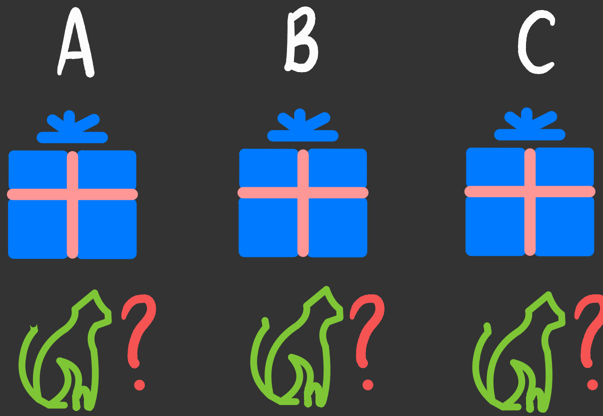
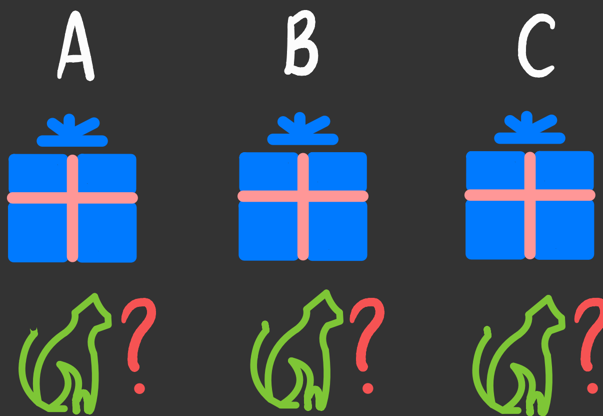
entailment and satisfiability

CSCI 373

suppose the cat is in box A or B



can we infer that the cat is in box A?

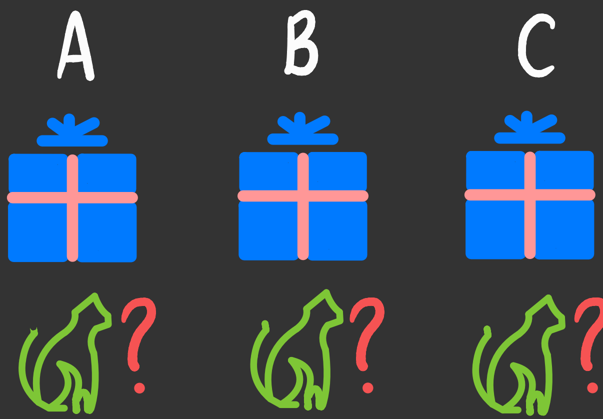


suppose the cat is in box A or B

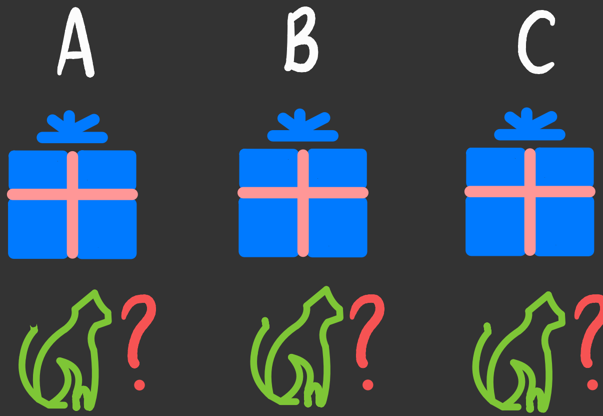
can we infer that the cat is in box A?

no.

suppose the cat is in box A



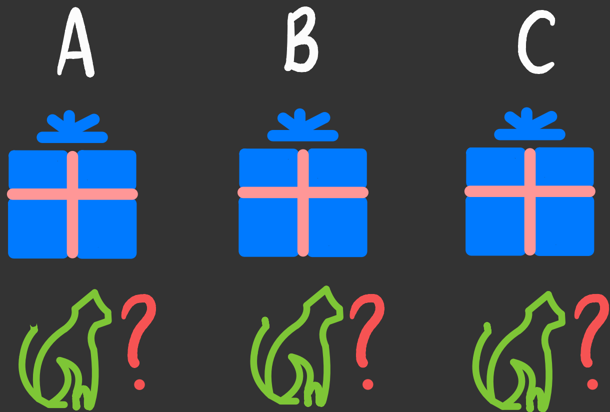
can we infer that the cat is in box A or B?



suppose the cat is in box A

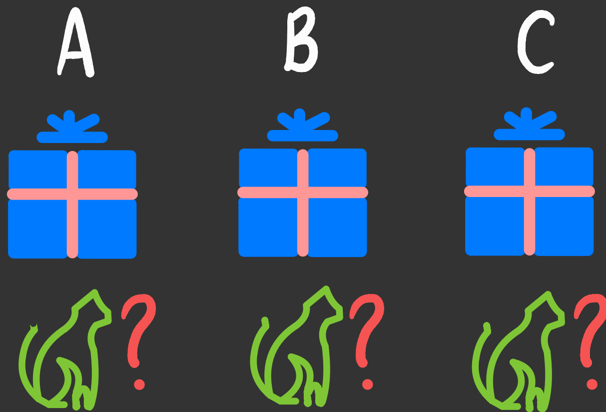
can we infer that the cat is in box A or B?

yes.



suppose the cat is in at least one box and is not in box A

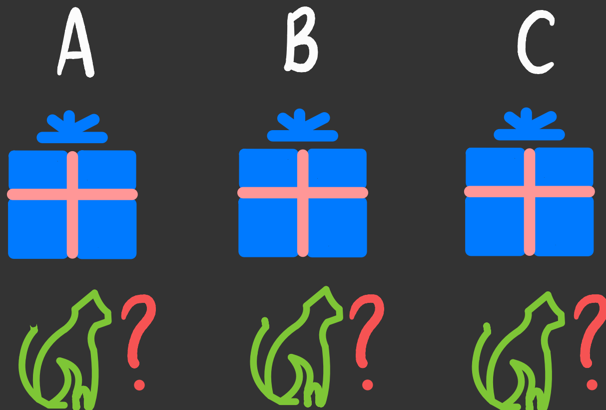
can we infer that the cat is in box B?



suppose the cat is in at least one box and is not in box A

can we infer that the cat is in box B?

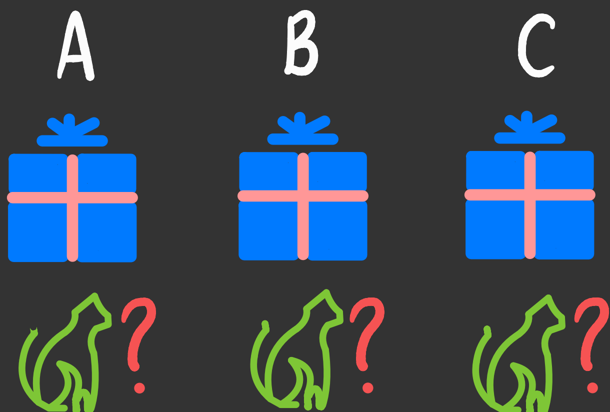
no.



suppose the cat is in at least one box and is not in box A

can we infer that the cat is in box B or C?

suppose the cat is in at least one box and is not in box A



can we infer that the cat is in box B or C?

yes.

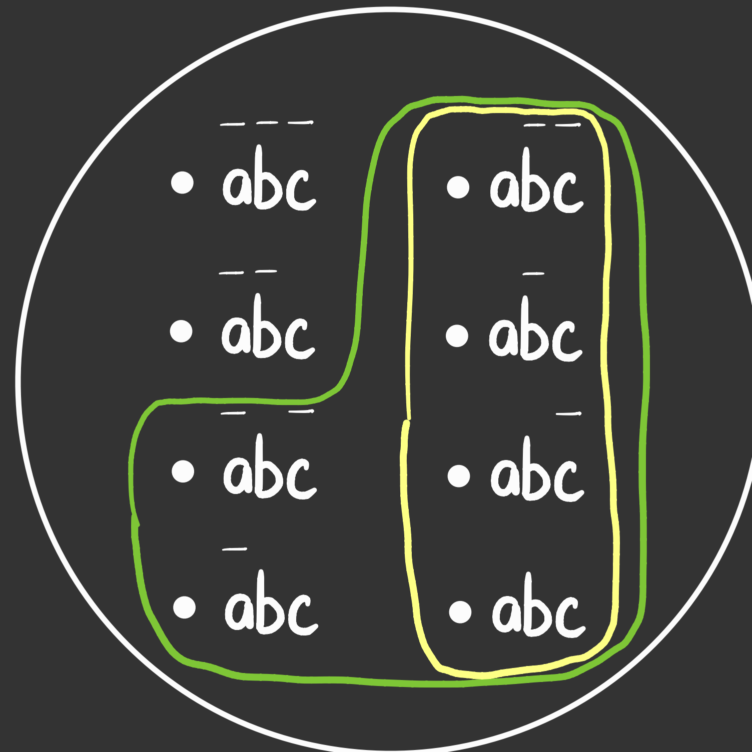
the cat is in box A or B

can we infer that the cat is in box A? no.

the cat is in box A or B

$\not\subseteq$

the cat is in box A



model space

$M(\{A, B, C\})$

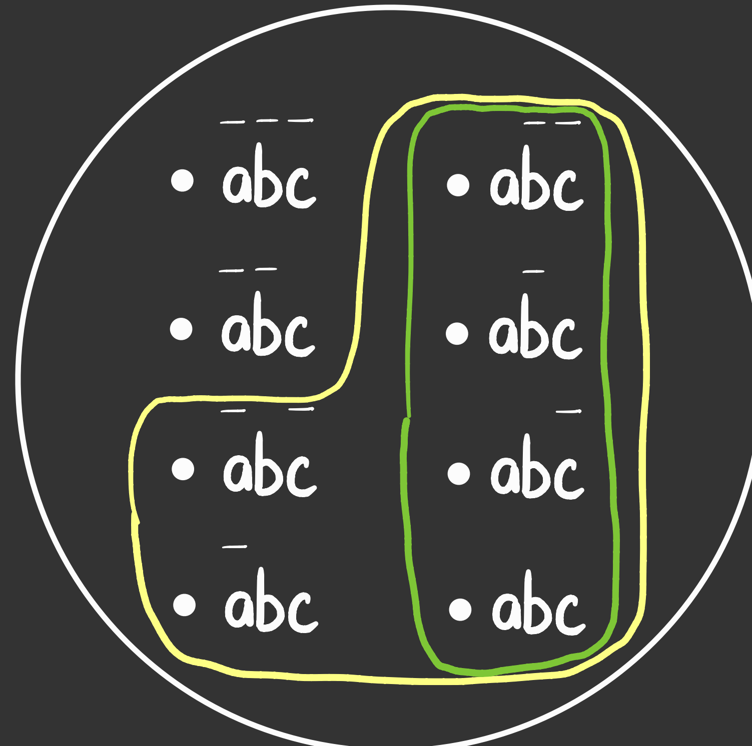
the cat is in box A

can we infer that the cat is in box A or B? yes.

the cat is in box A

$\subseteq$

the cat is in box A or B



model space

$M(\{A,B,C\})$

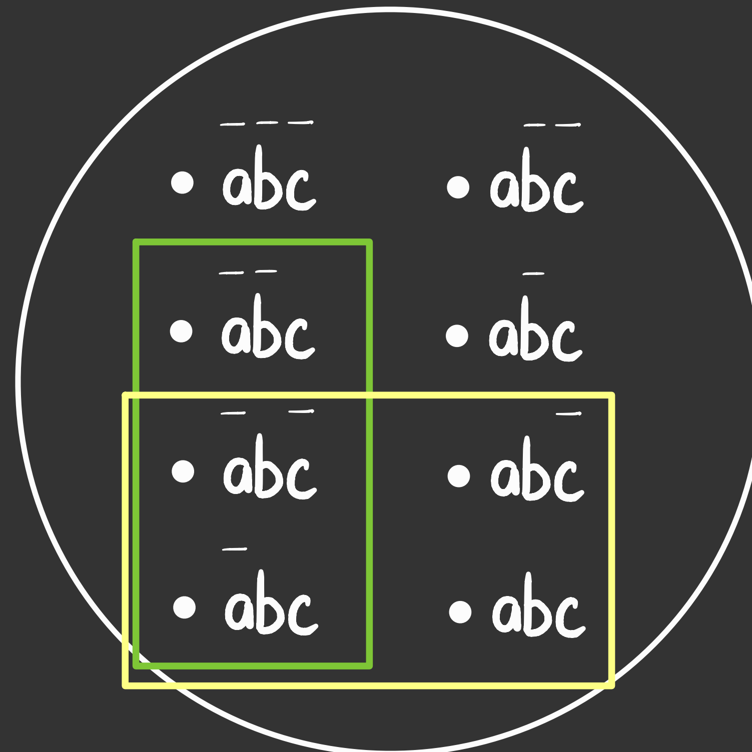
the cat is in at least one box and is not in box A

can we infer that the cat is in box B? no.

the cat is in at least one box and is not in box A

$\not\subseteq$

the cat is in box B



model space

$M(\{A, B, C\})$

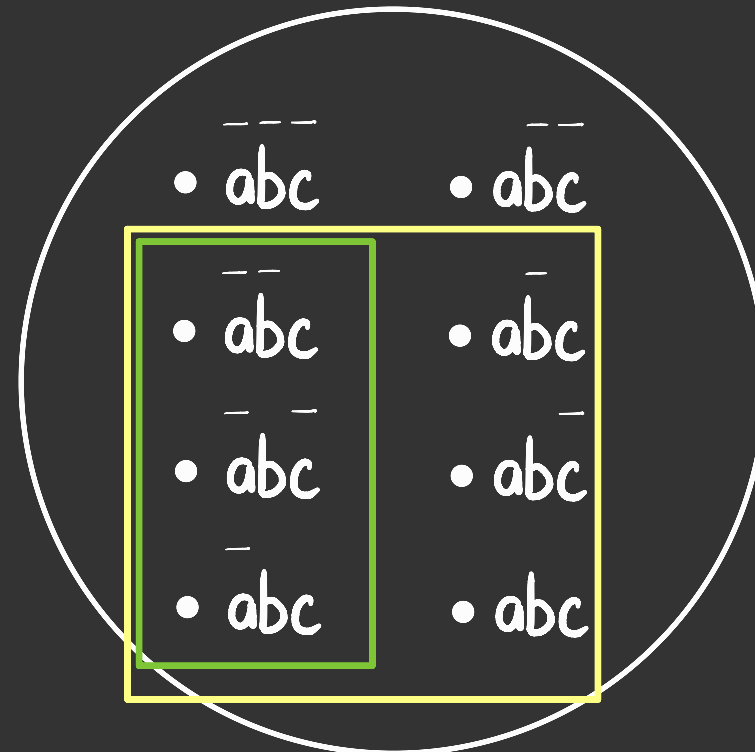
the cat is in at least one box and is not in box A

can we infer that the cat is in box B or C? yes.

the cat is in at least one box and is not in box A

$\subseteq$

the cat is in box B or C



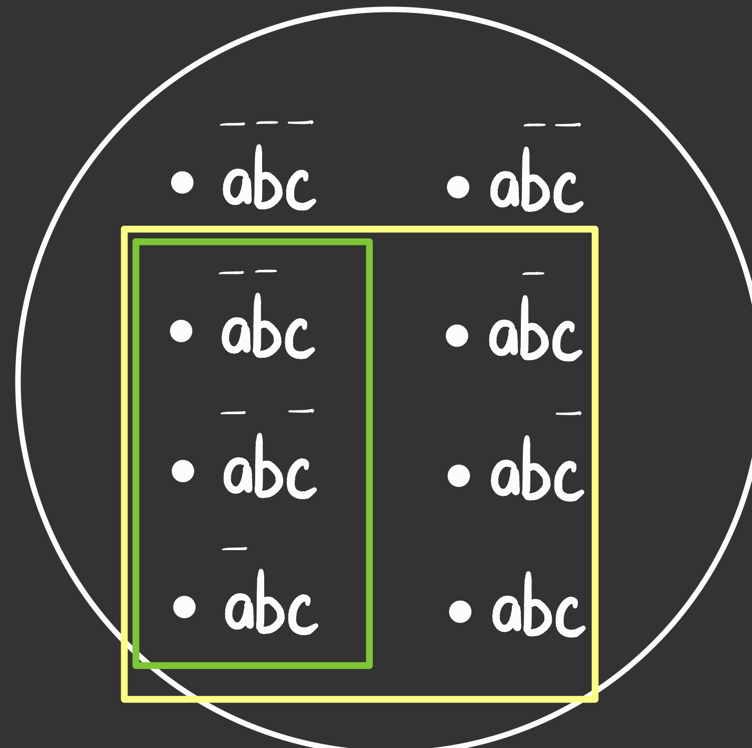
model space
$M(\{A, B, C\})$

suppose: $(A \vee B \vee C) \wedge \neg A$

can we infer: $B \vee C$?

yes, because

$$I((A \vee B \vee C) \wedge \neg A) \subseteq I(B \vee C)$$



model space

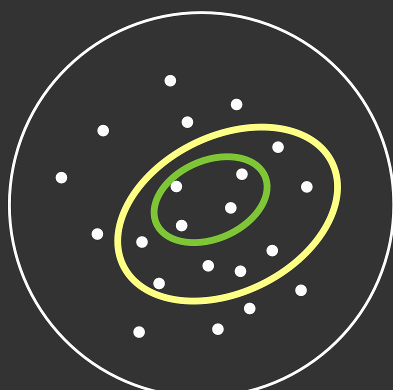
$M(\{A, B, C\})$

$\in L(\Sigma)$

$$\alpha \models \beta$$

if and only if

$$I(\alpha) \subseteq I(\beta)$$



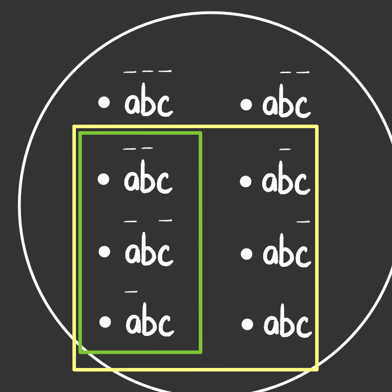
model space

$M(\Sigma)$

$$(A \vee B \vee C) \wedge \neg A \models B \vee C$$

if and only if

$$I((A \vee B \vee C) \wedge \neg A) \subseteq I(B \vee C)$$



model space

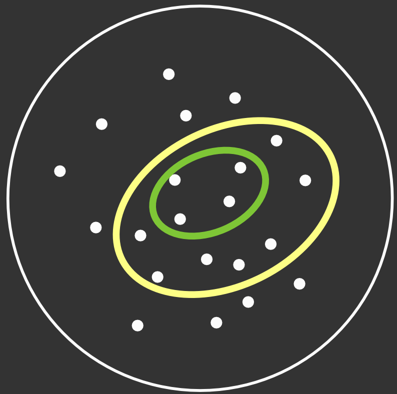
$M(\{A, B, C\})$

for $\alpha \in L(\Sigma)$, $\beta \in L(\Sigma)$

$$\alpha \models \beta$$

if and only if

$$I(\alpha) \subseteq I(\beta)$$



model space

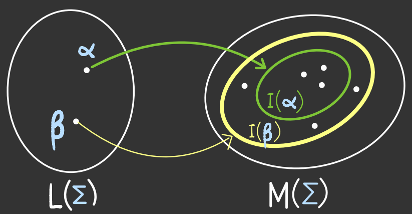
$M(\Sigma)$

this is called

entailment

"α entails β"

entailment



$\alpha \models \beta$

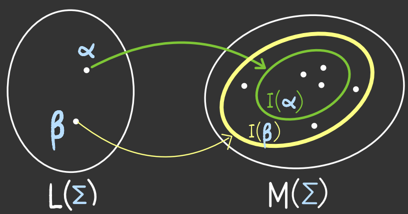
if and only if

$I(\alpha) \subseteq I(\beta)$

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

what is entailed by this board state?

entailment



$$\alpha \models \beta$$

if and only if

$$I(\alpha) \subseteq I(\beta)$$

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	4
			4	1	9			5
				8			7	9

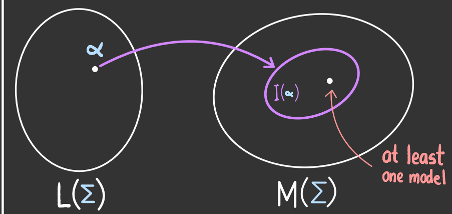
what is entailed by this board state?

?

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

is there a solution to this puzzle?

satisfiability



α is satisfiable

if and only if

$I(\alpha) \neq \{\}$

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

is there a solution to this puzzle?

a sentence $\alpha \in L(\Sigma)$ is satisfiable if its interpretation has at least one model, i.e.

$$I(\alpha) \neq \{\ \}$$

a sentence $\alpha \in L(\Sigma)$ is unsatisfiable if its interpretation has no models, i.e.

$$I(\alpha) = \{\ \}$$

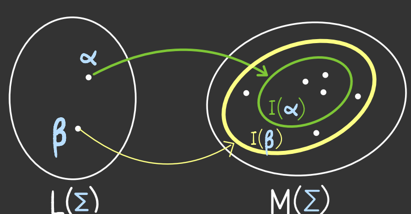
claim:

$$\alpha \models \beta$$

if and only if

$\alpha \land \neg \beta$ is unsatisfiable

entailment

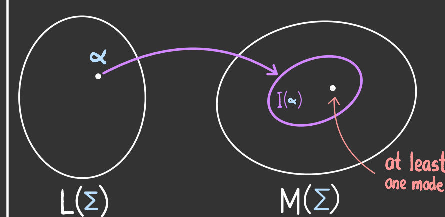


$$\alpha \models \beta$$

if and only if

$$I(\alpha) \subseteq I(\beta)$$

satisfiability



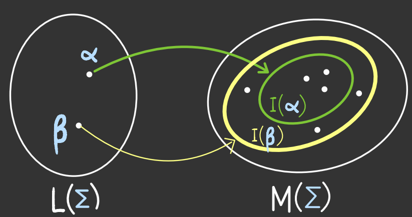
α is satisfiable

if and only if

$$I(\alpha) \neq \{\ \}$$

claim: $\alpha \models \beta$ if and only if $\alpha \land \neg \beta$ is unsatisfiable

entailment

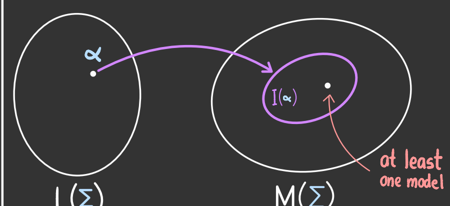


$\alpha \models \beta$

if and only if

$I(\alpha) \subseteq I(\beta)$

satisfiability



α is satisfiable

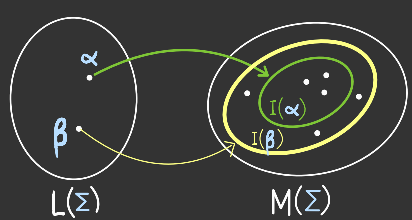
if and only if

$I(\alpha) \neq \{\,\}$

claim: $\alpha \models \beta$ if and only if $\alpha \wedge \neg \beta$ is unsatisfiable

in other words: $I(\alpha) \subseteq I(\beta)$ if and only if $\alpha \wedge \neg \beta$ is unsatisfiable

entailment



$\alpha \models \beta$

if and only if

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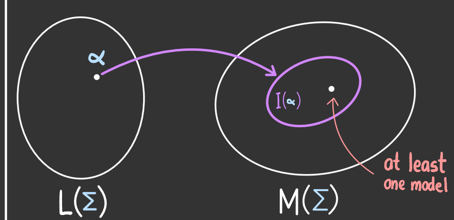
satisfiability



α is satisfiable

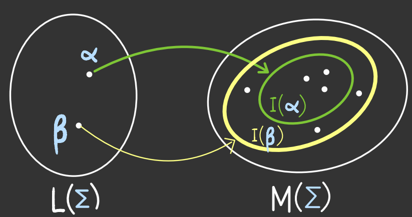
if and only if

$I(\alpha) \neq \{\}$

claim: $\alpha \models \beta$ if and only if $\alpha \wedge \neg\beta$ is unsatisfiable

in other words: $I(\alpha) \subseteq I(\beta)$ if and only if $I(\alpha \wedge \neg\beta) = \{\}$

entailment



$\alpha \models \beta$

if and only if

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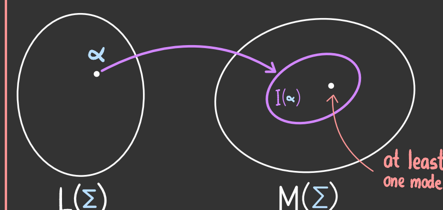
satisfiability



α is satisfiable

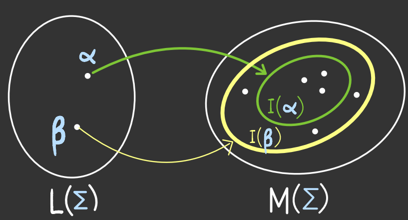
if and only if

$I(\alpha) \neq \{\}$

claim: $\alpha \models \beta$ if and only if $\alpha \land \neg \beta$ is unsatisfiable

in other words: $I(\alpha) \subseteq I(\beta)$ if and only if $I(\alpha) \cap \overline{I(\beta)} = \{\ \}$

entailment



$\alpha \models \beta$

if and only if

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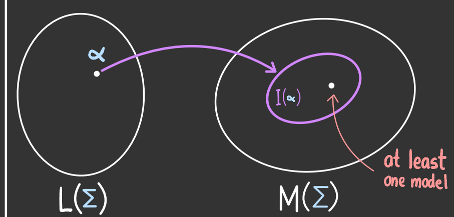
satisfiability



α is satisfiable

if and only if

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claim: $\alpha \vDash \beta$ if and only if $\alpha \wedge \neg \beta$ is unsatisfiable

in other words: $I(\alpha) \subseteq I(\beta)$ if and only if $I(\alpha) \cap \overline{I(\beta)} = \{\ \}$

proof ($\longrightarrow$): suppose $I(\alpha) \subseteq I(\beta)$

assume there exists model $m \in I(\alpha) \cap \overline{I(\beta)}$

so $m \in I(\alpha)$ and $m \in \overline{I(\beta)}$

so $m \in I(\alpha)$ and $m \notin I(\beta)$

so $I(\alpha) \nsubseteq I(\beta)$, which contradicts the premise

claim: $\alpha \models \beta$ if and only if $\alpha \wedge \neg \beta$ is unsatisfiable

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claim: $\alpha \models \beta$ if and only if $\alpha \wedge \neg\beta$ is unsatisfiable

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claim: $\alpha \models \beta$ if and only if $\alpha \wedge \neg \beta$ is unsatisfiable

in other words: $I(\alpha) \subseteq I(\beta)$ if and only if $I(\alpha) \cap \overline{I(\beta)} = \{\ \}$

proof ($\leftarrow$): suppose $I(\alpha) \cap \overline{I(\beta)} = \{\ \}$

consider a model $m \in I(\alpha)$

from the supposition, $m \notin \overline{I(\beta)}$

so $m \in I(\beta)$, implying that $I(\alpha) \subseteq I(\beta)$

claim: $\alpha \models \beta$ if and only if $\alpha \wedge \neg\beta$ is unsatisfiable

in other words: $I(\alpha) \subseteq I(\beta)$ if and only if $I(\alpha) \cap \overline{I(\beta)} = \{\ \}$

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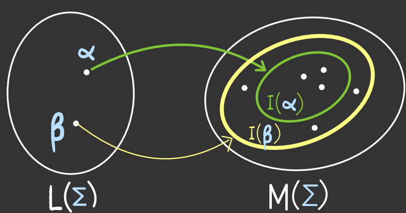
$\alpha \models \beta$

if and only if

$\alpha \wedge \neg \beta$ is unsatisfiable

entailment

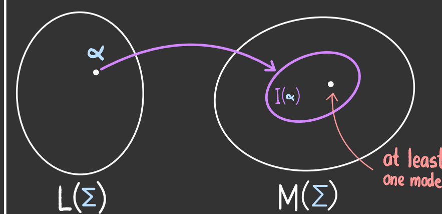


$\alpha \models \beta$

if and only if

$I(\alpha) \subseteq I(\beta)$

satisfiability



α is satisfiable

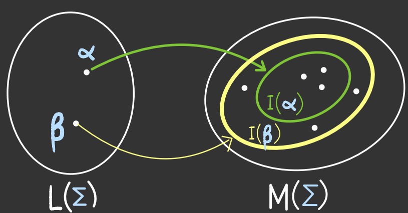
if and only if

$I(\alpha) \neq \{\ \}$

so if we can compute satisfiability,
then we can compute entailment

but I also: if we can compute entailment, then we can compute satisfiability

entailment

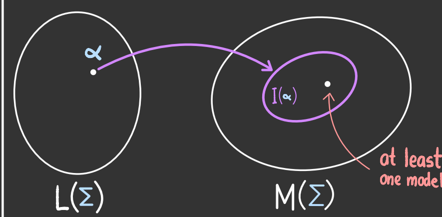


$\alpha \models \beta$

if and only if

$I(\alpha) \subseteq I(\beta)$

satisfiability



α is satisfiable

if and only if

$I(\alpha) \neq \{\}$

α is unsatisfiable

if and only if

$\alpha \models \bot$

claim: α is unsatisfiable if and only if $\alpha \models \bot$

proof:

$\longrightarrow$: suppose α is unsatisfiable.

thus $I(\alpha) = \{\}$

thus $I(\alpha) = I(\bot)$

thus $I(\alpha) \subseteq I(\bot)$

thus $\alpha \models \bot$

$\longleftarrow$: suppose $\alpha \models \bot$

thus $I(\alpha) \subseteq I(\bot)$

thus $I(\alpha) \subseteq \{\}$

thus $I(\alpha) = \{\}$

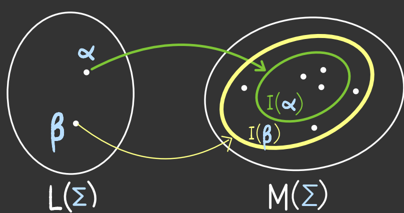
thus α is unsatisfiable.

$$\alpha \models \beta$$

if and only if

$\alpha \land \neg\beta$ is unsatisfiable

entailment

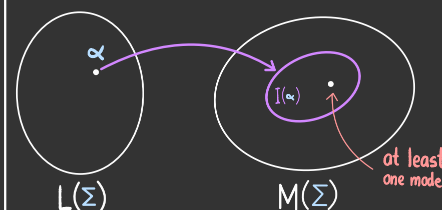


$$\alpha \models \beta$$

if and only if

$$I(\alpha) \subseteq I(\beta)$$

satisfiability



α is satisfiable

if and only if

$$I(\alpha) \neq \{\ \}$$

α is unsatisfiable

if and only if

$$\alpha \models \perp$$

if we can compute
entailment
using a satisfiability algorithm

and we can compute
satisfiability
using an entailment algorithm

which algorithm
should we implement?