definition:

$$T(V_1, V_2, \dots, V_n) = \sum_{j=1}^{n!} P_j [V_1, V_2, \dots, V_n] \cdot \theta \left(P_j \left[t_j - \sum_{i \neq j}^n t_i \right] \right) . \quad (108)$$

But this summation over j would not add up to the same value for $I(t)$ since it started in the order of one element in the sum. In this case we need a normalization in addition to T . From statistics we know a set of n different elements can be linear arranged in $n!$ ways. Coming in as a factor of $\frac{1}{n!}$ in the expressions later.

For a test we use $n = 3$

$$T(V_1, V_2, V_3) = \sum_{j=1}^{3!} P_j [V_1, V_2, V_3] \cdot \theta \left(P_j \left[t_j - \sum_{i \neq j}^3 t_i \right] \right) \quad (109a)$$

$$= V_1 V_2 V_3 \cdot \theta(t_1 - t_2 - t_3) \quad (109b)$$

$$+ V_1 V_3 V_2 \cdot \theta(t_1 - t_3 - t_2) \quad (109c)$$

$$+ V_2 V_1 V_3 \cdot \theta(t_2 - t_1 - t_3) \quad (109d)$$

$$+ V_2 V_3 V_1 \cdot \theta(t_2 - t_3 - t_1) \quad (109e)$$

$$+ V_3 V_1 V_2 \cdot \theta(t_3 - t_1 - t_2) \quad (109f)$$

$$+ V_3 V_2 V_1 \cdot \theta(t_3 - t_2 - t_1) \quad (109g)$$

Now we can apply it to new I and do a proof by Induction based on the number of V . The Induction start is the case of two V . In the Induction step we say in a mathematicians way that it works for at least one unspecified higher order. Let's call it k : (Note: $t_0 = t$)

$$I_k(t) = \prod_{a=1}^k \int_{-\infty}^{t_{a-1}} dt_a V_a = \frac{1}{k!} \left(\prod_{a=1}^k \int_{-\infty}^t dt_a \right) T(V_1, \dots, V_k). \quad (110)$$

Moving one increment higher in our 'chain' $k + 1$,

$$I_{k+1}(t) = \prod_{a=1}^{k+1} \int_{-\infty}^{t_{a-1}} dt_a V_a = \prod_{a=1}^k \int_{-\infty}^{t_{a-1}} dt_a V_a \cdot \int_{-\infty}^{t_k} dt_{k+1} V_{k+1} \quad (111)$$

Using the Induction Step and general definition for T :

$$I_{k+1}(t) = \frac{1}{k!} \left(\prod_{a=1}^k \int_{-\infty}^t dt_a \right) T(V_1, \dots, V_k) \cdot \int_{-\infty}^{t_k} dt_{k+1} V_{k+1} \quad (112)$$

$$I_{k+1}(t) = I_k(t) \cdot \int_{-\infty}^{t_k} dt_{k+1} V_{k+1} \quad (113)$$

This shows that the incrementation of k reduces to a multiplication with one more different element for the set. This increases the possible permutations by a factor of $k + 1$ resulting in $(k + 1)!$ in total. Giving us:

$$I_{k+1}(t) = \frac{1}{(k + 1)!} \left(\prod_{a=1}^{k+1} \int_{-\infty}^t dt_a \right) T(V_1, \dots, V_{k+1}) \quad (114)$$

Time ordering is a very grounded concept in field theory, it appears very naturally for expressing propagators in term of fields. The reason apart from pure mathematics is comprehensible. The different functions, functionals or fields are best organized for summarizing a scattering or interaction event if they are time-like sorted.

Furthermore it comes with a great advantage. All products in $T(\dots)$ do commute. Proof for two elements:

$$T(V_1, V_2) = V_1 V_2 \theta(t_1 - t_2) + V_2 V_1 \theta(t_2 - t_1) \quad (115a)$$

$$T(V_2, V_1) = V_2 V_1 \theta(t_2 - t_1) + V_1 V_2 \theta(t_1 - t_2) \quad (115b)$$

,since terms in sums always commutes,

$$T(V_1, V_2) = T(V_2, V_1) \quad (116)$$

In other words, the commutation relations say whether the subtraction of permutations of elements is zero or not. But in time ordering all permutations appear, we can rearrange the terms so subtraction of equal permutations happens. Therefore commutation in T holds.

5.C Anti-chronological time ordering

On a close inspection of the introduction of the Heaviside-function and its' insertion into the integral, we actually skipped a choice. If we just would have wanted to the overall boundaries the order of t_1 and t_2 in relation to V_1 and V_2 the insertion of θ could have been switched. This secretly led us to the definition of time-ordering or chronological time ordering as it is called more precisely. The other path would have resulted in anti-chronological time ordering:⁴

$$\bar{T}(V_1, V_2) = V_1 V_2 \theta(t_2 - t_1) + V_2 V_1 \theta(t_1 - t_2) \quad (117)$$

This section is not just to satisfy the observed readers but to proof a connection between both involving hermitian conjugation of the integrals, which can easily appear while doing picture transitions.

We choose the same starting point as for T :

$$\begin{aligned} I(t) &= \frac{1}{2} \int_{-\infty}^t dt_1 \int_{-\infty}^{t_1} dt_2 V_1 V_2 + \frac{1}{2} \int_{-\infty}^t dt_2 \int_{-\infty}^{t_2} dt_1 V_2 V_1 \\ &= \frac{1}{2!} \int_{-\infty}^t dt_1 \int_{-\infty}^{t_1} dt_2 T(V_1, V_2) \end{aligned}$$

$$I(t)^\dagger = \left(\frac{1}{2} \int_{-\infty}^t dt_1 \int_{-\infty}^{t_1} dt_2 V_1 V_2 + \frac{1}{2} \int_{-\infty}^t dt_2 \int_{-\infty}^{t_2} dt_1 V_2 V_1 \right)^\dagger \quad (118a)$$

$$= \frac{1}{2} \int_{-\infty}^t dt_1 \int_{-\infty}^{t_1} dt_2 (V_1 V_2)^\dagger + \frac{1}{2} \int_{-\infty}^t dt_2 \int_{-\infty}^{t_2} dt_1 (V_2 V_1)^\dagger \quad (118b)$$

$$= \frac{1}{2} \int_{-\infty}^t dt_1 \int_{-\infty}^{t_1} dt_2 V_2^\dagger V_1^\dagger + \frac{1}{2} \int_{-\infty}^t dt_2 \int_{-\infty}^{t_2} dt_1 V_1^\dagger V_2^\dagger \quad (118c)$$

, we observe that the switch of positions due to hermitian conjugation allows to use Eq. (117)

$$I(t)^\dagger = \frac{1}{2} \int_{-\infty}^t dt_1 \int_{-\infty}^{t_1} dt_2 V_2^\dagger V_1^\dagger \theta(t_1 - t_2) + \frac{1}{2} \int_{-\infty}^t dt_1 \int_{-\infty}^{t_1} dt_2 V_1^\dagger V_2^\dagger \theta(t_2 - t_1) \quad (118d)$$

$$\Rightarrow \frac{1}{2} \int_{-\infty}^t dt_1 \int_{-\infty}^{t_1} dt_2 \bar{T}(V_1^\dagger, V_2^\dagger) \quad (118e)$$

⁴earlier times to the left and later times to the right

Now we assume our total Hamiltonian $\hat{H} = \hat{H}_0 + \hat{V}$ is hermitian and \hat{V} won't lead to non fixed ground state energy \rightarrow vacuum instability

$$I(t)^\dagger = \frac{1}{2} \int_{-\infty}^t dt_1 \int_{-\infty}^t dt_2 \bar{T}(V_1, V_2) \quad (118f)$$

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