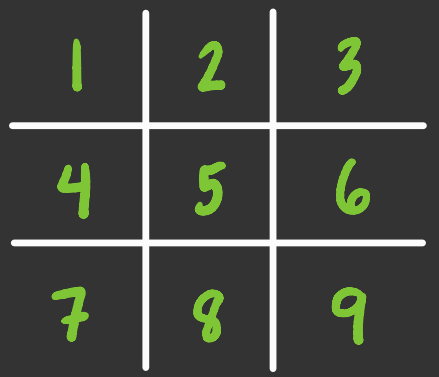
# tic-tac-toe workshop

CSCI 373

suppose we want to reason logically about

# tic-tac-toe




let $x_i$ be whether square $i$ has an ✕

let $o_i$ be whether square $i$ has an ○

let $w_\times$ be whether ✕ won

let $w_\circ$ be whether ○ won

# rules of tic-tac-toe

- no square has both X and O
- three X's in a row wins
- three O's in a row wins
- X and O can't both win

no square has both

$$
\begin{aligned}
&\neg(x_1 \wedge o_1) \\
\wedge\ &\neg(x_2 \wedge o_2) \\
\wedge\ &\neg(x_3 \wedge o_3) \\
\wedge\ &\neg(x_4 \wedge o_4) \\
\wedge\ &\neg(x_5 \wedge o_5) \\
\wedge\ &\neg(x_6 \wedge o_6) \\
\wedge\ &\neg(x_7 \wedge o_7) \\
\wedge\ &\neg(x_8 \wedge o_8) \\
\wedge\ &\neg(x_9 \wedge o_9)
\end{aligned}
$$

three X's in a row wins

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |
| 7 | 8 | 9 |

$$
\begin{aligned}
& ((x_1 \wedge x_2 \wedge x_3) \Rightarrow w_X) \\
\wedge\ & ((x_4 \wedge x_5 \wedge x_6) \Rightarrow w_X) \\
\wedge\ & ((x_7 \wedge x_8 \wedge x_9) \Rightarrow w_X) \\
\wedge\ & ((x_1 \wedge x_4 \wedge x_7) \Rightarrow w_X) \\
\wedge\ & ((x_2 \wedge x_5 \wedge x_8) \Rightarrow w_X) \\
\wedge\ & ((x_3 \wedge x_6 \wedge x_9) \Rightarrow w_X) \\
\wedge\ & ((x_1 \wedge x_5 \wedge x_9) \Rightarrow w_X) \\
\wedge\ & ((x_3 \wedge x_5 \wedge x_7) \Rightarrow w_X)
\end{aligned}
$$

three ○'s in a row wins.

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |
| 7 | 8 | 9 |

$$
\begin{aligned}
&((o_1 \wedge o_2 \wedge o_3) \Rightarrow w_{\bigcirc}) \\
\wedge\ &((o_4 \wedge o_5 \wedge o_6) \Rightarrow w_{\bigcirc}) \\
\wedge\ &((o_7 \wedge o_8 \wedge o_9) \Rightarrow w_{\bigcirc}) \\
\wedge\ &((o_1 \wedge o_4 \wedge o_7) \Rightarrow w_{\bigcirc}) \\
\wedge\ &((o_2 \wedge o_5 \wedge o_8) \Rightarrow w_{\bigcirc}) \\
\wedge\ &((o_3 \wedge o_6 \wedge o_9) \Rightarrow w_{\bigcirc}) \\
\wedge\ &((o_1 \wedge o_5 \wedge o_9) \Rightarrow w_{\bigcirc}) \\
\wedge\ &((o_3 \wedge o_5 \wedge o_7) \Rightarrow w_{\bigcirc})
\end{aligned}
$$


can't both

win

$$\neg(w_{\times} \wedge w_{\bigcirc})$$

# rules of tic-tac-toe

- no square has both X and O
- three X's in a row wins
- three O's in a row wins
- X and O can't both win

$$\neg(x_1 \wedge o_1) \wedge \neg(x_2 \wedge o_2) \wedge \neg(x_3 \wedge o_3) \wedge \neg(x_4 \wedge o_4) \wedge \neg(x_5 \wedge o_5) \wedge \neg(x_6 \wedge o_6) \wedge \neg(x_7 \wedge o_7) \wedge \neg(x_8 \wedge o_8) \wedge \neg(x_9 \wedge o_9)$$

$$((x_1 \wedge x_2 \wedge x_3) \Rightarrow w_X) \wedge ((x_4 \wedge x_5 \wedge x_6) \Rightarrow w_X) \wedge ((x_7 \wedge x_8 \wedge x_9) \Rightarrow w_X) \wedge ((x_1 \wedge x_4 \wedge x_7) \Rightarrow w_X) \wedge ((x_2 \wedge x_5 \wedge x_8) \Rightarrow w_X) \wedge ((x_3 \wedge x_6 \wedge x_9) \Rightarrow w_X) \wedge ((x_1 \wedge x_5 \wedge x_9) \Rightarrow w_X) \wedge ((x_3 \wedge x_5 \wedge x_7) \Rightarrow w_X)$$

$$((o_1 \wedge o_2 \wedge o_3) \Rightarrow w_O) \wedge ((o_4 \wedge o_5 \wedge o_6) \Rightarrow w_O) \wedge ((o_7 \wedge o_8 \wedge o_9) \Rightarrow w_O) \wedge ((o_1 \wedge o_4 \wedge o_7) \Rightarrow w_O) \wedge ((o_2 \wedge o_5 \wedge o_8) \Rightarrow w_O) \wedge ((o_3 \wedge o_6 \wedge o_9) \Rightarrow w_O) \wedge ((o_1 \wedge o_5 \wedge o_9) \Rightarrow w_O) \wedge ((o_3 \wedge o_5 \wedge o_7) \Rightarrow w_O)$$

$$\neg(w_X \wedge w_O)$$

# rules of tic-tac-toe

no square has both $X$ and $O$

and

three $X$'s in a row wins

and

three $O$'s in a row wins

and

$X$ and $O$ can't both win

$$\neg(x_1 \wedge o_1) \wedge \neg(x_2 \wedge o_2) \wedge \neg(x_3 \wedge o_3) \wedge \neg(x_4 \wedge o_4) \wedge \neg(x_5 \wedge o_5) \wedge \neg(x_6 \wedge o_6) \wedge \neg(x_7 \wedge o_7) \wedge \neg(x_8 \wedge o_8) \wedge \neg(x_9 \wedge o_9)$$

$$\wedge$$

$$((x_1 \wedge x_2 \wedge x_3) \Rightarrow w_X) \wedge ((x_4 \wedge x_5 \wedge x_6) \Rightarrow w_X) \wedge ((x_7 \wedge x_8 \wedge x_9) \Rightarrow w_X) \wedge ((x_1 \wedge x_4 \wedge x_7) \Rightarrow w_X) \wedge ((x_2 \wedge x_5 \wedge x_8) \Rightarrow w_X) \wedge ((x_3 \wedge x_6 \wedge x_9) \Rightarrow w_X) \wedge ((x_1 \wedge x_5 \wedge x_9) \Rightarrow w_X)$$

$$\wedge$$

$$((o_1 \wedge o_2 \wedge o_3) \Rightarrow w_O) \wedge ((o_4 \wedge o_5 \wedge o_6) \Rightarrow w_O) \wedge ((o_7 \wedge o_8 \wedge o_9) \Rightarrow w_O) \wedge ((o_1 \wedge o_4 \wedge o_7) \Rightarrow w_O) \wedge ((o_2 \wedge o_5 \wedge o_8) \Rightarrow w_O) \wedge ((o_3 \wedge o_6 \wedge o_9) \Rightarrow w_O) \wedge ((o_1 \wedge o_5 \wedge o_9) \Rightarrow w_O)$$

$$\wedge$$

$$\neg(w_X \wedge w_O)$$

# this is just a long conjunction

no square has both X and O

and

three X's in a row wins

and

three O's in a row wins

and

X and O can't both win

$$\neg(x_1 \wedge o_1) \wedge \neg(x_2 \wedge o_2) \wedge \neg(x_3 \wedge o_3) \wedge \neg(x_4 \wedge o_4) \wedge \neg(x_5 \wedge o_5) \wedge \neg(x_6 \wedge o_6) \wedge \neg(x_7 \wedge o_7) \wedge \neg(x_8 \wedge o_8) \wedge \neg(x_9 \wedge o_9)$$

$$\wedge$$

$$((x_1 \wedge x_2 \wedge x_3) \Rightarrow w_X) \wedge ((x_4 \wedge x_5 \wedge x_6) \Rightarrow w_X) \wedge ((x_7 \wedge x_8 \wedge x_9) \Rightarrow w_X) \wedge ((x_1 \wedge x_4 \wedge x_7) \Rightarrow w_X) \wedge ((x_2 \wedge x_5 \wedge x_8) \Rightarrow w_X) \wedge ((x_3 \wedge x_6 \wedge x_9) \Rightarrow w_X) \wedge ((x_1 \wedge x_5 \wedge x_9) \Rightarrow w_X)$$

$$\wedge$$

$$((o_1 \wedge o_2 \wedge o_3) \Rightarrow w_O) \wedge ((o_4 \wedge o_5 \wedge o_6) \Rightarrow w_O) \wedge ((o_7 \wedge o_8 \wedge o_9) \Rightarrow w_O) \wedge ((o_1 \wedge o_4 \wedge o_7) \Rightarrow w_O) \wedge ((o_2 \wedge o_5 \wedge o_8) \Rightarrow w_O) \wedge ((o_3 \wedge o_6 \wedge o_9) \Rightarrow w_O) \wedge ((o_1 \wedge o_5 \wedge o_9) \Rightarrow w_O)$$

$$\wedge$$

$$\neg(w_X \wedge w_O)$$

# this is just a long conjunction

$$\text{notBoth} = \{\neg(x_1 \wedge o_1), \neg(x_2 \wedge o_2), \neg(x_3 \wedge o_3), \neg(x_4 \wedge o_4), \neg(x_5 \wedge o_5), \neg(x_6 \wedge o_6), \neg(x_7 \wedge o_7), \neg(x_8 \wedge o_8), \neg(x_9 \wedge o_9)\}$$

$$\text{threeX} = \{((x_1 \wedge x_2 \wedge x_3) \Rightarrow w_{\times}), ((x_4 \wedge x_5 \wedge x_6) \Rightarrow w_{\times}), ((x_7 \wedge x_8 \wedge x_9) \Rightarrow w_{\times}), ((x_1 \wedge x_4 \wedge x_7) \Rightarrow w_{\times}), ((x_2 \wedge x_5 \wedge x_8) \Rightarrow w_{\times}), ((x_3 \wedge x_6 \wedge x_9) \Rightarrow w_{\times}), ((x_1 \wedge x_5 \wedge x_9) \Rightarrow w_{\times})\}$$

$$\text{threeO} = \{((o_1 \wedge o_2 \wedge o_3) \Rightarrow w_{\bigcirc}), ((o_4 \wedge o_5 \wedge o_6) \Rightarrow w_{\bigcirc}), ((o_7 \wedge o_8 \wedge o_9) \Rightarrow w_{\bigcirc}), ((o_1 \wedge o_4 \wedge o_7) \Rightarrow w_{\bigcirc}), ((o_2 \wedge o_5 \wedge o_8) \Rightarrow w_{\bigcirc}), ((o_3 \wedge o_6 \wedge o_9) \Rightarrow w_{\bigcirc}), ((o_1 \wedge o_5 \wedge o_9) \Rightarrow w_{\bigcirc})\}$$

$$\neg(w_{\times} \wedge w_{\bigcirc})$$

# this is just a long conjunction

$$\text{notBoth} = \{\neg(x_1 \wedge o_1), \neg(x_2 \wedge o_2), \neg(x_3 \wedge o_3), \neg(x_4 \wedge o_4), \neg(x_5 \wedge o_5), \neg(x_6 \wedge o_6), \neg(x_7 \wedge o_7), \neg(x_8 \wedge o_8), \neg(x_9 \wedge o_9)\}$$

$$\text{threeX} = \{((x_1 \wedge x_2 \wedge x_3) \Rightarrow w_\times), ((x_4 \wedge x_5 \wedge x_6) \Rightarrow w_\times), ((x_7 \wedge x_8 \wedge x_9) \Rightarrow w_\times), ((x_1 \wedge x_4 \wedge x_7) \Rightarrow w_\times), ((x_2 \wedge x_5 \wedge x_8) \Rightarrow w_\times), ((x_3 \wedge x_6 \wedge x_9) \Rightarrow w_\times), ((x_1 \wedge x_5 \wedge x_9) \Rightarrow w_\times)\}$$

$$\text{threeO} = \{((o_1 \wedge o_2 \wedge o_3) \Rightarrow w_\circ), ((o_4 \wedge o_5 \wedge o_6) \Rightarrow w_\circ), ((o_7 \wedge o_8 \wedge o_9) \Rightarrow w_\circ), ((o_1 \wedge o_4 \wedge o_7) \Rightarrow w_\circ), ((o_2 \wedge o_5 \wedge o_8) \Rightarrow w_\circ), ((o_3 \wedge o_6 \wedge o_9) \Rightarrow w_\circ), ((o_1 \wedge o_5 \wedge o_9) \Rightarrow w_\circ)\}$$

$$\left(\bigwedge_{\alpha \in \text{notBoth} \cup \text{threeX} \cup \text{threeO}} \alpha\right) \wedge \neg(w_\times \wedge w_\circ)$$

no square has both ✕ and ○

$$\neg(x_1 \wedge o_1) \wedge \neg(x_2 \wedge o_2) \wedge \neg(x_3 \wedge o_3) \wedge \neg(x_4 \wedge o_4) \wedge \neg(x_5 \wedge o_5) \wedge \neg(x_6 \wedge o_6) \wedge \neg(x_7 \wedge o_7) \wedge \neg(x_8 \wedge o_8) \wedge \neg(x_9 \wedge o_9)$$

define: $\text{notBoth}(i) = \{\neg(x_i \wedge o_i)\}$

three X's in a row wins

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |
| 7 | 8 | 9 |

$$((x_1 \wedge x_2 \wedge x_3) \Rightarrow w_X)$$
$$\wedge\ ((x_4 \wedge x_5 \wedge x_6) \Rightarrow w_X)$$
$$\wedge\ ((x_7 \wedge x_8 \wedge x_9) \Rightarrow w_X)$$
$$\wedge\ ((x_1 \wedge x_4 \wedge x_7) \Rightarrow w_X)$$
$$\wedge\ ((x_2 \wedge x_5 \wedge x_8) \Rightarrow w_X)$$
$$\wedge\ ((x_3 \wedge x_6 \wedge x_9) \Rightarrow w_X)$$
$$\wedge\ ((x_1 \wedge x_5 \wedge x_9) \Rightarrow w_X)$$
$$\wedge\ ((x_3 \wedge x_5 \wedge x_7) \Rightarrow w_X)$$

define: $lines = \{(1,2,3), (4,5,6), (7,8,9), (1,4,7), (2,5,8), (3,6,9), (1,5,9), (3,5,7)\}$

$$threeX = \bigcup_{(i,j,k) \in lines} (x_i \wedge x_j \wedge x_k) \Rightarrow w_X$$

three ○'s in a row

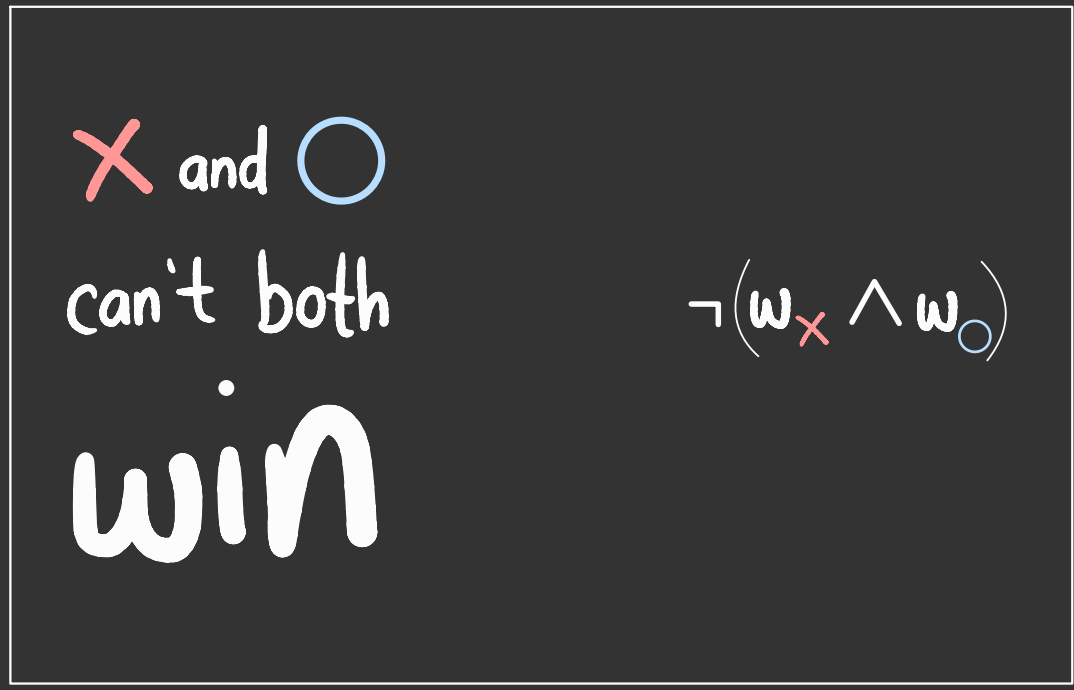
# wins

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |
| 7 | 8 | 9 |

$$
\begin{aligned}
&((o_1 \wedge o_2 \wedge o_3) \Rightarrow w_{\bigcirc}) \\
\wedge\ &((o_4 \wedge o_5 \wedge o_6) \Rightarrow w_{\bigcirc}) \\
\wedge\ &((o_7 \wedge o_8 \wedge o_9) \Rightarrow w_{\bigcirc}) \\
\wedge\ &((o_1 \wedge o_4 \wedge o_7) \Rightarrow w_{\bigcirc}) \\
\wedge\ &((o_2 \wedge o_5 \wedge o_8) \Rightarrow w_{\bigcirc}) \\
\wedge\ &((o_3 \wedge o_6 \wedge o_9) \Rightarrow w_{\bigcirc}) \\
\wedge\ &((o_1 \wedge o_5 \wedge o_9) \Rightarrow w_{\bigcirc}) \\
\wedge\ &((o_3 \wedge o_5 \wedge o_7) \Rightarrow w_{\bigcirc})
\end{aligned}
$$

define: $lines = \{(1,2,3), (4,5,6), (7,8,9), (1,4,7), (2,5,8), (3,6,9), (1,5,9), (3,5,7)\}$

$$threeO = \bigcup_{(i,j,k) \in lines} (o_i \wedge o_j \wedge o_k) \Rightarrow w_{\bigcirc}$$




seems fine

# rules of tic-tac-toe

no square has both X and O

$$\text{notBoth}(i) = \{\neg(x_i \wedge o_i)\}$$

three X's in a row wins

and

three O's in a row wins

$$\text{lines} = \{(1,2,3), (4,5,6), (7,8,9), (1,4,7), (2,5,8), (3,6,9), (1,5,9), (3,5,7)\}$$

$$\text{threeX} = \bigcup_{(i,j,k) \in \text{lines}} (x_i \wedge x_j \wedge x_k) \Rightarrow w_x$$

$$\text{threeO} = \bigcup_{(i,j,k) \in \text{lines}} (o_i \wedge o_j \wedge o_k) \Rightarrow w_o$$

X and O can't both win

$$\neg(w_x \wedge w_o)$$

# rules of tic-tac-toe

no square has both X and O

$$\text{notBoth}(i) = \{\neg(x_i \wedge o_i)\}$$

three X's in a row wins

and

three O's in a row wins

$$\text{lines} = \{(1,2,3),(4,5,6),(7,8,9),(1,4,7),(2,5,8),(3,6,9),(1,5,9),(3,5,7)\}$$

$$\text{threeX} = \bigcup_{(i,j,k) \in \text{lines}} (x_i \wedge x_j \wedge x_k) \Rightarrow w_x$$

$$\text{threeO} = \bigcup_{(i,j,k) \in \text{lines}} (o_i \wedge o_j \wedge o_k) \Rightarrow w_o$$

X and O can't both win

$$\neg(w_x \wedge w_o)$$

$$\left( \bigwedge_{\substack{i \in \{1,\ldots,9\} \\ s \in \text{notBoth}(i)}} s \right) \wedge \left( \bigwedge_{s \in \text{threeX} \cup \text{threeO}} s \right) \wedge \neg(w_x \wedge w_o)$$