

hidden markov models

CSCI
373

let's travel back to the
first days of fall
semester when we
were reconnecting with
all our old friends...



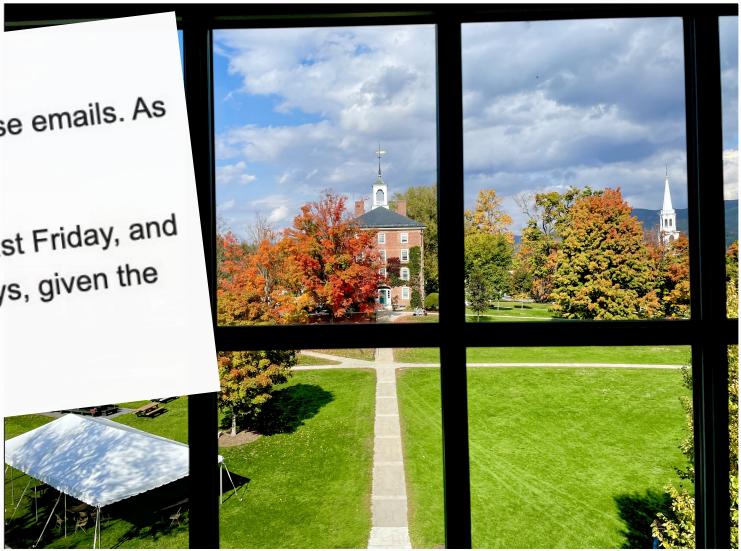
let's travel back to 11

Case counts

Starting now, we'll share case counts with you whenever I send these emails. As of today we have:

- **110 current student positives.** This is down from 165 last Friday, and the total is expected to decline further in the next few days, given the number of people almost done with isolation.
- **31 current faculty/staff positives.**

all our old friends...



like covid-19

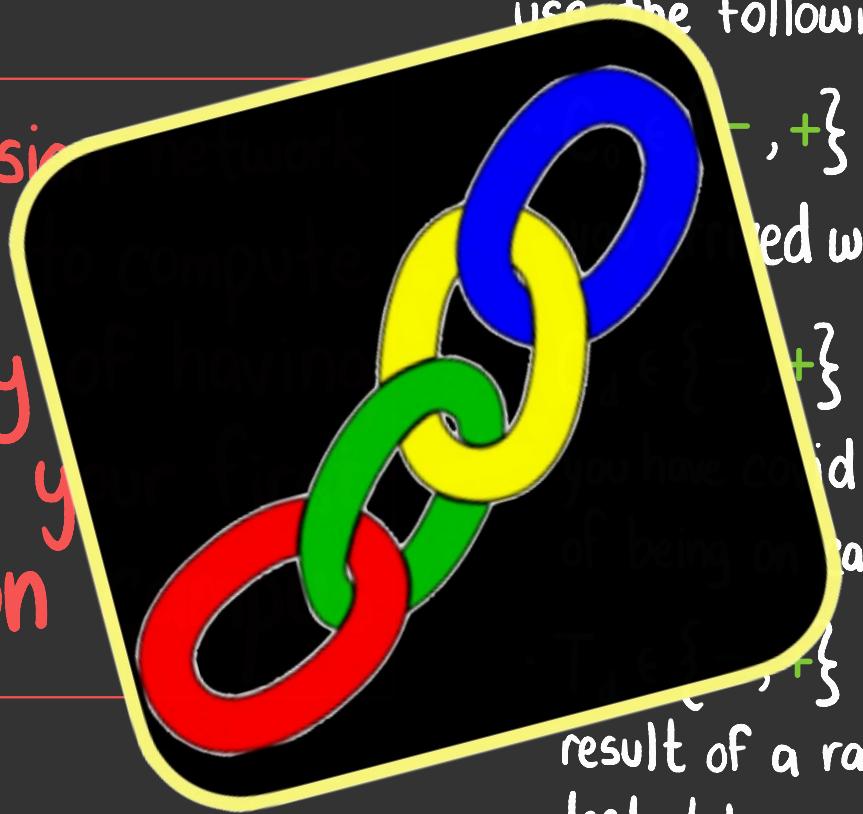
design a bayesian network
that allows us to compute
the probability of having
covid during your first
five days on campus

use the following variables:

- $C_0 \in \{-, +\}$ is whether you arrived with covid
- $C_d \in \{-, +\}$ is whether you have covid on day d of being on campus
- $T_d \in \{-, +\}$ is the result of a rapid antigen test taken on day d

design a bayesian
that allows us
the probability
covid during y
five days on

use the following variables:

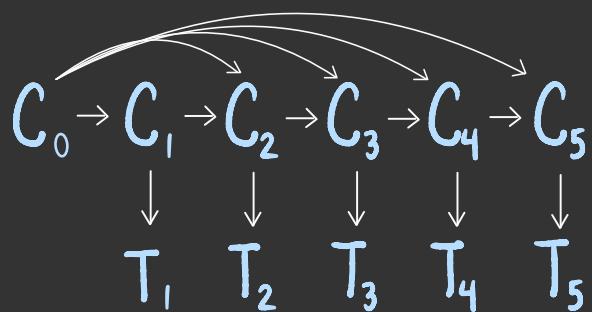


$\{ \cdot, + \}$ is whether
you have covid
 $\{ \cdot \}$ is whether
you have covid on day d
 $\{ \cdot, r \}$ is the
result of a rapid antigen
test taken on day d

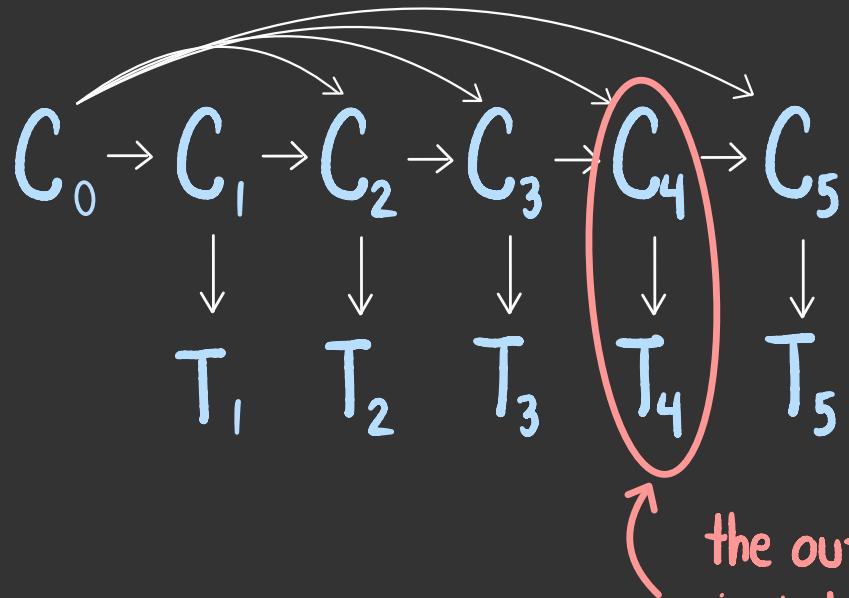

$$P(c_0, c_1, c_2, c_3, c_4, c_5, t_1, t_2, t_3, t_4, t_5) = P(t_5 | t_4, t_3, t_2, t_1, c_5, c_4, c_3, c_2, c_1, c_0) \\ \cdot P(t_4 | t_3, t_2, t_1, c_5, c_4, c_3, c_2, c_1, c_0) \\ \cdot P(t_3 | t_2, t_1, c_5, c_4, c_3, c_2, c_1, c_0) \\ \cdot P(t_2 | t_1, c_5, c_4, c_3, c_2, c_1, c_0) \\ \cdot P(t_1 | c_5, c_4, c_3, c_2, c_1, c_0) \\ \cdot P(c_5 | c_4, c_3, c_2, c_1, c_0) \\ \cdot P(c_4 | c_3, c_2, c_1, c_0) \\ \cdot P(c_3 | c_2, c_1, c_0) \\ \cdot P(c_2 | c_1, c_0) \\ \cdot P(c_1 | c_0) \\ \cdot P(c_0)$$


$$P(c_0, c_1, c_2, c_3, c_4, c_5, t_1, t_2, t_3, t_4, t_5) = P(t_5 | t_4, t_3, t_2, t_1, c_5, c_4, c_3, c_2, c_1, c_0) \\ \cdot P(t_4 | t_3, t_2, t_1, c_5, c_4, c_3, c_2, c_1, c_0) \\ \cdot P(t_3 | t_2, t_1, c_5, c_4, c_3, c_2, c_1, c_0) \\ \cdot P(t_2 | t_1, c_5, c_4, c_3, c_2, c_1, c_0) \\ \cdot P(t_1 | c_5, c_4, c_3, c_2, c_1, c_0) \\ \cdot P(c_5 | c_4, c_3, c_2, c_1, c_0) \\ \cdot P(c_4 | c_3, c_2, c_1, c_0) \\ \cdot P(c_3 | c_2, c_1, c_0) \\ \cdot P(c_2 | c_1, c_0) \\ \cdot P(c_1 | c_0) \\ \cdot P(c_0)$$

$$P(c_0, c_1, c_2, c_3, c_4, c_5, t_1, t_2, t_3, t_4, t_5)$$



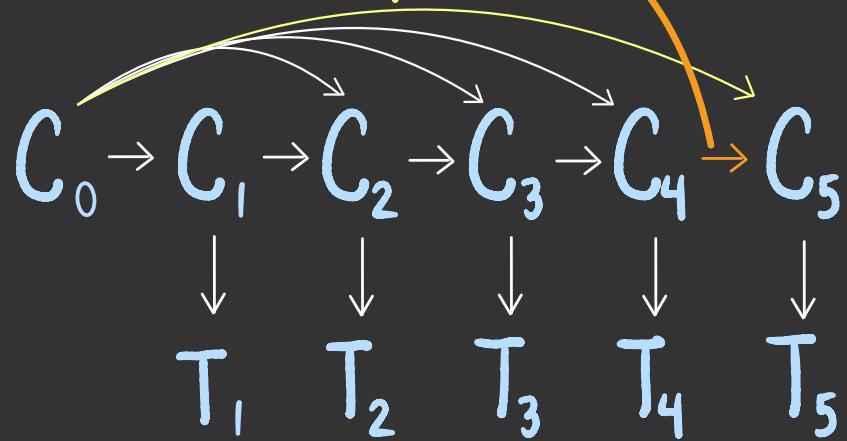
$$\begin{aligned}
 &= P(t_5 | t_4, t_3, t_2, t_1, c_5, c_4, c_3, c_2, c_1, c_0) \\
 &\cdot P(t_4 | t_3, t_2, t_1, c_5, c_4, c_3, c_2, c_1, c_0) \\
 &\cdot P(t_3 | t_2, t_1, c_5, c_4, c_3, c_2, c_1, c_0) \\
 &\cdot P(t_2 | t_1, c_5, c_4, c_3, c_2, c_1, c_0) \\
 &\cdot P(t_1 | c_5, c_4, c_3, c_2, c_1, c_0) \\
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 &\cdot P(c_2 | c_1, c_0) \\
 &\cdot P(c_1 | c_0) \\
 &\cdot P(c_0)
 \end{aligned}$$



the outcome of a test
is independent of the
other variables, given
our current covid status

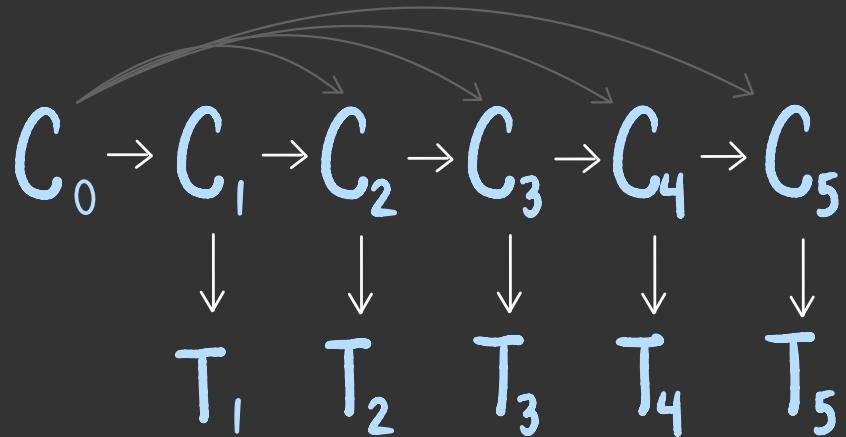
the likelihood of having covid on day d depends on

- whether we already have covid
- how long we've had it

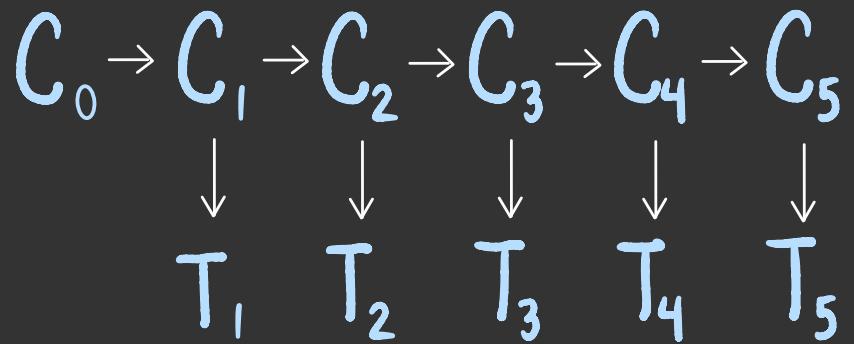


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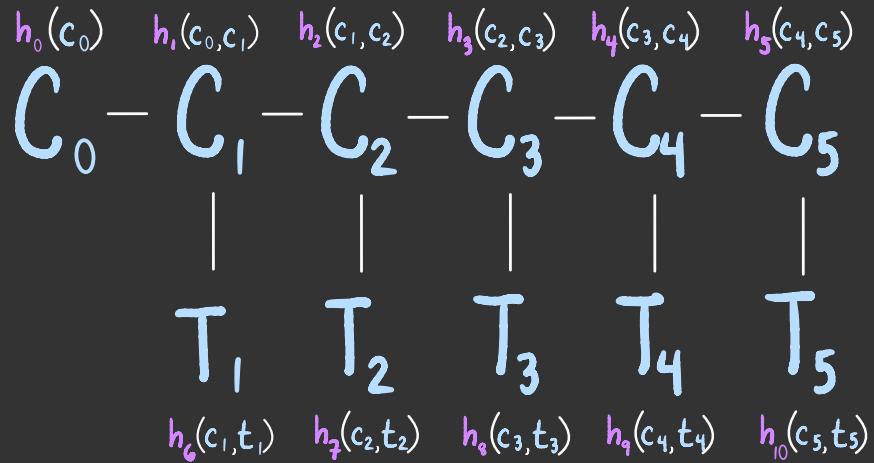


why might we make such a simplification?



compute the width of elimination order
 $C_0, T_1, C_1, T_2, C_2, T_3, C_3, T_4, C_4, T_5, C_5$

I. create an undirected graph where two variables are adjacent if they appear in a **common factor**



compute the width of elimination order

$C_0, T_1, C_1, T_2, C_2, T_3, C_3, T_4, C_4, T_5, C_5$

2. eliminate each variable by connecting all its neighbors

$$\begin{array}{ccccccc} h_0(c_0) & h_1(c_0, c_1) & h_2(c_1, c_2) & h_3(c_2, c_3) & h_4(c_3, c_4) & h_5(c_4, c_5) \\ C_0 - C_1 - C_2 - C_3 - C_4 - C_5 \\ | & | & | & | & | \\ T_1 & T_2 & T_3 & T_4 & T_5 \\ h_6(c_1, t_1) & h_7(c_2, t_2) & h_8(c_3, t_3) & h_9(c_4, t_4) & h_{10}(c_5, t_5) \end{array}$$

max clique
2

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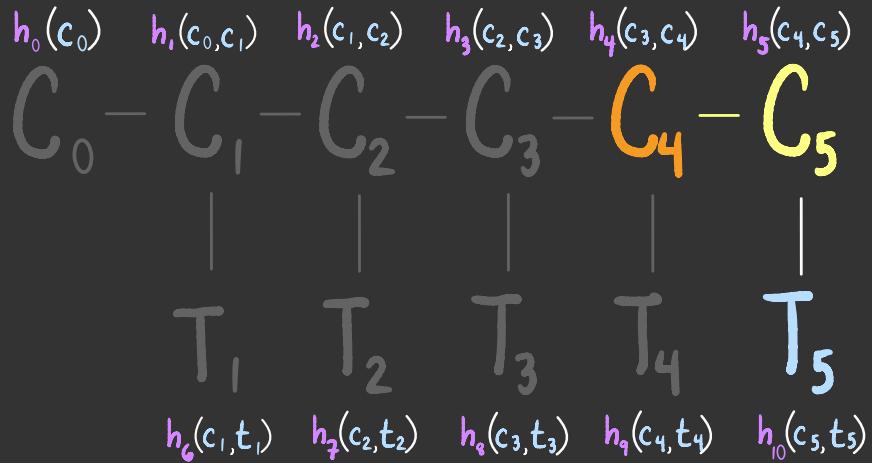
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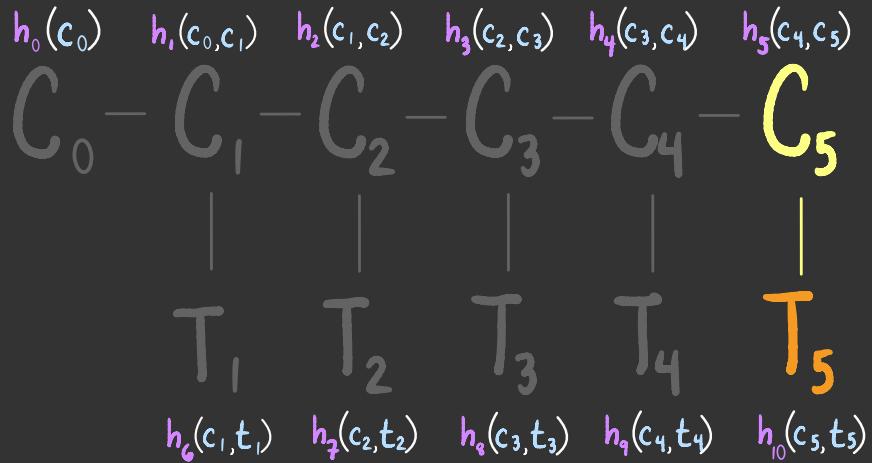


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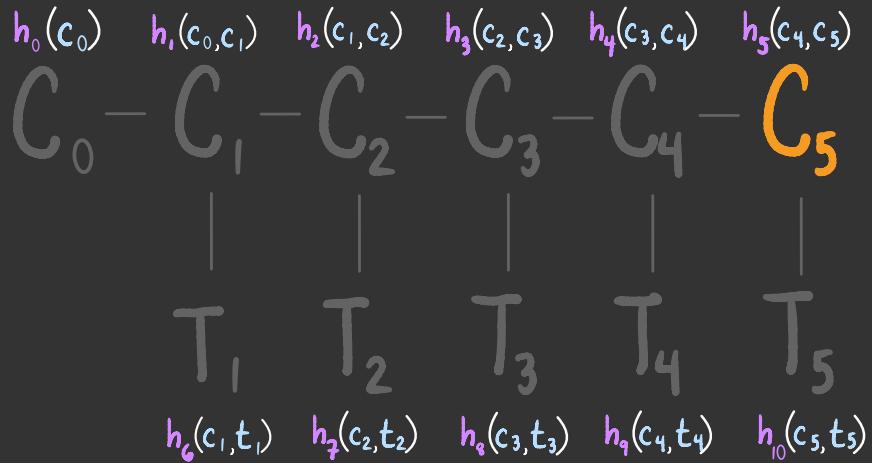


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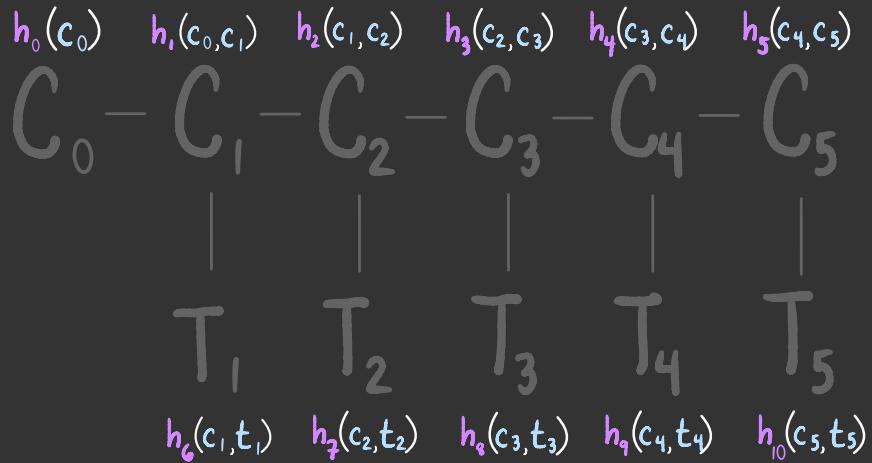


max clique
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2. eliminate each variable by connecting all its neighbors



width
2

compute the width of elimination order

$C_0, T_1, C_1, T_2, C_2, T_3, C_3, T_4, C_4, T_5, C_5$

recall:

the worst-case runtime of variable elimination is $O(nd^\omega)$, where

- n is the number of bayesian network variables
- d is the size of the largest variable domain
- ω is the width of the elimination order

for:

$$C_0 \rightarrow C_1 \rightarrow C_2 \rightarrow C_3 \rightarrow C_4 \rightarrow C_5$$
$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$
$$T_1 \quad T_2 \quad T_3 \quad T_4 \quad T_5$$

$$n = \boxed{?}$$

$$d = \boxed{?}$$

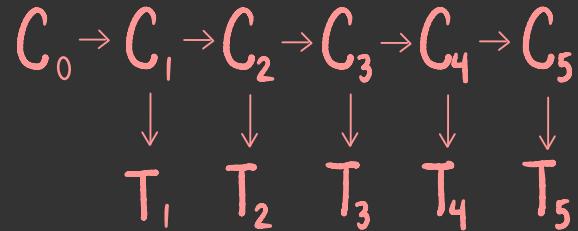
$$\omega = \boxed{?}$$

recall:

the worst-case runtime of variable elimination is $O(nd^\omega)$, where

- n is the number of bayesian network variables
- d is the size of the largest variable domain
- ω is the width of the elimination order

for:



$$n = 1 + 2 \cdot \frac{\text{number of days}}{m}$$

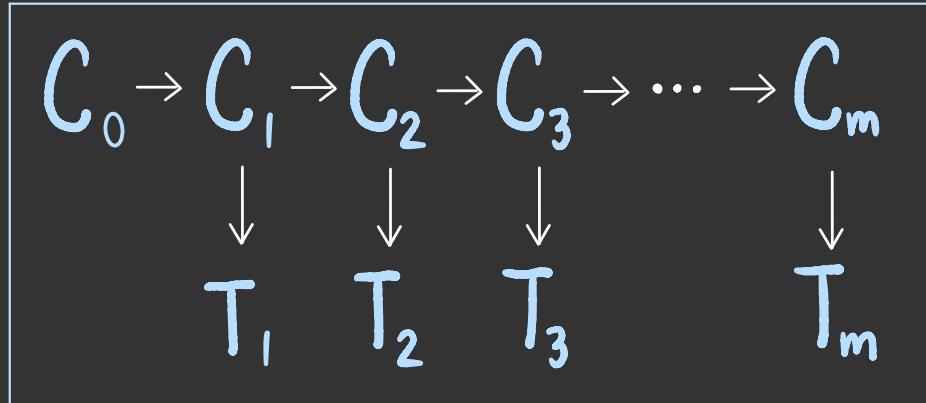
call this

$$d = 2$$

$$\omega = 2$$

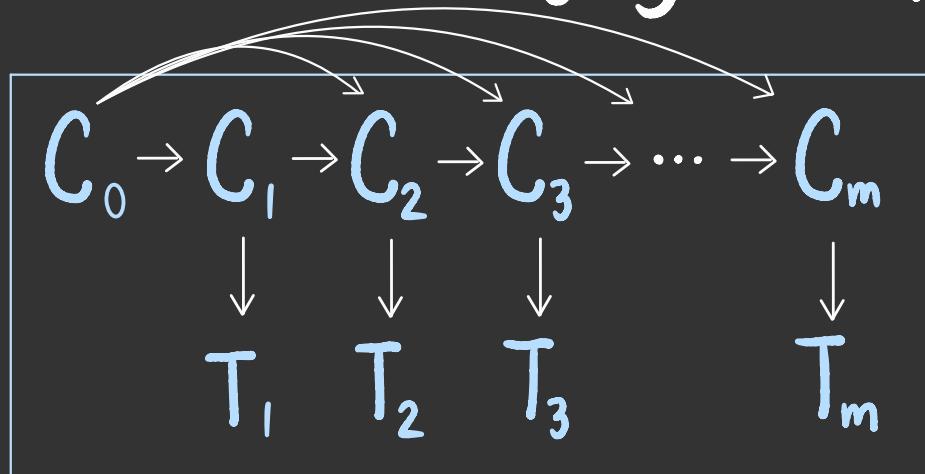
so the worst-case runtime is $O((1+2m)2^2) = O(4+8m) = O(m)$

so we can run variable elimination on any bayesian network with this graph structure

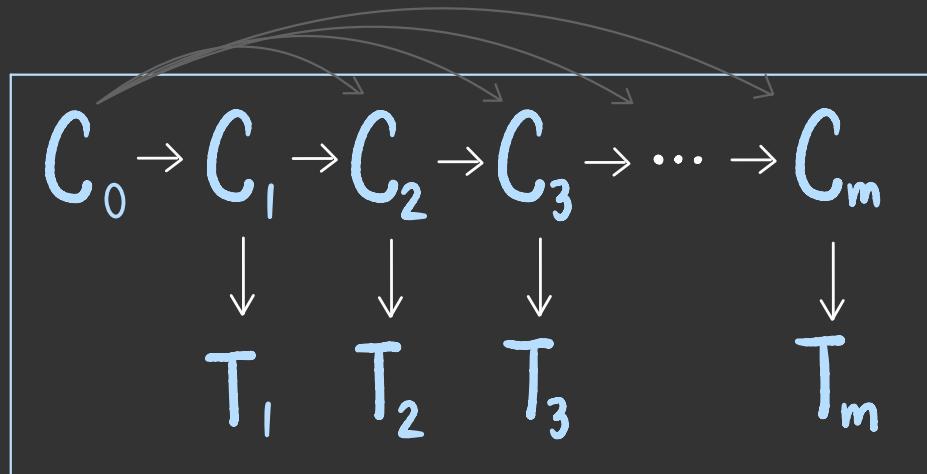


in time $O(m)$

this would not have been the case without
our earlier simplifying assumption

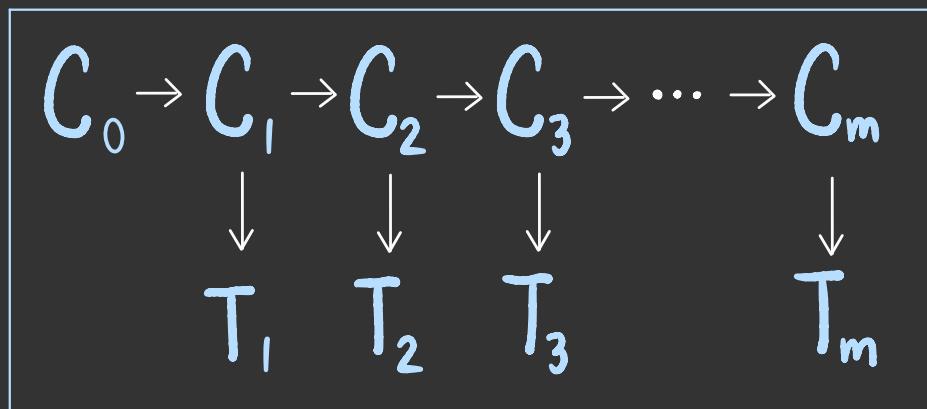


this assumption is called a markov assumption



namely, the probability of our current state depends only on the previous state

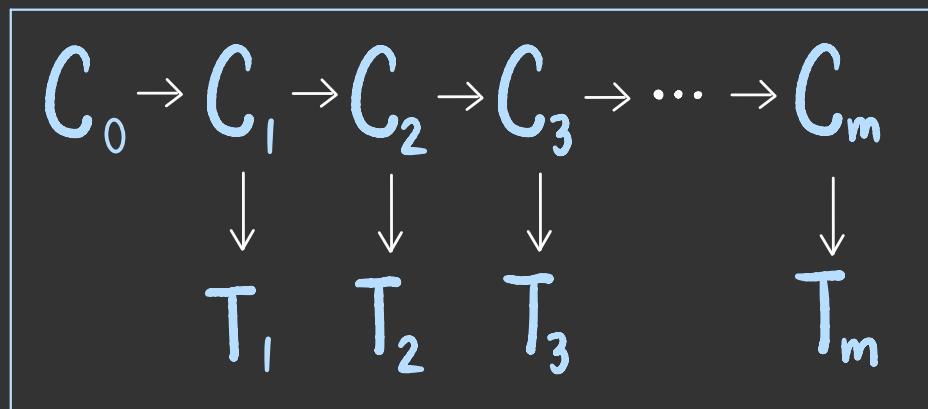
markov assumption : the probability of our current state depends only on the previous state



how would this be expressed as a conditional independence statement?

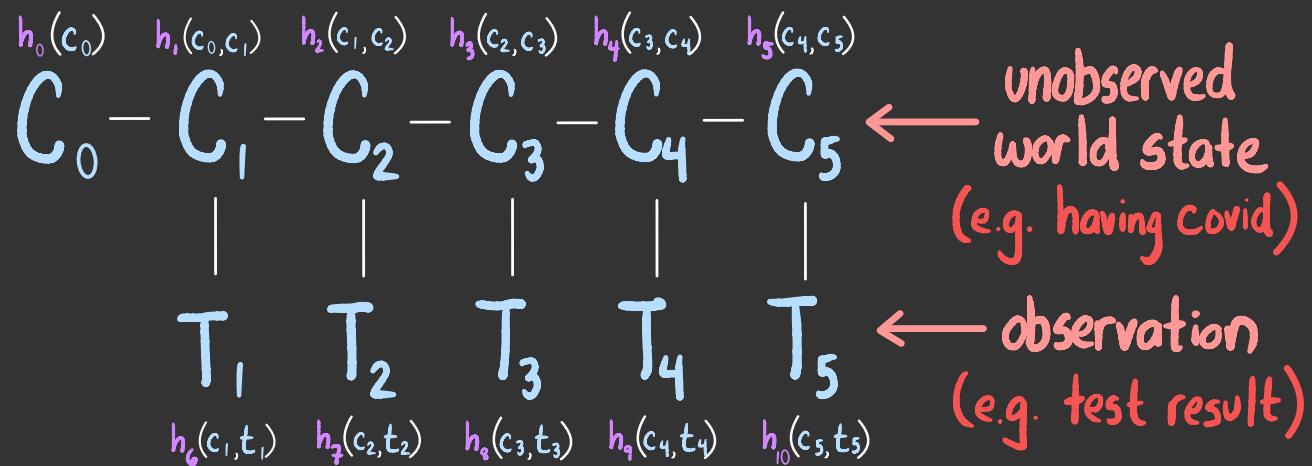


markov assumption : the probability of our current state depends only on the previous state



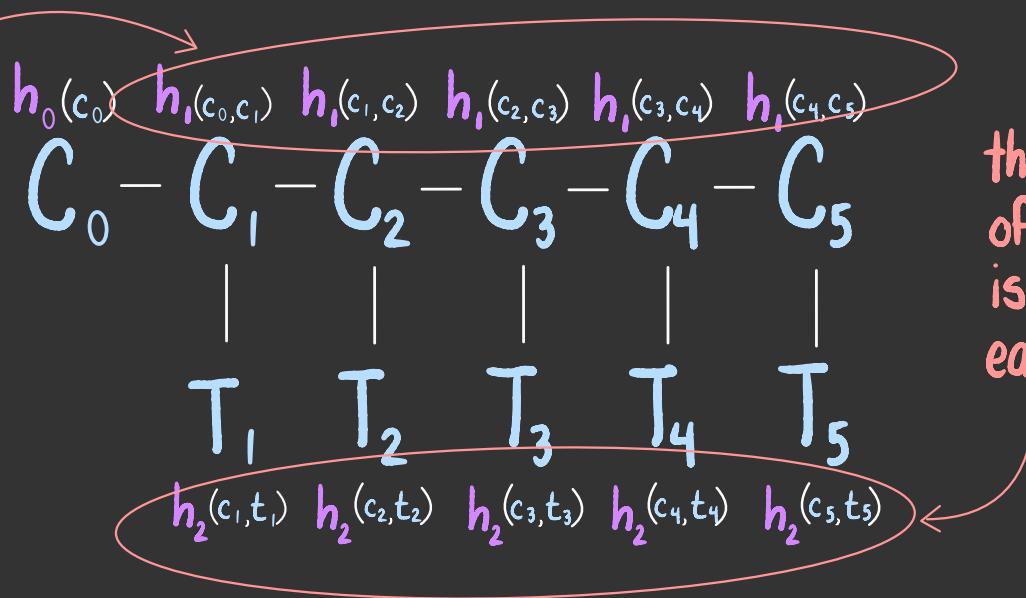
how would this be expressed as a conditional independence statement?

$C_i \perp\!\!\!\perp \{C_0, \dots, C_{i-2}\} \mid C_{i-1}$



this kind of bayesian network is popular
for temporal reasoning when we have
noisy observations about the world

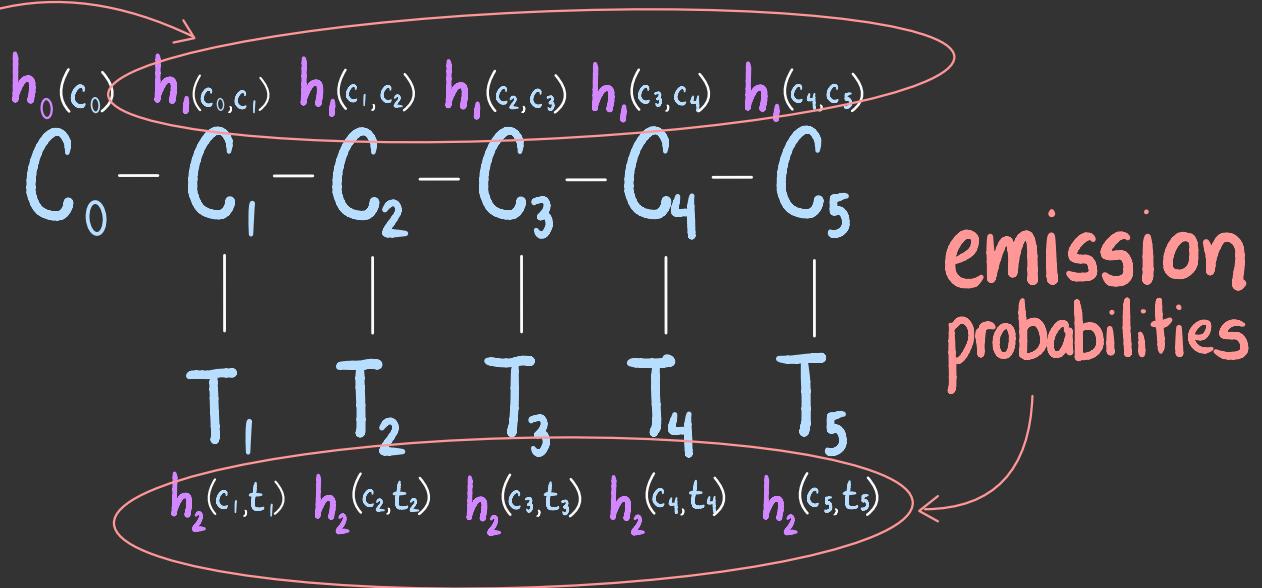
the probability
of contracting
covid is the
same at each
time step



the accuracy
of the test
is the same at
each time step

for such models, typically we make
the additional assumption of
time-invariance

transition
probabilities



such bayesian networks are called
hidden markov models