Completeness of resolution

CSCI 373

is $C_1 \wedge C_2 \wedge \cdots \wedge C_n$ unsatisfiable?

soundness

if we keep applying resolution to the clauses until we obtain $\perp$, then the sentence is unsatisfiable

✓

completeness

if we keep applying resolution to the clauses and never obtain $\perp$, then the sentence is satisfiable

✓

let's prove this

resolution

$$\left(\bigvee_{i=1}^{m} l_i\right) \bigwedge \left(\bigvee_{i=m+1}^{n} l_i\right) \vdash_R \bigvee_{\substack{i=1 \\ i\neq j,\, i\neq k}}^{n} l_i$$

where: $j \in \{1, \ldots, m\}$, $k \in \{m+1, \ldots, n\}$, $l_j = \overline{l_k}$

the resolution closure of a set S of clauses is the smallest set $RC(S)$ such that:

- if $c \in S$, then $c \in RC(S)$
- if $c_1, c_2 \in RC(S)$ and $c_1 \wedge c_2 \vdash_R c_3$, then $c_3 \in RC(S)$

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let $S = \{\neg A \vee \neg B \vee C,\ B \vee \neg D,\ \neg C\}$

$RC(S) =$?

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let $S = \{\neg A \vee \neg B \vee C, B \vee \neg D, \neg C\}$

$$RC(S) = \{\neg A \vee \neg B \vee C, B \vee \neg D, \neg C, \neg A \vee C \vee \neg D, \neg A \vee \neg B, \neg A \vee \neg D\}$$

is $c_1 \wedge c_2 \wedge \cdots \wedge c_n$ unsatisfiable?

completeness theorem:
if $\bot \notin RC(\{c_1, \ldots, c_n\})$,
then $c_1 \wedge c_2 \wedge \cdots \wedge c_n$
is satisfiable

completeness

if we keep applying resolution to the clauses and never obtain $\bot$, then the sentence is satisfiable ✓

let's prove this

completeness theorem:

if $\bot \notin RC(\{c_1, \ldots, c_n\})$, then $c_1 \wedge c_2 \wedge \cdots \wedge c_n$ is satisfiable

plan: we will show that if $\bot \notin RC(\{c_1, \ldots, c_n\})$, then we can construct a model m in the interpretation $I(c_1 \wedge c_2 \wedge \cdots \wedge c_n)$

greedy strategy:

assign 0 to the next signature var if it doesn't make a clause unsatisfiable

or

assign 1 to the next signature var if it doesn't make a clause unsatisfiable

or

stop

Σ	
A	$\mapsto$
B	$\mapsto$
C	$\mapsto$
D	$\mapsto$

model

$RC(S)$
$A \vee \neg B \vee D$
$B \vee C$
$\neg D$
$A \vee C \vee D$
$A \vee \neg B$
$A \vee C$

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$A \mapsto 0$	
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$C \mapsto$	
$D \mapsto$	

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RC(S)
$A \vee \neg B \vee D$
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greedy strategy:

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stop

Σ	
$A \mapsto 0$	
$B \mapsto 0$	
$C \mapsto 0$	
$D \mapsto$	

model

$RC(S)$
$A \vee \neg B \vee D$
$B \vee C$
$\neg D$
$A \vee C \vee D$
$A \vee \neg B$
$A \vee C$

makes these unsatisfiable

greedy strategy:

assign 0 to the next signature var if it doesn't make a clause unsatisfiable

or

assign 1 to the next signature var if it doesn't make a clause unsatisfiable

or

stop

Σ	
$A \mapsto 0$	
$B \mapsto 0$	
$C \mapsto 1$	
$D \mapsto$	
model	

$RC(S)$
$A \vee \neg B \vee D$
$B \vee C$
$\neg D$
$A \vee C \vee D$
$A \vee \neg B$
$A \vee C$

greedy strategy:

assign 0 to the next signature var if it doesn't make a clause unsatisfiable

or

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or

stop

Σ	
$A \mapsto 0$	
$B \mapsto 0$	
$C \mapsto 1$	
$D \mapsto 0$	

model

$RC(S)$
$A \vee \neg B \vee D$
$B \vee C$
$\neg D$
$A \vee C \vee D$
$A \vee \neg B$
$A \vee C$

greedy strategy:

assign 0 to the next signature var if it doesn't make a clause unsatisfiable

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or

stop

Σ
$A \mapsto 0$
$B \mapsto 0$
$C \mapsto 1$
$D \mapsto 0$
model

$RC(S)$
$A \vee \neg B \vee D$
$B \vee C$
$\neg D$
$A \vee C \vee D$
$A \vee \neg B$
$A \vee C$

all clauses satisfied

greedy strategy:

assign 0 to the next signature var if it doesn't make a clause unsatisfiable

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or

stop

Claim:

if $\perp \notin RC(S)$ then we never stop prematurely

$RC(S)$
$A \vee \neg B \vee D$
$B \vee C$
$\neg D$
$A \vee C \vee D$
$A \vee \neg B$
$A \vee C$

greedy strategy:

assign 0 to the next signature var if it doesn't make a clause unsatisfiable

or

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or

stop

assume this happens

Σ

$\sigma_1 \mapsto 0/1$

$\sigma_{k-1} \mapsto 0/1$

$\sigma_k \mapsto ?$

$\sigma_n \mapsto$

model

$RC(S)$

already unsatisfiable

$\neg\sigma_k \vee c'$

$\sigma_k \vee c''$

greedy strategy:

assign 0 to the next signature var if it doesn't make a clause unsatisfiable

or

assign 1 to the next signature var if it doesn't make a clause unsatisfiable

or

stop

Σ

$\sigma_1 \mapsto 0/1$

$\sigma_{k-1} \mapsto 0/1$

$\sigma_k \mapsto ?$

$\sigma_n \mapsto$

model

$RC(S)$
already unsatisfiable
$\neg\sigma_k \vee c'$
$\sigma_k \vee c''$
$c' \vee c''$

was already unsatisfiable

greedy strategy:

assign 0 to the next signature var if it doesn't make a clause unsatisfiable

or

assign 1 to the next signature var if it doesn't make a clause unsatisfiable

or

stop

contradiction!

Σ	
σ_1	$\mapsto 0/1$
	?
σ_{k-1}	$\mapsto 0/1$
σ_k	$\mapsto$
σ_n	$\mapsto$
model	

$RC(S)$
already unsatisfiable
$\neg\sigma_k \vee c'$
$\sigma_k \vee c''$
$c' \vee c''$

was already unsatisfiable

completeness theorem:

if $\perp \notin RC(\{c_1, \ldots, c_n\})$, then $c_1 \wedge c_2 \wedge \cdots \wedge c_n$ is satisfiable

plan: we will show that if $\perp \notin RC(\{c_1, \ldots, c_n\})$, then we can construct a model m in the interpretation $I(c_1 \wedge c_2 \wedge \cdots \wedge c_n)$ ✓

is $C_1 \wedge C_2 \wedge \cdots \wedge C_n$ unsatisfiable?

soundness

if we keep applying resolution to the clauses until we obtain $\perp$, then the sentence is **unsatisfiable**

completeness

if we keep applying resolution to the clauses and never obtain $\perp$, then the sentence is **satisfiable**