PROPOSITIONAL LOGIC

① Try to answer the following:

"The longest day of the year (in terms of number of daylight hours) occurs during which month?"

It's likely you might have a reflex answer to this question:

But then we might follow up with the question:

"In Australia, the longest day of the year occurs during which month?"

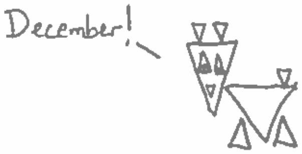
② At this point, some reasoning is required.

Australia is in the Southern hemisphere.

In the Southern hemisphere, the summer solstice occurs in December.

The summer solstice is the longest day of the year.

Thus, in Australia, the longest day of the year is in December.

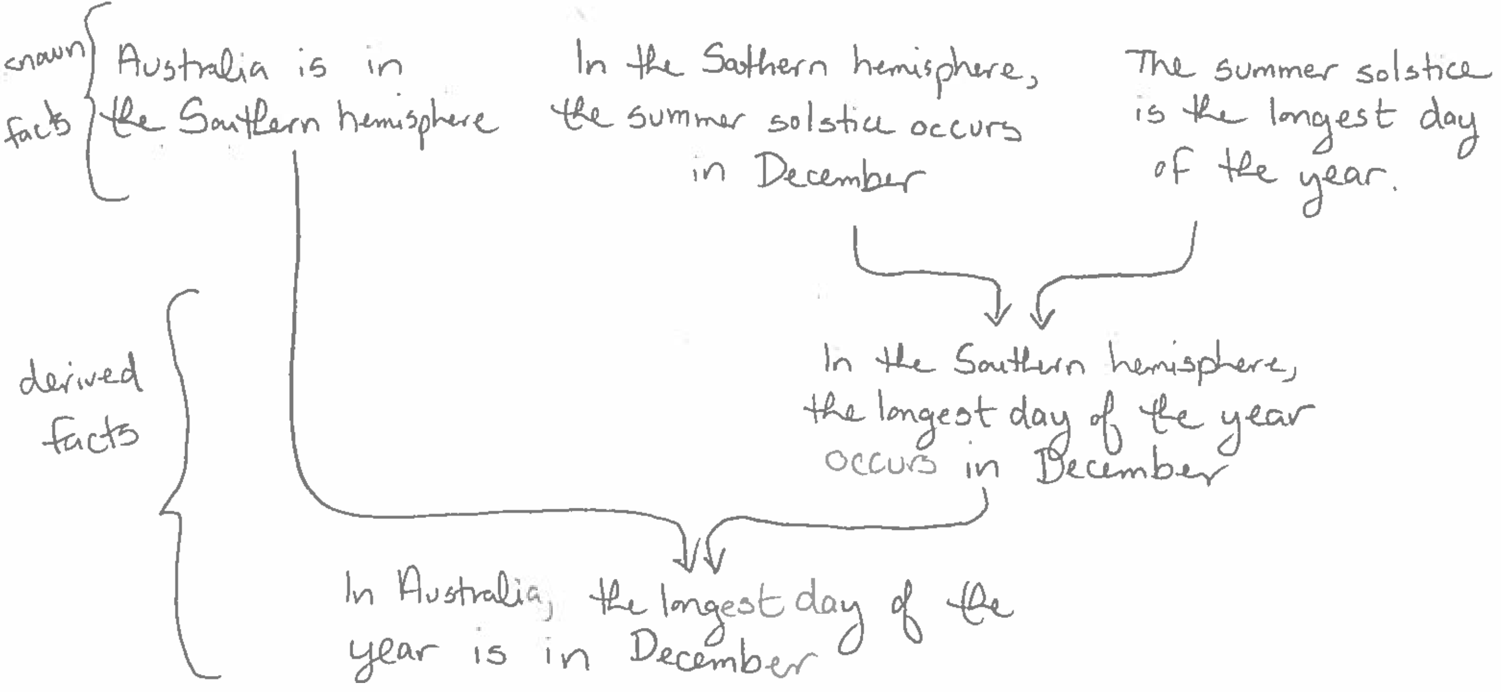


(3) What just happened? Well, one (unlikely) theory of cognition is that we just have a big table that stores answers to all questions we might be asked:

Place	Longest Day Occurs
Canada	June
Egypt	June
China	June
Australia	December
Fiji	December

Of course, then for any city, town, geographic feature, etc. we'd need a separate table entry.

(4) A more acceptable theory is that we do store some set of facts, but we can derive more facts through the application of rules of inference.


known facts
Australia is in the Southern hemisphere
In the Southern hemisphere, the summer solstice occurs in December
The summer solstice is the longest day of the year.
derived facts
In the Southern hemisphere, the longest day of the year occurs in December
In Australia, the longest day of the year is in December

⑤ This reasoning seems very natural, but what are we actually doing? Can we create a mathematical model that replicates this process? This is what the study of logic attempts to do.

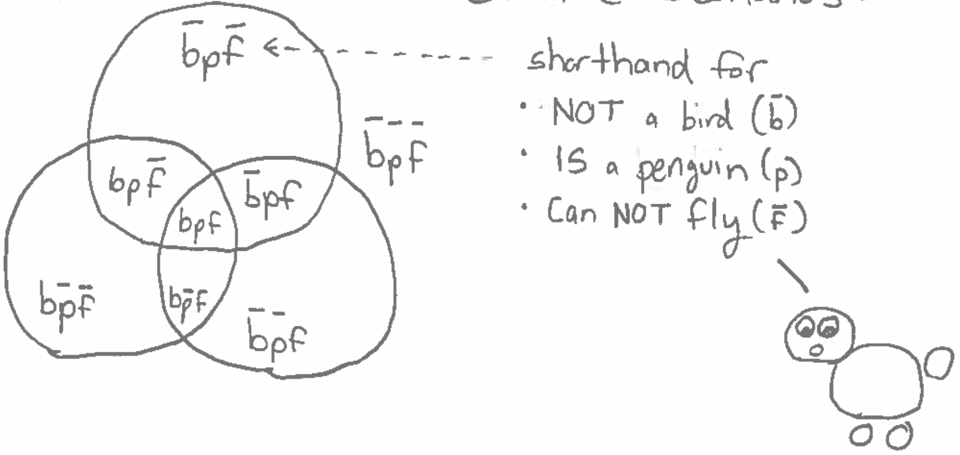
⑥ We'll begin by focusing on if-then statements. Consider the following simple example:

If you are a penguin, then you cannot fly.
Suppose you can fly.
Are you a penguin?

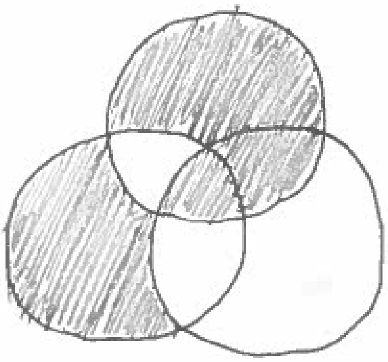
⑦ Fundamentally, we can boil this down into eight "possible worlds" that consider three criteria:

- whether you are a BIRD
- whether you are a PENGUIN
- whether you can FLY

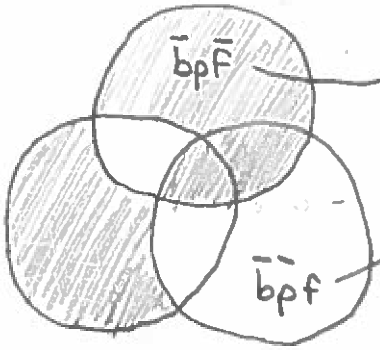
Using a Venn diagram, we can visualize the scenarios:



⑧ We might consider some scenarios to be impossible. For instance, $\bar{b}p\bar{f}$ is impossible because a penguin is a bird by definition (so it can't be a penguin but not a bird). We could color in impossible scenarios:



⑨ But it's more standard to use a truth table:



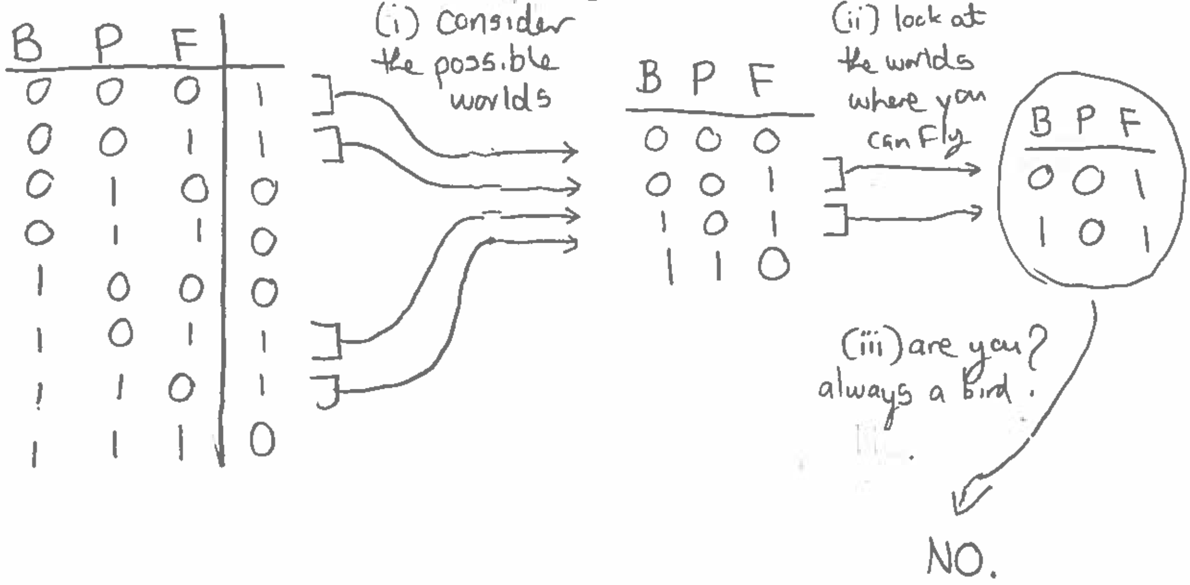
BIRD	PENGUIN	FLY	
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	0

how many possible truth tables?

PROPOSITIONAL LOGIC

(10) A truth table can answer many questions:

- if you can fly, does that mean you are a bird?



B	P	F	
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	0

(i) consider the possible worlds

B	P	F
0	0	0
0	0	1
1	0	1
1	1	0

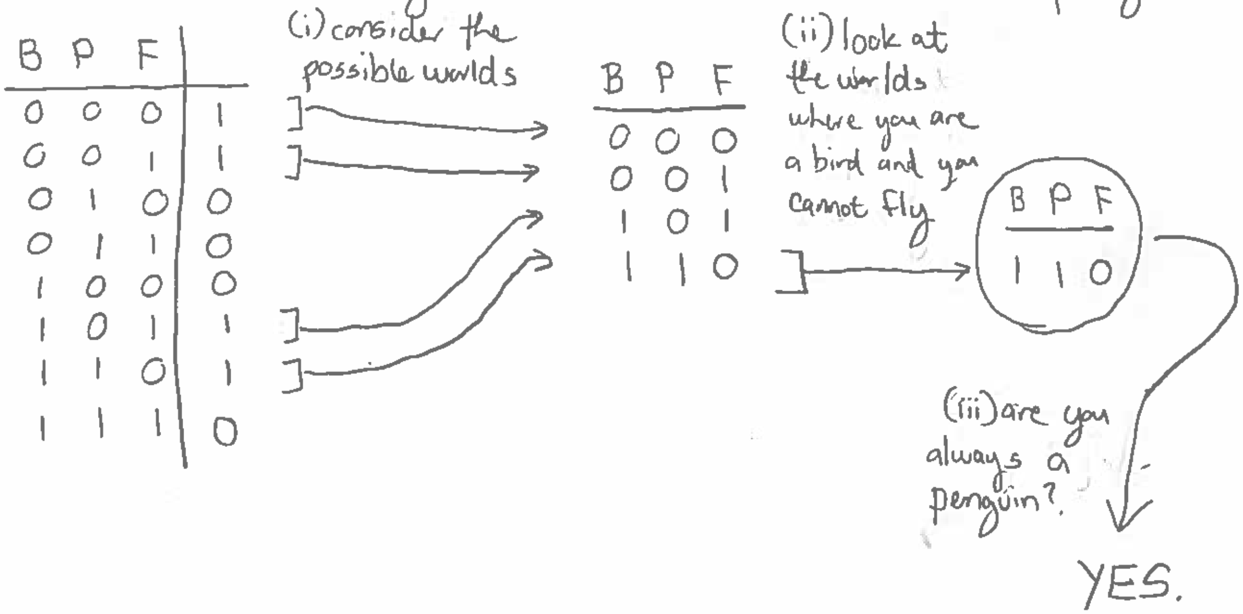
(ii) look at the worlds where you can Fly

B	P	F
0	0	1
1	0	1

(iii) are you always a bird?

NO.

- if a bird cannot fly, does that mean it's a penguin?



B	P	F	
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	0

(i) consider the possible worlds

B	P	F
0	0	0
0	0	1
1	0	1
1	1	0

(ii) look at the worlds where you are a bird and you cannot fly

B	P	F
1	1	0

(iii) are you always a penguin?

YES.

PROPOSITIONAL LOGIC*

(11) Unfortunately, it's impractical to represent knowledge with truth tables:

A_1	A_2	$\cdots$	A_n	
0	0		0	
0	0		1	
0	0		0	
0	0		1	
$\vdots$	$\vdots$		$\vdots$	
1	1		0	
1	1		1	

2^n rows = too many to fit in memory for moderately large n

(12) Propositional logic allows us to represent knowledge more compactly and answer questions more efficiently. Essentially, it is a language for which every <u>sentence</u> encodes a truth table. We can view it as an encoding of the space of truth tables.

e.g. $B \wedge P$ encodes

B	P	F	
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

Language is a choice of what we prioritize as important (and atomic). There is a reason why the word "in" is short and the word "disestablishmentarianism" is long.

$(B \wedge \neg P) \Rightarrow F$ encodes

B	P	F	
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

PROPOSITIONAL LOGIC

13. Let's define the language. First, we assume an alphabet Σ of symbols. For our animal scenario, this wauld be $\Sigma = \{B, P, F\}$. Often this alphabet Σ is refered to as a <u>signature</u>.

For a signature Σ, the propositional language $\mathcal{L}(\Sigma)$ is the minimal set of strings s.t.:

- for all $\sigma \in \Sigma$, $\sigma \in \mathcal{L}(\Sigma)$
- $\text{True} \in \mathcal{L}(\Sigma)$
- $\text{False} \in \mathcal{L}(\Sigma)$
- if $\alpha \in \mathcal{L}(\Sigma)$, then $\neg\alpha \in \mathcal{L}(\Sigma)$
- if $\alpha \in \mathcal{L}(\Sigma)$ and $\beta \in \mathcal{L}(\Sigma)$, then:
  - $(\alpha \wedge \beta) \in \mathcal{L}(\Sigma)$
  - $(\alpha \vee \beta) \in \mathcal{L}(\Sigma)$
  - $(\alpha \Rightarrow \beta) \in \mathcal{L}(\Sigma)$
  - $(\alpha \Leftrightarrow \beta) \in \mathcal{L}(\Sigma)$

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14. Each string in $\mathcal{L}(\Sigma)$ is called a <u>sentence</u> of $\mathcal{L}(\Sigma)$ (or sometimes a <u>farmula</u>). Examples include:

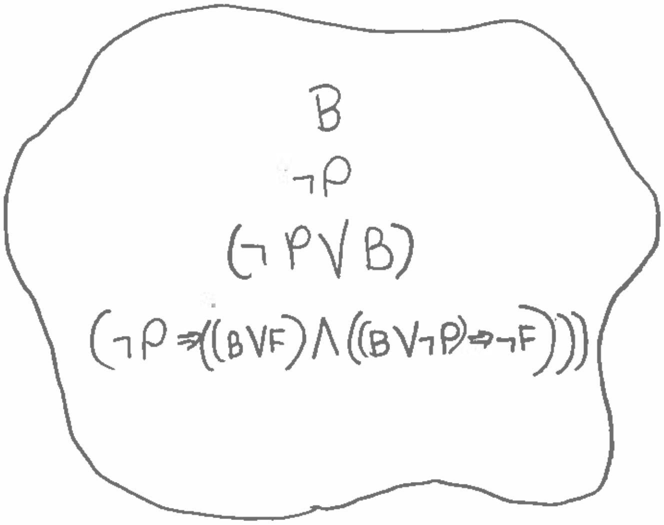
$$B$$

$$\neg P$$

$$(\neg P \vee B)$$

$$(\neg P \Rightarrow (B \vee F) \wedge ((B \vee \neg P) \Rightarrow \neg F))$$

(15) So far we have our language (untethered to meaning) and the meanings we'd like to encode:



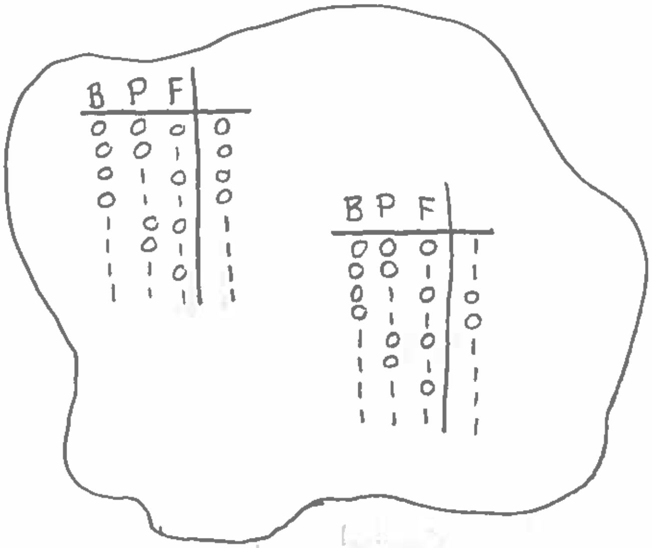
B

$\neg P$

$(\neg P \vee B)$

$(\neg P \Rightarrow ((B \vee F) \wedge ((B \vee \neg P) \Rightarrow \neg F)))$

language $\mathcal{L}(\Sigma)$
"syntax"



B	P	F	
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

B	P	F	
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

truth tables $\tau(\Sigma)$
"semantics"

(16) To make this into a logic, we need to define an interpretation function $\mathbb{I}: \mathcal{L}(\Sigma) \mapsto \tau(\Sigma)$ that assigns a truth table to each sentence in our language.

First, define a <u>model</u> as a function from signature Σ to $\{0,1\}$. Essentially a model is a raw of a truth table (without the truth value). The space of models is written $M(\Sigma)$.

B	P	F		
0	0	0	←	model
0	0	1	←	model
0	1	0	←	model
0	1	1	←	⋮
1	0	0	←	
1	0	1	←	
1	1	0	←	
1	1	1	←	

PROPOSITIONAL LOGIC

(17) Define the interpretation of a sentence in $\mathcal{L}(\Sigma)$ as:

- $I(\text{True}) = M(\Sigma)$
- $I(\text{False}) = \emptyset$
- for all $\sigma \in \Sigma$, $I(\sigma) = \{m \in M(\Sigma) \mid m(\sigma) = 1\}$
- for all $\alpha \in \mathcal{L}(\Sigma)$, $I(\neg\alpha) = M(\Sigma) - I(\alpha)$
- for all $\alpha, \beta \in \mathcal{L}(\Sigma)$:
  - $I((\alpha \wedge \beta)) = I(\alpha) \cap I(\beta)$
  - $I((\alpha \vee \beta)) = I(\alpha) \cup I(\beta)$
  - $I((\alpha \Rightarrow \beta)) = I((\neg\alpha \vee \beta))$
  - $I((\alpha \Leftrightarrow \beta)) = I(\alpha \Rightarrow \beta) \cap I(\beta \Rightarrow \alpha)$

(18) For instance, let's interpret $((B \wedge \neg P) \Rightarrow F)$, i.e. "non-penguin birds can fly"

$$I\Big(((B \wedge \neg P) \Rightarrow F)\Big)$$

$$= I\Big((\neg(B \wedge \neg P) \vee F)\Big)$$

$$= I\Big(\neg(B \wedge \neg P)\Big) \cup I(F)$$

$$= \Big(M(\Sigma) - I\big((B \wedge \neg P)\big)\Big) \cup I(F)$$

$$= \Big(M(\Sigma) - \big(I(B) \cap I(\neg P)\big)\Big) \cup I(F)$$

$$= \left(\left\{\begin{matrix} 000 \\ 001 \\ 010 \\ 011 \\ 100 \\ 101 \\ 110 \\ 111 \end{matrix}\right\} - \left(\left\{\begin{matrix} 100 \\ 101 \\ 110 \\ 111 \end{matrix}\right\} \cap \left\{\begin{matrix} 000 \\ 001 \\ 100 \\ 101 \end{matrix}\right\}\right)\right) \cup \left\{\begin{matrix} 001 \\ 011 \\ 101 \\ 111 \end{matrix}\right\}$$

$$= \left(M(\Sigma) - \left\{\begin{matrix} 100 \\ 101 \end{matrix}\right\}\right) \cup \left\{\begin{matrix} 001 \\ 011 \\ 101 \\ 111 \end{matrix}\right\}$$

$$= \left\{\begin{matrix} 000 \\ 001 \\ 010 \\ 011 \\ 110 \\ 111 \end{matrix}\right\} \cup \left\{\begin{matrix} 001 \\ 011 \\ 101 \\ 111 \end{matrix}\right\}$$

$$= \left\{\begin{matrix} 000 \\ 001 \\ 010 \\ 011 \\ 101 \\ 110 \\ 111 \end{matrix}\right\}$$

(19) Now that we've associated an interpretation to each logical sentence, we can work in either the language (syntax) space or the truth table (semantic) space:

English	syntax space	semantic space
"every penguin is a bird"	$P \Rightarrow B$	see first table
"penguins cannot fly"	$P \Rightarrow \neg F$	see second table

B	P	F	
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

B	P	F	
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	0

syntax space: can be much more compact!

semantic space: always the same size

What if the signature is larger?

20 Because it is usually more concise to express knowledge in the language space, that's where we mainly want to live. But that also means we want a way to "reason" in that space, i.e. to answer questions like:

Given that:

- every bird can fly
- penguins cannot fly

Is a penguin not a bird?

Let's see how to represent such a question in both the syntax and the semantic space:

syntax space

Thus we define entailment in the following way:

$$(B \Rightarrow F) \wedge (P \Rightarrow \neg F) \models P \Rightarrow \neg B$$

iff $I\big((B \Rightarrow F) \wedge (P \Rightarrow \neg F)\big) \subseteq I(P \Rightarrow \neg B)$

$$\boxed{\text{general form: } \alpha \models \beta \text{ iff } I(\alpha) \subseteq I(\beta)}$$

semantic space

B	P	F	
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	0

B	P	F	
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

every bird can fly and penguins cannot ⊆ a penguin is not a bird

in every model where every bird can fly and penguins cannot, a penguin is not a bird.

21 The fundamental computational question of propositional logic is, given two propositional sentences $\alpha, \beta \in \mathcal{L}(\Sigma)$, does it hold that $\alpha \models \beta$? We pursue this in the next lesson.