Big Data Computing

Master's Degree in Computer Science

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SAPIENZA
UNIVERSITÀ DI ROMA

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Recap from Last Lecture(s)

- High-dimensional naïve representation (i.e., feature space) of text data

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- High-dimensional naïve representation (i.e., feature space) of text data
- Clustering high-dimensional data may be problematic
  - Due to the curse of dimensionality
- Many other data sources (e.g., images) share the same issue
- Good news! High-dimensionality is often not real!
  - Due to the way in which we observe/collect data

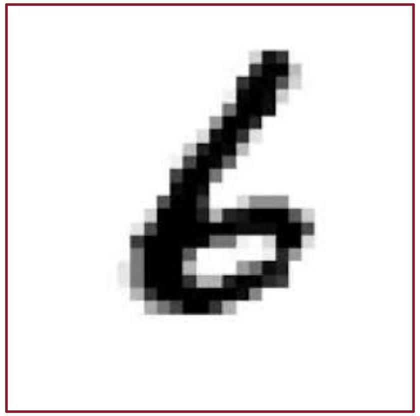
DIMENSIONALITY REDUCTION

Modeled vs. True Dimensionality

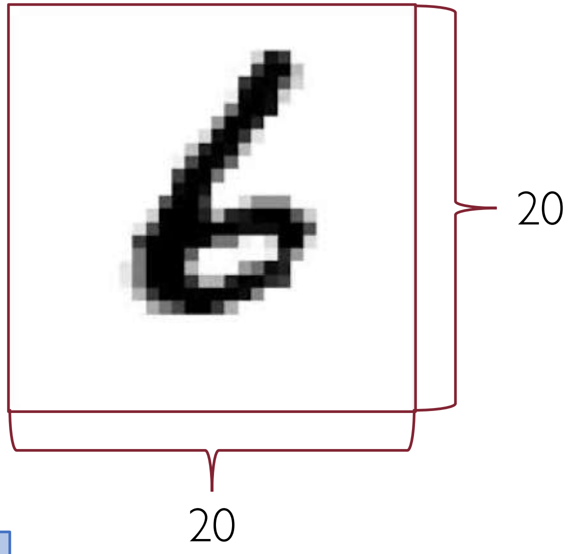
Example

Handwritten digit recognition

Modeled vs. True Dimensionality



Modeled vs. True Dimensionality

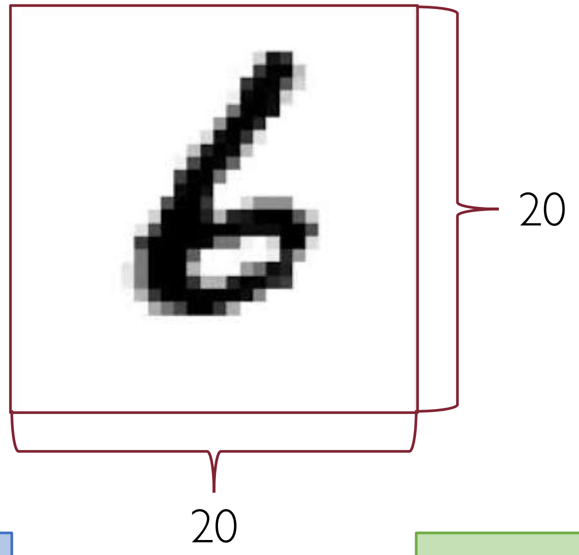


Modeled dimensionality

Each digit represented by 20x20 bitmap

400-dimensional binary vector

Modeled vs. True Dimensionality



Modeled dimensionality

Each digit represented by 20x20 bitmap

400-dimensional binary vector

True dimensionality

Actual digits just cover a tiny fraction of all this huge space

Small variations of the pen-stroke

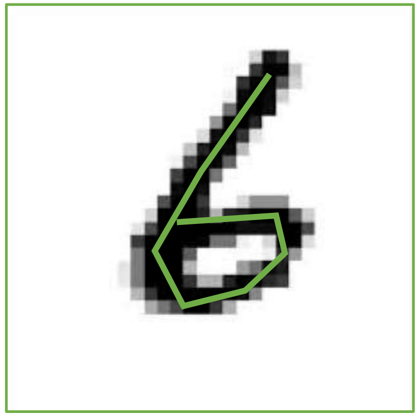
Modeled vs. True Dimensionality

Random samples from 400-d space

Modeled vs. True Dimensionality

Random samples from 400-d space

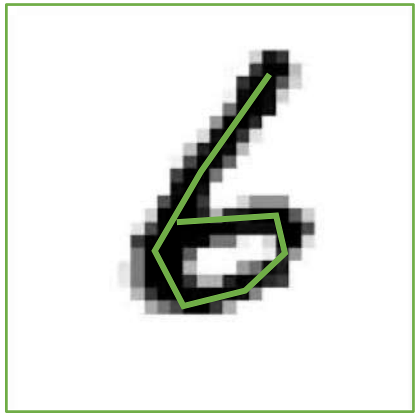
True digits living in a 400-d space



Modeled vs. True Dimensionality

Random samples from 400-d space

True digits living in a 400-d space



We model data (i.e., digits) as very high dimensional…

… In fact, they are not so

The Curse of Dimensionality

As dimensionality grows fewer examples in each region of the feature space
(assuming # examples is constant)

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The number of examples must grow exponentially with dimensionality if we want to maintain the same "density"

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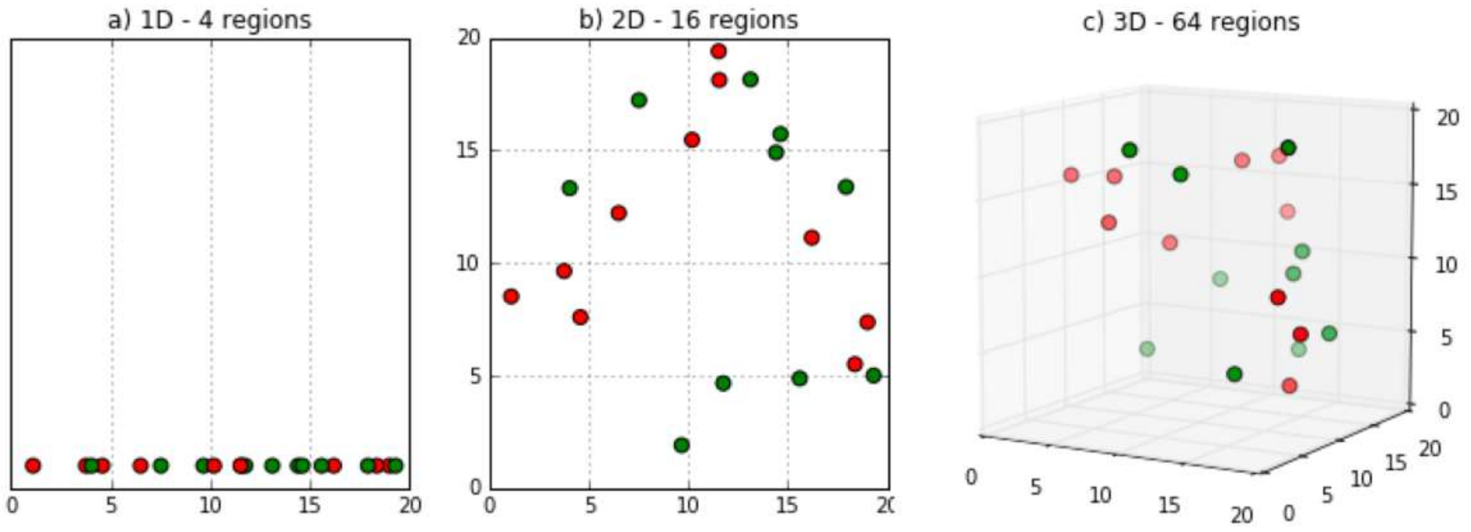

Dealing with High Dimensionality

3 possible approaches

Feature Engineering
(using domain knowledge)

e.g., SIFT in computer vision

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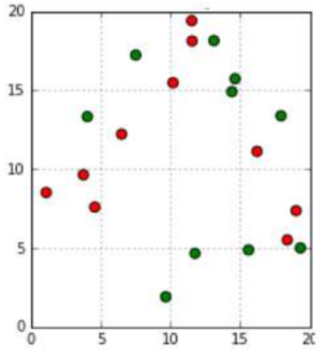
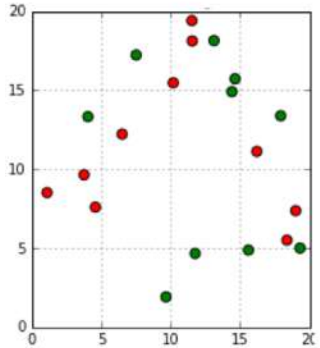
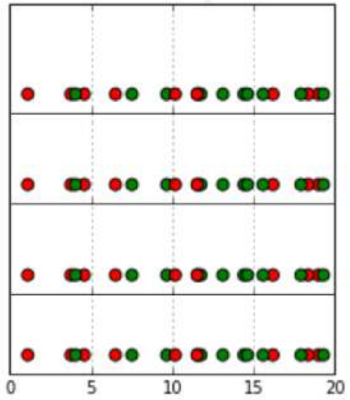
Making Assumptions



Dealing with High Dimensionality: Assumptions

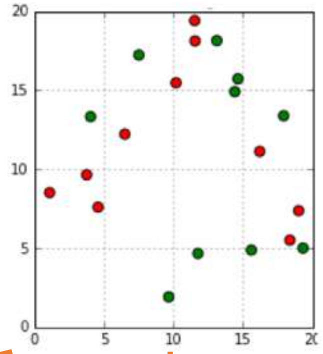


independence



Count along each dimension separately

Dealing with High Dimensionality: Assumptions



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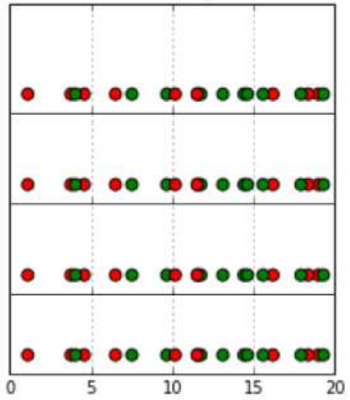
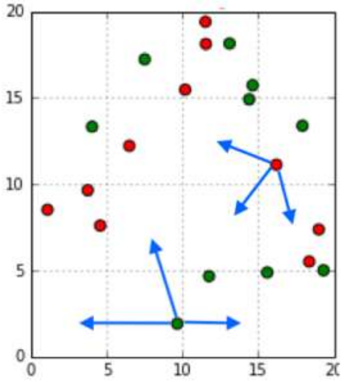


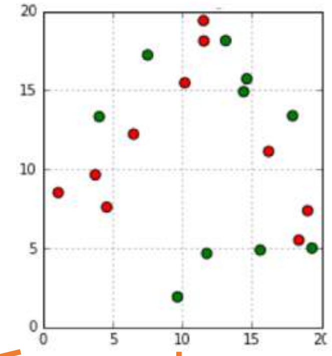
Count along each dimension separately

smoothness



Propagate counts to neighboring regions

Dealing with High Dimensionality: Assumptions



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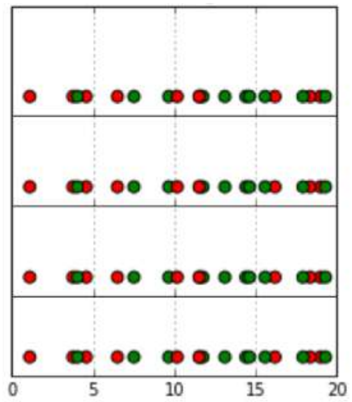

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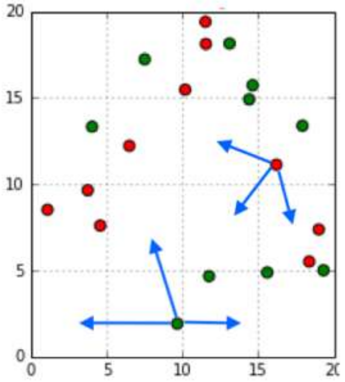
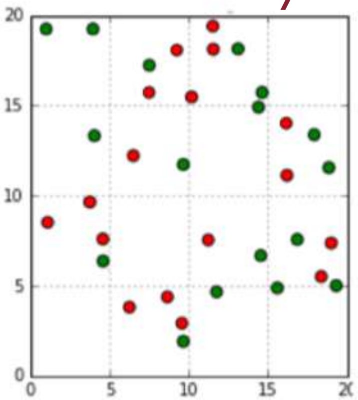


Propagate counts to neighboring regions

simmetry



Invariance to the order of dimensions

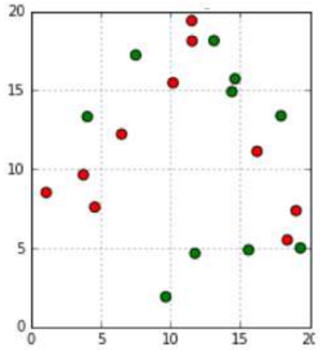
Dealing with High Dimensionality

3 possible approaches

Feature Engineering
(using domain knowledge)

e.g., SIFT in computer vision

Making Assumptions



Reduce Dimensionality

Create a new set of dimensions (i.e., variables)

Dimensionality Reduction

- A technique to unveil the actual (i.e., meaningful) dimensions of data
- A pre-processing step for representing data with fewer features
- Preserve as much "structure" of the data as possible
- Retained structure must be discriminative affecting data separability

"structure" here means **variance**

Dimensionality Reduction

2 main approaches

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Feature Selection

Pick a subset of the original dimensions that are good predictors (e.g., using information gain)

$x_1, x_2, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_{d-1}, x_d$

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Feature Extraction

Build a new set of $k < d$ dimensions as a (linear) combination of the originals

$e_1, e_2, \ldots, e_k$

$e_i = f(x_1, x_2, \ldots, x_d)$

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Dimensionality reduction technique based on feature extraction

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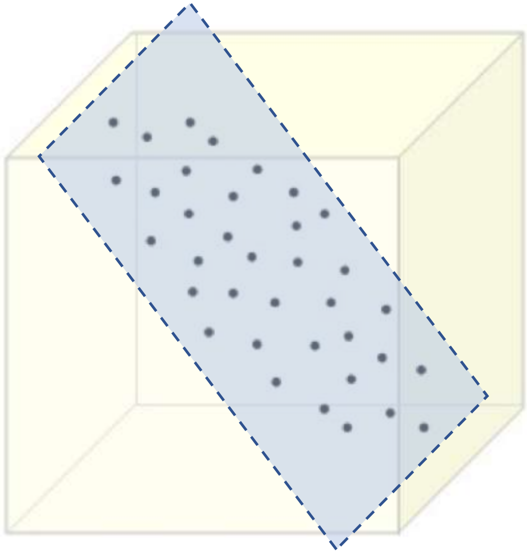
Example

A 3-d set of points embedded into a 2-d hyperplane



Principal Component Analysis (PCA)

PCA defines a set of principal components as follows:

- 1st: direction of the greatest variance of data
- 2nd: perpendicular to 1st and greatest variance of what's left
- … and so on until d

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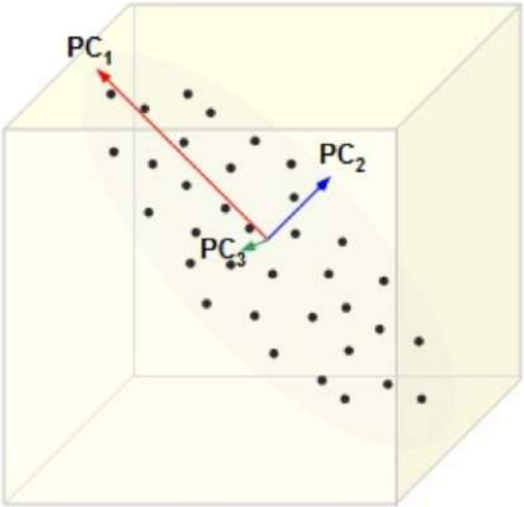
The top k < d components become the new dimensions



e.g., k = 2

Principal Component Analysis (PCA)

PC_1 and PC_2 are the top-2 principal components

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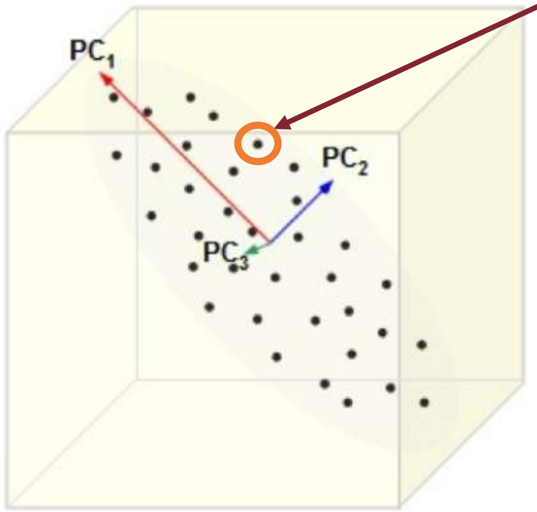
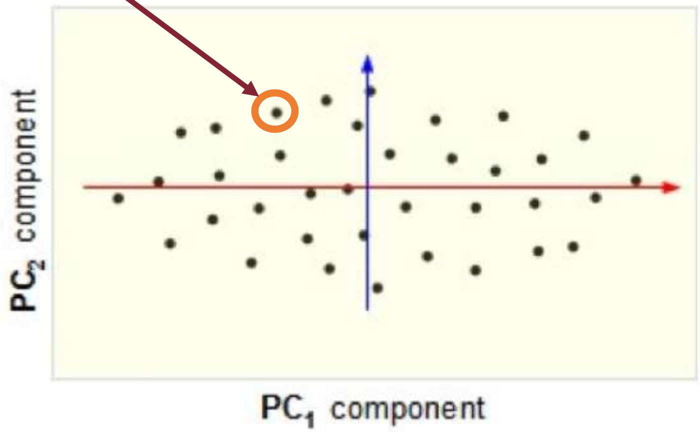
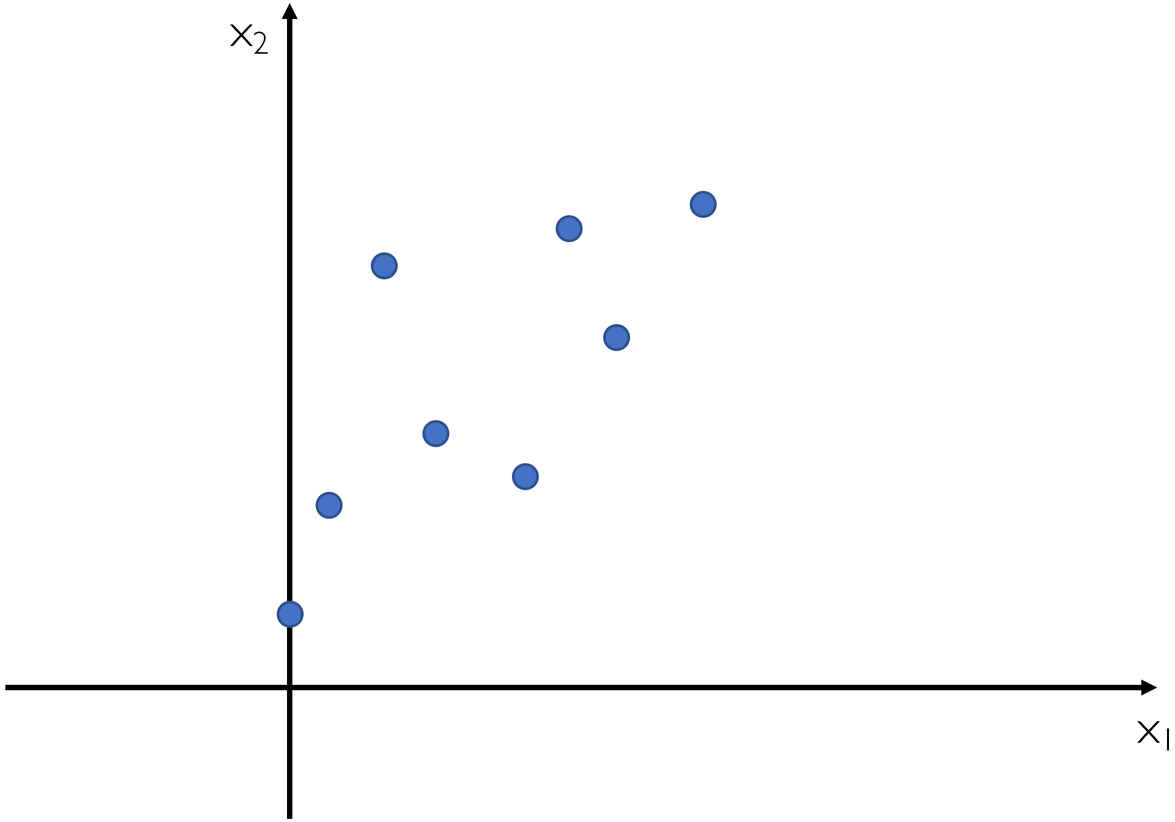
Change the coordinates of every point according to the new dimensions





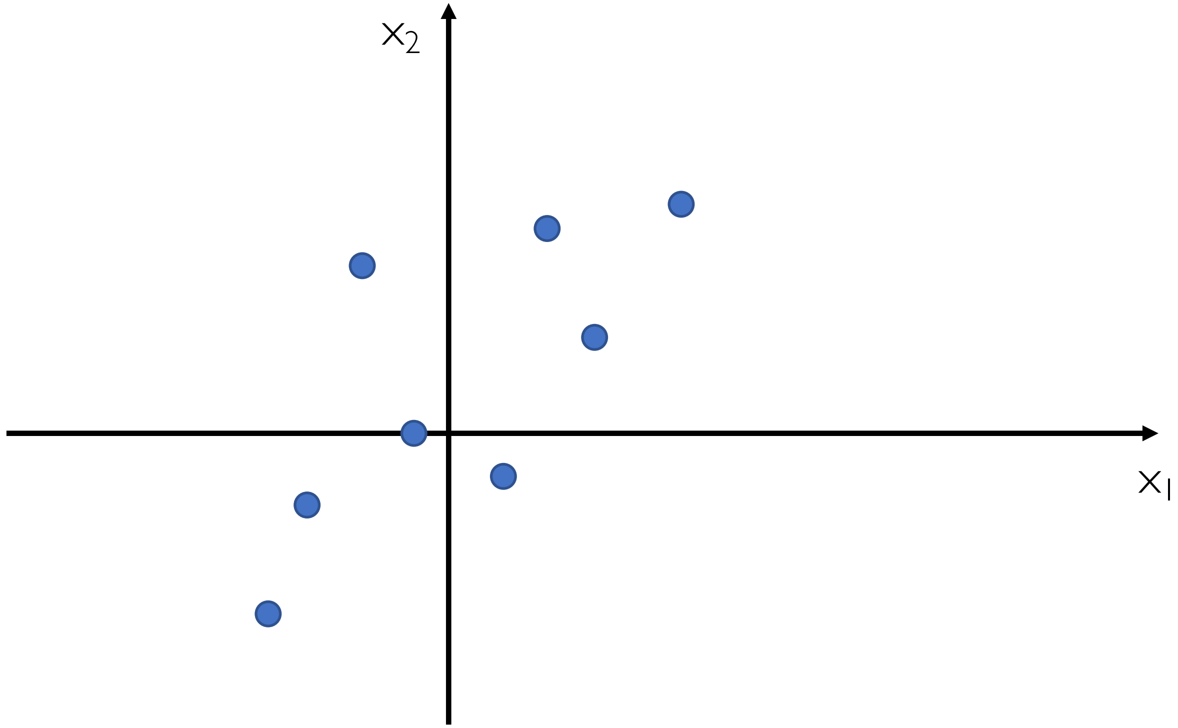
Why Do We Look for Greatest Variance?

Example: Reduce 2-dimensional data to 1-d



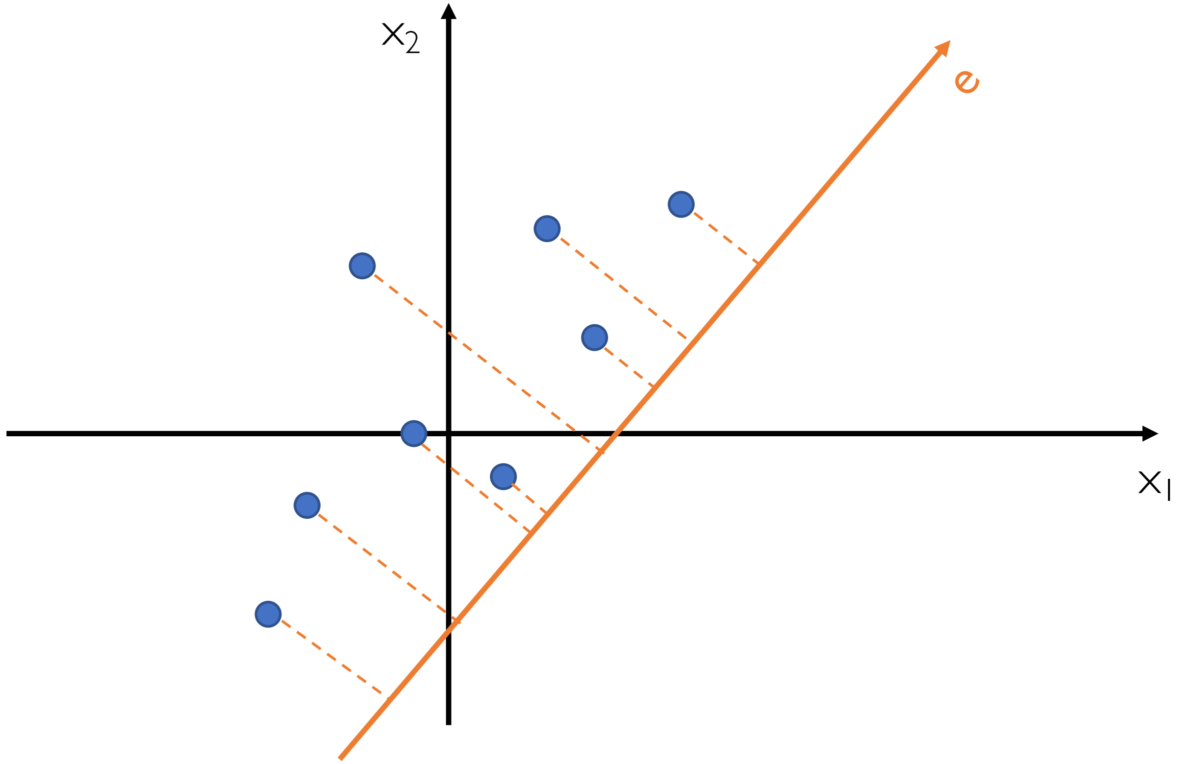
Why Do We Look for Greatest Variance?

First of all, let's center the points around the mean along x_1 and x_2



Why Do We Look for Greatest Variance?

Map, i.e., project (x_1, x_2) to a new single dimension axis e



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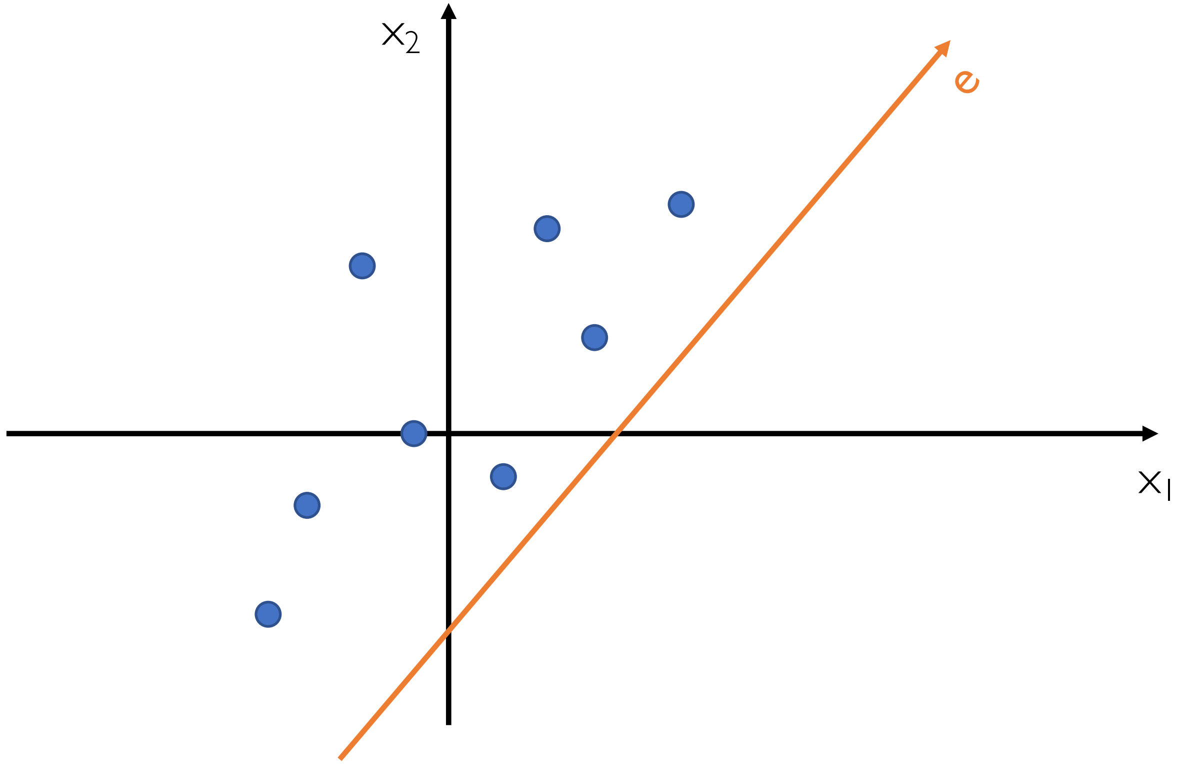

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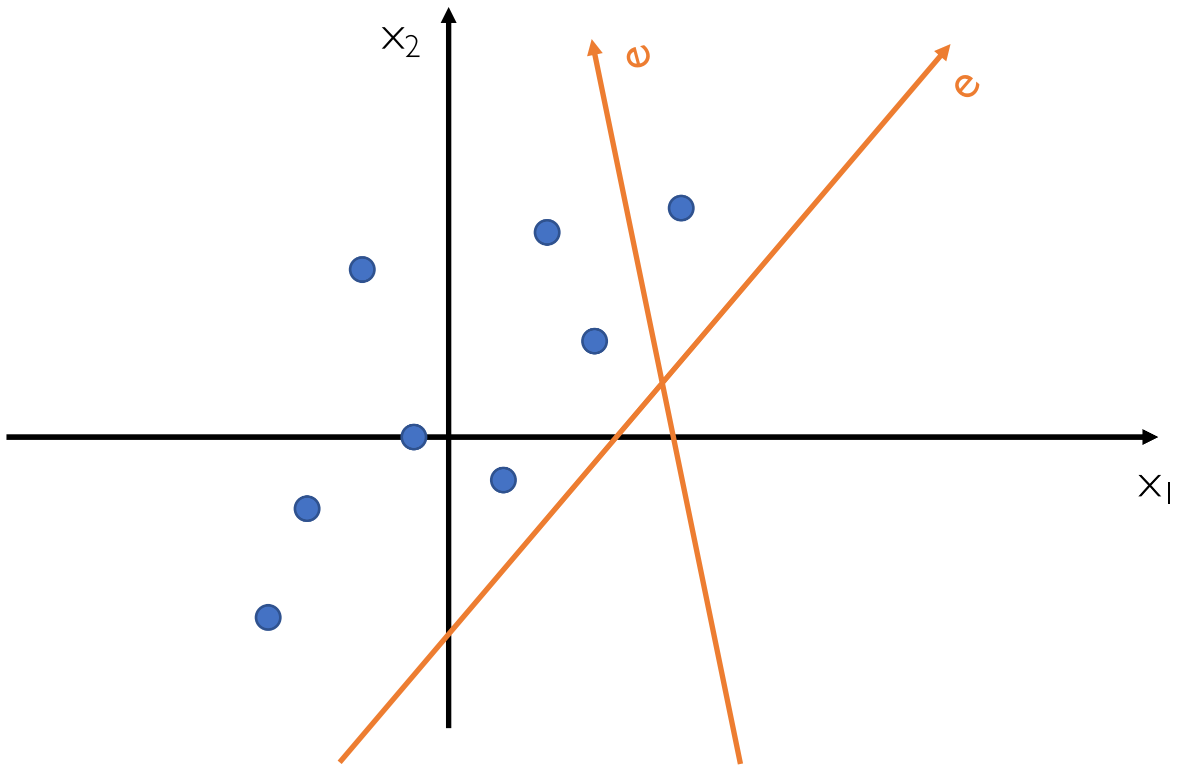

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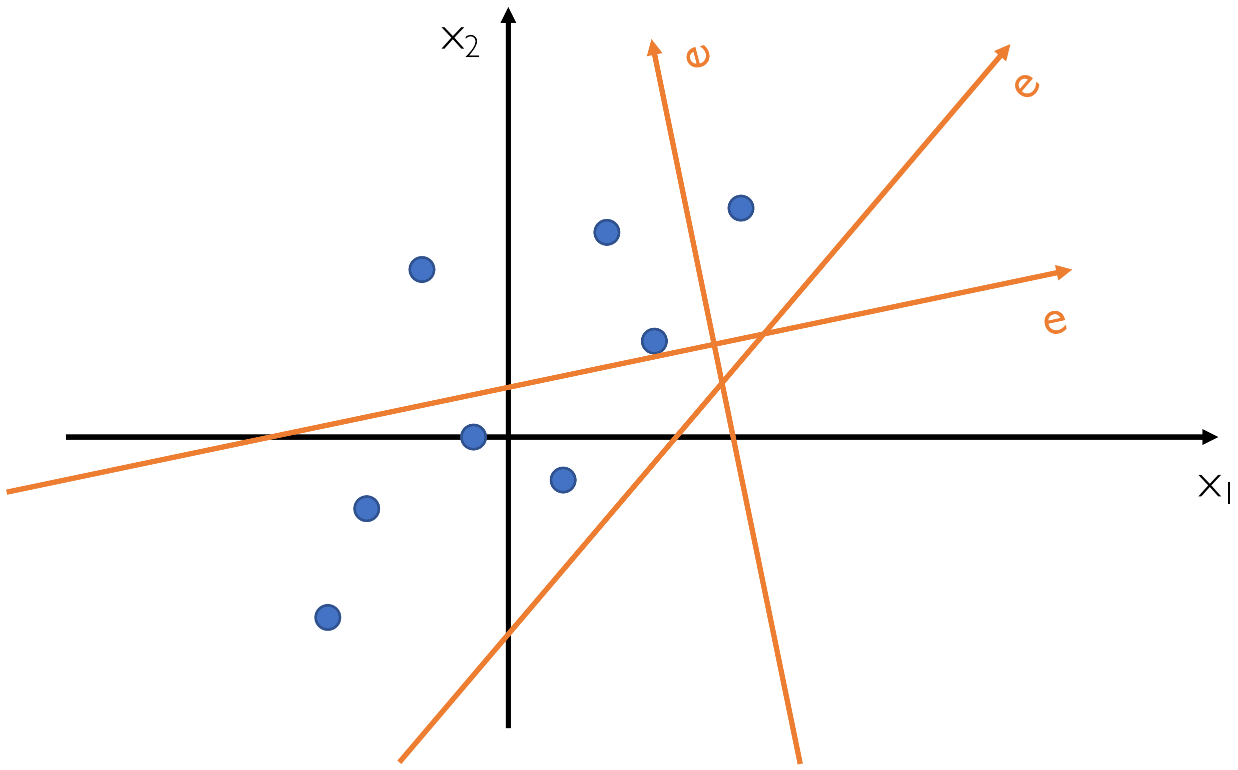

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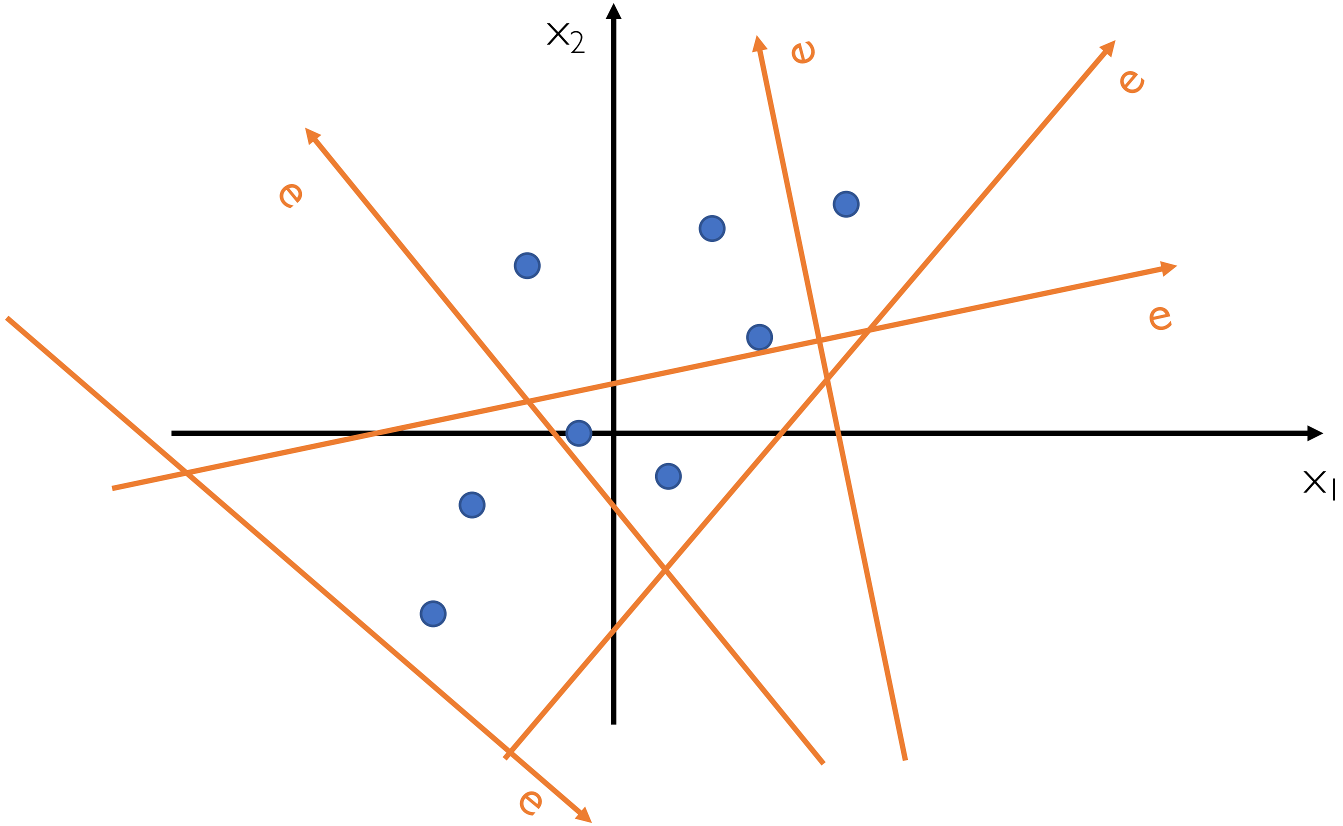
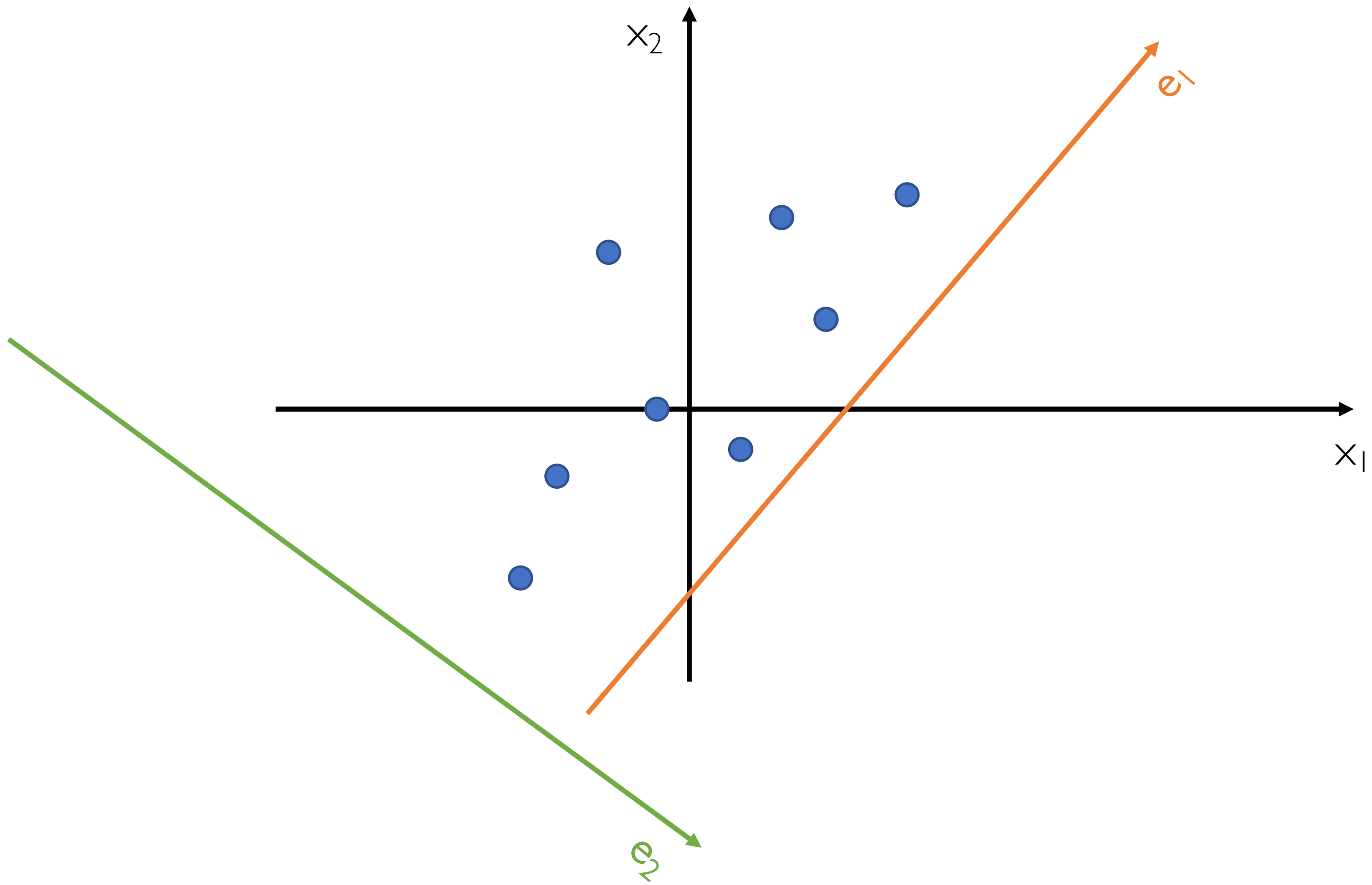


infinitely many mappings from (x_1, x_2) to a new axis e

Why Do We Look for Greatest Variance?

Let's consider 2 different mappings e_1 and e_2



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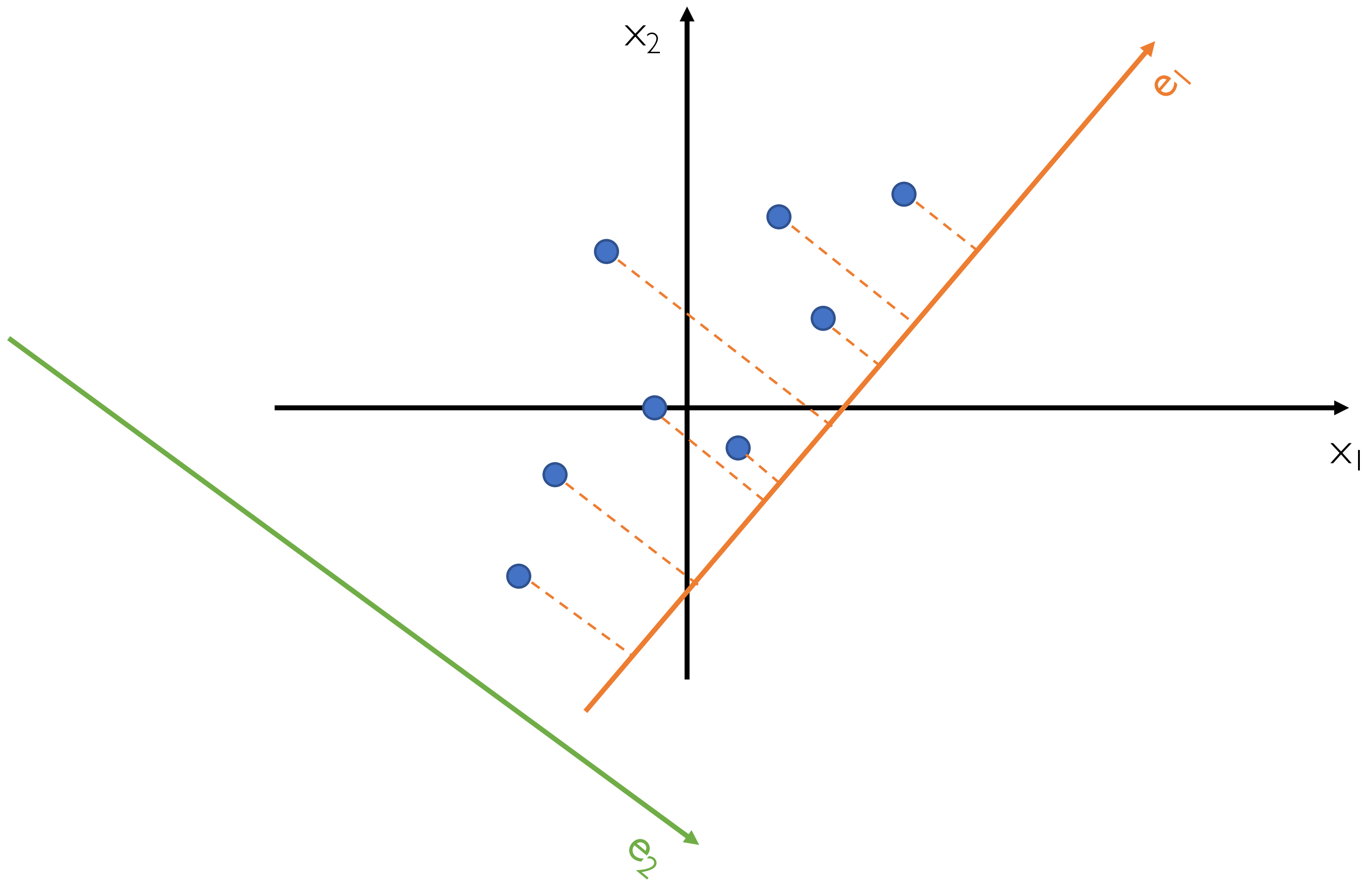

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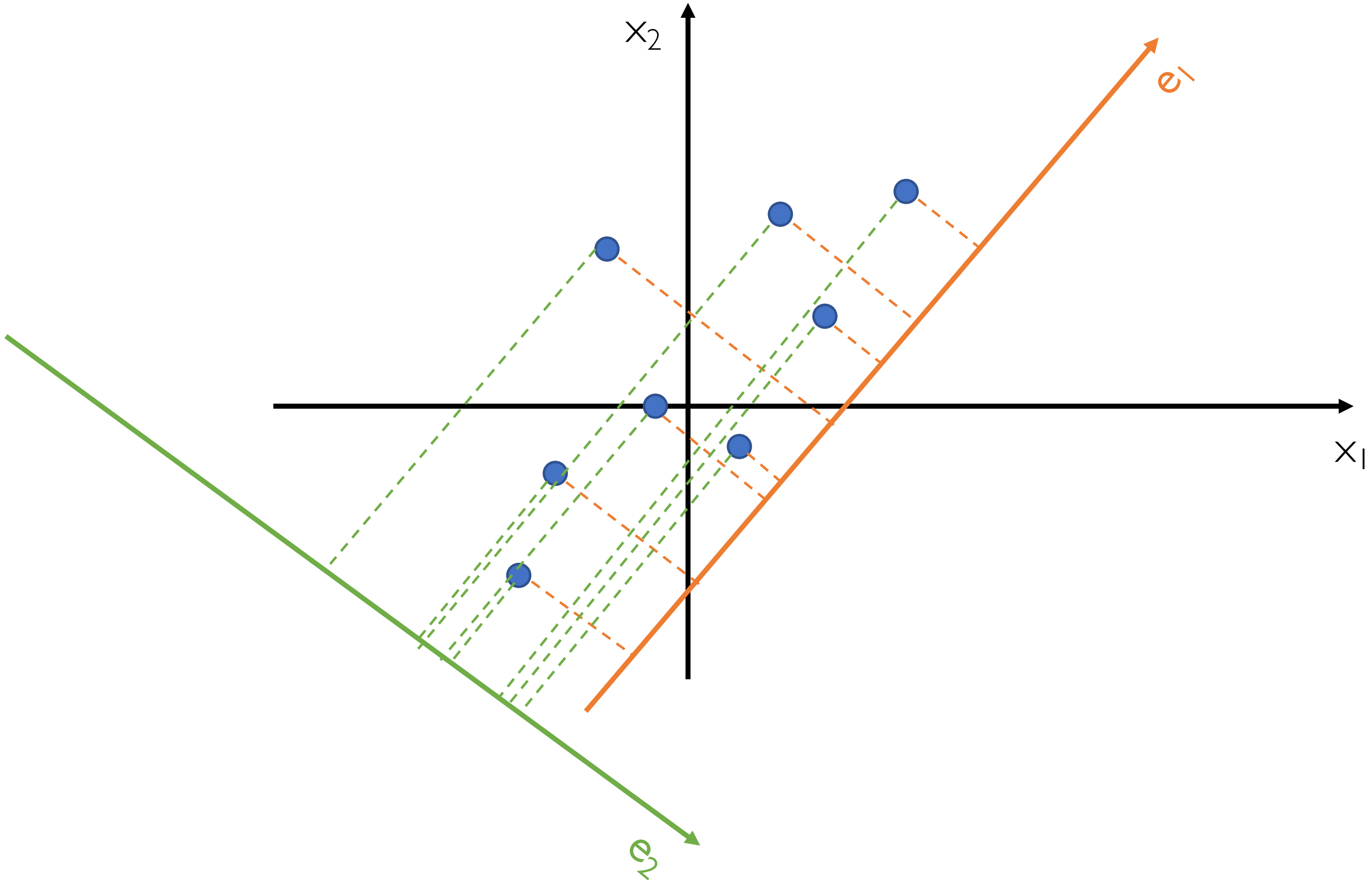
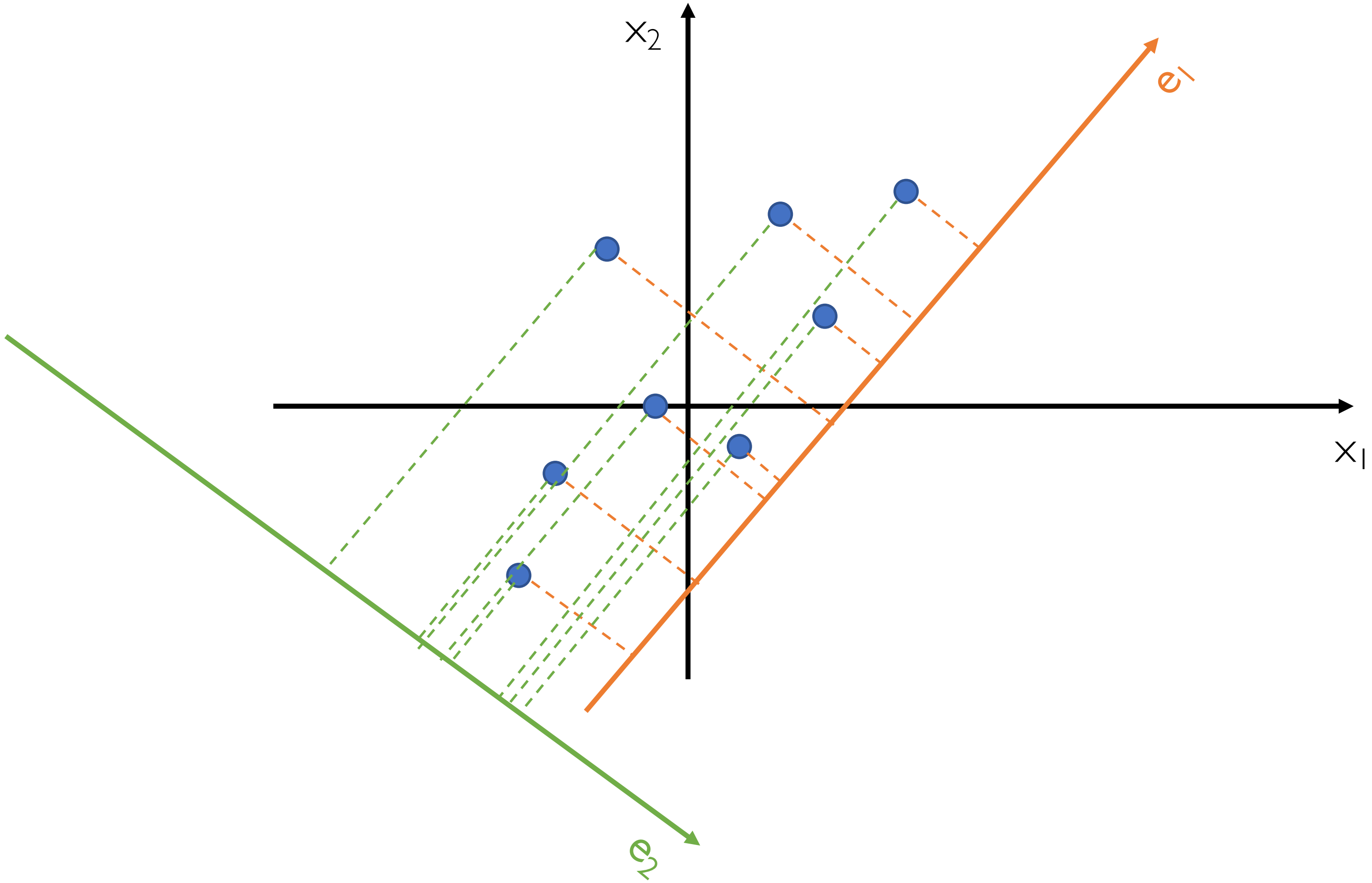


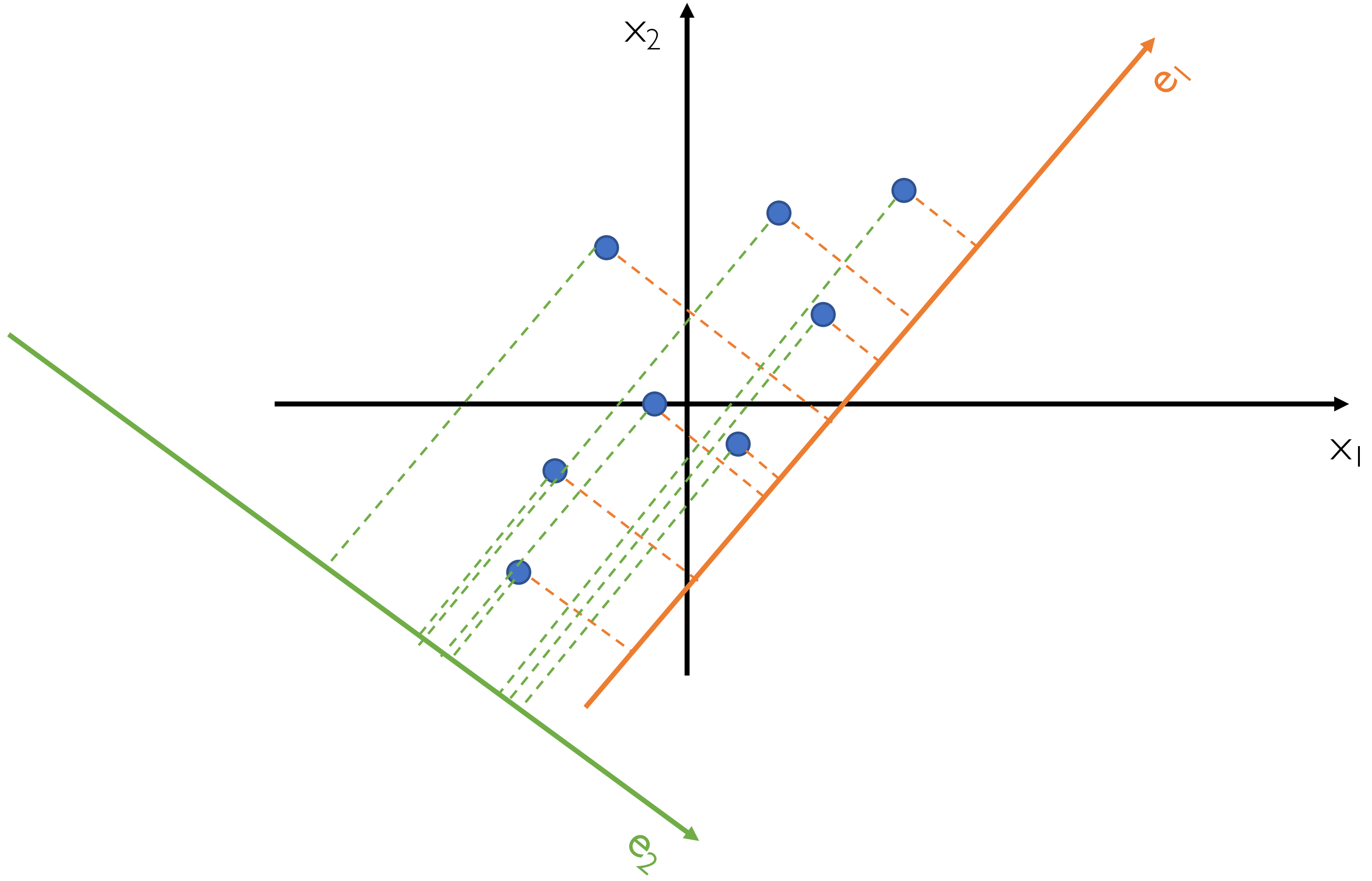
Why Do We Look for Greatest Variance?

Which one is better?



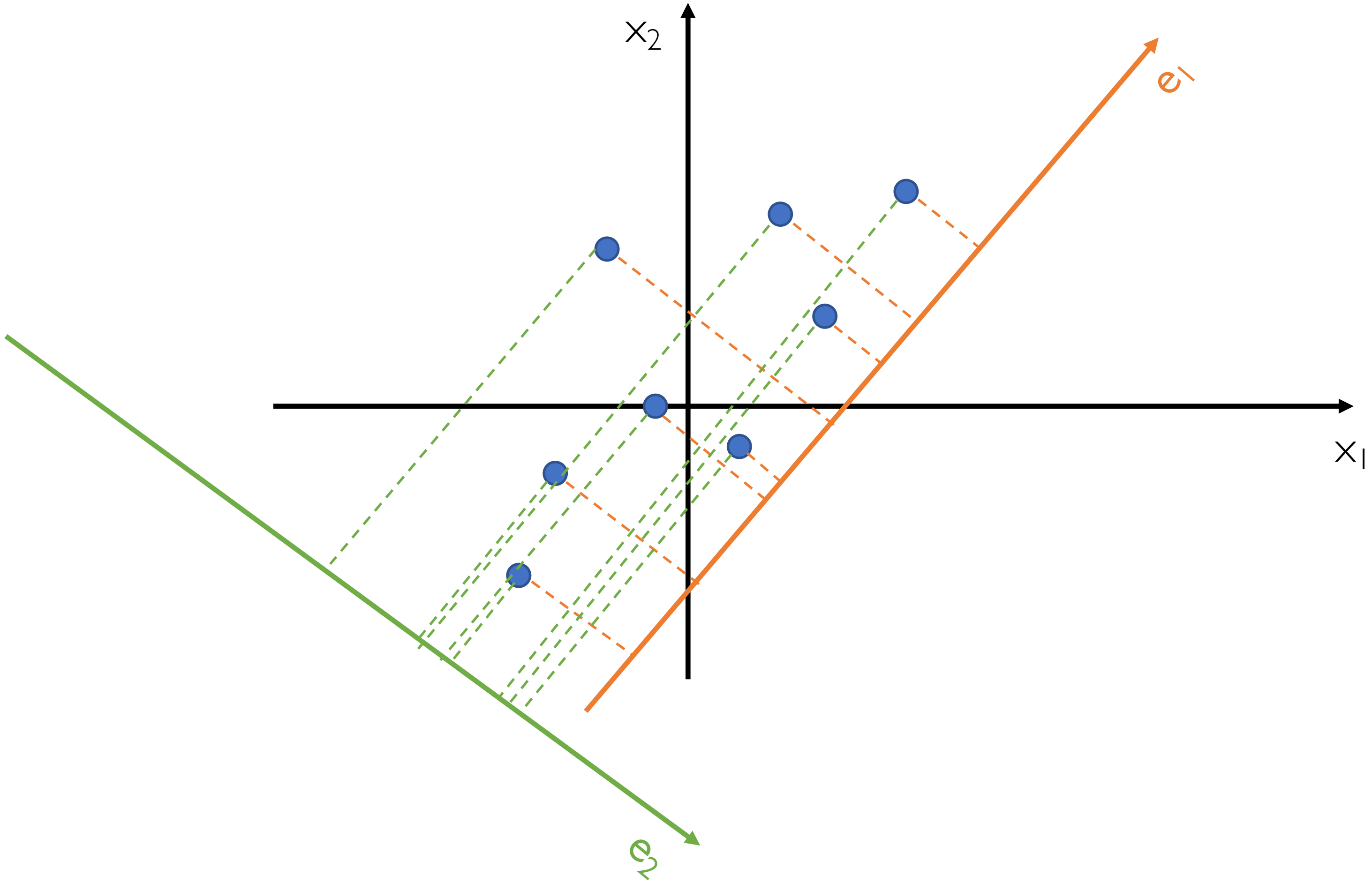
Why Do We Look for Greatest Variance?

Points projected onto e_1 look more spread-out than onto e_2



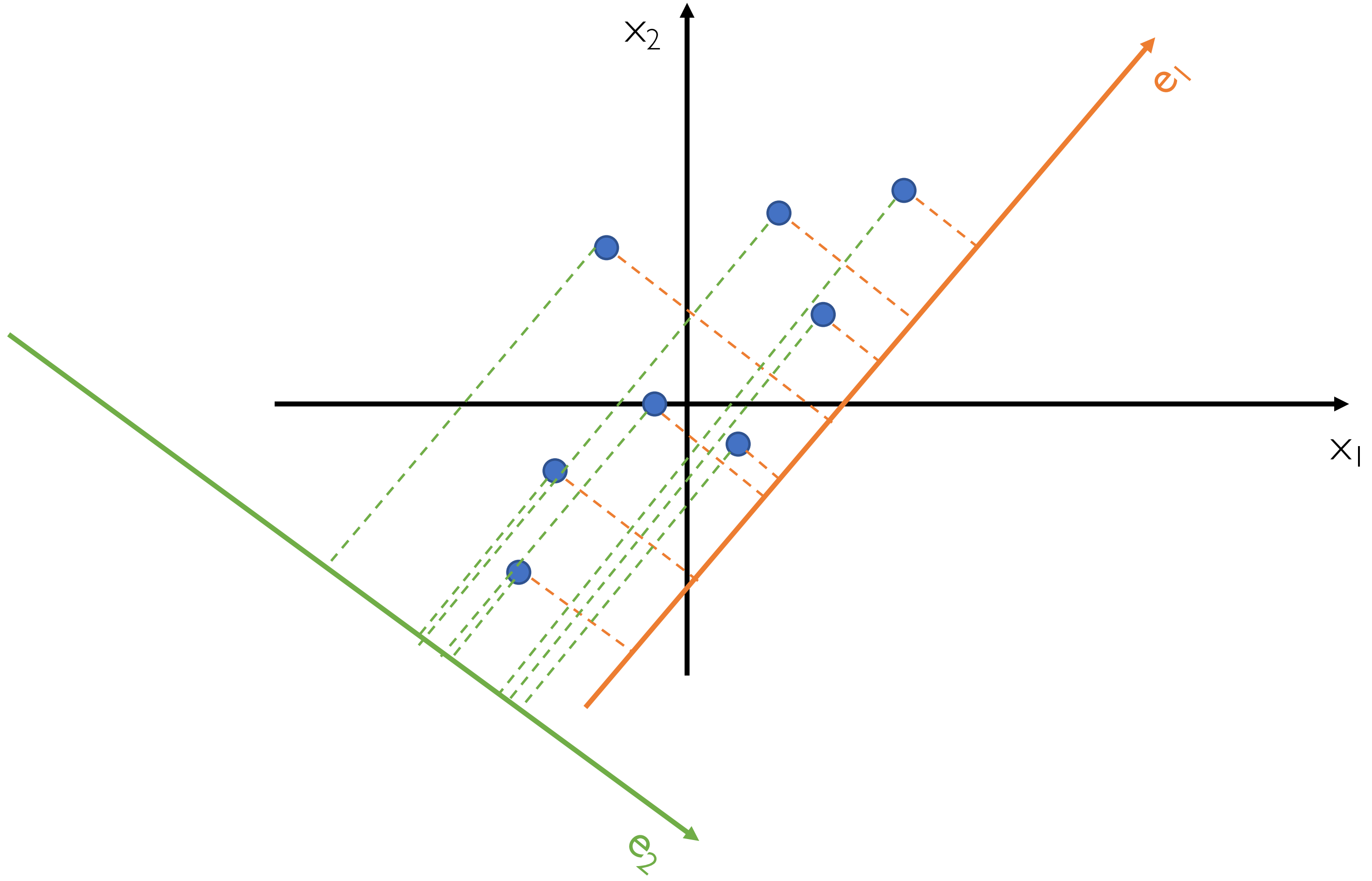
Why Do We Look for Greatest Variance?

The variance along e_1 is larger than along e_2



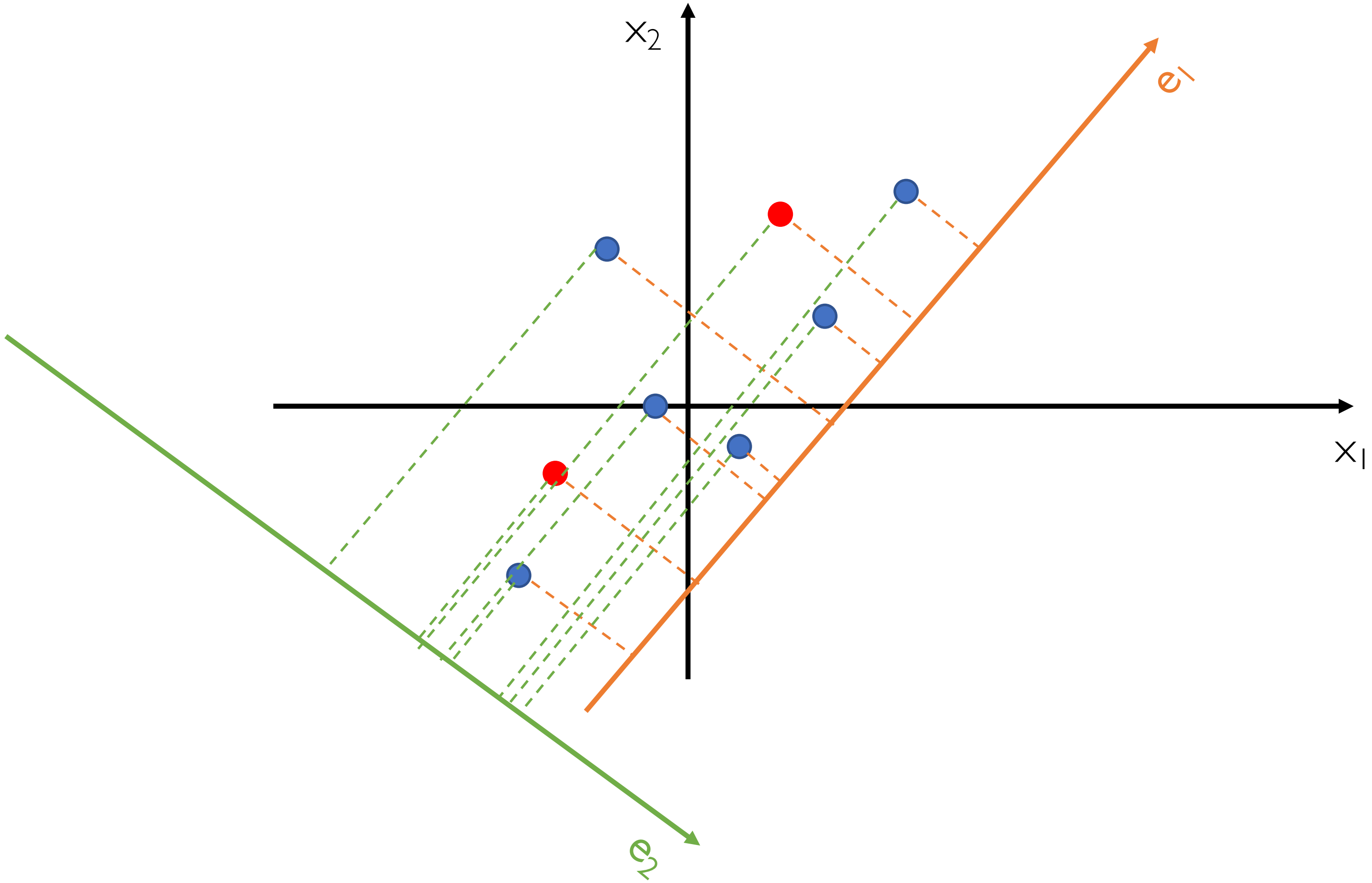
Why Do We Look for Greatest Variance?

Why is that good?



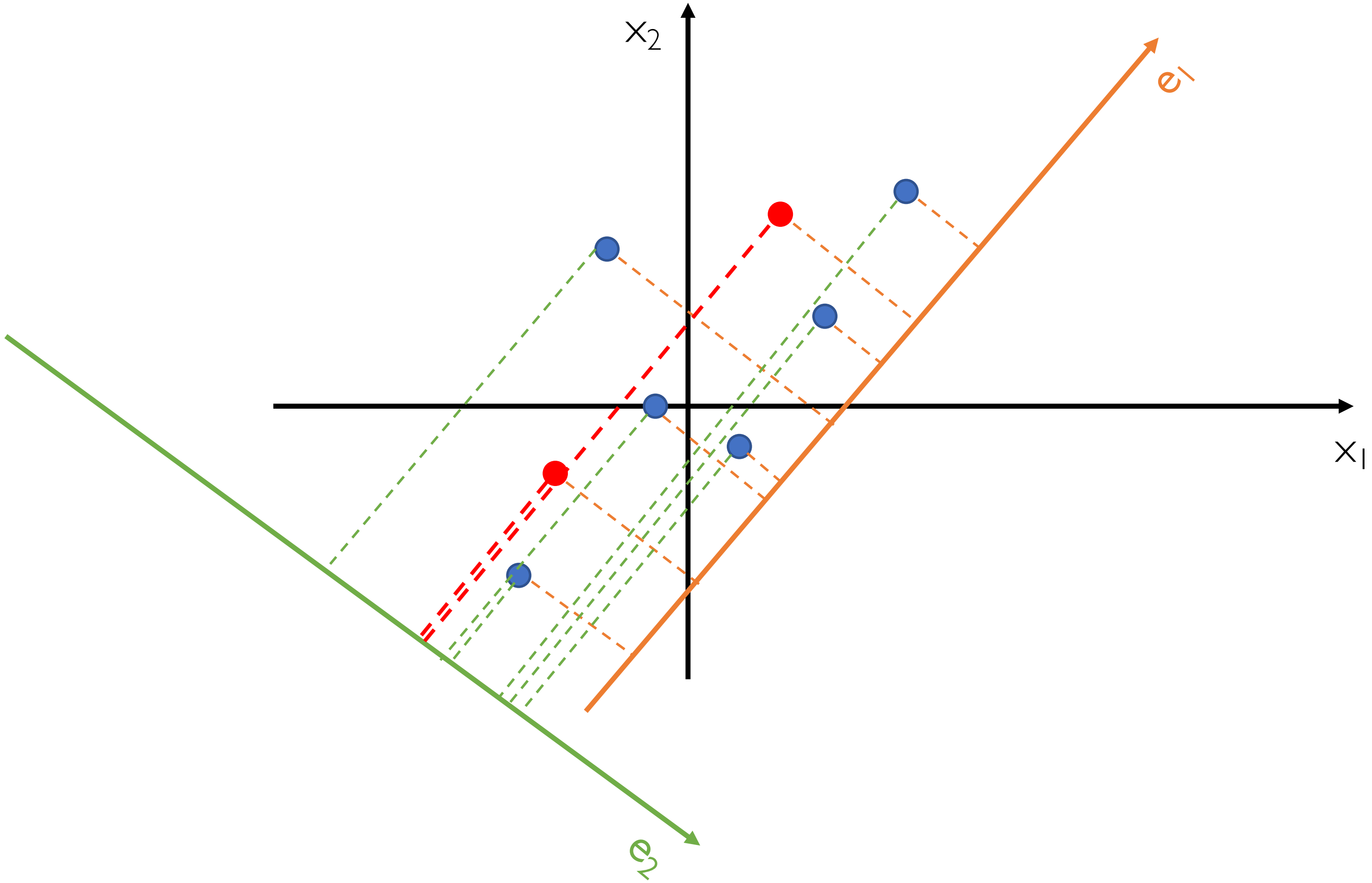
Why Do We Look for Greatest Variance?

Consider the 2 red points below



Why Do We Look for Greatest Variance?

On (x_1, x_2) far away from each other, end up close if projected onto e_2



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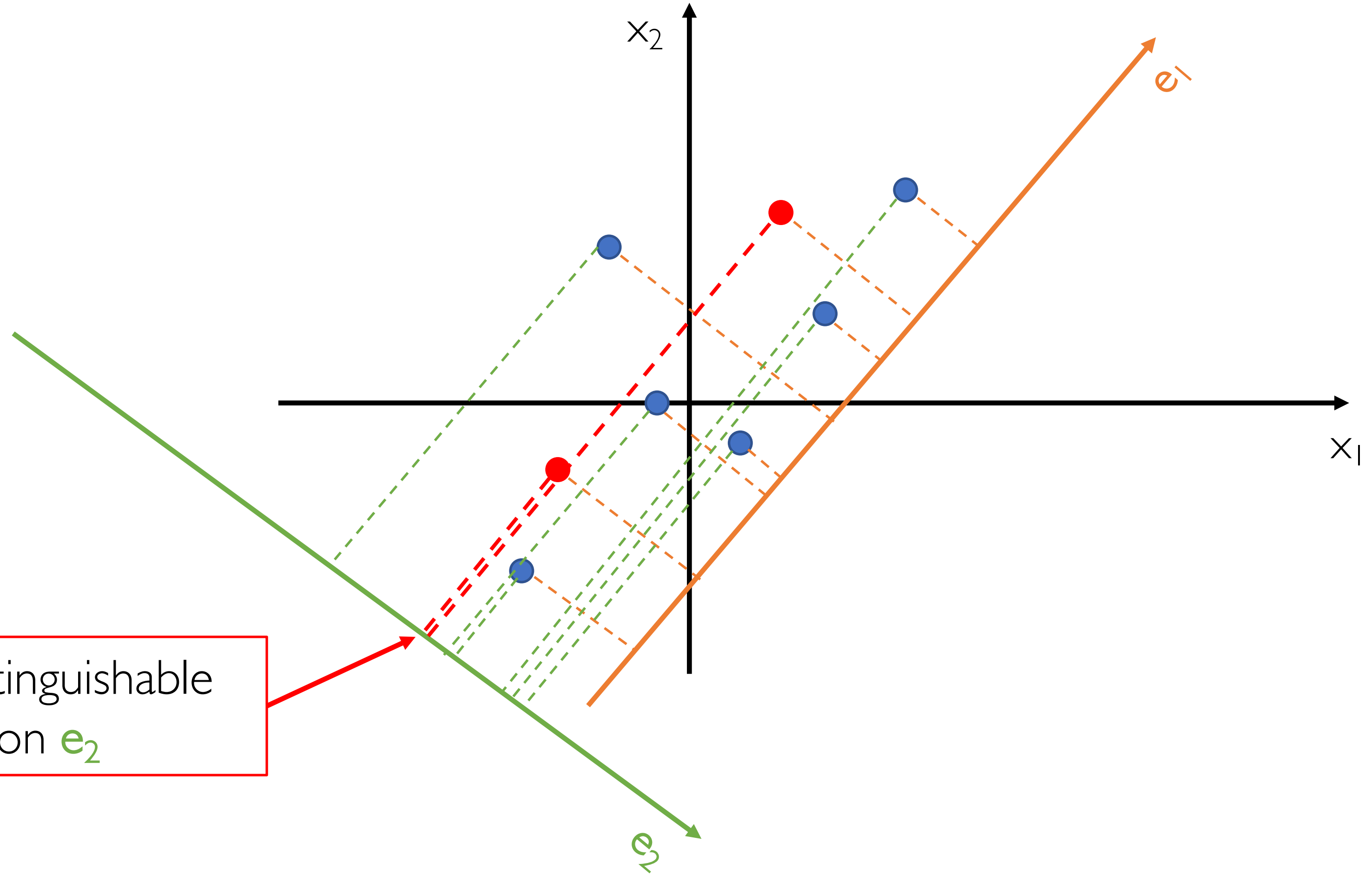
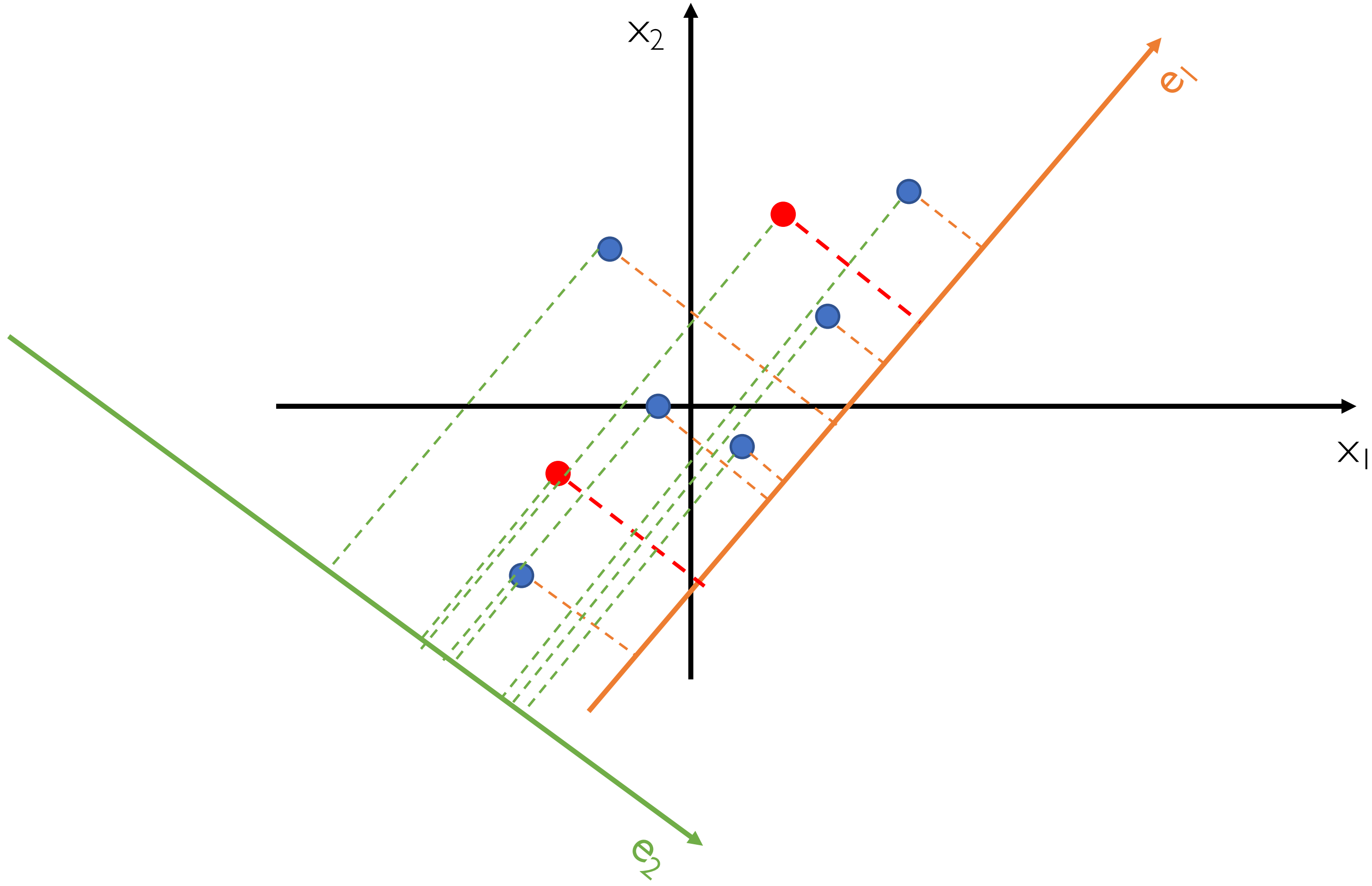


Almost indistinguishable points on e_2

Why Do We Look for Greatest Variance?

If projected onto e_1 they better preserve their distance



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- Minimize distances between points as measured on (x_1, x_2) space and those measured on **e**

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⇩

Solution

Pick e so as to maximize variance of projected data

Variance of a Random Variable

- The variance of a random variable X measures how far a set of (random) numbers are spread out from their mean value

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- The variance of a random variable X measures how far a set of (random) numbers are spread out from their mean value
- Formally, it is the expected value of the squared deviation from its mean

$$\mathrm{Var}(X) = \mathrm{E}[(X - \mu)^2]$$

$$\text{where } \mu = \mathrm{E}[X]$$

Covariance of Two Random Variables

- A measure of the joint variability of two random variables X and Y
  - Do X and Y increase/decrease together, or when one increases/decreases the other decreases/increases?

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  - Do X and Y increase/decrease together, or when one increases/decreases the other decreases/increases?
- Formally, it is the expected value of the product of their deviations from their individual means

$$\mathrm{Cov}(X, Y) = \mathrm{E}[(X - \mu_X)(Y - \mu_Y)]$$

$$\boxed{\mathrm{Cov}(X, X) = \mathrm{Var}(X)}$$

where $\mu_X = \mathrm{E}[X]$ and $\mu_Y = \mathrm{E}[Y]$

Covariance Matrix

- Given a random vector $\mathbf{X} = (X_1, \ldots, X_d)$ its covariance matrix K is a dxd square matrix with the covariance between each pair of elements

Covariance Matrix

- Given a random vector $\mathbf{X} = (X_1, \ldots, X_d)$ its covariance matrix K is a dxd square matrix with the covariance between each pair of elements
- In the matrix diagonal there are variances, i.e., the covariance of each element with itself

$$\mathrm{K}[i, j] = \mathrm{Cov}(X_i, X_j)$$

Covariance Matrix of Original Dimensions

- The original set of dimensions is a random vector $\mathbf{X} = (X_1, \ldots, X_d)$

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- In our example, $d = 2$ and $\mathbf{X} = (X_1, X_2)$
- The covariance matrix K is a 2-by-2 matrix
- To ease the covariance computation, we center each data point at zero
  - Subtracting the mean of each attribute/dimension
  - The mean of each dimension becomes then 0

Covariance Matrix of Original Dimensions

Let n be the total number of data points: $\mathbf{x}_1, \ldots, \mathbf{x}_n$

Each data point is represented by a (x_1, x_2) pair

$\mathbf{x}_i = (x_{i,1}, x_{i,2})$

We associate 2 random variables X_1, X_2 to each dimension, and we compute:

$$\mu_1 = \mathrm{E}[X_1] = \frac{1}{n}\sum_{i=1}^{n} x_{i,1}$$

$$\mu_2 = \mathrm{E}[X_2] = \frac{1}{n}\sum_{i=1}^{n} x_{i,2}$$

$$\mathbf{x}_i = (x_{i,1} - \mu_1, x_{i,2} - \mu_2)$$

Covariance Matrix of Original Dimensions

Let us rewrite each data point $\mathbf{x}_i$ as follows:

$\mathbf{x}_i = (x'_{i,1}, x'_{i,2})$ where:

$x'_{i,1} = x_{i,1} - \mu_1; x'_{i,2} = x_{i,2} - \mu_2$

$$\mu_1^{\text{new}} = \mathrm{E}[X_1] = \frac{1}{n}\sum_{i=1}^{n} x'_{i,1} = \frac{1}{n}\sum_{i=1}^{n}(x_{i,1} - \mu_1)$$

$$\mu_2^{\text{new}} = \mathrm{E}[X_2] = \frac{1}{n}\sum_{i=1}^{n} x'_{i,2} = \frac{1}{n}\sum_{i=1}^{n}(x_{i,2} - \mu_2)$$

Covariance Matrix of Original Dimensions

$$\mu_1^{\text{new}} = \frac{1}{n}\sum_{i=1}^{n}(x_{i,1}-\mu_1) = \frac{1}{n}\Bigg(\underbrace{\sum_{i=1}^{n} x_{i,1}}_{n\mu_1} - \underbrace{\sum_{i=1}^{n} \mu_1}_{n\mu_1}\Bigg) = 0$$

$$\mu_2^{\text{new}} = \frac{1}{n}\sum_{i=1}^{n}(x_{i,2}-\mu_2) = \frac{1}{n}\Bigg(\underbrace{\sum_{i=1}^{n} x_{i,2}}_{n\mu_2} - \underbrace{\sum_{i=1}^{n} \mu_2}_{n\mu_2}\Bigg) = 0$$

0-mean

Covariance Matrix of Original Dimensions

Scaling data so as to have 0-mean on all dimensions
allow computing covariance much easily

$$\mathrm{Cov}(X_1, X_2) = \mathrm{E}[(X_1 - \underbrace{\mu_1^{\mathrm{new}}}_{=0})(X_2 - \underbrace{\mu_2^{\mathrm{new}}}_{=0})] = \mathrm{E}[X_1 X_2]$$

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Scaling data so as to have 0-mean on all dimensions
allow computing covariance much easily

$$\mathrm{Cov}(X_1, X_2) = \mathrm{E}[(X_1 - \underbrace{\mu_1^{\mathrm{new}}}_{=0})(X_2 - \underbrace{\mu_2^{\mathrm{new}}}_{=0})] = \mathrm{E}[X_1 X_2]$$

As a consequence, the covariance matrix is also easier to compute!

Covariance Matrix of Original Dimensions

Let's assume the following is our 2-by-2 covariance matrix

$$\begin{array}{c c} & \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} 2 & 4/5 \\ 4/5 & 3/5 \end{bmatrix} \end{array}$$

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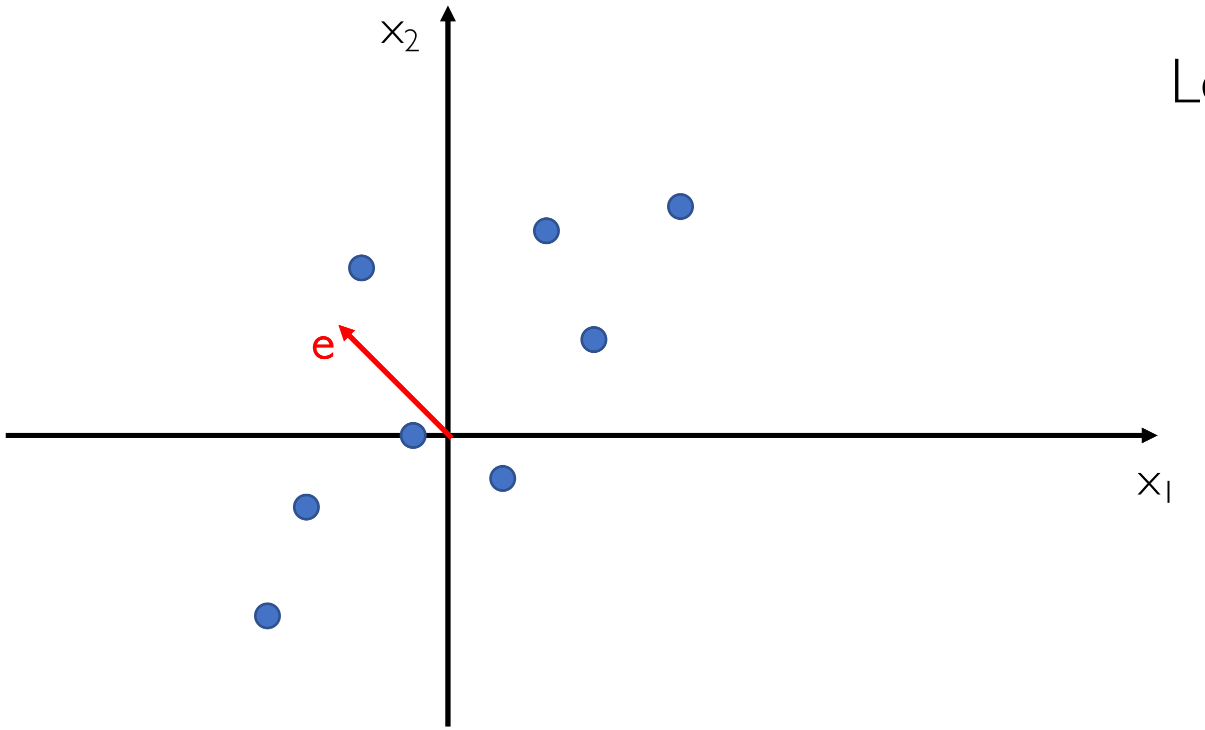
$$\boxed{\mathrm{Cov}(X_2, X_2) = \mathrm{Var}(X_2) = \frac{1}{n}\sum_{i=1}^{n} (x'_{i,2})^2}$$

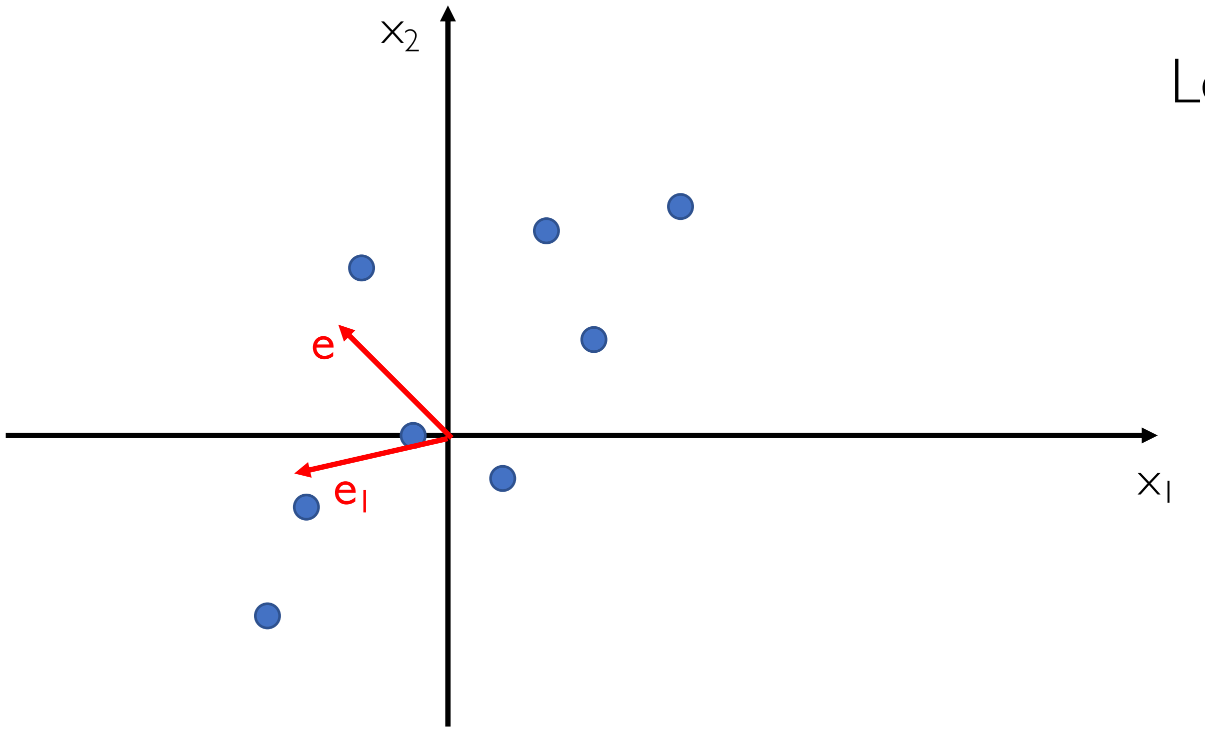
Covariance Matrix of Original Dimensions



Let's multiply our 2-by-2 covariance matrix K by a random vector e = (-1,1)

$$\underbrace{\begin{bmatrix} 2 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\mathbf{e}} =$$

Covariance Matrix of Original Dimensions



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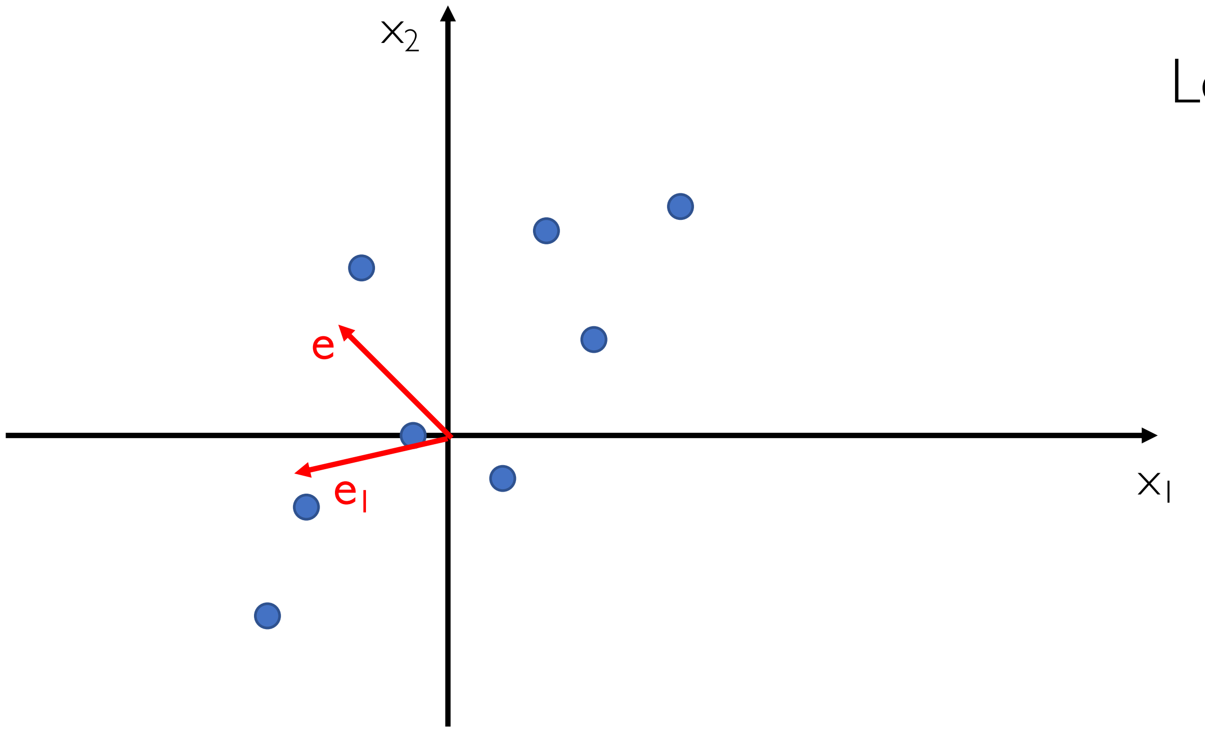

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$\boxed{\text{slope = 1/-1 = -1}}$

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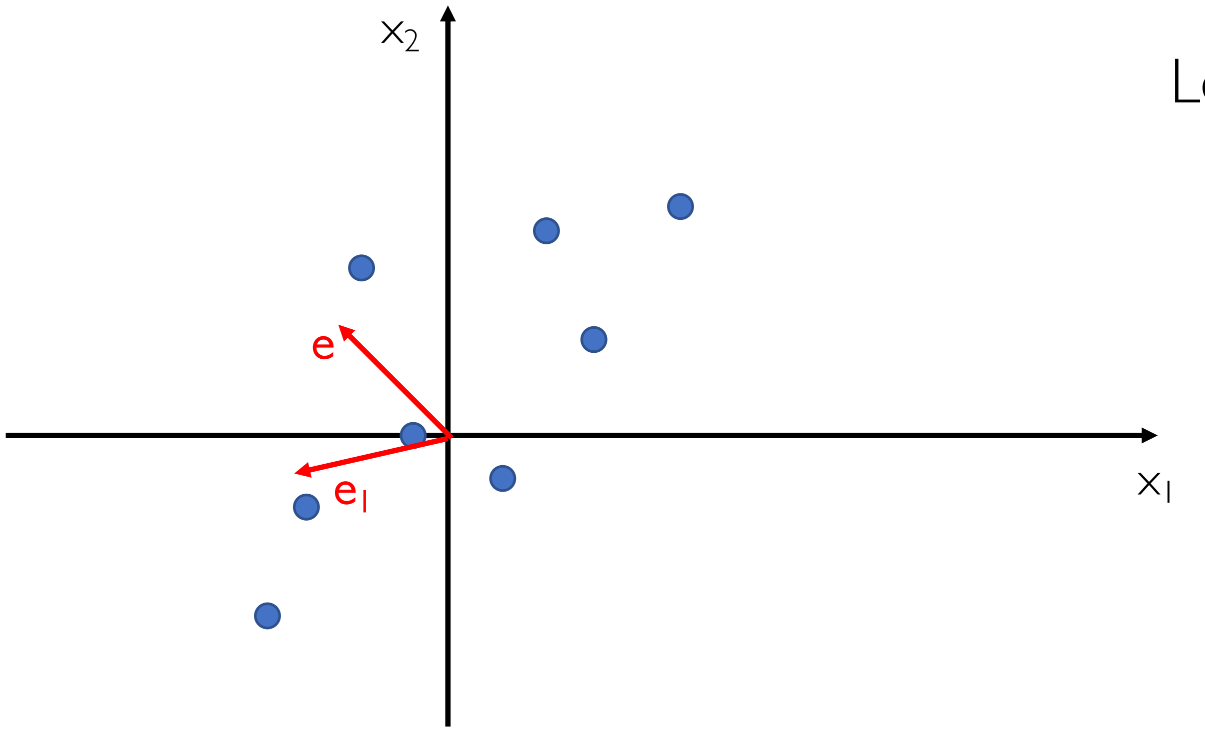

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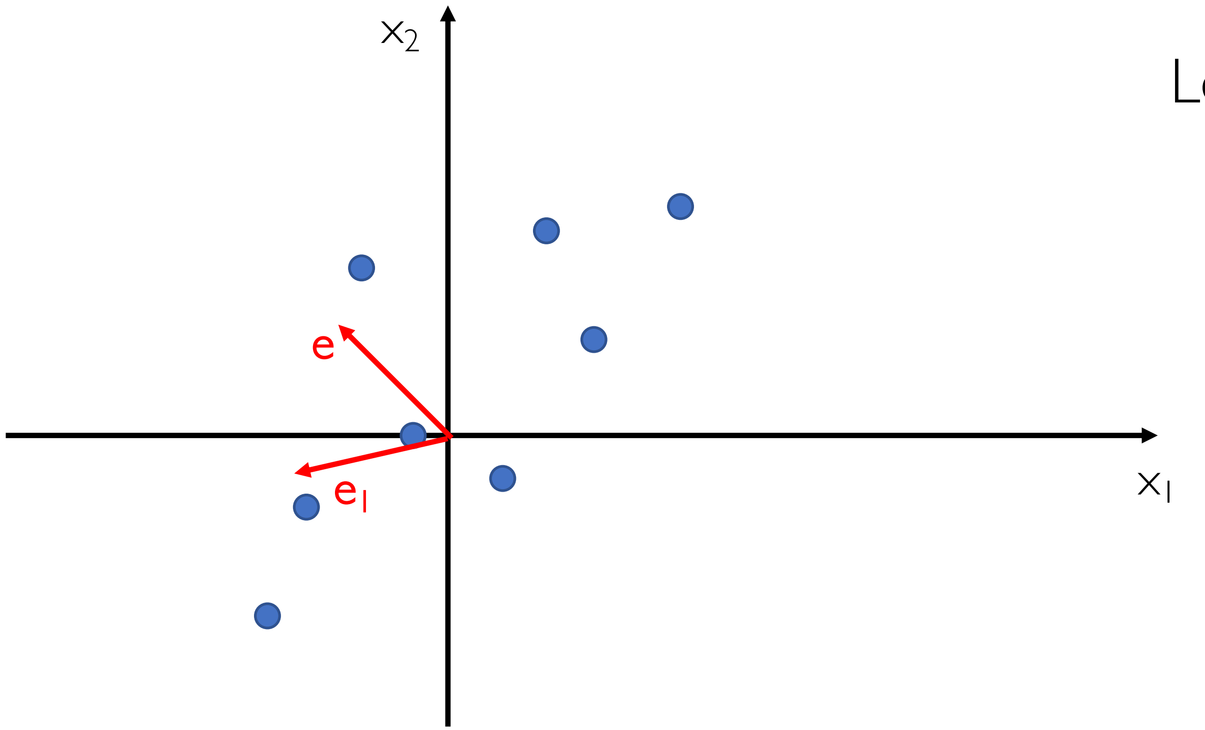
slope = 1/-1 = -1

new slope = -(1/5)/-(6/5) = 1/6

Covariance Matrix of Original Dimensions



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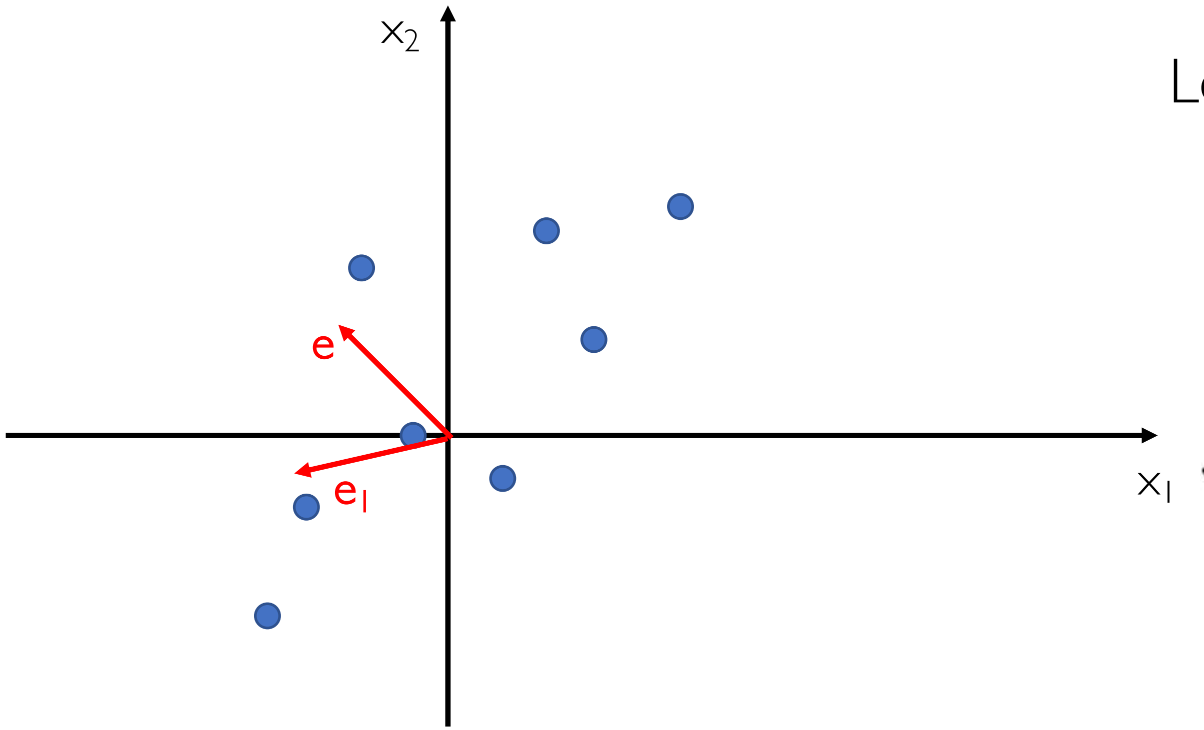
Turns towards the direction of the greatest variance

new slope = -(1/5)/-(6/5) = 1/6

Covariance Matrix of Original Dimensions



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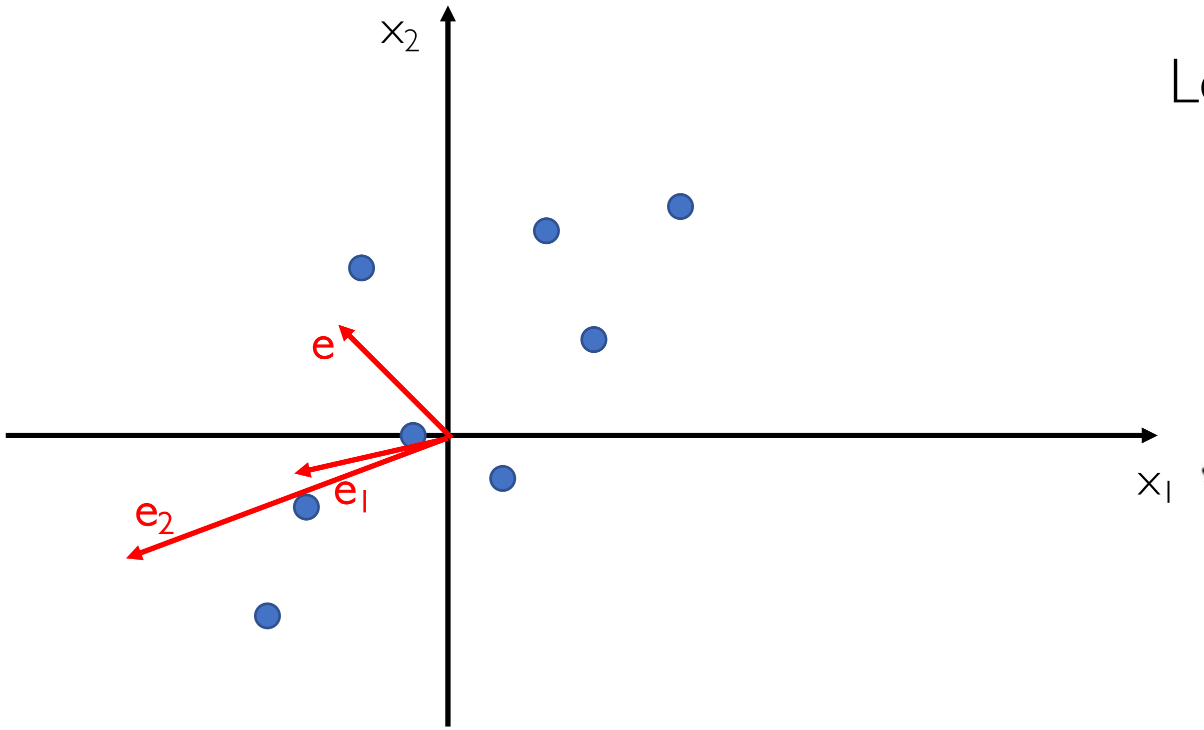
Covariance Matrix of Original Dimensions



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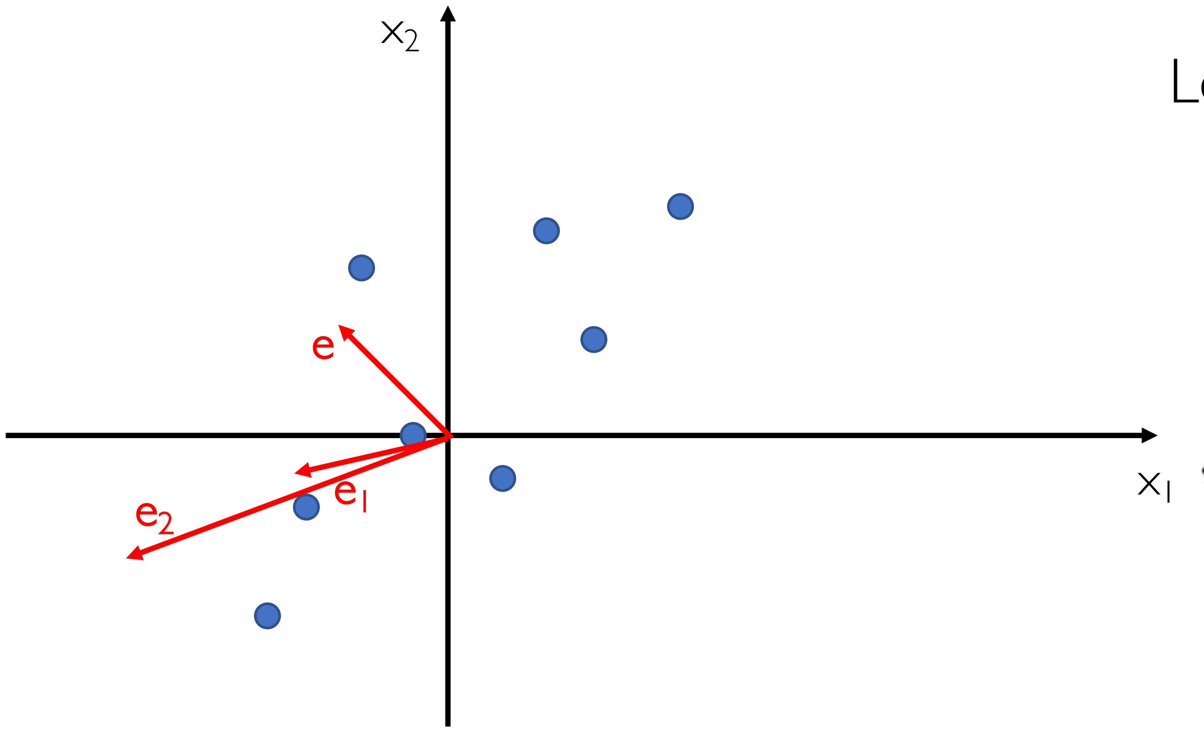

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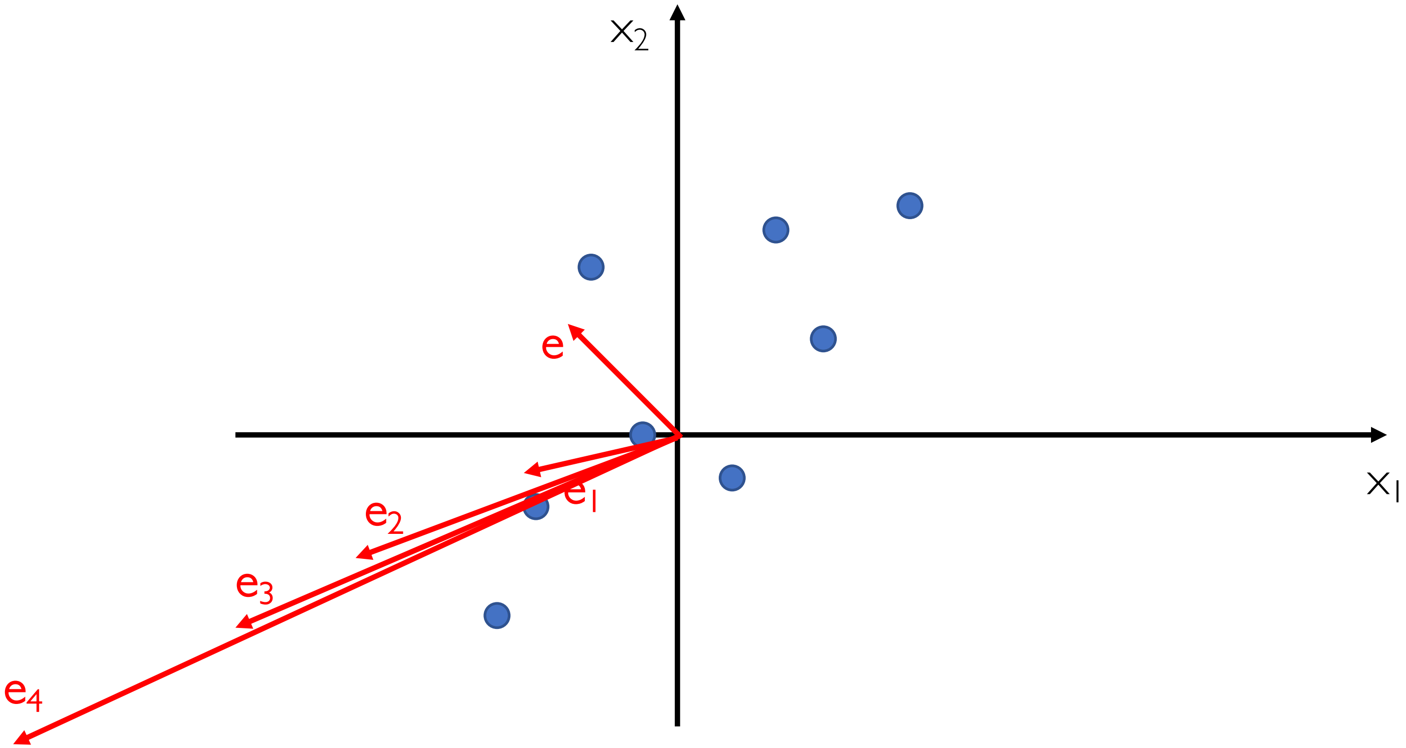
slope = 1/6

new slope = 27/64

Turns towards the direction of the greatest variance

Covariance Matrix of Original Dimensions

If we keep doing this the resulting vector is getting longer and turns towards the direction of the largest variance



Covariance Matrix of Original Dimensions

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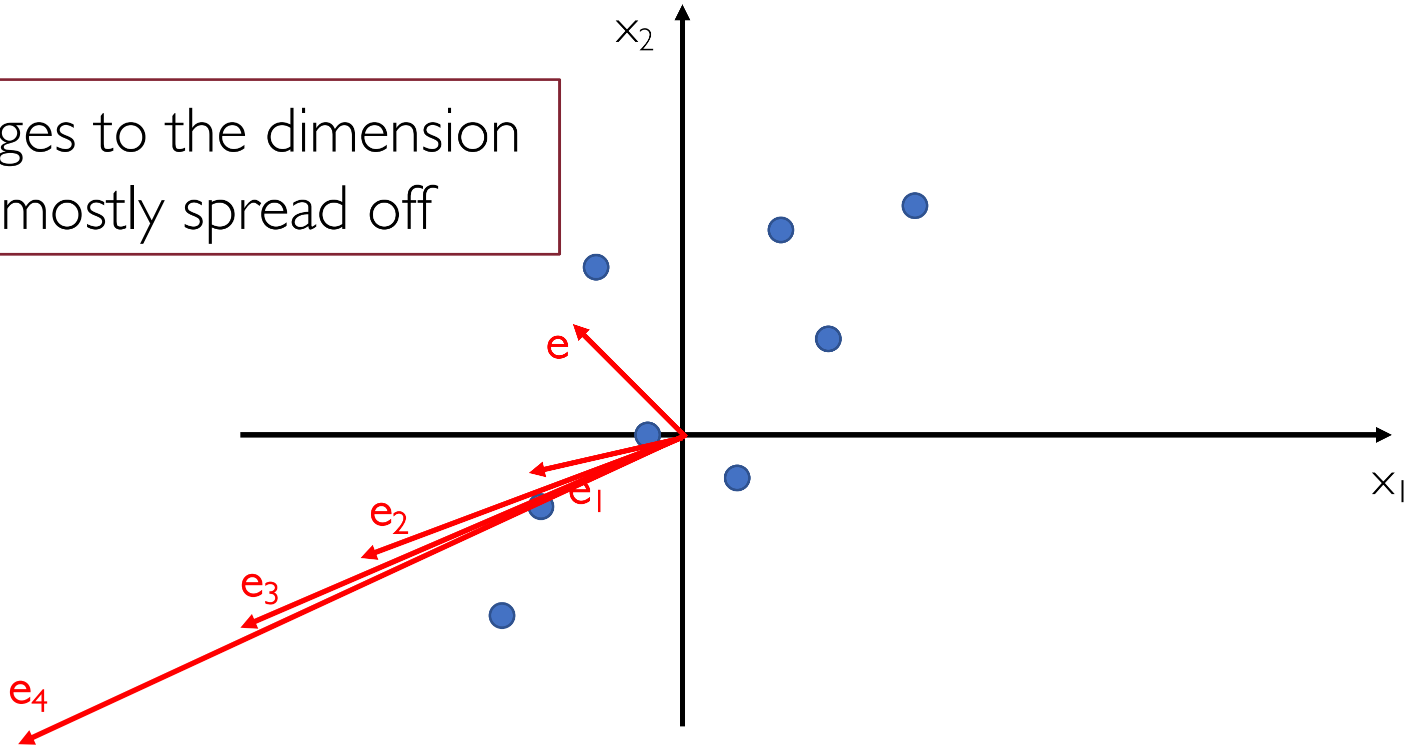
The slope converges to the dimension where data is mostly spread off



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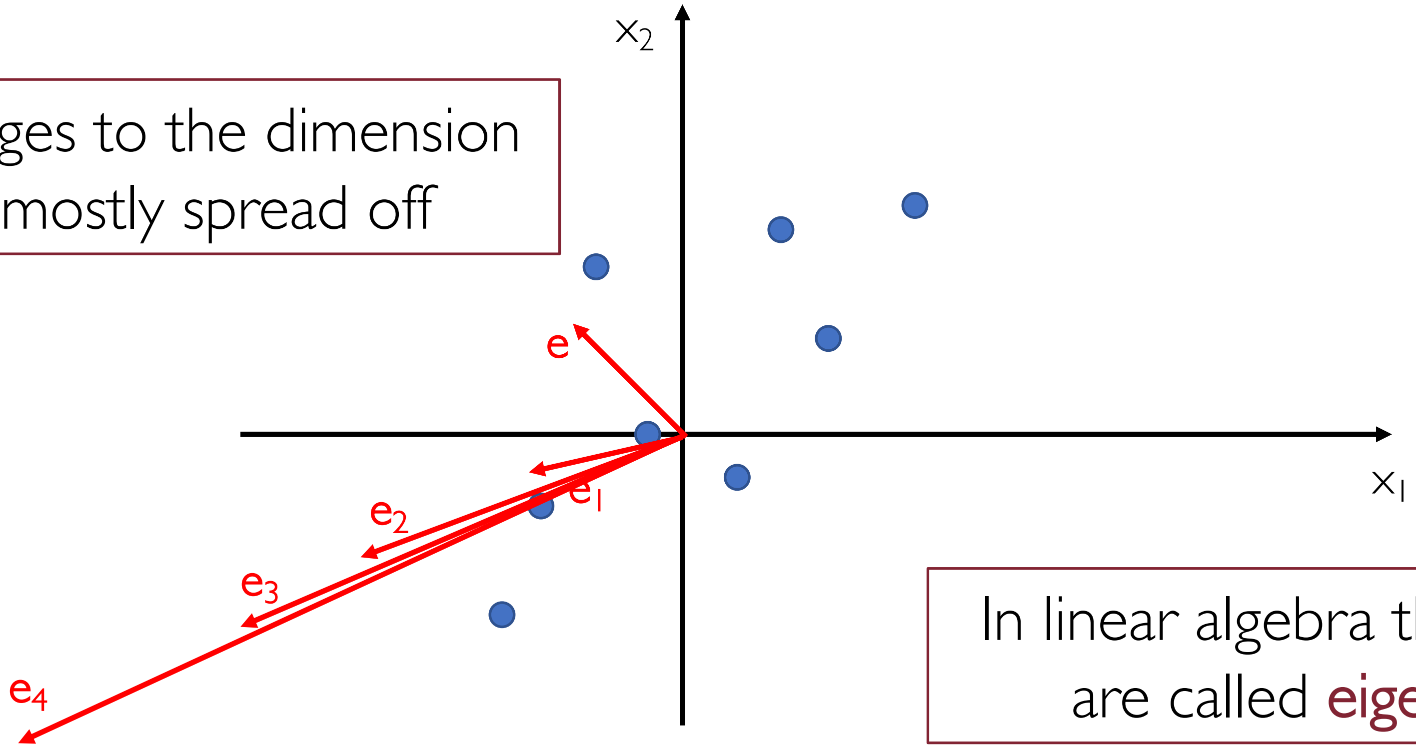

In linear algebra those vectors are called eigenvectors

Eigenvectors of the Covariance Matrix

$$\boxed{K\mathbf{e} = \lambda\mathbf{e}}$$

Eigenvectors of the Covariance Matrix

linear transformation

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eigenvector

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When you multiply a matrix by an eigenvector **e** the resulting vector does not change its direction, but it is only scaled by a factor λ (eigenvalue)

Eigenvectors of the Covariance Matrix

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When you multiply a matrix by an eigenvector **e** the resulting vector does not change its direction, but it is only scaled by a factor λ (eigenvalue)

In other words, eigenvectors encapsulate all the relevant information to describe a linear transformation (in our case, represented by the covariance matrix K)

Eigenvectors of the Covariance Matrix

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eigenvector

eigenvalue

When you multiply a matrix by an eigenvector e the resulting vector does not change its direction, but it is only scaled by a factor λ (eigenvalue)

Principal Components

eigenvectors of the covariance matrix with the largest eigenvalues

How Do We Compute Eigenvectors?

Remember that we want to solve for **e** the following:

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$$K\mathbf{e} = \lambda\mathbf{e}$$

We can rewrite the system of equations above as:

$$K\mathbf{e} - \lambda\mathbf{e} = 0 \Rightarrow (K - \lambda I)\mathbf{e} = 0$$

I is the identity matrix

How Do We Compute Eigenvectors?

We therefore resort to solve the following homogeneous system:

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e = 0

The only way for the homogeneous system above to have a non-trivial solution is for its matrix (K – λI) to be non-invertible, otherwise:

$$\cancel{(K - \lambda I)}\cancel{(K - \lambda I)^{-1}}\boxed{\mathbf{e} = 0}\cancel{(K - \lambda I)^{-1}}$$

How Do We Compute Eigenvectors?

A square matrix is invertible iff its determinant is not equal to 0

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$\Downarrow$

If the determinant of the matrix $(K - \lambda I)$ is equal to 0, it is non-invertible

How Do We Compute Eigenvectors?

A square matrix is invertible iff its determinant is not equal to 0

⇩

If the determinant of the matrix $(K - \lambda I)$ is equal to 0, it is non-invertible

⇩

The corresponding homogeneous system will have a non-trivial solution

How Do We Compute Eigenvectors?

1. Find the eigenvalues by solving for λ: det(K – λI) = 0

$$\det\left(\underbrace{\begin{bmatrix} 2-\lambda & 4/5 \\ 4/5 & 3/5-\lambda \end{bmatrix}}_{K-\lambda I}\right) = 0$$

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$$(2-\lambda)(3/5-\lambda)-(4/5)(4/5) = \lambda^2-13/5\lambda+14/25$$

$$\boxed{\lambda^2 - 13/5\lambda + 14/25 = 0}$$

characteristic equation of K

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characteristic equation of K

$$\boxed{\lambda_1 = \frac{13+\sqrt{113}}{10} \approx 2.36;} \quad \boxed{\lambda_2 = \frac{13-\sqrt{113}}{10} \approx 0.24}$$

How Do We Compute Eigenvectors?

2. Plug each eigenvalue in to find the corresponding eigenvector

$$\underbrace{\begin{bmatrix} 2 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} e_{1,1} \\ e_{1,2} \end{bmatrix}}_{\mathbf{e_1}} = \lambda_1 \underbrace{\begin{bmatrix} e_{1,1} \\ e_{1,2} \end{bmatrix}}_{\mathbf{e_1}}$$

$$\underbrace{\begin{bmatrix} 2 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} e_{2,1} \\ e_{2,2} \end{bmatrix}}_{\mathbf{e_2}} = \lambda_2 \underbrace{\begin{bmatrix} e_{2,1} \\ e_{2,2} \end{bmatrix}}_{\mathbf{e_2}}$$

How Do We Compute Eigenvectors?

Let's see what happens for λ_1

$$\begin{cases} 2e_{1,1} + 4/5e_{1,2} = \frac{13+\sqrt{113}}{10}e_{1,1} \\ 4/5e_{1,1} + 3/5e_{1,2} = \frac{13+\sqrt{113}}{10}e_{1,2} \end{cases}$$

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$$\begin{cases} 2e_{1,1} + 0.8e_{1,2} = 2.36e_{1,1} \\ 0.8e_{1,1} + 0.6e_{1,2} = 2.36e_{1,2} \end{cases}$$

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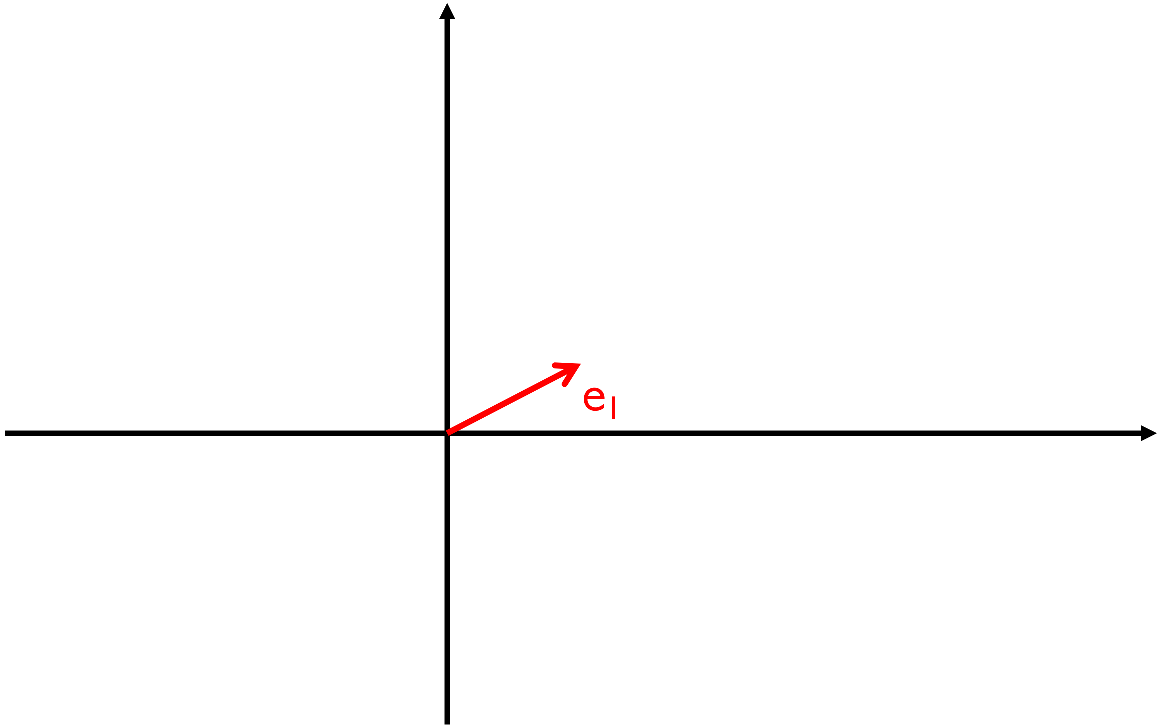
$$\boxed{e_{1,1} \approx 2.2e_{1,2}}$$

The system has infintely many solutions

How Do We Compute Eigenvectors?

Any vector which satisfies the relationship above works!



$$\begin{bmatrix} 1.1 \\ 0.5 \end{bmatrix}$$

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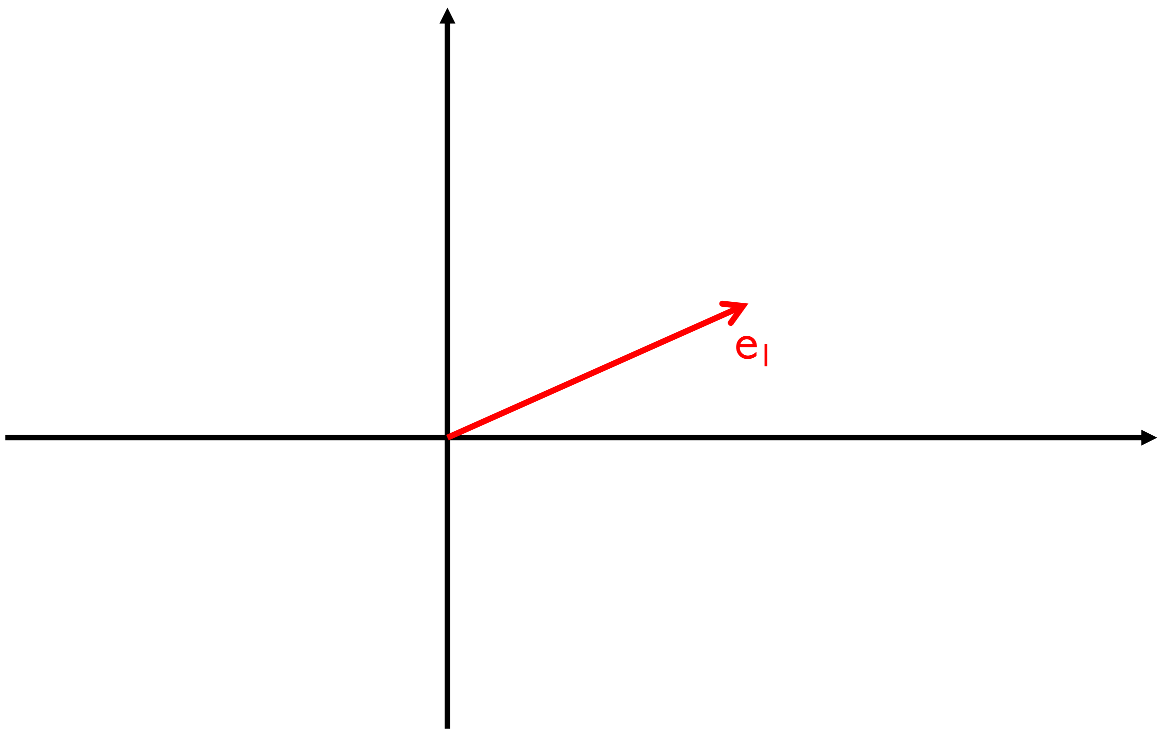

$$\begin{bmatrix} 2.2 \\ 1 \end{bmatrix}$$

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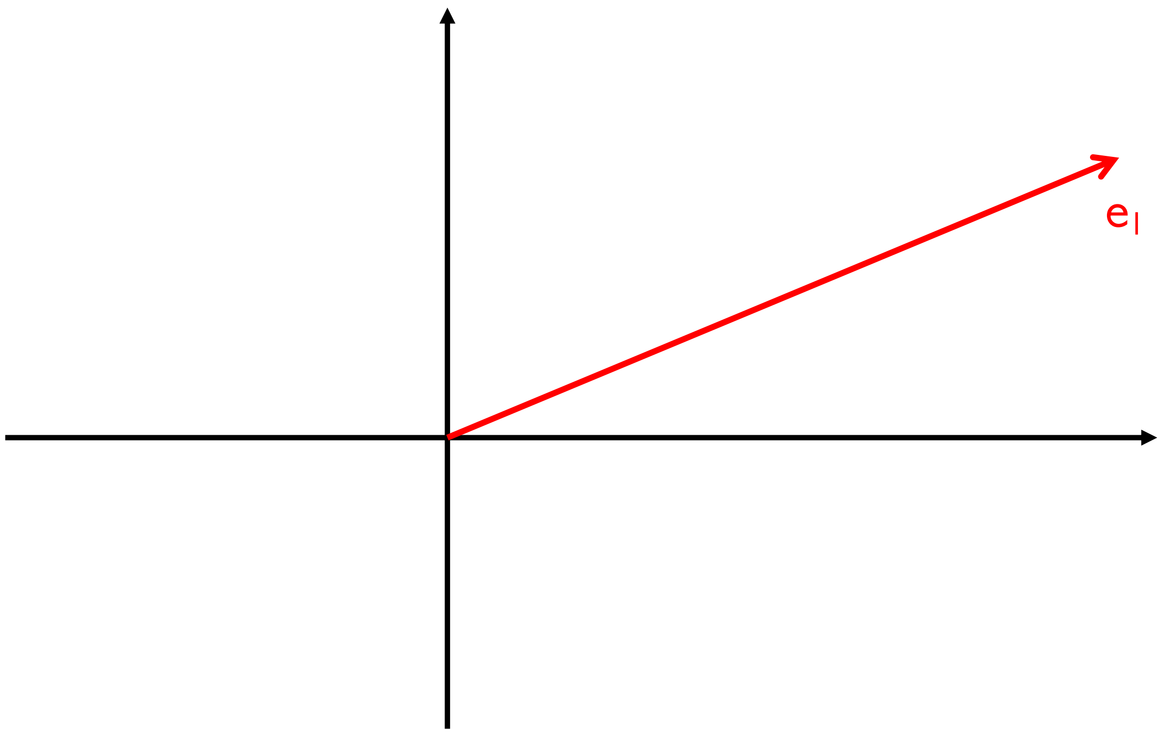
How Do We Compute Eigenvectors?

Any vector which satisfies the relationship above works!

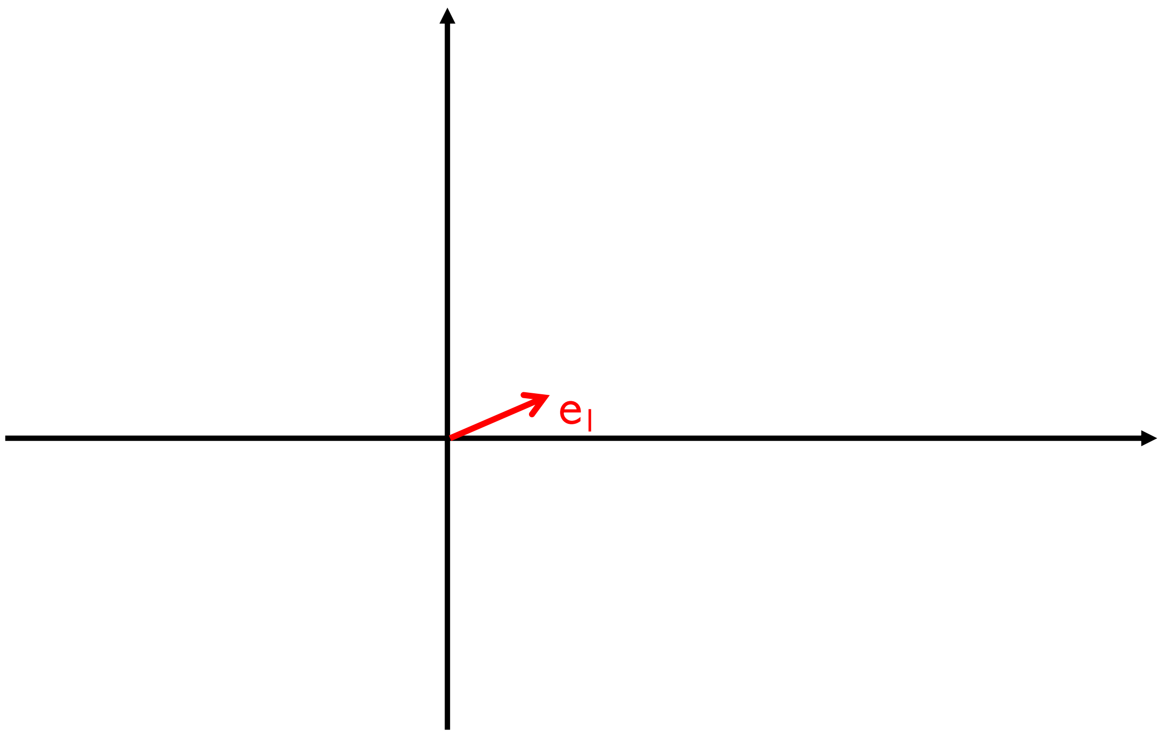


$$\begin{bmatrix} 4.4 \\ 2 \end{bmatrix}$$

$$e_{1,1} \approx 2.2 e_{1,2}$$

How Do We Compute Eigenvectors?

By convention, we restrict to $||\mathbf{e}_1|| = 1$

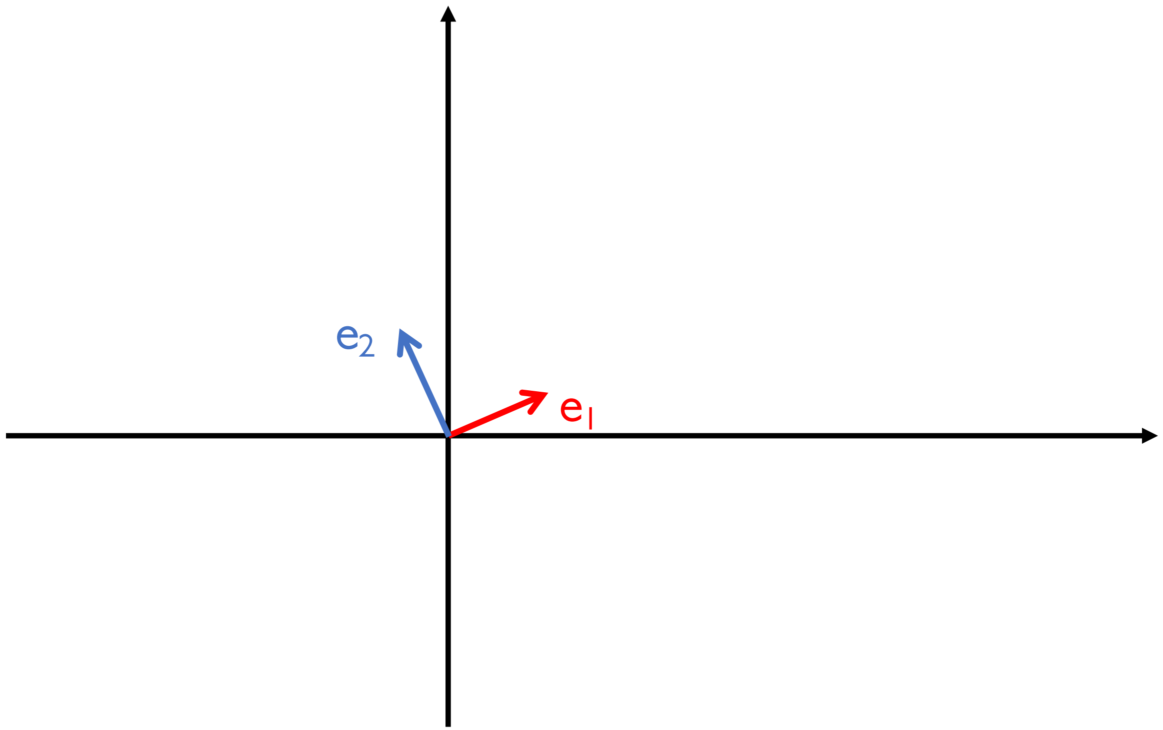


$$\begin{bmatrix} 0.91 \\ 0.41 \end{bmatrix}$$

$$e_{1,1} \approx 2.2 e_{1,2}$$

How Do We Compute Eigenvectors?

The second eigenvector e_2 can be found by plugging in the smaller eigenvalue λ_2



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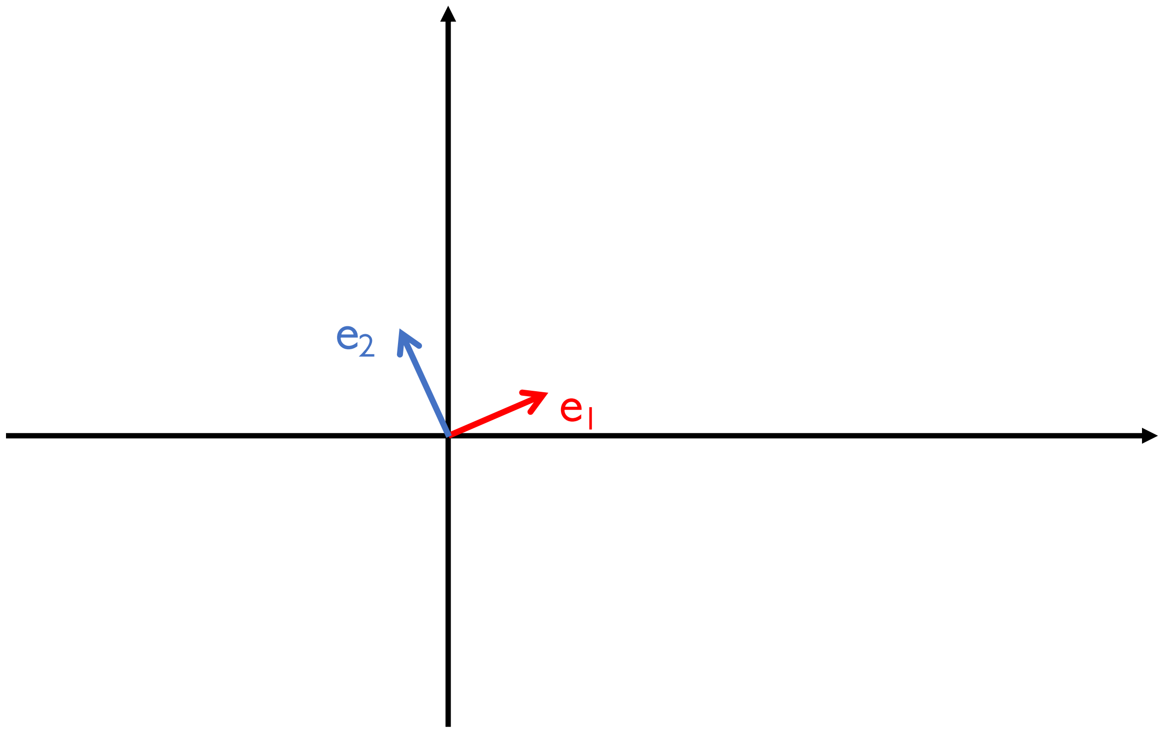

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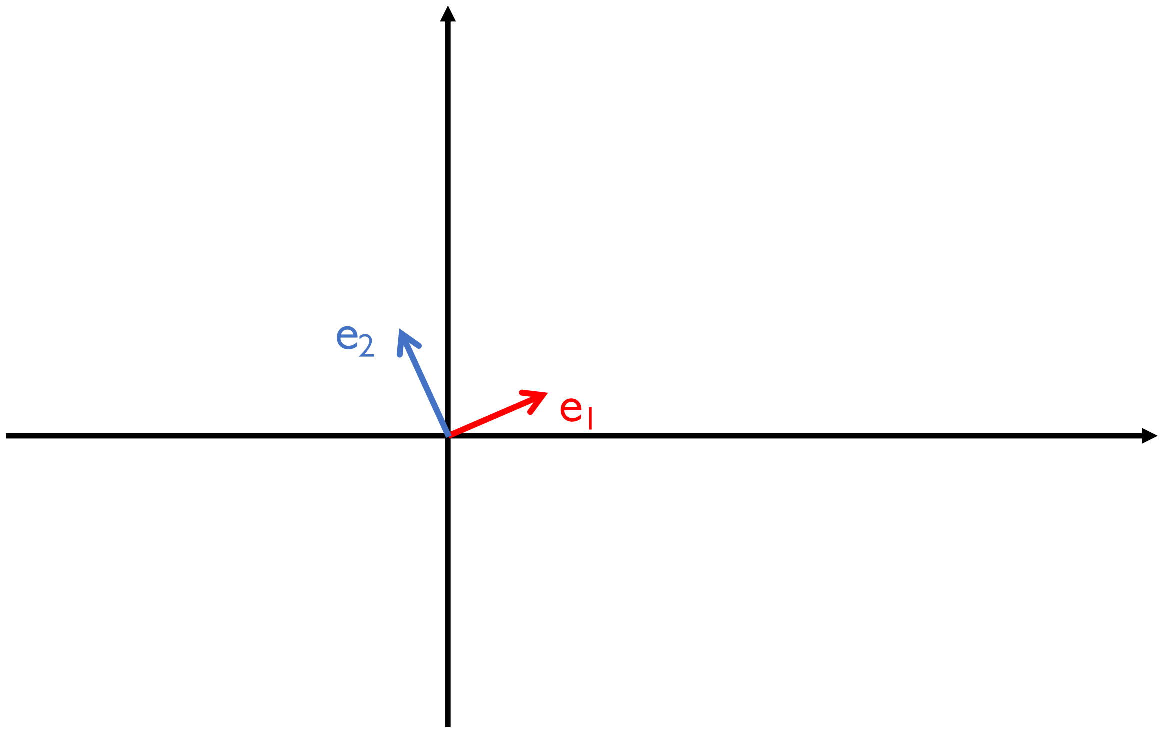

e_1 and e_2 are the new coordinate system replacing the original x_1 and x_2

$$\mathbf{e_1} = \begin{bmatrix} 0.91 \\ 0.41 \end{bmatrix} \quad \mathbf{e_2} = \begin{bmatrix} -0.41 \\ 0.91 \end{bmatrix}$$

Principal Components

$$\mathbf{e}_1 = \begin{bmatrix} 0.91 \\ 0.41 \end{bmatrix} \mathbf{e}_2 = \begin{bmatrix} -0.41 \\ 0.91 \end{bmatrix}$$

e_1 is the 1st principal component as it is the eigenvector corresponding to the largest eigenvalue

e_2 is the 2nd principal component as it is the eigenvector corresponding to the smallest eigenvalue

Projecting to New Dimensions: 2-d Case

- e_1 and e_2 identify our new coordinate system (principal components)

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- Let $\mathbf{x} = (x_1, x_2)$ be a point (i.e., a vector) in the original $(\mathbf{x}_1, \mathbf{x}_2)$-space

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- Let $\mathbf{x} = (x_1, x_2)$ be a point (i.e., a vector) in the original $(\mathbf{x}_1, \mathbf{x}_2)$-space

$$\Downarrow$$

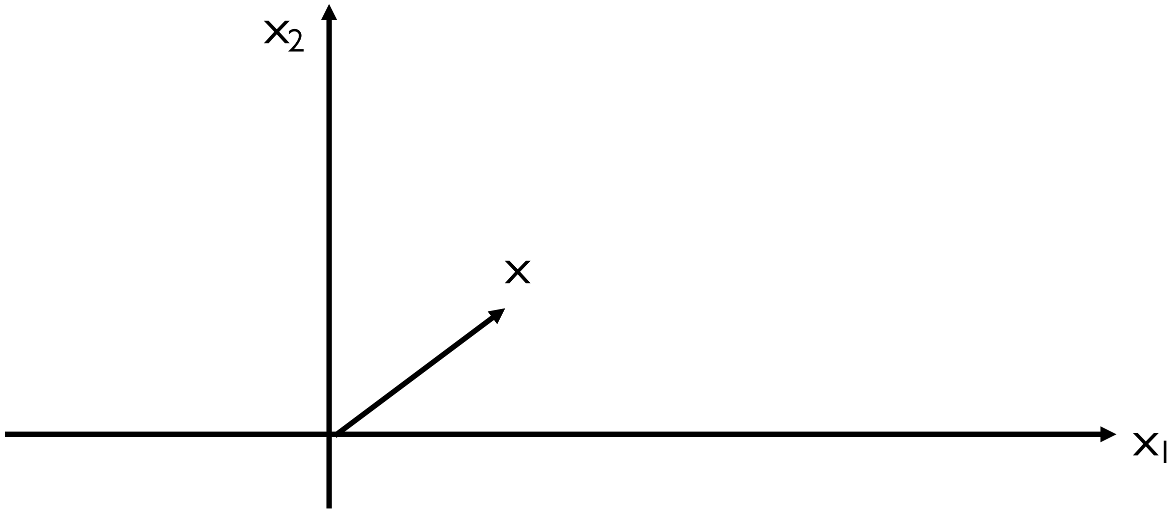
Goal

We want to represent $\mathbf{x}$ in the new $(\mathbf{e}_1, \mathbf{e}_2)$-coordinate system

Projecting to New Dimensions: 2-d Case

1. Center **x** around the mean of each dimension

$$\mathbf{x}' = \mathbf{x} - \boldsymbol{\mu} = (x_1 - \mu_1, x_2 - \mu_2)$$



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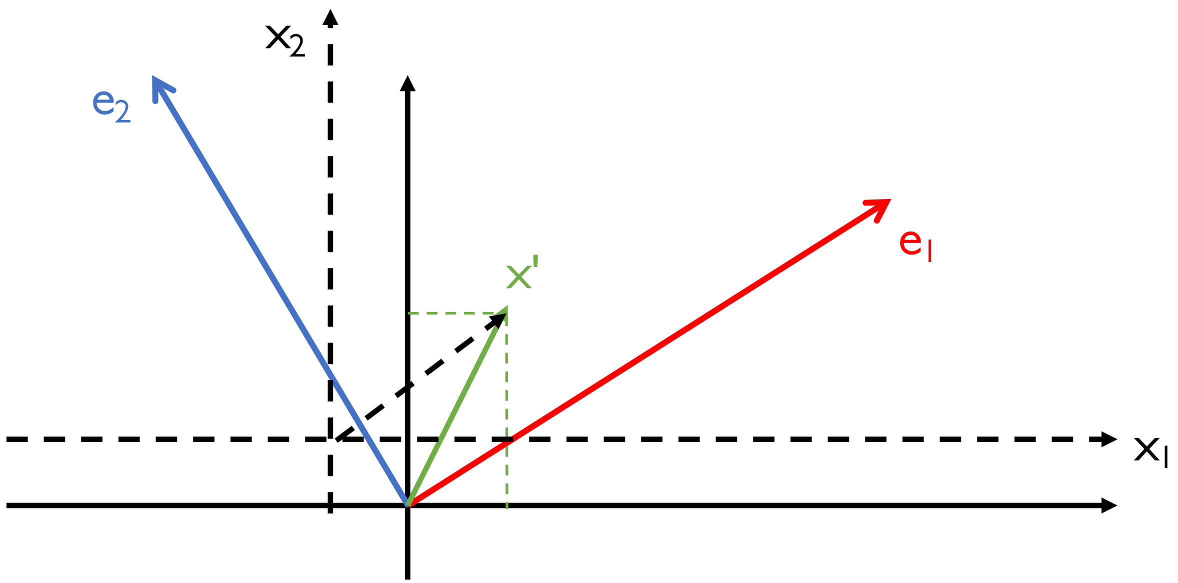
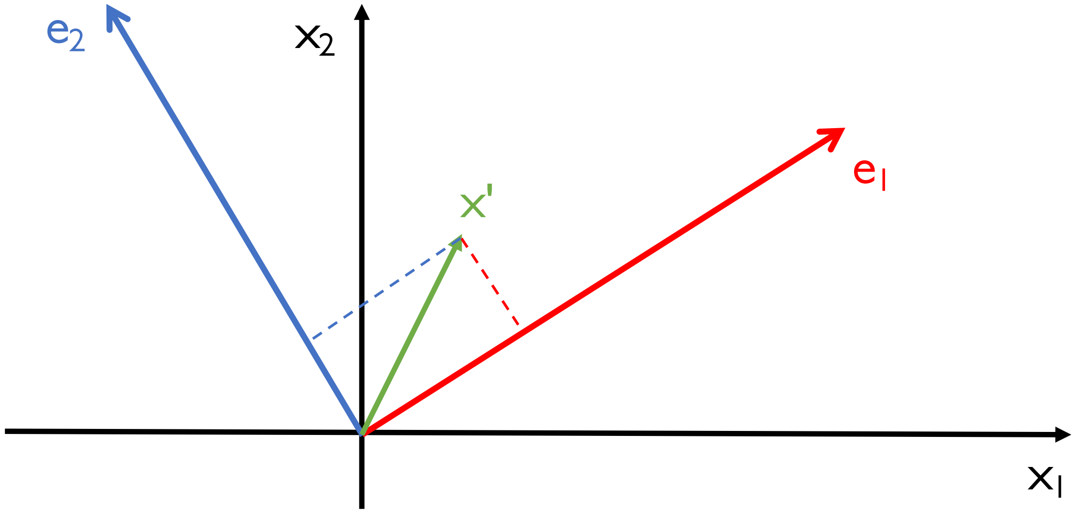


Projecting to New Dimensions: 2-d Case

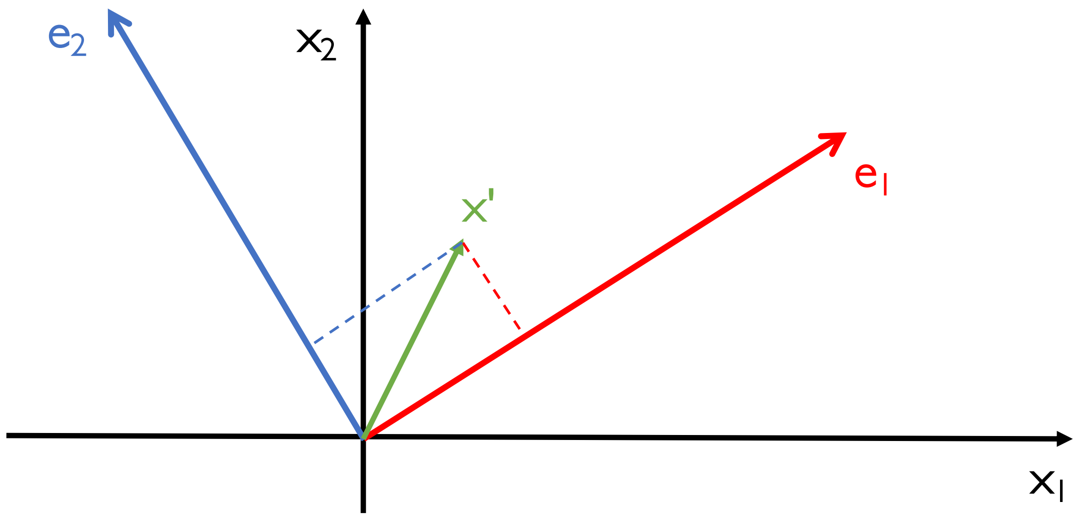
2. Project **x'** on each dimension $\mathbf{e}_1$ and $\mathbf{e}_2$

$$\mathbf{x}' = \underbrace{(x'_1, x'_2)}_{\text{coordinates of } \mathbf{x}' \text{ in the } (\mathbf{e}_1,\mathbf{e}_2)\text{-space}} = (\mathbf{x}'^T\mathbf{e}_1, \mathbf{x}'^T\mathbf{e}_2)$$



Projecting to New Dimensions: 2-d Case

Why the dot product?



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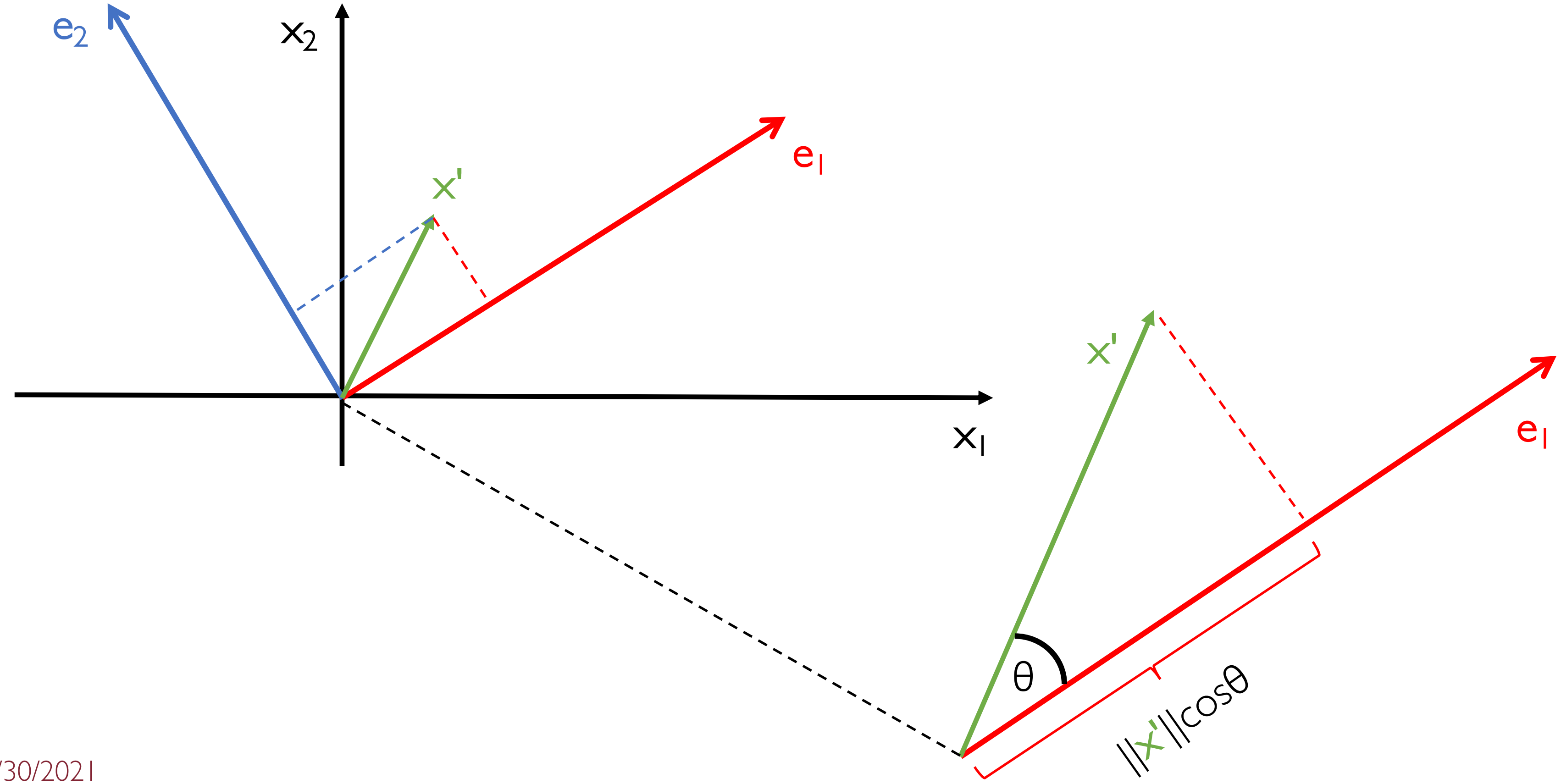
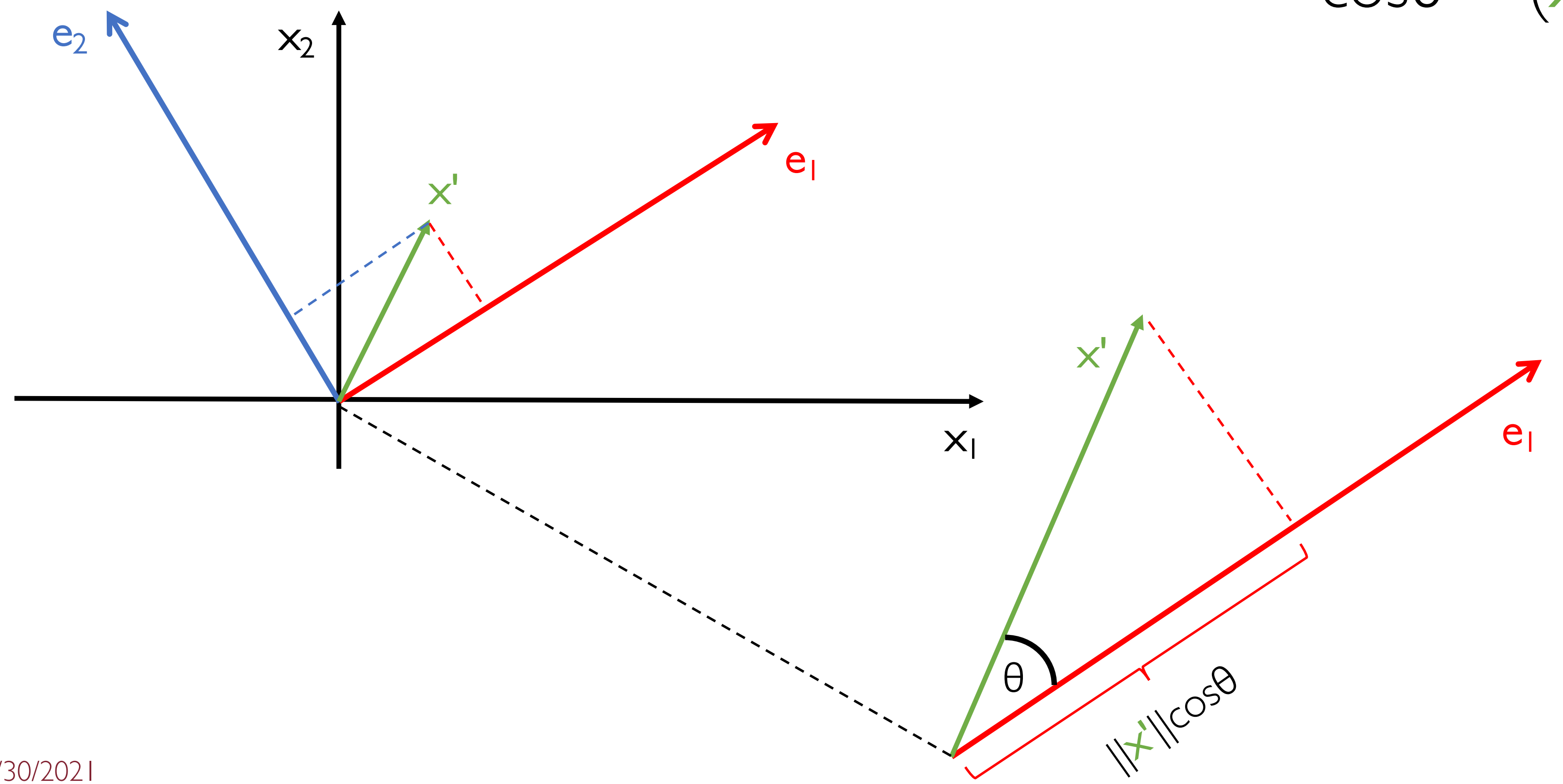


Projecting to New Dimensions: 2-d Case

Why the dot product?

$\cos\theta = (x'e_1)/||x'||\,||e_1||$



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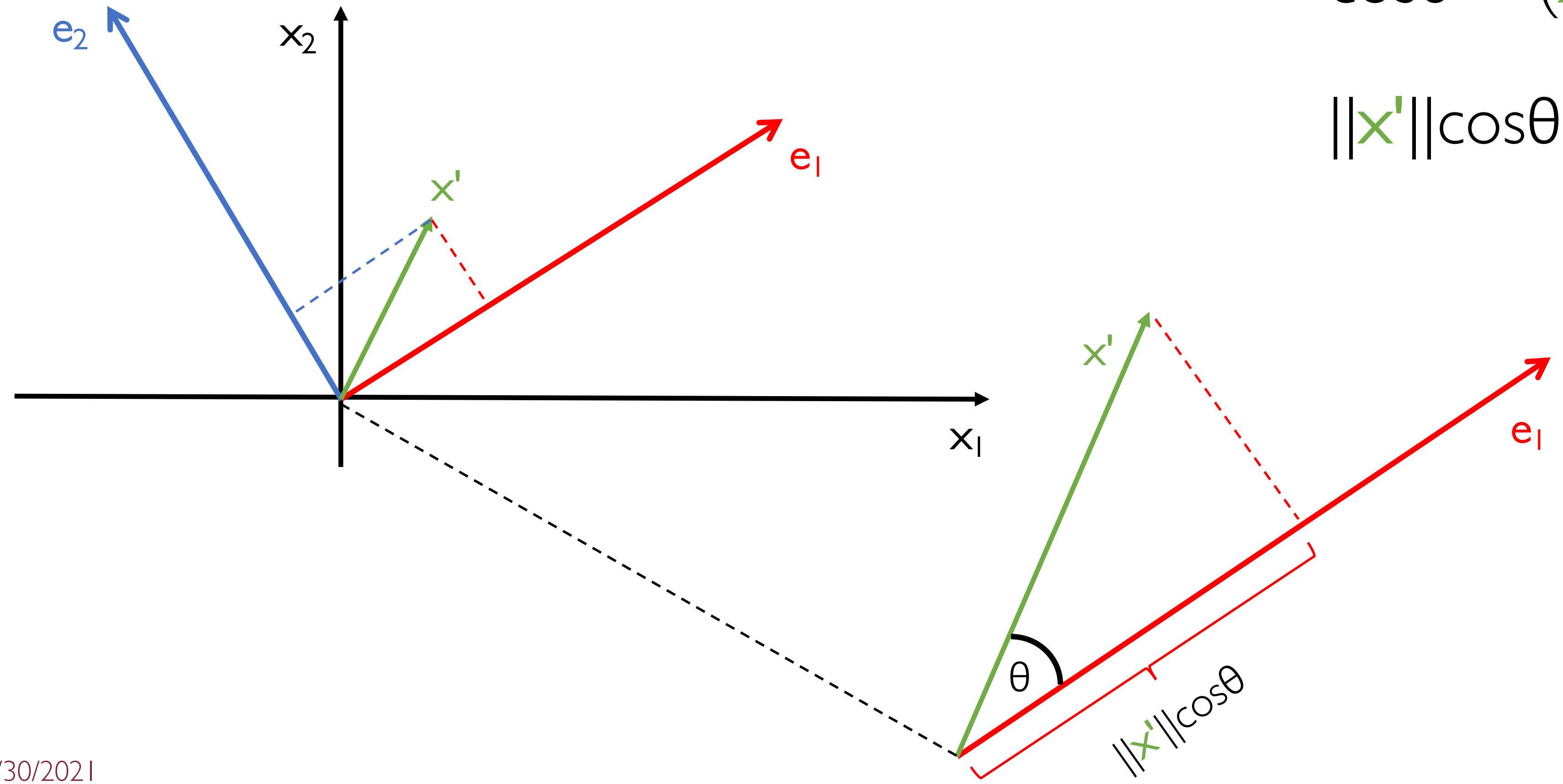
$||x'||\cos\theta = x'e_1/||e_1||$



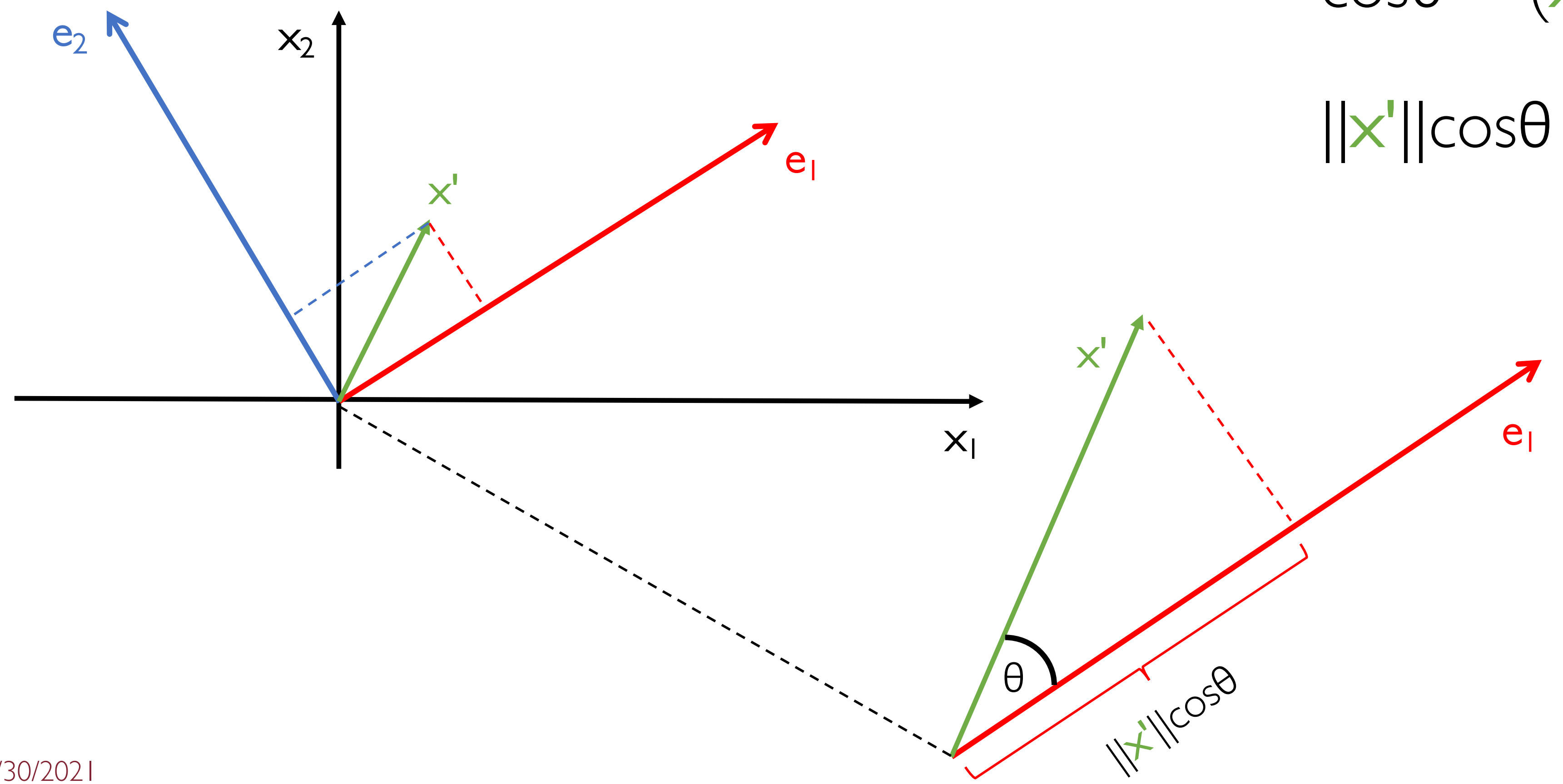
Projecting to New Dimensions: 2-d Case

Why the dot product?

$$\cos\theta = (x'e_1)/\|x'\|\|e_1\|$$

$$\|x'\|\cos\theta = x'e_1/\|e_1\|$$

$\|e_1\| = 1$



Projecting to New Dimensions: 2-d Case

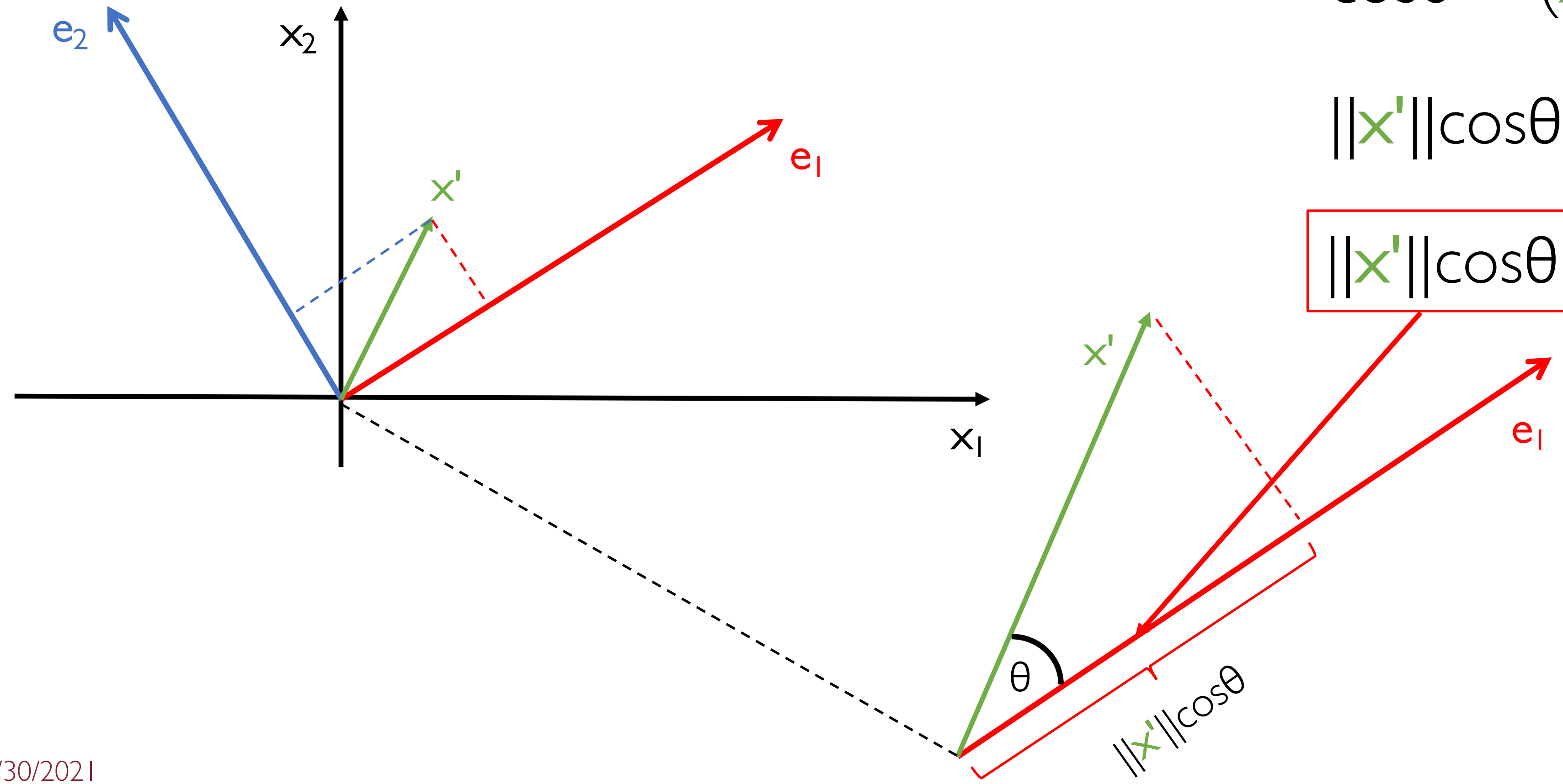
Why the dot product?

$\cos\theta = (x'e_1)/||x'||||e_1||$

$||x'||\cos\theta = x'e_1/||e_1||$

$\boxed{||x'||\cos\theta = x'e_1}$

$||e_1|| = 1$



Projecting to New Dimensions: 2-d Case

The new coordinates of the original data point **x** according to the eigenvectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are as follows:

$$\mathbf{x}' = \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = \begin{bmatrix} \mathbf{x}'^T\mathbf{e}_1 \\ \mathbf{x}'^T\mathbf{e}_2 \end{bmatrix} = \begin{bmatrix} (x_1 - \mu_1)e_{1,1} + (x_2 - \mu_2)e_{1,2} \\ (x_1 - \mu_1)e_{2,1} + (x_2 - \mu_2)e_{2,2} \end{bmatrix}$$

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1. Mean centering

$$\mathbf{x}' = \mathbf{x} - \boldsymbol{\mu} = \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \\ \vdots \\ x_d - \mu_d \end{bmatrix}$$

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2. Projection to principal components

$$\mathbf{x}' = \begin{bmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_k \end{bmatrix} = \begin{bmatrix} (\mathbf{x}-\boldsymbol{\mu})^T \mathbf{e}_1 \\ (\mathbf{x}-\boldsymbol{\mu})^T \mathbf{e}_2 \\ \vdots \\ (\mathbf{x}-\boldsymbol{\mu})^T \mathbf{e}_k \end{bmatrix} = \begin{bmatrix} (x_1-\mu_1)e_{1,1} + (x_2-\mu_2)e_{1,2} + \ldots + (x_d-\mu_d)e_{1,d} \\ (x_1-\mu_1)e_{2,1} + (x_2-\mu_2)e_{2,2} + \ldots + (x_d-\mu_d)e_{2,d} \\ \vdots \\ (x_1-\mu_1)e_{k,1} + (x_2-\mu_2)e_{k,2} + \ldots + (x_d-\mu_d)e_{k,d} \end{bmatrix}$$

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More details available here:
https://github.com/gtolomei/big-data-computing/raw/master/extra/Notes_on_Principal_Component_Analysis.pdf

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Pick the subset of k eigenvectors that "explain" the most variance

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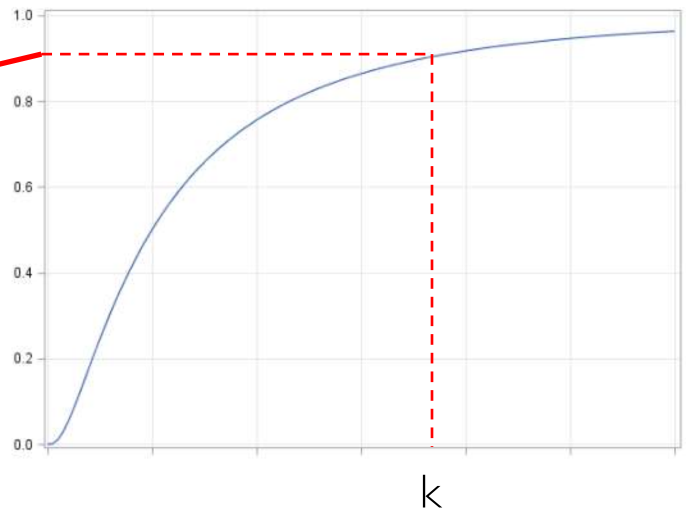
2. Pick the first k eigenvectors that explain x% of the total variance

$$\frac{\sum_{i=1}^{k} \lambda_i}{\sum_{i=1}^{d} \lambda_i} \leq 1$$

e.g., x = 90÷95%



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Solution

Normalize each dimension to 0-mean and 1-std-deviation

$$z = \frac{x - \mu}{\sigma}$$

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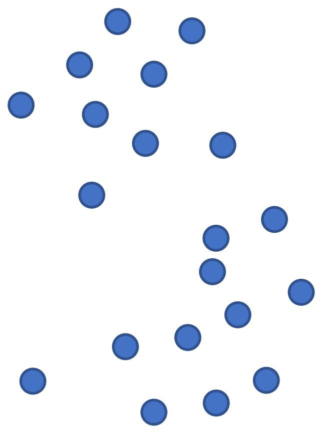

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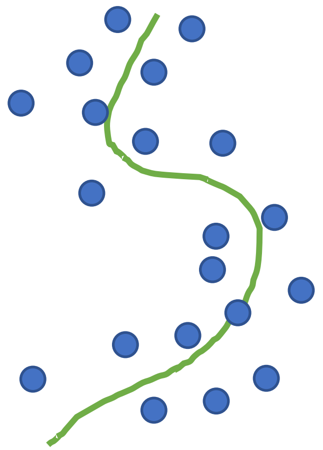

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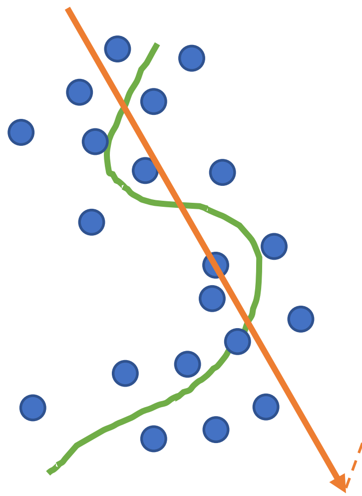

PCA will find a straight line and will not mimic non-linearity

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- PCA is a dimensionality reduction technique which allows representing high-dimensional data into low-dimensional linear subspace
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- PCA can be very expensive for many very large-scale applications
- If data do not live on a linear subspace PCA may not work well