proof-based satisfiability

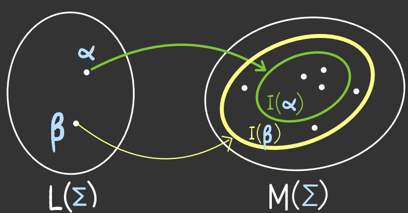
CSCI 373

α is unsatisfiable

if and only if

$\alpha \models \bot$

entailment

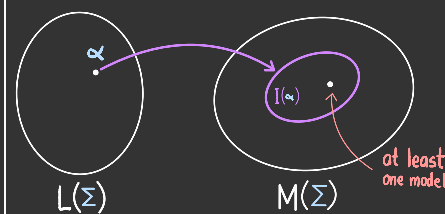


$\alpha \models \beta$

if and only if

$I(\alpha) \subseteq I(\beta)$

satisfiability



α is satisfiable

if and only if

$I(\alpha) \neq \{\,\}$

$\alpha \models \beta$

if and only if

$\alpha \wedge \neg\beta$ is unsatisfiable

α is unsatisfiable

if and only if

$\alpha \models \perp$

this gives us an alternative strategy for determining whether a sentence is unsatisfiable:

prove that it entails $\perp$

let's prove two helpful results

1

for any sentences $\alpha, \beta, \gamma \in L(\Sigma)$:

if $\alpha \models \beta$, then $\alpha \land \gamma \models \beta$

entailment is not affected by additional knowledge

2

for any sentences $\alpha, \beta \in L(\Sigma)$:

if $\alpha \models \beta$, then $\alpha \land \beta \equiv \alpha$

learning something we already know doesn't change our mental state

entailment is not affected by additional knowledge

for any sentences $\alpha, \beta, \gamma \in L(\Sigma)$:

if $\alpha \models \beta$, then $\alpha \wedge \gamma \models \beta$

proof: suppose $\alpha \models \beta$, thus $I(\alpha) \subseteq I(\beta)$.

$$
\begin{aligned}
I(\alpha \wedge \gamma) &= I(\alpha) \cap I(\gamma) \\
&\subseteq I(\beta) \cap I(\gamma) \\
&\subseteq I(\beta)
\end{aligned}
$$

so $\alpha \wedge \gamma \models \beta$

entailment is not affected by additional knowledge

$\boxed{\begin{array}{l}\text{for any sentences } \alpha, \beta, \gamma \in L(\Sigma): \\ \text{if } \alpha \models \beta, \text{ then } \alpha \wedge \gamma \models \beta\end{array}}$

proof: suppose $\alpha \models \beta$, thus $I(\alpha) \subseteq I(\beta)$.

$$\begin{aligned} I(\alpha \wedge \gamma) &= I(\alpha) \cap I(\gamma) \\ &\subseteq I(\beta) \cap I(\gamma) \\ &\subseteq I(\beta) \end{aligned}$$

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$I(\alpha \wedge \beta) = I(\alpha) \cap I(\beta) = I(\alpha)$.

so $\alpha \wedge \beta \equiv \alpha$

is this unsatisfiable?

$$(\neg A \vee \neg B) \wedge (A \vee C) \wedge B \wedge \neg C$$

is this unsatisfiable?

$$\underbrace{(\neg A \lor \neg B) \land (A \lor C)} \land B \land \neg C$$

this entails $\neg B \lor C$.

why?

case analysis on a satisfying model m.

either $m(A)=0$: $(\neg A \vee \neg B) \wedge (A \vee C) \wedge B \wedge \neg C$

this literal is unsatisfied

so $m(C)=1$

or $m(A)=1$: $(\neg A \vee \neg B) \wedge (A \vee C) \wedge B \wedge \neg C$

this literal is unsatisfied

so $m(B)=0$

therefore, $m(B)=0$ or $m(C)=1$. $\neg B \vee C$

is this unsatisfiable?

$$(\neg A \vee \neg B) \wedge (A \vee C) \wedge B \wedge \neg C$$

by case analysis: $(\neg A \vee \neg B) \wedge (A \vee C) \models \neg B \vee C$

is this unsatisfiable?

$$(\neg A \vee \neg B) \wedge (A \vee C) \wedge B \wedge \neg C$$

by case analysis: $(\neg A \vee \neg B) \wedge (A \vee C) \models \neg B \vee C$

this also means: $\underbrace{(\neg A \vee \neg B) \wedge (A \vee C)}_{\alpha} \wedge \underbrace{B \wedge \neg C}_{\gamma} \models \underbrace{\neg B \vee C}_{\beta}$

because

for any sentences $\alpha, \beta, \gamma \in L(\Sigma)$:

if $\alpha \models \beta$, then $\alpha \wedge \gamma \models \beta$

entailment is not affected by additional knowledge

is this unsatisfiable?

$$(\neg A \lor \neg B) \land (A \lor C) \land B \land \neg C$$

by case analysis: $(\neg A \lor \neg B) \land (A \lor C) \models \neg B \lor C$

this also means: $\underbrace{(\neg A \lor \neg B) \land (A \lor C) \land B \land \neg C}_{\alpha} \models \underbrace{\neg B \lor C}_{\beta}$

and moreover: $(\neg A \lor \neg B) \land (A \lor C) \land B \land \neg C$

because

$$\equiv (\neg A \lor \neg B) \land (A \lor C) \land B \land \neg C \land (\neg B \lor C)$$

$$\boxed{\text{if } \alpha \models \beta, \text{ then } \alpha \land \beta \equiv \alpha}$$

learning something we already know doesn't change our mental state

if some subset of clauses...

$$\boxed{(\neg A \vee \neg B) \wedge (A \vee C)} \wedge B \wedge \neg C$$

entails another clause:

$$(\neg A \vee \neg B) \wedge (A \vee C) \models \boxed{\neg B \vee C}$$

then we can conjoin the entailed clause to the original sentence without changing its interpretation

$$(\neg A \vee \neg B) \wedge (A \vee C) \wedge B \wedge \neg C \boxed{\wedge (\neg B \vee C)} \equiv (\neg A \vee \neg B) \wedge (A \vee C) \wedge B \wedge \neg C$$

is $(\neg A \vee \neg B) \wedge (A \vee C) \wedge B \wedge \neg C$ unsatisfiable?

clauses

$\neg A \vee \neg B$

$A \vee C$

B

$\neg C$

is $(\neg A \vee \neg B) \wedge (A \vee C) \wedge B \wedge \neg C$ unsatisfiable?

clauses

$\boxed{\neg A \vee \neg B}$

$\boxed{A \vee C}$

B

$\neg C$

$$\overline{(\neg A \vee \neg B)} \wedge \underline{(A \vee C)} \models (\neg B \vee C)$$

so add $(\neg B \vee C)$
to our clauses

is $(\neg A \vee \neg B) \wedge (A \vee C) \wedge B \wedge \neg C$ unsatisfiable?

clauses

$\neg A \vee \neg B$

$A \vee C$

B

$\neg C$

$\neg B \vee C$

$$\boxed{\text{is } (\neg A \lor \neg B) \land (A \lor C) \land B \land \neg C \text{ unsatisfiable?}}$$

clauses

$\neg A \lor \neg B$

$A \lor C$

$\boxed{B}$

$\neg C$

$\boxed{\neg B \lor C}$

$$\underline{B} \land \underline{(\neg B \lor C)} \models C$$

so add C

to our clauses

$$\boxed{\text{is } (\neg A \vee \neg B) \wedge (A \vee C) \wedge B \wedge \neg C \text{ unsatisfiable?}}$$

clauses

$\neg A \vee \neg B$

$A \vee C$

B

$\neg C$

$\neg B \vee C$

C

is $(\neg A \vee \neg B) \wedge (A \vee C) \wedge B \wedge \neg C$ unsatisfiable?

clauses

$\neg A \vee \neg B$

$A \vee C$

B

$\boxed{\neg C}$

$\neg B \vee C$

$\boxed{C}$

$C \wedge \neg C \models \bot$

so add $\bot$ to our clauses

is $(\neg A \lor \neg B) \land (A \lor C) \land B \land \neg C$ unsatisfiable?

clauses

$\neg A \lor \neg B$

$A \lor C$

B

$\neg C$

$\neg B \lor C$

C

$\bot$

at this point, we have established that the original sentence is logically equivalent to the conjunction of clauses on the left

is $(\neg A \vee \neg B) \wedge (A \vee C) \wedge B \wedge \neg C$ unsatisfiable?

clauses

$\neg A \vee \neg B$

$A \vee C$

B

$\neg C$

$\neg B \vee C$

C

$\perp$

but anything conjoined with $\perp$ has an empty **interpretation**

$$I(\alpha \wedge \perp) = I(\alpha) \cap I(\perp)$$

$$= I(\alpha) \cap \{\}$$

$$= \{\}$$

is $(\neg A \vee \neg B) \wedge (A \vee C) \wedge B \wedge \neg C$ unsatisfiable?

clauses

$\neg A \vee \neg B$

$A \vee C$

B

$\neg C$

$\neg B \vee C$

C

$\bot$

anything conjoined with $\bot$ is

unsatisfiable

$$I(\alpha \wedge \bot) = I(\alpha) \cap I(\bot)$$
$$= I(\alpha) \cap \{\}$$
$$= \{\}$$

is $(\neg A \vee \neg B) \wedge (A \vee C) \wedge B \wedge \neg C$ unsatisfiable?

clauses

$\neg A \vee \neg B$

$A \vee C$

B

$\neg C$

$\neg B \vee C$

C

$\perp$

therefore

$(\neg A \vee \neg B) \wedge (A \vee C) \wedge B \wedge \neg C$

is unsatisfiable

is $c_1 \wedge c_2 \wedge \cdots \wedge c_n$ unsatisfiable?

clauses

c_1

c_2

$\vdots$

c_n

strategy:

keep finding new clauses that are entailed by some subset of our current clauses until we entail $\perp$

is $c_1 \wedge c_2 \wedge \cdots \wedge c_n$ unsatisfiable?

clauses

c_1

c_2

$\vdots$

c_n

strategy:

keep finding new clauses that are entailed by some subset of our current clauses until we entail $\perp$

but how?