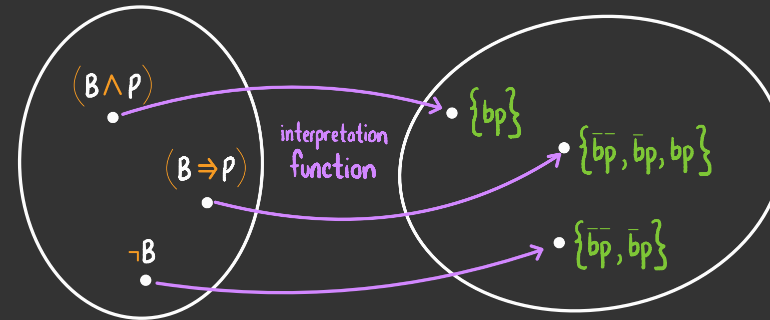
conjunctive normal form

CSCI 373

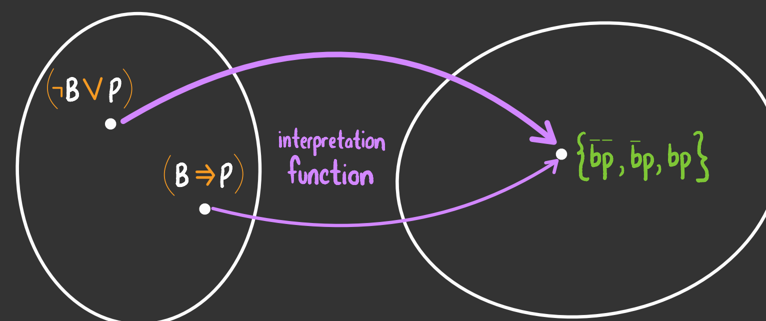
propositional logic

is **unambiguous**:



each sentence maps to a unique truth table

it is, however, **redundant**:

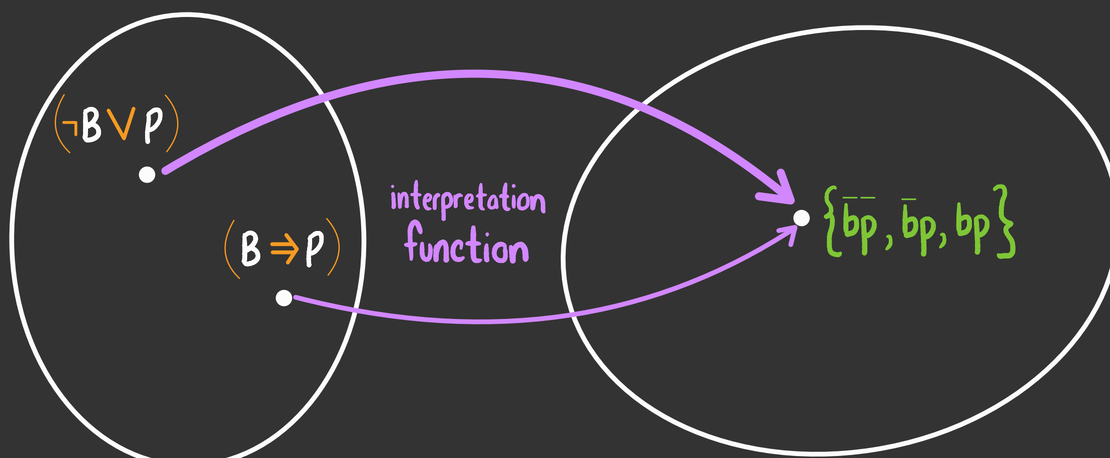


multiple sentences map to the same truth table two sentences are logically equivalent if they have the same interpretation

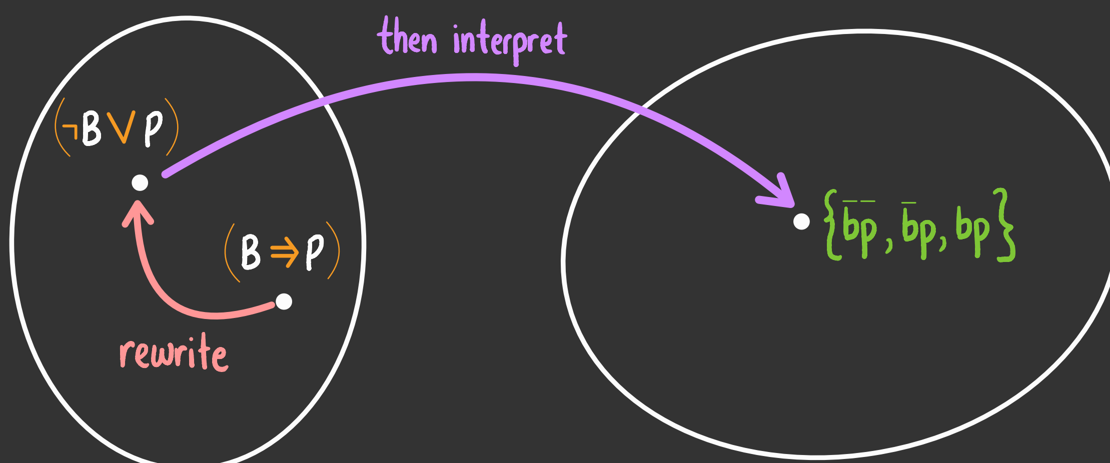
in other words,

"logically equivalent"

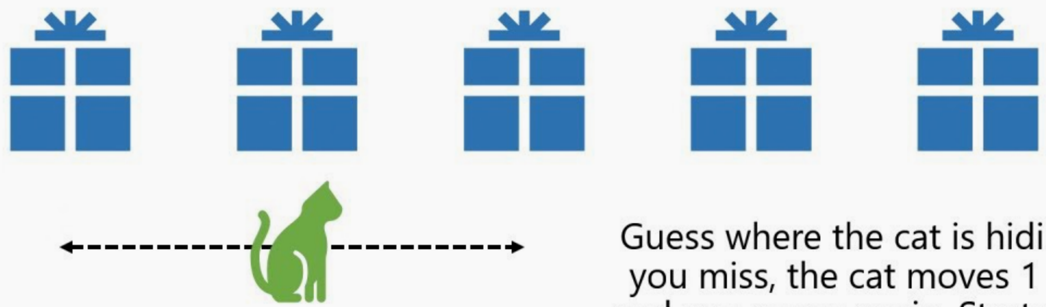
for $\alpha, \beta \in L(\Sigma)$: $\alpha \equiv \beta$ iff $I(\alpha) = I(\beta)$



in principle, it can be convenient to rewrite a sentence as a logically equivalent sentence that is simpler to deal with computationally



Microsoft Interview Question
Can You Find The Hiding Cat?



Guess where the cat is hiding. If you miss, the cat moves 1 box, and you guess again. Strategy to the cat for sure in fewest guesses?

A B C

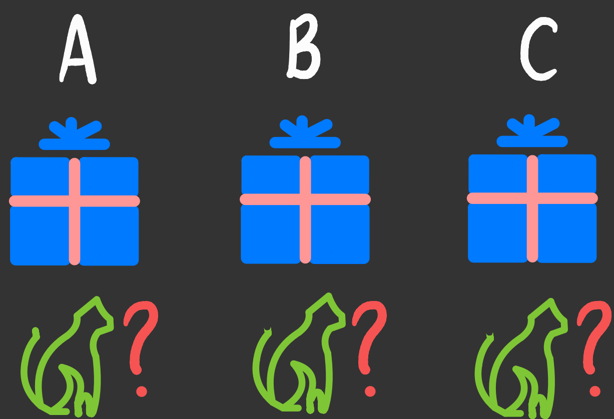
A is whether the cat is in box A

B is whether the cat is in box B

C is whether the cat is in box C

"the cat is in exactly one box"

$$\boxed{\text{express in } L(\{A, B, C\})}$$



A is whether the cat is in box A
B is whether the cat is in box B
C is whether the cat is in box C

"the cat is in exactly **one box**"

$$
\begin{aligned}
&(A \lor B \lor C) \\
\land &(\neg A \lor \neg B) \\
\land &(\neg A \lor \neg C) \\
\land &(\neg B \lor \neg C)
\end{aligned}
$$

conjunction of clauses

$$(A \lor B \lor C) \land (\neg A \lor \neg B) \land (\neg A \lor \neg C) \land (\neg B \lor \neg C)$$

clauses

this convenient form is called

conjunctive normal form (cnf)

literals

$$(A \vee B \vee C) \wedge (\neg A \vee \neg B) \wedge (\neg A \vee \neg C) \wedge (\neg B \vee \neg C)$$

- $\text{literals}(\Sigma) = \Sigma \cup \{\neg\sigma \mid \sigma \in \Sigma\}$

literals

$$(A \lor B \lor C) \land (\neg A \lor \neg B) \land (\neg A \lor \neg C) \land (\neg B \lor \neg C)$$

clauses

- $\text{literals}(\Sigma) = \Sigma \cup \{\neg\sigma \mid \sigma \in \Sigma\}$
- $\text{clauses}(\Sigma) = \{\bot\} \cup \left\{\bigvee_{i=1}^{m} \ell_i \;\middle|\; m \geq 1,\ \ell_i \in \text{literals}(\Sigma)\right\}$

$$\underbrace{\underbrace{(A \vee B \vee C)} \wedge \underbrace{(\neg A \vee \neg B)} \wedge \underbrace{(\neg A \vee \neg C)} \wedge \underbrace{(\neg B \vee \neg C)}}_{\text{cnf sentence}}$$

- $\text{literals}(\Sigma) = \Sigma \cup \{\neg\sigma \mid \sigma \in \Sigma\}$
- $\text{clauses}(\Sigma) = \{\bot\} \cup \left\{ \bigvee_{i=1}^{m} \ell_i \;\middle|\; m \geq 1,\ \ell_i \in \text{literals}(\Sigma) \right\}$
- $\text{cnf}(\Sigma) = \{\top\} \cup \left\{ \bigwedge_{i=1}^{n} c_i \;\middle|\; n \geq 1,\ c_i \in \text{clauses}(\Sigma) \right\}$

any propositional sentence
can be rewritten as a
logically equivalent
cnf sentence

1. eliminate implications

pattern	rewrite as
$\alpha \Rightarrow \beta$	$\neg\alpha \vee \beta$
until no more rewrites possible	

2. move $\neg$ inward

pattern	rewrite as
$\neg\neg\alpha$	α
$\neg(\alpha \vee \beta)$	$\neg\alpha \wedge \neg\beta$
$\neg(\alpha \wedge \beta)$	$\neg\alpha \vee \neg\beta$
until no more rewrites possible	

3. distribute

pattern	rewrite as
$\alpha \wedge (\beta \vee \gamma)$	$(\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$
$(\alpha \vee \beta) \wedge \gamma$	$(\alpha \wedge \gamma) \vee (\beta \wedge \gamma)$
$\alpha \vee (\beta \wedge \gamma)$	$(\alpha \vee \beta) \wedge (\alpha \vee \gamma)$
$(\alpha \wedge \beta) \vee \gamma$	$(\alpha \vee \gamma) \wedge (\beta \vee \gamma)$
until no more rewrites possible	

$$(B \vee C) \Rightarrow A$$

$$\neg(B \vee C) \vee A$$

$$(\neg B \wedge \neg C) \vee A$$

$$(\neg B \vee A) \wedge (\neg C \vee A)$$

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$\alpha \vee (\beta \wedge \gamma)$	$(\alpha \vee \beta) \wedge (\alpha \vee \gamma)$
$(\alpha \wedge \beta) \vee \gamma$	$(\alpha \vee \gamma) \wedge (\beta \vee \gamma)$
until no more rewrites possible	

for all of these, the rewrite is logically equivalent to the original sentence e.g.

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until no more rewrites possible	

for all of these, the rewrite is logically equivalent to the original sentence

e.g. $\neg(\alpha \wedge \beta) \equiv \neg\alpha \vee \neg\beta$

because: $I(\neg(\alpha \wedge \beta)) = ?$

$= ?$

$= ?$

$= ?$

$= I(\neg\alpha \vee \neg\beta)$

$$\top \xrightarrow{I} M(\Sigma)$$
$$\bot \xrightarrow{I} \{\}$$
$$\sigma \xrightarrow{I} \{m \mid m \in M(\Sigma),\ m(\sigma) = 1\}$$
$$\neg\alpha \xrightarrow{I} \overline{I(\alpha)}$$
$$(\alpha \wedge \beta) \xrightarrow{I} I(\alpha) \cap I(\beta)$$
$$(\alpha \vee \beta) \xrightarrow{I} I(\alpha) \cup I(\beta)$$
$$(\alpha \Rightarrow \beta) \xrightarrow{I} \overline{I(\alpha)} \cup I(\beta)$$

e.g. $\neg(\alpha \wedge \beta) \equiv \neg\alpha \vee \neg\beta$

because: $I(\neg(\alpha \wedge \beta)) = \overline{I(\alpha \wedge \beta)}$

$=$

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$\top$	$\xrightarrow{I}$	$M(\Sigma)$
$\bot$	$\xrightarrow{I}$	$\{\}$
σ	$\xrightarrow{I}$	$\{m \mid m \in M(\Sigma), m(\sigma) = 1\}$
$\neg\alpha$	$\xrightarrow{I}$	$\overline{I(\alpha)}$
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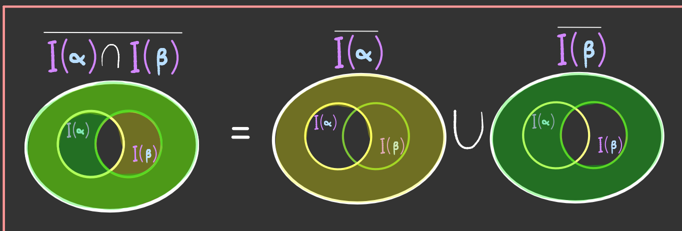

$\top$	$\xrightarrow{I}$	$M(\Sigma)$
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$\neg\alpha$	$\xrightarrow{I}$	$\overline{I(\alpha)}$
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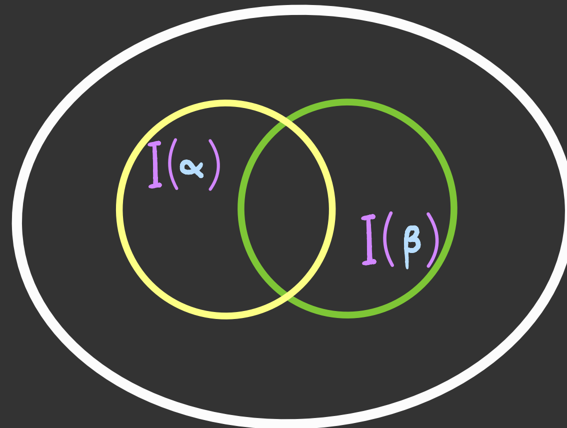
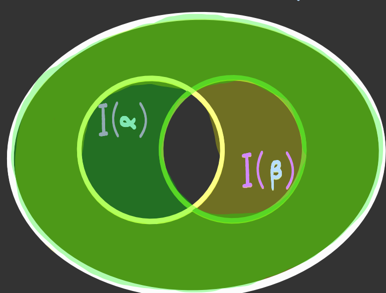
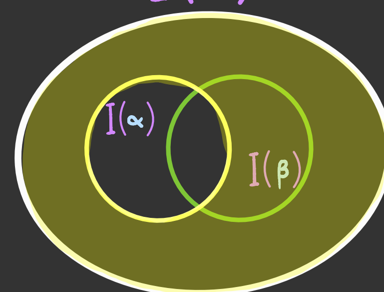
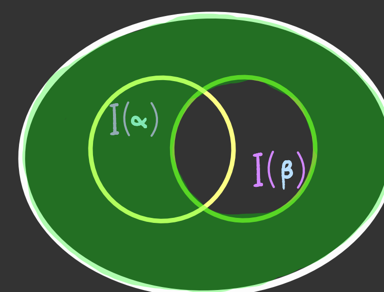


$$\overline{I(\alpha)\cap I(\beta)} = \overline{I(\alpha)} \cup \overline{I(\beta)}$$







e.g. $\neg(\alpha \wedge \beta) \equiv \neg\alpha \vee \neg\beta$

because: $I(\neg(\alpha \wedge \beta)) = \overline{I(\alpha \wedge \beta)}$

$$= \overline{I(\alpha) \cap I(\beta)}$$

$$= \overline{I(\alpha)} \cup \overline{I(\beta)}$$

$$=$$

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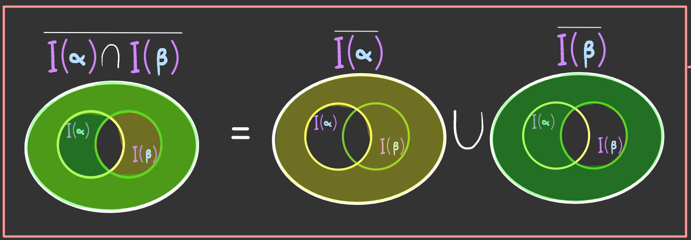

$$
\begin{array}{lcl}
\top & \xrightarrow{I} & M(\Sigma) \\
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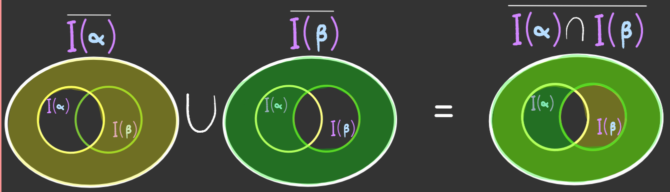

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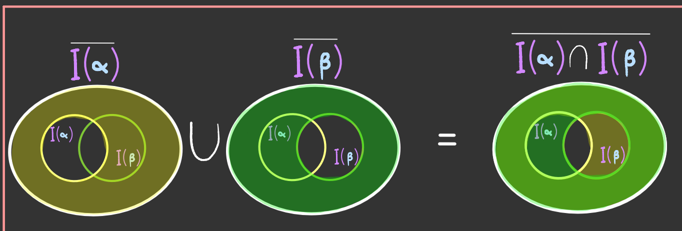

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until no more rewrites possible

we can similarly show each rewrite is logically equivalent to the original sentence since any propositional sentence can be rewritten as a logically equivalent cnf sentence, we can focus on reasoning using cnf sentences