# monty hall workshop

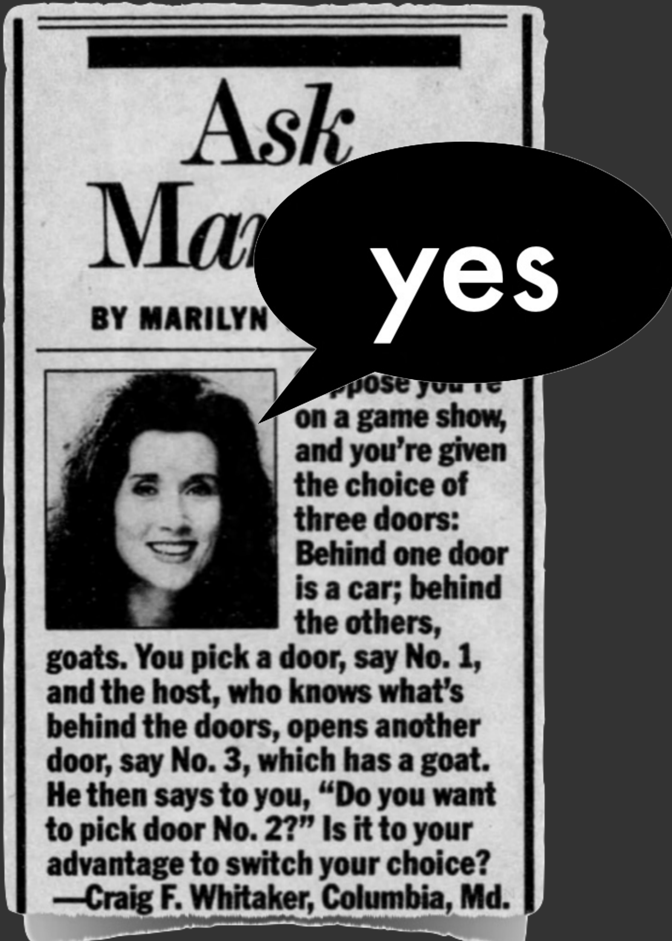
CSCI 373


# Ask Marilyn™

BY MARILYN VOS SAVANT

Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?
—Craig F. Whitaker, Columbia, Md.


should you switch?



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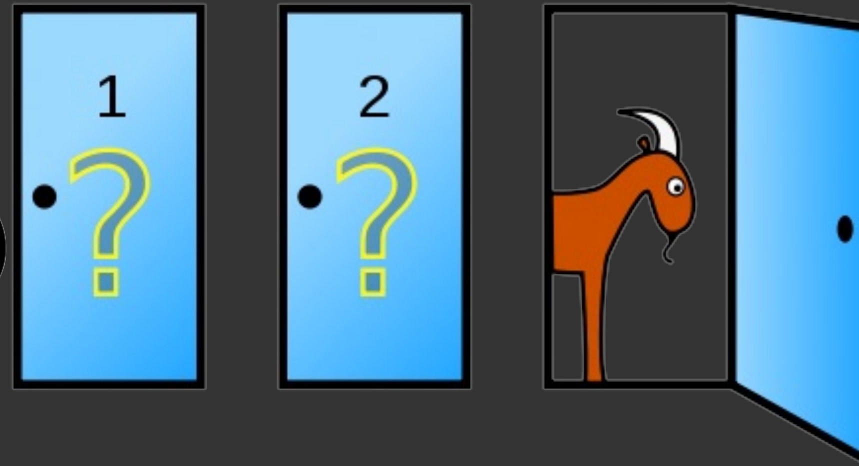


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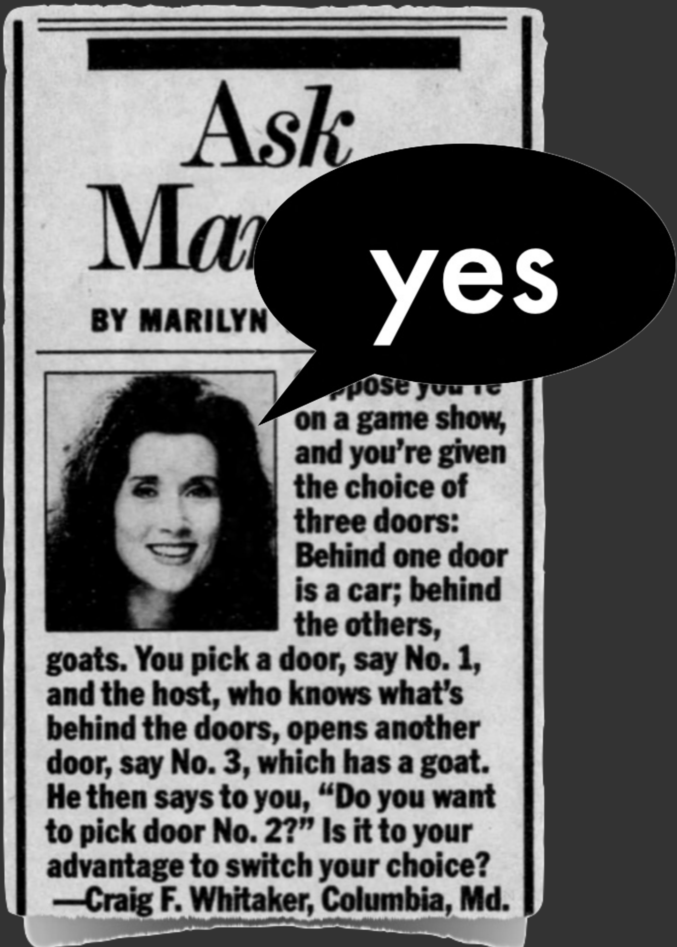


> You blew it, and you blew it big! Since you seem to have difficulty grasping the basic principle at work here, I'll explain. After the host reveals a goat, you now have a one-in-two chance of being correct. Whether you change your selection or not, the odds are the same. There is enough mathematical illiteracy in this country, and we don't need the world's highest IQ propagating more. Shame!
>
> Scott Smith, University of Florida[3]

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**—Craig F. Whitaker, Columbia, Md.**

I have been a faithful reader of your column, and I have not, until now, had any reason to doubt you. However, in this matter (for which I do have expertise), your answer is clearly at odds with the truth.
***James Rauff, Ph.D.***
***Millikin University***

May I suggest that you obtain and refer to a standard textbook on probability before you try to answer a question of this type again?
***Charles Reid, Ph.D.***
***University of Florida***

I am sure you will receive many letters on this topic from high school and college students. Perhaps you should keep a few addresses for help with future columns.
***W. Robert Smith, Ph.D.***
***Georgia State University***

You are utterly incorrect about the game show question, and I hope this controversy will call some public attention to the serious national crisis in mathematical education. If you can admit your error, you will have contributed constructively towards the solution of a deplorable situation. How many irate mathematicians are needed to get you to change your mind?
***E. Ray Bobo, Ph.D.***
***Georgetown University***

I am in shock that after being corrected by at least three mathematicians, you still do not see your mistake.
***Kent Ford***
***Dickinson State University***

Maybe women look at math problems differently than men.
***Don Edwards***
***Sunriver, Oregon***

You are the goat!
***Glenn Calkins***
***Western State College***

You made a mistake, but look at the positive side. If all those Ph.D.'s were wrong, the country would be in some very serious trouble.
***Everett Harman, Ph.D.***
***U.S. Army Research Institute***

# Monty Hall problem refuses to die

To the Editor,

Further regarding the controversy which refuses to die over the "Monty Hall Problem:" The letters from Robert DiStefano and J.T. Kroenert which appeared on Aug. 6 both contained the same error. They incorrectly assigned two separate independent random outcomes to the same event. Namely, if you correctly guess that the car is behind door #1 on the first try, it doesn't matter whether Monty exposes door #2 or door #3, because he knows in advance where everything is. Marilyn vos Savant is right; the odds of winning never become 50-50, and the New York Times News Service graphic provided with the original story is indeed correct.

The answer can be explained with less math and more common sense when you realize that Monty, like any good stage magician, is directing your attention away from what's really happening — a process of elimination (probabilistically speaking). You are always more likely to get a goat than a car on the first try (two-thirds of the time, in fact), so two out of three times, it works this way:

1. The first door you pick (say, #1) hides a goat, so the other two hide one goat and one car.

2. This means that Monty has only one goat left, and he obligingly shows you where it is (say, #3).

3. This means the car is behind the remaining door (#2).

By switching from door #1 to door #2, you win two out of three times, because those two out of three times, you first picked a goat. The winning contestants are thinking in terms of eliminating goats rather than winning cars.

Gerry Maynard,
North Kingstown

★ ★ ★

To the Editor,

I am writing concerning Monty Hall's Problem. In a recent article in this paper, Marilyn vos Savant claimed that the contestant would increase his odds of winning from $\frac{1}{3}$ to $\frac{2}{3}$ by switching from his original choice. Subsequently, two letters appeared in which her claim was rebutted but the conclusions in these letters are based on incorrect reasoning.

Robert DiStefano asks, "What could possibly be the explanation for why the contestants pick the wrong door on their first choice twice as often as they pick the correct door?" The answer to this is quite simple. There are two goats to choose, but only one car. Consider a can containing one red and two blue marbles. If I reach into the can and randomly draw one marble, I have a $\frac{1}{3}$ chance of choosing the red marble, which is similar to choosing the car in the original problem.

Now, suppose I reach into the can of three marbles again and take one out, but this time I do not look at it. I allow another person to look into the can and pull out a blue marble. What are the odds that I have the red marble? By J. T. Kroenert's reasoning the odds should be $\frac{1}{2}$, but clearly they are $\frac{1}{3}$. There were three marbles in the can (two blue and one red) when I chose, so my odds are $\frac{1}{3}$, regardless of whether or not someone draws a blue marble after I have chosen. The same is true for the doors. Once the contestant has chosen a door (with a $\frac{1}{3}$ chance of choosing the car), all other actions after the selection are irrelevant, and the odds of winning the car remain $\frac{1}{3}$ if the contestant does not deviate from his original selection. This fact alone disproves the two theories printed Aug. 6.

Moving to the scenario in which the contestant always switches his choice, again consider the marbles. I select one marble and then allow the "host" to remove a blue marble. I discard the marble in my hand and take the last marble from the can. What are the odds that this one is red?

There are two distinct events which may occur. First, $\frac{1}{3}$ of the time, the red marble will be the first selected, leaving two blue marbles in the can. This means I will always end up holding a blue marble. However, $\frac{2}{3}$ of the time, I will draw a blue marble first. The "host" also takes a blue marble. Both blue marbles are now removed, one in my hand, and one in the "host's" hand, leaving the red marble as my second selection. Thus, when switching my choice, I have raised my odds of winning from $\frac{1}{3}$ to $\frac{2}{3}$. The same reasoning applies to the goats and car.

As Mr. DiStefano points out, there are two combinations which may happen as a result of the contestant first choosing the red marble (or the car). Looking at the figures provided in this letter, you will notice that the car is picked first two out of four times ($\frac{1}{2}$), but the above demonstrates that the car should be picked first in only $\frac{1}{3}$ of the cases. The fallacy of his reasoning lies in the assumption that the two different combinations formed by selecting the car first each occur just as frequently as the other two. This is not the case; since the car is only first chosen $\frac{1}{3}$ of the time, all permutations of this occurrence must only count for $\frac{1}{3}$ of all possibilities, not half as Mr. DiStefano's and Mr. Kroenert's diagrams imply.

Although the conclusion stands by itself, its validity can be demonstrated by means of a computer simulation. A program was written which plays the game 10,000 times in a matter of minutes. The computation showed that not switching resulted in a win only 33.2 percent of the time, whereas switching resulted in a win 66.8 percent of the time. If you are not convinced by this, I challenge you to play out the scenario a number of times and see for yourself that switching doors does indeed make a significant difference.

Stephen N. Freund, Barrington

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To the Editor,

Another explanation of the Monty Hall problem that has been used successfully with some math-anxious or math-phobic people is as follows. The problem can be simulated by taking a deck of cards and removing any one ace and the two jokers. For the math-weary or those with a keen interest in this problem, I recommend you get a deck of cards and follow along.

The ace will equal the car and the two jokers will equal the two goats. Two people are needed — player H (Host Monty) and player C (the contestant). H shuffles the three cards and places them face down in no particular order. C picks one card randomly but does not turn it over. Hidden from C's view, H looks at the two unpicked cards and exposes a joker (at least one of the unpicked cards must be a joker). C gets to keep his original card or switch to the remaining unexposed card. This completes the simulation.

Now I will use the above simulation to find the theoretical probability of picking an ace if C elects to switch his original pick. Initially, after C picks the first card, the probability that he picked the ace is 1 out of 3 or $\frac{1}{3}$. The probability that the ace is one of the two unpicked cards is (the sum of the probability that each is an ace) $\frac{1}{3}$ $\frac{1}{3} = \frac{2}{3}$, or 2 out of 3.

We now have two groups of cards; the single picked card with a probability of being an ace equal to $\frac{1}{3}$ and the pair of remaining unpicked cards with a probability that one of them is an ace is $\frac{2}{3}$. At least one of the unpicked cards must be a joker, yet that does not change the probability that one is an ace from $\frac{2}{3}$.

When H exposes one of the unpicked cards as a joker, H is visually affirming what is already known — one of the two unpicked cards is a joker. The probability that the ace is still in the pair of unpicked cards still remains the same, equal to $\frac{2}{3}$. With the joker exposed, therefore the probability of the remaining unpicked, unexposed card being an ace is still equal to $\frac{2}{3}$. Hence, if C switches to the unpicked, unexposed card the probability of that card being an ace is $\frac{2}{3}$.

Thus the theoretical probability of picking an ace after a switch is equal to $\frac{2}{3}$. Note the theoretical probability of not switching and getting the ace would still be $\frac{1}{3}$.

To find the probability by experiment, my wife and I performed the switch-card simulation 100 times. Always switching resulted in an ace 70 out 100 times or an experimental probability of 70/100=.7, very close to the theoretical probability of $\frac{2}{3}$, approximately equal to .666667.

Enclosed is a computer printout that simulates the Monty Hall problem with the computer always switching. For 100,000 tries a switch resulted in an ace 66,621 times out of a 100,000, for a probability of .66621.

Obviously, an experiment repeated enough times should yield a probability close to the theoretical probability. For the skeptics who still do not agree with the above results, get out your deck of cards, find a partner and try the card simulation. I recommend you do the simulation at least 50 times.

I am glad that the Journal-Bulletin has run the sequence of articles and letters pertaining to the Monty Hall problem. The newspaper has stimulated, at least in a mathematical sense, the minds of many people.

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*The author is chairman of the mathematics department at Tolman Senior High School in Pawtucket.*

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Providence Journal, 8/20/90

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Providence Journal, 8/20/90

assume the contestant's initial choice is door 1
and that the contestant's strategy is to switch

let $C$ be the door with the car

let $G$ be the door that the host opens to reveal a goat

let $F$ be the contestant's final choice

let $W$ be whether the contestant wins

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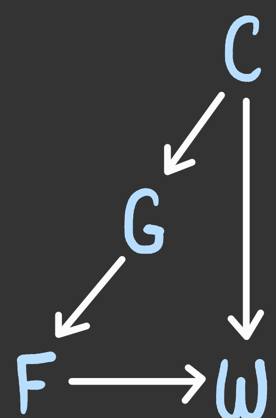


$$P(w,f,g,c) = P(w \mid f, c) \cdot P(f \mid g) \cdot P(g \mid c) \cdot P(c)$$

find some conditional independence relationships

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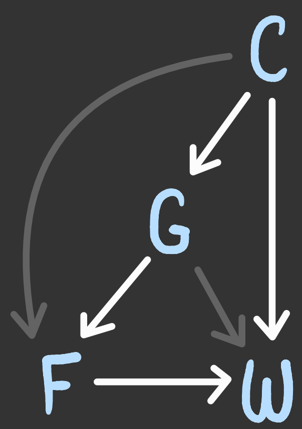


$$
\begin{aligned}
P(w,f,g,c) &= P(w \mid f,g,c) \\
&\cdot P(f \mid g,c) \\
&\cdot P(g \mid c) \\
&\cdot P(c)
\end{aligned}
$$

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$$
\begin{tikzcd}
 & C \arrow[dl] \arrow[dd] \\
G \arrow[d] & \\
F \arrow[r] & W
\end{tikzcd}
$$

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next, we need to specify these distributions

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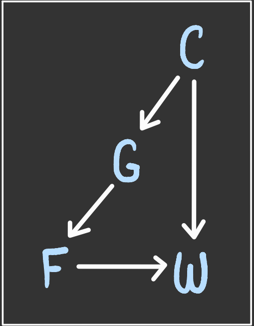


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| $f$ | $c$ | $w$ | $P(w \mid f,c)$ |
|---|---|---|---|

how many rows ?

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$$
\begin{tikzcd}
 & & C \arrow[dl] \arrow[dd] \\
 & G \arrow[dl] & \\
F \arrow[rr] & & W
\end{tikzcd}
$$

$$P(w,f,g,c) = P(w \mid f,c) \cdot P(f \mid g) \cdot P(g \mid c) \cdot P(c)$$

| $f$ | $c$ | $w$ | $P(w \mid f,c)$ |
|---|---|---|---|
| 1 | 1 | yes | ? |
| 1 | 1 | no | ? |
| 1 | 2 | yes | ? |
| 1 | 2 | no | ? |
| 1 | 3 | yes | ? |
| 1 | 3 | no | ? |
| 2 | 1 | yes | ? |
| 2 | 1 | no | ? |
| 2 | 2 | yes | ? |
| 2 | 2 | no | ? |
| 2 | 3 | yes | ? |
| 2 | 3 | no | ? |
| 3 | 1 | yes | ? |
| 3 | 1 | no | ? |
| 3 | 2 | yes | ? |
| 3 | 2 | no | ? |
| 3 | 3 | yes | ? |
| 3 | 3 | no | ? |

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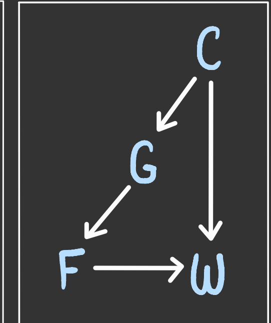


$$P(w, f, g, c) = P(w \mid f, c) \cdot P(f \mid g) \cdot P(g \mid c) \cdot P(c)$$

| f | c | w | P(w\|f,c) |
|---|---|---|---|
| 1 | 1 | yes | 1 |
| 1 | 1 | no | 0 |
| 1 | 2 | yes | 0 |
| 1 | 2 | no | 1 |
| 1 | 3 | yes | 0 |
| 1 | 3 | no | 1 |
| 2 | 1 | yes | 0 |
| 2 | 1 | no | 1 |
| 2 | 2 | yes | 1 |
| 2 | 2 | no | 0 |
| 2 | 3 | yes | 0 |
| 2 | 3 | no | 1 |
| 3 | 1 | yes | 0 |
| 3 | 1 | no | 1 |
| 3 | 2 | yes | 0 |
| 3 | 2 | no | 1 |
| 3 | 3 | yes | 1 |
| 3 | 3 | no | 0 |

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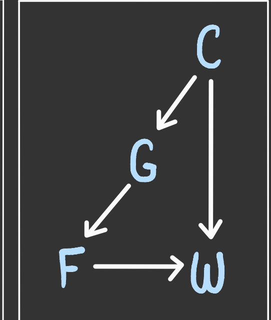


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| g | f | $P(f \mid g)$ |
|---|---|---|
| 2 | 1 | $\boxed{?}$ |
| 2 | 2 | $\boxed{?}$ |
| 2 | 3 | $\boxed{?}$ |
| 3 | 1 | $\boxed{?}$ |
| 3 | 2 | $\boxed{?}$ |
| 3 | 3 | $\boxed{?}$ |

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F \arrow[rr] & & W
\end{tikzcd}$$

$$P(w, f, g, c) = P(w \mid f, c) \cdot P(f \mid g) \cdot P(g \mid c) \cdot P(c)$$

| $g$ | $f$ | $P(f \mid g)$ |
|---|---|---|
| 2 | 1 | 0 |
| 2 | 2 | 0 |
| 2 | 3 | 1 |
| 3 | 1 | 0 |
| 3 | 2 | 1 |
| 3 | 3 | 0 |

assume the contestant's initial choice is door 1
and that the contestant's strategy is to switch

let C be the door with the car
let G be the door that the host opens to reveal a goat
let F be the contestant's final choice
let W be whether the contestant wins

$$
\begin{tikzcd}
 & & C \arrow[dl] \arrow[dd] \\
 & G \arrow[dl] & \\
F \arrow[rr] & & W
\end{tikzcd}
$$

$$
P(w,f,g,c) = P(w \mid f,c) \cdot P(f \mid g) \cdot P(g \mid c) \cdot P(c)
$$

| $c$ | $g$ | $P(g \mid c)$ |
|---|---|---|
| 1 | 2 | $\boxed{?}$ |
| 1 | 3 | $\boxed{?}$ |
| 2 | 2 | $\boxed{?}$ |
| 2 | 3 | $\boxed{?}$ |
| 3 | 2 | $\boxed{?}$ |
| 3 | 3 | $\boxed{?}$ |

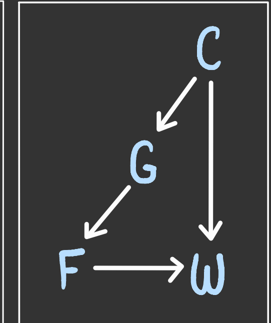
assume the contestant's initial choice is door 1
and that the contestant's strategy is to switch

let $C$ be the door with the car

let $G$ be the door that the host opens to reveal a goat

let $F$ be the contestant's final choice

let $W$ be whether the contestant wins




$$P(w, f, g, c) = P(w \mid f, c) \cdot P(f \mid g) \cdot P(g \mid c) \cdot P(c)$$

| $c$ | $g$ | $P(g \mid c)$ |
|---|---|---|
| 1 | 2 | 0.5 |
| 1 | 3 | 0.5 |
| 2 | 2 | 0 |
| 2 | 3 | 1 |
| 3 | 2 | 1 |
| 3 | 3 | 0 |

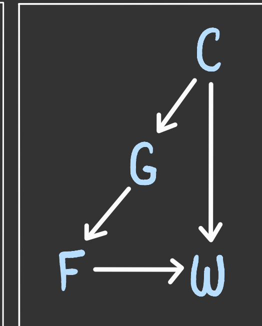
assume the contestant's initial choice is door 1
and that the contestant's strategy is to switch

let C be the door with the car

let G be the door that the host opens to reveal a goat

let F be the contestant's final choice

let W be whether the contestant wins




$$P(w,f,g,c) = P(w \mid f,c) \cdot P(f \mid g) \cdot P(g \mid c) \cdot P(c)$$

| C | P(c) |
|---|---|
| 1 | ? |
| 2 | ? |
| 3 | ? |

assume the contestant's initial choice is door 1
and that the contestant's strategy is to switch

let $C$ be the door with the car

let $G$ be the door that the host opens to reveal a goat

let $F$ be the contestant's final choice

let $W$ be whether the contestant wins

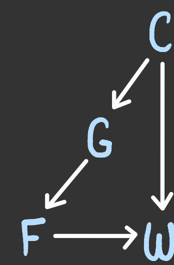
$$\begin{tikzcd} & & C \arrow[dl] \arrow[dd] \\ & G \arrow[dl] & \\ F \arrow[rr] & & W \end{tikzcd}$$

| $C$ | $P(c)$ |
|---|---|
| 1 | $0.\overline{33}$ |
| 2 | $0.\overline{33}$ |
| 3 | $0.\overline{33}$ |

$$P(w, f, g, c) = P(w \mid f, c) \cdot P(f \mid g) \cdot P(g \mid c) \cdot P(c)$$

assume the contestant's initial choice is door 1
and that the contestant's strategy is to switch

let C be the door with the car
let G be the door that the host opens to reveal a goat
let F be the contestant's final choice
let W be whether the contestant wins




$$P(w,f,g,c) = P(w \mid f,c) \cdot P(f \mid g) \cdot P(g \mid c) \cdot P(c)$$

| f | c | w | |
|---|---|---|---|
| 1 | 1 | yes | 1 |
| 1 | 1 | no | 0 |
| 1 | 2 | yes | 0 |
| 1 | 2 | no | 1 |
| 1 | 3 | yes | 0 |
| 1 | 3 | no | 1 |
| 2 | 1 | yes | 0 |
| 2 | 1 | no | 1 |
| 2 | 2 | yes | 1 |
| 2 | 2 | no | 0 |
| 2 | 3 | yes | 0 |
| 2 | 3 | no | 1 |
| 3 | 1 | yes | 0 |
| 3 | 1 | no | 1 |
| 3 | 2 | yes | 0 |
| 3 | 2 | no | 1 |
| 3 | 3 | yes | 1 |
| 3 | 3 | no | 0 |

| g | f | |
|---|---|---|
| 2 | 1 | 0 |
| 2 | 2 | 0 |
| 2 | 3 | 1 |
| 3 | 1 | 0 |
| 3 | 2 | 1 |
| 3 | 3 | 0 |

| c | g | |
|---|---|---|
| 1 | 2 | 0.5 |
| 1 | 3 | 0.5 |
| 2 | 2 | 0 |
| 2 | 3 | 1 |
| 3 | 2 | 1 |
| 3 | 3 | 0 |

| c | |
|---|---|
| 1 | 0.33 |
| 2 | 0.33 |
| 3 | 0.33 |

$$P(w, f, g, c) = P(w \mid f, c) \cdot P(f \mid g) \cdot P(g \mid c) \cdot P(c)$$

| c | |
|---|---|
| 1 | 0.33 |
| 2 | 0.33 |
| 3 | 0.33 |

| c | g | |
|---|---|---|
| 1 | 2 | 0.5 |
| 1 | 3 | 0.5 |
| 2 | 2 | 0 |
| 2 | 3 | 1 |
| 3 | 2 | 1 |
| 3 | 3 | 0 |


| g | f | |
|---|---|---|
| 2 | 1 | 0 |
| 2 | 2 | 0 |
| 2 | 3 | 1 |
| 3 | 1 | 0 |
| 3 | 2 | 1 |
| 3 | 3 | 0 |

| f | c | w | |
|---|---|---|---|
| 1 | 1 | yes | 1 |
| 1 | 1 | no | 0 |
| 1 | 2 | yes | 0 |
| 1 | 2 | no | 1 |
| 1 | 3 | yes | 0 |
| 1 | 3 | no | 1 |
| 2 | 1 | yes | 0 |
| 2 | 1 | no | 1 |
| 2 | 2 | yes | 1 |
| 2 | 2 | no | 0 |
| 2 | 3 | yes | 0 |
| 2 | 3 | no | 1 |
| 3 | 1 | yes | 0 |
| 3 | 1 | no | 1 |
| 3 | 2 | yes | 0 |
| 3 | 2 | no | 1 |
| 3 | 3 | yes | 1 |
| 3 | 3 | no | 0 |

assume the contestant's initial choice is door 1 and that the contestant's strategy is to switch

let C be the door with the car

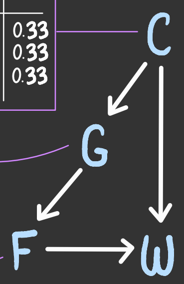
let G be the door that the host opens to reveal a goat

let F be the contestant's final choice

let W be whether the contestant wins

| c | |
|---|---|
| 1 | 0.33 |
| 2 | 0.33 |
| 3 | 0.33 |

| c | g | |
|---|---|---|
| 1 | 2 | 0.5 |
| 1 | 3 | 0.5 |
| 2 | 2 | 0 |
| 2 | 3 | 1 |
| 3 | 2 | 1 |
| 3 | 3 | 0 |




| g | f | |
|---|---|---|
| 2 | 1 | 0 |
| 2 | 2 | 0 |
| 2 | 3 | 1 |
| 3 | 1 | 0 |
| 3 | 2 | 1 |
| 3 | 3 | 0 |

| f | c | w | |
|---|---|---|---|
| 1 | 1 | yes | 1 |
| 1 | 1 | no | 0 |
| 1 | 2 | yes | 0 |
| 1 | 2 | no | 1 |
| 1 | 3 | yes | 0 |
| 1 | 3 | no | 1 |
| 2 | 1 | yes | 0 |
| 2 | 1 | no | 1 |
| 2 | 2 | yes | 1 |
| 2 | 2 | no | 0 |
| 2 | 3 | yes | 0 |
| 2 | 3 | no | 1 |
| 3 | 1 | yes | 0 |
| 3 | 1 | no | 1 |
| 3 | 2 | yes | 0 |
| 3 | 2 | no | 1 |
| 3 | 3 | yes | 1 |
| 3 | 3 | no | 0 |

# should the contestant switch?

how do we express this as a probability query?

your answer here

assume the contestant's initial choice is door 1
and that the contestant's strategy is to switch

let C be the door with the car

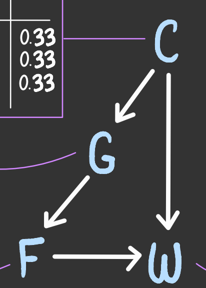
let G be the door that the host opens to reveal a goat

let F be the contestant's final choice

let W be whether the contestant wins

| c | |
|---|---|
| 1 | 0.33 |
| 2 | 0.33 |
| 3 | 0.33 |

| c | g | |
|---|---|---|
| 1 | 2 | 0.5 |
| 1 | 3 | 0.5 |
| 2 | 2 | 0 |
| 2 | 3 | 1 |
| 3 | 2 | 1 |
| 3 | 3 | 0 |




| g | f | |
|---|---|---|
| 2 | 1 | 0 |
| 2 | 2 | 0 |
| 2 | 3 | 1 |
| 3 | 1 | 0 |
| 3 | 2 | 1 |
| 3 | 3 | 0 |

| f | c | w | |
|---|---|---|---|
| 1 | 1 | yes | 1 |
| 1 | 1 | no | 0 |
| 1 | 2 | yes | 0 |
| 1 | 2 | no | 1 |
| 1 | 3 | yes | 0 |
| 1 | 3 | no | 1 |
| 2 | 1 | yes | 0 |
| 2 | 1 | no | 1 |
| 2 | 2 | yes | 1 |
| 2 | 2 | no | 0 |
| 2 | 3 | yes | 0 |
| 2 | 3 | no | 1 |
| 3 | 1 | yes | 0 |
| 3 | 1 | no | 1 |
| 3 | 2 | yes | 0 |
| 3 | 2 | no | 1 |
| 3 | 3 | yes | 1 |
| 3 | 3 | no | 0 |

# should the contestant

# switch?

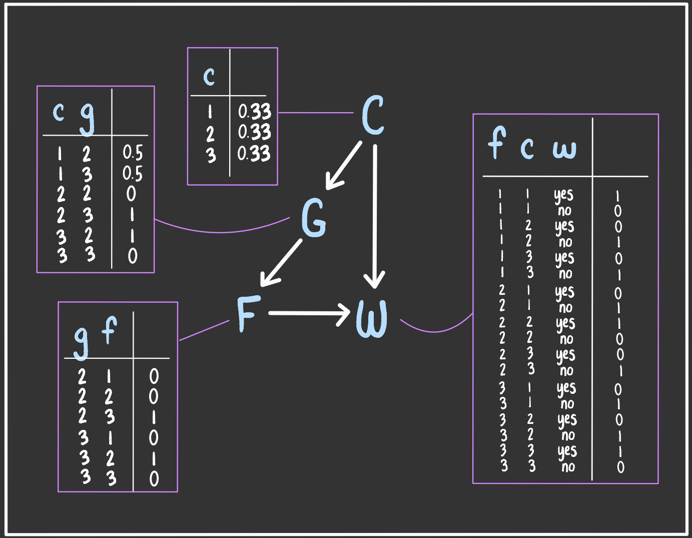
how do we express this as a probability query?

$$P(w)$$

$P(w) = ?$

marginal distribution over W

a representation of the joint distribution




| c | |
|---|---|
| 1 | 0.33 |
| 2 | 0.33 |
| 3 | 0.33 |

| c | g | |
|---|---|---|
| 1 | 2 | 0.5 |
| 1 | 3 | 0.5 |
| 2 | 2 | 0 |
| 2 | 3 | 1 |
| 3 | 2 | 1 |
| 3 | 3 | 0 |

| g | f | |
|---|---|---|
| 2 | 1 | 0 |
| 2 | 2 | 0 |
| 2 | 3 | 1 |
| 3 | 1 | 0 |
| 3 | 2 | 1 |
| 3 | 3 | 0 |

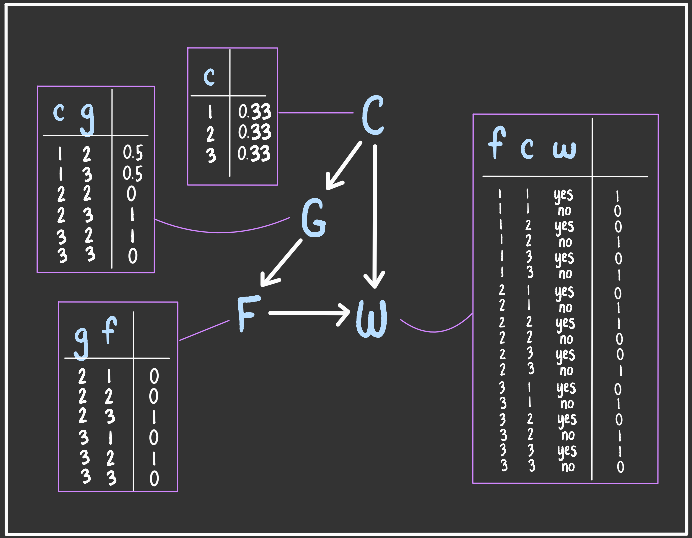
| f | c | w | |
|---|---|---|---|
| 1 | 1 | yes | 1 |
| 1 | 1 | no | 0 |
| 1 | 2 | yes | 0 |
| 1 | 2 | no | 1 |
| 1 | 3 | yes | 0 |
| 1 | 3 | no | 1 |
| 2 | 1 | yes | 0 |
| 2 | 1 | no | 1 |
| 2 | 2 | yes | 1 |
| 2 | 2 | no | 0 |
| 2 | 3 | yes | 0 |
| 2 | 3 | no | 1 |
| 3 | 1 | yes | 0 |
| 3 | 1 | no | 1 |
| 3 | 2 | yes | 0 |
| 3 | 2 | no | 1 |
| 3 | 3 | yes | 1 |
| 3 | 3 | no | 0 |

how do we express a marginal probability in terms of a joint probability?

$$P(w) = \sum_{c} \sum_{f} \sum_{g} P(w, f, g, c)$$

marginal distribution over W

a representation of the joint distribution




| c | |
|---|---|
| 1 | 0.33 |
| 2 | 0.33 |
| 3 | 0.33 |

| c | g | |
|---|---|---|
| 1 | 2 | 0.5 |
| 1 | 3 | 0.5 |
| 2 | 2 | 0 |
| 2 | 3 | 1 |
| 3 | 2 | 1 |
| 3 | 3 | 0 |

| g | f | |
|---|---|---|
| 2 | 1 | 0 |
| 2 | 2 | 0 |
| 2 | 3 | 1 |
| 3 | 1 | 0 |
| 3 | 2 | 1 |
| 3 | 3 | 0 |

| f | c | w | |
|---|---|---|---|
| 1 | 1 | yes | 1 |
| 1 | 1 | no | 0 |
| 1 | 2 | yes | 0 |
| 1 | 2 | no | 1 |
| 1 | 3 | yes | 0 |
| 1 | 3 | no | 1 |
| 2 | 1 | yes | 0 |
| 2 | 1 | no | 1 |
| 2 | 2 | yes | 1 |
| 2 | 2 | no | 0 |
| 2 | 3 | yes | 0 |
| 2 | 3 | no | 1 |
| 3 | 1 | yes | 0 |
| 3 | 1 | no | 1 |
| 3 | 2 | yes | 0 |
| 3 | 2 | no | 1 |
| 3 | 3 | yes | 1 |
| 3 | 3 | no | 0 |

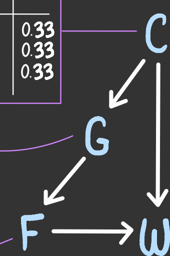
$$P(w) = \sum_c \sum_f \sum_g P(w, f, g, c)$$

$$= \sum_c \sum_f \sum_g P(w \mid f, c) P(f \mid g) P(g \mid c) P(c)$$

| c | |
|---|---|
| 1 | 0.33 |
| 2 | 0.33 |
| 3 | 0.33 |

| c | g | |
|---|---|---|
| 1 | 2 | 0.5 |
| 1 | 3 | 0.5 |
| 2 | 2 | 0 |
| 2 | 3 | 1 |
| 3 | 2 | 1 |
| 3 | 3 | 0 |

| g | f | |
|---|---|---|
| 2 | 1 | 0 |
| 2 | 2 | 0 |
| 2 | 3 | 1 |
| 3 | 1 | 0 |
| 3 | 2 | 1 |
| 3 | 3 | 0 |




| f | c | w | |
|---|---|---|---|
| 1 | 1 | yes | 1 |
| 1 | 1 | no | 0 |
| 1 | 2 | yes | 0 |
| 1 | 2 | no | 1 |
| 1 | 3 | yes | 0 |
| 1 | 3 | no | 1 |
| 2 | 1 | yes | 0 |
| 2 | 1 | no | 1 |
| 2 | 2 | yes | 1 |
| 2 | 2 | no | 0 |
| 2 | 3 | yes | 0 |
| 2 | 3 | no | 1 |
| 3 | 1 | yes | 0 |
| 3 | 1 | no | 1 |
| 3 | 2 | yes | 0 |
| 3 | 2 | no | 1 |
| 3 | 3 | yes | 1 |
| 3 | 3 | no | 0 |

$$P(w) = \sum_c \sum_f \sum_g P(w, f, g, c)$$

$$= \sum_c \sum_f \sum_g \underbrace{P(w \mid f, c)}_{h_1(f,c,w)} \underbrace{P(f \mid g)}_{h_2(g,f)} \underbrace{P(g \mid c)}_{h_3(c,g)} \underbrace{P(c)}_{h_4(c)}$$

these are all just **multivariable functions**

$$\begin{tikzcd} C \arrow[d] \arrow[ddr] & \\ G \arrow[d] & \\ F \arrow[r] & W \end{tikzcd}$$

$h_4$

| c | |
|---|---|
| 1 | 0.33 |
| 2 | 0.33 |
| 3 | 0.33 |

$h_3$

| c | g | |
|---|---|---|
| 1 | 2 | 0.5 |
| 1 | 3 | 0.5 |
| 2 | 2 | 0 |
| 2 | 3 | 1 |
| 3 | 2 | 1 |
| 3 | 3 | 0 |

$h_2$

| g | f | |
|---|---|---|
| 2 | 1 | 0 |
| 2 | 2 | 0 |
| 2 | 3 | 1 |
| 3 | 1 | 0 |
| 3 | 2 | 1 |
| 3 | 3 | 0 |

$h_1$

| f | c | w | |
|---|---|---|---|
| 1 | 1 | yes | 1 |
| 1 | 1 | no | 0 |
| 1 | 2 | yes | 0 |
| 1 | 2 | no | 1 |
| 1 | 3 | yes | 0 |
| 1 | 3 | no | 1 |
| 2 | 1 | yes | 0 |
| 2 | 1 | no | 1 |
| 2 | 2 | yes | 1 |
| 2 | 2 | no | 0 |
| 2 | 3 | yes | 0 |
| 2 | 3 | no | 1 |
| 3 | 1 | yes | 0 |
| 3 | 1 | no | 1 |
| 3 | 2 | yes | 0 |
| 3 | 2 | no | 1 |
| 3 | 3 | yes | 1 |
| 3 | 3 | no | 0 |

# these are all just multivariable functions

$P(g \mid c)$

$h_3(c, g)$

$h_3$

| c | g | |
|---|---|---|
| 1 | 2 | 0.5 |
| 1 | 3 | 0.5 |
| 2 | 2 | 0 |
| 2 | 3 | 1 |
| 3 | 2 | 1 |
| 3 | 3 | 0 |

| $c$ | $g$ | $P(g \mid c)$ |
|---|---|---|
| 1 | 2 | 0.5 |
| 1 | 3 | 0.5 |
| 2 | 2 | 0 |
| 2 | 3 | 1 |
| 3 | 2 | 1 |
| 3 | 3 | 0 |

these are all just multivariable **functions**

| $c$ | $g$ | $h_3(c,g)$ $\cancel{P(g \mid c)}$ |
|---|---|---|
| 1 | 2 | 0.5 |
| 1 | 3 | 0.5 |
| 2 | 2 | 0 |
| 2 | 3 | 1 |
| 3 | 2 | 1 |
| 3 | 3 | 0 |

these are all just multivariable functions so computing a probability boils down to computing a sum of products

$$P(w) = \sum_c \sum_f \sum_g P(w \mid f, c)\, P(f \mid g)\, P(g \mid c)\, P(c)$$

so computing a probability boils down to computing a sum of products

$$P(w) = \sum_{c} \sum_{f} \sum_{g} \overset{h_1(f,c,w)}{\cancel{P(w \mid f,c)}}\ \overset{h_2(g,f)}{\cancel{P(f \mid g)}}\ \overset{h_3(c,g)}{\cancel{P(g \mid c)}}\ \overset{h_4(c)}{\cancel{P(c)}}$$

$$P(w) = \sum_{c} \sum_{f} \sum_{g} h_1(f, c, w) h_2(g, f) h_3(c, g) h_4(c)$$

$$P(w) = \sum_{c} \sum_{f} \sum_{g} \underline{h_1(f, c, w)}\, h_2(g, f)\, h_3(c, g)\, \underline{h_4(c)}$$

not all of the functions

involve g

$$P(w) = \sum_c \sum_f \sum_g h_1(f,c,w)\,h_2(g,f)\,h_3(c,g)\,h_4(c)$$

$$= \sum_c \sum_f h_1(f,c,w)\,h_4(c) \sum_g h_2(g,f)\,h_3(c,g)$$

$$\boxed{\sum_x k f(x) = k \sum_x f(x)}$$

e.g. $\sum_{x=1}^{3} 10x^2 = 10\cdot 1^2 + 10\cdot 2^2 + 10\cdot 3^2 = 10(1^2 + 2^2 + 3^2) = 10 \sum_{x=1}^{3} x^2$

$$P(w) = \sum_c \sum_f \sum_g h_1(f,c,w)\,h_2(g,f)\,h_3(c,g)\,h_4(c)$$

$$= \sum_c \sum_f h_1(f,c,w)\,h_4(c) \boxed{\sum_g h_2(g,f)\,h_3(c,g)}$$

this subexpression is a

**function**

of which variables?

your answer

**here**

$$P(w) = \sum_c \sum_f \sum_g h_1(f,c,w)\,h_2(g,f)\,h_3(c,g)\,h_4(c)$$

$$= \sum_c \sum_f h_1(f,c,w)\,h_4(c) \boxed{\sum_g h_2(g,f)\,h_3(c,g)}$$

this subexpression is a **function** of which variables?

f, c

```python
def summation(f, c):
    result = 0
    for g in [2, 3]:
        result += h2(g, f) + h3(c, g)
```

$$P(w) = \sum_c \sum_f \sum_g h_1(f,c,w)\, h_2(g,f)\, h_3(c,g)\, h_4(c)$$

$$= \sum_c \sum_f h_1(f,c,w)\, h_4(c) \sum_g h_2(g,f)\, h_3(c,g)$$

$$= \sum_c \sum_f h_1(f,c,w)\, h_4(c)\; h_5(f,c) \qquad \text{let } h_5(f,c) = \sum_g h_2(g,f)\, h_3(c,g)$$

```
def h5(f, c):
    result = 0
    for g in [2, 3]:
        result += h2(g, f) + h3(c, g)
```

$$P(w) = \sum_c \sum_f \sum_g h_1(f,c,w)\, h_2(g,f)\, h_3(c,g)\, h_4(c)$$

$$= \sum_c \sum_f h_1(f,c,w)\, h_4(c) \sum_g h_2(g,f)\, h_3(c,g)$$

$$= \sum_c \sum_f h_1(f,c,w)\, h_4(c)\; h_5(f,c) \longleftarrow \text{let } h_5(f,c) = \sum_g h_2(g,f)\, h_3(c,g)$$

```
h5(1, 1)
h5(1, 2)
h5(1, 3)
h5(2, 1)
h5(2, 2)
h5(2, 3)
h5(3, 1)
h5(3, 2)
h5(3, 3)
```

compute and store all 9 values of this function $\longrightarrow$

| F | C | $h_5(f,c)$ |
|---|---|---|
| 1 | 1 | 0 |
| 1 | 2 | 0 |
| 1 | 3 | 0 |
| 2 | 1 | .5 |
| 2 | 2 | 1 |
| 2 | 3 | 0 |
| 3 | 1 | .5 |
| 3 | 2 | 0 |
| 3 | 3 | 1 |

$$P(w) = \sum_c \sum_f \sum_g h_1(f,c,w)\, h_2(g,f)\, h_3(c,g)\, h_4(c)$$

$$= \sum_c \sum_f h_1(f,c,w)\, h_4(c) \sum_g h_2(g,f)\, h_3(c,g)$$

$$= \sum_c \sum_f h_1(f,c,w)\, h_4(c)\;\; h_5(f,c) \longleftarrow \text{let } \underbrace{h_5(f,c)}_{9 \text{ computed values}} = \sum_g h_2(g,f)\, h_3(c,g)$$

$$P(w) = \sum_c \sum_f \sum_g h_1(f,c,w)\, h_2(g,f)\, h_3(c,g)\, h_4(c)$$

$$= \sum_c \sum_f h_1(f,c,w)\, h_4(c) \sum_g h_2(g,f)\, h_3(c,g)$$

$$= \sum_c \sum_f h_1(f,c,w)\, h_4(c)\; h_5(f,c) \longleftarrow \text{let } \underbrace{h_5(f,c)}_{9 \text{ computed values}} = \sum_g h_2(g,f)\, h_3(c,g)$$

$$= \sum_c h_4(c) \boxed{\sum_f h_1(f,c,w)\, h_5(f,c)}$$

this subexpression is a

# function

of which variables?

$$P(w) = \sum_c \sum_f \sum_g h_1(f,c,w)\,h_2(g,f)\,h_3(c,g)\,h_4(c)$$

$$= \sum_c \sum_f h_1(f,c,w)\,h_4(c) \sum_g h_2(g,f)\,h_3(c,g)$$

$$= \sum_c \sum_f h_1(f,c,w)\,h_4(c)\;h_5(f,c) \longleftarrow \text{let } \underbrace{h_5(f,c)}_{9 \text{ computed values}} = \sum_g h_2(g,f)\,h_3(c,g)$$

$$= \sum_c h_4(c) \sum_f h_1(f,c,w)\;h_5(f,c)$$

$$= \sum_c h_4(c)\,h_6(w,c) \longleftarrow \text{let } h_6(w,c) = \sum_f h_1(f,c,w)\,h_5(f,c)$$

compute and store all 6 values of this function →

| W | C | $h_6(w,s,c)$ |
|---|---|---|
| yes | 1 | 0 |
| yes | 2 | 1 |
| yes | 3 | 1 |
| no | 1 | 1 |
| no | 2 | 0 |
| no | 3 | 0 |

$$P(w) = \sum_c \sum_f \sum_g h_1(f,c,w)\,h_2(g,f)\,h_3(c,g)\,h_4(c)$$

$$= \sum_c \sum_f h_1(f,c,w)\,h_4(c) \sum_g h_2(g,f)\,h_3(c,g)$$

$$= \sum_c \sum_f h_1(f,c,w)\,h_4(c)\;h_5(f,c) \longleftarrow \text{let } \underbrace{h_5(f,c)}_{9 \text{ computed values}} = \sum_g h_2(g,f)\,h_3(c,g)$$

$$= \sum_c h_4(c) \sum_f h_1(f,c,w)\;h_5(f,c)$$

$$= \sum_c h_4(c)\,h_6(w,c) \longleftarrow \text{let } \underbrace{h_6(w,c)}_{6 \text{ computed values}} = \sum_f h_1(f,c,w)\,h_5(f,c)$$

$$P(w) = \sum_c \sum_f \sum_g h_1(f,c,w)\, h_2(g,f)\, h_3(c,g)\, h_4(c)$$

$$= \sum_c \sum_f h_1(f,c,w)\, h_4(c) \sum_g h_2(g,f)\, h_3(c,g)$$

$$= \sum_c \sum_f h_1(f,c,w)\, h_4(c)\, h_5(f,c) \longleftarrow \text{let } \underbrace{h_5(f,c)}_{9 \text{ computed values}} = \sum_g h_2(g,f)\, h_3(c,g)$$

$$= \sum_c h_4(c) \sum_f h_1(f,c,w)\, h_5(f,c)$$

$$\boxed{= \sum_c h_4(c)\, h_6(w,c)} \longleftarrow \text{let } \underbrace{h_6(w,c)}_{6 \text{ computed values}} = \sum_f h_1(f,c,w)\, h_5(f,c)$$

this subexpression is a **function** of which variables?

$$
\begin{aligned}
P(w) &= \sum_c \sum_f \sum_g h_1(f,c,w)\, h_2(g,f)\, h_3(c,g)\, h_4(c) \\
&= \sum_c \sum_f h_1(f,c,w)\, h_4(c) \sum_g h_2(g,f)\, h_3(c,g) \\
&= \sum_c \sum_f h_1(f,c,w)\, h_4(c)\, h_5(f,c) \\
&= \sum_c h_4(c) \sum_f h_1(f,c,w)\, h_5(f,c) \\
&= \sum_c h_4(c)\, h_6(w,c) \\
&= h_7(w)
\end{aligned}
$$

let $\underbrace{h_5(f,c)}_{9 \text{ computed values}} = \sum_g h_2(g,f)\, h_3(c,g)$

let $\underbrace{h_6(w,c)}_{6 \text{ computed values}} = \sum_f h_1(f,c,w)\, h_5(f,c)$

let $\underbrace{h_7(w)}_{2 \text{ computed values}} = \sum_c h_4(c)\, h_6(w,c)$

$$P(w) = \sum_c \sum_f \sum_g h_1(f,c,w)\,h_2(g,f)\,h_3(c,g)\,h_4(c)$$

$$= \sum_c \sum_f h_1(f,c,w)\,h_4(c) \sum_g h_2(g,f)\,h_3(c,g)$$

$$= \sum_c \sum_f h_1(f,c,w)\,h_4(c)\,h_5(f,c)$$

let $\underbrace{h_5(f,c)}_{9 \text{ computed values}} = \sum_g h_2(g,f)\,h_3(c,g)$

$$= \sum_c h_4(c) \sum_f h_1(f,c,w)\,h_5(f,c)$$

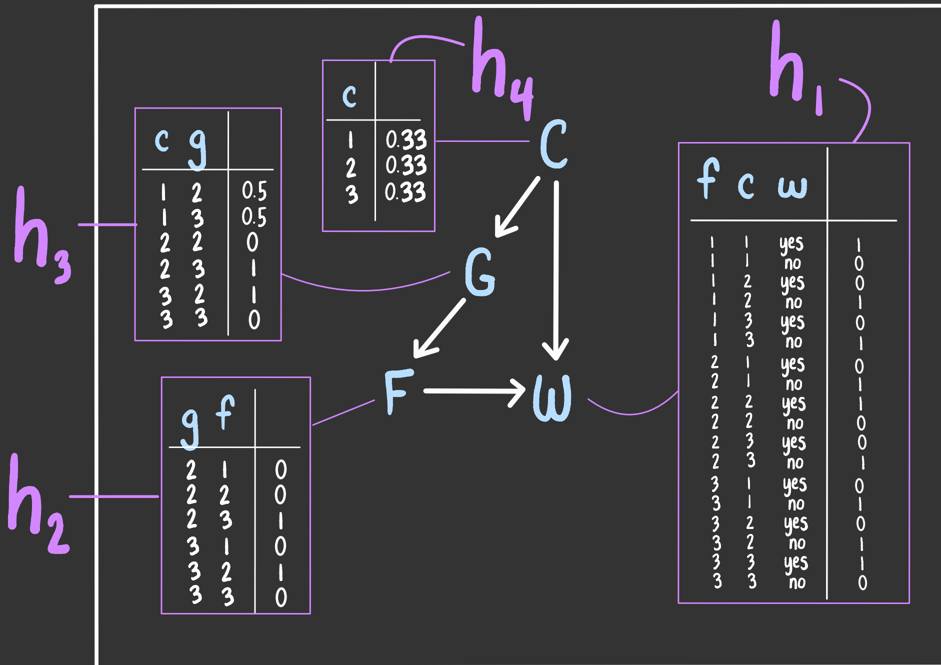
$$= \sum_c h_4(c)\,h_6(w,c)$$

let $\underbrace{h_6(w,c)}_{6 \text{ computed values}} = \sum_f h_1(f,c,w)\,h_5(f,c)$

$$= h_7(w)$$

let $\underbrace{h_7(w)}_{2 \text{ computed values}} = \sum_c h_4(c)\,h_6(w,c)$

we've computed the marginal!




$h_4$:

| c | |
|---|---|
| 1 | 0.33 |
| 2 | 0.33 |
| 3 | 0.33 |

$h_3$:

| c | g | |
|---|---|---|
| 1 | 2 | 0.5 |
| 1 | 3 | 0.5 |
| 2 | 2 | 0 |
| 2 | 3 | 1 |
| 3 | 2 | 1 |
| 3 | 3 | 0 |

$h_2$:

| g | f | |
|---|---|---|
| 2 | 1 | 0 |
| 2 | 2 | 0 |
| 2 | 3 | 1 |
| 3 | 1 | 0 |
| 3 | 2 | 1 |
| 3 | 3 | 0 |

$h_1$:

| f | c | w | |
|---|---|---|---|
| 1 | 1 | yes | 1 |
| 1 | 1 | no | 0 |
| 1 | 2 | yes | 0 |
| 1 | 2 | no | 1 |
| 1 | 3 | yes | 0 |
| 1 | 3 | no | 1 |
| 2 | 1 | yes | 0 |
| 2 | 1 | no | 1 |
| 2 | 2 | yes | 1 |
| 2 | 2 | no | 0 |
| 2 | 3 | yes | 0 |
| 2 | 3 | no | 1 |
| 3 | 1 | yes | 0 |
| 3 | 1 | no | 1 |
| 3 | 2 | yes | 0 |
| 3 | 2 | no | 1 |
| 3 | 3 | yes | 1 |
| 3 | 3 | no | 0 |

compute these

let $\underbrace{h_5(f,c)}_{9 \text{ computed values}} = \sum_g h_2(g,f)\,h_3(c,g)$

let $\underbrace{h_6(w,c)}_{6 \text{ computed values}} = \sum_f h_1(f,c,w)\,h_5(f,c)$

let $\underbrace{h_7(w)}_{2 \text{ computed values}} = \sum_c h_4(c)\,h_6(w,c)$

| f | c | w | $h_1$ |
|---|---|---|---|
| 1 | 1 | yes | 1 |
| 1 | 1 | no | 0 |
| 1 | 2 | yes | 0 |
| 1 | 2 | no | 1 |
| 1 | 3 | yes | 0 |
| 1 | 3 | no | 1 |
| 2 | 1 | yes | 0 |
| 2 | 1 | no | 1 |
| 2 | 2 | yes | 1 |
| 2 | 2 | no | 0 |
| 2 | 3 | yes | 0 |
| 2 | 3 | no | 1 |
| 3 | 1 | yes | 0 |
| 3 | 1 | no | 1 |
| 3 | 2 | yes | 0 |
| 3 | 2 | no | 1 |
| 3 | 3 | yes | 1 |
| 3 | 3 | no | 0 |

| g | f | $h_2$ |
|---|---|---|
| 2 | 1 | 0 |
| 2 | 2 | 0 |
| 2 | 3 | 1 |
| 3 | 1 | 0 |
| 3 | 2 | 1 |
| 3 | 3 | 0 |

| c | g | $h_3$ |
|---|---|---|
| 1 | 2 | 0.5 |
| 1 | 3 | 0.5 |
| 2 | 2 | 0 |
| 2 | 3 | 1 |
| 3 | 2 | 1 |
| 3 | 3 | 0 |

| c | $h_4$ |
|---|---|
| 1 | 0.33 |
| 2 | 0.33 |
| 3 | 0.33 |

compute these

let $\underbrace{h_5(f,c)}_{9 \text{ computed values}} = \sum_g h_2(g,f)\,h_3(c,g)$

let $\underbrace{h_6(w,c)}_{6 \text{ computed values}} = \sum_f h_1(f,c,w)\,h_5(f,c)$

let $\underbrace{h_7(w)}_{2 \text{ computed values}} = \sum_c h_4(c)\,h_6(w,c)$

to compute $P(w) = h_7(w)$

compute $h_5(f, c)$

| F | C | $h_5(f, c)$ |
|---|---|---|
| 1 | 1 | ? |
| 1 | 2 | ? |
| 1 | 3 | ? |
| 2 | 1 | ? |
| 2 | 2 | ? |
| 2 | 3 | ? |
| 3 | 1 | ? |
| 3 | 2 | ? |
| 3 | 3 | ? |

compute $h_6(w, c)$

| W | C | $h_6(w, c)$ |
|---|---|---|
| yes | 1 | ? |
| yes | 2 | ? |
| yes | 3 | ? |
| no | 1 | ? |
| no | 2 | ? |
| no | 3 | ? |

compute $h_7(w)$

| W | $h_7(w)$ |
|---|---|
| yes | ? |
| no | ? |

to compute $P(w) = h_7(w)$

compute $h_5(f, c)$

| F | C | $h_5(f, c)$ |
|---|---|---|
| 1 | 1 | 0 |
| 1 | 2 | 0 |
| 1 | 3 | 0 |
| 2 | 1 | .5 |
| 2 | 2 | 1 |
| 2 | 3 | 0 |
| 3 | 1 | .5 |
| 3 | 2 | 0 |
| 3 | 3 | 1 |

compute $h_6(w, c)$

| W | C | $h_6(w, c)$ |
|---|---|---|
| yes | 1 | 0 |
| yes | 2 | 1 |
| yes | 3 | 1 |
| no | 1 | 1 |
| no | 2 | 0 |
| no | 3 | 0 |

compute $h_7(w)$

| W | $h_7(w)$ |
|---|---|
| yes | $\frac{2}{3}$ |
| no | $\frac{1}{3}$ |

to compute $P(w) = h_7(w)$

compute $h_5(f, c)$

| F | C | $h_5(f,c)$ |
|---|---|---|
| 1 | 1 | 0 |
| 1 | 2 | 0 |
| 1 | 3 | 0 |
| 2 | 1 | .5 |
| 2 | 2 | 1 |
| 2 | 3 | 0 |
| 3 | 1 | .5 |
| 3 | 2 | 0 |
| 3 | 3 | 1 |

compute $h_6(w, c)$

| W | C | $h_6(w,c)$ |
|---|---|---|
| yes | 1 | 0 |
| yes | 2 | 1 |
| yes | 3 | 1 |
| no | 1 | 1 |
| no | 2 | 0 |
| no | 3 | 0 |

compute $h_7(w)$

| W | $h_7(w)$ |
|---|---|
| yes | $\frac{2}{3}$ |
| no | $\frac{1}{3}$ |

you should switch!