Empirical Methods in Finance

Project 2

Groupe 11

Yann Boulben Meyer : 1941993
Rémy David : 17833641
Marko :
Elio : 2095104630

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1 Static allocation

Q.1.1

We want to maximize this expression :

$$\max_{\tilde{\alpha}} \ \mu_p - \frac{\lambda}{2}\sigma_p^2$$

$$\begin{aligned} With \quad \mu_p &= \tilde{\alpha}'\mu + \left(1 - \tilde{\alpha}'e\right)R_f \\ \sigma_p^2 &= \tilde{\alpha}'\Sigma\tilde{\alpha} \end{aligned}$$

By substituting μ_p and σ_p^2 the new expression becomes :

$$\max_{\tilde{\alpha}} \ \tilde{\alpha}'\mu + \left(1 - \tilde{\alpha}'e\right)R_f - \frac{\lambda}{2}\tilde{\alpha}'\Sigma\tilde{\alpha}$$

By taking the first-order condition, we find:

$$\frac{\partial}{\partial\tilde{\alpha}} : \ \mu - R_f e - \lambda\Sigma\tilde{\alpha} = 0$$

$$\Longleftrightarrow \quad \tilde{\alpha}^* = \frac{1}{\lambda}\Sigma^{-1}(\mu - R_f e)$$

The extended form in our case becomes :

$$\begin{pmatrix} \tilde{\alpha}_s^* \\ \tilde{\alpha}_b^* \end{pmatrix} = \frac{1}{\lambda}\Sigma^{-1}\left(\begin{pmatrix} \mu_s \\ \mu_b \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} R_f\right)$$

Finally, to have the weight invested in the risk-free rate, we compute :

$$1 - e'\tilde{\alpha}^* = 1 - \tilde{\alpha}_s^* - \tilde{\alpha}_b^*$$

Q.1.2

By estimating the expected returns as well as the covariance matrix, we applied the formula we found earlier to compute the optimal weights for the two different cases of risk-aversion (λ). The results are summarized in the following table :

Weights \ Risk aversion	*λ = 2*	*λ = 10*
S&P 500	***1.119974***	***0.223995***
MSCI Pacific	***-0.048201***	***-0.009640***
Risk-free	***-0.071773***	***0.785645***

We can see that for a low risk-aversion (λ = 2) the weight in the Risk-free rate is negative, which means we borrow money to invest more in risky assets, increasing market exposure.

In the case of a high risk-aversion (λ = 10), we still long the S&P 500 and short the MSCI Pacific but the weights are reduced to invest more in the risk-free asset, lowering market exposure.

Moreover, among the risky assets, the proportions invested remain the same (ratio S&P 500/MSCI is equal) in both cases since they are optimal, only the scale changes.

2 Estimation of a GARCH model

Q.2.1

To compute the excess return of the S&P500 and the MSCI Pacific, we subtracted the weekly risk-free rate from the weekly returns and squared the results to obtain the squared excess returns.

In order to perform the Kolmogorov-Smirnov test for normality, we first decided to standardize our data to improve the accuracy of our results. We conducted a normality test for the excess returns and a chi-squared test for the excess returns squared. Indeed, if we assume that the data follows a normal distribution the squared data would follow a chi-squared distribution.

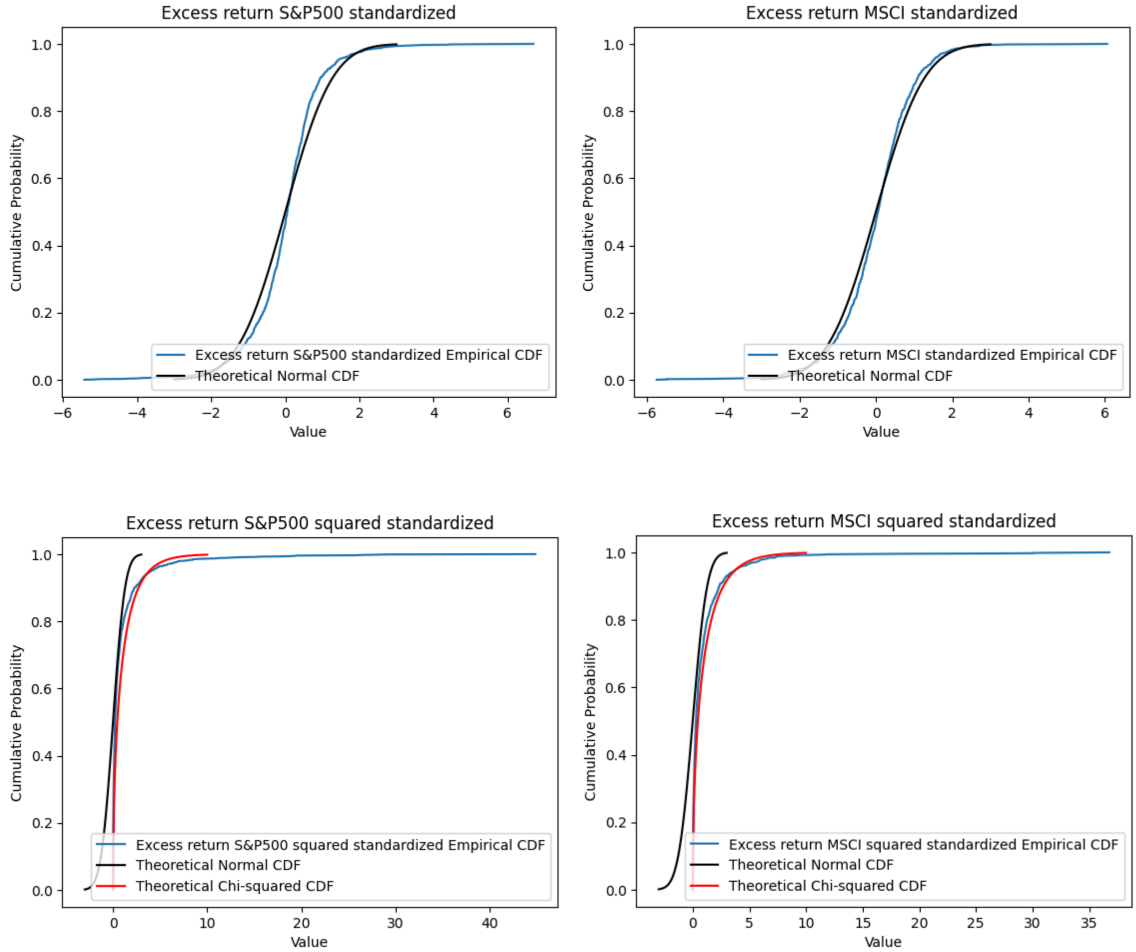
Kolmogorov-smirnov test

	KS Statistic (Normal)	*P-value (Normal)*	*KS Statistic (Chi-squared)*	*P-value (Chi-squared)*
Excess return S&P500 standardized	***0.0836***	***0.0000***		
Excess return S&P500 squared standardized			***0.1522***	***0.0000***
Excess return MSCI standardized	***0.0592***	***0.0004***		
Excess return MSCI squared standardized			***0.0951***	***0.0000***

The null hypothesis H_0 is that the data follows the same distribution as the one specified, and the alternative hypothesis is that they are different.

Since the p-value is extremely close to 0 for every test (especially the tests on the squared excess return), we can conclude with a very high significance threshold (at least 1%) that the excess returns for both do not follow a normal distribution and that the squared excess returns do not follow a chi-squared distribution.

To provide a more visualize result, we compared the CDF of the excess returns with the normal CDF as well as the CDF of the squared excess returns with the chi-squared CDF :



Concerning the auto-correlations, we performed a Ljung Box test with four lags. The goal is to determine the significance of autocorrelation in simple and squared excess returns up to 4 lags.

Ljung Box test

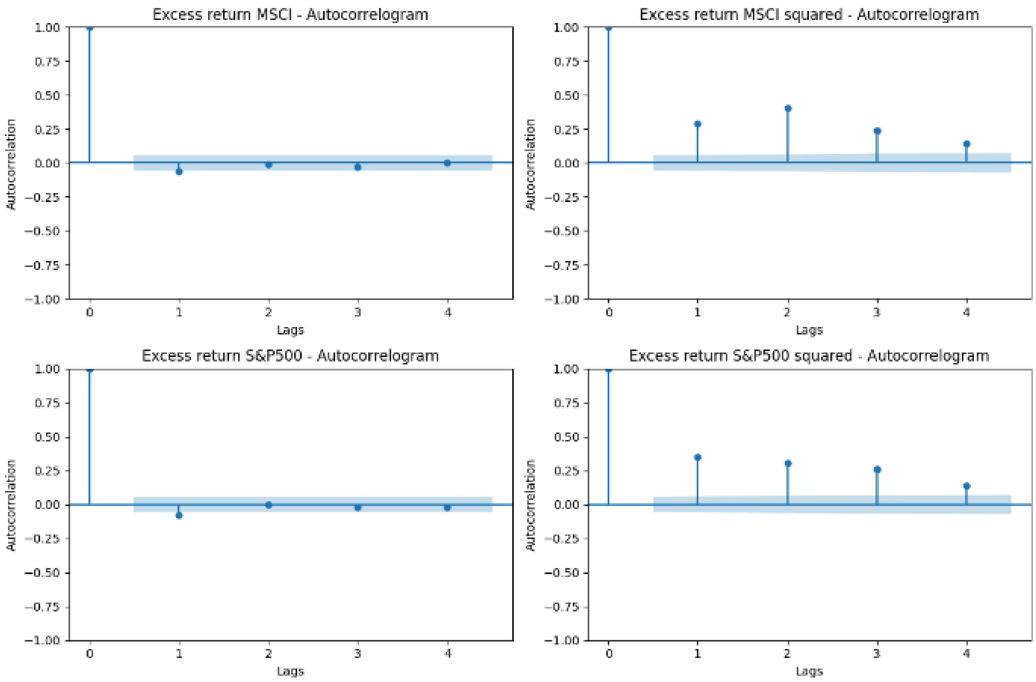
	S&P500				**MSCI Pacific**			
	Excess return		*(Excess return)^2*		*Excess return*		*(Excess return)^2*	
Lags	*Lb stat*	*P value*	*Lb stat*	*P value*	*Lb stat*	*P value*	*Lb stat*	*P value*
1	**6.9879**	**0.0082**	**146.0443**	**0.0000**	**4.4399**	**0.0351**	**99.7040**	**0.0000**
2	**6.9898**	**0.0303**	**257.3772**	**0.0000**	**4.5451**	**0.1031**	**296.2396**	**0.0000**
3	**7.5310**	**0.0568**	**338.0809**	**0.0000**	**5.5027**	**0.1385**	**366.0730**	**0.0000**
4	**8.1012**	**0.0879**	**361.4950**	**0.0000**	**5.5030**	**0.2395**	**389.6070**	**0.0000**

The null hypothesis is that there is no autocorrelation in the data up to a specified lag, and the alternative hypothesis is that there is autocorrelation.

Concerning the excess returns, the autocorrelation is significant at the first lag but decreases as we go further back in time. The first two lags for S&P500 are significant to an α=5%, and only the first lag for the MSCI with the same α. This is consistent with the behavior observed in financial markets, as a shock has less and less impact over time.

The excess squared returns of the S&P500 and MSCI show an extremely significant autocorrelation at all lags tested, indicating a strong dependence on volatility. However, excess returns show significant autocorrelation only in the very short term.

Same as above, we provided an autocorrelogram for each variable to illustrate our results:



The idea behind these tests is to analyze our data and find a model that fits and explains their behaviors. A high significance of the first lag for the excess returns can be represented by an autoregressive model of order 1. For the squared excess returns, a high autocorrelation significance of the lags means a high dependance of the variance on past variances, implying heteroskedasticity, which can be represented by a GARCH model.

Q.2.2

To estimate an AR_1 model on stock and bond returns, we fit our data to the following equation:

$$R_{i,t+1} = \alpha_i + \rho_i R_{i,t} + \varepsilon_{i,t+1} \text{ for i} = \text{S\&}P500,\ MSCI\ Pacific$$

Results of the AR_1

	S&P500	MSCI Pacific
α_i	***0.0019*****	***0.0013****
	(0.001)	***(0.001)***
ρ_i	***-0.0774******	***-0.0615*****
	(0.029)	***(0.029)***
** p < 0.1, ** p < 0.5, *** p < 0.01*		

Considering the significance of our model, we can see differences between the two securities. Indeed, while both parameters are significant for the S&P500 with an α=5%, only the autoregressive coefficient ρ is significant (α=5%) for MSCI Pacific.

The parameter ρ represents the impact of R_{t-1} on $R_{t.}$ This means that past returns will negatively impact the following returns, which is consistent with financial return behavior since a high increase(decrease) tends to be followed by a high decrease(increase). Finally, the constant represents the expected weekly return without the influence of the past return.

Concerning the goodness of fit, for both securities we obtain a very small R^2 and adjusted R^2 (less than 1%). The model explains less than 1% of the variability in the weekly returns of S&P500 and MSCI, which suggests that the model is too simple to capture the dynamics of the data. Thus, we will move to the GARCH model.

Q.2.3

By using the results of our AR_1 model in the previous question, we extracted the residuals $\varepsilon_{i,t+1}=\sigma_{i,t+1}z_{i,t+1}$ (we assume that z follows a normal distribution), which represent the part unexplained by the AR_1, and performed a GARCH model on it. The $GARCH_{(1,1)}$ can be used to identify volatility clusters, which are typical characteristics of financial returns and can be described by the following equation :

$$\sigma^2_{i,t+1} = \omega_i + \alpha_i \varepsilon^2_{i,t} + \beta_i \sigma^2_{i,t}$$

Results of the $GARCH_{1,1}$ estimation

	S&P500	MSCI Pacific
ω_i	***3.443e-05****	***4.4171e-05****
	(9.817e-06)	***(1.91e-05)***
α_i	***0.240207****	***0.163553****
	(4.642e-02)	***(4.697e-02)***
β_i	***0.719687****	***0.779210****
	(4.073e-02)	***(5.956e-02)***

** p < 0.1, ** p < 0.5, *** p < 0.01*

All the parameters for both securities are very significant (with a threshold<1% except for $\omega_{MSCI,}$ with a threshold of 5%), indicating a good predictability power of our model. The constant term ω represents the average variance, which is very low for both securities. The β suggests a persistence in volatility, meaning that past volatility shocks affect the volatility today. This clarifies why our AR_1 had difficulties explaining the behavior of the returns since it supposes constant volatility. As we can see, this impact of past volatility is stronger for the MSCI Pacific than the S&P500.

Finally, the α indicates that the past squared residuals have a significance on current volatility.

Moreover, the sum of α + β is an essential parameter to understand the behavior of the volatility. If the sum is equal to 1, it indicates that past volatility shocks are persistent over time, which means the presence of a unit root. Hence, the model would not be stationary since the variance does not revert to a long-term mean.

Considering our result, it does not seem to be the case for our securities. To verify this assumption, we conduct a test where the null hypothesis is α + β = 1 while the alternative hypothesis is α + β < 1. To do so, we decided to perform a Wald test, which is described by this equation:

$$W = T\left(c\left(\hat{\theta}\right) - q\right)' \left[\left(\frac{\partial c\left(\hat{\theta}\right)}{\partial \hat{\theta}'}\right) \Sigma \left(\frac{\partial c\left(\hat{\theta}\right)}{\partial \hat{\theta}'}\right)'\right]^{-1} \left(c\left(\hat{\theta}\right) - q\right)$$

The asymptotic distribution of W is a $\chi^2(k)$, where k is the number of restrictions in c, hence 1 in our case. The results are summarized below:

Wald test

	S&P500	MSCI Pacific
Wald stat	***2867.95***	***4439.15***
P-value	***0.00000***	***0.00000***

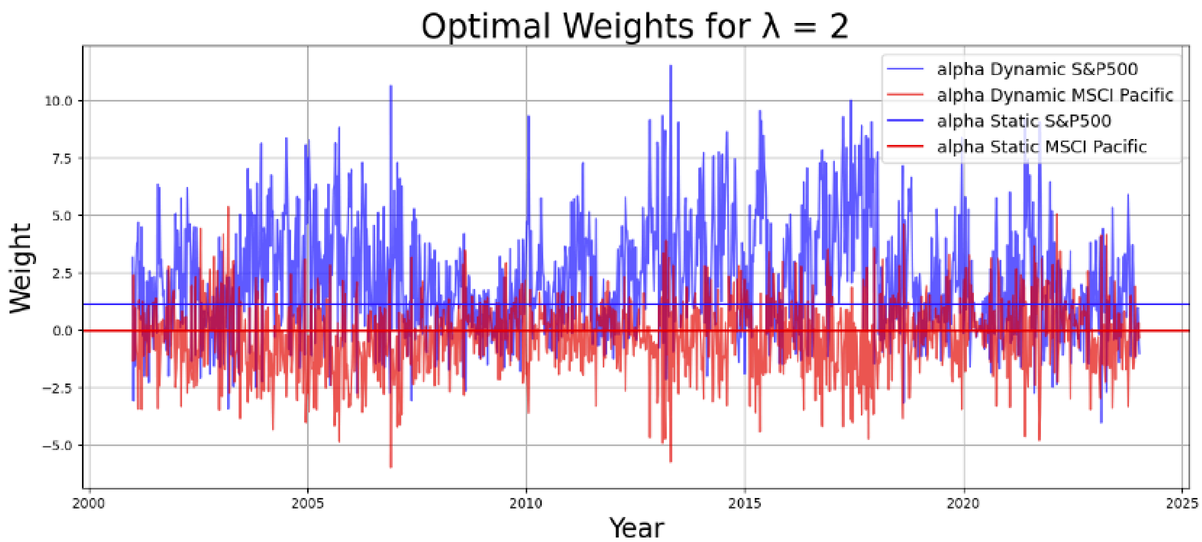
Considering the p-value, we can reject the null hypothesis and affirm with a very high significance (threshold<1%) that α + β < 1 for both securities, implying a stationarity of our model.

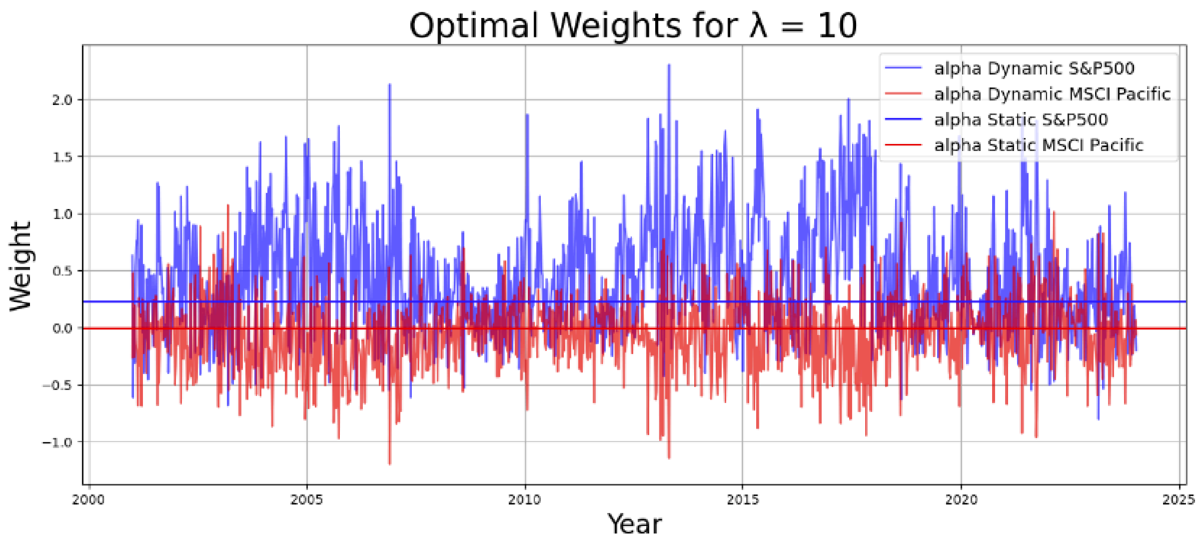
3 Dynamic allocation

Q.3.1

By assuming a constant correlation of residuals over time, we can compute the covariance matrix and maximize the mean-variance to compute the optimal weights for each t.

The following graphs illustrate the changes in the weights of the dynamic portfolio over time for two different levels λ of risk aversion:



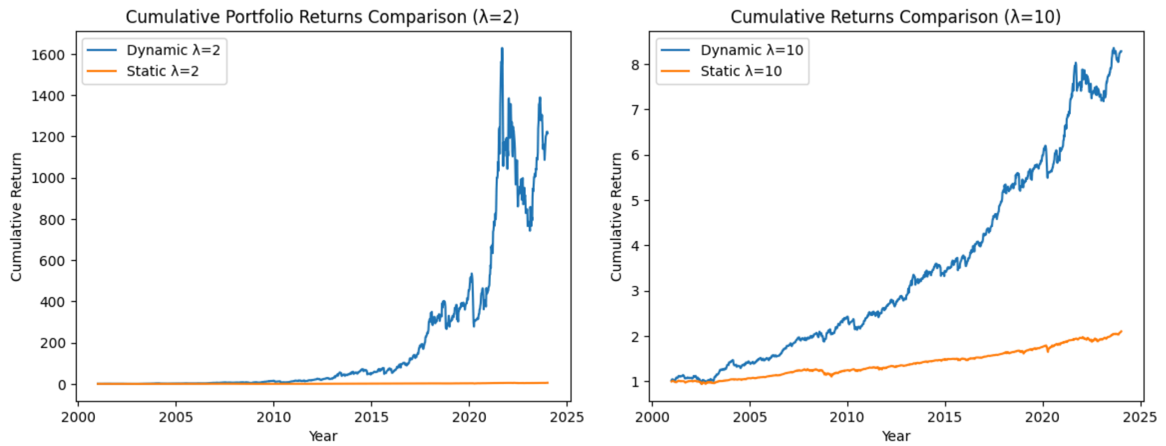


In both cases, the dynamic portfolio weights are extremely volatile, especially the one with λ_2. It's no wonder since a lower λ indicates a lower risk aversion.
Moreover, the weights can be very high the past week and very low the current week, which supports our explanation of the autoregressive coefficient ρ of the AR_1 model.
Finally, for λ_2, the weights invested in S&P500 can exceed 10, involving a position ten times our wealth, which is too risky to be applied in real life.

Q.3.2

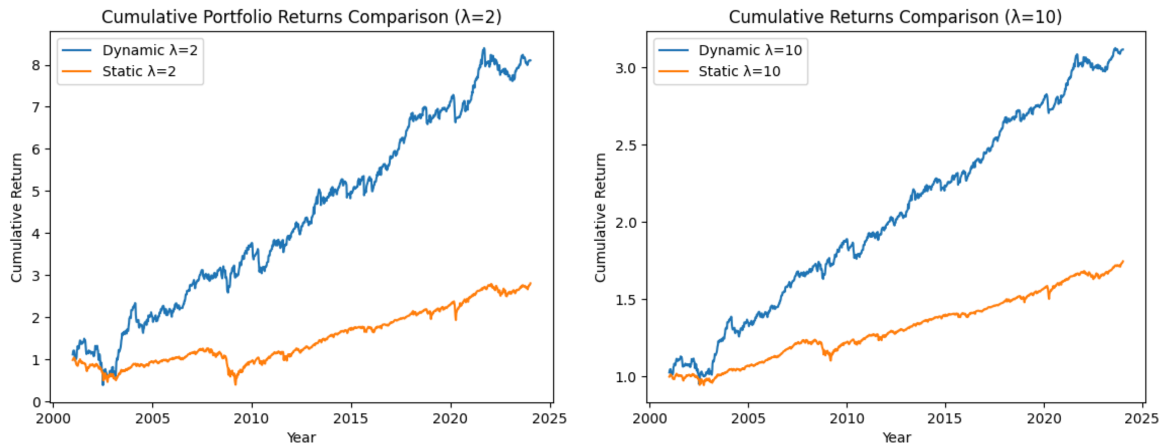
By computing the portfolio returns with our optimal weights, we can calculate the cumulative returns of the portfolios and compare the static portfolio with the dynamic in the two levels of risk aversion:

Cumulative return of the portfolios



As we can see, the dynamic portfolio λ=2 is explosive, hence we added the log to have a better view:

Log cumulative return of the portfolios



The dynamic portfolio significantly outperformed the static portfolio in both levels of risk aversion. Moreover, the λ_2 portfolios performed better than the λ_{10} portfolios in static and dynamic. All these return differences are explained by the risk. The less risk-averse we are, the more we take leverage in positions. Moreover, as explained earlier, dynamic portfolios imply a bigger exposure to market risk. Since more risk means higher potential returns, dynamic allocation is the best strategy to reach the highest return and would always be selected if no other constraint is added.

However, a high exposure implies high potential losses. As we can see, for the dynamic portfolio with $\lambda_{2,}$ the cumulative return moves frenetically, ranging from 1600 to 800. Moreover, we don't take into account the transaction costs. Indeed, in the static case, we only take two positions for the entire investment strategy, while in the dynamic, we take higher positions every week. This will result in very high fees, making the dynamic strategy less attractive than expected.

Q.3.3

In order to be indifferent between dynamic and static portfolios, we would need to charge a transaction fee of 24% for λ_2 and 0.6% for λ_{10}. The difference lies in the unrivaled performance of the dynamic portfolios. Indeed, while the dynamic portfolio with λ_{10} ended with a performance of 800%, the λ_{10} had a total return of 120000%. Since the transaction fees are generally <1%, the dynamic portfolio λ_2 would still be attractive compared to the dynamic portfolio $\lambda_{10,}$ which has a much smaller gain margin. However, as we said earlier, the dynamic λ_2 is unrealistic since the exposure on the market is too big considering the position sizes.

4 Computing the VaR of a portfolio

Q.4.1

After computing the daily returns of our portfolios, we worked on the negative return (loss) to find the unconditional value at risk (VaR) with a confidence interval of θ=99%. The unconditional VaR is the minimum potential loss depending on a specific probability and horizon. In our case, It is the minimum possible negative return we can have in a day with a probability of 1%. In other words, each day, we have a 99% chance not to exceed this negative return. We talk about unconditional VaR because it does not depend on specific market conditions.

To compute it, we have to take the overall distribution of our losses and find the 99% quantile. Indeed, since we negated the returns, the big losses are not anymore on the left tails of the probability density function but on the right tails. Hence, in this case, the unconditional quantile of 99% equals the unconditional VaR of 99%.

We assume a normal distribution, so we first need to find the quantile 99% of a normal distribution $\mathbb{N}(0,1)$, which we call $z_{0.99,}$ and apply a destandardization by using the unconditional mean and volatility of the portfolio losses:

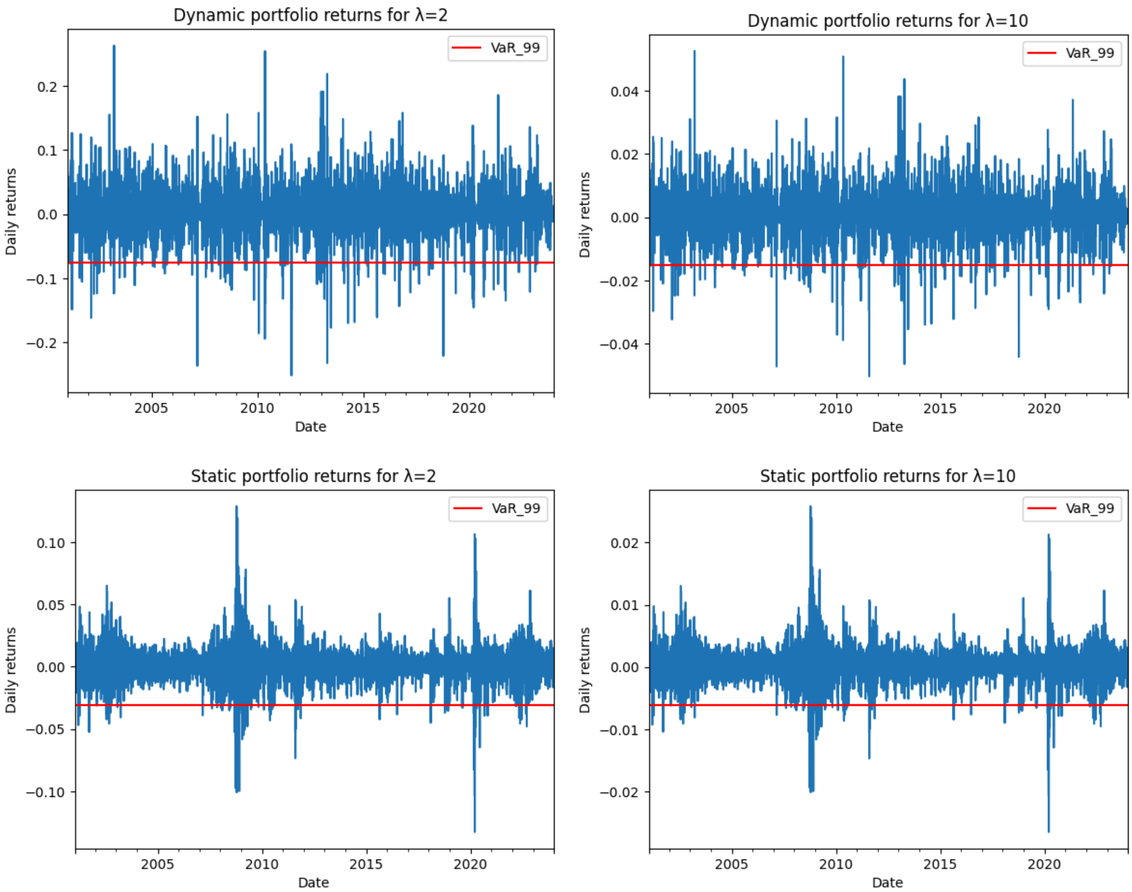
$$VaR_{p,\theta=99\%}^{(Uncond)} = \overline{L_p} + \sigma_p \cdot z_{0.99}$$

Moments and unconditional Value at risk

	Mean	Variance	Value at risk$_{0.99}$
Static portfolio loss λ_2	**-0.000392**	**0.000180**	**0.030823**
Static portfolio loss λ_{10}	**-0.000115**	**0.000007**	**0.006127**
Dynamic portfolio loss λ_2	**-0.001601**	**0.001113**	**0.076025**
Dynamic portfolio loss λ_{10}	**-0.000357**	**0.000045**	**0.015168**

As we can see, for the same levels of risk aversion, dynamic portfolios have a higher VaR than static portfolios, which corroborates our explanation of the market exposure of dynamic portfolios due to higher position sizes. Furthermore, a lower level of risk aversion naturally leads to an increase in risk and, consequently, a higher VaR.

Unconditional Value at risk



Q.4.2

In this question, we use an AR_1-$GARCH_{1,1}$ model to describe the dynamics of our portfolio losses. Thus we assume a non-constant volatility and a non-constant mean for each day.
The conditional VaR or expected shortfall (ES) differs slightly from the VaR. While the VaR gives a threshold for which 1% of our daily loss returns will exceed it, the ES will provide the expected value of the return in the case where the 1% threshold is exceeded. Here, we want the temporal evolution of the portfolios' VaR, meaning we have to compute the expected shortfall for each day.

To do so, we first need to compute the expected mean using the AR_1 parameters and the expected variance with the $GARCH_{1,1}$ for each day. Then, we have to find the temporal evolution of the conditional quantile of a specific probability (θ=99%) of the 1-day loss.

In order to do that, we have to find the distribution of the losses for each day and compute the 99% quantiles. Since we assumed that losses follow a normal distribution, the distribution for each day would be an $\mathbb{N}(\mu_{p,t+1},\sigma^2_{p,t})$, with $\mu_{p,t+1}$ and $\sigma^2_{p,t}$ representing the expected mean and variance of a 1-day loss, respectively. The quantiles can be computed with the same formula we used in Q4.1 and could be represented as the unconditional VaR for each day.

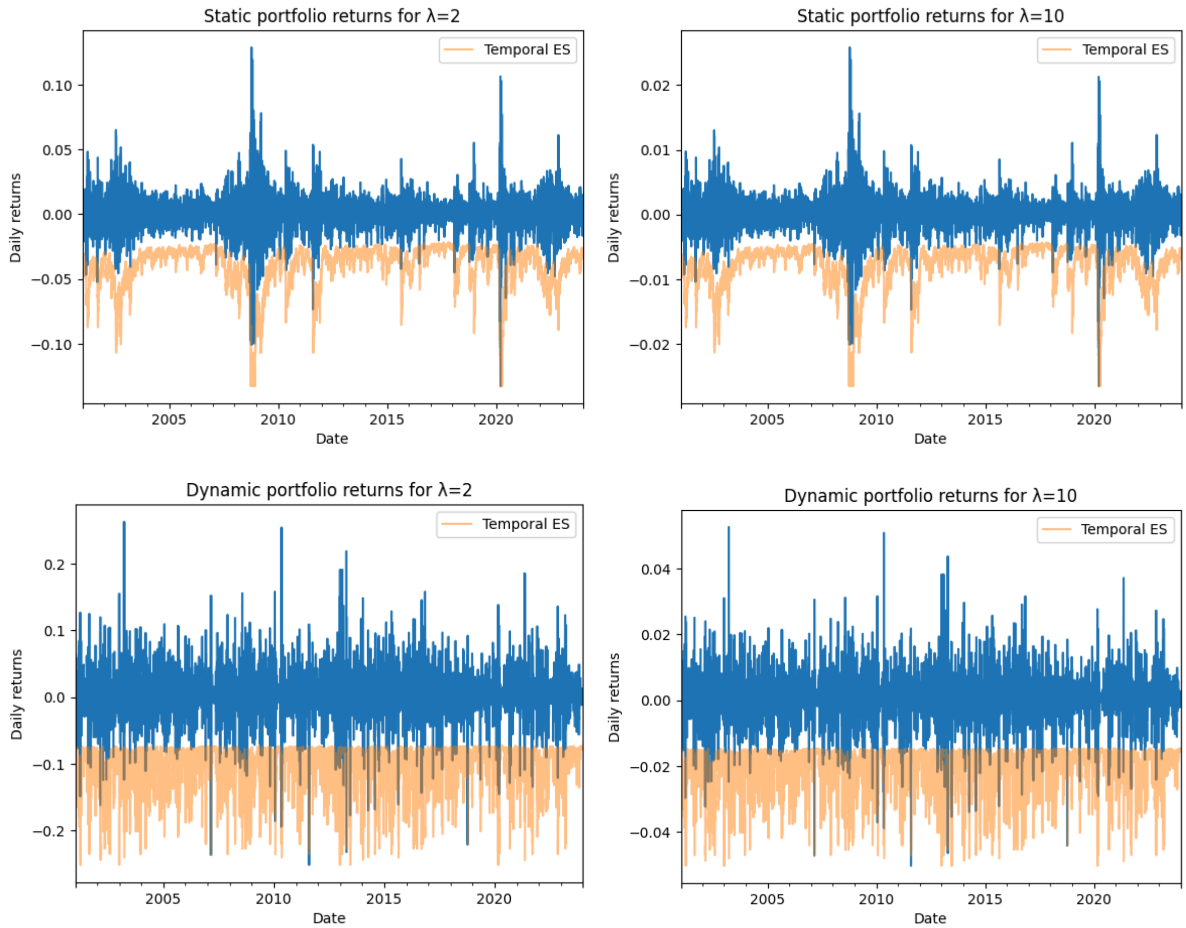
Finally, we can compute the temporal ES_t by taking the average of the 99% quantiles that exceed the quantile q_t.

Results of the AR_1-$GARCH_{1,1}$ estimation

	AR_1 parameters		*$GARCH_{1,1}$ parameters*		
	α_i	*ρ_i*	*ω_i*	*α_i*	*β_i*
Static portfolio loss λ_2	***-0.0004***** ***(0.000)***	***-0.1299****** ***(0.013)***	**2.617e-06***** **(5.413e-07)**	**0.114817***** **(1.168e-02)**	**0.868680***** **(1.231e-02)**
Static portfolio loss λ_{10}	***-0.0001****** ***(3.44e-05)***	***-0.1302****** ***(0.013)***	**1.051e-07***** **(2.166e-08)**	**0.114622***** **(1.167e-02)**	**0.868765***** **(1.230e-02)**
Dynamic portfolio loss λ_2	***-0.0018****** ***(0.000)***	***-0.1523****** ***(0.013)***	**1.780e-04***** **(25.523e-06)**	**0.367908***** **(3.653e-02)**	**0.520910***** **(3.155e-02)**
Dynamic portfolio loss λ_{10}	***-0.0004****** ***(8.54e-05)***	***-0.1524****** ***(0.013)***	**7.115e-06***** **(1.020e-06)**	**0.368238***** **(3.655e-02)**	**0.520845***** **(3.160e-02)**

** $p < 0.1$, ** $p < 0.5$, *** $p < 0.01$*

Conditional Value at Risk$_{(GARCH)}$



When we look at the graphs, the temporal expected shortfall (TES) is smoother for the static portfolios than the dynamic ones. Since static portfolios have, by definition, fixed weights over time, the risk variation would only be driven by change in volatility, while for the dynamic portfolios, the risk variation is also driven by allocation changes. This double variation leads to a higher sensitivity of the dynamic portfolios to market conditions, hence risk measures.

Moreover, since risk aversion only changes the scale of the portfolio returns, the temporal expected shortfall is rescaled in accordance.

Q.4.3

GEV parameters of the daily losses

	Location ω	Scale ψ	Shape ξ
Static portfolio loss λ_2	**2.526554**	**0.725756**	**0.006400**
Static portfolio loss λ_{10}	**2.522825**	**0.729319**	**0.007609**
Dynamic portfolio loss λ_2	**2.519197**	**0.817373**	**0.155086**
Dynamic portfolio loss λ_{10}	**2.517864**	**0.817355**	**0.156579**

Using the maximum likelihood estimation method, we found the parameters of our generalized extreme values model (GEV) stored in the table above.

The first parameter ϖ is the location; it determines the central tendency of the distribution. An increase in the location will shift the distribution to the right while decreasing the opposite. For the portfolios, the results are nearly the same, meaning that the maximums of each quarter of our standardized data are, on average, around 2.50, while the overall average is 0 since the data are standardized.

The second parameter ψ is the scale. It rules the spread of the distribution and has to be positive. As we can see, the dynamic portfolios have a larger ψ, indicating more variability, which is consistent with their nature.

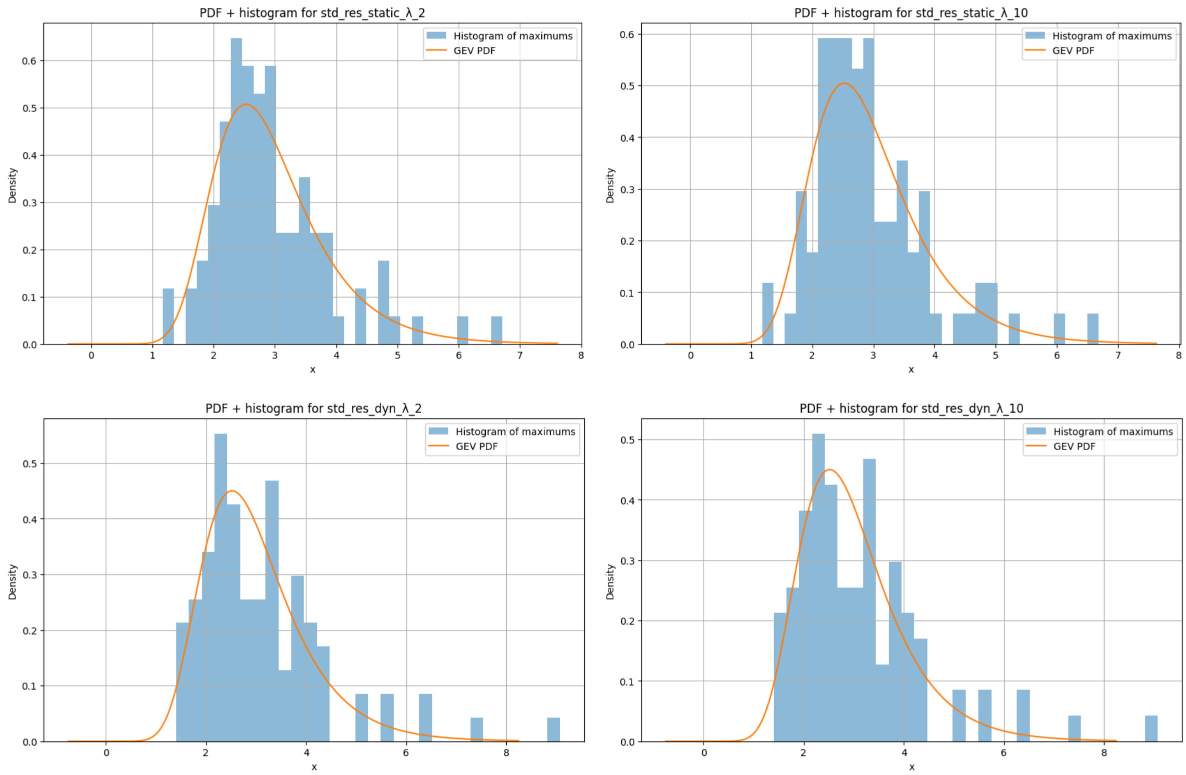
The last parameter, ξ, known as the shape, is the most important one. It defines the tail behavior of the distribution. The distribution is classified into three categories: the Fréchet ($\xi > 0$), the Gumbel ($\xi = 0$), and the Weibull distribution ($\xi < 0$).

For the static portfolios, the value is really close to 0, implying a Gumbel distribution characterized by an exponential tail. This distribution looks the most like the normal distribution. This means that the extreme losses are not as frequent or as large as they could be if the data were following a heavy-tailed distribution.

Concerning the dynamic portfolios, the positive shape implies a Fréchet described by a heavy right-tail. This means that there is a significant probability of extremely large values.

The distributions are consistent with the characteristics of the portfolios. Indeed, the dynamic portfolio is much riskier than the static; hence, it is normal to have more chances of extreme values.

PDF and histogram of the maximums m_τ



Q.4.4

To compute the θ = 99% quantile for the distribution of the maximum $m_{\tau,}$ we can use the following equations depending on the shape's value:

$$\hat{q}_\theta = \hat{\varpi} + \frac{\hat{\psi}}{\hat{\xi}}\left[(-\ln(\theta))^{-\xi} - 1\right] \text{ if } \xi > 0$$

$$\hat{q}_\theta = \hat{\varpi} - \hat{\psi}\ln(-\ln(\theta)) \text{ if } \xi = 0$$

Once we have done that, we can deduce the 99% quantile of the standardized data. Indeed, we assumed that standardized residuals are iid. Hence, the relation between the probability of the standardized and maximums would be $\theta^{*}=\theta^{1/N}$ with N the number of observations in a sub-sample. This relation leads to the following quantile equations :

$$\hat{q}^*_\theta = \hat{\varpi} + \frac{\hat{\psi}}{\hat{\xi}}\left[(-N\ln(\theta))^{-\xi} - 1\right] \text{ if } \xi > 0$$

$$\hat{q}^*_\theta = \hat{\varpi} - \hat{\psi}\ln(-N\ln(\theta)) \text{ if } \xi = 0$$

⚠ Note: Since we split data quarterly according to the date, our N is not equal to 60 but 65. We divided the number of observations $z_{p,t+1}$ by the number of maximums: 5989/92 ≈ 65.

Quantile of the distributions (θ = 99%)

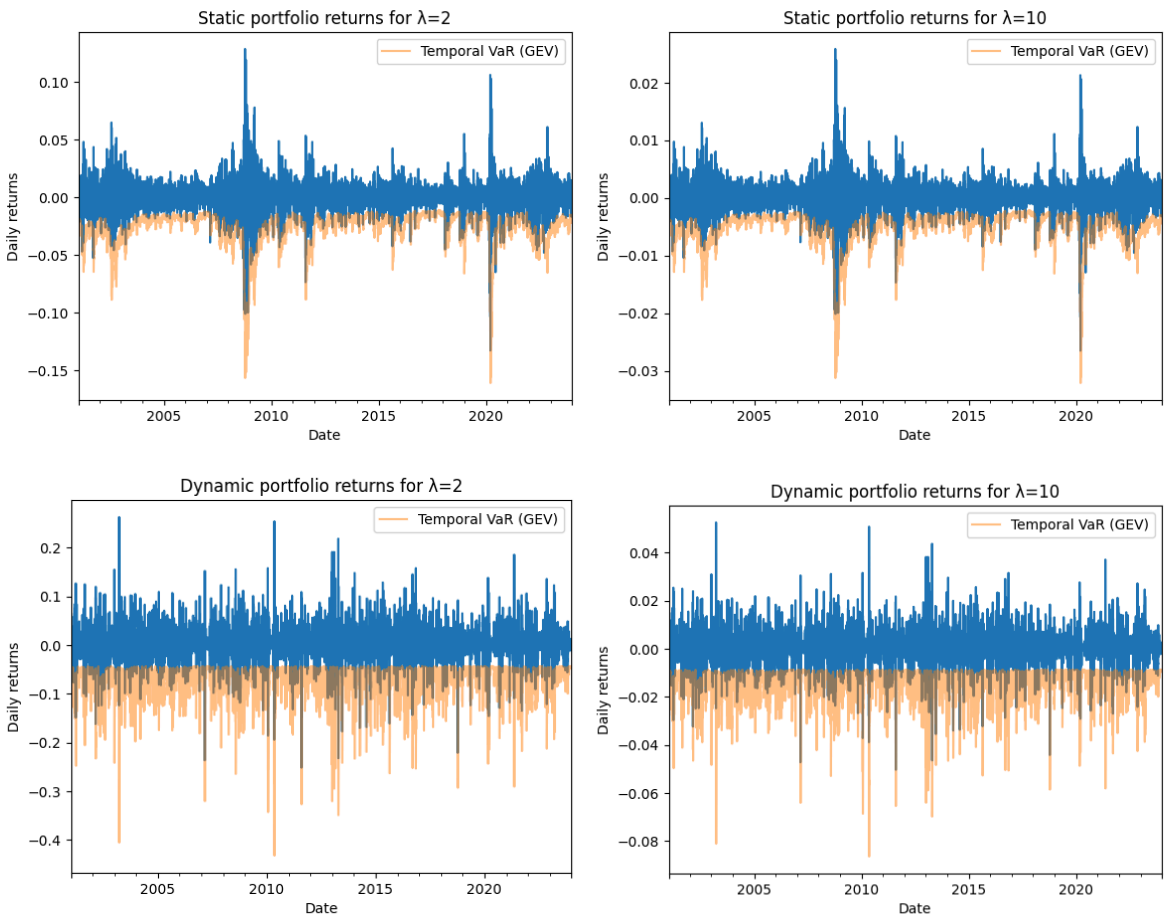
	Quantile m_τ	*Quantile $z_{p,t+1}$*
Static portfolio loss λ_2	***5.865139***	***2.834461***
Static portfolio loss λ_{10}	***5.877801***	***2.832244***
Dynamic portfolio loss λ_2	***8.005504***	***2.877636***
Dynamic portfolio loss λ_{10}	***8.025159***	***2.876410***

This can be interpreted as the minimum value exceeded in 1% of cases. We assumed that the losses follow a normal distribution; hence, the standardized residuals should follow a $\mathbb{N}(0,1)$ with $z_{0.99} \approx 2.32$, which is lower than our results. This difference results in the fact that the losses don't really follow a normal distribution, and these nuances are captured by the GEV model.

Q.4.5

To compute the temporal evolution of the loss distribution's 99% quantile, we have to use the same formula as in Q4.1, except this time, we replace $z_{0.99}$ with the 99% quantile of the distribution of the $z_{p,t+1}$ computed in Q4.4. Since we are not talking about expected shortfall, the quantile just computed is the $VaR^{(GEV)}$.

Value at risk$_{(GEV)}$ of the daily return portfolios



At first glance, the results seem pretty similar to the VaR$^{(GARCH),}$ with smoother temporal VaR for the static portfolios than the dynamic ones. However, when we look closer, we can see that the value at risk has less volatility than the VaR$^{(GARCH)}$, indicating better accuracy.

Q.4.6

To compare the different VaRs, we have to know the assumptions and understand the methods used.

The VaR$^{(uncond)}$ assumes that the losses are iid and that the distribution remains constant over time, which is not the case. It's very easy to implement, but it ignores a lot of parameters. Indeed, we know that financial return distributions generally have fat tails. This is entirely ignored since we assume a normal distribution. Moreover, for the same reason, a constant mean and variance disregard the potential changes in market situations and volatility clustering. Hence, the VaR$^{(uncond)}$ is just a “line”, which doesn't vary over time and fails to show the dependency of the losses over time.

Concerning the VaR$^{(GARCH)}$, we integrate the AR$_1$ and the GARCH$_{1,1}$ model. This is much better since we can capture the losses' autocorrelation and volatility clustering. This leads to a more dynamic and realistic modeling. Unlike the $VaR^{(uncond)}$, it adapts to new information, thus providing a more accurate measure of risk. Moreover, the AR_1 model considers the mean-reversion behavior in financial returns. However, we supposed that the residuals of the AR_1 model follow a normal distribution, which is a problem in capturing the tail's behavior.

Finally, the $VaR^{(GEV)}$ is defined with the GEV model. This model takes into account the autocorrelation and volatility clustering by using the expected mean and variance from the AR_1-$GARCH_{1,1}$ model. Furthermore, this model is specifically designed to model the tail behavior of the distribution of extreme values. Since we are looking for $VaR_{0.99,}$ which is located in the extreme values, the $VaR^{(GEV)}$ is probably the best estimator we can have.