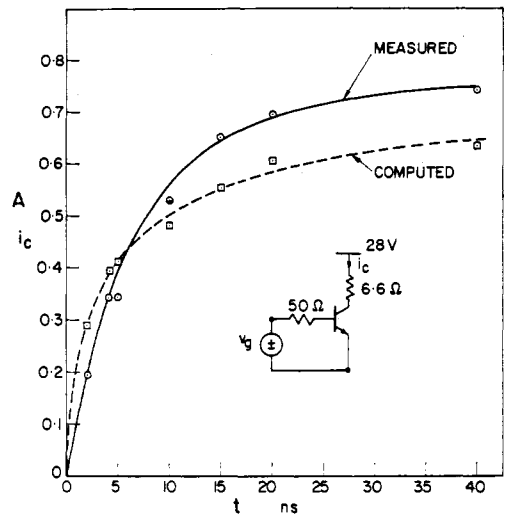

Fig. 2. Measured and computed time response i_c to a step input for the circuit shown. Note: The L/R time constant due to load lead inductance is about 0.7 ns and is not included in the computed curve.

TABLE II

MEASURED AND COMPUTED VALUES OF SMALL-SIGNAL CURRENT GAIN AS A FUNCTION OF FREQUENCY FOR THE CIRCUIT OF FIG. 2 WITH A DC BIAS $v_0 = 1.8$ V

Frequency cycles/s	1	10^7	3×10^7	10^8	3×10^8
Computed h_{fe}	21.5	20.3	13.8	6.35	2.6
Measured h_{fe}	27	27	21	8.5	2.0

where the values a_{ij} are obtained from the tabulated points. Applying Laplace transformation to (3), we get

$$I_{c_1}(s) = (a_{11} + a_{31}s)V_{cb_1}(s) + (a_{12} + a_{32}s)V_{eb_1}(s) - a_{31}v_{cb_1}(t_0) - a_{32}v_{eb_1}(t_0)$$
$$I_{e_1}(s) = (a_{21} + a_{41}s)V_{cb_1}(s) + (a_{22} + a_{42}s)V_{eb_1}(s) - a_{41}v_{cb_1}(t_0) - a_{42}v_{eb_1}(t_0) \quad (4)$$

where $v_{cb_1}(t_0)$ and $v_{eb_1}(t_0)$ represent the initial conditions. Equations (4) can now be easily used to formulate the equations of the circuit in which the transistor is used. In our analysis method we use nodal formulation. Note that for a multisectional model, each section is characterized by an equation of form (4), but with possibly different values of a_{ij}, depending on the linear region in which the section is operating. Similarly, the values of the base resistances R_{bi} depend on the operating region.

The problem of finding the transient response can now be summarized as follows: starting at time $t = t_0$ with the circuit operating in a known initial region on the PWL characteristics, and with known initial conditions (the initial operating region and the initial conditions can be obtained by finding the dc solution of the circuit at time $t = t_0$ [2]), we find the response of the circuit within the operating region using the numerical inverse Laplace technique. When a region boundary is crossed, the circuit element values and initial conditions are updated to correspond to the new operating region, and the circuit is again solved, until the time interval in which the response is required has expired.

EXAMPLE

The above method has been implemented as part of the WATAND computer program system [6]. The transistor circuit shown in Fig. 2 was both simulated using the computer program and tested in the laboratory. Fig. 2 also shows the measured and computed time response of the collector current i_c for a 2-V step input, v_0. Table II shows measured and computed values of the small-signal current gain as a function of frequency for given dc bias conditions. The input data to the program consisted of device physical data (mask dimensions, sheet resistivities, junction depths, and emitter region carrier lifetime). It is felt that the results are good, when it is remembered that there is a considerable spread in the real device physical parameters. The generation of the device characteristic tables required on execution time of about 11 s and a core memory of about 60 kbytes on an IBM 360/75 computer, and the circuit analysis required about 6 s with 80-kbyte core memory.

It should be pointed out that the previously described method is also suitable for linear analyses. The small-signal parameters of the transistors in a given circuit are automatically available in any operating region, and small-signal analyses can be performed at a minimal cost.

ACKNOWLEDGMENT

The authors gratefully acknowledge device fabrication data supplied by Thomson CSF, Dept. DMH, Orsay, France, for the transistor used in the examples presented in this work.

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Diffraction Calculations Using Fast Fourier Transform Methods

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***Abstract*—By using a coordinate transformation based on Gaussian-beam theory, we can apply efficient fast Fourier transform (FFT) computational methods to diffraction problems involving spherically diverging or converging waves as well as to quasi-collimated beams of radiation.**

I. INTRODUCTION

We present a coordinate transformation based on Gaussian-beam theory that makes it possible to apply spatial frequency methods, and especially the very efficient fast Fourier transform (FFT), to diffraction problems involving spherically diverging and converging waves, by transforming these problems into equivalent collimated-beam problems. This transformation should be of considerable value for optical diffraction, antenna-pattern, and acoustic propagation calculations.

II. THE SPATIAL-FREQUENCY TRANSFORM METHOD

In solving diffraction problems within the scalar-wave and paraxial approximations, one seeks solutions to the paraxial wave equation [1]

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - 2jk\frac{\partial u}{\partial z} = 0 \quad (1)$$

where the scalar-wave amplitude is of the form $u(x, y, z) \exp(j\omega t -$

Manuscript received September 10, 1973.
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jkz), and the z dependence of $u(x, y, z)$ is assumed slow compared to $k \equiv 2\pi/\lambda$. Given the input wave function $u_1(x, y)$ at an input plane z_1, one can calculate the output wave $u_2(x, y)$ at a later plane z_2 by first forming the spatial-frequency Fourier transform of the input wave, i.e.,

$$P_1(\nu_x, \nu_y) = \int\int u_1(x, y) \exp(j2\pi\nu_x x + j2\pi\nu_y y)dx\,dy. \quad (2)$$

The spatial-frequency or plane-wave components are then propagated to the output plane z_2 by writing (in the Fresnel approximation)

$$P_2(\nu_x, \nu_y) = P_1(\nu_x, \nu_y) \exp\left[j(\pi\lambda)(\nu_x^2 + \nu_y^2)(z_2 - z_1)\right]. \quad (3)$$

Finally, the output wave is then recovered by the inverse Fourier transformation

$$u_2(x, y) = \int\int P_2(\nu_x, \nu_y) \exp(-j2\pi\nu_x x - j2\pi\nu_y y)d\nu_x d\nu_y. \quad (4)$$

This spatial-frequency approach to diffraction and beam-propagation problems is by now well known [2], [3].

III. Difficulties with Spherical-Wave Cases

For efficient numerical calculations, the Fourier transformations (2) and (4) can be carried out using the FFT [4]. The spatial-frequency approach combined with FFT should provide a faster and more accurate calculational procedure than, for example, direct numerical evaluation of the equivalent Huygens integral calculation. The computational efficiency of the FFT procedure is reduced, however, if the optical beam or acoustic wave under consideration is primarily a spherically diverging or converging beam, in which some comparatively slowly varying transverse amplitude and phase structure multiplies a basically spherical wavefront.

This difficulty may be expressed in two ways. The FFT process, straightforwardly applied, requires that the same equally spaced discrete sampling grid be employed at both transverse planes z_1 and z_2. The transverse width over which the wave function is expanded must be wide enough to cover the function at the wider end of the beam (plus a guard band), while the spacing between sample points in the grid must be small enough to describe the function with adequate accuracy at the narrower end of the beam. Together these criteria increase the number of sample points that must be used in the discrete Fourier transformations.

To put this in another way, the spherical-wave factor in the converging or diverging beam increases the spread of the transform in wave-vector space (i.e., in the ν_x and ν_y domains) without a corresponding increase in real information. The increased spread in the frequency domain then requires more closely spaced samples in the real space coordinates.

IV. The Coordinate Transformation

We give, in this section, a coordinate transformation for avoiding the spherical-wave difficulties just described. A less general form of this transformation has been suggested earlier by Bradley and Hermann [5]. The coordinate transformation is carried out by first extracting an arbitrary lowest order Gaussian spherical wave

$$[1/q(z)] \exp\left[-jk(x^2 + y^2)/2q(z)\right]$$

from the function $u(x, y, z)$ by writing

$$u(x, y, z) = [v(x', y', z')/q(z)] \exp\left[-jk(x^2 + y^2)/2q(z)\right] \quad (5)$$

where $q(z)$ is the complex curvature of the Gaussian beam, governed by $dq(z)/dz = 1$. If z is measured from a beam waist, then $q(z) = z + jb$, where $b \equiv \pi w_0^2/\lambda$ is the confocal parameter for a Gaussian beam with waist spot size w_0 [1].

The remaining functional variation $v(x', y', z')$ is then expressed in terms of the complex-transformed coordinates

$$x'(x, z) = \frac{\alpha x}{q(z)} = \frac{\alpha x}{z + jb}$$

$$y'(y, z) = \frac{\alpha y}{q(z)} = \frac{\alpha y}{z + jb}$$

$$\frac{dz'(z)}{dz} = \frac{\alpha^2}{q^2(z)}$$

$$z_2' - z_1' = \frac{\alpha^2(z_2 - z_1)}{(z_1 + jb)(z_2 + jb)} \quad (6)$$

where α is an arbitrary complex constant. When this transform is carried through, it is found that the function $v(x', y', z')$ then obeys

$$\frac{\partial^2 v}{\partial x'^2} + \frac{\partial^2 v}{\partial y'^2} - 2jk\frac{\partial v}{\partial z'} = 0 \quad (7)$$

i.e., exactly the same paraxial wave equation as for $u(x, y, z)$ but now in the primed coordinate space. Equation (7) can then be solved for $v(x', y', z')$ by the same transform methods (2)–(4) operating in the primed coordinate space. However, because an underlying beam divergence can be transformed out, the variation of $v(x', y', z')$ in the primed coordinate system can be slower and more confined than the variation of $u(x, y, z)$ in the real coordinate system.

V. Discussion: Two Limiting Cases

The solution and interpretation of (7) in the primed coordinate space is made more difficult by the complex values of the primed coordinates that correspond to real values of the real coordinates x, y, z. It may be that in some cases known analytic solutions to the paraxial wave equation can be extended into the primed coordinate space by analytic continuation, depending upon the analyticity of the input functions $u_1(x, y)$ and $v_1(x, y)$ and the locations of their singularities. We have not as yet explored this point.

The primed coordinates can be made entirely real, however, so that standard Fourier methods and especially the FFT can be applied to (7), in either of two limiting cases. In the first and essentially trivial case, we take the arbitrary confocal parameter b to be much longer than any distance z over which solutions of (1) are to be obtained. We also take the arbitrary constant α to have a purely imaginary value, say, $\alpha = jB$. The coordinate transformation (5), (6) then becomes nothing more than a trivial linear scaling of the real coordinate system by a scale factor B/b.

The opposite limiting case provides the main motivation for this letter. For problems involving spherically diverging or converging waves, let the origin of the z coordinate be located at the focus of the spherical wave, and let b be chosen much smaller than the magnitudes of any of the distances z at which $u(x, y, z)$ is to be calculated. Let α also be chosen purely real. The coordinate transformations are then given by

$$x' = \frac{\alpha x}{z}$$

$$y' = \frac{\alpha y}{z}$$

$$z_2' - z_1' = \alpha^2 \frac{z_2 - z_1}{z_1 z_2}. \quad (8)$$

A grid of constant width in the primed x', y' system now corresponds to a grid in the real x, y space that is linearly expanding with distance z (or linearly contracting for $z < 0$). A function $u(x, y, z)$ diverging linearly with distance in the real coordinate system will have a near-constant width in the primed coordinate system, and the function $v(x', y')$ can be efficiently calculated using a fixed-grid FFT method. This expanding-grid transformation has recently been applied with excellent results to an unstable optical-resonator mode calculation [6], where diverging spherical waves are inherent in the physical problem.

VI. Discussion: The Fresnel Number of a Spherical Wave

Consider a diverging spherical wave with radius of curvature R_1 propagating from an aperture of width $2a$ at plane z_1 on out to $z_2 \to \infty$. Let us transform this problem into the primed coordinate system and then evaluate the Fresnel number of the resulting transformed problem. The input coordinate in this case will have the value $z_1 = R_1$, with $z_2 \to \infty$, and with $b \ll R_1$. If we use primes to denote all quantities in the transformed coordinate system, then the transverse width of the input function $v_1(x', y')$ in the primed coordinate system will be $2a' = 2\alpha a/z_1 = 2\alpha a/R_1$, and the propagation distance from z_1' to z_2' will be $L' \equiv z_2' - z_1' = \alpha^2(z_2 - z_1)/z_1 z_2 = \alpha^2/R_1$ in the limit as $z_2 \to \infty$. The

Fresnel number in the primed space will then be given by

$$N' = \frac{a'^2}{L'\lambda} = \frac{a^2}{R_1\lambda}. \tag{9}$$

Thus the propagation of a diverging wave of initial radius R_1 and width $2a$ all the way out to infinity in real space is exactly equivalent to the propagation in the primed space of a bounded quasi-collimated beam through a much shorter distance defined by a Fresnel number $N' = a^2/R_1\lambda$. This number is, of course, the ratio of the purely geometrical divergence a/R_1 to the "diffraction divergence" λ/a of the aperture. A large Fresnel number N' means a short collimated-beam propagation distance in the primed coordinate space.

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A Simple Generalization of Luenberger Method for Pole Assignment

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***Abstract*—A simple generalization of the Luenberger method for pole assignment is introduced which allows one to obtain a greater number of feedback laws giving both the desired poles to the system and the choice of a larger number of solutions to the designer.**

I. Introduction

In the synthesis of control systems it is of great interest to determine a feedback law so that the system assumes arbitrary dynamics. In this field, important results have been obtained by Wonham [1], Anderson and Luenberger [2], Simon and Mitter [3]. The common basic idea of these papers is the use of proper canonic forms in order to avoid the troubles due to the nonlinearity of the equation system which gives the terms of the feedback matrix necessary for a given pole configuration; unfortunately, these methods give only a limited number of *all* the possible solutions. In the present letter we introduce a simple method for enlarging the number of solutions obtainable by the pole-assignment procedure introduced by Luenberger.

II. Pole Assignment Procedure

Let us consider the linear stationary system:

$$\dot{x} = Ax + Bu \tag{1}$$

where x and u are, respectively, the $(n\times 1)$ state vector and the $(r\times 1)$ control vector, A and B are matrices of proper dimensions. As is well known, the problem of pole assigning implies determination of the elements of a matrix K such that the matrix $(A+BK)$ has the desired eigenvalues. More precisely, given the arbitrary set of scalars $S = \{s_1, s_2, \cdots, s_n\}$ such that every s_i with $\mathrm{Im}\,[s_i] \neq 0$, appears with the corresponding complex conjugates, the $(r\times n)$ elements of K must be determined so that the following identity holds:

$$\det\,(A + BK - Is) = \prod_{i=1}^{n} (s_i - s). \tag{2}$$

Equation (2) is equivalent to an n-equation nonlinear system, with $n\times r$ unknowns, which has an infinite number of solutions.

Before introducing the main results of this letter, let us give some definitions and make some preliminary considerations. First of all, defining

$$B = [b_1 \mid b_2 \mid \cdots \mid b_i \mid \cdots \mid b_r]$$

(where b_i is the ith column of the matrix B), the following definition can be given:

Definition: For a given system of the type (1), let us call elementary subspace of controllability $\mathcal{R}_{r_i}$ the controllability subspace corresponding to every input vector $b_i (i = 1, 2, \cdots, r)$.

Obviously, if the system is controllable, one obtains:

$$\mathcal{R}^n = \mathcal{R}_{r_1} \cup \mathcal{R}_{r_2} \cup \cdots \cup \mathcal{R}_{r_i} \cup \cdots \cup \mathcal{R}_{r_f}, \qquad \text{where } f \leq r.$$

Let us note that such elementary subspaces of controllability are not only cyclic subspaces, but also of dynamic noninteraction [6] for the system (1). In fact, let us remind that a noninteraction subspace is nothing but a controllability subspace for which the controls have been restricted to a given subspace of control; in this case $\mathcal{R}_{r_i}$ is the subspace reachable by means of the control law $u(t)$ having all components null except the ith.

These elementary subspaces are the main tool for the pole-assignment procedure presented in [1], [2]; namely, one can choose any subset (of this set of subspaces) which overlaps the whole space state and determines the corresponding feedback matrix, which gives the system a preassigned pole configuration. Let us indicate by t the number of all feedback matrices which can be found with such a procedure; obviously, this number will coincide with the number of all the possible previously mentioned subsets, that is, with the number of all possible different combinations of inputs which completely control the system (in fact, all the input vectors b_i, $i = 1, 2, \cdots, r$, may not be necessary for the system to be controllable). However, the previously mentioned set of elementary subspaces of controllability do not contain the totality of cyclic subspaces of dynamic noninteraction for the system (1) and it can be seen that other sets of such subspaces do exist and hence further matrices for the pole assignment can be found.

First of all, let us find a way to obtain these sets; to this end, if we determine the least A-invariant subspaces (see Theorem A-3 in the Appendix) and we add them in every possible way we obtain the totality of invariant subspaces.[1] Using Definition (A-2) (from the Appendix) and Theorem (A-4) (from the Appendix) we can select between these invariant subspaces those that are both cyclic and of dynamic noninteraction for the system (1).

In conclusion, let us assume that we have determined any set V_j made up of any number l of noninteraction cyclic subspaces $\mathcal{S}_q$, where $q = 1, 2, \cdots, l$, such that their union contains the whole state space and none of them can be contained by the union of others, that is:

$$\bigcup_{q\neq p} \mathcal{S}_q \neq \bigcup_{q=1}^{l} \mathcal{S}_q = \mathcal{R}^n \qquad (l \leq r)$$

where $\mathcal{S}_q$ and $\mathcal{S}_p$ are any elements of V_j.

According to Theorems (A-1) and (A-2) (in the Appendix), we can build up, column by column, an $(r\times r)$ matrix G_j such that the noninteraction cyclic subspaces of V_j become elementary subspaces of controllability for the system (A, BG_j), that is (A, B_j) if we define $B_j = BG_j$.[2]

As can be seen in [1] and [2], for a completely controllable system like (A, B_j) it is always possible to obtain t transformation matrices T_{ji}; each putting the system into forms of the type:

Manuscript received November 10, 1972; revised July 11, 1973. This work was supported by the Consiglio Nazionale delle Ricerche, Italy.

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[1] According to Theorem (A-3) (in the Appendix) and to the uniqueness of the Jordan canonical form, the subspaces thus found are the totality of the A-invariant subspaces.

[2] Roughly, one can say that the matrix G_j makes a kind of preprocessing in order to change the elementary subspaces of controllability.