Specialization report

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1 Introduction

Quantum field theory is a set of methods and concepts that allow us to describe elementary particle processes. Of great interest are derived quantities like cross-sections σ or decay rates Γ as most common comparisons to experiments. From the side of theoretical physics, they can be extracted by calculating the amplitudes of the related correlation functions. Such calculations require pertubative approaches. However the starting points for setting up correlations functions like a Lagrangian, operators and states are described and formulated in the way of the Heisenberg picture without any parts that allow for pertubation theory. The main goal of this report is the derivation of the solution to this dilemma. The Gell-Mann Low formula. It allows transition to pictures inside the correlation functions without loss of information by including the scattering operator S. This S allows to be calculated pertubatively and therefore also the actual correlation functions. We will working out a few handy techniques and methods on the way to Gell-Mann Low formula. This includes discussing the more common Schrödinger, Heisenberg and Interaction picture. Followed up by In and *out* pictures which Gell-Mann Low formula allows easy transition inside the corr. function. These are based on boundary conditions imposing free motion on particles along them. We derive them from experimental and physical point of views.

2 Pictures in Quantum Mechanics

2.1 Schrödinger picture and Heisenberg picture

The choice of a picture always requires to establish the states but also the corresponding operators. In the Schrödinger picture the operators are time-independent but the wavefunctions are time dependent. The time evolution of a state vector is controlled by the Schrödinger equation. Let $|\Psi(t)\rangle$ denote a state vector at time t. It satisfies

$$i\hbar\frac{\partial}{\partial t}\left|\Psi_S(t)\right\rangle = \hat{H}\left|\Psi_S(t)\right\rangle, \tag{1}$$

where $\hat{H}$ is the Hamiltonian of the system. When assuming it time independent, the solution of Eq. (1) can be formally written as

$$\left|\Psi_S(t)\right\rangle = \hat{U}(t-t_0)\left|\Psi_S(t_0)\right\rangle \tag{2}$$

with $\hat{U}(t-t_0) = e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}$ the time evolution operator, which satisfies the differential equation

$$i\hbar\partial_t \hat{U}(t-t_0) = \hat{H}\hat{U}(t-t_0). \tag{3}$$

Under the general assumption of the Hamiltonian being hermitian $\hat{U}(t-t_0)$ is also an unitary operator, meaning:

$$\begin{aligned}&\hat{U}(t-t_0) \times \hat{U}^{\dagger}(t-t_0) = e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}e^{\frac{i}{\hbar}\hat{H}(t-t_0)} = 1\\&= \hat{U}(t-t_0) \times \hat{U}^{-1}(t-t_0)\end{aligned} \tag{4}$$

Going back to Eq. (2) we see $|\Psi_S(t_0)\rangle$ is a ket of $t = t_0$. We shall generally take $t_0 = 0$ and write

$$|\Psi_S(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t} |\Psi_H\rangle \,. \tag{5}$$

The state on the right-hand side has no longer time dependence. This defines the state in the Heisenberg picture.

The above two pictures differ between each other in the way of storing the time dependence. In the Schrödinger picture only the states carry such a dependence, whereas in the Heisenberg picture only operators has this possibility. To verify this statement we study the matrix element of an operator in the Schrödinger picture

$$\langle \Psi'_S(t)| \hat{A}_S |\Psi_S(t)\rangle = \langle \Psi'_H| e^{\frac{i}{\hbar}t\hat{H}} \hat{A}_S e^{-\frac{i}{\hbar}t\hat{H}} |\Psi_H\rangle \,, \tag{6}$$

where Eq. (5) has been used. As a consequence,

$$\hat{A}_H(t) = e^{\frac{i}{\hbar}t\hat{H}} \hat{A}_S e^{-\frac{i}{\hbar}t\hat{H}} = \hat{U}(t)^{-1}\hat{A}_S\hat{U}(t). \tag{7}$$

This new operator $\hat{A}_H(t)$ in combination with the state $|\Psi_H\rangle$ defines the Heisenberg picture. Observe that the time evolution of $\hat{A}_H(t)$ is dictated by an equation that follows from differentiating the equation above with respect to t:

$$\frac{d}{dt}\hat{A}_H(t) = \frac{i}{\hbar}\hat{H}e^{\frac{i}{\hbar}t\hat{H}}\hat{A}_S e^{-\frac{i}{\hbar}t\hat{H}} + e^{\frac{i}{\hbar}t\hat{H}}\hat{A}_S\left(-\frac{i}{\hbar}\hat{H}\right)e^{-\frac{i}{\hbar}t\hat{H}}. \tag{8}$$

Here we have used the time evolution equation (3). Hence,

$$\begin{aligned}\frac{d}{dt}\hat{A}_H(t) &= \frac{i}{\hbar}\hat{U}(t)^{-1}\hat{H}\hat{A}_S\hat{U}(t) - \frac{i}{\hbar}\hat{U}(t)^{-1}\hat{A}_S\hat{H}\hat{U}(t)\\&= \frac{i}{\hbar}\hat{U}(t)^{-1}\hat{H}\underbrace{\hat{U}(t)\hat{U}(t)^{-1}}_{=1}\hat{A}_S\hat{U}(t) - \frac{i}{\hbar}\hat{U}(t)^{-1}\hat{A}_S\underbrace{\hat{U}(t)\hat{U}(t)^{-1}}_{=1}\hat{H}\hat{U}(t).\end{aligned} \tag{9}$$

The inserted 1 allows us to express Eq. (9) in term of operators in the Heisenberg picture.

$$\frac{d}{dt}\hat{A}_H(t) = \frac{i}{\hbar}\hat{H}_H(t)\hat{A}_H(t) - \frac{i}{\hbar}\hat{A}_H(t)\hat{H}_H(t), \tag{10}$$

where $\hat{H}_H(t)$ is the respective Hamiltonian in the Heisenberg picture. Therefore:

$$i\frac{d}{dt}\hat{A}_H(t) = \frac{1}{\hbar}\left[\hat{A}_H(t), \hat{H}_H(t)\right]. \tag{11}$$

2.2 Interaction picture

A third picture can be introduced: the Interaction picture (sometimes called the Dirac picture). We will see very shortly that, in the Interacting picture both the states and the respective operators are time dependent. Let us suppose that the Hamiltonian in the Schrödinger picture can be splitted as follows $\hat{H} = \hat{H}_0 + \hat{V}$. Normally $\hat{H}_0$ describe the free motion of a system, whereas $\hat{V}$ represents its interaction, which could be with an external source. Although it often used in a perturbative approach, the Interaction picture does not require $\hat{V}$ to be small as compared with $\hat{H}_0$. Inserting this decomposition of $\hat{H}$ in the unitary operator introduced below Eq. (2):

$$\hat{U}(t) = e^{-\frac{i}{\hbar}t\hat{H}} = e^{-\frac{i}{\hbar}t\left(\hat{H}_0+V\right)} = e^{-\frac{i}{\hbar}t\hat{H}_0}\hat{\Omega}_I(t) \tag{12}$$

This expression helps us to establish a formula from which operators and states in the interaction picture can be defined[1]. For this, consider a matrix element $\langle\Psi'_S(t)|\,\hat{A}_S\,|\Psi_S(t)\rangle$. Taking into account Eq. (5) and (12) we find

$$\langle\Psi'_S(t)|\,\hat{A}_S\,|\Psi_S(t)\rangle = \langle\Psi'_H|\,(e^{-\frac{i}{\hbar}t\hat{H}_0}\Omega_I(t))^\dagger\hat{A}_S e^{-\frac{i}{\hbar}t\hat{H}_0}\Omega_I(t)\,|\Psi_H\rangle \tag{13a}$$

$$= \langle\Psi'_H|\,\Omega_I(t)^{-1}\hat{A}_I(t)\Omega_I(t)\,|\Psi_H\rangle\,. \tag{13b}$$

Here the operator in the interaction picture reads

$$\hat{A}_I(t) = e^{+\frac{i}{\hbar}t\hat{H}_0}\hat{A}_S e^{-\frac{i}{\hbar}t\hat{H}_0}, \tag{14}$$

whereas a corresponding state in this picture is

$$|\Psi_I(t)\rangle = \Omega_I(t)\,|\Psi_H\rangle\,. \tag{15}$$

[1]By substituting Eq. (12) in Eq. (4) one can see directly that $\Omega_I(t)$ is also unitary

At the level of operators, the connection between the Interaction and the Heisenberg picture is established by inverting Eq. (7) and inserting the resulting $\hat{A}_S$ into Eq. (14). This leads to

$$\hat{A}_I(t) = e^{+\frac{i}{\hbar}t\hat{H}_0}\hat{U}(t)\hat{A}_H(t)\hat{U}(t)^{-1}e^{-\frac{i}{\hbar}t\hat{H}_0}. \tag{16a}$$

$$= e^{+\frac{i}{\hbar}t\hat{H}_0}e^{-\frac{i}{\hbar}t\hat{H}}\hat{A}_H(t)e^{\frac{i}{\hbar}t\hat{H}}e^{-\frac{i}{\hbar}t\hat{H}_0}, \tag{16b}$$

ending with

$$\hat{A}_I(t) = \hat{\Omega}_I(t)\hat{A}_H(t)\hat{\Omega}_I(t)^{-1}. \tag{17}$$

The time evolution equation for $\hat{A}_I(t)$ can be found as done for $\hat{A}_H(t)$ [see below Eq. (7)]:

$$i\hbar\frac{\partial}{\partial t}\hat{A}_I = \left[\hat{A}_I, \hat{H}_0\right]. \tag{18}$$

Furthermore, an equation for $\hat{\Omega}_I(t)$ can be determined. To this end we invert Eq. (12) and express $\hat{\Omega}_I(t) = e^{\frac{i}{\hbar}t\hat{H}_0}\hat{U}(t)$. Afterwards we differentiate with respect to times:

$$i\hbar\partial_t\hat{\Omega}_I(t) = e^{\frac{i}{\hbar}t\hat{H}_0}\left(i\hbar\partial_t\hat{U}(t)\right) - \hat{H}_0e^{\frac{i}{\hbar}t\hat{H}_0}\hat{U}(t) \tag{19a}$$

$$= \hat{H}e^{\frac{i}{\hbar}t\hat{H}_0}\hat{U}(t) - \hat{H}_0e^{\frac{i}{\hbar}t\hat{H}_0}\hat{U}(t), \tag{19b}$$

where Eq. (3) has been used. Using the definition of $\hat{\Omega}_I(t)$ we end up with

$$i\hbar\frac{\partial}{\partial t}\hat{\Omega}_I(t) = \hat{V}_I(t)\hat{\Omega}_I(t). \tag{20}$$

To find a well defined solution, an initial condition is needed. In Eq. (12) we see that at $t = 0$, the time evolution operator reduces to $U(0) = 1$, and from this the following condition $\Omega_I(0) = 1$ arises. We remark that $V_I(t)$ in the Interaction picture as introduced above does not require V to be of any specific form but can still be applied in presence of external sources.

2.3 The *in* and *out* picture: External currents

Consider the set-up of most experiments in elementary particle and nuclear physics. Several particles approach each other from a macroscopic scale and interact in a microscopic section comparable to the Compton wavelength of the incoming particles. On this scale vacuum fluctuations are no longer negligible for the dynamic of the involved particles and make them impossible to distinguish between each other. As a result, the products of the interaction spread up to a macroscopic distances and the distinguishability between outgoing particles is admitted. Therefore, at such asymptotically distances, the description of the incoming and outgoing multi-particle states can be approached by direct products of single-particle effectively non-interacting states.

To bring this concept into our formulation let's consider the action of a scalar field Φ with mass $m = m_0 c/\hbar$ coupled to an external source $j(\underline{x}, t)$[2]:

$$I = \int d^4x \; \mathcal{L}(\Phi, \dot{\Phi}, j) = \int d^4x \left(\frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} m^2 \Phi^2 - \Phi j \right). \tag{21}$$

Taking the functional derivative with respect to Φ and setting it to zero, we obtain the equation of motion

$$\left(\partial^2 + m^2\right) \Phi = j. \tag{22}$$

To proceed, we quantize our field in a box of volume V and length L. The classical field and its canonical momentum $\Pi = \partial \mathcal{L}/\partial \dot{\Phi}(\underline{x}, t) = \dot{\Phi}(\underline{x}, t)$ are then promoted to operators $\hat{\Phi}(\underline{x}, t)$ and $\hat{\Pi}(\underline{x}, t)$ in the Heisenberg picture. Satisfying the equal-time commutator:

$$\left[\hat{\Phi}(\underline{x}, t), \hat{\Pi}(\underline{x}', t)\right] = i\delta^3(\underline{x} - \underline{x}'). \tag{23}$$

We then expand the field operator as follows:

$$\hat{\Phi}(\underline{x}, t) = \sum_{\underline{k}} \hat{q}_{\underline{k}}(t) u_{\underline{k}}(\underline{x}). \tag{24}$$

The 3 dim. wave vector $\underline{k}$ for the modes is represented by $\underline{k} = \frac{2\pi}{L}(n_x, n_y, n_z)$ with $n_i \in \mathbb{Z}$. In this separated time and space dependency, we choose the Fourier basis for $u_{\underline{k}}(\underline{x})$

$$u_{\underline{k}}(\underline{x}) = \frac{1}{L^{3/2}} e^{i\underline{k}\cdot\underline{x}}, \tag{25}$$

where the volume L^3 provides the required normalization. We remark that $u_{\underline{k}}(\underline{x})$ constitutes an orthonormalized basis in the Hilbert space

$$\int d^3x \; u^*_{\underline{k}'}(\underline{x}) u_{\underline{k}}(\underline{x}) = \delta_{\underline{k},\underline{k}'} \tag{26}$$

[2]From now on we will work in natural units and set $c = \hbar = 1$

$$\sum_{\underline{k}} u^*_{\underline{k}}(\underline{x}) u_{\underline{k}}(\underline{x}') = \delta^3\left(\underline{x} - \underline{x}'\right). \tag{27}$$

We now substitute Eq. (25) into the equation of motion (22) . As a consequence

$$\sum_{\underline{k}} \left[\ddot{\hat{q}}_{\underline{k}}(t) u_{\underline{k}}(\underline{x}) + \underline{k}^2 \hat{q}_{\underline{k}}(t) u_{\underline{k}}(\underline{x}) + m^2 \hat{q}_{\underline{k}}(t) u_{\underline{k}}(\underline{x})\right] = j(\underline{x}, t). \tag{28}$$

To get an equation for $\hat{q}_{\underline{k}}(t)$ alone we need to get rid of $u_{\underline{k}}(\underline{x})$ and remove the space dependence in the current. Multiplying with $u^*_{\underline{k}'}(\underline{x})$ and integrating over the whole space we find,

$$\sum_{\underline{k}} \left[\int d^3x \; u^*_{\underline{k}'} u_{\underline{k}} \left(\ddot{\hat{q}}_{\underline{k}}(t) + \left(\underline{k}^2 + m^2\right) \hat{q}_{\underline{k}}(t)\right)\right] = \underbrace{\int d^3x \; j(\underline{x}, t) \frac{1}{\sqrt{V}} e^{i\underline{k}\cdot\underline{x}}}_{=\tilde{j}(\underline{k}, t)}. \tag{29}$$

After using the orthonormality relation (26) this expression reduces to

$$\ddot{\hat{q}}_{\underline{k}}(t) + \omega^2_{\underline{k}} \hat{q}_{\underline{k}}(t) = \tilde{j}(\underline{k}, t), \tag{30}$$

where $\omega^2_{\underline{k}} = (\underline{k}^2 + m^2)$ is the energy of the particle in mode $\underline{k}$.

We now make the assumption that the current vanishes outside a finite time interval,

$$j(\underline{k}, t) \to 0 \text{ for } t \to \pm\infty. \tag{31}$$

As a consequence one can distinguish between early and late times. For early time Eq. (30) approaches the homogeneous differential equation. We will call its asymptotic solution by $\hat{q}_{\underline{k}}(t) \to \hat{q}_{k,in}(t)$. Explicitly,

$$\hat{q}_{\underline{k},in}(t) \approx \frac{1}{2\omega_{\underline{k}}} \left(\hat{a}_{\underline{k},in} e^{-i\omega_{\underline{k}} t} + \hat{a}^\dagger_{\underline{k},in} e^{i\omega_{\underline{k}} t}\right), \quad t \to -\infty, \tag{32}$$

where $\hat{a}_{\underline{k},in}$ denotes the annihilation operator, whereas $\hat{a}^\dagger_{\underline{k},in}$ is the corresponding creation operator. Their commutator is

$$\left[\hat{a}_{\underline{k},in}, \hat{a}^\dagger_{\underline{k}',in}\right] = \delta_{\underline{k},\underline{k}'}. \tag{33}$$

At late times Eq. (30) also reduces to a homogeneous type. In this case the asymptotic solution $\hat{q}_{\underline{k}}(t) \to \hat{q}_{k,out}(t)$ reads

$$\hat{q}_{\underline{k},out}(t) \approx \frac{1}{2\omega_{\underline{k}}} \left(\hat{a}_{\underline{k},out} e^{-i\omega_{\underline{k}} t} + \hat{a}^\dagger_{\underline{k},out} e^{i\omega_{\underline{k}} t}\right), \quad t \to +\infty. \tag{34}$$

The solution for $\hat{q}_{\underline{k}}(t)$, at times for which $j(\underline{x},t)$ is active, would then consist of the homogeneous solution plus a term containing the current:

$$\hat{q}_{\underline{k}}(t) = \hat{q}_{\underline{k},in}(t) + \frac{1}{\omega_{\underline{k}}} \int_{-\infty}^{t} dt' \sin\left[\omega_{\underline{k}}(t-t')\right] \tilde{j}(\underline{k},t'), \tag{35}$$

where $\bar{j}_{\underline{k}}(\omega_{\underline{k}}) = \displaystyle\int_{-\infty}^{\infty} dt \tilde{j}(\underline{k},t) e^{i\omega_{\underline{k}}t}$ is the temporal Fourier transform of the current. For late times $t \to +\infty$ the expression above approaches to

$$\hat{q}_{\underline{k},out}(t) \approx \hat{q}_{\underline{k},in}(t) + \frac{1}{\omega_{\underline{k}}} \int_{-\infty}^{\infty} dt' \sin\left[\omega_{\underline{k}}(t-t')\right] \tilde{j}(\underline{k},t'). \tag{36}$$

After splitting the sinus function, we find

$$\hat{q}_{\underline{k},out}(t) = \hat{q}_{\underline{k},in}(t) - \frac{i}{2\omega_{\underline{k}}} e^{i\omega_{\underline{k}}t}\, \bar{j}_{\underline{k}}(-\omega_{\underline{k}}) + \frac{i}{2\omega_{\underline{k}}} e^{-i\omega_{\underline{k}}t}\, \bar{j}_{\underline{k}}(\omega_{\underline{k}}), \tag{37}$$

From this equation we can obtain the connection between creation and annihilation operators associated with the asymptotically far fields $t \to \pm\infty$. In compact notation

$$\hat{a}_{\underline{k},out} = \hat{a}_{\underline{k},in} + i\bar{j}_{\underline{k}}(\omega_{\underline{k}}), \tag{38a}$$

$$\hat{a}^{\dagger}_{\underline{k},out} = \hat{a}^{\dagger}_{\underline{k},in} - i\bar{j}_{\underline{k}}(-\omega_{\underline{k}}). \tag{38b}$$

This shows that, in the presence of an external current, the two sets of second quantization operators are not the same. Therefore we need to differ between the corresponding *in* and *out* eigenstates. Particularly, it has to be stated that the vacua also differ in this scenario.

It is important to stress, that the full solution $\hat{q}_{\underline{k}}(t)$ found in Eq. (35) has to be understood in the Heisenberg picture. From this we can proceed as shown in section **2.2**. We split the Hamiltonian as done there: $\hat{H} = \hat{H}_0 + \hat{V}$.

$$\hat{H}_0(\Phi,\Pi) = \int d^3x \left[\frac{1}{2}\hat{\Pi}^2 + \frac{1}{2}(\nabla\hat{\Phi})^2 + \frac{1}{2}m^2\hat{\Phi}^2\right], \tag{39}$$

$$\hat{V}(\Phi) = \int d^3x\, j\hat{\Phi}. \tag{40}$$

Expressing both field operators in terms of the Fourier basis given in (24), and using the orthonormality relation Eq. (26), as well as the reality condition of the field for $\hat{q}_{-\underline{k}}(t) = \hat{q}^{\star}_{\underline{k}}(t)$ we can express the Hamiltonian as follows:

$$\hat{H}_0(q, \dot{q}) = \sum_{\underline{k}} \left\{ \frac{1}{2} \dot{\hat{q}}^2_{\underline{k}}(t) + \frac{1}{2} \omega^2_{\underline{k}} \hat{q}^2_{\underline{k}}(t) \right\}, \tag{41}$$

$$\hat{V}(q) = \sum_{\underline{k}} \tilde{j}(\underline{k}, t) \hat{q}_{\underline{k}}(t). \tag{42}$$

From this form we go to the Interaction picture. In the present context, the potential V_I appearing in Eq. (20) reads:

$$\hat{V}_I(q_I) = \sum_{\underline{k}} \tilde{j}(\underline{k}, t) \hat{q}_{\underline{k}_I}(t), \tag{43}$$

where we used Eq. (17) to transform $\hat{q}_{\underline{k}}(t)$ into the Interaction picture

$$\hat{q}_{\underline{k}_I}(t) = \hat{\Omega}_I(t) \hat{q}_{\underline{k}}(t) \hat{\Omega}_I^{-1}(t). \tag{44}$$

To have a well defined operator $\hat{\Omega}_I(t)$ we need conditions for any Ω so that $\Omega \to 1$ as stated for the Interaction picture in Eq. (20) which is at the moment mostly depended on the current j. The early time condition at $t = -\infty$ defines the *in* picture in reminiscence to the first asymptotic solution given in Eq (32) and it writes:

$$i \frac{\partial}{\partial t} \hat{\Omega}_{in}(t) = \hat{V}_{in}(t) \hat{\Omega}_{in}(t), \tag{45}$$

where the initial condition $\hat{\Omega}_{in}(-\infty) = 1$ has to be fulfilled. Contrary to the previous case the operator of the *out* picture will satisfy the differential equation:

$$i \frac{\partial}{\partial t} \hat{\Omega}_{out}(t) = \hat{V}_{out}(t) \hat{\Omega}_{out}(t), \tag{46}$$

with $\hat{\Omega}_{out}(+\infty) = 1$.

3 Scattering operator

3.1 Solutions for the Interaction, *in* and *out* picture

In this section we solve the differential equations for the various pictures established in section 2.2 and 2.3. We start with the Interaction picture depended on t' and integrate both sides of Eq. (20). For $t > 0$ its left-hand side gives:

$$\int_0^t \mathrm{d}t' i \frac{\partial}{\partial_{t'}} \hat{\hat{\Omega}}_I(t') = i\left[\hat{\Omega}_I(t) - 1\right], \tag{47}$$

where the initial condition $\hat{\Omega}_I(0) = 1$ has been used. With this formula and the integral over the right-hand side of (20), we find an expression for $\hat{\Omega}_I(t)$.

$$\hat{\Omega}_I(t) = 1 - i\int_0^t \mathrm{d}t' \hat{V}_I(t')\hat{\Omega}_I(t'), \tag{48}$$

since the expression has an $\hat{\Omega}_I(t)$ on the other side we will go on by an iterative approach.

$$\begin{aligned}\hat{\Omega}_I(t) &= 1 - i\int_0^t \mathrm{d}t' \hat{V}_I(t') \cdot \left(1 - i\int_0^{t'} \mathrm{d}t'' \hat{V}_I(t'')\hat{\Omega}_I(t'')\right) \\ &= 1 - i\int_0^t \mathrm{d}t' \hat{V}_I(t') + i^2\int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \hat{V}_I(t'')\hat{\Omega}_I(t'').\end{aligned} \tag{49}$$

The iteration increments the power of i and the number of integrals. By repeating the operation described above we can write

$$\hat{\Omega}_I(t) = \sum_{n=0}^{\infty} (-i)^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \ldots \int_0^{t_{n-1}} \mathrm{d}t_n \hat{V}_I(t_1) \cdot \ldots \cdot \hat{V}_I(t_n). \tag{50}$$

A problematic aspect of this series are the different integral limits. Each term introduces a new t_i and keeps the previous t_{i-1} as an integral variable which forces us to solve them in a strict order. To circumvent this formal aspect we will perform some additional operations. Let us consider the term from Eq. (50) containing the product of two interactions :

$$I(t) = \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \hat{V}_I(t_1)\hat{V}(t_2). \tag{51}$$

By developing the change of variable $t_2 \longleftrightarrow t_1$,[3] this integral can be written as

$$I(t) = \int_0^t \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_1 \hat{V}_I(t_2)\hat{V}_I(t_1). \tag{52}$$

[3]The Jacobian of this change of variable is the unity

We find an alternative representation of $I(t)$ by adding (51) and (52):

$$I(t) = \frac{1}{2}\int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \hat{V}_I(t_1)\hat{V}_I(t_2) + \frac{1}{2}\int_0^t \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_1 \hat{V}_I(t_2)\hat{V}_I(t_1). \tag{53}$$

In order to have a common integration limit t, we introduce the chronological time ordering.

$$T(\hat{V}_I(t_1), \hat{V}_I(t_2)) = \hat{V}_I(t_1)\ \hat{V}_I(t_2)\ \theta(t_1 - t_2)\ +\ \hat{V}_I(t_2)\ \hat{V}_I(t_1)\ \theta(t_2 - t_1). \tag{54}$$

The chronological time ordering sets operators depending of earlier times to the right and later to the left. The the Heaviside-Step-function is 0 for negative values of its argument and 1 when it becomes positive. By subtracting t_1 and t_2 in the argument of the step functions we are able to switch between the two terms in Eq. (53) and extending the integral limits to t, since it sets terms to zero for negative arguments. Therefore no change appears in the result of the integral by extending the limit. We used for $t_1 > t_2 \to \theta(t_1 - t_2)$ and for $t_2 > t_1 \to \theta(t_2 - t_1)$. By applying Eq. (54) at Eq. (53), we find the desired notation:

$$I(t) = \frac{1}{2}\int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_2 \hat{V}_1\ \hat{V}_2\ \theta(t_1 - t_2) + \frac{1}{2}\int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_2\ \hat{V}_2\ \hat{V}_1\ \theta(t_2 - t_1), \tag{55}$$

$$I(t) = \frac{1}{2!}\int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_2 T(\hat{V}_1, \hat{V}_2). \tag{56}$$

This case of two interactions is generalized to terms involving $\hat{V}$ n-times in the appendix 5.A. By labelling t using numbers instead of primes, the solution to $\hat{\Omega}_I(t)$ given in Eq. (50) can be written in the time-ordered form:

$$\hat{\Omega}_I(t) = \frac{(-i)^n}{n!}\int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_2 \ldots \int_0^t \mathrm{d}t_n T\left\{\hat{V}_I(t_1), \ldots, \hat{V}_I(t_n)\right\}. \tag{57}$$

Observe that this expression is a non-pertubative result, which can be written as:

$$\hat{\Omega}_I(t) = T\left(e^{-i\int_0^t \mathrm{d}t' \hat{V}_I(t')}\right), \quad \text{for } t \geq 0. \tag{58}$$

To not limit the Interaction picture only to positive t values, we need to a complementary expression for only negative t. Assuming $t < 0$. This change the integral in Eq. (47) to:

$$\int_t^0 \mathrm{d}t' i \frac{\partial}{\partial_{t'}} \hat{\Omega}_I(t') = i\left[1 - \hat{\Omega}_I(t)\right]. \tag{59}$$

Alongside performing in the integral of the right hand side of Eq. (20) in the new limits, we find:

$$\hat{\Omega}_I(t) = 1 + i\int_t^0 \mathrm{d}t' \hat{V}_I(t')\hat{\Omega}_I(t'). \tag{60}$$

From this our infinite sum expression still holds up to an different sign:

$$\hat{\Omega}_I(t) = \sum_{n=0}^{\infty} i^n \int_t^0 \mathrm{d}t_1 \int_{t_1}^0 \mathrm{d}t_2 \ldots \int_{tn-1}^0 \mathrm{d}t_n \hat{V}_I(t_1) \cdot \ldots \cdot \hat{V}_I(t_n). \tag{61}$$

The key difference now stands in the negativity of all t and a logical order for them would prefer later times to the right, coming closer to 0. This requires the anti-chronological time ordering:

$$\bar{T}(\hat{V}(t_1), \hat{V}(t_2)) = \hat{V}(t_2)\ \hat{V}(t_1)\ \theta(t_1 - t_2)\ +\ \hat{V}(t_1)\ \hat{V}(t_2)\ \theta(t_2 - t_1). \tag{62}$$

A generalized expression containing n $\hat{V}$ is given in appendix 5.B. Using it similar as before:

$$\hat{\Omega}_I(t) = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int_t^0 \mathrm{d}t_1 \int_t^0 \mathrm{d}t_2 \ldots \int_t^0 \mathrm{d}t_n \bar{T}\left\{\hat{V}_I(t_1), \ldots, \hat{V}_I(t_n)\right\}. \tag{63}$$

As a consequence We arrive at the second expression:

$$\hat{\Omega}_I(t) = \bar{T}\left(e^{i\int_t^0 \mathrm{d}t' \hat{V}_I(t')}\right), \text{ for } t < 0. \tag{64}$$

Often the integration borders are flipped to have a the same sign as Eq. (58)

$$\hat{\Omega}_I(t) = \bar{T}\left(e^{-i\int_0^t \mathrm{d}t' \hat{V}_I(t')}\right), \text{ for } t < 0. \tag{65}$$

A notation for $\hat{\Omega}_I(t)$ without specifying the values of t can be derived again using Heaviside-Step-functions:

$$\hat{\Omega}_I(t) = T\left(e^{-i\int_0^t \mathrm{d}t' \hat{V}_I(t')}\right)\theta(t) + \bar{T}\left(e^{-i\int_0^t \mathrm{d}t' \hat{V}_I(t')}\right)\theta(-t). \tag{66}$$

For the *in* picture we proceed in an almost identical fashion to the Interaction picture for $t > 0$. Only the lower boundary in the integral is changed to $-\infty$ as it is the asymptotic condition of this picture. This resolves the need for a two term solution. After resumation, we obtain:

$$\hat{\Omega}_{in}(t) = T\left(e^{-i\int_{-\infty}^{t} \mathrm{d}t' \hat{V}_{in}(t')}\right). \tag{67}$$

The *out* picture on the other hand follows the derivation of the expression for $t < 0$. We start at Eq. (59) with ∞ instate of 0. Here we argue t being smaller then ∞ needs one change of sign like before and anti-chronological ordering $\bar{T}$ introduced in Eq. (62), since t only coming closer to the limit as it runs.

$$\hat{\Omega}_{out}(t) = \sum_{n=0}^{\infty} \frac{(i)^n}{n!} \int_t^{\infty} \mathrm{d}t_1 \int_t^{\infty} \mathrm{d}t_2 \dots \int_t^{\infty} \mathrm{d}t_n \bar{T}\left\{\hat{V}_{out}(t_1), \dots, \hat{V}_{out}(t_n)\right\}. \tag{68}$$

Performing a flip in the integral limits we conclude:

$$\hat{\Omega}_{out}(t) = \bar{T}\left(e^{-i\int_{\infty}^{t} \mathrm{d}t' \hat{V}_{in}(t')}\right). \tag{69}$$

3.2 Connections

The necessity for further developing connections between the *in* and *out* picture can be stressed in different ways. Let us take a look at general eigenstates associated to these pictures, $|in\rangle$ and $|out\rangle$. In section 2.3 we saw that the creation and annihilation operators are linked to these pictures and each other in Eq. (38) via an external current. Extending this to the eigenstates we need a new piece in our repertoire. We start with a state in the Heisenberg picture and do a transformation to the *in* picture following the same way as Eq. (15) for the Interaction picture.

$$|in\rangle = \hat{\Omega}_{in}(t) |\Psi_H\rangle . \tag{70}$$

Likewise for the *out* picture we write :

$$|out\rangle = \hat{\Omega}_{out}(t) |\Psi_H\rangle . \tag{71}$$

Combining both:

$$|in\rangle = \hat{\Omega}_{in}(t)\hat{\Omega}_{out}^{-1}(t) |out\rangle . \tag{72}$$

This product is called the scattering operator S. It connects any state

$$|in\rangle = S |out\rangle . \tag{73}$$

By taken t to the initial conditions of the pictures we obtain two more expressions. Which we will use to verify explicit expressions for S.

$$S = \hat{\Omega}_{in}(t)\hat{\Omega}_{out}^{-1}(t) \tag{74a}$$
$$= \hat{\Omega}_{in}(\infty) \tag{74b}$$
$$= \hat{\Omega}_{out}(-\infty)^{-1}. \tag{74c}$$

Let's consider the first equality of (74) by inserting the expressions given in Eq. (67),(69),

$$S = T\left(e^{-i\int_{-\infty}^{t} \mathrm{d}t' \hat{V}_{in}(t')}\right) \cdot \left[\bar{T}\left(e^{-i\int_{\infty}^{t} \mathrm{d}t'' \hat{V}_{out}(t'')}\right)\right]^{-1}. \tag{75}$$

As stated under Eq. (12) $\hat{\Omega}_I(t)$ is unitary. Since we established in section3.1 $\hat{\Omega}_{in}(t)$ and $\hat{\Omega}_{out}(t)$ mainly differ in the integral limits, this property carries over. So $\hat{\Omega}_{out}(t)^{-1} = \hat{\Omega}_{out}(t)^{\dagger}$, therefore:

$$S = T\left(e^{-i\int_{-\infty}^{t} \mathrm{d}t' \hat{V}_{in}(t')}\right) \cdot \left[\bar{T}\left(e^{-i\int_{\infty}^{t} \mathrm{d}t'' \hat{V}_{out}(t'')}\right)\right]^{\dagger}. \tag{76}$$

The hermitian conjugation of anti-chronological time ordering operators turns it into the chronological time ordering of same operators, as long as they are hermitian. Noted that this can be seen immediately by hermitian conjugate Eq. (62)

$$\left[\bar{T}(\hat{V}(t_n)\ldots\hat{V}(t_1))\right]^{\dagger} = T(\hat{V}(t_1)\ldots\hat{V}(t_n)), \tag{77}$$

observe that the operators' positions are switched. Yet the Heaviside-step-functions are unchanged which gives directly the definition in Eq. (54).

$$S = T\left(e^{-i\int_{-\infty}^{t} \mathrm{d}t' \hat{V}_{in}(t')}\right) \cdot T\left(e^{+i\int_{\infty}^{t} \mathrm{d}t'' \hat{V}_{out}(t'')}\right). \tag{78}$$

The lack of overlap in the integral limits allows us to fuse T:

$$S = T\left(e^{-i\int_{-\infty}^{t} \mathrm{d}t' \hat{V}_{in}(t')} \cdot e^{+i\int_{\infty}^{t} \mathrm{d}t'' \hat{V}_{out}(t'')}\right), \tag{79}$$

here the commutation in T allow ease use ofthe Baker-Campbell-Hausdorff-formula and we flipped the limits in the integral over $\hat{V}_{out}(t'')$.

$$S = T\left(e^{-i\int_{-\infty}^{t} \mathrm{d}t' \hat{V}_{in}(t') - i\int_{t}^{\infty} \mathrm{d}t'' \hat{V}_{out}(t'')}\right) \tag{80}$$

We see the overlap makes S time-independent. Therefore t is set to ∞:

$$S = T\left(e^{-i\int_{-\infty}^{\infty}\mathrm{d}t'\hat{V}_{in}(t')-i\int_{\infty}^{\infty}\mathrm{d}t''\hat{V}_{out}(t'')}\right) = T\left(e^{-i\int_{-\infty}^{\infty}\mathrm{d}t'\hat{V}_{in}(t')}\right) = \hat{\Omega}_{in}(\infty). \tag{81}$$

Or we choose $-\infty$:

$$S = T\left(e^{-i\int_{-\infty}^{-\infty}\mathrm{d}t'\hat{V}_{in}(t')-i\int_{-\infty}^{\infty}\mathrm{d}t''\hat{V}_{out}(t'')}\right) = T\left(e^{-i\int_{-\infty}^{\infty}\mathrm{d}t'\hat{V}_{out}(t')}\right) = \hat{\Omega}_{out}(-\infty)^{-1}. \tag{82}$$

The time independence and integral over all t makes them equal.

$$S = T\left(e^{-i\int_{-\infty}^{\infty}\mathrm{d}t\hat{V}_{I}(t)}\right). \tag{83}$$

The next set of relations for S to be verified are:

$$\begin{aligned}\hat{\Omega}_{in}(t)\hat{\Omega}_{I}(t)^{-1} &= S\ \hat{\Omega}_{I}(\infty)^{-1} &\text{(84a)}\\ &= \hat{\Omega}_{in}(0). &\text{(84b)}\end{aligned}$$

The first one on the right hand side is just 1 seen directly by the definition $S = \hat{\Omega}_I(\infty)$. Using the exponential formula for the left hand sight leads to:

$$\begin{aligned}\hat{\Omega}_{in}(t)\hat{\Omega}_{I}(t)^{-1} &= T\left(e^{-i\int_{-\infty}^{t}\mathrm{d}t'\hat{V}_{in}(t')}\right)\left(\bar{T}\left(e^{-i\int_{0}^{t}\mathrm{d}t'\hat{V}_{I}(t')}\right)\right)^{-1} &\text{(85a)}\\ &\underset{\mathrm{T}\to\bar{\mathrm{T}}}{=} T\left(e^{-i\int_{-\infty}^{t}\mathrm{d}t'\hat{V}_{in}(t')}\right)T\left(e^{+i\int_{0}^{t}\mathrm{d}t'\hat{V}_{I}(t')}\right) &\text{(85b)}\\ &= T\left(e^{-i\int_{-\infty}^{t}\mathrm{d}t'\hat{V}_{in}(t')}e^{+i\int_{0}^{t}\mathrm{d}t'\hat{V}_{I}(t')}\right) &\text{(85c)}\\ &\underset{\mathrm{CBH}}{=} T\left(e^{-i\int_{-\infty}^{t}\mathrm{d}t'\hat{V}_{in}(t')-i\int_{t}^{0}\mathrm{d}t'\hat{V}_{I}(t')}\right) &\text{(85d)}\\ &= \hat{\Omega}_{in}(0)\hat{\Omega}_{I}(0) \underset{\hat{\Omega}_{\mathrm{I}}(0)=1}{=} \hat{\Omega}_{in}(0) &\text{(85e)}\end{aligned}$$

This last expression is time independent and is also 1 by looking at the differential equation for $\hat{\Omega}_{in}(t)$.

The last expression we need is :

$$\hat{\Omega}_{in}(t) = \bar{T}\left(e^{i\int_{t}^{\infty}\mathrm{d}t'\hat{V}_{in}(t')}\right)S. \tag{86}$$

First it must satisfy the differential equation for $\hat{\Omega}_{in}(t)$ given in Eq. (45):

$$i\partial_t\hat{\Omega}_{in}(t) = i\partial_t\left(\sum_n i^n\int_t^\infty \mathrm{d}t_1\hat{V}_{in}(t_1)\dots\int_{t_{n-1}}^\infty \mathrm{d}t_n\hat{V}_{in}(t_n)\right)S, \tag{87a}$$

by making an integration by part we derive,

$$i\partial_t\hat{\Omega}_{in}(t) = i\left(\sum_n i^n\partial_t\left[\bar{\hat{V}}_{in}(\infty)-\bar{\hat{V}}_{in}(t)\right]\int_{t_1}^\infty \mathrm{d}t_2\hat{V}_{in}(t_2)\dots\int_{t_{n-1}}^\infty \mathrm{d}t_n\hat{V}_{in}(t_n)\right)S. \tag{87b}$$

Now a partial differentiate in respect to t is required:

$$i\partial_t\hat{\Omega}_{in}(t) = i\left(\sum_n i^n\left[0-\hat{V}_{in}(t)\right]\int_{t_1}^\infty \mathrm{d}t_2\hat{V}_{in}(t_2)\dots\int_{t_{n-1}}^\infty \mathrm{d}t_n\hat{V}_{in}(t_n)\right)S, \tag{87c}$$

$$= \hat{V}_{in}(t)\sum_n\frac{i^{n-1}}{(n-1)!}\int_{t_1}^\infty \mathrm{d}t_2\dots\int_{t_{n-1}}^\infty \mathrm{d}t_n\bar{T}\left(\hat{V}_{in}(t_2)\dots\hat{V}_{in}(t_n)\right)S. \tag{87d}$$

We again write the expression compactly and since t being the earliest time in the integral and anti-chronological time ordering can be applied we are allowed to move $\hat{V}_{in}(t)$ out of T.

$$\begin{aligned} i\partial_t\hat{\Omega}_{in}(t) &= \bar{T}\left(\hat{V}_{in}(t)e^{i\int_t^\infty \mathrm{d}t'\hat{V}_{in}(t')}\right)S \\ &= \hat{V}_{in}(t)\bar{T}\left(e^{i\int_t^\infty \mathrm{d}t'\hat{V}_{in}(t')}\right)S. \end{aligned} \tag{88}$$

One also has to provide $\hat{\Omega}_{in}(-\infty) = 1$:

$$\hat{\Omega}_{in}(t) = \bar{T}\left(e^{i\int_{-\infty}^\infty \mathrm{d}t'\hat{V}_{in}(t')}\right)S \tag{89a}$$

$$= (\hat{\Omega}_{in}(\infty))^\dagger S \tag{89b}$$

$$= S^\dagger S = S^{-1}S = 1 \tag{89c}$$

This last step required S to be unitary. This follows directly as in the expression in Eq. (115) only unitary operators have been used.

3.3 Wick theorem and vacuum stability

The Wick theorem as a means to evaluate correlation functions, can be established by further investigating S. We choose $\sum_{\underline{k}} \int dt\ \bar{j}_{\underline{k}}(t)\hat{q}_{\underline{k},in}(t)$ in the chronological time-ordered exponential function of S and using expressions found in section 2.3.
Explicitly,

$$S = T\left[\exp\left(i\sum_{\underline{k}}\int dt\ \bar{j}_{\underline{k}}(t)\hat{q}_{\underline{k},in}(t)\right)\right] \tag{90a}$$

$$\overset{\text{Eq.(32)}}{=} \quad T\left[\exp\left(i\sum_{\underline{k}}\int dt\ \bar{j}_{\underline{k}}(t)\frac{1}{2\omega_{\underline{k}}}\left(\hat{a}_{\underline{k},in}e^{-i\omega_{\underline{k}}t} + \hat{a}^{\dagger}_{\underline{k},in}e^{i\omega_{\underline{k}}t}\right)\right)\right]. \tag{90b}$$

using the the CBH-formula allows us to separate, to simplify we use the temporal Fourier transformation given below Eq. (35):

$$\begin{aligned} S = &\exp\left(i\sum_{\underline{k}}\bar{j}_{\underline{k}}(\omega_{\underline{k}})\frac{1}{2\omega_{\underline{k}}}\hat{a}^{\dagger}_{\underline{k},in}\right) \\ &\times \exp\left(i\sum_{\underline{k}}\bar{j}_{\underline{k}}(-\omega_{\underline{k}})\frac{1}{2\omega_{\underline{k}}}\hat{a}_{\underline{k},in}\right) \\ &\times \exp\left(-\frac{1}{2}\left[i\sum_{\underline{k}}\bar{j}_{\underline{k}}(\omega_{\underline{k}})\frac{1}{2\omega_{\underline{k}}}\hat{a}^{\dagger}_{\underline{k},in}, i\sum_{\underline{k}}\bar{j}_{\underline{k}}(-\omega_{\underline{k}})\frac{1}{2\omega_{\underline{k}}}\hat{a}_{\underline{k},in}\right]\right), \end{aligned} \tag{91}$$

the Fourier transformation allowed us to drop T. Now we move all common factors out of the commutator,

$$\begin{aligned} S = &\exp\left(i\sum_{\underline{k}} \bar{j}_{\underline{k}}(\omega_{\underline{k}})\frac{1}{2\omega_{\underline{k}}}\hat{a}^{\dagger}_{\underline{k},in}\right) \\ &\times \exp\left(i\sum_{\underline{k}} \bar{j}_{\underline{k}}(-\omega_{\underline{k}})\frac{1}{2\omega_{\underline{k}}}\hat{a}_{\underline{k},in}\right) \\ &\times \exp\left(\sum_{\underline{k}} \frac{-1}{4\omega_{\underline{k}}}\left[i\bar{j}_{\underline{k}}(\omega_{\underline{k}})\hat{a}^{\dagger}_{\underline{k},in}, i\bar{j}_{\underline{k}}(-\omega_{\underline{k}})\hat{a}_{\underline{k},in}\right]\right), \end{aligned} \tag{92}$$

with the commutation relation given in Eq. (33) we obtain:

$$\begin{aligned} S = &\exp\left(i\sum_{\underline{k}} \frac{1}{2\omega_{\underline{k}}}\bar{j}_{\underline{k}}(\omega_{\underline{k}})\hat{a}^{\dagger}_{\underline{k},in}\right) \\ &\times \exp\left(i\sum_{\underline{k}} \frac{1}{2\omega_{\underline{k}}}\bar{j}_{\underline{k}}(-\omega_{\underline{k}})\hat{a}_{\underline{k},in}\right) \\ &\times \exp\left(-\sum_{\underline{k}} \frac{1}{4\omega_{\underline{k}}}|\bar{j}_{\underline{k}}(\omega_{\underline{k}})|^2\right). \end{aligned} \tag{93}$$

In this expression the creation operators $\hat{a}^{\dagger}_{\underline{k},in}$ are placed to the left of the annihilation operators $\hat{a}_{\underline{k},in}$. This is known as "normal" ordering. To keep it this way and making other expressions easier, we want to denote a notational symbol to force them to stay like it.

Calling it the "normal ordering" we write in a general form:

$$:aa^{\dagger}: \quad = \quad :a^{\dagger}a: \quad = a^{\dagger}a. \tag{94}$$

Take notice of the effect on the commutator:

$$\begin{aligned} :\left[\hat{a}_{\underline{k},in}, \hat{a}^{\dagger}_{\underline{k},in}\right]: \quad &= \quad :\hat{a}_{\underline{k},in}\hat{a}^{\dagger}_{\underline{k},in}: - :\hat{a}^{\dagger}_{\underline{k},in}\hat{a}_{\underline{k},in}: \\ &= \quad :\hat{a}^{\dagger}_{\underline{k},in}\hat{a}_{\underline{k},in}: - :\hat{a}^{\dagger}_{\underline{k},in}\hat{a}_{\underline{k},in}: \\ &= 0 \end{aligned} \tag{95}$$

Following this in our case the CBH-formula becomes trivial inside of normal ordering and we can write:

$$
\begin{aligned}
&\exp\left(i\sum_{\underline{k}}\frac{1}{2\omega_{\underline{k}}}\bar{j}_{\underline{k}}(\omega_{\underline{k}})\hat{a}^{\dagger}_{\underline{k},in}\right)\times\exp\left(i\sum_{\underline{k}}\frac{1}{2\omega_{\underline{k}}}\bar{j}_{\underline{k}}(-\omega_{\underline{k}})\hat{a}_{\underline{k},in}\right)\\
&=\;:\exp\left(i\sum_{\underline{k}}\frac{1}{2\omega_{\underline{k}}}\bar{j}_{\underline{k}}(\omega_{\underline{k}})\hat{a}^{\dagger}_{\underline{k},in}+i\sum_{\underline{k}}\frac{1}{2\omega_{\underline{k}}}\bar{j}_{\underline{k}}(-\omega_{\underline{k}})\hat{a}_{\underline{k},in}\right):,
\end{aligned}
\tag{96}
$$

this is again our starting point from Eq. (113),

$$
=\;:\exp\left(i\sum_{\underline{k}}\int dt\;\bar{j}_{\underline{k}}(t)\hat{q}_{\underline{k},in}(t)\right): \tag{97}
$$

The third exponential function in Eq. (115) can be formulated differently by reversing the temporal Fourier transformation. The exponential functions of the transformation $e^{-iw_{\underline{k}}t}$ will be joined and we take the absolute value of the different times t, t' :

$$
\begin{aligned}
&\exp\left(-\sum_{\underline{k}}\frac{1}{4\omega_{\underline{k}}}|\bar{j}_{\underline{k}}(\omega_{\underline{k}})|^2\right)\\
&=\exp\left(-\sum_{\underline{k}}\frac{1}{4\omega_{\underline{k}}}\int dt\int dt'\;\bar{j}_{\underline{k}}(t)e^{-iw_{\underline{k}}|t-t'|}\bar{j}_{\underline{k}}(t')\right),
\end{aligned}
\tag{98}
$$

,we now introduce the Feynman Green's function for one mode :

$$
G_{\underline{k}}(t-t')=\frac{1}{2\omega_{\underline{k}}}e^{-i\omega_{\underline{k}}|t-t'|}. \tag{99}
$$

By taking Eq. (120),(121) and (122) into account we found the Wick theorem for modes of a real scalar field.

$$
\begin{aligned}
&T\left[\exp\left(i\sum_{\underline{k}}\int dt\;\bar{j}_{\underline{k}}(t)\hat{q}_{\underline{k},in}(t)\right)\right]\\
&=\;:\exp\left(i\sum_{\underline{k}}\int dt\;\bar{j}_{\underline{k}}(t)\hat{q}_{\underline{k},in}(t)\right):\times\exp\left(-\frac{1}{2}\sum_{\underline{k}}\int dt\int dt'\;\bar{j}_{\underline{k}}(t)G_k(t-t')\bar{j}_{\underline{k}}(t')\right).
\end{aligned}
\tag{100}
$$

One can easily derive this expression in terms of the scalar field Φ. Heuristicly speaking one replaces the modes with the fields, the integrals with $4-$dim. ones and the Green's function with the scalar propagator $\triangle(x-x')$. Which is the Green's function to the free equation of motion of the real scalar field. This theorem in combination with functional derivatives respect the currents of the regarded theory and taking the vacuum expectation value, allows us to find the desired correlations functions.

Beside obtaining the Wick theorem, we can use Eq. (116) to find the probability for staying in the ground state for one mode:

$$p_{0,k} = |\langle 0_{out}|0_{in}\rangle|^2 = |\langle 0_{in}|S|0_{in}\rangle|^2 \tag{101a}$$

$$= \left| \langle 0_{in}| e^{\dfrac{i}{2\omega_{\underline{k}}}\bar{j}_{\underline{k}}(\omega_{\underline{k}})\hat{a}^\dagger_{\underline{k},in}} e^{\dfrac{i}{2\omega_{\underline{k}}}\bar{j}_{\underline{k}}(-\omega_{\underline{k}})\hat{a}_{k,in}} e^{-\dfrac{1}{4\omega_{\underline{k}}}|\bar{j}_{\underline{k}}(-\omega_{\underline{k}})|^2} |0_{in}\rangle \right|^2 . \tag{101b}$$

The annihilation operator will return a 0 in the exponent. Therefore only one factor remain important. The probabilty while taking to account any mode transition just involves the integral over all k:

$$p_0 = \exp\left\{ -\int \frac{d^3k}{(2\pi)^3} \left| \frac{\bar{j}_{\underline{k}}(\omega_{\underline{k}})}{\sqrt{2\omega_{\underline{k}}}} \right|^2 \right\} . \tag{102}$$

This confirms our statement about an unstable vacuum. The negative sign in the exponent translates to smaller probability at higher external currents. So the vacuum can change it's state if a current is > 0.

4 Gell-Mann Low formula

To motivate Gell-Mann Low formula as the important asset, a common way of application and requirement will be laid out.

First, the formula allows us to transform a polynomial , chronological (or anti-chronological) ordered set of operators in the Heisenberg picture to the three pictures with initial conditions. There won't be any picture related unitary operator $\hat{\Omega}$ only by the scattering operator.

This strikes as a fundamental step for dealing with the pictures as S can be evaluated pertubativly and nothing else in the formula up to this point with the same accuracy. Furthermore this transition needs be made very early when working on many topics of quantum field theory. As most of the time, one would begin with classical mechanics to get to QFT. Starting with the action of your problem in term of classical fields and then apply second quantisation to promote them to operators in the Heisenberg picture. This would be the point to transition and one needs the Gell-Mann Low formula. Recalling Eq. (17) we can write:

$$Q_H(t) = \hat{\Omega}_I(t)^{-1} Q_I(t)\ \hat{\Omega}_I(t). \tag{103}$$

Since the proof didn't require any specifications on $\hat{\Omega}$ as it only remains as the difference to the Heisenberg operator U. Allowing us to write:

$$Q_H(t) = \hat{\Omega}_{in}(t)^{-1} Q_{in}(t)\ \hat{\Omega}_{in}(t). \tag{104}$$

Expressing $\hat{\Omega}_{in}$ by S as seen above:

$$Q_H(t) = \left(\bar{T}\left(e^{i\int_t^\infty \mathrm{d}t' \hat{V}_{in}(t')}\right) S\right)^{-1} Q_{in}(t) T\left(e^{-i\int_{-\infty}^t \mathrm{d}t' \hat{V}_{in}(t')}\right) \tag{105}$$

being a unitary operator, we replace -1 by $\dagger$ and apply hermitian conjugation

$$\begin{aligned} Q_H(t) &= \left(\bar{T}\left(e^{i\int_t^\infty \mathrm{d}t' \hat{V}_{in}(t')}\right) S\right)^{\dagger} Q_{in}(t) T\left(e^{-i\int_{-\infty}^t \mathrm{d}t' \hat{V}_{in}(t')}\right) && \text{(106a)} \\ &= S^{\dagger}\left(\bar{T}\left(e^{i\int_t^\infty \mathrm{d}t' \hat{V}_{in}(t')}\right)\right)^{\dagger} Q_{in}(t) T\left(e^{-i\int_{-\infty}^t \mathrm{d}t' \hat{V}_{in}(t')}\right) && \text{(106b)} \\ &= S^{-1} T\left(e^{-i\int_t^\infty \mathrm{d}t' \hat{V}_{in}(t')}\right) Q_{in}(t) T\left(e^{-i\int_{-\infty}^t \mathrm{d}t' \hat{V}_{in}(t')}\right). && \text{(106c)} \end{aligned}$$

This step used the unitarity of S itself and the appendix for transition to chronological time ordering. Following that it is allowed to merge T due to zero overlap in the boundaries and correct order

$$Q_H(t) = S^{-1} T\left(e^{-i\int_t^\infty \mathrm{d}t' \hat{V}_{in}(t')} Q_{in}(t) e^{-i\int_{-\infty}^t \mathrm{d}t' \hat{V}_{in}(t')}\right). \tag{107}$$

The position of Q_{in} is still not ideal. Changing it will require commutations. Using the infinite series for e and removing time ordering will reduce the problem to commutations between Q_{in} and $\hat{V}_{in}$.

$$Q_H(t) = S^{-1}\left(\sum_n (-i)^n \int_{-\infty}^t \mathrm{d}t_1' \int_t^\infty \mathrm{d}t_1' \dots \int_{-\infty}^{t_{n-1}} \mathrm{d}t_n' \int_{t_{n-1}}^\infty \mathrm{d}t_n' \hat{V}_{in}(t_1') \dots \hat{V}_{in}(t_n') Q_{in}(t)\right). \tag{108}$$

We see Q_{in} and $\hat{V}_{in}$ can commute without any problem since Q_{in} depends on t and $t \neq t'$. Moving it to the left and reapply T as well as e:

$$Q_H(t) = S^{-1}\left(\sum_n (-i)^n Q_{in}(t) \int_{-\infty}^{t} \mathrm{d}t'_1 \int_{t}^{\infty} \mathrm{d}t'_1 \ldots \int_{-\infty}^{t_{n-1}} \mathrm{d}t'_n \int_{t_{n-1}}^{\infty} \mathrm{d}t'_n \hat{V}_{in}(t'_1) \ldots \hat{V}_{in}(t'_n)\right) \tag{109a}$$

$$Q_H(t) = S^{-1} T\left(Q_{in}(t) e^{-i\int_{-\infty}^{\infty} \mathrm{d}t' \hat{V}_{in}(t')}\right). \tag{109b}$$

Identifying the exponential function as S, the result is:

$$Q_H(t) = S^{-1} T\left(Q_{in}(t) S\right). \tag{110}$$

Next will be for more than one operator. Starting with the left side in a non trivial time ordering and applying the transformation for each operator:

$$T\left(Q_H(t_1) Q_H(t_2) \ldots\right) = T\left(\prod_j \hat{\Omega}_{in}(t_j)^{-1} Q_{in}(t_j) \hat{\Omega}_{in}(t_j)\right) \tag{111a}$$

,inserting the expression with S for a $\hat{\Omega}$ depending on t_j ,

$$= T\left(\prod_j S^{-1} T\left(e^{-i\int_{t_j}^{\infty} \mathrm{d}t'_j \hat{V}_{in}(t'_j)}\right) Q_{in}(t_j) T\left(e^{-i\int_{-\infty}^{t_j} \mathrm{d}t'_j \hat{V}_{in}(t'_j)}\right)\right) \tag{111b}$$

,the series and commutation follows the same argumentation as for a single Q,

$$T\left(Q_H(t_1) Q_H(t_2) \ldots\right) = T\left(\prod_j S^{-1} Q_{in}(t_j) S\right). \tag{112}$$

Moving S outside of T gives the final form of the Gell-Mann Low formula:

$$T\left(Q_H(t_1) Q_H(t_2) \ldots\right) = S^{-1} T\left(Q_{in}(t_1) Q_{in}(t_2) \ldots S\right). \tag{113}$$

To proof S can be outside of chronological time ordering we look at $T(S^{-1})$:

$$T(S^{-1}) = T(S^{\dagger}) \tag{114a}$$

,using the expression of S in term of the In-picture and apply hermitian conjugation,

$$T(S^{-1}) = T\left(\bar{T}\left(e^{i\int_{-\infty}^{\infty} \mathrm{d}t' \hat{V}_{in}(t')}\right)\right) \tag{114b}$$

,e^x as a series,

$$= T\left(\sum_n \frac{(i)^n}{n!}\int_{-\infty}^{\infty}\mathrm{d}t_1\int_{-\infty}^{\infty}\mathrm{d}t_2\ldots\int_{-\infty}^{\infty}\mathrm{d}t_n\bar{T}\left(\hat{V}_{in}(t_1)\ldots\hat{V}_{in}(t_n)\right)\right) \tag{114c}$$

and removing the anti-chronological time ordering,

$$= T\left(\sum_n \frac{(i)^n}{n!}i^n\int_{-\infty}^{\infty}\mathrm{d}t_1\int_{-\infty}^{\infty}\mathrm{d}t_2\ldots\int_{-\infty}^{\infty}\mathrm{d}t_n\hat{V}_{in}(t_1)\ldots\hat{V}_{in}(t_n)\right). \tag{114d}$$

The boundaries make every integral independent from each other so rearranging t_1 to t_n wouldn't change the result

$$\rightarrow T(S^{-1}\ldots) = S^{-1}T(\ldots)$$

5 Conclusion

In this report we have presented a set of fundamental concepts and methods. These are essential for starting to study Quantum field theory. We began with well-known pictures from quantum mechanics and introduced the In and Out picture as consequences of working with asymptotic states. They lead us to the important connecting element between the pictures, the scattering operator. The connections allowed us to derive the Gell-Mann Low formula, which plays a major part in the second quantisation process one would do in quantum field theories.

5.A Chronological time ordering

To expand our concept to cases of more then two t we start by advancing T. In general it consists of a summation of all permutations P of a given set multiplied by Heaviside functions. These Heaviside functions have arguments with the negative summation of the t in the same permutation P. We keep the number of brackets low by denoting $\hat{V}_I(t_1) \to \hat{V}_1$

Definition:

$$T(\hat{V}_1, \hat{V}_2, \ldots, \hat{V}_n) = \sum_{j=1}^{n!} P_j \left[\hat{V}_1, \hat{V}_2, \ldots, \hat{V}_n\right] \cdot \theta \left(P_j \left[t_j - \sum_{i \neq j}^{n} t_i \right] \right) \quad . \tag{115}$$

But this summation over j would not add up to the same value for $I(t)$ since it started in the order of one element in the sum.In this case we need a normalization in addition to T . From statistics we know a set of n different elements can be linear arranged in $n!$ ways.Coming in as a factor of $\frac{1}{n!}$ in the expressions later.

For a test we use $n = 3$

$$T(\hat{V}_1, \hat{V}_2, \hat{V}_3) = \sum_{j=1}^{3!} P_j \left[\hat{V}_1, \hat{V}_2, \hat{V}_3\right] \cdot \theta \left(P_j \left[t_j - \sum_{i \neq j}^{3} t_i \right] \right) \tag{116a}$$

$$= \hat{V}_1 \hat{V}_2 \hat{V}_3 \cdot \theta(t_1 - t_2 - t_3) \tag{116b}$$

$$+ \hat{V}_1 \hat{V}_3 \hat{V}_2 \cdot \theta(t_1 - t_3 - t_2) \tag{116c}$$

$$+ \hat{V}_2 \hat{V}_1 \hat{V}_3 \cdot \theta(t_2 - t_1 - t_3) \tag{116d}$$

$$+ \hat{V}_2 \hat{V}_3 \hat{V}_1 \cdot \theta(t_2 - t_3 - t_1) \tag{116e}$$

$$+ \hat{V}_3 \hat{V}_1 \hat{V}_2 \cdot \theta(t_3 - t_1 - t_2) \tag{116f}$$

$$+ \hat{V}_3 \hat{V}_2 \hat{V}_1 \cdot \theta(t_3 - t_2 - t_1) \tag{116g}$$

Now we can apply it to new I and do a proof by Induction based on the number of $\hat{V}$. The Induction start is the case of two $\hat{V}$. In the Induction step we say in a mathematicians way that it works for at least one unspecified higher order. Let's call it k : (*Note:* $t_0 = t$)

$$I_k(t) = \prod_{a=1}^{k} \int_{-\infty}^{t_{a-1}} \mathrm{d}t_a \; \hat{V}_a = \frac{1}{k!} (\prod_{a=1}^{k} \int_{-\infty}^{t} \mathrm{d}t_a \;) T(\hat{V}_1, \ldots, \hat{V}_k). \tag{117}$$

Moving one increment higher in our 'chain' $k+1$,

$$I_{k+1}(t) = \prod_{a=1}^{k+1} \int_{-\infty}^{t_{a-1}} \mathrm{d}t_a \ \hat{V}_a = \prod_{a=1}^{k} \int_{-\infty}^{t_{a-1}} \mathrm{d}t_a \ \hat{V}_a \cdot \int_{-\infty}^{t_k} \mathrm{d}t_{k+1} \ \hat{V}_{k+1} \tag{118}$$

Using the Induction Step and general definition for T:

$$I_{k+1}(t) = \frac{1}{k!} (\prod_{a=1}^{k} \int_{-\infty}^{t} \mathrm{d}t_a \ ) T(\hat{V}_1, \ldots, \hat{V}_k) \cdot \int_{-\infty}^{t_k} \mathrm{d}t_{k+1} \ \hat{V}_{k+1} \tag{119}$$

$$I_{k+1}(t) = I_k(t) \cdot \int_{-\infty}^{t_k} \mathrm{d}t_{k+1} \ \hat{V}_{k+1} \tag{120}$$

This shows that the incrementation of k reduces to a multiplication with one more different element for the set. This increases the possible permutations by a factor of $k+1$ resulting in $(k+1)!$ in total. Giving us:

$$I_{k+1}(t) = \frac{1}{(k+1)!} (\prod_{a=1}^{k+1} \int_{-\infty}^{t} \mathrm{d}t_a \ ) T(\hat{V}_1, \ldots, \hat{V}_{k+1}) \tag{121}$$

Time ordering is a very grounded concept in field theory, it appears very naturally for expressing propagators in term of fields. The reason apart from pure mathematics is comprehensible. The different functions, functionals or fields are best organized for summarizing a scattering or interaction event if they are time-like sorted.
Furthermore it comes with a great advantage. All products in $T(\ldots)$ do commute. Proof for two elements:

$$T(\hat{V}_1, \hat{V}_2) = \hat{V}_1 \ \hat{V}_2 \ \theta(t_1 - t_2) \ + \ \hat{V}_2 \ \hat{V}_1 \ \theta(t_2 - t_1) \tag{122a}$$

$$T(\hat{V}_2, \hat{V}_1) = \hat{V}_2 \ \hat{V}_1 \ \theta(t_2 - t_1) \ + \ \hat{V}_1 \ \hat{V}_2 \ \theta(t_1 - t_2) \tag{122b}$$

,since terms in sums always commutes,

$$T(\hat{V}_1, \hat{V}_2) = T(\hat{V}_2, \hat{V}_1) \tag{123}$$

In other words, the commutation relations say whether the subtraction of permutations of elements is zero or not. But in time ordering all permutations appear, we can rearrange the terms so subtraction of equal permutations happens. Therefore commutation in T holds.

5.B Anti-chronological time ordering

On a close inspection of the introduction of the Heaviside-function and its' insertion into the integral, we actually skipped a choice. If we just would have wanted to the overall boundaries the order of t_1 and t_2 in relation to $\hat{V}_1$ and $\hat{V}_2$ the insertion of θ could have been switched. This secretly led us to the definition of time-ordering or chronological time ordering as it is called more precisely. The other path would have resulted in anti-chronological time ordering:[4]

$$\bar{T}(\hat{V}_1, \hat{V}_2) = \hat{V}_1\ \hat{V}_2\ \theta(t_2 - t_1)\ +\ \hat{V}_2\ \hat{V}_1\ \theta(t_1 - t_2) \tag{124}$$

This section is not just to satisfy the observed readers but to proof a connection between both involing hermitian conjugation of the integrals, which can easily appear while doing picture transitions.
We choose the same starting point as for T:

$$\begin{aligned} I(t) &= \frac{1}{2}\int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 \hat{V}_1\hat{V}_2 + \frac{1}{2}\int_{-\infty}^{t} \mathrm{d}t_2 \int_{-\infty}^{t_2} \mathrm{d}t_1 \hat{V}_2\hat{V}_1 \\ &= \frac{1}{2!}\int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t} \mathrm{d}t_2 T(\hat{V}_1, \hat{V}_2). \end{aligned}$$

$$I(t)^\dagger = \left(\frac{1}{2}\int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 \hat{V}_1\hat{V}_2 + \frac{1}{2}\int_{-\infty}^{t} \mathrm{d}t_2 \int_{-\infty}^{t_2} \mathrm{d}t_1 \hat{V}_2\hat{V}_1\right)^\dagger \tag{125a}$$

$$= \frac{1}{2}\int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 \left(\hat{V}_1\hat{V}_2\right)^\dagger + \frac{1}{2}\int_{-\infty}^{t} \mathrm{d}t_2 \int_{-\infty}^{t_2} \mathrm{d}t_1 \left(\hat{V}_2\hat{V}_1\right)^\dagger \tag{125b}$$

$$= \frac{1}{2}\int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 \hat{V}_2^\dagger\hat{V}_1^\dagger + \frac{1}{2}\int_{-\infty}^{t} \mathrm{d}t_2 \int_{-\infty}^{t_2} \mathrm{d}t_1 \hat{V}_1^\dagger\hat{V}_2^\dagger, \tag{125c}$$

we observe that the switch of positions due to hermitian conjugation allows to use Eq. (111)

$$I(t)^\dagger = \frac{1}{2}\int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t} \mathrm{d}t_2 \hat{V}_2^\dagger\hat{V}_1^\dagger\theta(t_1 - t_2) + \frac{1}{2}\int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t} \mathrm{d}t_2 \hat{V}_1^\dagger\hat{V}_2^\dagger\theta(t_2 - t_1) \tag{125d}$$

$$\Rightarrow \frac{1}{2}\int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t} \mathrm{d}t_2 \bar{T}(\hat{V}_1^\dagger, \hat{V}_2^\dagger) \tag{125e}$$

[4]earlier times to the left and later times to the right

Now we assume our total Hamiltonian $\hat{H} = \hat{H}_0 + \hat{\hat{V}}$ is hermitian and $\hat{\hat{V}}$ won't lead to non fixed ground state energy

$$I(t)^\dagger = \frac{1}{2} \int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t} \mathrm{d}t_2 \bar{T}(\hat{V}_1, \hat{V}_2) \tag{125f}$$

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