Big Data Computing

Master's Degree in Computer Science

2024-2025

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Goal: The "Importance" of a Node

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- Derive such a score from the structural properties of the graph only (i.e., via **link analysis**)
- Exploit the fact that the Web is an example of a **scale-free network**

Computing Node Importance

Several link analysis approaches to compute web page importance

- PageRank
- Hubs and Authorities (HITS)
- Personalized PageRank
- Web Spam Detection

PageRank

One Slide PageRank

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- Assigns a numerical score to each web page with the purpose of indicating its relative importance within the whole collection

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PageRank's Intuition: Links as Votes

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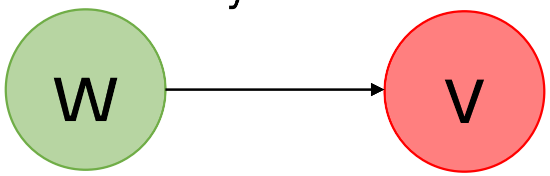

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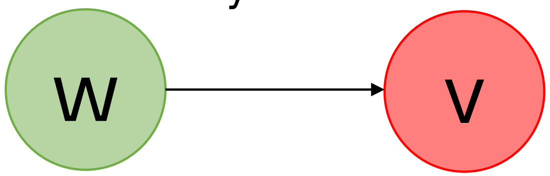

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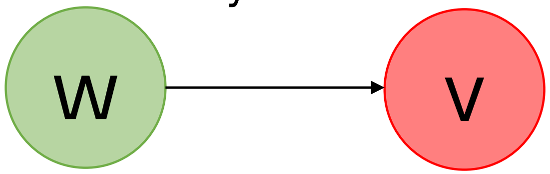

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Different web pages have different in-degree (scale-free network)

www.stanford.edu has more than 23K in-links

www.uniroma1.it/~tolomei has one or two in-links!

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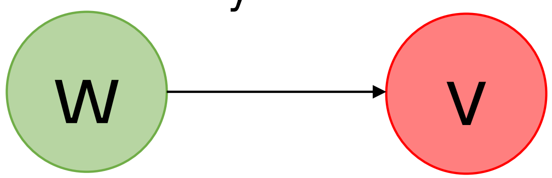

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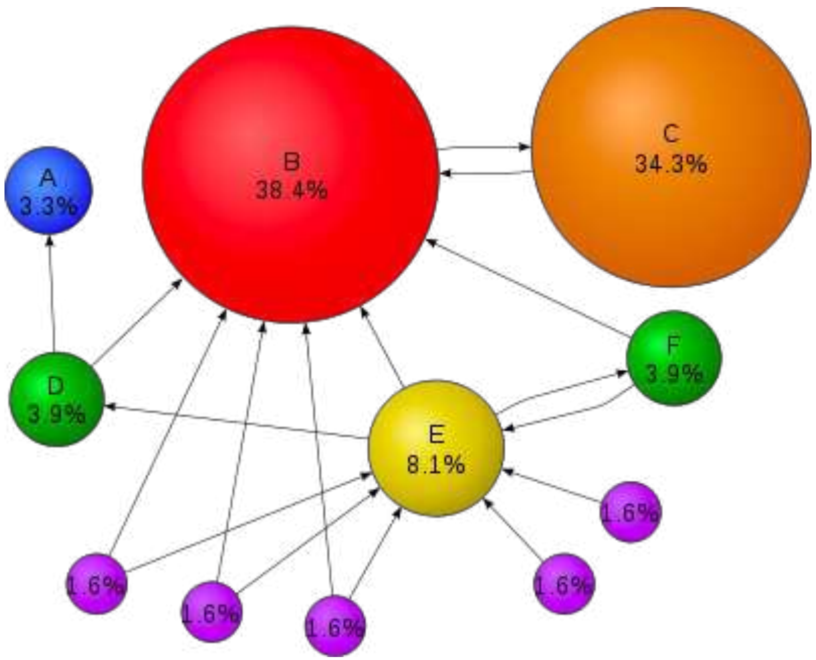
Recursive definition

PageRank Scores: Example



Circle size proportional to the node importance

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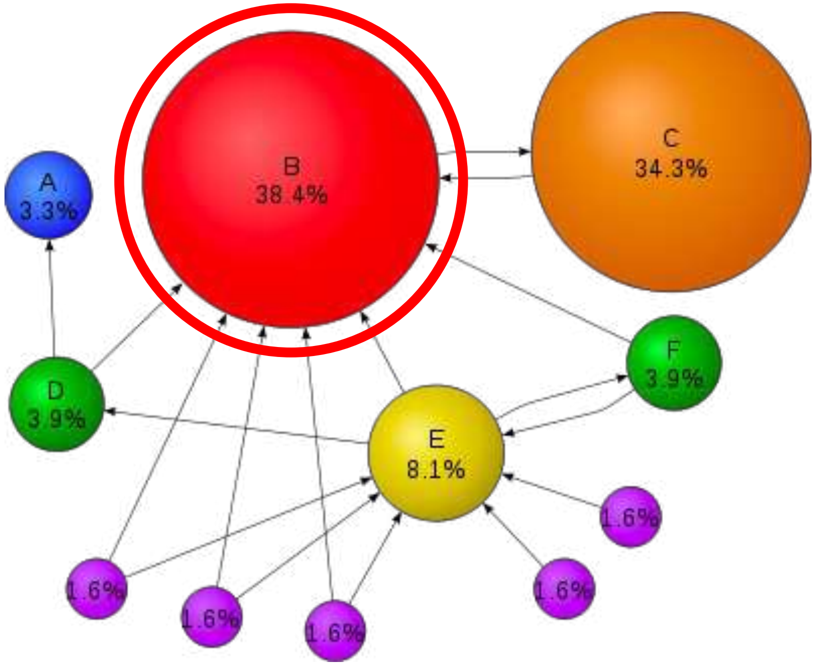

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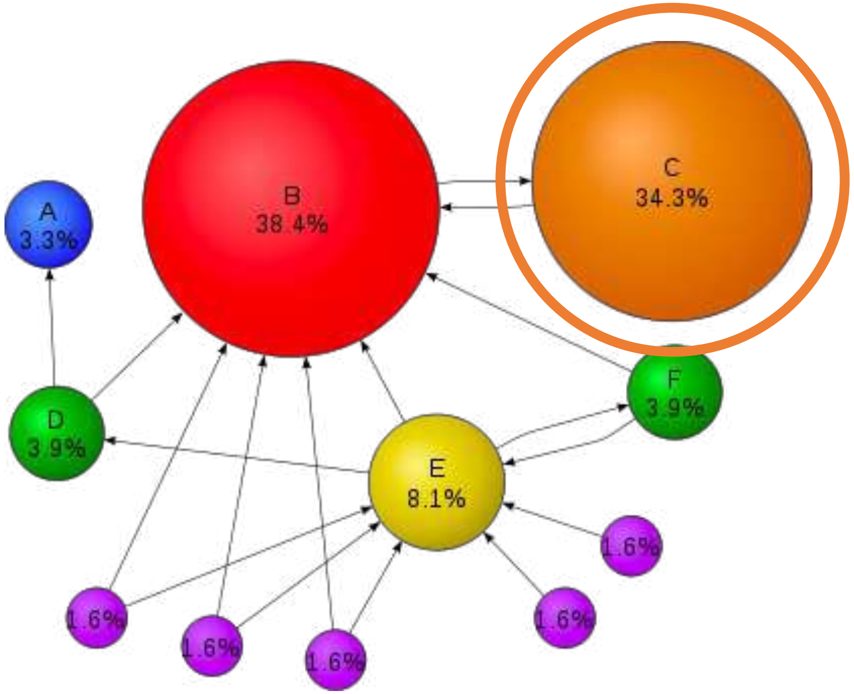

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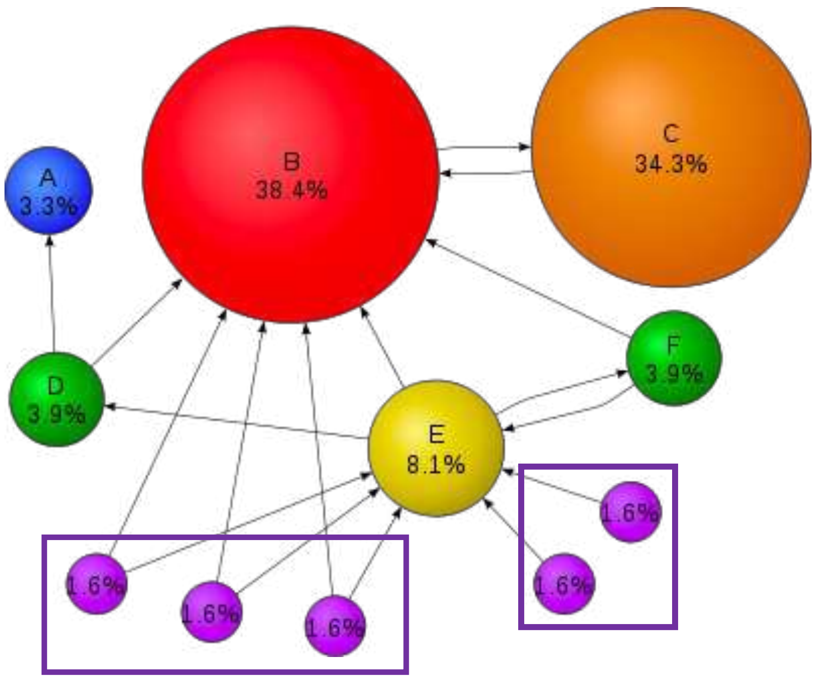

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Many other less important nodes

PageRank: Prelminaries

$G = (V, E)$ The Web Graph $|V| = N$ Number of Nodes (pages)

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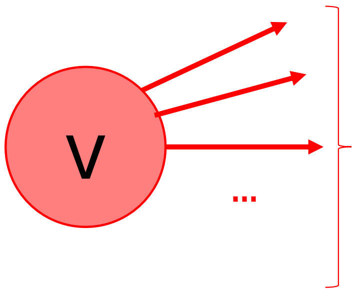

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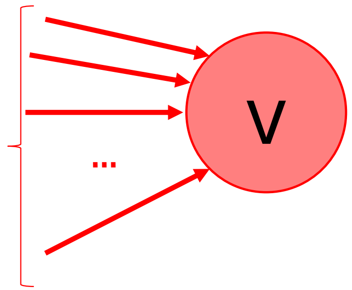

PageRank: First Simple Recursive Formulation

Each link's vote to a page v is proportional to the importance of the source page w, which the link comes from

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If a page w has importance r_w and out-degree o_w, each out-link will get an equal proportion of the importance, i.e., r_w/o_w

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If a page w has importance r_w and out-degree o_w, each out-link will get an equal proportion of the importance, i.e., r_w/o_w

Each page v's importance can be computed just as the sum of votes of all its incoming links (i.e., in-degree)

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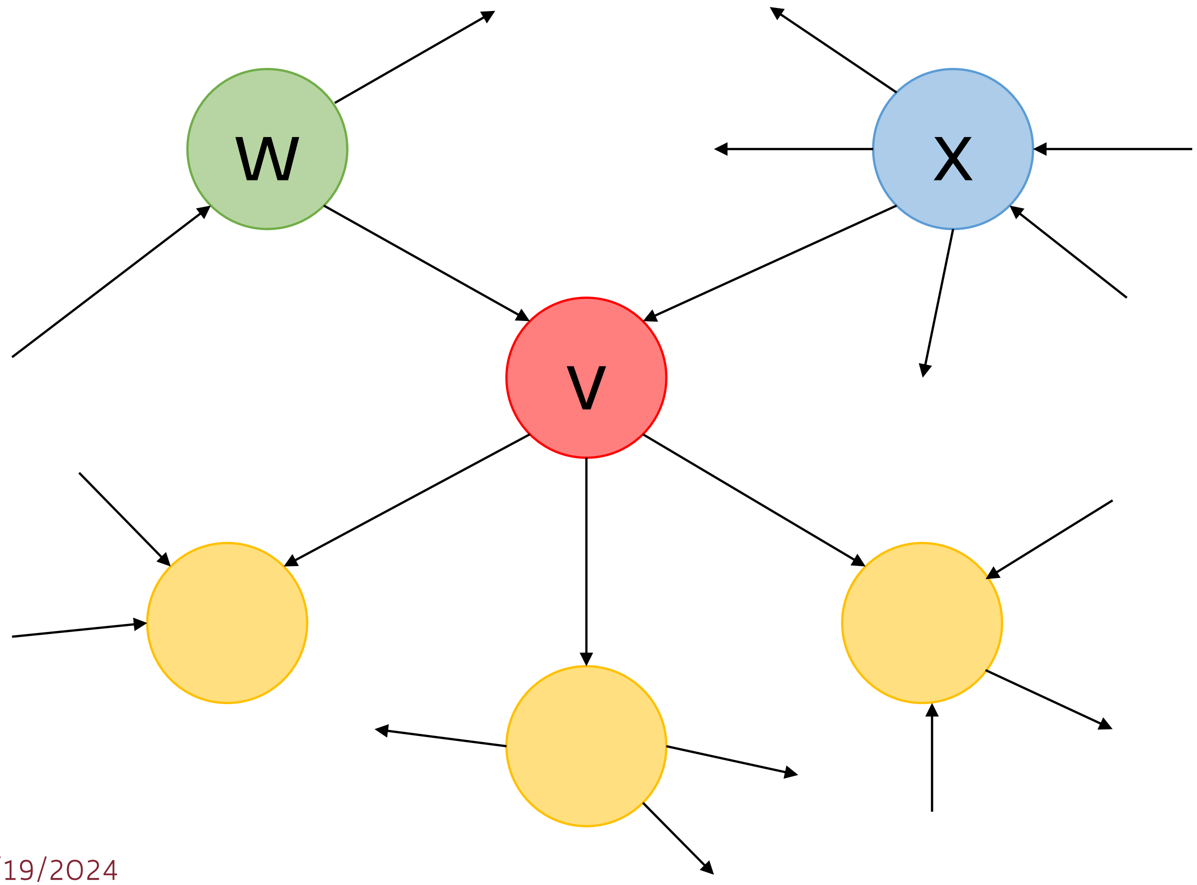

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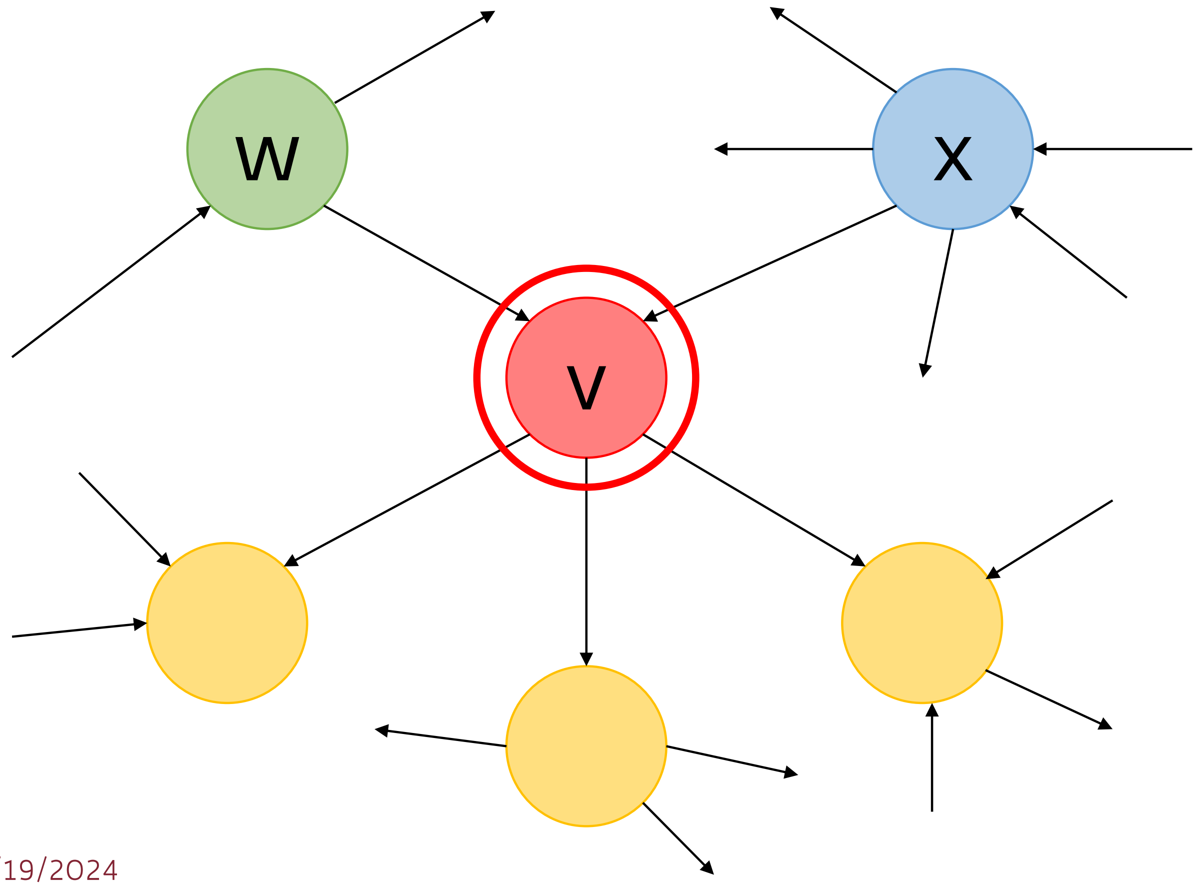
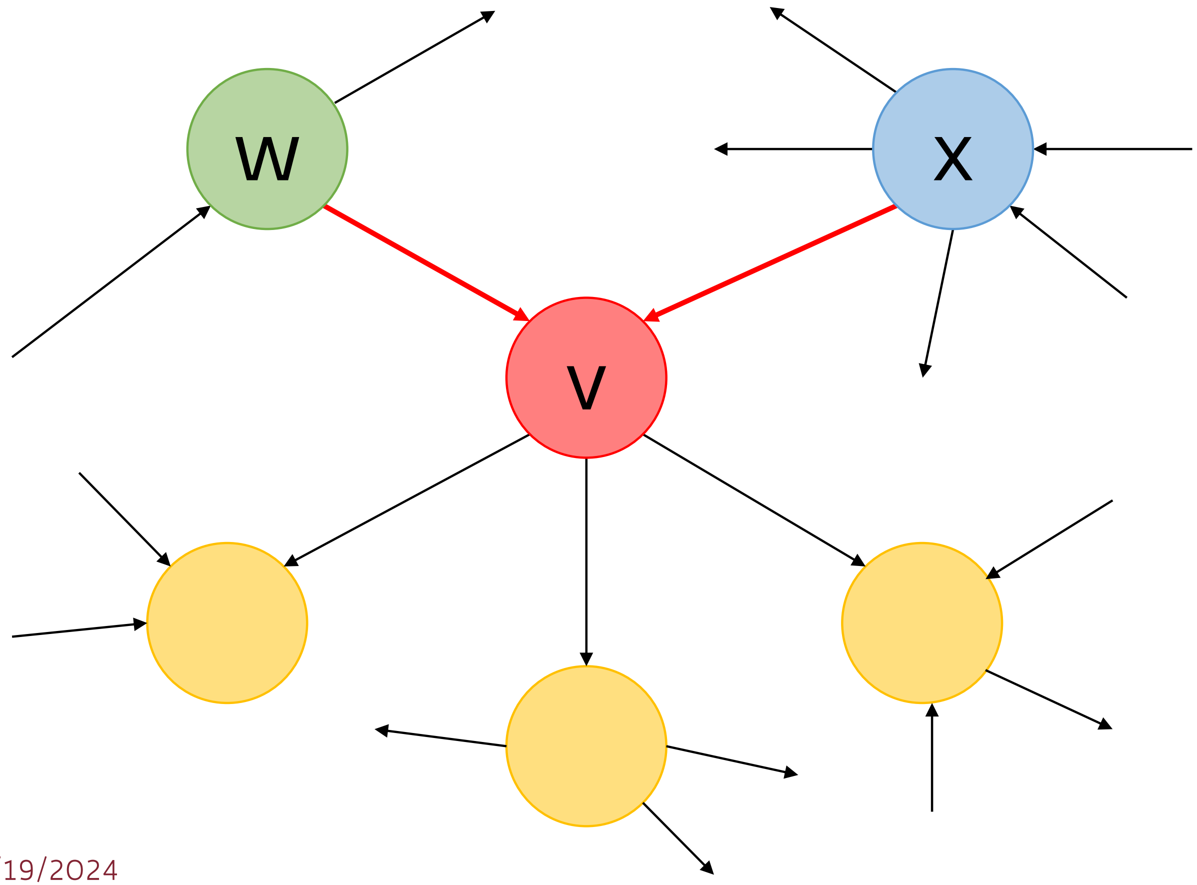


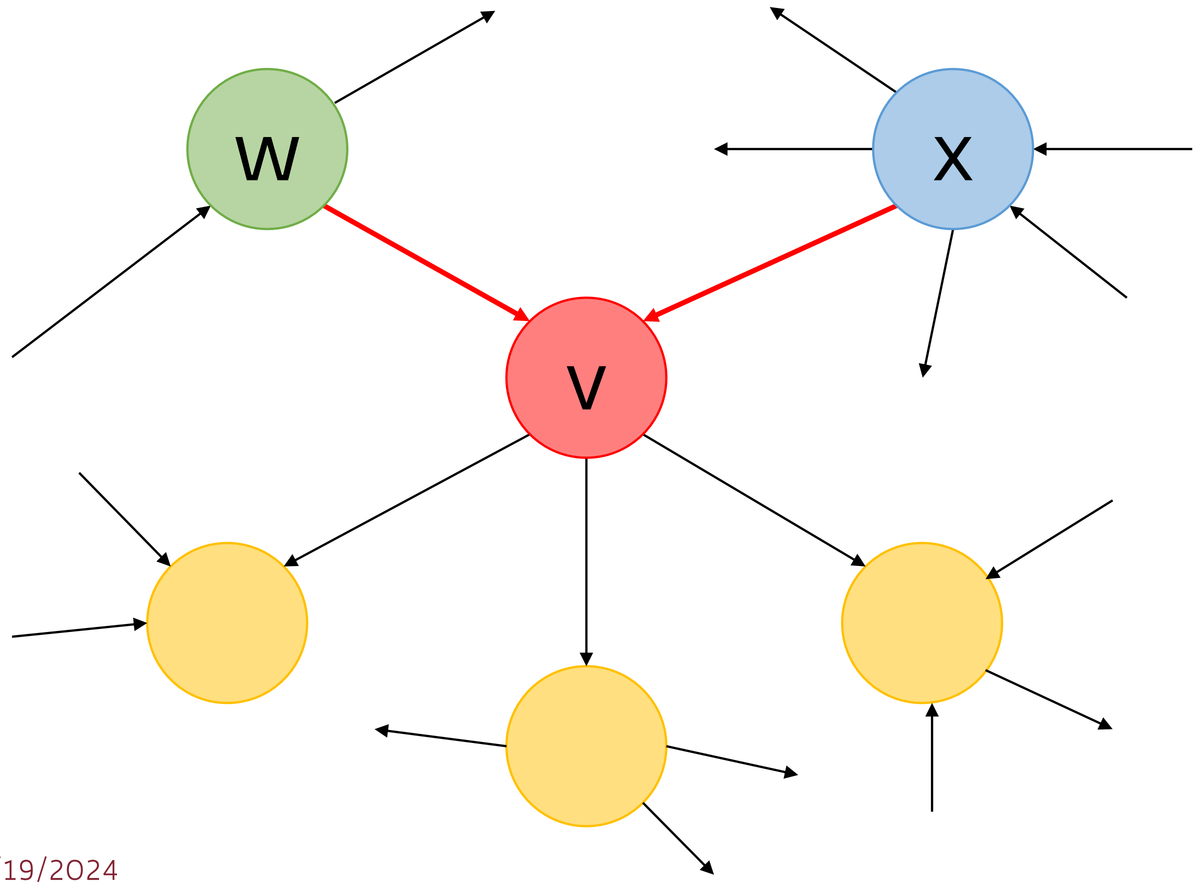
What is r_v?

PageRank: First Simple Recursive Formulation



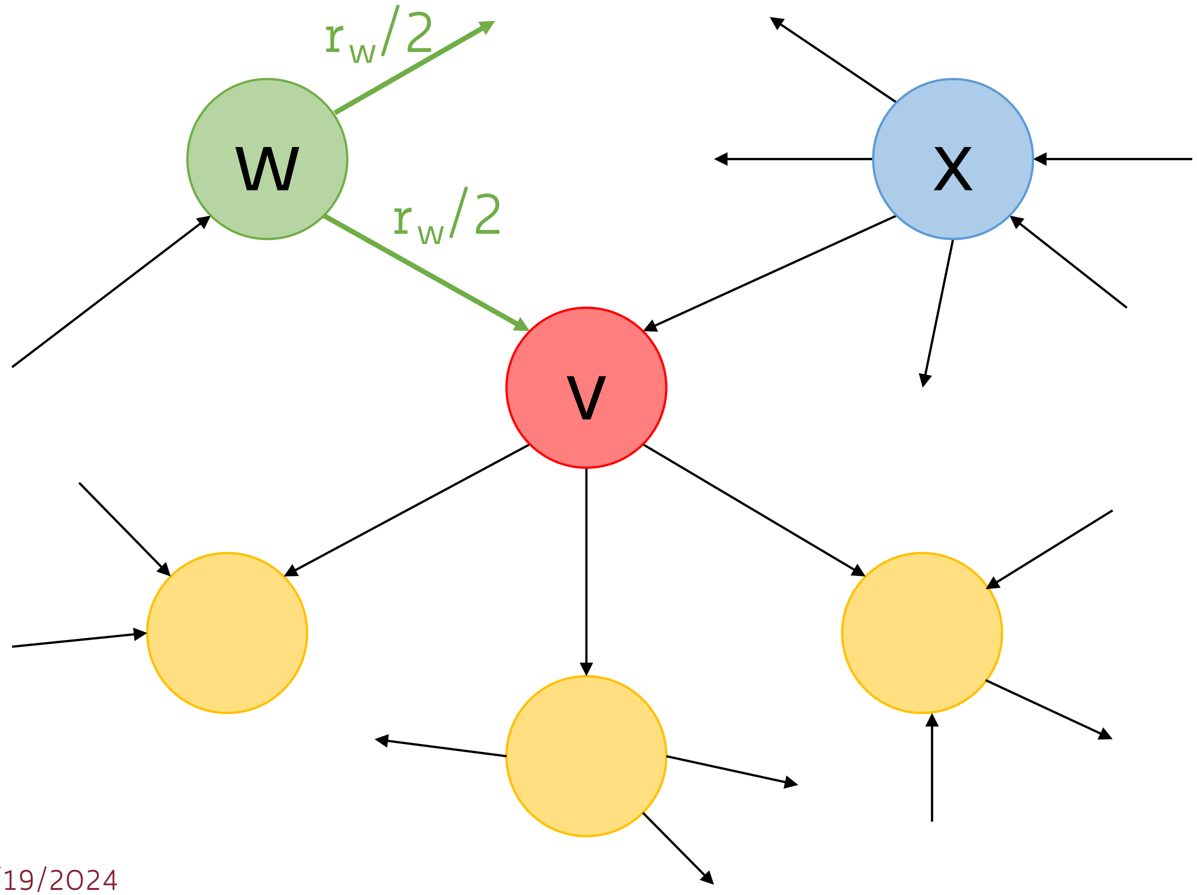
Suppose v has only 2 in-links coming from w and x

PageRank: First Simple Recursive Formulation



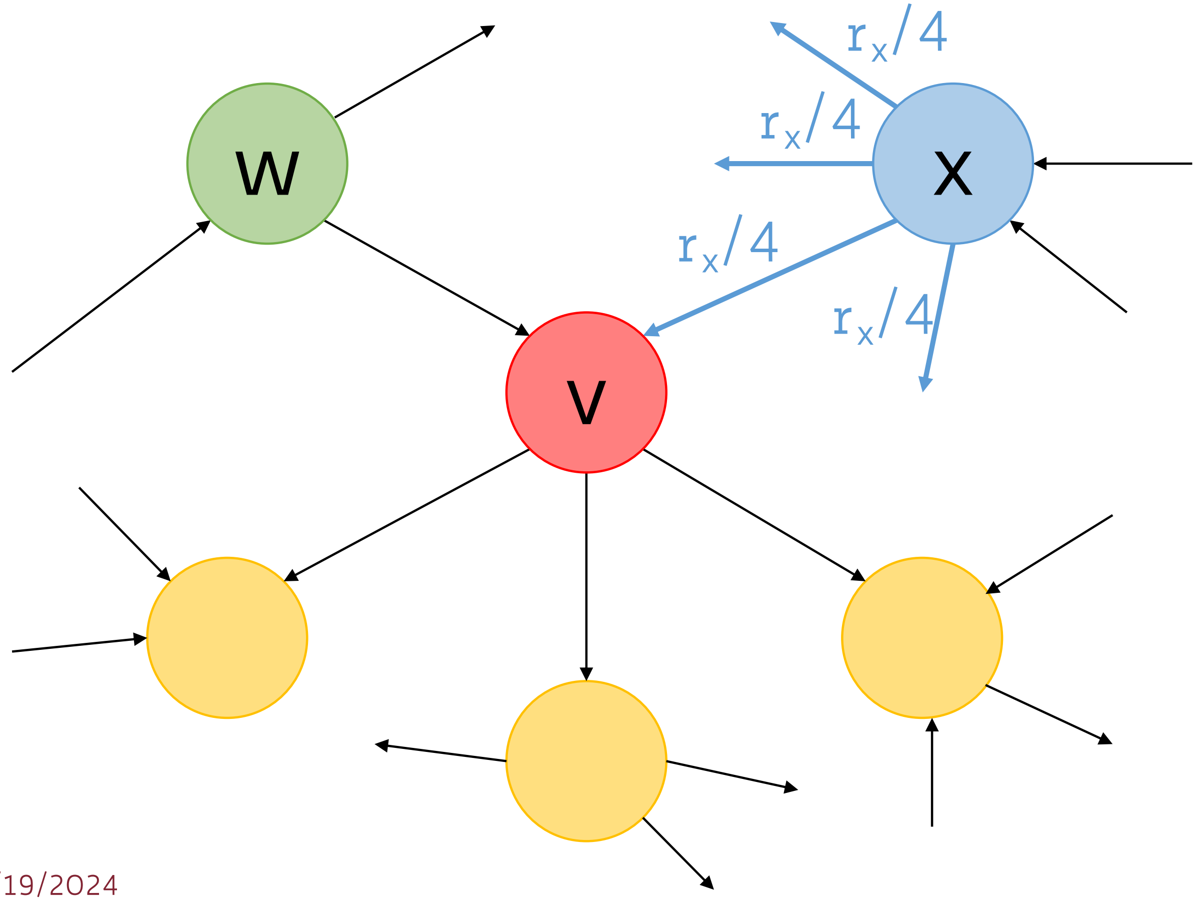
We must compute the in-link's vote from w and from x

PageRank: First Simple Recursive Formulation



The importance of page w (r_w) is distributed across each of its 2 outgoing links

PageRank: First Simple Recursive Formulation



The importance of page x (r_x) is distributed across each of its 4 outgoing links

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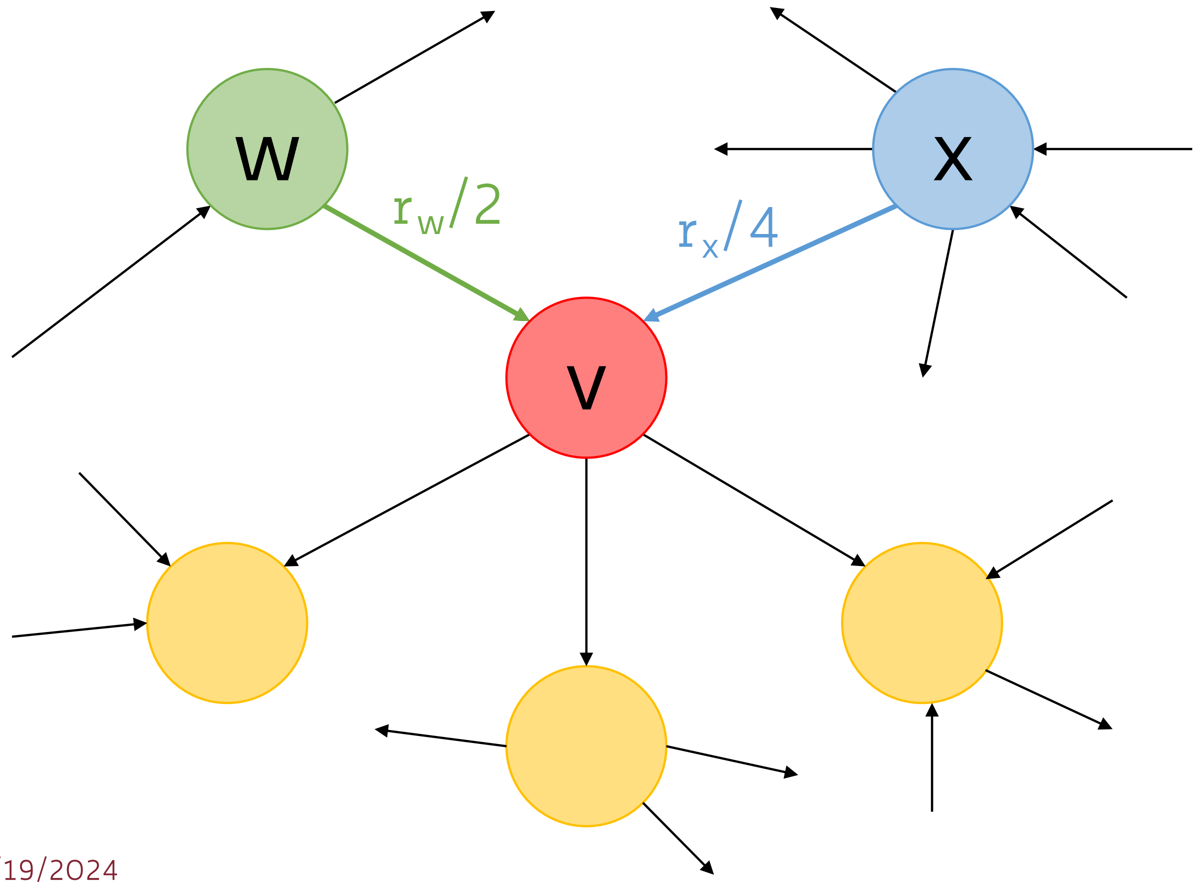

The importance of page v (r_v) is just the sum of its incoming links' votes

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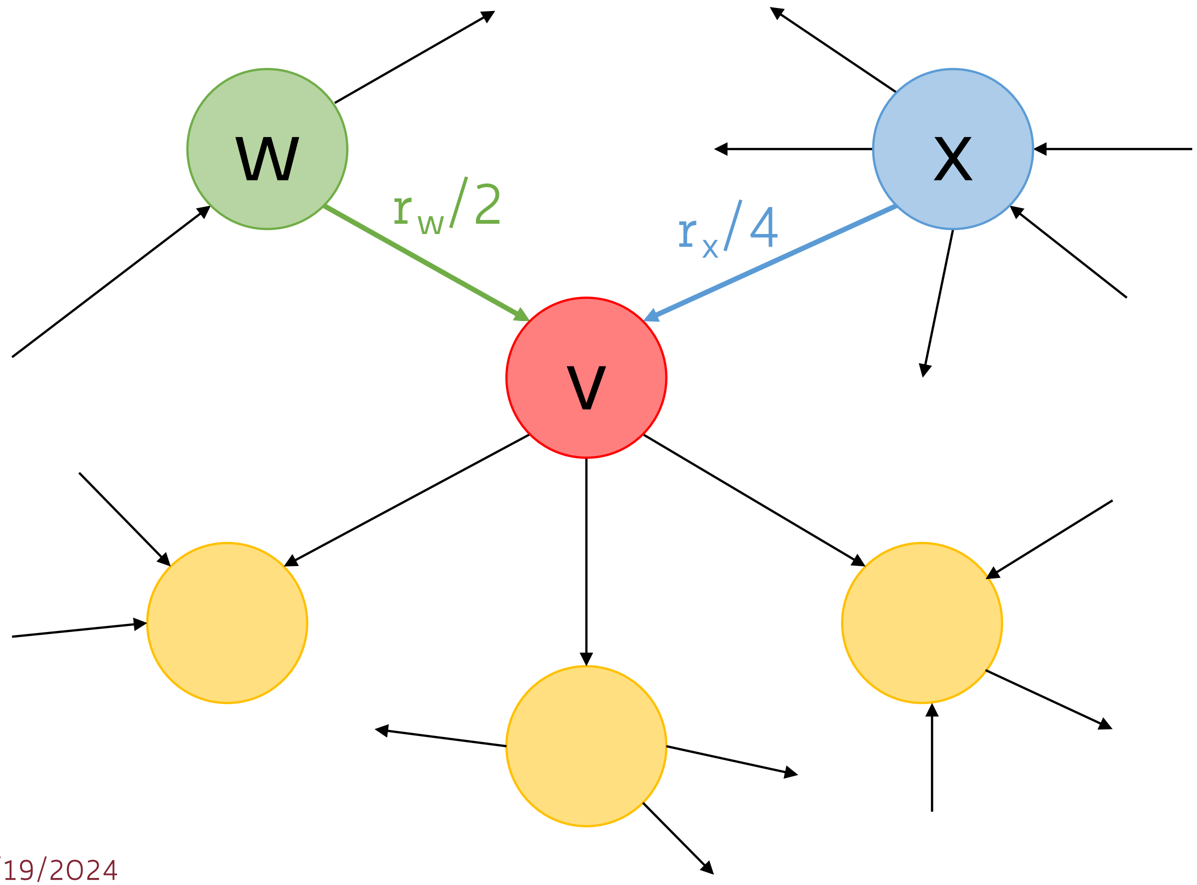

The importance of page v (r_v) is just the sum of its incoming links' votes

$$\boxed{r_v = r_w/2 + r_x/4}$$

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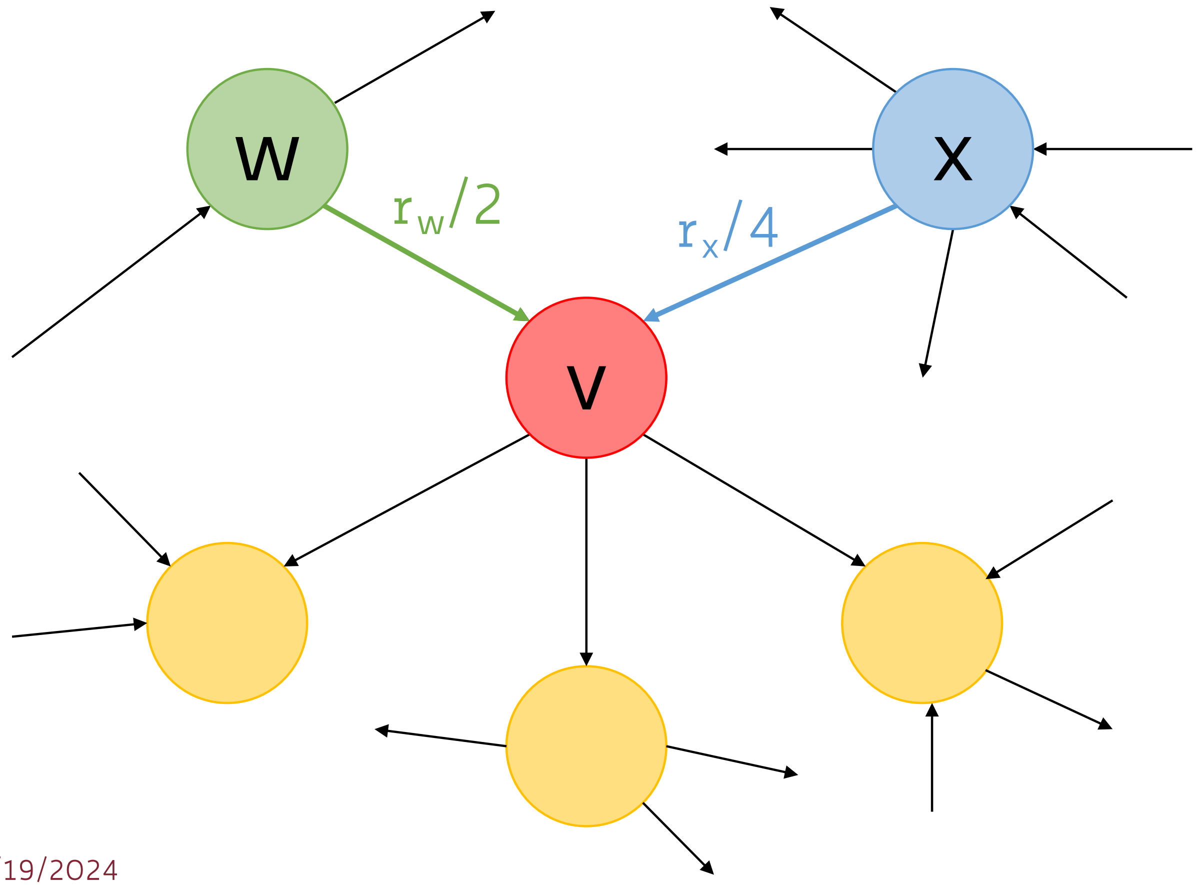

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$$\boxed{r_v = \sum_{u \in I_v} \frac{r_u}{o_u}}$$

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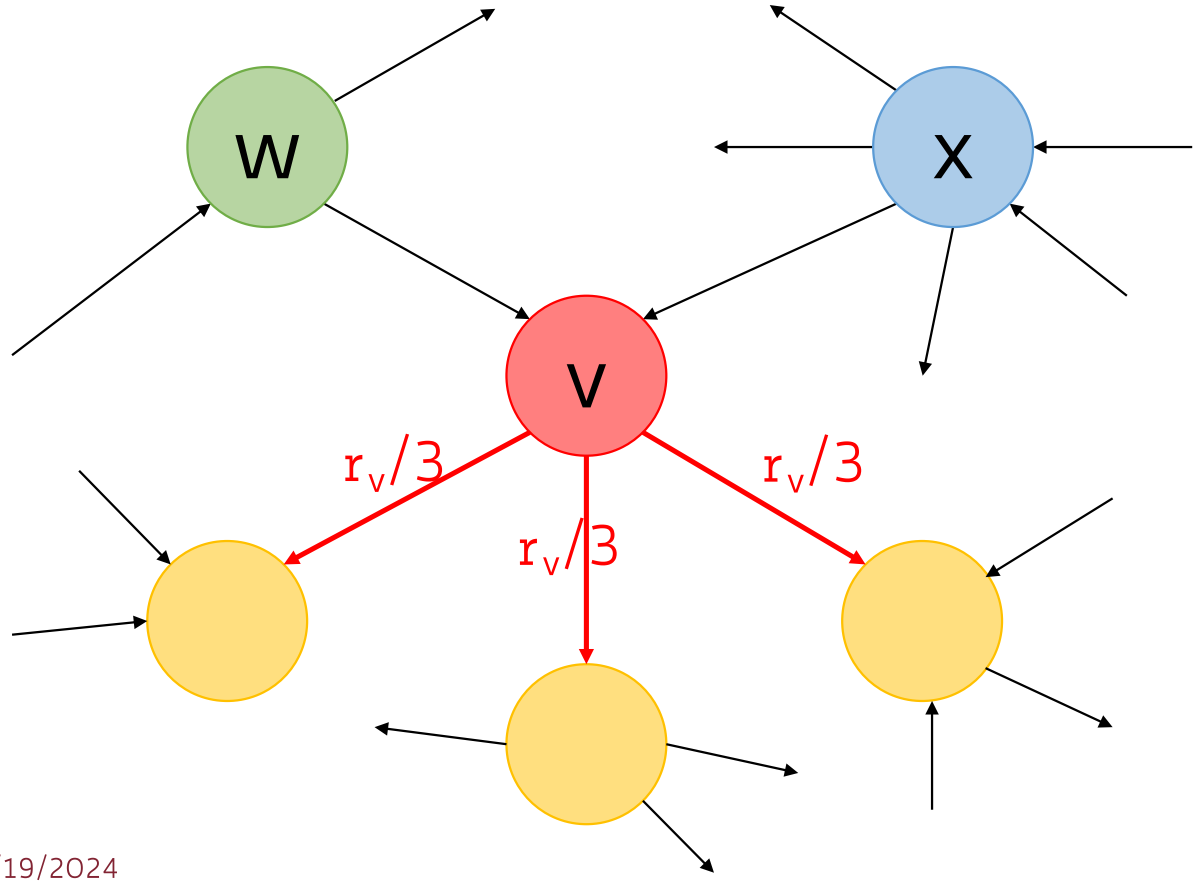

Similarly, page v uniformly distributes its importance r_v to its outgoing links

PageRank's Interpretations

$\boxed{\text{2 main perspectives}}$

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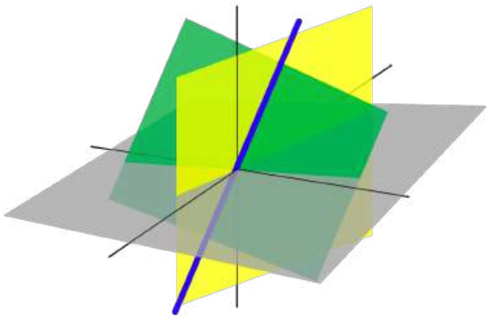
2 main perspectives

Linear Algebra



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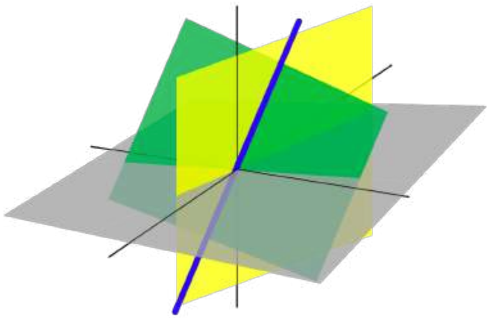

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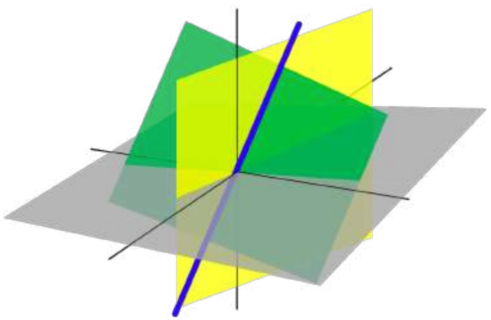
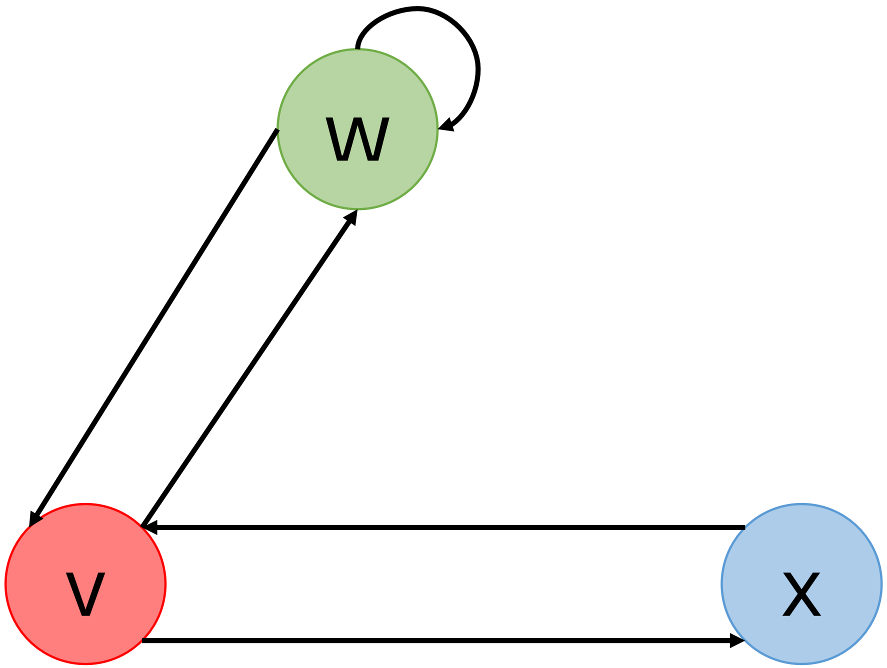


Probabilistic

PageRank: The "Flow" Model



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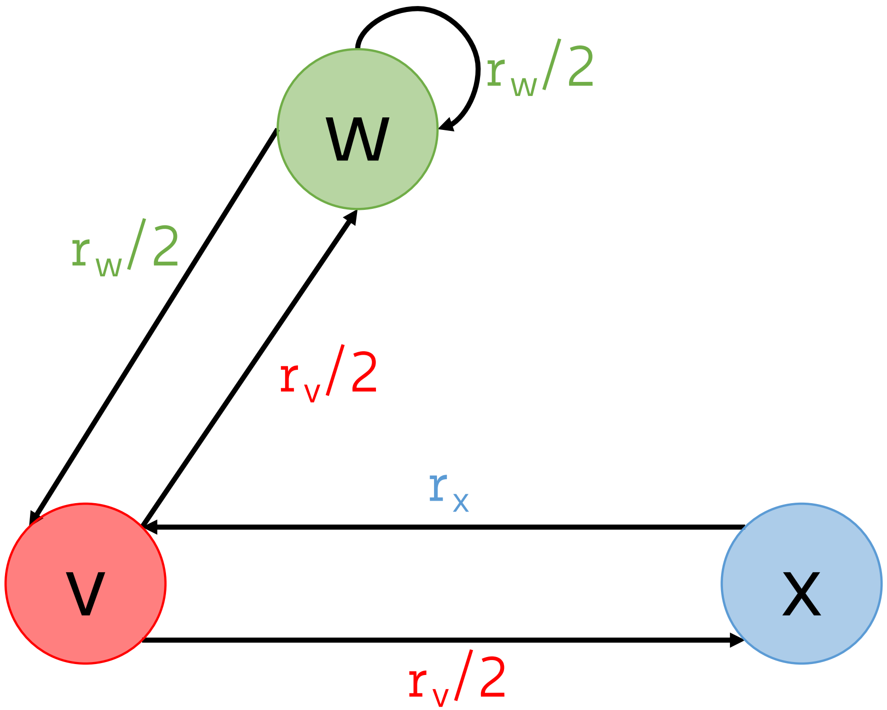

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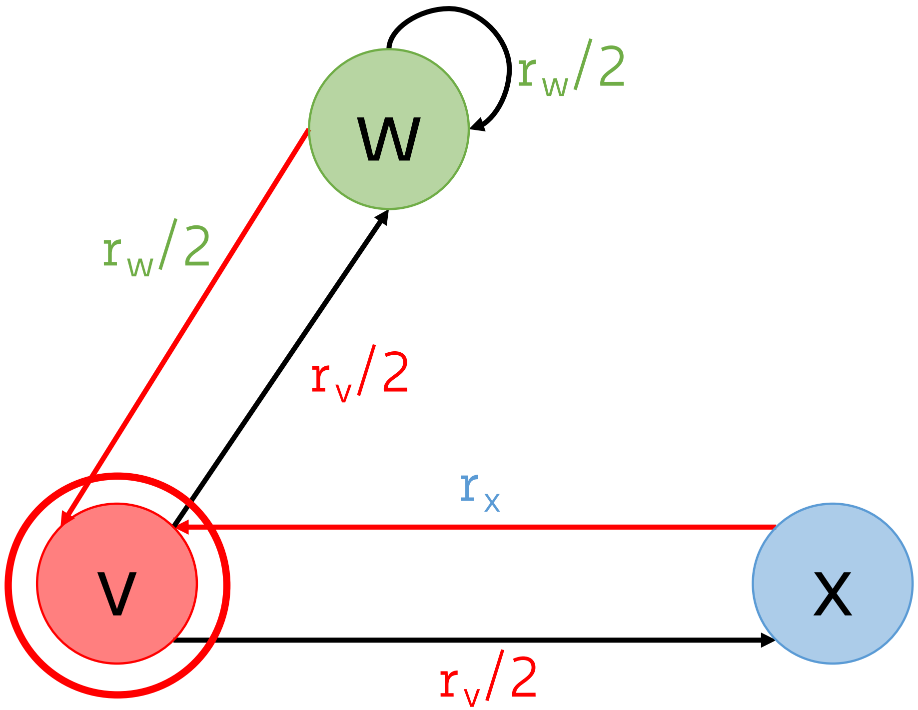

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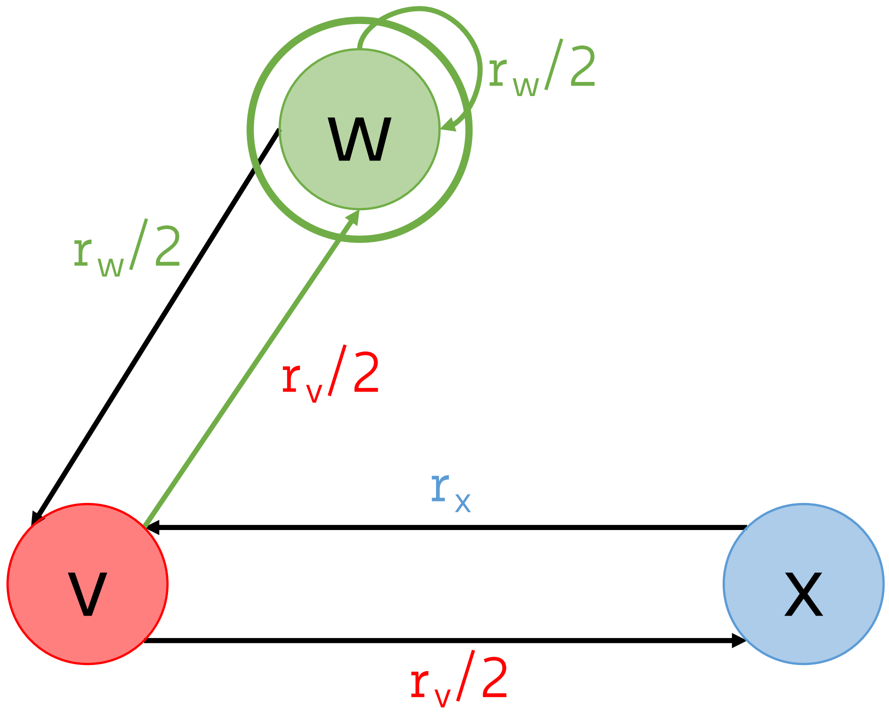

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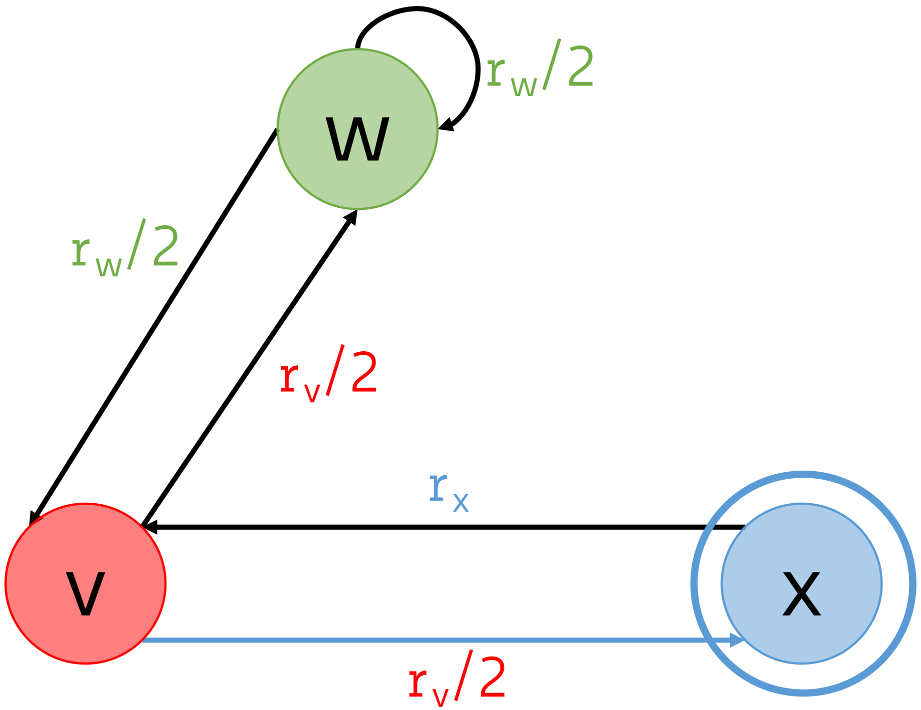

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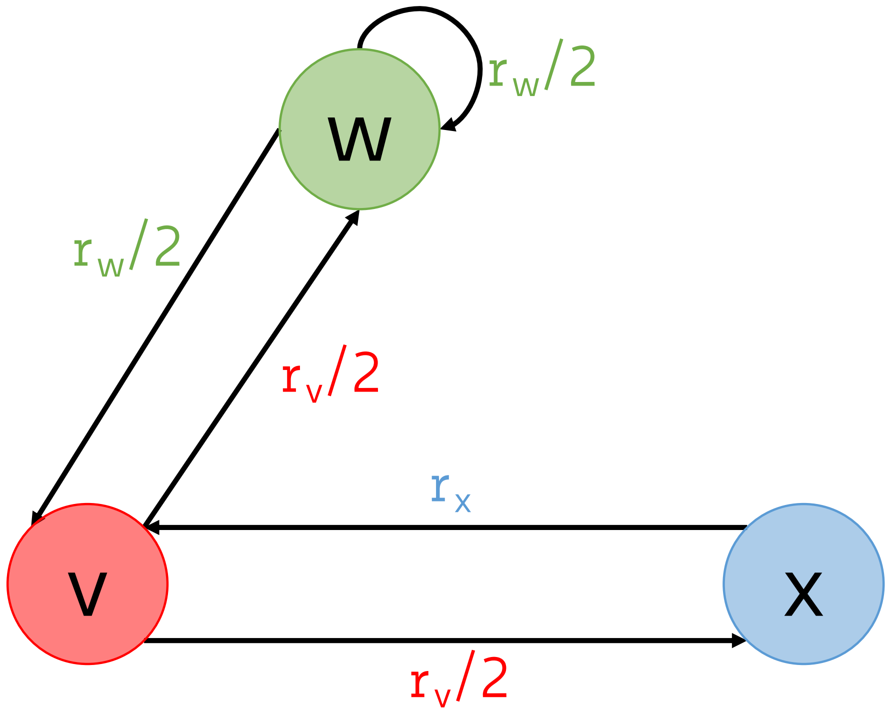

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"Flow" Equations

Solving the System of "Flow" Equations

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3 equations with 3 unknowns: r_v, r_w, and r_x

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⇓

No unique solution!
Infinitely many apart from a constant scale factor

Solving the System of "Flow" Equations

$$\begin{cases} r_v = r_w/2 + r_x \\ r_w = r_v/2 + r_w/2 \\ r_x = r_v/2 \\ \boxed{r_v + r_w + r_x = 1} \end{cases}$$

Additional constraint (equation) enforces the uniqueness of the solution

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This may work for very small systems of linear equations
(e.g., using Gaussian elimination)

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In the case of web pages we might have 100s of billions of equations!

We need a new formulation

PageRank: The Matrix Formulation

Represent the Web graph of documents $G=(V, E)$ s.t. $|V|=N$ as a column stochastic matrix M of size $N \times N$

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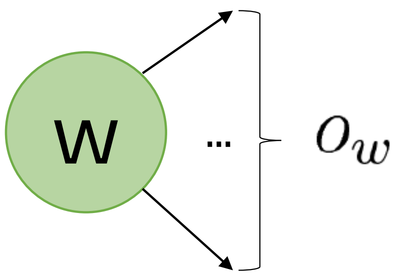

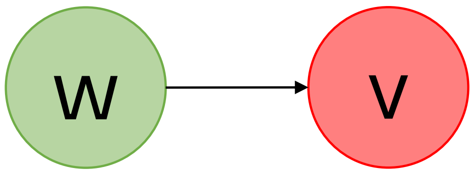

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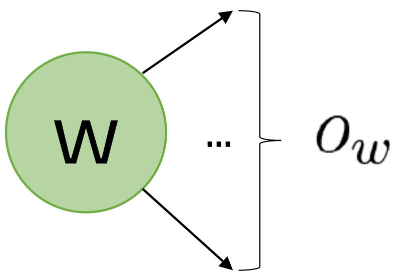

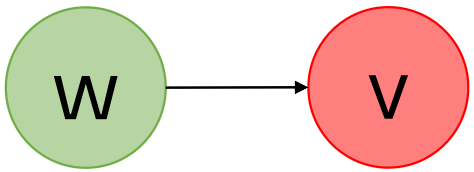

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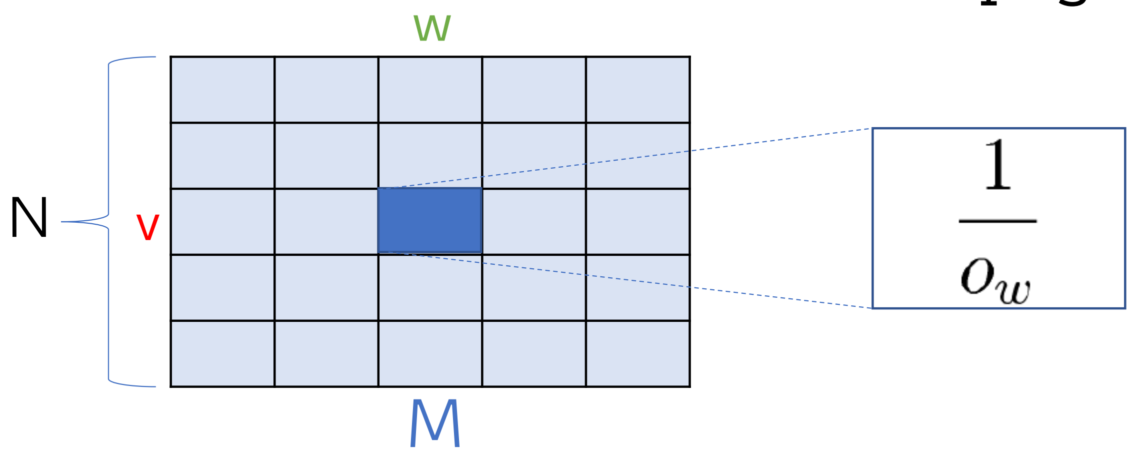

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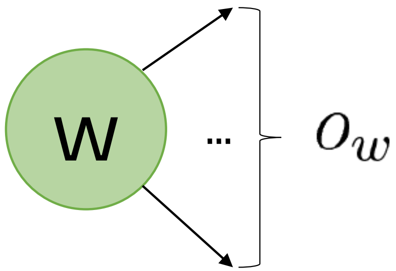
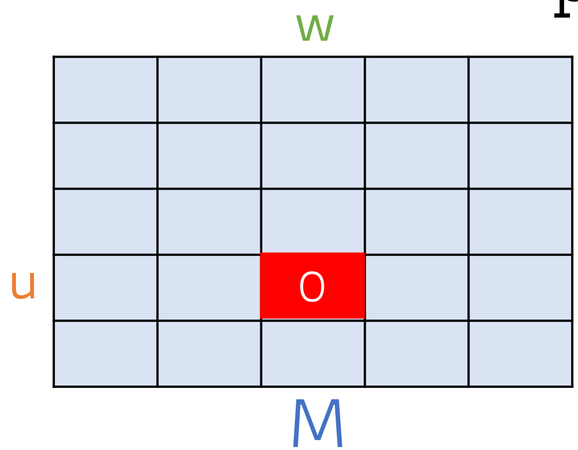


For any other page u which w is not pointing to M[u, w] = 0



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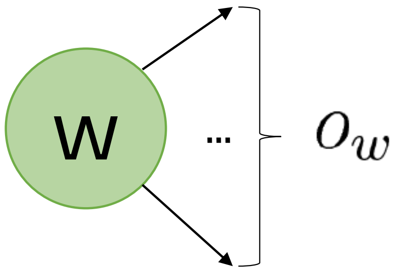

M is column stochastic because, by design, each of its column sums up to 1

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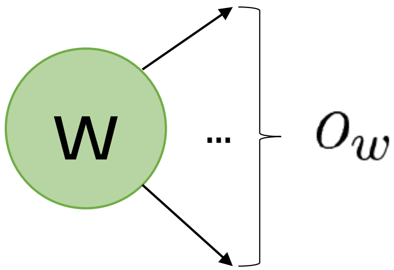

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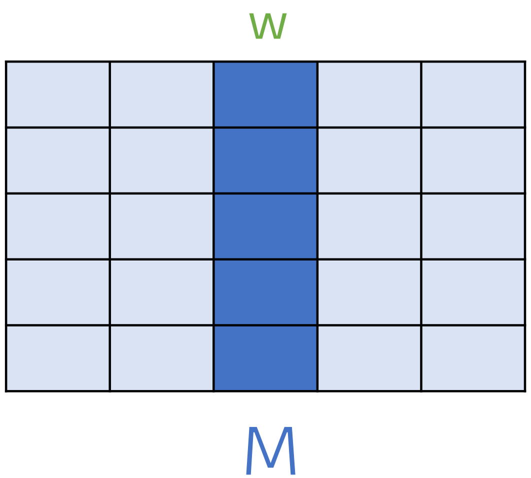

The w-th column will contain o_w <= N non-zero entries, each evaluating to $1/o_w$

$$\sum_{v=1}^{N} m_{v,w} = o_w \times \frac{1}{o_w} = 1$$

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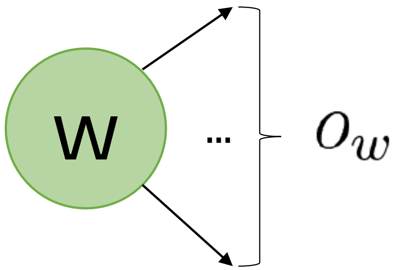

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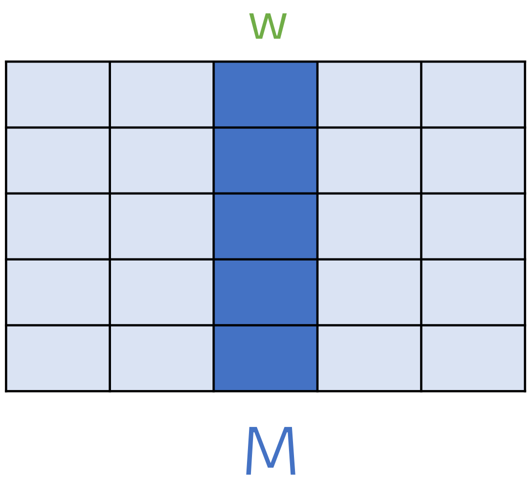

Note:
We are implicitly assuming there exists at least one outgoing link from each node

A Formal View of the Matrix M

$$\mathbf{A}_{N\times N} \quad a_{v,w} = \begin{cases} 1 & \text{if } w \in O_v \\ 0 & \text{otherwise} \end{cases}$$ Traditional adjacency matrix

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Traditional adjacency matrix

$$\mathbf{L}_{N\times N} \quad l_{v,w} = \begin{cases} o_v & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}$$

Diagonal matrix of out-degrees

A Formal View of the Matrix M

$$\mathbf{A}_{N\times N} \quad a_{v,w} = \begin{cases} 1 & \text{if } w \in O_v \\ 0 & \text{otherwise} \end{cases}$$ Traditional adjacency matrix

$$\mathbf{L}_{N\times N} \quad l_{v,w} = \begin{cases} o_v & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}$$ Diagonal matrix of out-degrees

$$\mathbf{M}_{N\times N} \quad m_{v,w} = \begin{cases} \frac{1}{o_w} & \text{if } v \in O_w \\ 0 & \text{otherwise} \end{cases}$$ Column stochastic matrix

$$\boxed{\mathbf{M} = (\mathbf{L}^{-1}\mathbf{A})^T}$$

PageRank: The Matrix Formulation

$\mathbf{r}$ Nx1 rank vector with an entry for each page

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r_v

Rank score of page v

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All the rank scores must sum up to 1

PageRank: The Matrix Formulation

$\mathbf{r}$ Nx1 rank vector with an entry for each page

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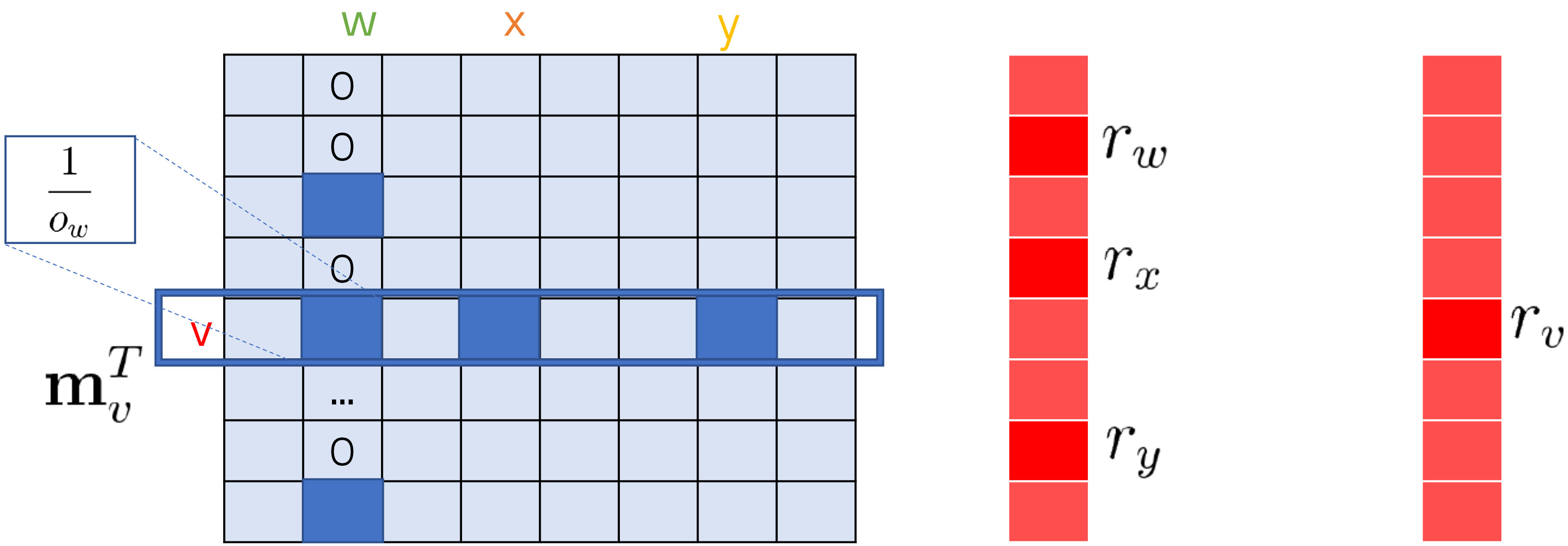
All the rank scores must sum up to 1

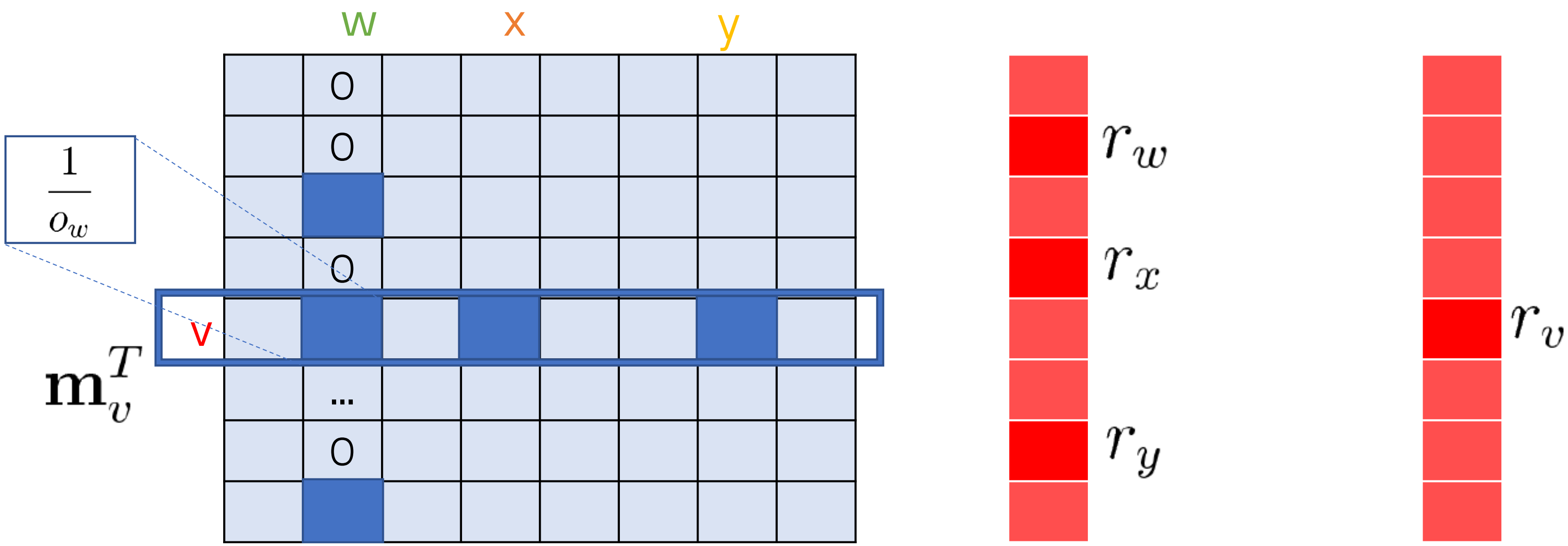
$$r_v = \sum_{w \in I_v} \frac{r_w}{o_w} \quad \Longrightarrow \quad \mathbf{r} = \mathbf{M}\mathbf{r}$$

Flow equations in matrix form

PageRank: The Matrix Formulation

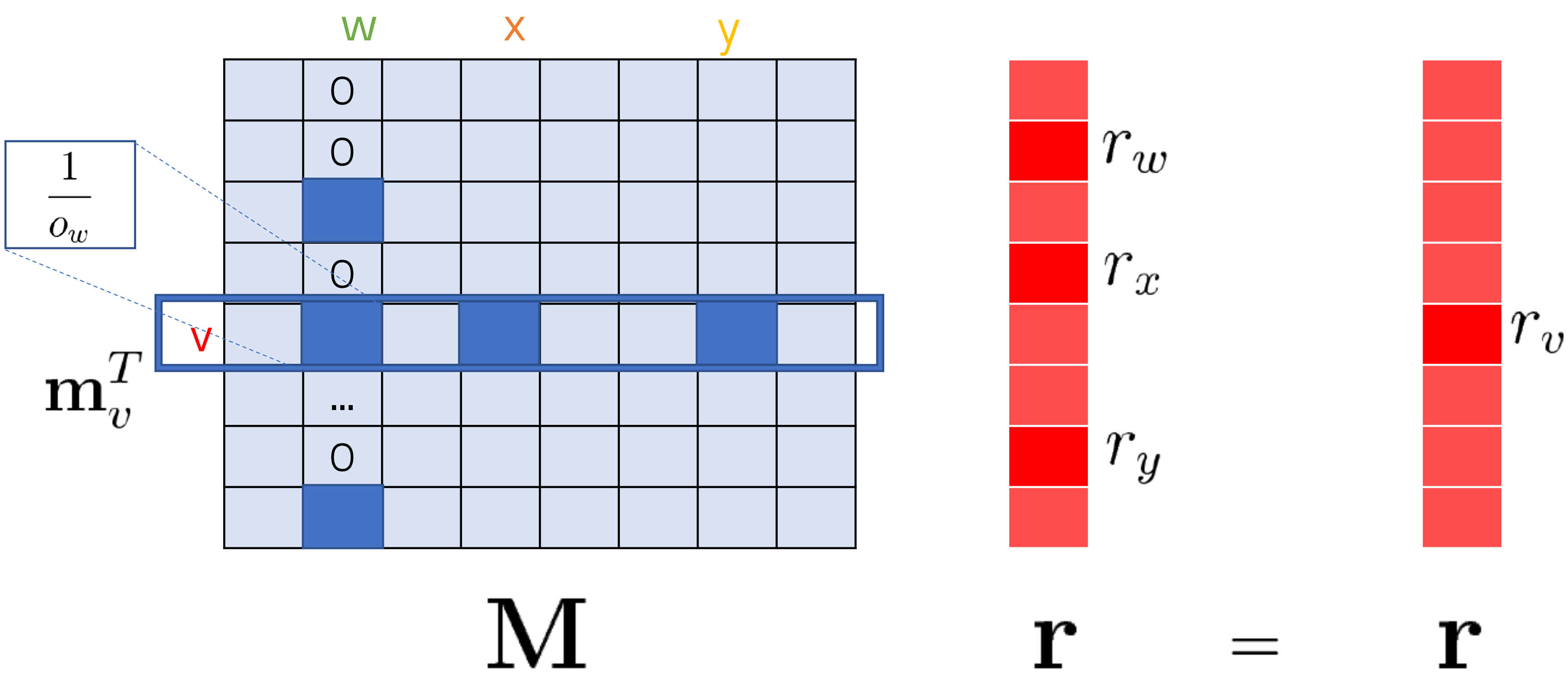


PageRank: The Matrix Formulation



$$r_v = \mathbf{m}_v^T \cdot \mathbf{r} = \sum_{w=1}^{N} m_{v,w} \times r_w = \sum_{w=1}^{N} \frac{1}{o_w} \times r_w = \sum_{w=1}^{N} \frac{r_w}{o_w} = \sum_{w \in I_v} \frac{r_w}{o_w}$$

PageRank: The Matrix Formulation



PageRank: The Eigenvector Formulation

$$\boxed{\mathbf{M}\mathbf{r} = \mathbf{r}}$$

Doesn't it look familiar?

PageRank: The Eigenvector Formulation

$$\boxed{\mathbf{Mr} = \mathbf{r}}$$

Doesn't it look familiar?

$$\mathbf{Ax} = \lambda \mathbf{x}$$

x is an eigenvector

λ is an eigenvalue

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x is an eigenvector

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So, the rank vector r is an eigenvector of the matrix M

In fact, r is the eigenvector corresponding to the eigenvalue λ = 1

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For a fixed eigenvalue, eigenvectors are just scalar multiples of each other

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Since PageRank should reflect only the relative importance of the nodes, choose $r = r^*$ as the eigenvector whose entries sum up to 1

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Since PageRank should reflect only the relative importance of the nodes, choose $r = r^*$ as the eigenvector whose entries sum up to 1

This may be referred to as the probabilistic eigenvector corresponding to the eigenvalue $\lambda = 1$

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We know from linear algebra theory that for any stochastic matrix M its largest eigenvalue is λ = 1

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PageRank: The Eigenvector Formulation

$$\boxed{\mathbf{Mr} = \mathbf{r}}$$

We know from linear algebra theory that for any stochastic matrix M its largest eigenvalue is $\lambda = 1$

Therefore, $r = r^*$ is the principal eigenvector of M (i.e., the eigenvector associated with the largetst eigenvalue)

Note:

So far, we have assumed that M is (column) stochastic yet this may not be the case for the general Web graph...

PageRank: Quick Recap

We start from "flow" equations

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We reformulate the system of linear equations using linear algebra
(i.e., stochastic matrix M and rank vector r)

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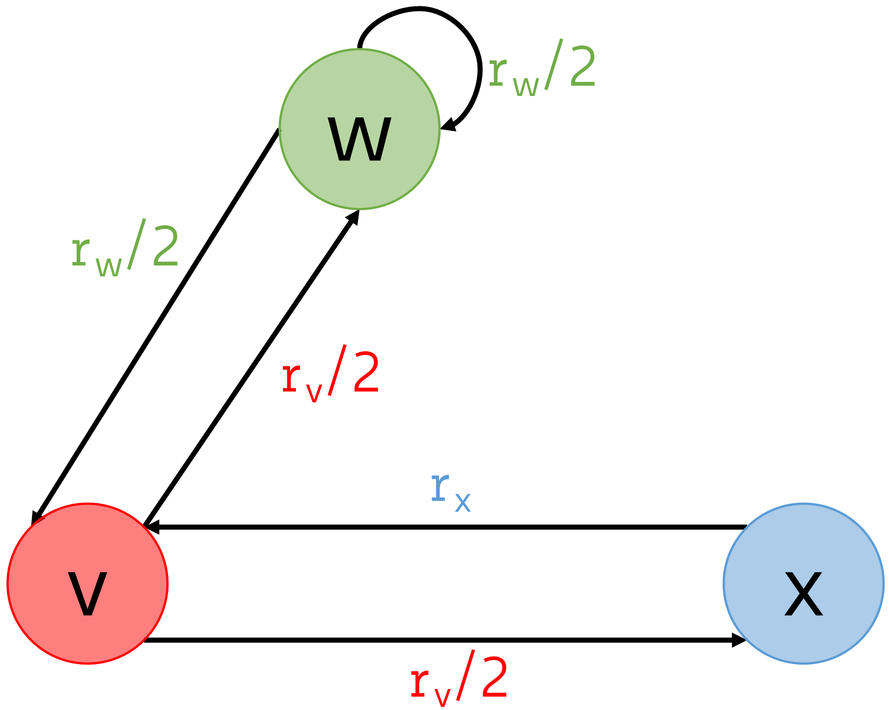
⇓

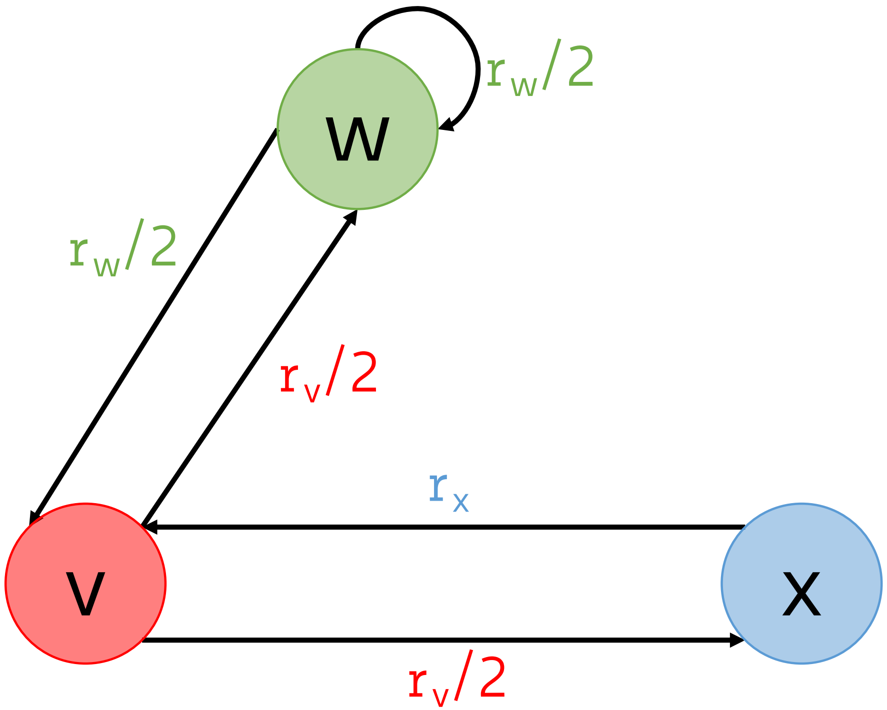
We know how to solve this efficiently using power iteration method

PageRank: The "Flow" Model



$$\begin{cases} r_v = r_w/2 + r_x \\ r_w = r_v/2 + r_w/2 \\ r_x = r_v/2 \end{cases}$$

PageRank: The "Flow" Model



$$\begin{cases} r_v = r_w/2 + r_x \\ r_w = r_v/2 + r_w/2 \\ r_x = r_v/2 \end{cases}$$

$$\mathbf{r} = \mathbf{M}\,\mathbf{r}, \qquad \mathbf{M} = \begin{bmatrix} 0 & 1/2 & 1 \\ 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \end{bmatrix}$$

PageRank: Power Iteration Method

At the beginning, we assume all pages have the same rank score, uniformly distributed across the N pages

$$\textbf{init:}\ \ t = 0; \mathbf{r}(t) = (1/N, 1/N, \ldots, 1/N)^T$$

PageRank: Power Iteration Method

Keep updating the rank vector r until convergence

init: $t = 0; \mathbf{r}(t) = (1/N, 1/N, \ldots, 1/N)^T$

repeat:

$$\mathbf{r}(t+1) = \mathbf{M}\mathbf{r}(t)$$

until $\delta(\mathbf{r}(t+1), \mathbf{r}(t)) < \epsilon$

$\epsilon > 0$

PageRank: Power Iteration Method

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repeat:

$$\mathbf{r}(t+1) = \mathbf{M}\mathbf{r}(t)$$

$$\textbf{until}\ \delta(\mathbf{r}(t+1), \mathbf{r}(t)) < \epsilon$$

$$\epsilon > 0$$

$$\delta(\mathbf{r}(t+1), \mathbf{r}(t)) = |\mathbf{r}(t+1) - \mathbf{r}(t)|$$

or

$$\delta(\mathbf{r}(t+1), \mathbf{r}(t)) = ||\mathbf{r}(t+1) - \mathbf{r}(t)||$$

Power Iteration Method: Example

$$
\begin{cases}
r_v = r_w/2 + r_x \\
r_w = r_v/2 + r_w/2 \\
r_x = r_v/2
\end{cases}
$$

$$
\mathbf{r} = \underbrace{\begin{bmatrix} 0 & 1/2 & 1 \\ 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \end{bmatrix}}_{\mathbf{M}} \mathbf{r}
$$

Power Iteration Method: Example

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$$\mathbf{r} = \mathbf{M}\,\mathbf{r}$$

0	1/2	1
1/2	1/2	0
1/2	0	0

1/3
1/3
1/3

$\mathbf{r}(0)$

Power Iteration Method: Example

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0	1/2	1
1/2	1/2	0
1/2	0	0

$$\mathbf{r}(0) = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} \quad \mathbf{r}(1) = \mathbf{M}\mathbf{r}(0) = \begin{bmatrix} 3/6 \\ 1/3 \\ 1/6 \end{bmatrix}$$

Power Iteration Method: Example

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$$\underset{\mathbf{r}}{\begin{bmatrix} \ \\ \ \\ \ \end{bmatrix}} = \underset{\mathbf{M}}{\begin{bmatrix} 0 & 1/2 & 1 \\ 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \end{bmatrix}} \underset{\mathbf{r}}{\begin{bmatrix} \ \\ \ \\ \ \end{bmatrix}}$$

$$\underset{\mathbf{r}(0)}{\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}} \quad \underset{\mathbf{r}(1) = \mathbf{Mr}(0)}{\begin{bmatrix} 3/6 \\ 1/3 \\ 1/6 \end{bmatrix}} \quad \underset{\mathbf{r}(2) = \mathbf{Mr}(1)}{\begin{bmatrix} 1/3 \\ 5/12 \\ 3/12 \end{bmatrix}}$$

Power Iteration Method: Example

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$$\underset{\mathbf{r}(0)}{\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}} \quad \underset{\mathbf{r}(1)=\mathbf{M}\mathbf{r}(0)}{\begin{bmatrix} 3/6 \\ 1/3 \\ 1/6 \end{bmatrix}} \quad \underset{\mathbf{r}(2)=\mathbf{M}\mathbf{r}(1)}{\begin{bmatrix} 1/3 \\ 5/12 \\ 3/12 \end{bmatrix}} \quad \ldots \quad \underset{\mathbf{r}(t+1)=\mathbf{M}\mathbf{r}(t)}{\begin{bmatrix} 6/15 \\ 6/15 \\ 3/15 \end{bmatrix}} \quad \begin{matrix} 2/5 \\ 2/5 \\ 1/5 \end{matrix}$$

Power Iteration Method: Example

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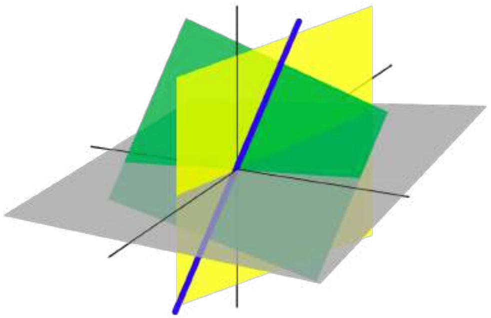
We came up with the same set of solutions for r_v, r_w, and r_x without explicitly solving the system of equations

PageRank's Interpretations

2 main perspectives

Linear Algebra



Probabilistic

Random Walk Interpretation of Page Rank

Imagine a random surfer navigating through the pages of the Web graph

Random Walk Interpretation of Page Rank

Initially, at time t=0 the surfer can be on any web page

www.donuts.com

www.krustyburger.com

www.duffbeer.com

...

www.moes.com

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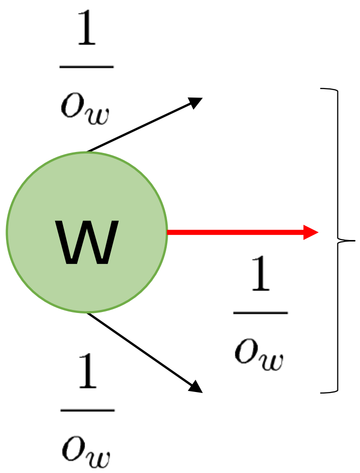
Each web page has equal probability 1/N to be chosen as starting point

Random Walk Interpretation of Page Rank

At any given time t, the surfer is on some web page w

Random Walk Interpretation of Page Rank

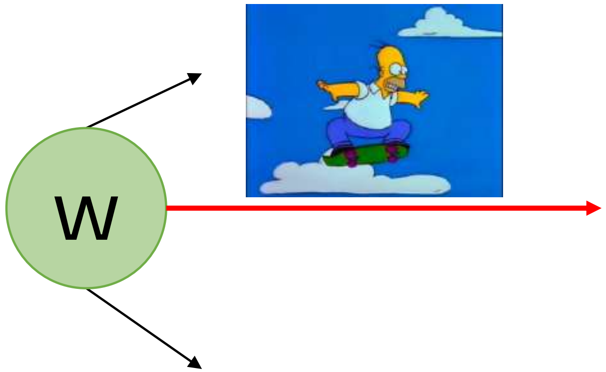
At time t+1, the surfer follows one of the outgoing links from web page w, chosen uniformly at random



o_w out-degree of node w

Random Walk Interpretation of Page Rank

The surfer ends up into some other web page v pointed by w



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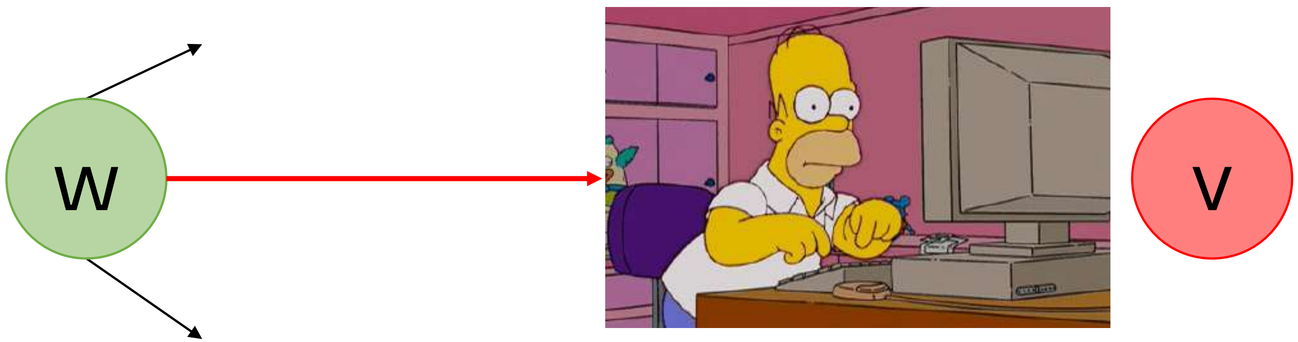

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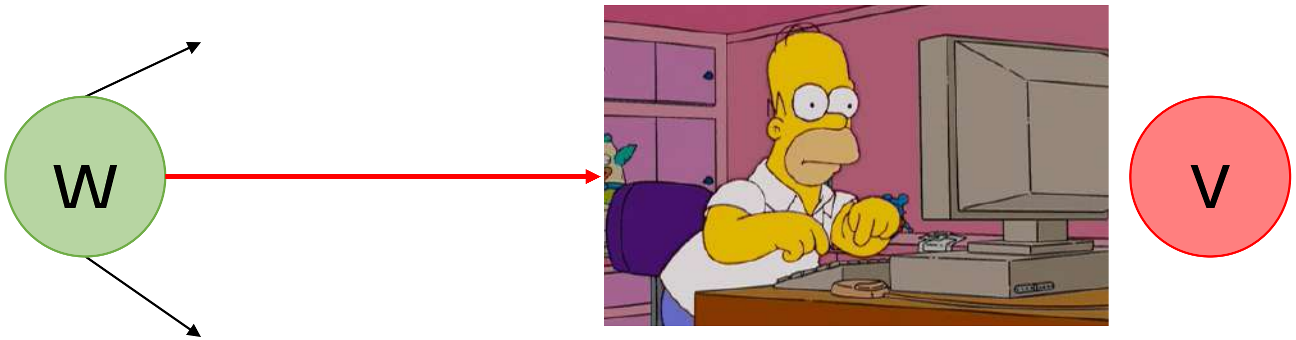

This process repeats indefinitely and is known as random walk

Transition Matrix M

$$\mathbf{M}_{N\times N} \quad m_{v,w} = \begin{cases} \frac{1}{o_w} & \text{if } v \in O_w \\ 0 & \text{otherwise} \end{cases}$$ Column stochastic matrix

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The v-th, w-th entry of M indicates the probability of a random surfer moving from page w to page v

Such a matrix describes a Markov chain over the finite state space V of nodes (i.e., pages) of the Web graph

Random Walk Interpretation of Page Rank

X Discrete-Valued Random Variable taking on |V| = N possible values

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N-dimensional stochastic (i.e., probability) vector associated with X

$$\mathbf{p} \subseteq \mathbb{R}^N = (P(X=1), \ldots, P(X=w), \ldots, P(X=N))^T$$

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Probability distribution over web pages at time t

Random Walks as Markov Chains

Random Walks are also known as stochastic processes with Markov property (i.e., Markov chains)

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The transition probability of moving to the next state depends only on the present state and not on the previous states

$$P(X_{t+1} = v | X_1 = x_1, X_2 = x_2, \ldots, X_t = x_t) = P(X_{t+1} = v | X_t = x_t)$$

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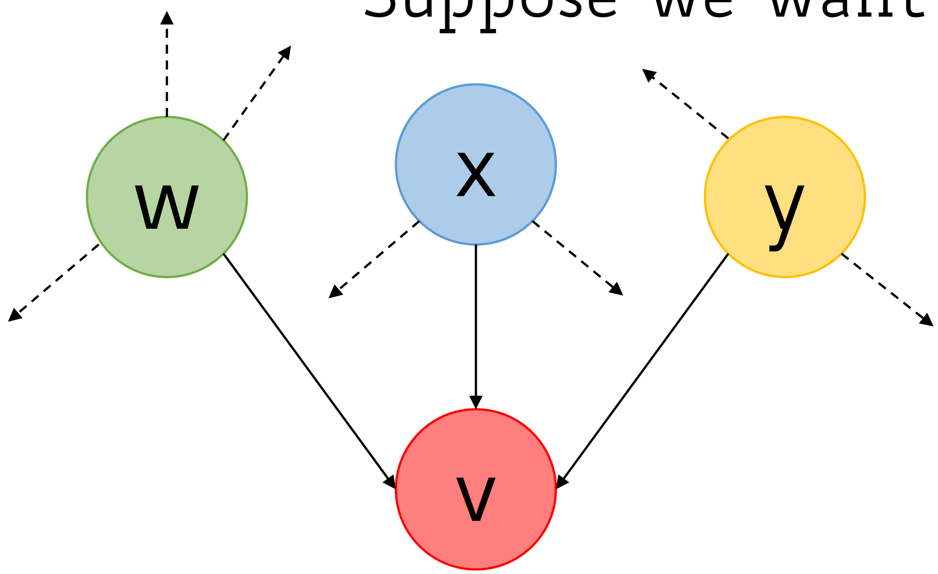
$$P(X_{t+1} = v | X_1 = x_1, X_2 = x_2, \ldots, X_t = x_t) = P(X_{t+1} = v | X_t = x_t)$$

The probability that the random surfer will be on page v at time t+1 depends only on where the surfer was at time t

Random Walk Interpretation of Page Rank

Where is the random surfer at time t+1 knowing where he was at time t?

Suppose we want to estimate $P(X_{t+1} = v)$



Assume v has only 3 incoming links from w, x, and y

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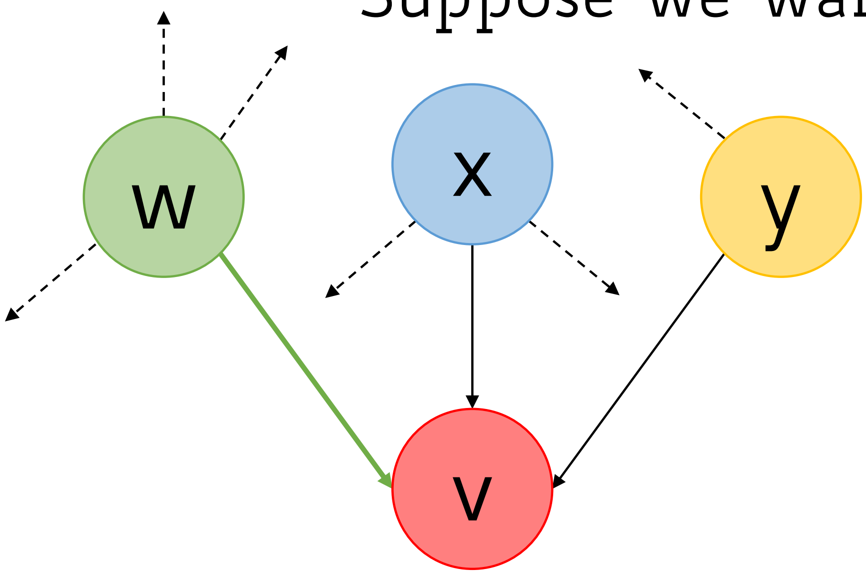

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$$P(X_{t+1} = v) = \boxed{P(X_t = w, Z_w = v) +}$$

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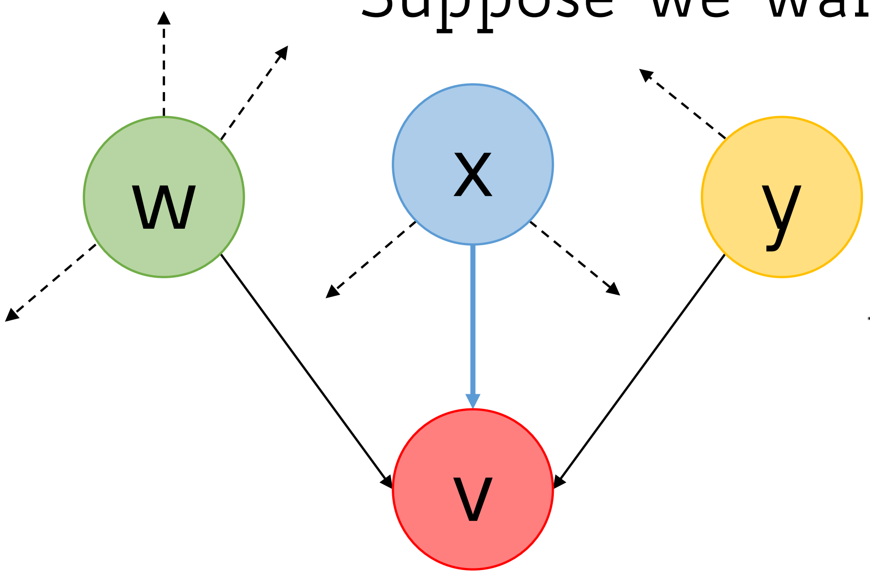

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$$\boxed{P(X_t = x, Z_x = v) +}$$

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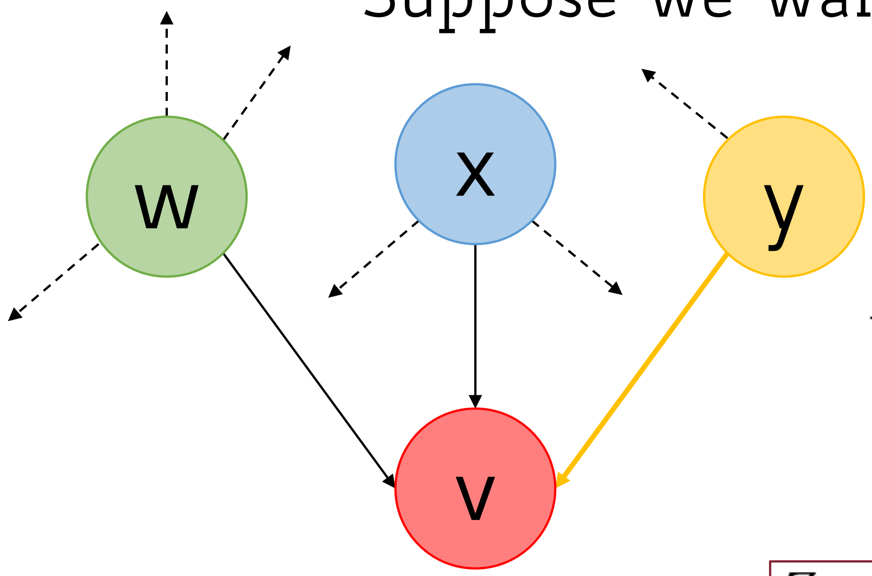
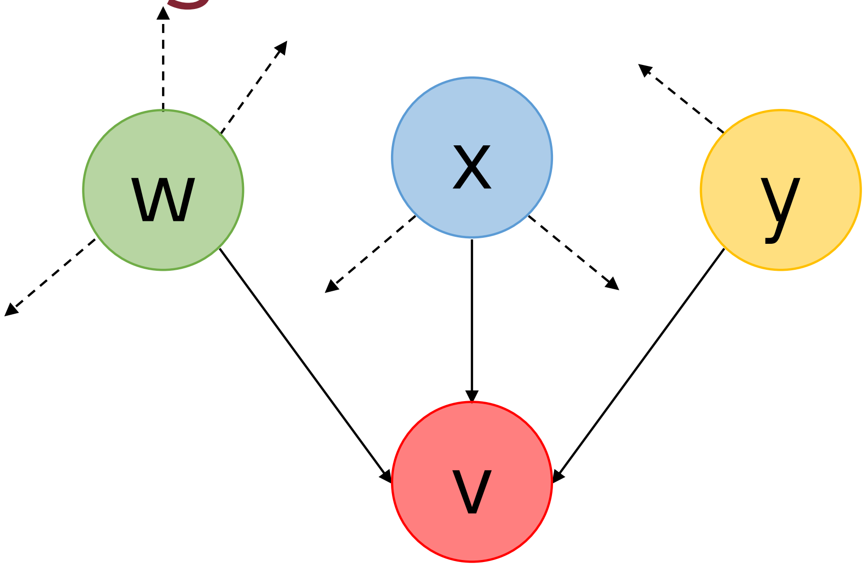


Assume v has only 3 incoming links from w, x, and y

$$P(X_{t+1} = v) = P(X_t = w, Z_w = v) +$$
$$P(X_t = x, Z_x = v) +$$
$$\boxed{P(X_t = y, Z_y = v)}$$

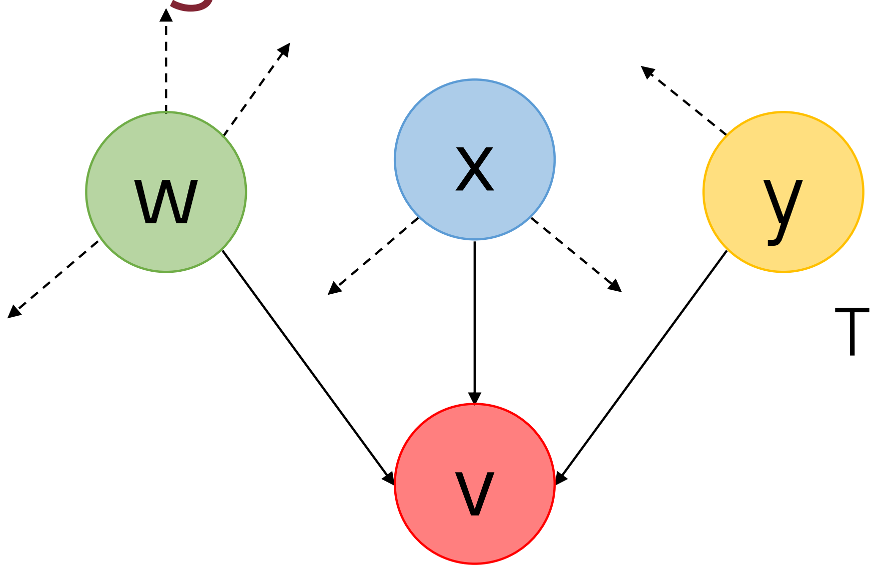
$$\boxed{Z_u \sim \text{Uniform}(1, o_u)}$$

Random Walk Interpretation of Page Rank



$$\boxed{\mathbf{p}(t+1) = \mathbf{M}\mathbf{p}(t)}$$

Random Walk Interpretation of Page Rank

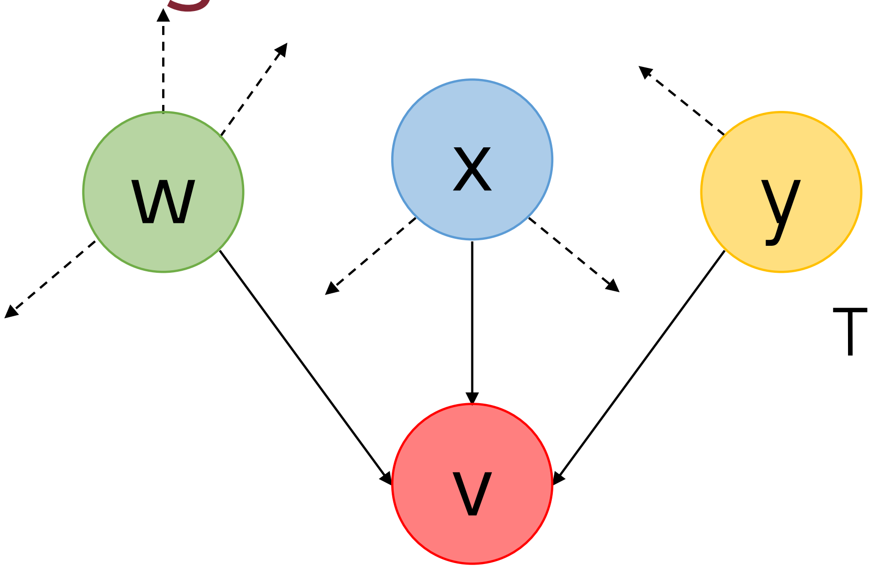


$$\boxed{\mathbf{p}(t+1) = \mathbf{M}\mathbf{p}(t)}$$

This resembles our PageRank equation

$$\boxed{\mathbf{r}(t+1) = \mathbf{M}\mathbf{r}(t)}$$

Random Walk Interpretation of Page Rank



$$\boxed{\mathbf{p}(t+1) = \mathbf{M}\mathbf{p}(t)}$$

This resembles our PageRank equation

$$\boxed{\mathbf{r}(t+1) = \mathbf{M}\mathbf{r}(t)}$$

Solving the former is equivalent to solving the latter!

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The probability that page v will be visited after one step corresponds to the v-th entry of p(1), obtained as:

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The probability that page v will be visited after one step corresponds to the v-th entry of p(1), obtained as:

$$\mathbf{p}(1) = \mathbf{M}\mathbf{p}(0)$$

More generally, the probability of visiting any web page after t steps is:

$$\boxed{\mathbf{p}(t) = \mathbf{M}^t\mathbf{p}(0)}$$

Random Walk Interpretation of Page Rank

$$\mathbf{p}(0) = (\underbrace{1/N}_{P(X_0=1)}, \ldots, \underbrace{1/N}_{P(X_0=w)}, \ldots, \underbrace{1/N}_{P(X_0=N)})^T$$

Random Walk Interpretation of Page Rank

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Random Walk Interpretation of Page Rank

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$$\mathbf{p}(1) = \mathbf{M}\mathbf{p}(0)$$

$$\mathbf{p}(2) = \mathbf{M}\mathbf{p}(1) = \underbrace{\mathbf{M} \times \mathbf{M}}_{\mathbf{M}^2}\mathbf{p}(0)$$

Random Walk Interpretation of Page Rank

$$\mathbf{p}(0) = (\underbrace{1/N}_{P(X_0=1)}, \ldots, \underbrace{1/N}_{P(X_0=w)}, \ldots, \underbrace{1/N}_{P(X_0=N)})^T$$

$$\mathbf{p}(1) = \mathbf{M}\mathbf{p}(0)$$

$$\mathbf{p}(2) = \mathbf{M}\mathbf{p}(1) = \underbrace{\mathbf{M} \times \mathbf{M}}_{\mathbf{M}^2}\mathbf{p}(0)$$

$$\vdots$$

$$\mathbf{p}(k) = \mathbf{M}\mathbf{p}(k-1) = \underbrace{\mathbf{M} \times \mathbf{M} \times \ldots \times \mathbf{M}}_{\mathbf{M}^k}\mathbf{p}(0)$$

$$\vdots$$

Random Walk Interpretation of Page Rank

$$\{\mathbf{p}(t)\}_{t=0,1,\ldots,T}$$

Discrete Stochastic Process

Markov chain

$$\mathbf{p}(0) = (\underbrace{1/N}_{P(X_0=1)}, \ldots, \underbrace{1/N}_{P(X_0=w)}, \ldots, \underbrace{1/N}_{P(X_0=N)})^T$$

$$\mathbf{p}(1) = \mathbf{M}\mathbf{p}(0)$$

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p* is the stationary distribution of the random walk

Equivalence between Formulations

Linear Algebra

Probabilistic

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System of linear "flow" equations

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So the PageRank vector r^* corresponds to the stationary distribution p^* for the random walk on the graph encoded by M!

Equivalence between Formulations

So the PageRank vector r^* corresponds to the stationary distribution p^* for the random walk on the graph encoded by M!

Intuitively, the PageRank vector indicates for each web page the probability that a random surfer will eventually get to that page

Hang on a Second...

Linear Algebra

Probabilistic

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How do we know that the power iteration method always converge to r^*?

existence

How do we know that r^* is unique?

uniqueness

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existence and uniqueness of r^* (p^*) are guaranteed under certain conditions on the matrix M

Existence and Uniqueness of PageRank

If M is a column stochastic matrix with all positive entries:
- $\lambda = 1$ is an eigenvalue of M with multiplicity one
- $\lambda = 1$ is the largest eigenvalue of M
- There exists a unique (right) eigenvector r^* associated with the eigenvalue $\lambda = 1$ with the sum of its entries equal to 1

Perron-Frobenius theorem (circa 1910)

Existence and Uniqueness of PageRank

If M is a column stochastic matrix with all positive entries, then M has a unique steady-state vector p^* such that for any p(0)

$\mathbf{p}(t) = \mathbf{M}^t\mathbf{p}(0)$ converges to $\mathbf{p}^*$ as $t \longrightarrow \infty$

Perron-Frobenius theorem (circa 1910)

Existence and Uniqueness of PageRank

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The steady-state vector is the unique eigenvector associated with the largest eigenvalue $\lambda = 1$

Are We Done, Then?

Problem: We cannot apply the Perron-Frobenius theorem to the matrix M as we originally defined it

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$$\mathbf{M}_1 = \begin{bmatrix} 0.6 & 0.5 & 0 \\ 0.4 & 0.3 & 1 \\ 0 & 0.2 & 0 \end{bmatrix} \qquad \mathbf{M}_2 = \begin{bmatrix} 0.6 & 0.5 & 0.1 \\ 0.2 & 0.3 & 0.4 \\ 0.2 & 0.2 & 0.5 \end{bmatrix}$$

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Both M_1 and M_2 are column stochastic, but only M_2 is positive

So? Should We Give Up?

Here is where Brin and Page, in fact Google, comes in!

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We show how they fixed the issues with the original definition of M to accommodate for the heterogeneity of the Web graph

By doing so, we know that a solution to our PageRank problem exists and is unique!

Google's PageRank

Problems with Original PageRank Formulation

We cannot directly apply the Perron-Frobenius theorem to the original Web graph matrix M

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Then we discuss how Brin and Page fixed this in their seminal paper which sets up the rising of Google

Problems with Original PageRank Formulation

2 main issues to solve:

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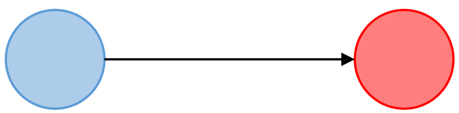
Dead End

Pages with no outlinks cause PageRank to leak out



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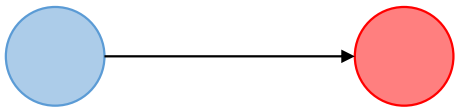
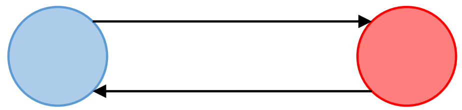


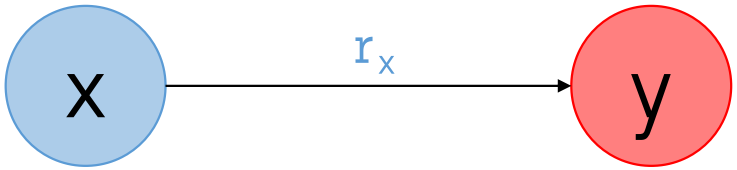
Spider Trap

Not every node is reachable and PageRank gets eventually absorbed by a few pages



The "Dead End" Problem (Dangling Nodes)

Example:



$$r_y = r_x$$

M

0	0
1	0

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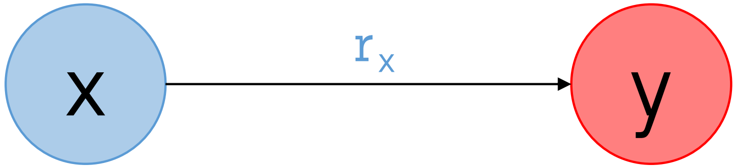

$$r_y = r_x$$

$$\mathbf{M}$$

0	0
1	0

When a web page has no outgoing links (dangling node) the resulting column vector in the matrix M is not stochastic anymore!

Previously, we assumed each web page has at least one outgoing link, and therefore M was stochastic

The "Dead End" Problem (Dangling Nodes)

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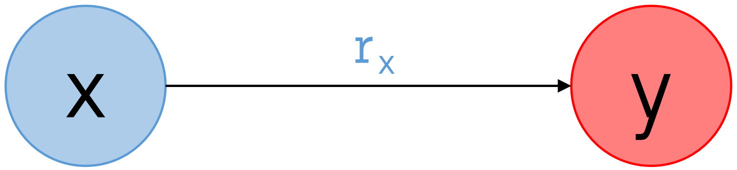

$$r_y = r_x$$

M

0	0
1	0

Assume the following initialization for r:

$$\mathbf{r}(0) = \begin{bmatrix} r_x^{(0)} \\ r_y^{(0)} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

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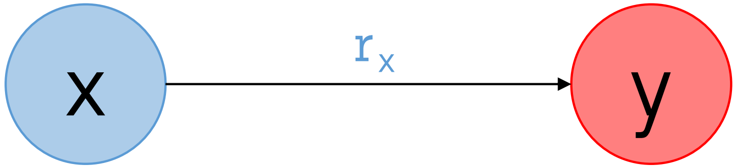

$$r_y = r_x$$

$$\mathbf{M} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\mathbf{r}(1) = \mathbf{M} \quad \mathbf{r}(0)$$

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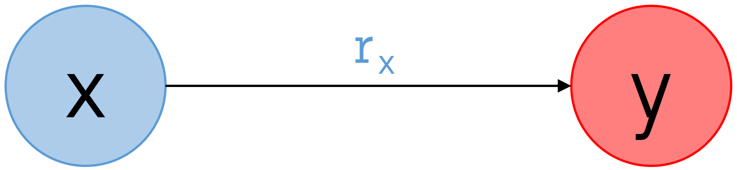

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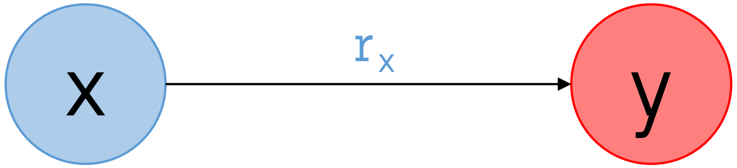


$$r_y = r_x$$

$\mathbf{M}$

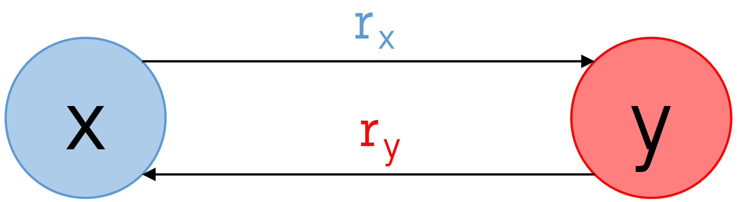
0	0
1	0

$$\mathbf{r}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{r}(1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \mathbf{r}(2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \dots \quad \mathbf{r}(t-1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \mathbf{r}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The PageRank vector vanishes to 0!

The "Spider Trap" Problem

Example:



$$\begin{cases} r_x = r_y \\ r_y = r_x \end{cases}$$

$$\mathbf{M}$$

0	1
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The "Spider Trap" Problem

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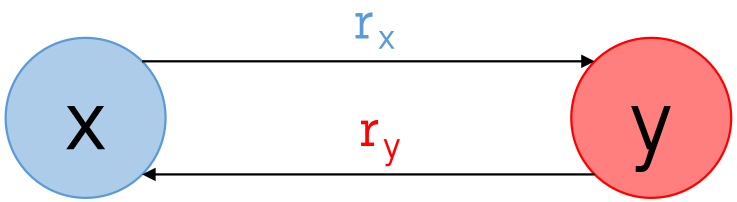

$$\begin{cases} r_x = r_y \\ r_y = r_x \end{cases}$$

$$\mathbf{M} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

M is column stochastic non-negative (but not strictly positive)
Does PageRank converge regardless of the initialization of r?

The "Spider Trap" Problem

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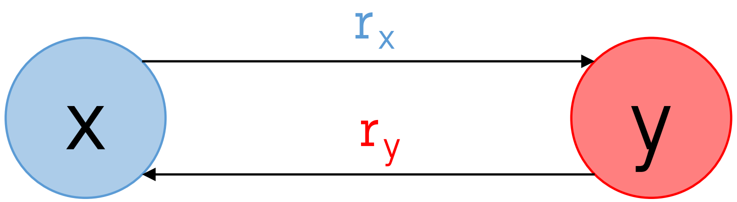

$$\begin{cases} r_x = r_y \\ r_y = r_x \end{cases}$$

$\mathbf{M}$

0	1
1	0

Assume the same initialization as before for r:

$$\mathbf{r}(0) = \begin{bmatrix} r_x^{(0)} \\ r_y^{(0)} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

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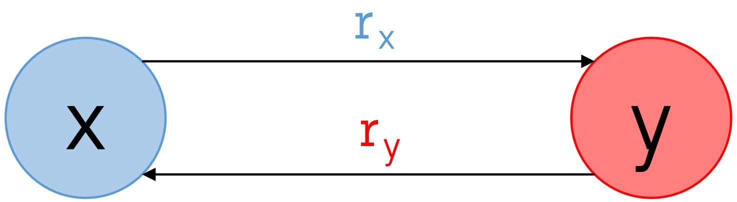

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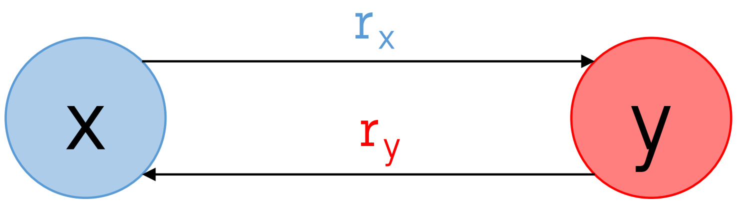

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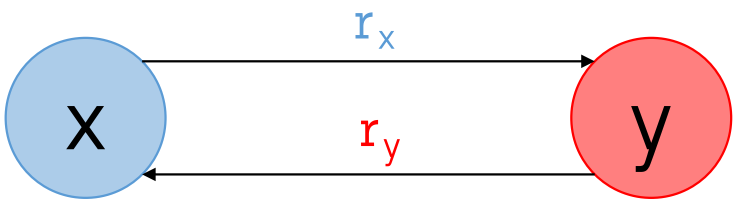

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The PageRank vector keeps alternating its components and never converges!

Problems with Original PageRank Formulation

2 main issues to solve:

Dead End	Spider Trap
Pages with no outlinks cause PageRank to leak out	Not every node is reachable and PageRank gets eventually absorbed by a few pages

Problems with Original PageRank Formulation

2 main issues to solve:

Dead End

Pages with no outlinks cause PageRank to leak out

M is not column stochastic as some nodes have no outlinks

Spider Trap

Not every node is reachable and PageRank gets eventually absorbed by a few pages

Problems with Original PageRank Formulation

2 main issues to solve:

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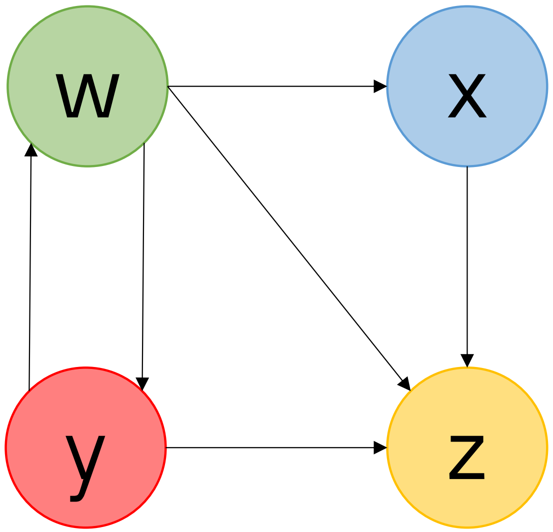
Pages with no outlinks cause PageRank to leak out

Spider Trap

Not every node is reachable and PageRank gets eventually absorbed by a few pages

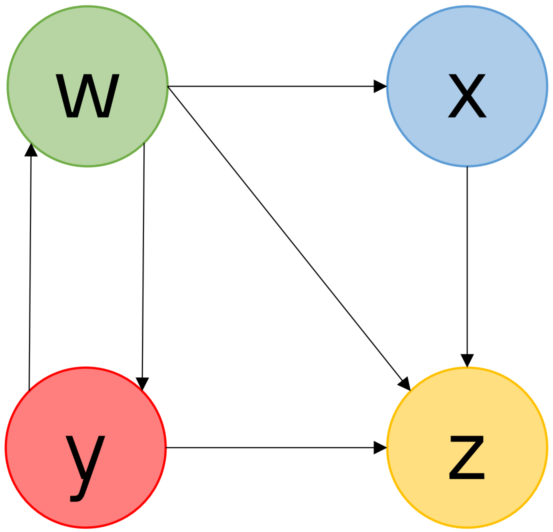
M is stochastic but not strictly positive

Deal with Dangling Nodes



$$
\begin{array}{cc}
 & \begin{array}{cccc} w & x & y & z \end{array} \\
\mathbf{M} = \begin{array}{c} w \\ x \\ y \\ z \end{array} & \begin{bmatrix} 0 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1 & 1/2 & 0 \end{bmatrix}
\end{array}
$$

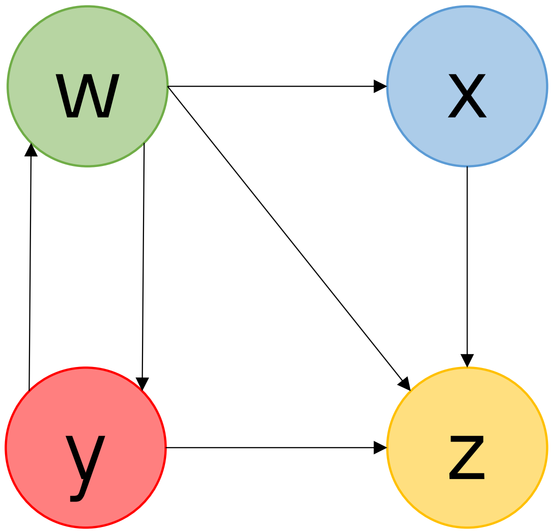
Deal with Dangling Nodes



$$\mathbf{M} = \begin{array}{c} \\ w \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{bmatrix} 0 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1 & 1/2 & 0 \end{bmatrix} \end{array}$$

z is a dangling node

Deal with Dangling Nodes



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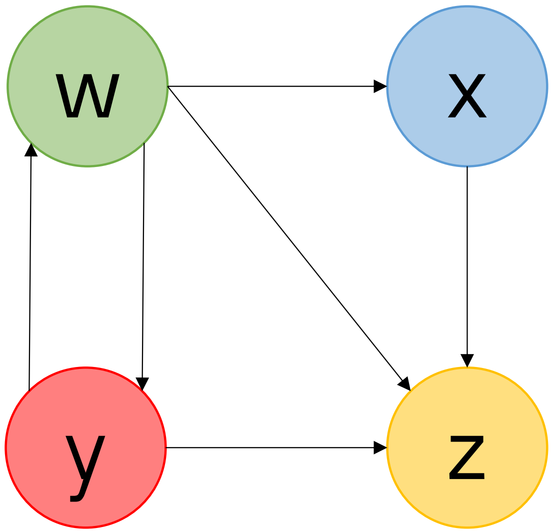
M is not (column) stochastic

Deal with Dangling Nodes



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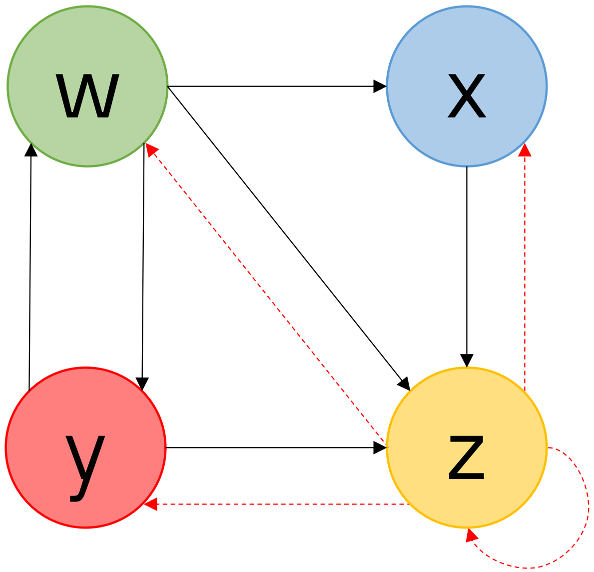
If we apply simplified PageRank to M the rank vector r will eventually vanish to 0

Deal with Dangling Nodes



$$\mathbf{M}' = \begin{array}{c} \\ w \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{bmatrix} 0 & 0 & 1/2 & 1/4 \\ 1/3 & 0 & 0 & 1/4 \\ 1/3 & 0 & 0 & 1/4 \\ 1/3 & 1 & 1/2 & 1/4 \end{bmatrix} \end{array}$$

Solution: Teleporting

Create artificial links from any dangling node to any other node

Deal with Dangling Nodes: Teleporting

This adjustment is justified by modeling the behaviour of a web surfer

Deal with Dangling Nodes: Teleporting

This adjustment is justified by modeling the behaviour of a web surfer

After reading a page with no out-going link, jump to a page picked uniformly at random amongst the N

Deal with Dangling Nodes: Teleporting

Initially, we set $\mathbf{M}_{N \times N}$ $m_{v,w} = \begin{cases} \frac{1}{o_w} & \text{if } v \in O_w \\ 0 & \text{otherwise} \end{cases}$

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Now we change it to $\mathbf{M}'_{N\times N}$ $m'_{v,w} = \begin{cases} \frac{1}{o_w} & \text{if } v \in O_w \\ \frac{1}{N} & \text{if } \sum_{v=1}^{N} m_{v,w} = 0 \\ 0 & \text{otherwise} \end{cases}$

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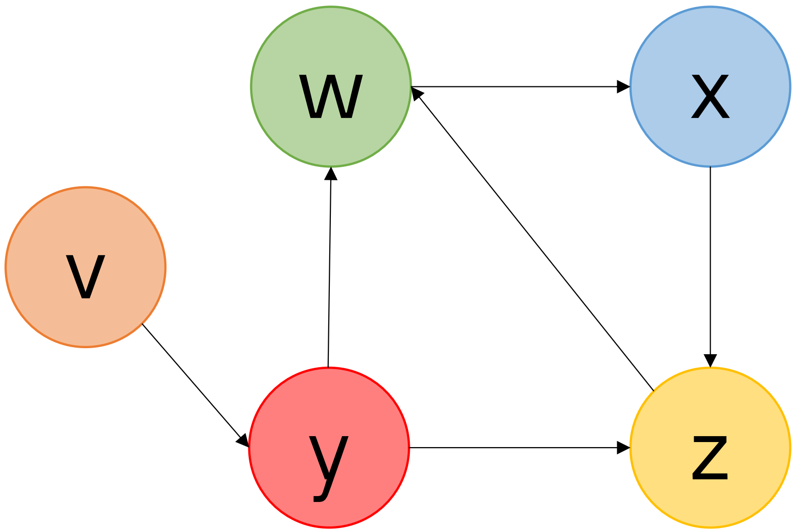
$$\boxed{\mathbf{M} \rightsquigarrow \mathbf{M}'}$$

This transformation allows M' to be column stochastic

Deal with Spider Traps



$$\mathbf{M} = \begin{array}{c} \\ v \\ w \\ x \\ y \\ z \end{array} \!\!\begin{array}{c} \begin{array}{ccccc} v & w & x & y & z \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1/2 & 0 \end{bmatrix} \end{array}$$

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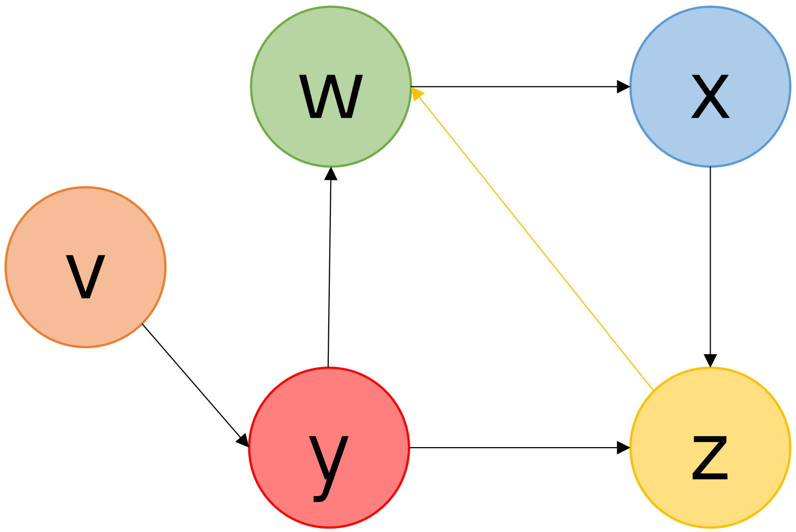

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z is not a dangling node anymore

M is (column) stochastic

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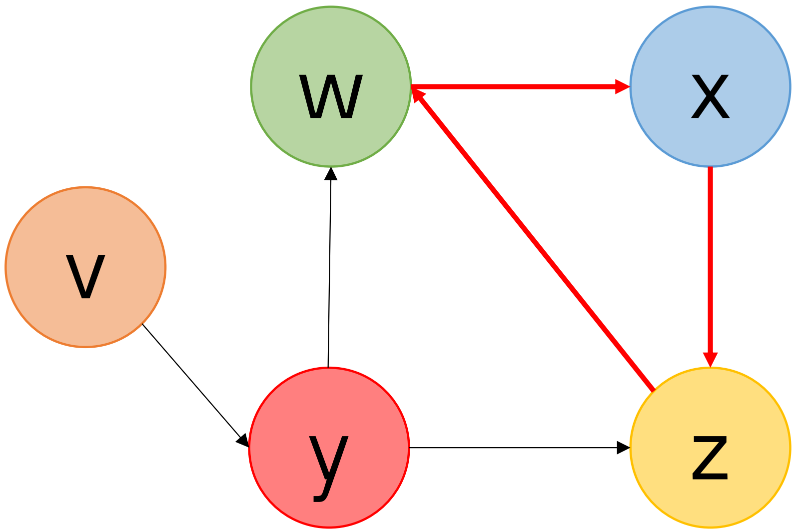

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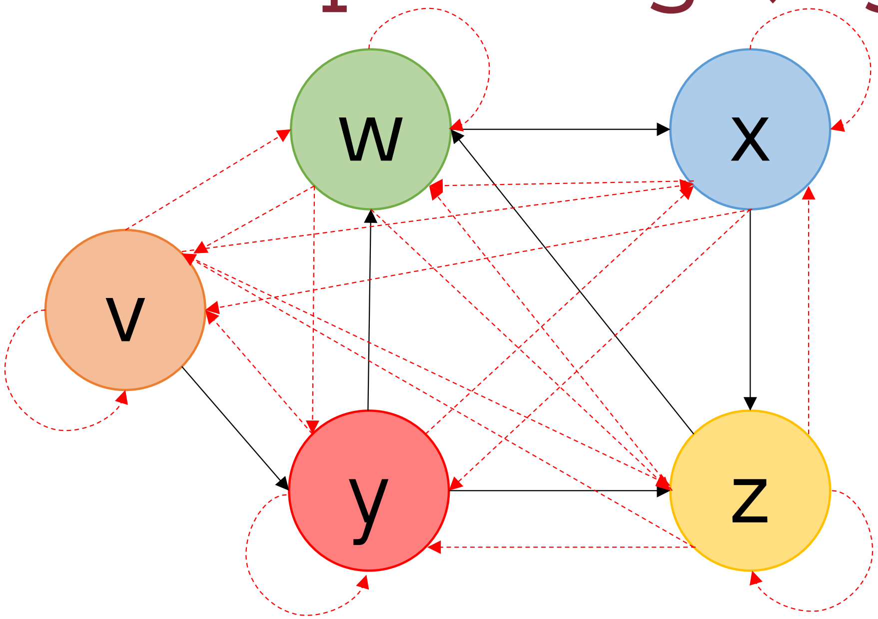
If we apply simplified PageRank to M some entries of the rank vector r will eventually drop to 0, as we get stuck in w, x, z

Deal with Spider Traps: Teleporting (Again!)



$$\mathbf{M}' = \begin{array}{c} \\ v \\ w \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{ccccc} v & w & x & y & z \end{array} \\ \begin{bmatrix} 0.03 & 0.03 & 0.03 & 0.03 & 0.03 \\ 0.03 & 0.03 & 0.03 & 0.455 & 0.88 \\ 0.03 & 0.88 & 0.03 & 0.03 & 0.03 \\ 0.88 & 0.03 & 0.03 & 0.03 & 0.03 \\ 0.03 & 0.03 & 0.88 & 0.455 & 0.03 \end{bmatrix} \end{array}$$

Solution: Probabilistic Teleporting

Create artificial links from each node to every other node and follow each of it with probability (1-d)/N

Deal with Spider Traps: Probabilistic Teleporting

To avoid the surfer to get stuck in a spider trap

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On each page w the surfer will either follow one of its outgoing links with probability d or jump to another page with probability (1-d)

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d is called damping factor

d = 0.85 in the original Google formulation

The Google's PageRank Formulation

$$\mathbf{M}_{N \times N} \quad m_{v,w} = \begin{cases} \frac{1}{o_w} & \text{if } v \in O_w \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{\mathbf{M} \rightsquigarrow \mathbf{M}'}$$

Ensure the matrix is stochastic

$$\mathbf{M}'_{N \times N} \quad m'_{v,w} = \begin{cases} \frac{1}{o_w} & \text{if } v \in O_w \\ \frac{1}{N} & \text{if } \sum_{v=1}^{N} m_{v,w} = 0 \\ 0 & \text{otherwise} \end{cases}$$

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$$\mathbf{G} = d\mathbf{M}' + \frac{1-d}{N} \underbrace{\begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix}}_{\mathbf{1}_{N\times N}}$$

$$\boxed{\mathbf{M}' \rightsquigarrow \mathbf{G}}$$

Ensure the matrix is strictly positive

Why Does Teleporting Solve Our Problem?

$$\mathbf{G} = d\mathbf{M}' + \frac{1-d}{N} \underbrace{\begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix}}_{\mathbf{1}_{N \times N}}$$

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The Perron-Frobenius theorem now applies to G and guarantees the existence (convergence) and uniqueness of the steady-state eigenvector r^*

$$\mathbf{r}(t) = \mathbf{G}^t\mathbf{r}(0)$$

$$\mathbf{r} \rightsquigarrow \mathbf{r}^* \text{ as } t \longrightarrow \infty$$

How Do We Actually Compute PageRank?

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Problem:
G represents a fully-connected graph with a huge number of nodes (web pages)

G is a dense matrix

How Do We Actually Compute PageRank?

Assuming the number of web pages in the graph is $N=10^9$

G will have N^2 entries = 10^{18}

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Note: The Web contains far more than $N=10^9$ pages!

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$$\mathbf{r} = \mathbf{G}\mathbf{r}$$

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Approximately 10 links per web page reduces the amount of memory required to store M' by a factor of 8 w.r.t. G (10^{10} vs. 10^{18} entries)

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We can work with M' rather than G

Re-Arrange the Equation

$$\boxed{\mathbf{r} = d\mathbf{M}'\mathbf{r} + \left[\frac{1-d}{N}\right]_{N\times 1}}$$

At each iteration we can compute PageRank vector as follows:

1. $\mathbf{r}(t+1) = d\mathbf{M}'\mathbf{r}(t)$
2. $\mathbf{r}(t+1) = \mathbf{r}(t+1) + \left[\frac{1-d}{N}\right]_{N\times 1}$

Add the constant (1-d)/N to each component of r(t+1)

PageRank: Pseudocode

Algorithm: PageRank

Input : A directed Web graph $G = (V, E)$, where $|V| = N$ and its associated matrix $\mathbf{M}_{N\times N}$ defined as follows: $\mathbf{M}_{v,w} = \frac{1}{o_w}$ if w points to v, 0 otherwise ($o_w = |O_w|$ where $O_w = \{x \in V : (w, x) \in E\}$);
A *damping factor* $d \in (0, 1)$;
A *tolerance* $\epsilon > 0$.

Output: The PageRank vector $\mathbf{r}^*_{N\times 1}$

Init : $t \leftarrow 0; \mathbf{r}(t) \leftarrow \left(\frac{1}{N}, \ldots, \frac{1}{N}\right);$

repeat
  $t \leftarrow t + 1;$
  /* Compute the temporary PageRank score of every page v */
  for $i \leftarrow 1$ **to** N **do**
    $r_v^{\text{tmp}}(t) \leftarrow \sum_{w \in I_v} \frac{r_w(t-1)}{o_w}$; /* $r_v^{\text{tmp}}(t) = 0$ if v has no in-links */
  end
  /* Adjust the PageRank score of each page v with *teleporting* */
  for $i \leftarrow 1$ **to** N **do**
    $r_v(t) \leftarrow d \times r_v^{\text{tmp}}(t) + \frac{1-d}{N};$
  end
until $|\mathbf{r}(t) - \mathbf{r}(t-1)| < \epsilon$
return $\mathbf{r}^* = \mathbf{r}(t);$

Take-Home Message of Today

- We present an example of **link analysis** algorithm: **PageRank**

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- We present an example of **link analysis** algorithm: PageRank
- Goal: Find an **importance score** for each web page
- Represent the Web graph as a matrix **M**, where a link from page w to v is a **vote** from w to v
- 2 different yet equivalent approaches:
  - Linear Algebra → Matrix eigenvector
  - Probabilistic → Stationary distribution of Markov chain (**random walk**)

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- The **existence** (convergence) and **uniqueness** of PageRank is guaranteed only for certain matrices $\mathbf{M}$ (Perron-Frobenius theorem)

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- The Web graph is **disconnected** and may contain **no-exit loops**
- Google solution: **probabilistic teleport links**
- Still efficiently computable from the original, sparse matrix M