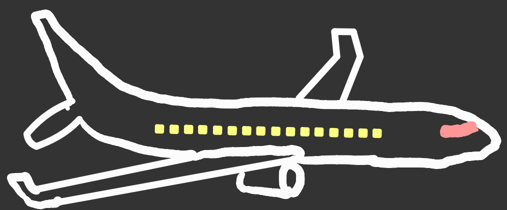
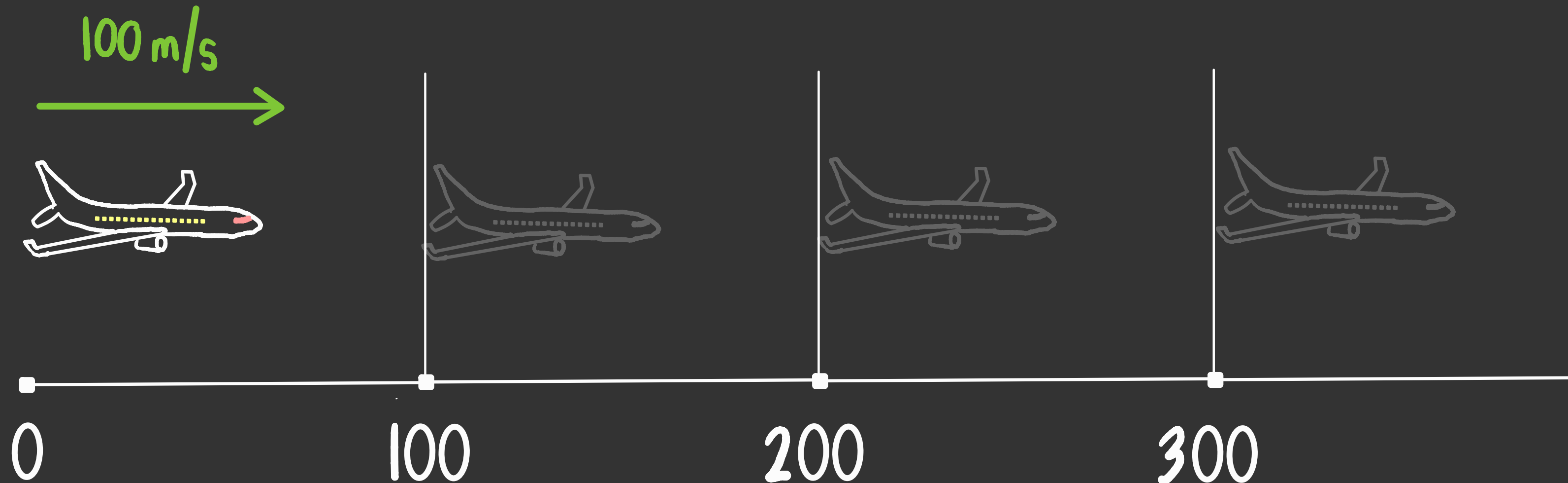
kalman filters

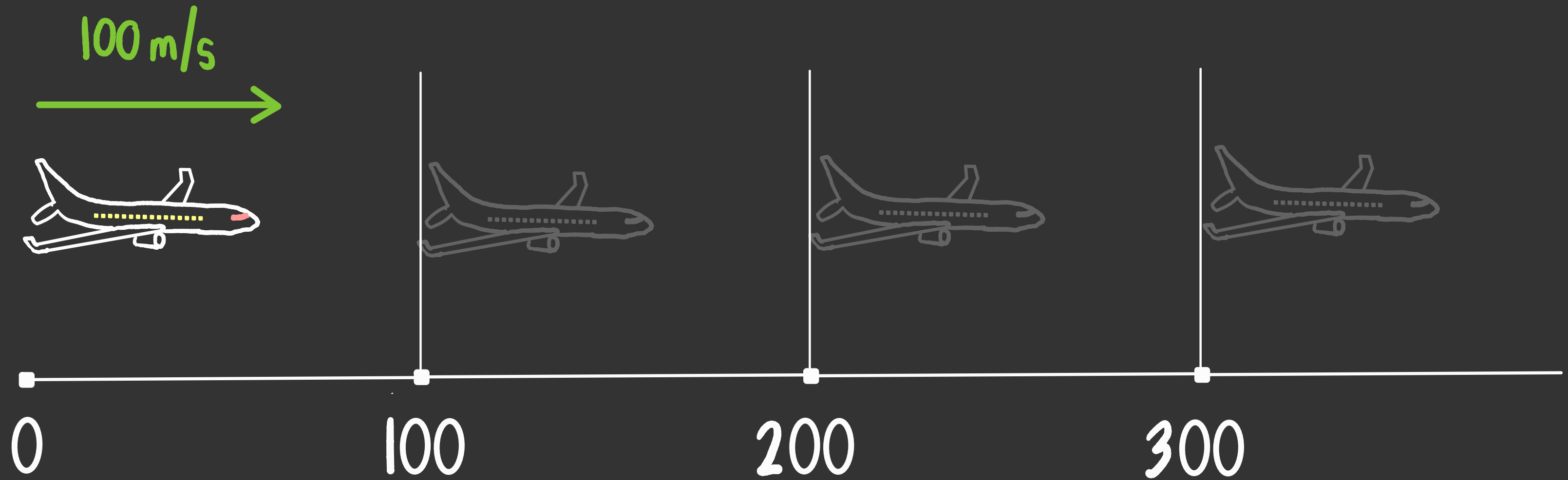
CSCI 373

consider the problem of tracking an **airplane** at cruising altitude

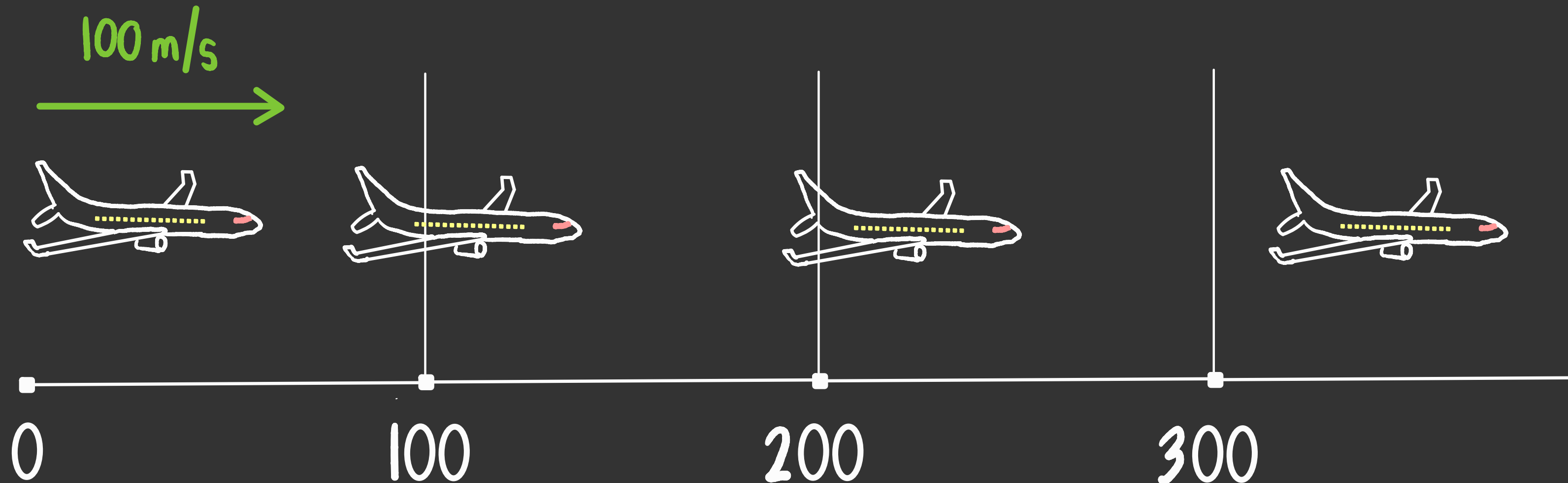




strategy 1: we know this plane flies at an average velocity of 100 meters per second, so we predict that the plane is at position 100t after t seconds



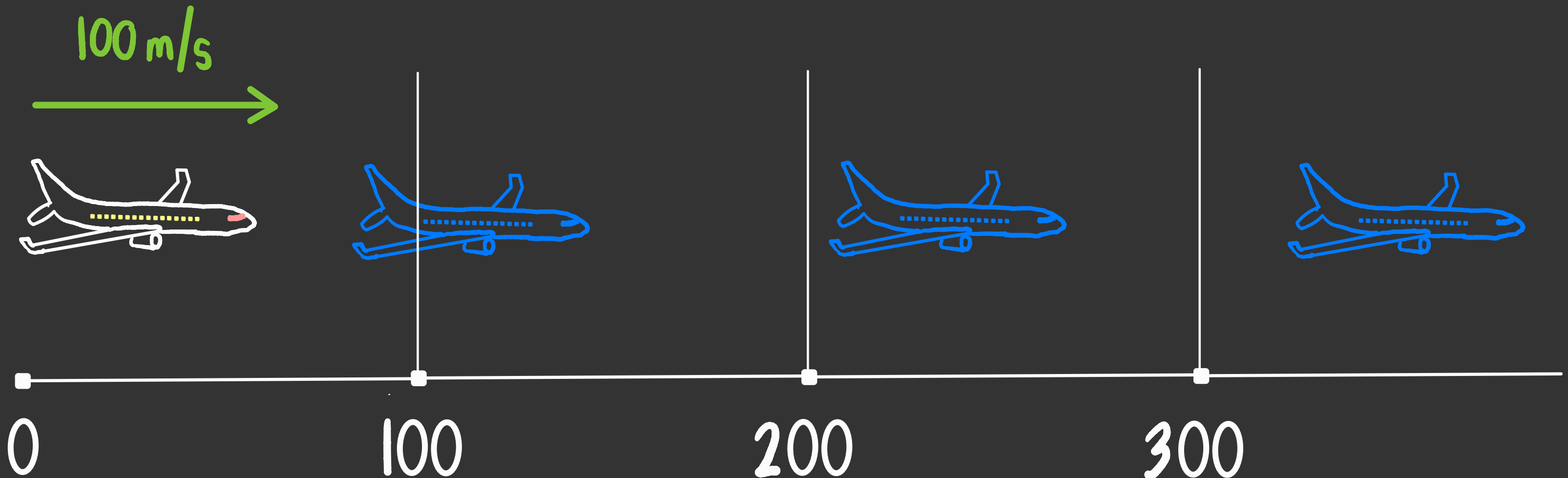
what is a drawback of this strategy?



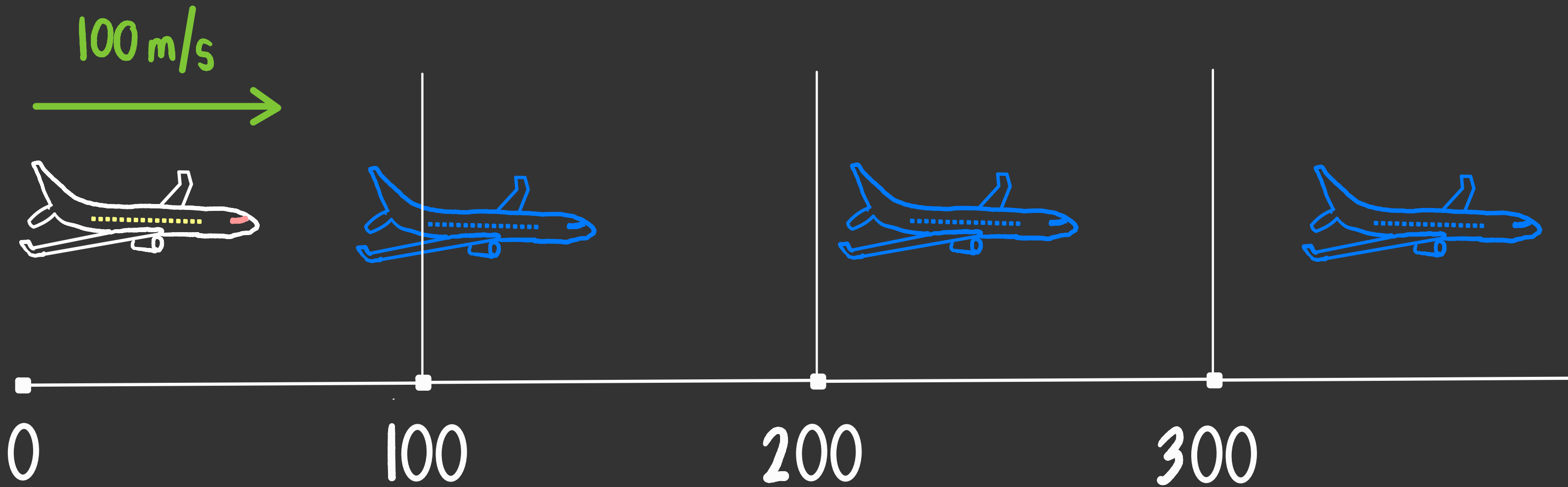
an average velocity of 100 meters per second

does not mean

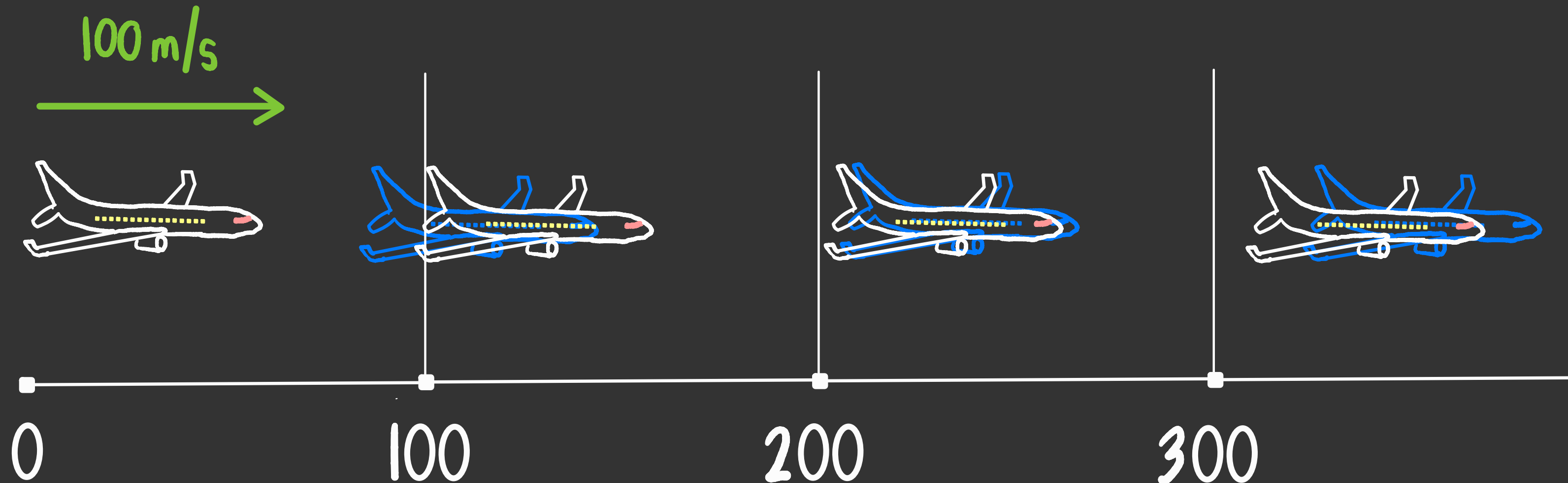
a constant velocity of 100 meters per second



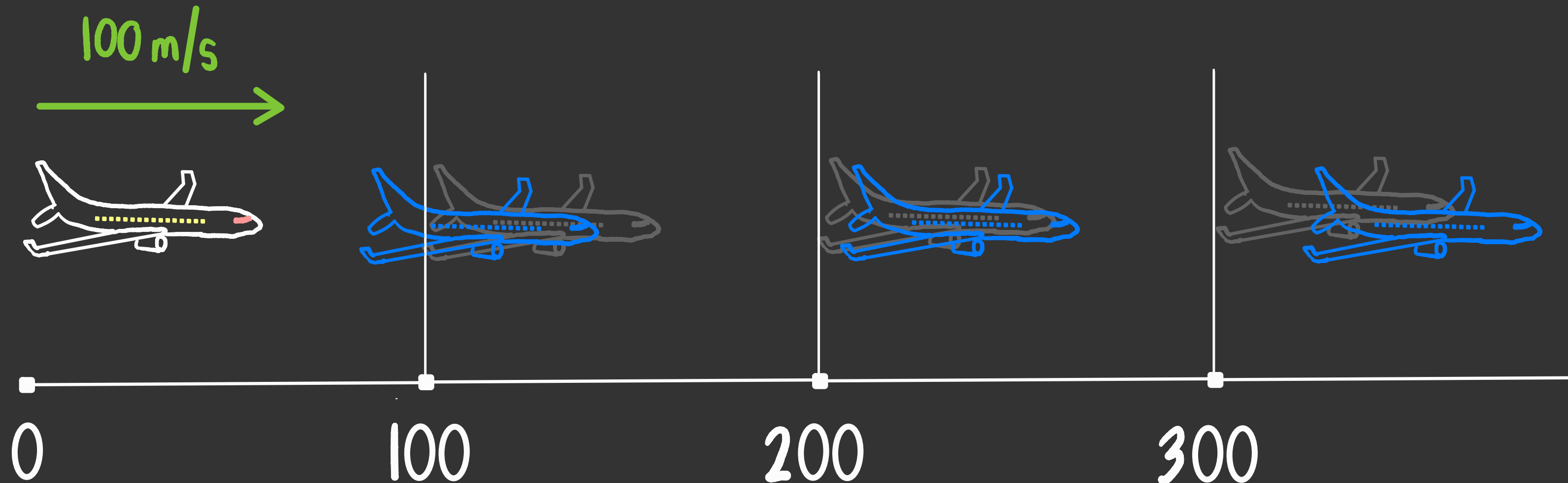
strategy 2: use **radar** to determine the plane's position

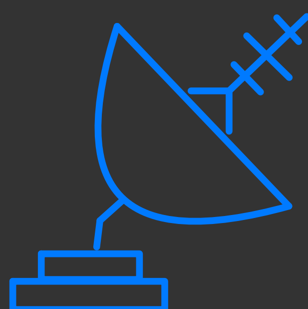


what is a drawback of this strategy?

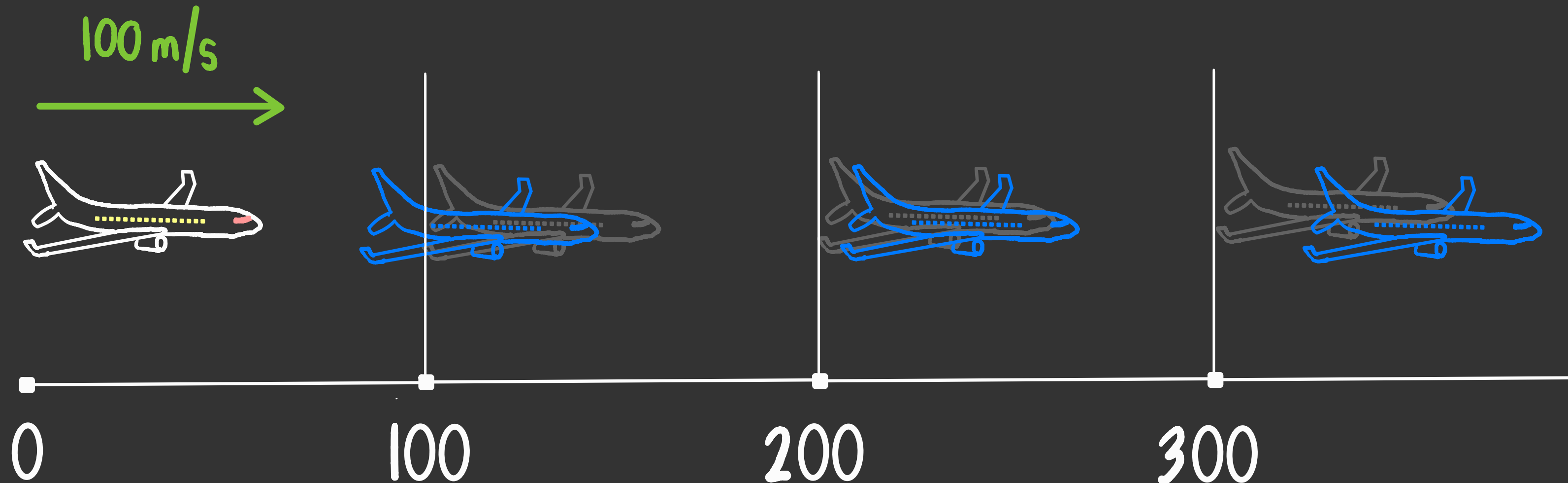


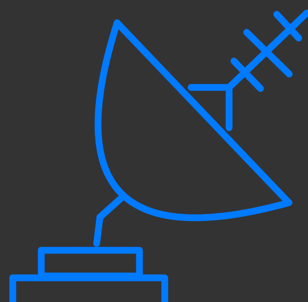
radar is a noisy sensor





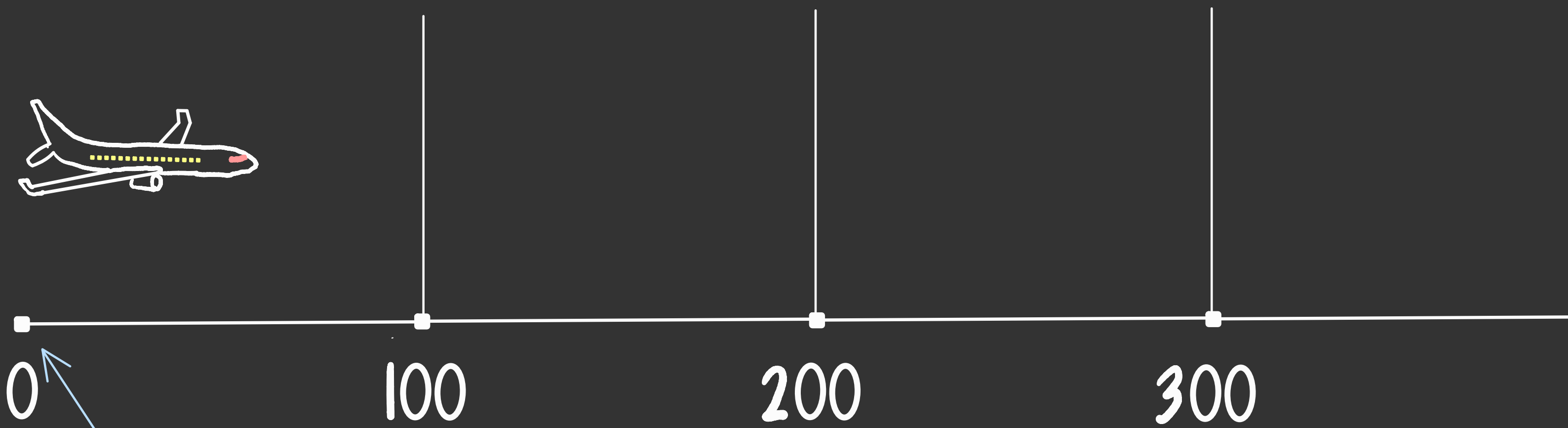
strategy 3: use both strategies 1 and 2!



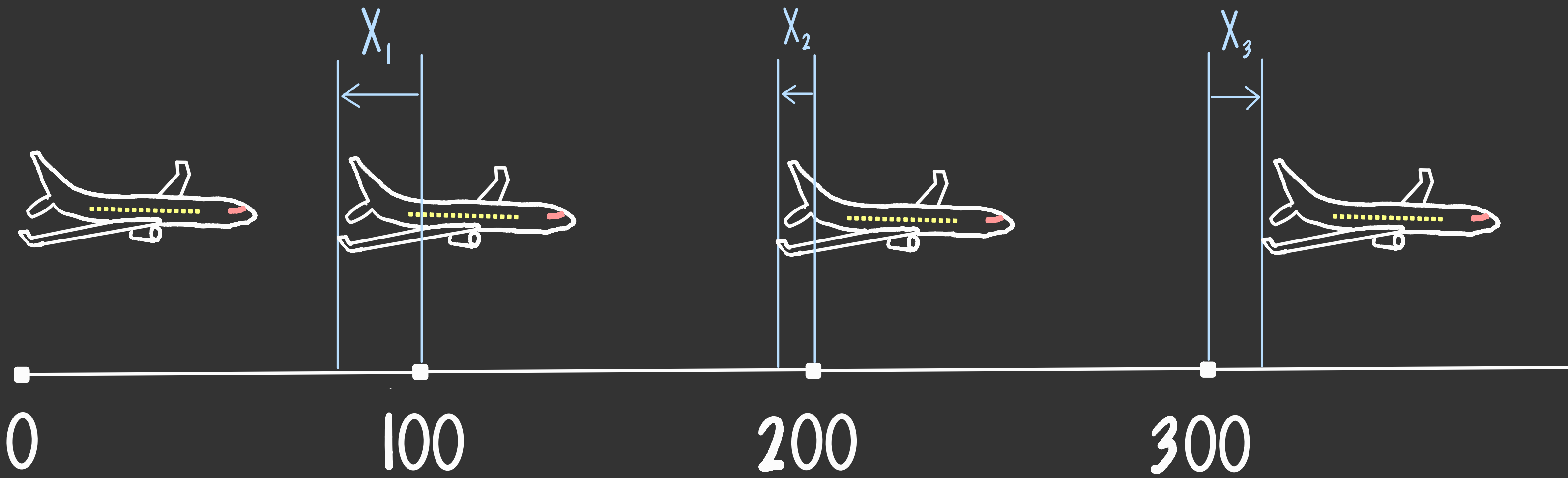


strategy 3: use both strategies 1 and 2!

but how can we fuse the two?

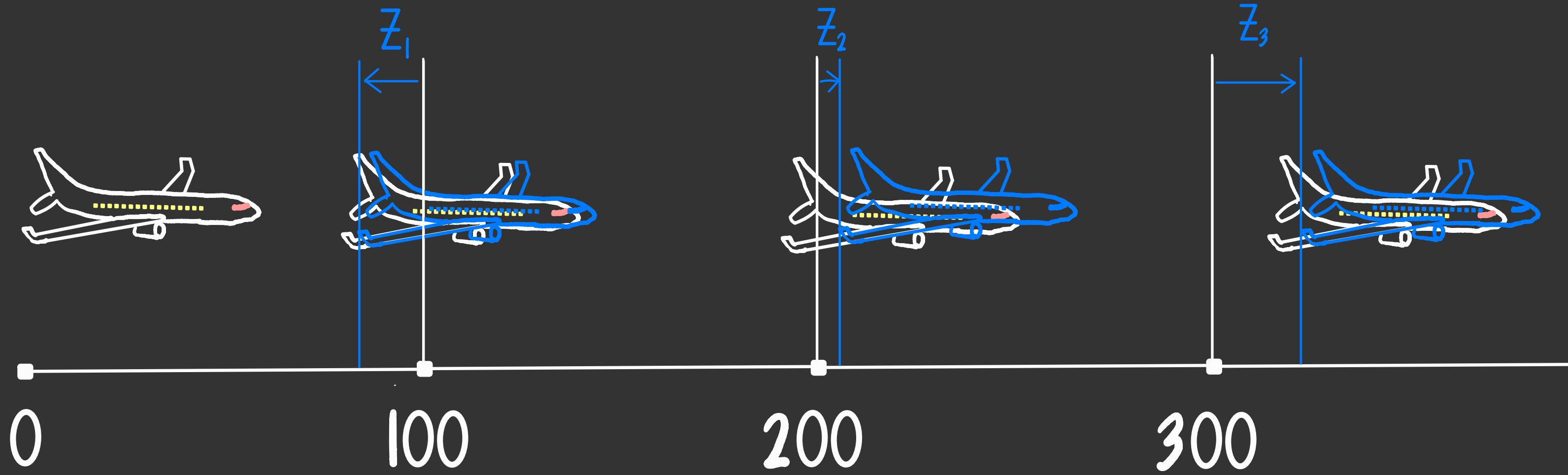


let X_0 be the starting position of the plane



let X_0 be the starting position of the plane

let X_t be the position of the plane after t seconds minus the expected position after t seconds



let X_0 be the starting position of the plane

let X_t be the position of the plane after t seconds minus the expected position after t seconds

let Z_t be the radar reading at time t minus the expected position

create a bayesian network using these variables

let X_0 be the starting position of the plane

let X_t be the plane's position after t seconds relative to the expected position

let Z_t be the radar reading at time t relative to the expected position

$$
\begin{tikzcd}
X_0 \arrow[r] & X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & X_3 \arrow[d] \\
& Z_1 & Z_2 & Z_3
\end{tikzcd}
$$

let X_0 be the starting position of the plane

let X_t be the plane's position after t seconds relative to the expected position

let Z_t be the radar reading at time t relative to the expected position

want to compute:

$$\boxed{P(x_3 \mid z_1, z_2, z_3)}$$

$$\begin{tikzcd}
X_0 \arrow[r] & X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & X_3 \arrow[d] \\
 & Z_1 & Z_2 & Z_3
\end{tikzcd}$$

maybe observed — let X_0 be the starting position of the plane

unobserved — let X_t be the plane's position after t seconds relative to the expected position

observed — let Z_t be the radar reading at time t relative to the expected position

want to compute:

$$P(x_3 \mid z_1, z_2, z_3)$$

$$\begin{tikzcd}
X_0 \arrow[r] & X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & X_3 \arrow[d] \\
 & Z_1 & Z_2 & Z_3
\end{tikzcd}$$

what kind of bayesian network does this look like?

want to compute:

$$P(x_3 \mid z_1, z_2, z_3)$$

$$\begin{tikzcd} X_0 \arrow[r] & X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & X_3 \arrow[d] \\ & Z_1 & Z_2 & Z_3 \end{tikzcd}$$

this has the same graph as a hidden markov model, but the variables are continuous not discrete

want to compute:

$$P(x_3 \mid z_1, z_2, z_3)$$

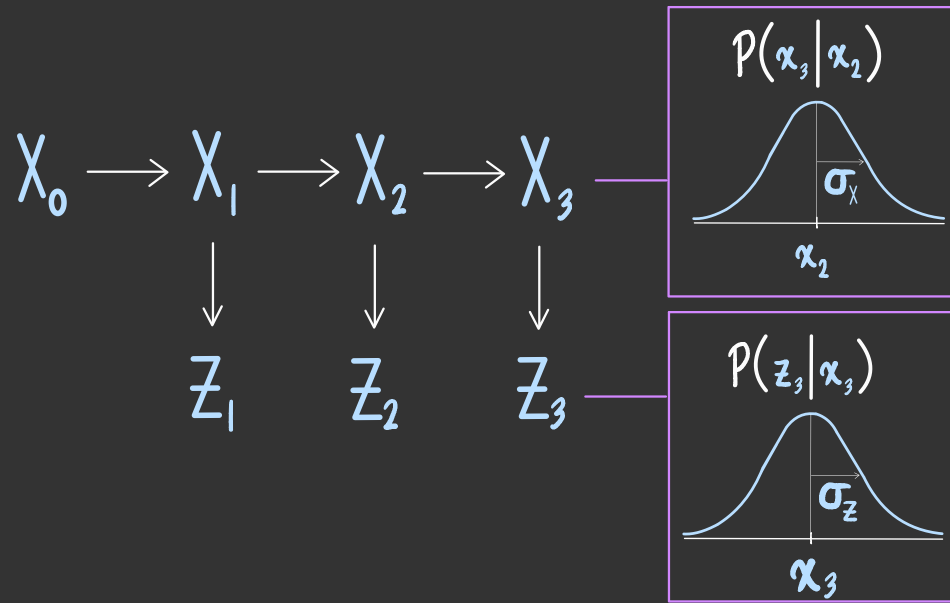
$$
\begin{tikzcd}
X_0 \arrow[r] & X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & X_3 \arrow[d] \\
 & Z_1 & Z_2 & Z_3
\end{tikzcd}
$$

x_2	x_3	$P(x_3 \mid x_2)$
0	0	.
0	0.17	.
0.29	0	.
⋮	⋮	⋮

so we can't represent the conditional probabilities as tables

want to compute:

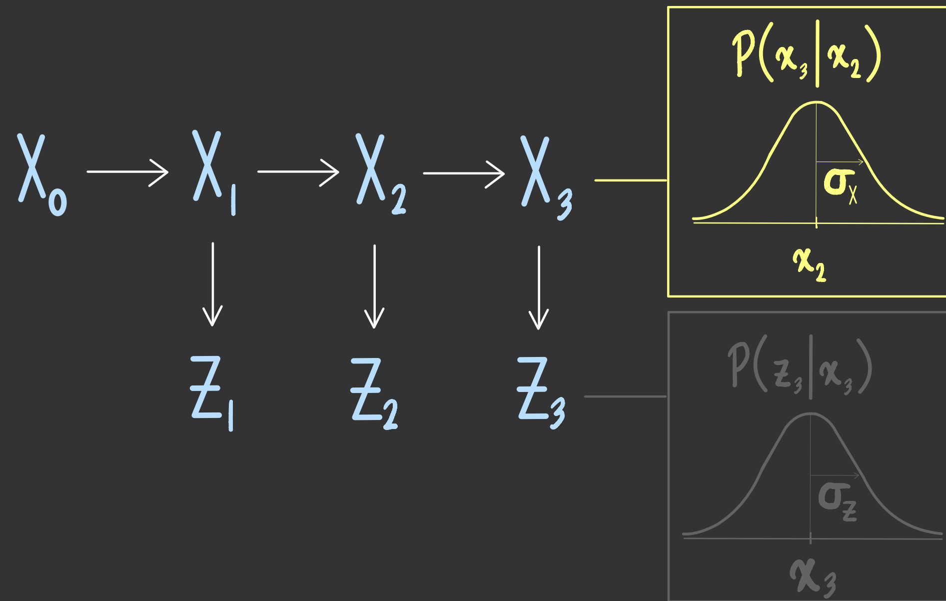
$$P(x_3 \mid z_1, z_2, z_3)$$



let's try **this** instead

want to compute:

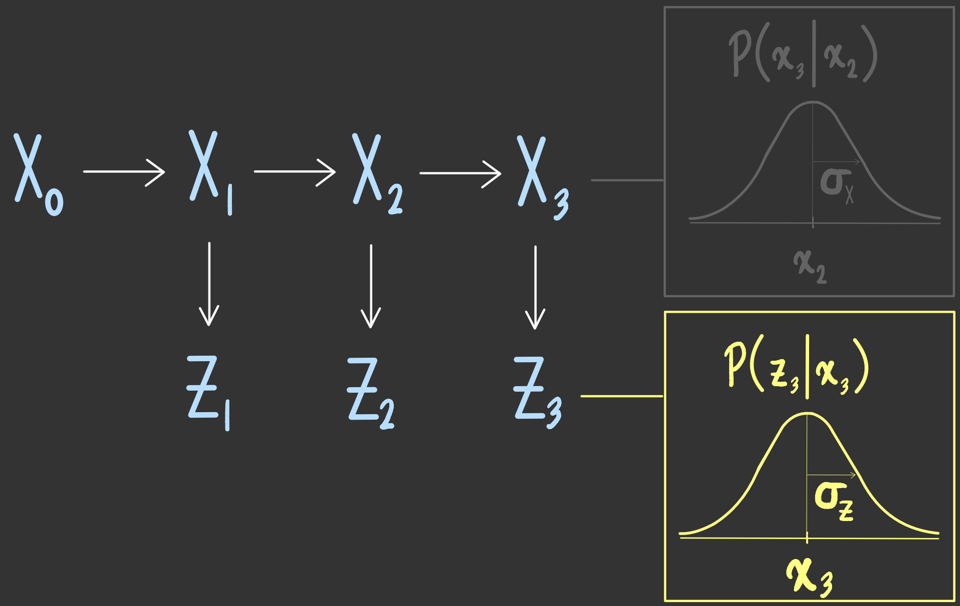
$$P(x_3 \mid z_1, z_2, z_3)$$



the plane's position at time 3 relative to its expected position is normally distributed with mean x_2 and standard deviation σ_X

want to compute:

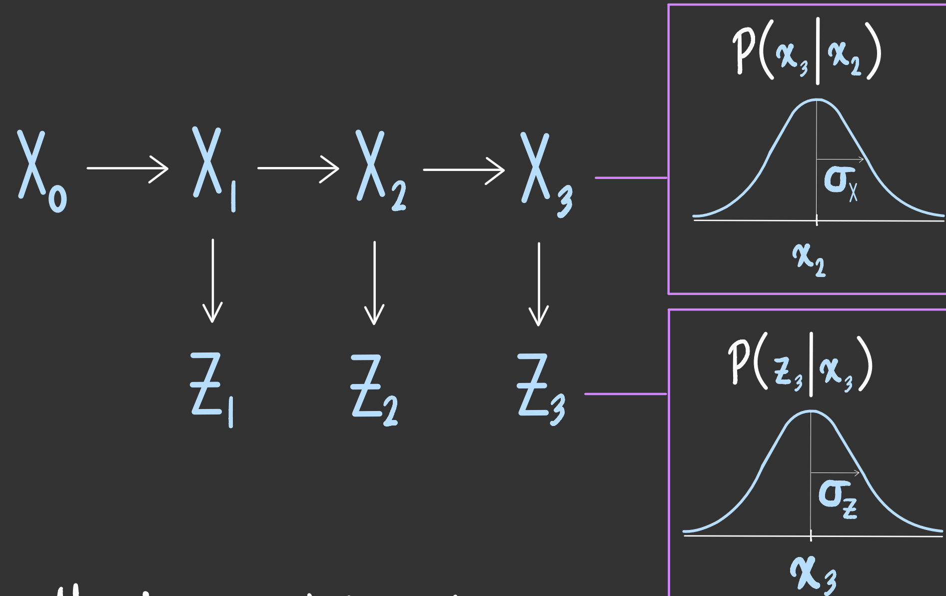
$P(x_3 \mid z_1, z_2, z_3)$



the radar reading at time 3 is normally distributed with mean x_3 and standard deviation σ_Z

want to compute:

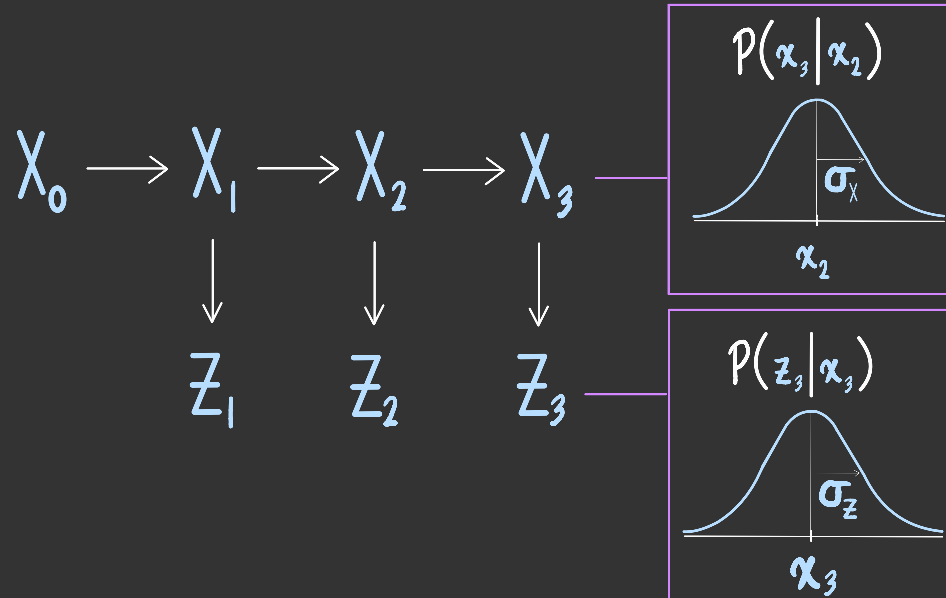
$$P(x_3 \mid z_1, z_2, z_3)$$



but without probability tables, we can't compute probabilities using variable elimination or the junction tree algorithm

want to compute:

$$P(x_3 \mid z_1, z_2, z_3)$$



we need to use **calculus**

interlude:

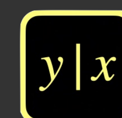
the rules of continuous probability

how do these translate to continuous variables?

given variables $X_1, \ldots, X_n$ with domains $D(X_i)$
and joint probability $P(x_1, \ldots, x_n)$

$$P(x_1, \ldots, x_m) = \sum_{x_{m+1} \in D(X_{m+1})} \cdots \sum_{x_n \in D(X_n)} P(x_1, \ldots, x_n)$$

$$P(x_{m+1}, \ldots, x_n \mid x_1, \ldots, x_m) = \frac{P(x_1, \ldots, x_n)}{P(x_1, \ldots, x_m)}$$



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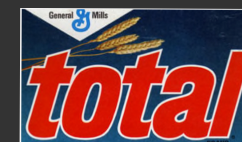


how do these translate to continuous variables?

given variables $X_1, \ldots, X_n$ with domains $D(X_i)$
and joint probability $P(x_1, \ldots, x_n)$

how do we sum over all real numbers?

$$P(x_1, \ldots, x_m) = \sum_{x_{m+1} \in D(X_{m+1})} \cdots \sum_{x_n \in D(X_n)} P(x_1, \ldots, x_n)$$

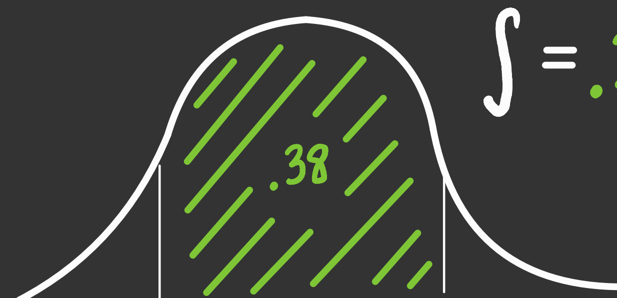
$\sum = .38$

how do these translate to continuous variables?

given variables $X_1, \dots, X_n$ with domains $D(X_i)$
and joint probability $P(x_1, \dots, x_n)$

$$P(x_1, \dots, x_m) = \int \cdots \int P(x_1, \dots, x_n)\, dx_{m+1} \cdots dx_n$$

$\sum = .38$



$\int = .38$

how do these translate to continuous variables?

given variables $X_1, \dots, X_n$ with domains $D(X_i)$
and joint probability $P(x_1, \dots, x_n)$

$$\boxed{P(x_1, \dots, x_m) = \int \cdots \int P(x_1, \dots, x_n)\, dx_{m+1} \cdots dx_n}$$

$$P(x_{m+1}, \dots, x_n \mid x_1, \dots, x_m) = \frac{P(x_1, \dots, x_n)}{P(x_1, \dots, x_m)}$$

$y \mid x$

how do these translate to continuous variables?

given variables $X_1, \dots, X_n$ with domains $D(X_i)$
and joint probability $P(x_1, \dots, x_n)$

$$P(x_1, \dots, x_m) = \int \cdots \int P(x_1, \dots, x_n)\, dx_{m+1} \cdots dx_n$$

$$P(x_{m+1}, \dots, x_n \mid x_1, \dots, x_m) = \frac{P(x_1, \dots, x_n)}{P(x_1, \dots, x_m)}$$

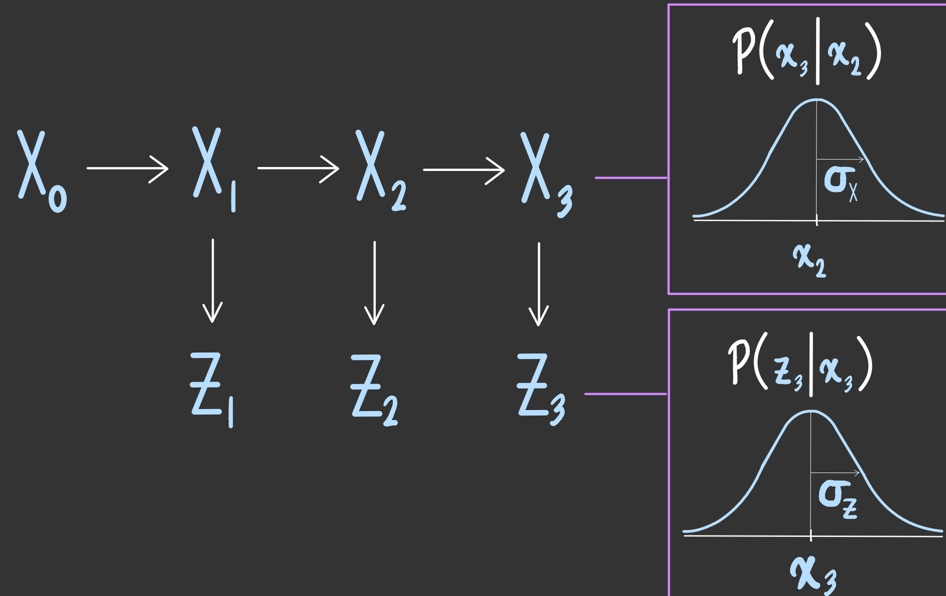
$y \mid x$

end of

interlude

want to compute:

$$P(x_3 \mid x_0, z_1, z_2, z_3)$$



we need to use calculus

$$P(x_3 \mid x_0, z_1, z_2, z_3) \overset{y|x}{=} \boxed{?}$$

$$P(x_3 \mid x_0, z_1, z_2, z_3) \overset{\boxed{y|x}}{=} \frac{P(x_3, x_0, z_1, z_2, z_3)}{P(x_0, z_1, z_2, z_3)}$$

$$P(x_3 \mid x_0, z_1, z_2, z_3) \overset{y|x}{=} \frac{P(x_3, x_0, z_1, z_2, z_3)}{P(x_0, z_1, z_2, z_3)} \overset{\text{total}}{=} \frac{P(x_3, x_0, z_1, z_2, z_3)}{\int P(x_3, x_0, z_1, z_2, z_3)\, dx_3}$$

$$
\begin{aligned}
P(x_3 \mid x_0, z_1, z_2, z_3) &= \frac{P(x_3, x_0, z_1, z_2, z_3)}{P(x_0, z_1, z_2, z_3)} \\
&= \frac{P(x_3, x_0, z_1, z_2, z_3)}{\int P(x_3, x_0, z_1, z_2, z_3)\, dx_3} \\
&= \frac{P(z_3 \mid x_3, x_0, z_1, z_2)\, P(x_3 \mid x_0, z_1, z_2)\, P(z_2 \mid z_1, x_0)\, P(z_1 \mid x_0)\, P(x_0)}{\int P(z_3 \mid x_3, x_0, z_1, z_2)\, P(x_3 \mid x_0, z_1, z_2)\, P(z_2 \mid z_1, x_0)\, P(z_1 \mid x_0)\, P(x_0)\, dx_3}
\end{aligned}
$$

$$
\begin{aligned}
P(x_3 \mid x_0, z_1, z_2, z_3) &= \frac{P(x_3, x_0, z_1, z_2, z_3)}{P(x_0, z_1, z_2, z_3)} \\
&= \frac{P(x_3, x_0, z_1, z_2, z_3)}{\int P(x_3, x_0, z_1, z_2, z_3)\, dx_3} \\
&= \frac{P(z_3 \mid x_3, x_0, z_1, z_2)\, P(x_3 \mid x_0, z_1, z_2)\, P(z_2 \mid z_1, x_0)\, P(z_1 \mid x_0)\, P(x_0)}{\int P(z_3 \mid x_3, x_0, z_1, z_2)\, P(x_3 \mid x_0, z_1, z_2)\, P(z_2 \mid z_1, x_0)\, P(z_1 \mid x_0)\, P(x_0)\, dx_3} \\
&= \frac{P(z_2 \mid z_1, x_0)\, P(z_1 \mid x_0)\, P(x_0)\;\; P(z_3 \mid x_3, x_0, z_1, z_2)\, P(x_3 \mid x_0, z_1, z_2)}{P(z_2 \mid z_1, x_0)\, P(z_1 \mid x_0)\, P(x_0) \int P(z_3 \mid x_3, x_0, z_1, z_2)\, P(x_3 \mid x_0, z_1, z_2)\, dx_3}
\end{aligned}
$$

$$
\begin{aligned}
P(x_3 \mid x_0, z_1, z_2, z_3) &= \frac{P(x_3, x_0, z_1, z_2, z_3)}{P(x_0, z_1, z_2, z_3)} \\
&= \frac{P(x_3, x_0, z_1, z_2, z_3)}{\int P(x_3, x_0, z_1, z_2, z_3)\, dx_3} \\
&= \frac{P(z_3 \mid x_3, x_0, z_1, z_2)\, P(x_3 \mid x_0, z_1, z_2)\, P(z_2 \mid z_1, x_0)\, P(z_1 \mid x_0)\, P(x_0)}{\int P(z_3 \mid x_3, x_0, z_1, z_2)\, P(x_3 \mid x_0, z_1, z_2)\, P(z_2 \mid z_1, x_0)\, P(z_1 \mid x_0)\, P(x_0)\, dx_3} \\
&= \frac{P(z_2 \mid z_1, x_0)\, P(z_1 \mid x_0)\, P(x_0)\; P(z_3 \mid x_3, x_0, z_1, z_2)\, P(x_3 \mid x_0, z_1, z_2)}{P(z_2 \mid z_1, x_0)\, P(z_1 \mid x_0)\, P(x_0) \int P(z_3 \mid x_3, x_0, z_1, z_2)\, P(x_3 \mid x_0, z_1, z_2)\, dx_3} \\
&= \frac{P(z_3 \mid x_3, x_0, z_1, z_2)\, P(x_3 \mid x_0, z_1, z_2)}{\int P(z_3 \mid x_3, x_0, z_1, z_2)\, P(x_3 \mid x_0, z_1, z_2)\, dx_3}
\end{aligned}
$$

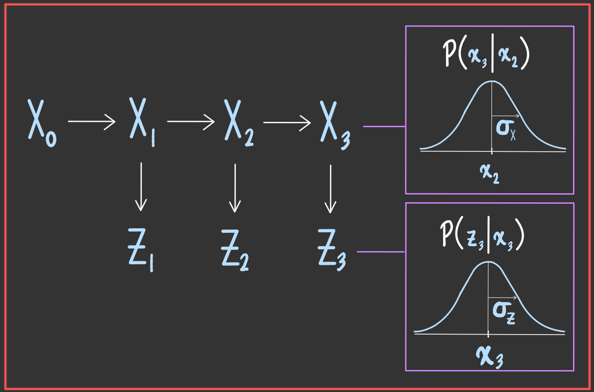
$$
\begin{aligned}
P(x_3 \mid x_0, z_1, z_2, z_3) &= \frac{P(x_3, x_0, z_1, z_2, z_3)}{P(x_0, z_1, z_2, z_3)} \\
&= \frac{P(x_3, x_0, z_1, z_2, z_3)}{\int P(x_3, x_0, z_1, z_2, z_3)\, dx_3} \\
&= \frac{P(z_3 \mid x_3, x_0, z_1, z_2)\, P(x_3 \mid x_0, z_1, z_2)\, P(z_2 \mid z_1, x_0)\, P(z_1 \mid x_0)\, P(x_0)}{\int P(z_3 \mid x_3, x_0, z_1, z_2)\, P(x_3 \mid x_0, z_1, z_2)\, P(z_2 \mid z_1, x_0)\, P(z_1 \mid x_0)\, P(x_0)\, dx_3} \\
&= \frac{P(z_2 \mid z_1, x_0)\, P(z_1 \mid x_0)\, P(x_0)\ P(z_3 \mid x_3, x_0, z_1, z_2)\, P(x_3 \mid x_0, z_1, z_2)}{P(z_2 \mid z_1, x_0)\, P(z_1 \mid x_0)\, P(x_0) \int P(z_3 \mid x_3, x_0, z_1, z_2)\, P(x_3 \mid x_0, z_1, z_2)\, dx_3} \\
&= \frac{P(z_3 \mid x_3, x_0, z_1, z_2)\, P(x_3 \mid x_0, z_1, z_2)}{\int P(z_3 \mid x_3, x_0, z_1, z_2)\, P(x_3 \mid x_0, z_1, z_2)\, dx_3}
\end{aligned}
$$

$$
f(x) = \frac{g(x)}{\int_{-\infty}^{\infty} g(x)\, dx}
$$

$$
\begin{aligned}
P(x_3 \mid x_0, z_1, z_2, z_3) &= \frac{P(x_3, x_0, z_1, z_2, z_3)}{P(x_0, z_1, z_2, z_3)} \\
&= \frac{P(x_3, x_0, z_1, z_2, z_3)}{\int P(x_3, x_0, z_1, z_2, z_3)\, dx_3} \\
&= \frac{P(z_3 \mid x_3, x_0, z_1, z_2)\, P(x_3 \mid x_0, z_1, z_2)\, P(z_2 \mid z_1, x_0)\, P(z_1 \mid x_0)\, P(x_0)}{\int P(z_3 \mid x_3, x_0, z_1, z_2)\, P(x_3 \mid x_0, z_1, z_2)\, P(z_2 \mid z_1, x_0)\, P(z_1 \mid x_0)\, P(x_0)\, dx_3} \\
&= \frac{P(z_2 \mid z_1, x_0)\, P(z_1 \mid x_0)\, P(x_0)\;\; P(z_3 \mid x_3, x_0, z_1, z_2)\, P(x_3 \mid x_0, z_1, z_2)}{P(z_2 \mid z_1, x_0)\, P(z_1 \mid x_0)\, P(x_0) \int P(z_3 \mid x_3, x_0, z_1, z_2)\, P(x_3 \mid x_0, z_1, z_2)\, dx_3} \\
&= \frac{P(z_3 \mid x_3, x_0, z_1, z_2)\, P(x_3 \mid x_0, z_1, z_2)}{\int P(z_3 \mid x_3, x_0, z_1, z_2)\, P(x_3 \mid x_0, z_1, z_2)\, dx_3} \\
&= \;\propto P(z_3 \mid x_3, x_0, z_1, z_2)\, P(x_3 \mid x_0, z_1, z_2)
\end{aligned}
$$

$$P(x_3 \mid x_0, z_1, z_2, z_3) = \propto \underline{P(z_3 \mid x_3, x_0, z_1, z_2)}\,P(x_3 \mid x_0, z_1, z_2)$$

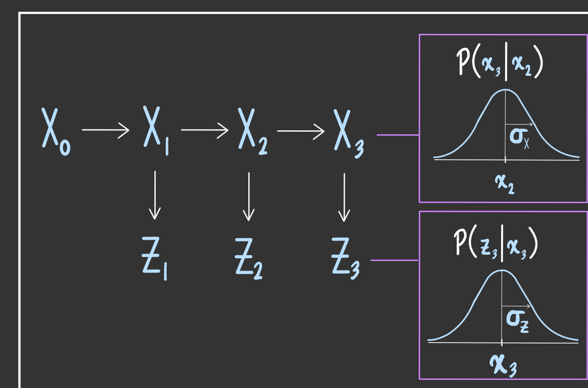
what does the bayesian network tell us?



$$P(x_3 \mid x_0, z_1, z_2, z_3) = \propto P(z_3 \mid x_3, x_0, z_1, z_2)\, P(x_3 \mid x_0, z_1, z_2)$$

$$= \propto P(z_3 \mid x_3)\, P(x_3 \mid x_0, z_1, z_2)$$

$$\begin{tikzcd}
X_0 \arrow[r] & X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & X_3 \arrow[d] \\
& Z_1 & Z_2 & Z_3
\end{tikzcd}$$



$$P(x_3 \mid x_0, z_1, z_2, z_3) = \propto P(z_3 \mid x_3, x_0, z_1, z_2)\, P(x_3 \mid x_0, z_1, z_2)$$

$$= \propto P(z_3 \mid x_3)\, P(x_3 \mid x_0, z_1, z_2)$$

$$= \propto P(z_3 \mid x_3) \int P(x_3 \mid x_2, x_0, z_1, z_2)\, P(x_2 \mid x_0, z_1, z_2)\, dx_2$$

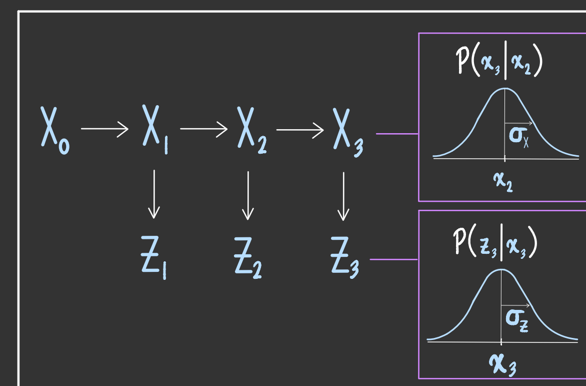
a conditional distribution is still a distribution

$$P(x_3 \mid x_0, z_1, z_2) = Q(x_3)$$

$$\overset{\text{total}}{=} \int Q(x_2, x_3)\, dx_2$$

$$\overset{y|x}{=} \int Q(x_3 \mid x_2)\, Q(x_2)\, dx_2$$

$$= \int P(x_3 \mid x_2, x_0, z_1, z_2)\, P(x_2 \mid x_0, z_1, z_2)\, dx_2$$

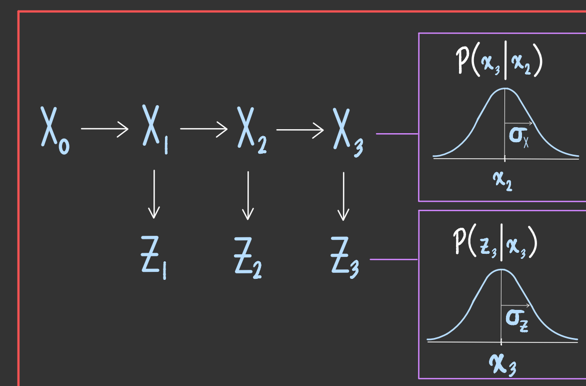


$$P(x_3 \mid x_0, z_1, z_2, z_3) = \propto P(z_3 \mid x_3, x_0, z_1, z_2)\, P(x_3 \mid x_0, z_1, z_2)$$

$$= \propto P(z_3 \mid x_3)\, P(x_3 \mid x_0, z_1, z_2)$$

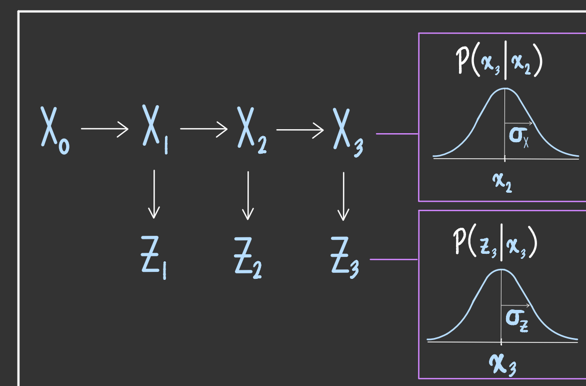
$$= \propto P(z_3 \mid x_3) \int \underline{P(x_3 \mid x_2, x_0, z_1, z_2)}\, P(x_2 \mid x_0, z_1, z_2)\, dx_2$$

what does the bayesian network tell us?



$$
\begin{aligned}
P(x_3 \mid x_0, z_1, z_2, z_3) &= \propto P(z_3 \mid x_3, x_0, z_1, z_2)\, P(x_3 \mid x_0, z_1, z_2) \\
&= \propto P(z_3 \mid x_3)\, P(x_3 \mid x_0, z_1, z_2) \\
&= \propto P(z_3 \mid x_3) \int P(x_3 \mid x_2, x_0, z_1, z_2)\, P(x_2 \mid x_0, z_1, z_2)\, dx_2 \\
&= \propto P(z_3 \mid x_3) \int P(x_3 \mid x_2)\, \underbrace{P(x_2 \mid x_0, z_1, z_2)}_{\text{can't simplify further}}\, dx_2
\end{aligned}
$$

$$
\begin{tikzcd}
X_0 \arrow[r] & X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & X_3 \arrow[d] \\
 & Z_1 & Z_2 & Z_3
\end{tikzcd}
$$

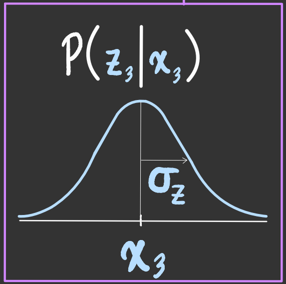


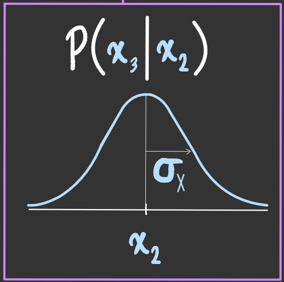
$$P(x_3 \mid x_0, z_1, z_2, z_3) = \propto P(z_3 \mid x_3, x_0, z_1, z_2)\, P(x_3 \mid x_0, z_1, z_2)$$

$$= \propto P(z_3 \mid x_3)\, P(x_3 \mid x_0, z_1, z_2)$$

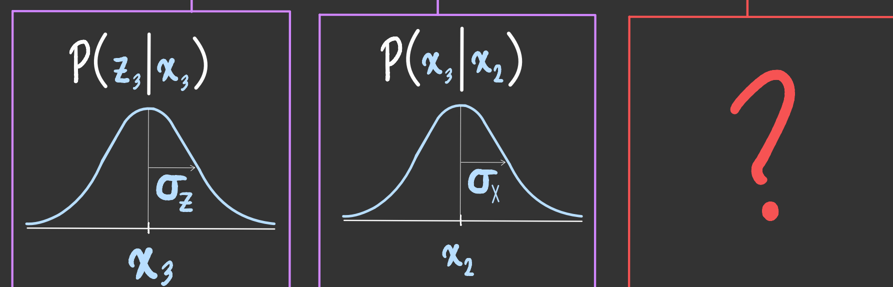
$$= \propto P(z_3 \mid x_3) \int P(x_3 \mid x_2, x_0, z_1, z_2)\, P(x_2 \mid x_0, z_1, z_2)\, dx_2$$

$$= \propto P(z_3 \mid x_3) \int P(x_3 \mid x_2)\, P(x_2 \mid x_0, z_1, z_2)\, dx_2$$

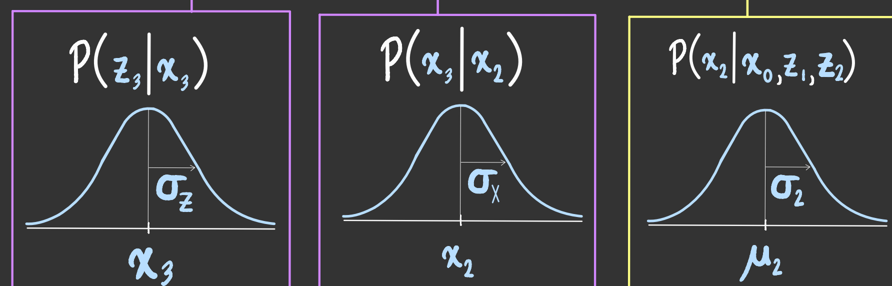




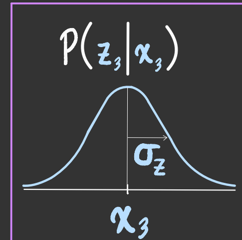
$$
\begin{aligned}
P(x_3 \mid x_0, z_1, z_2, z_3) &= \alpha\, P(z_3 \mid x_3, x_0, z_1, z_2)\, P(x_3 \mid x_0, z_1, z_2) \\
&= \alpha\, P(z_3 \mid x_3)\, P(x_3 \mid x_0, z_1, z_2) \\
&= \alpha\, P(z_3 \mid x_3) \int P(x_3 \mid x_2, x_0, z_1, z_2)\, P(x_2 \mid x_0, z_1, z_2)\, dx_2 \\
&= \alpha\, P(z_3 \mid x_3) \int P(x_3 \mid x_2)\, P(x_2 \mid x_0, z_1, z_2)\, dx_2
\end{aligned}
$$

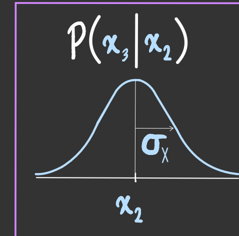


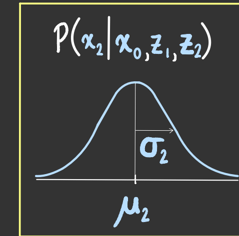
$$
\begin{aligned}
P(x_3 \mid x_0, z_1, z_2, z_3) &= \propto P(z_3 \mid x_3, x_0, z_1, z_2)\, P(x_3 \mid x_0, z_1, z_2) \\
&= \propto P(z_3 \mid x_3)\, P(x_3 \mid x_0, z_1, z_2) \\
&= \propto P(z_3 \mid x_3) \int P(x_3 \mid x_2, x_0, z_1, z_2)\, P(x_2 \mid x_0, z_1, z_2)\, dx_2 \\
&= \propto P(z_3 \mid x_3) \int P(x_3 \mid x_2)\, P(x_2 \mid x_0, z_1, z_2)\, dx_2
\end{aligned}
$$



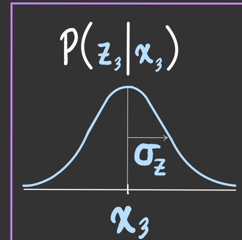
let's assume this is normally distributed with mean μ_2 and standard deviation σ_2

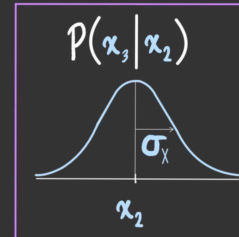


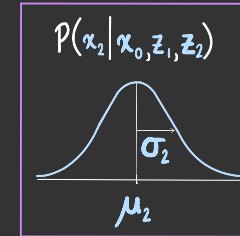




$$P(x_3|x_0,z_1,z_2,z_3) = \alpha \cdot P(z_3|x_3) \int P(x_3|x_2) \cdot P(x_2|x_0,z_1,z_2)\,dx_2$$



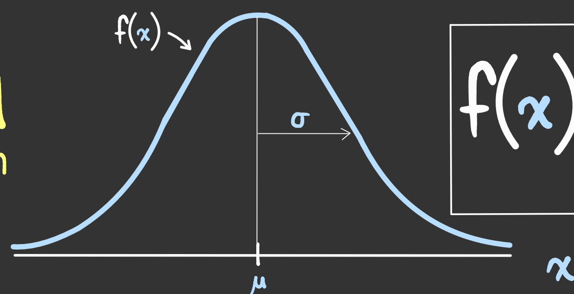




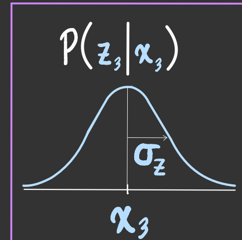
$$P(x_3 | x_0, z_1, z_2, z_3) = \alpha \cdot P(z_3 | x_3) \int P(x_3 | x_2) \cdot P(x_2 | x_0, z_1, z_2)\, dx_2$$

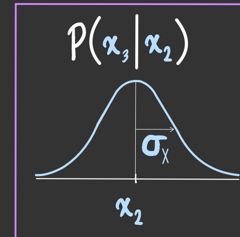
$$= \alpha \frac{1}{\sigma_z \sqrt{2\pi}} e^{-\frac{(z_3 - x_3)^2}{2\sigma_z^2}} \int \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{(x_3 - x_2)^2}{2\sigma_x^2}} \frac{1}{\sigma_2 \sqrt{2\pi}} e^{-\frac{(x_2 - \mu_2)^2}{2\sigma_2^2}}\, dx_2$$

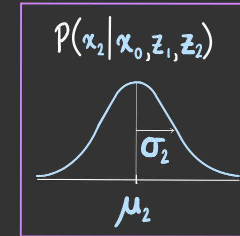
normal
distribution



$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

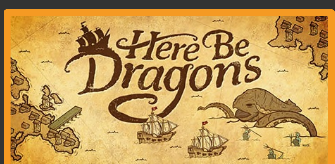






$$P(x_3 \mid x_0, z_1, z_2, z_3) = \alpha \cdot P(z_3 \mid x_3) \int P(x_3 \mid x_2) \cdot P(x_2 \mid x_0, z_1, z_2)\, dx_2$$

$$= \alpha \frac{1}{\sigma_z \sqrt{2\pi}} e^{-\frac{(z_3 - x_3)^2}{2\sigma_z^2}} \int \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{(x_3 - x_2)^2}{2\sigma_x^2}} \frac{1}{\sigma_2 \sqrt{2\pi}} e^{-\frac{(x_2 - \mu_2)^2}{2\sigma_2^2}}\, dx_2$$



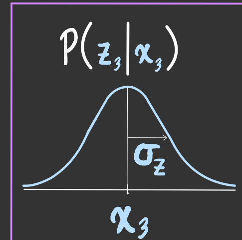
$$\vdots$$

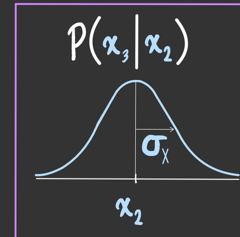
$$= \frac{1}{\sigma_3 \sqrt{2\pi}} e^{\frac{-(x_3 - \mu_3)^2}{2\sigma_3^2}}$$

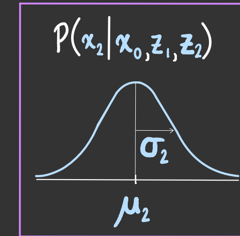
where

$$\mu_3 = \frac{(\sigma_2^2 + \sigma_x^2) z_3 + \sigma_z^2 \mu_2}{\sigma_2^2 + \sigma_x^2 + \sigma_z^2}$$

$$\sigma_3^2 = \frac{(\sigma_2^2 + \sigma_x^2) \sigma_z^2}{\sigma_2^2 + \sigma_x^2 + \sigma_z^2}$$

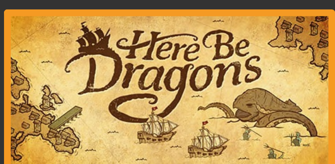






$$P(x_3|x_0,z_1,z_2,z_3) = \propto \cdot P(z_3|x_3) \int P(x_3|x_2) \cdot P(x_2|x_0,z_1,z_2)\, dx_2$$

$$= \propto \frac{1}{\sigma_z\sqrt{2\pi}} e^{-\frac{(z_3-x_3)^2}{2\sigma_z^2}} \int \frac{1}{\sigma_x\sqrt{2\pi}} e^{-\frac{(x_3-x_2)^2}{2\sigma_x^2}} \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{(x_2-\mu_2)^2}{2\sigma_2^2}}\, dx_2$$



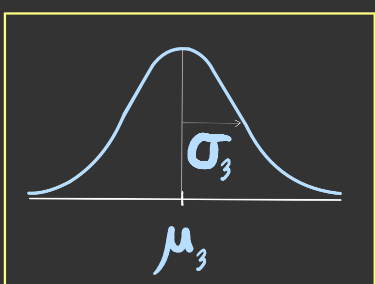
$$\vdots$$

$$= \frac{1}{\sigma_3\sqrt{2\pi}} e^{\frac{-(x_3-\mu_3)^2}{2\sigma_3^2}}$$

where

$$\mu_3 = \frac{(\sigma_2^2+\sigma_x^2)z_3 + \sigma_z^2\mu_2}{\sigma_2^2+\sigma_x^2+\sigma_z^2}$$

$$\sigma_3^2 = \frac{(\sigma_2^2+\sigma_x^2)\sigma_z^2}{\sigma_2^2+\sigma_x^2+\sigma_z^2}$$

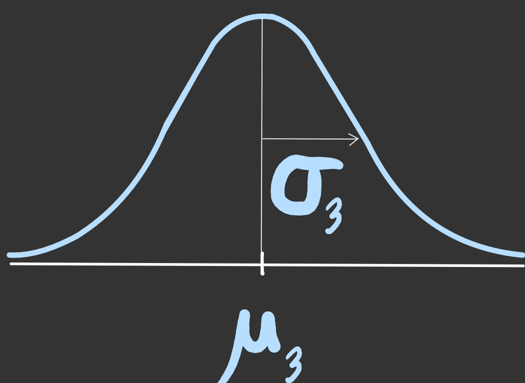


another normal distribution!

so if $P(x_2 \mid x_0, z_1, z_2)$ is a normal distribution,

then so is $P(x_3 \mid x_0, z_1, z_2, z_3)$

$P(x_3 \mid x_0, z_1, z_2, z_3)$



where

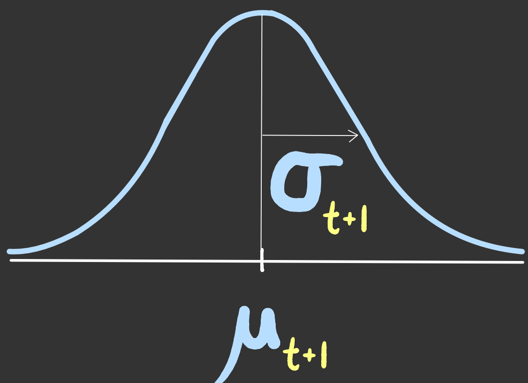
$$\mu_3 = \frac{(\sigma_2^2 + \sigma_x^2) z_3 + \sigma_z^2 \mu_2}{\sigma_2^2 + \sigma_x^2 + \sigma_z^2}$$

$$\sigma_3^2 = \frac{(\sigma_2^2 + \sigma_x^2) \sigma_z^2}{\sigma_2^2 + \sigma_x^2 + \sigma_z^2}$$

so if $P(x_t | x_0, z_1, \ldots, z_t)$ is a normal distribution,

then so is $P(x_{t+1} | x_0, z_1, \ldots, z_{t+1})$

$P(x_{t+1} | x_0, z_1, \ldots, z_{t+1})$



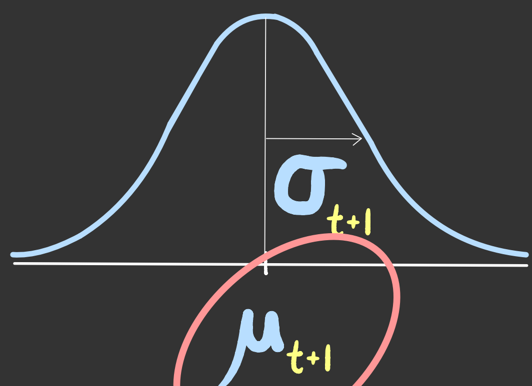
where

$$\mu_{t+1} = \frac{(\sigma_t^2 + \sigma_x^2) z_{t+1} + \sigma_z^2 \mu_t}{\sigma_t^2 + \sigma_x^2 + \sigma_z^2}$$

$$\sigma_{t+1}^2 = \frac{(\sigma_t^2 + \sigma_x^2) \sigma_z^2}{\sigma_t^2 + \sigma_x^2 + \sigma_z^2}$$

and the most likely relative position of the plane is just the **mean** of the distribution

$$P(x_{t+1} \mid x_0, z_1, \ldots, z_{t+1})$$



where

$$\mu_{t+1} = \frac{(\sigma_t^2 + \sigma_x^2) z_{t+1} + \sigma_z^2 \mu_t}{\sigma_t^2 + \sigma_x^2 + \sigma_z^2}$$

$$\sigma_{t+1}^2 = \frac{(\sigma_t^2 + \sigma_x^2)\sigma_z^2}{\sigma_t^2 + \sigma_x^2 + \sigma_z^2}$$

if $P(x_t \mid x_0, z_1, \ldots, z_t)$ is a normal distribution,

then so is $P(x_{t+1} \mid x_0, z_1, \ldots, z_{t+1})$

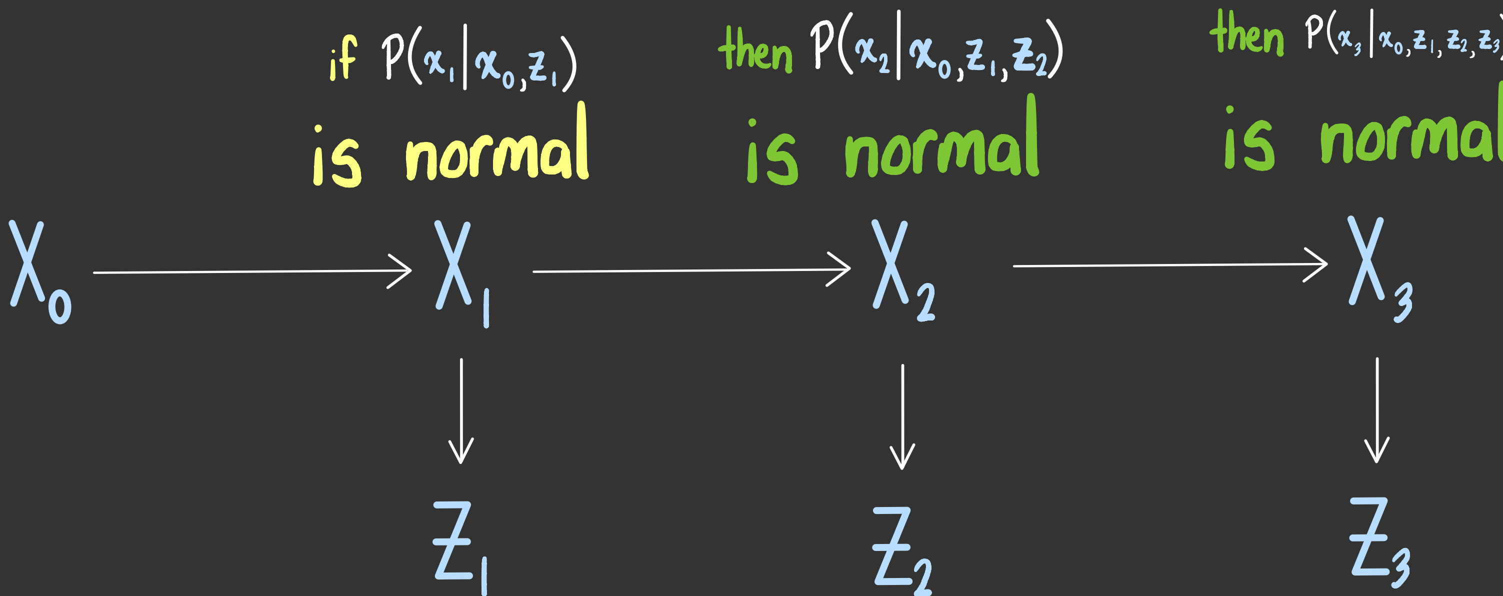
if $P(x_2 \mid x_0, z_1, z_2)$ is normal

then $P(x_3 \mid x_0, z_1, z_2, z_3)$ is normal

$$
\begin{tikzcd}
X_0 \arrow[r] & X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & X_3 \arrow[d] \\
 & Z_1 & Z_2 & Z_3
\end{tikzcd}
$$

if $P(x_t \mid x_0, z_1, \dots, z_t)$ is a normal distribution,

then so is $P(x_{t+1} \mid x_0, z_1, \dots, z_{t+1})$



if $P(x_t | x_0, z_1, \dots, z_t)$ is a normal distribution,

then so is $P(x_{t+1} | x_0, z_1, \dots, z_{t+1})$

if $P(x_0)$ is normal

then $P(x_1 | x_0, z_1)$ is normal

then $P(x_2 | x_0, z_1, z_2)$ is normal

then $P(x_3 | x_0, z_1, z_2, z_3)$ is normal

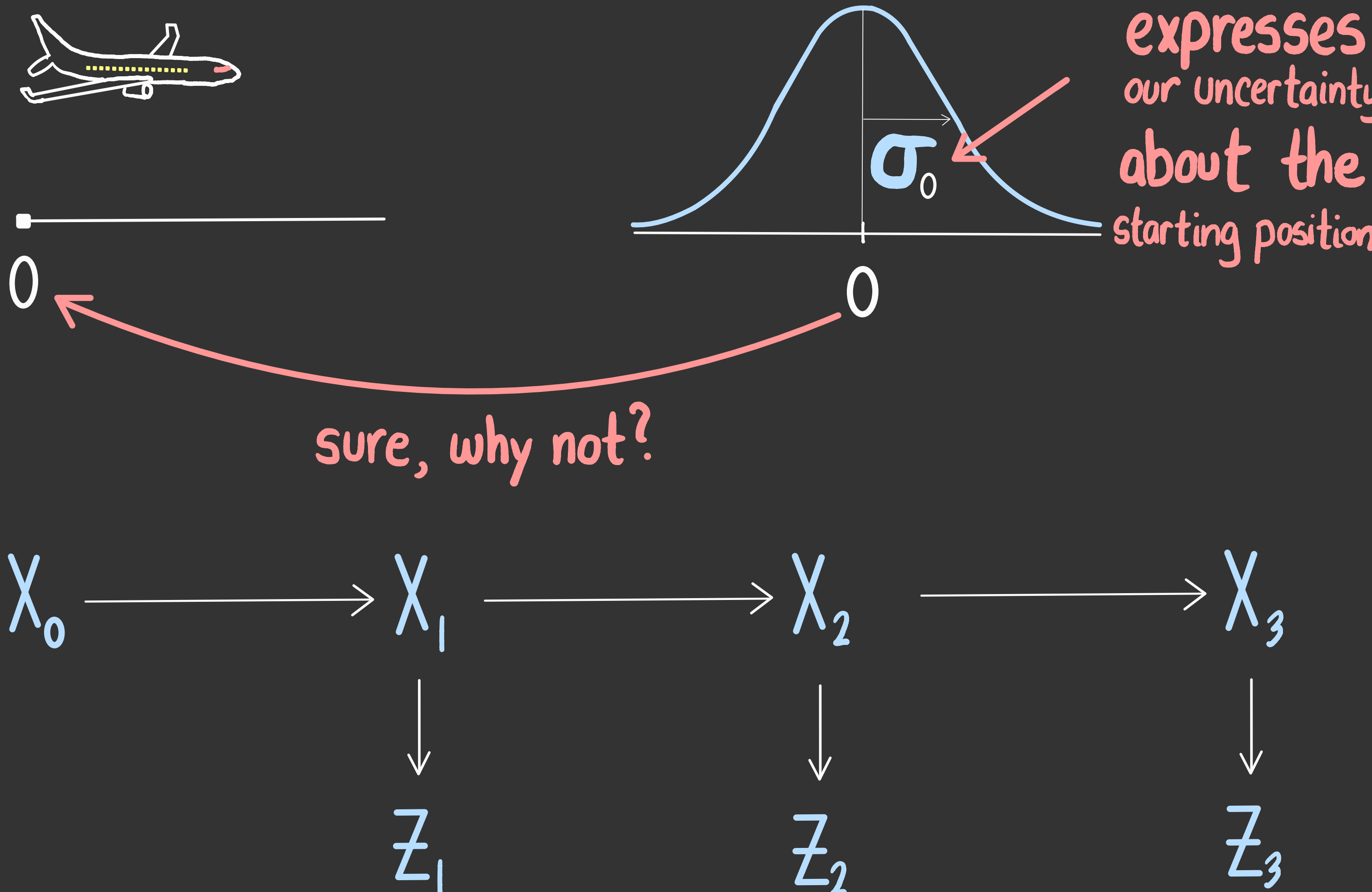
$$\begin{tikzcd}
X_0 \arrow[r] & X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & X_3 \arrow[d] \\
 & Z_1 & Z_2 & Z_3
\end{tikzcd}$$

if $P(x_t \mid x_0, z_1, \dots, z_t)$ is a normal distribution,

then so is $P(x_{t+1} \mid x_0, z_1, \dots, z_{t+1})$

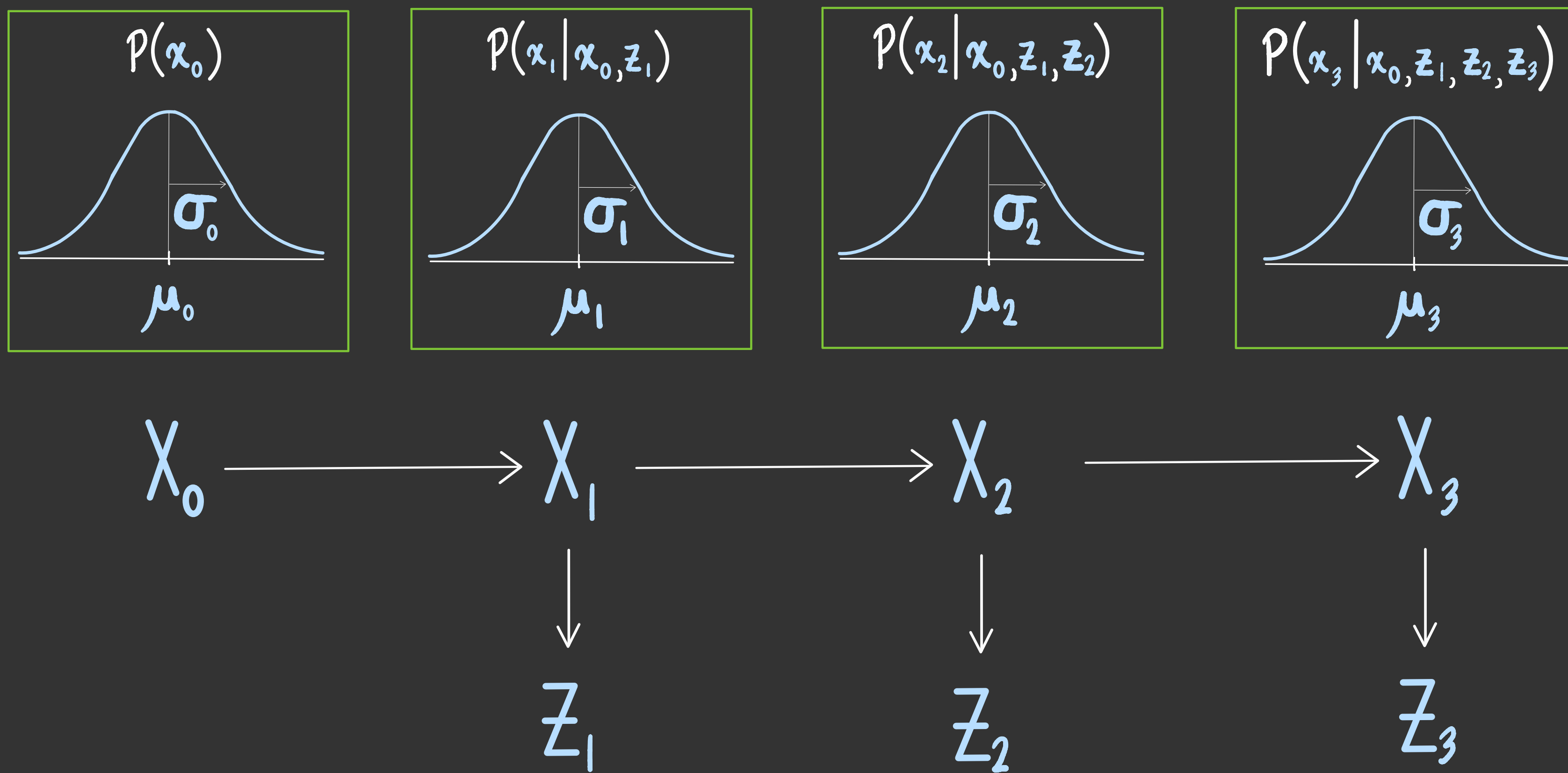
is $P(x_0)$ normal?

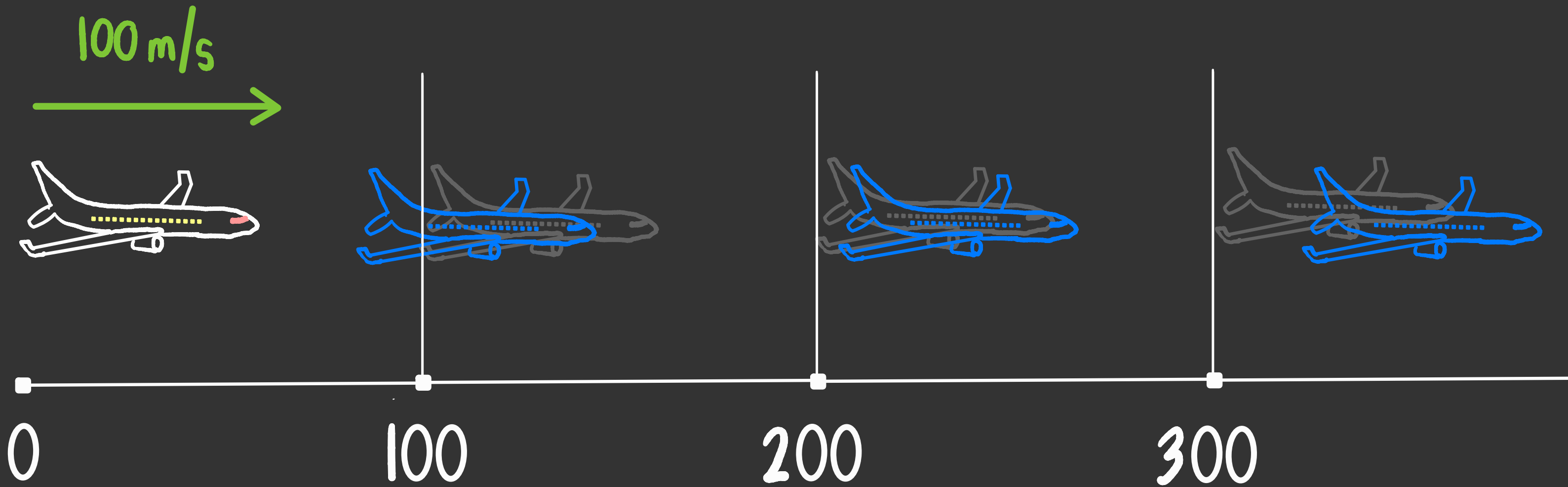
$$
\begin{tikzcd}
X_0 \arrow[r] & X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & X_3 \arrow[d] \\
 & Z_1 & Z_2 & Z_3
\end{tikzcd}
$$



σ_0
expresses our uncertainty about the starting position
0
0
sure, why not?
X_0
X_1
X_2
X_3
Z_1
Z_2
Z_3

we can keep efficiently updating our estimate of the plane's position at each time step

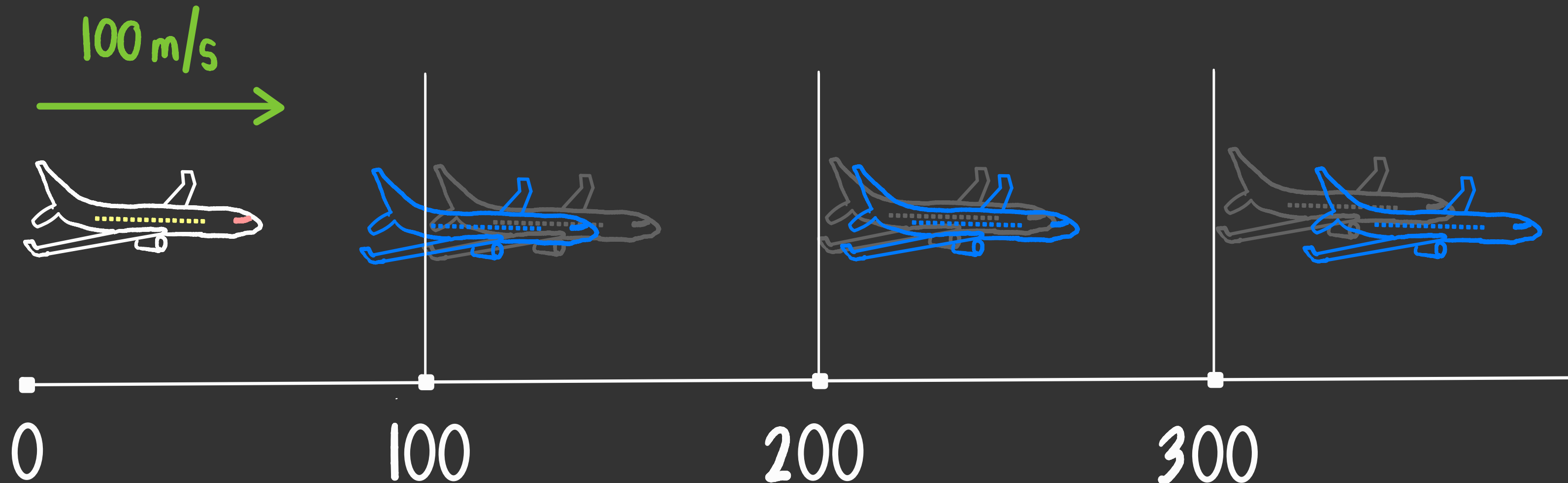




this approach fuses:

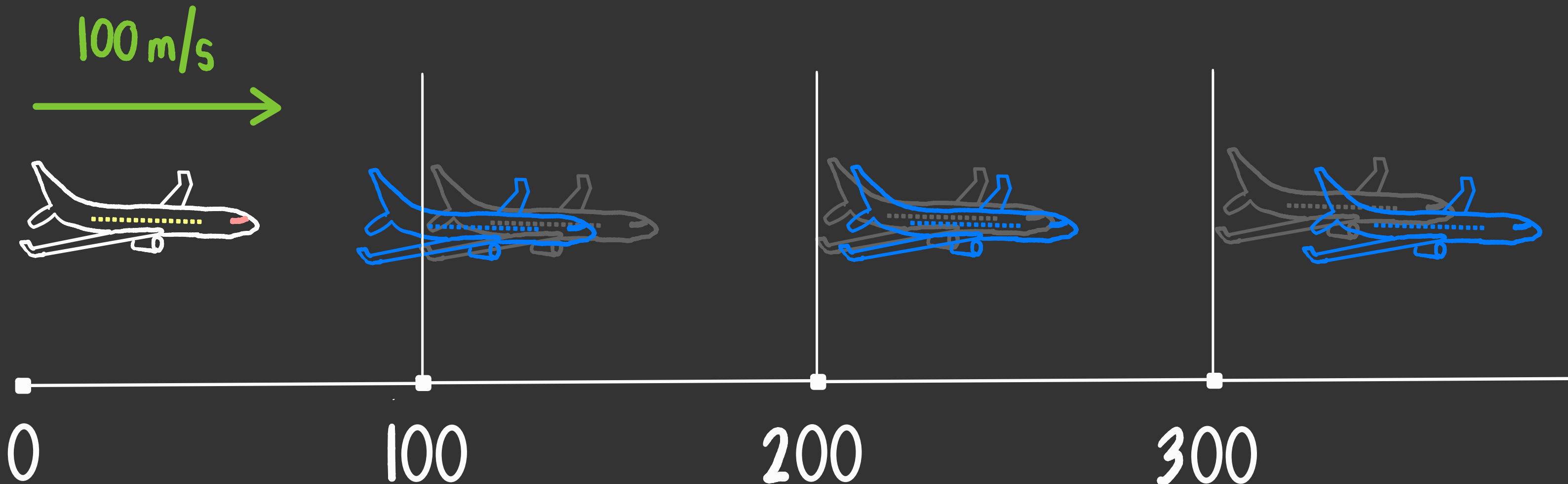
- our prior knowledge of the velocity
- our noisy radar observations

to produce a probabilistically optimal prediction



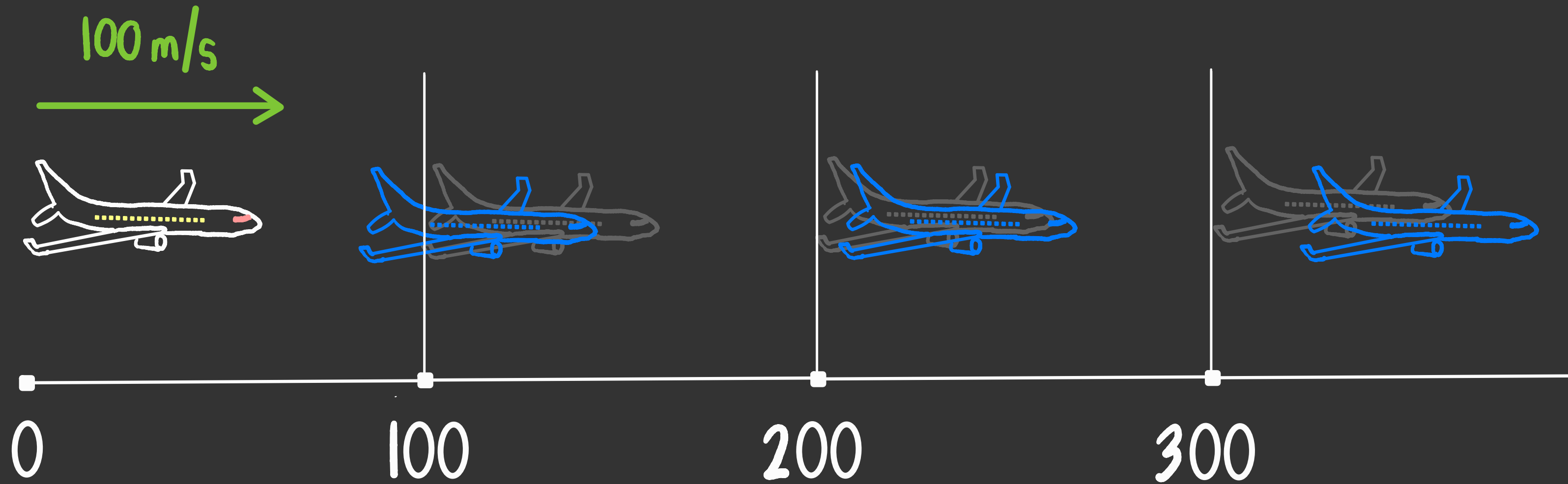
$$\mu_{t+1} = \frac{(\sigma_t^2 + \sigma_x^2) z_{t+1} + \sigma_z^2 \mu_t}{\sigma_t^2 + \sigma_x^2 + \sigma_z^2}$$

our previous estimate of the relative position



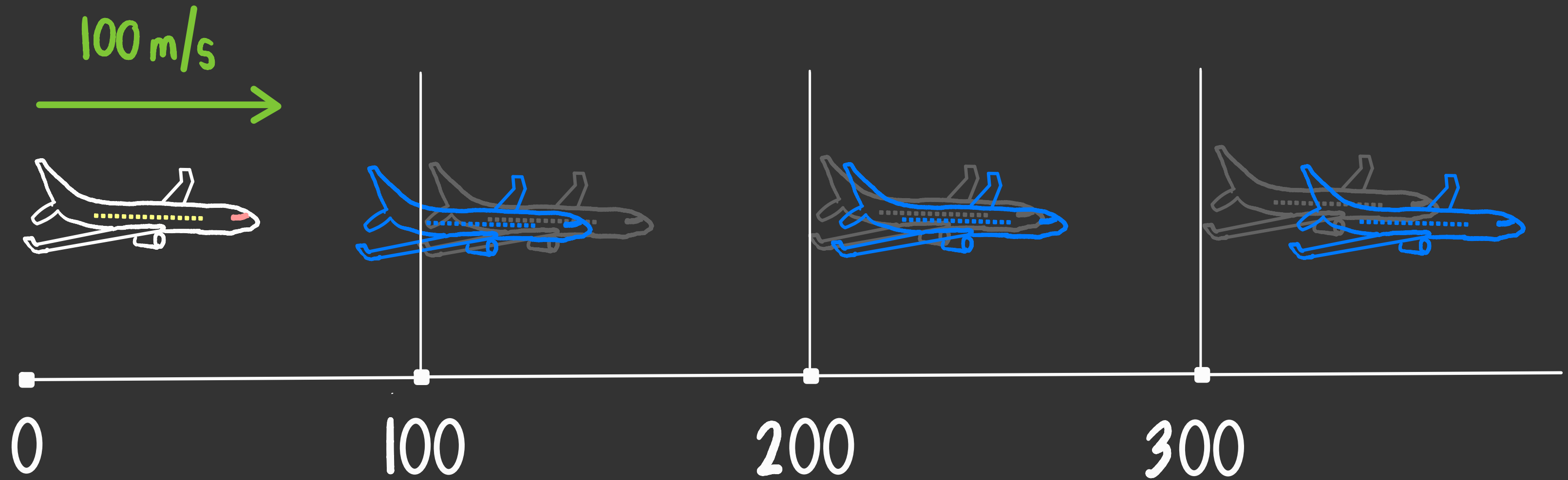
$$\mu_{t+1} = \frac{(\sigma_t^2+\sigma_x^2)\,z_{t+1} + \sigma_z^2\,\mu_t}{\sigma_t^2+\sigma_x^2+\sigma_z^2}$$

our radar observation



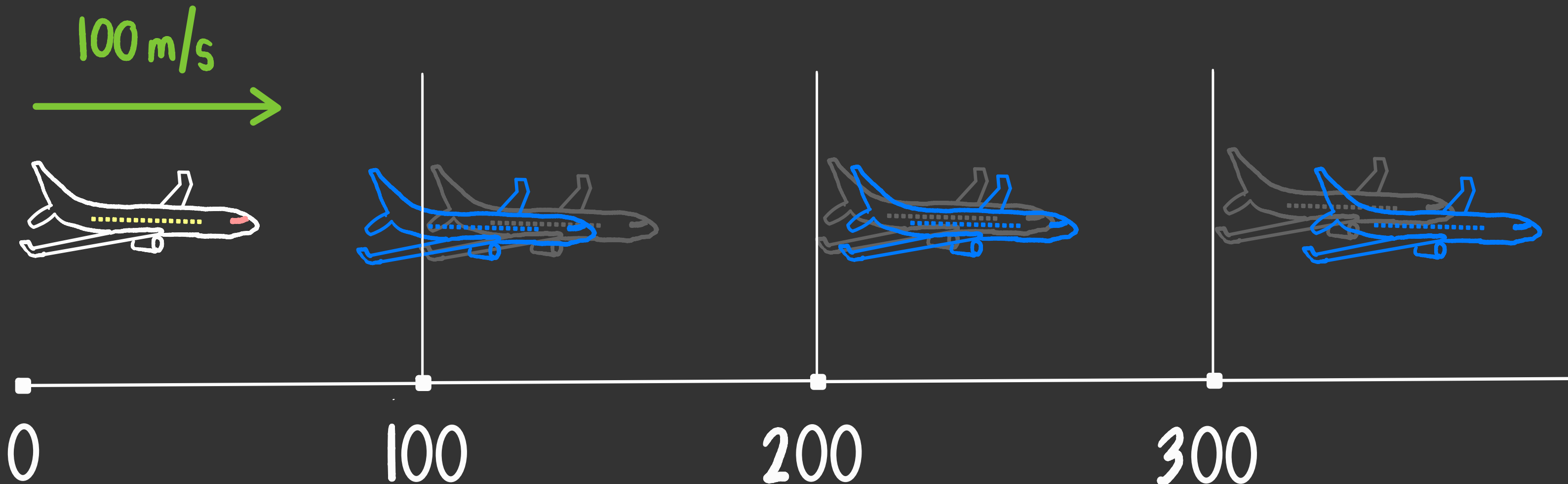
$$\mu_{t+1} = \frac{\sigma_t^2 + \sigma_x^2}{\sigma_t^2 + \sigma_x^2 + \sigma_z^2} z_{t+1} + \frac{\sigma_z^2}{\sigma_t^2 + \sigma_x^2 + \sigma_z^2} \mu_t$$

this information is weighted by the different "noises"



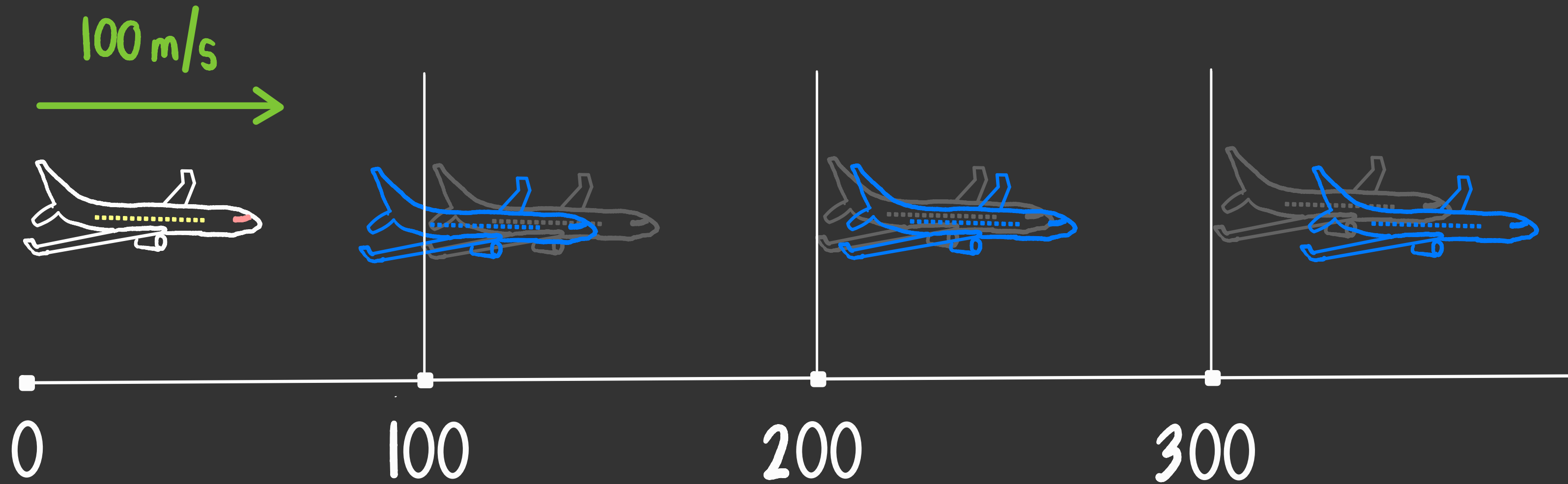
$$\mu_{t+1} = \frac{\sigma_t^2 + \sigma_x^2}{\sigma_t^2 + \sigma_x^2 + \sigma_z^2} z_{t+1} + \frac{\sigma_z^2}{\sigma_t^2 + \sigma_x^2 + \sigma_z^2} \mu_t$$

the less variance in velocity, the less we rely on the radar observations



$$\mu_{t+1} = \frac{\sigma_t^2 + \sigma_x^2}{\sigma_t^2 + \sigma_x^2 + \sigma_z^2} z_{t+1} + \frac{\sigma_z^2}{\sigma_t^2 + \sigma_x^2 + \sigma_z^2} \mu_t$$

the less noisy the radar, the less we rely on the estimate of the previous position



$$\mu_{t+1} = \frac{\sigma_t^2 + \sigma_x^2}{\sigma_t^2 + \sigma_x^2 + \sigma_z^2} z_{t+1} + \frac{\sigma_z^2}{\sigma_t^2 + \sigma_x^2 + \sigma_z^2} \mu_t$$

this model is called a **kalman filter**