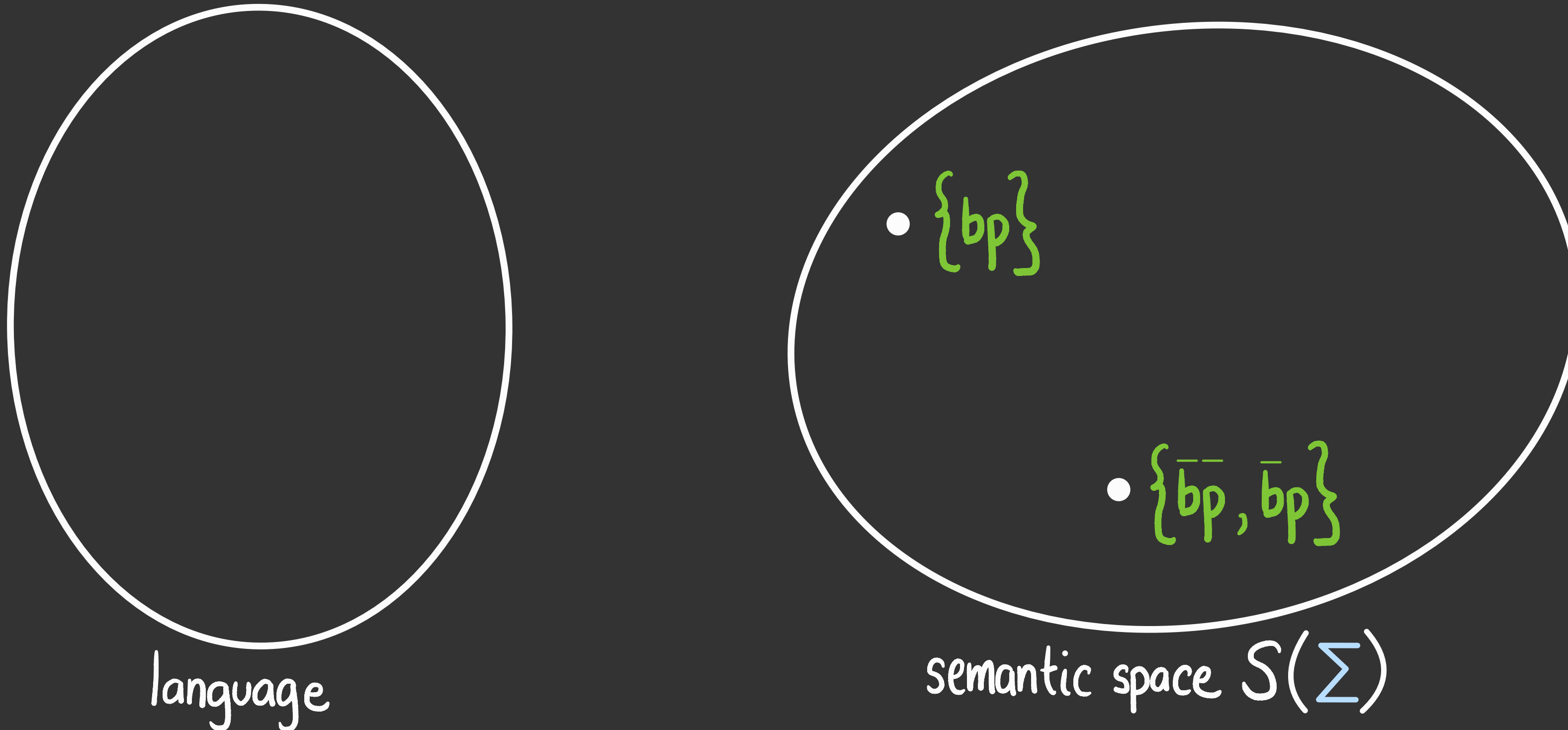
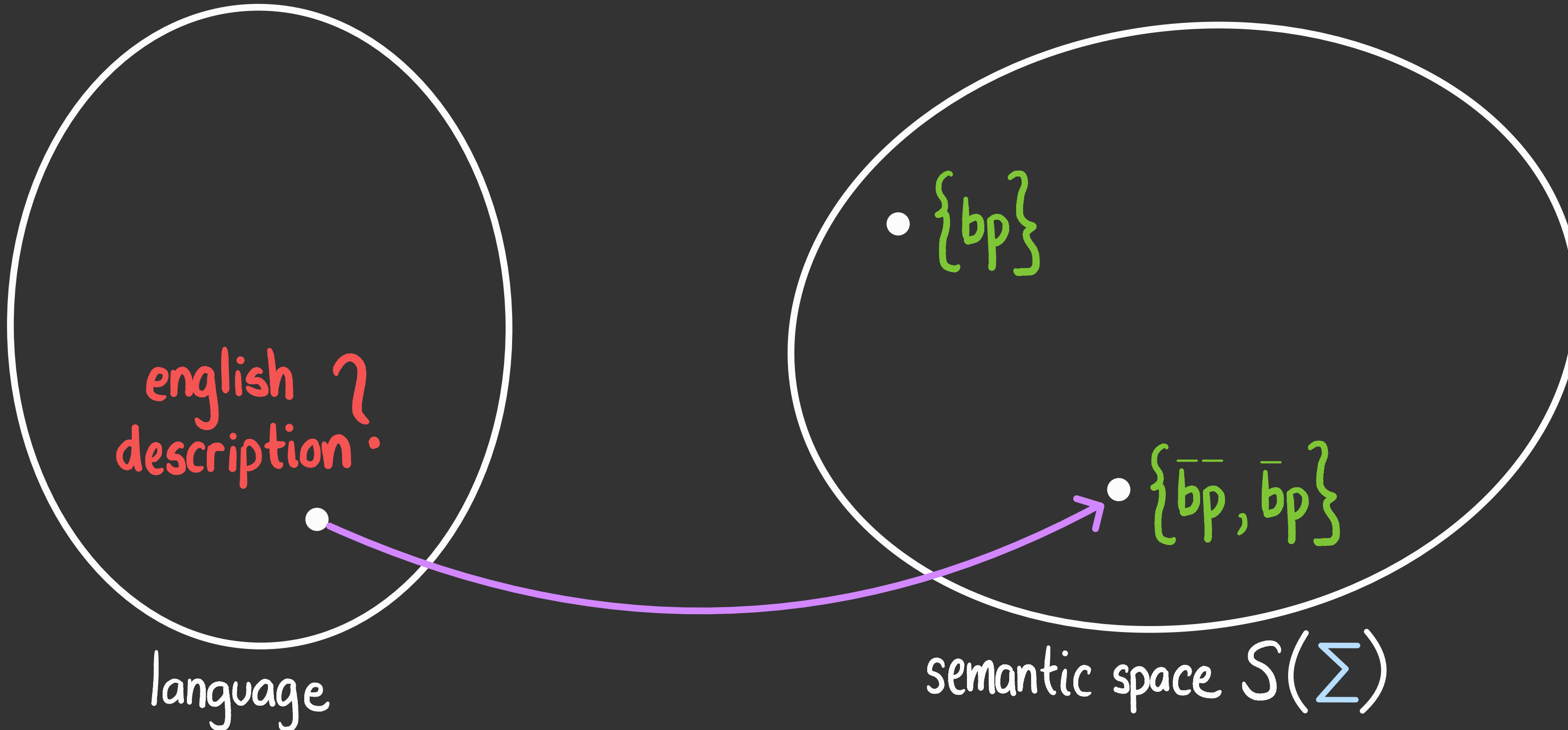
propositional logic

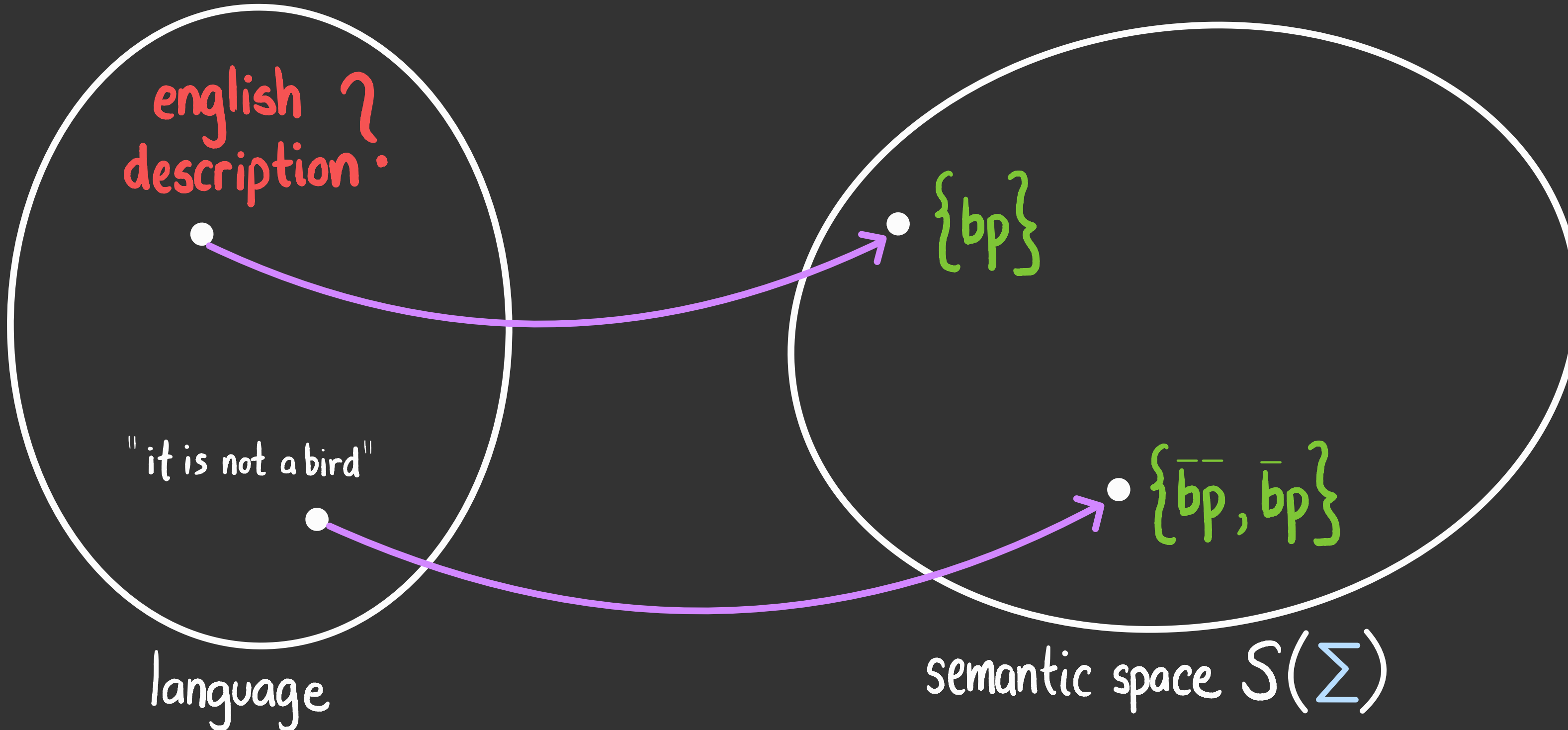
CSCI 373



next we construct a language to represent elements of the semantic space



next we construct a **language** to represent elements of the semantic space



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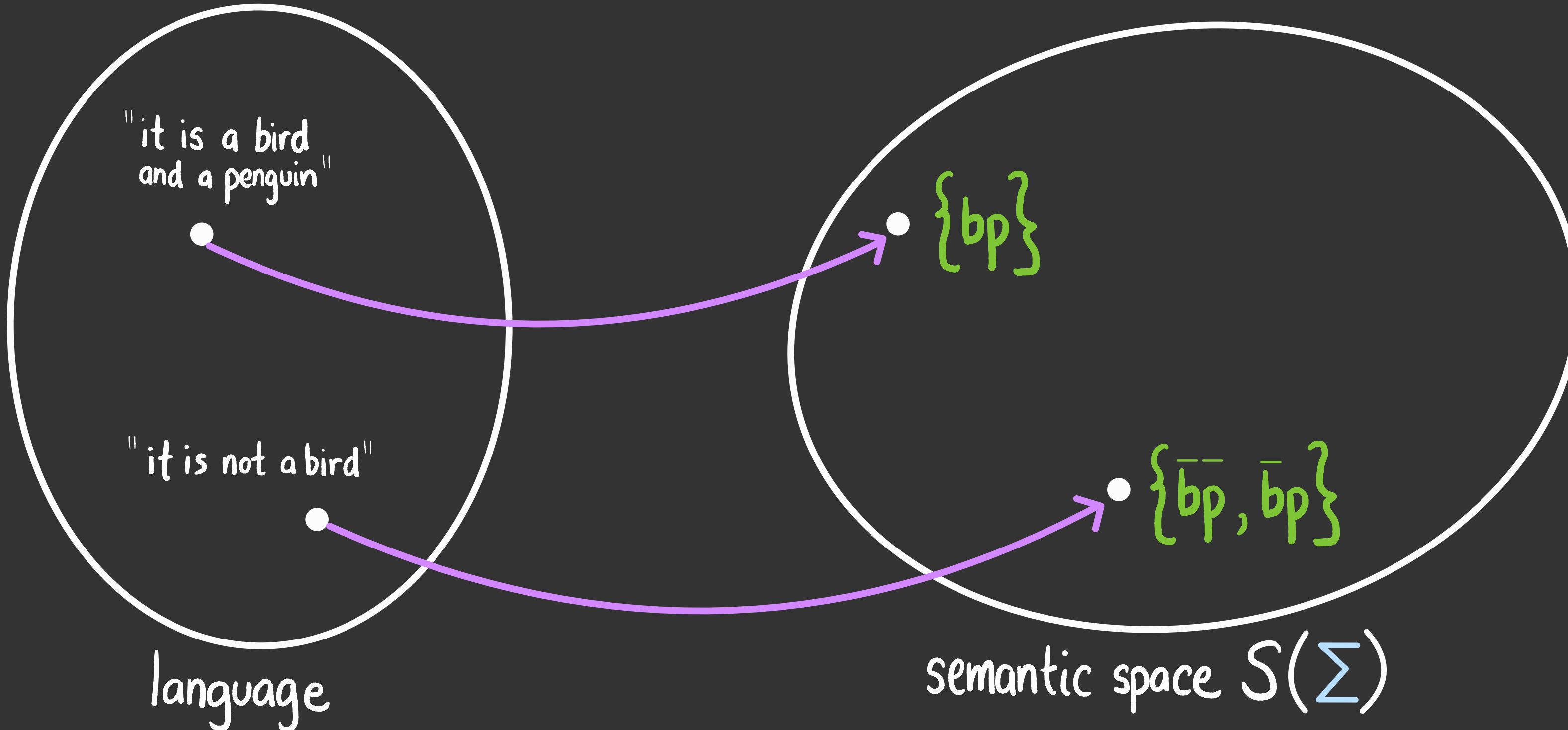

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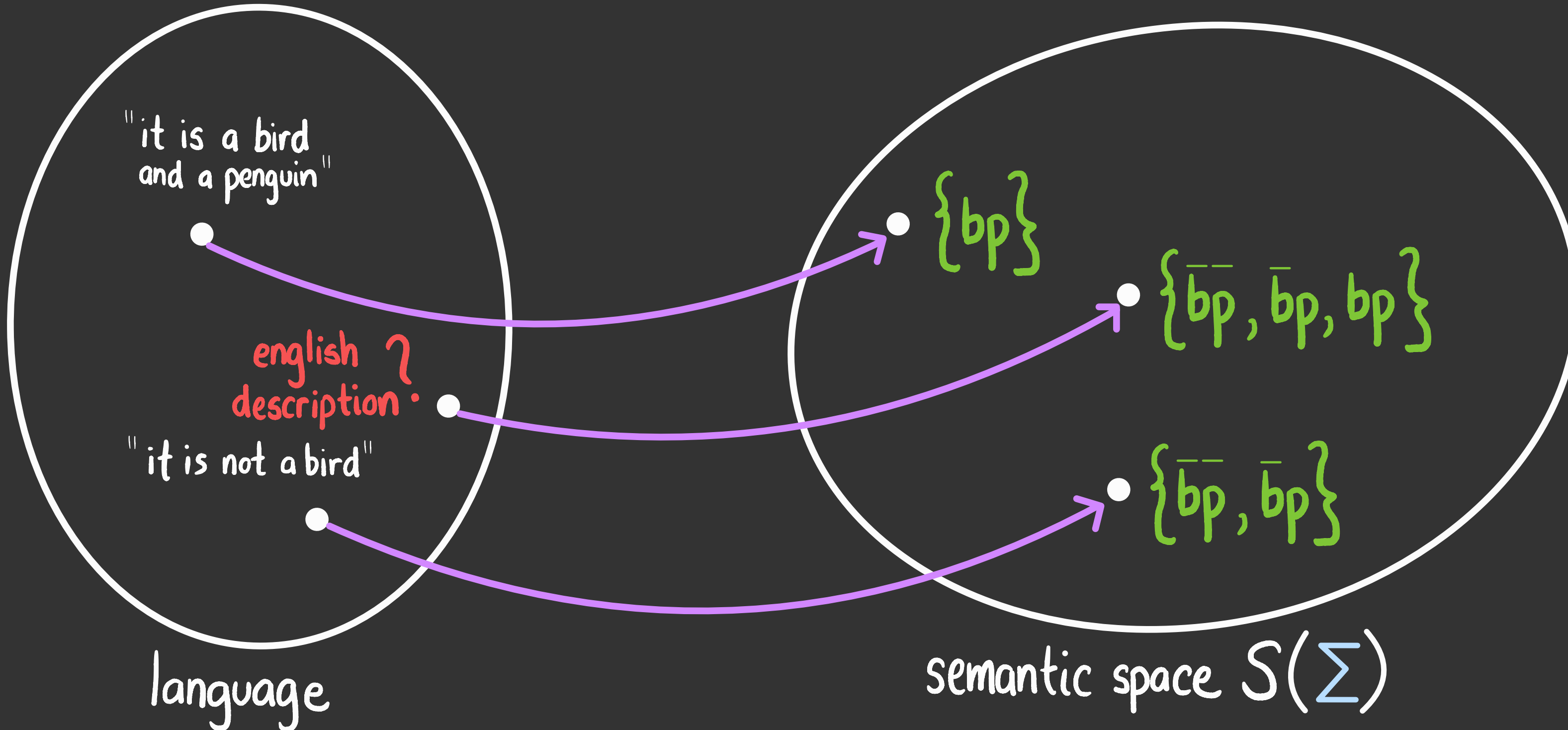
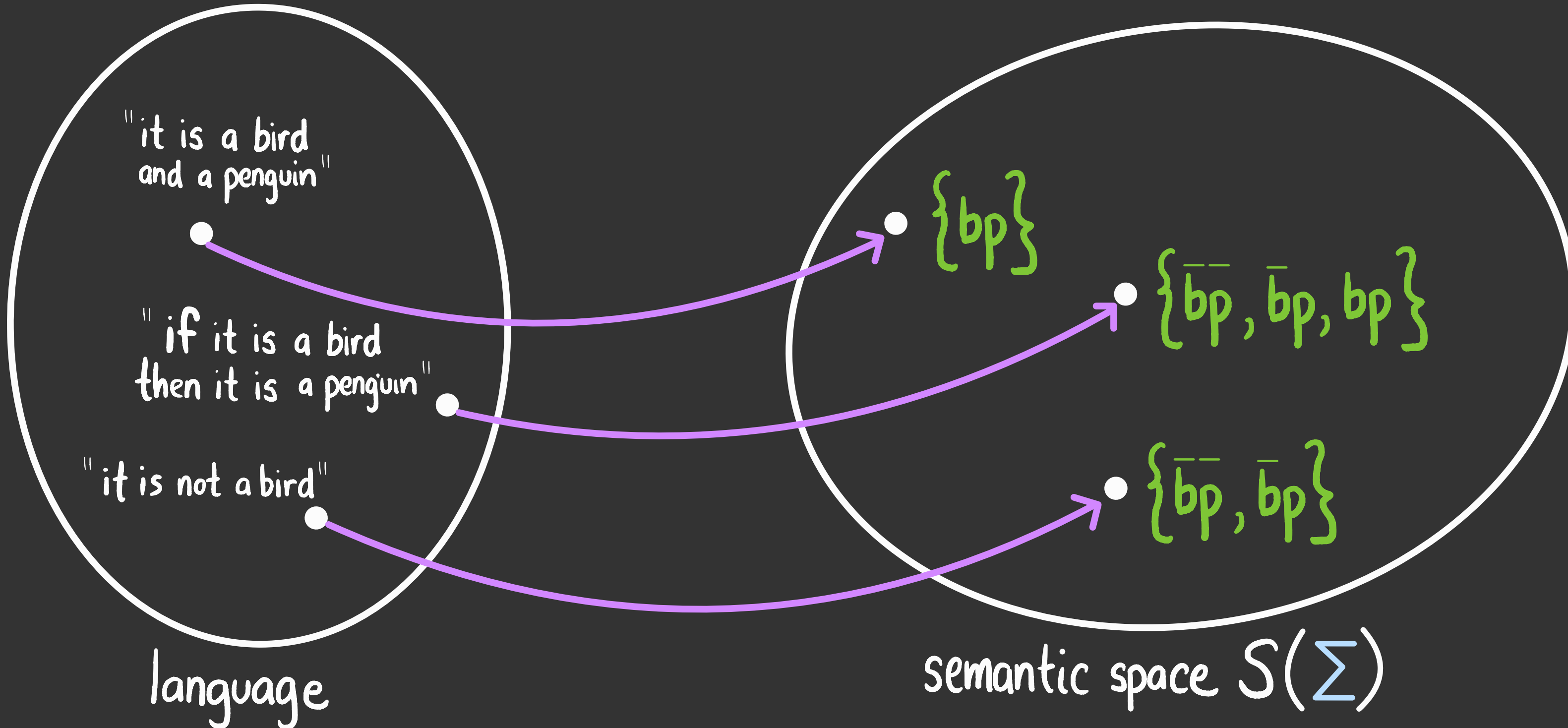
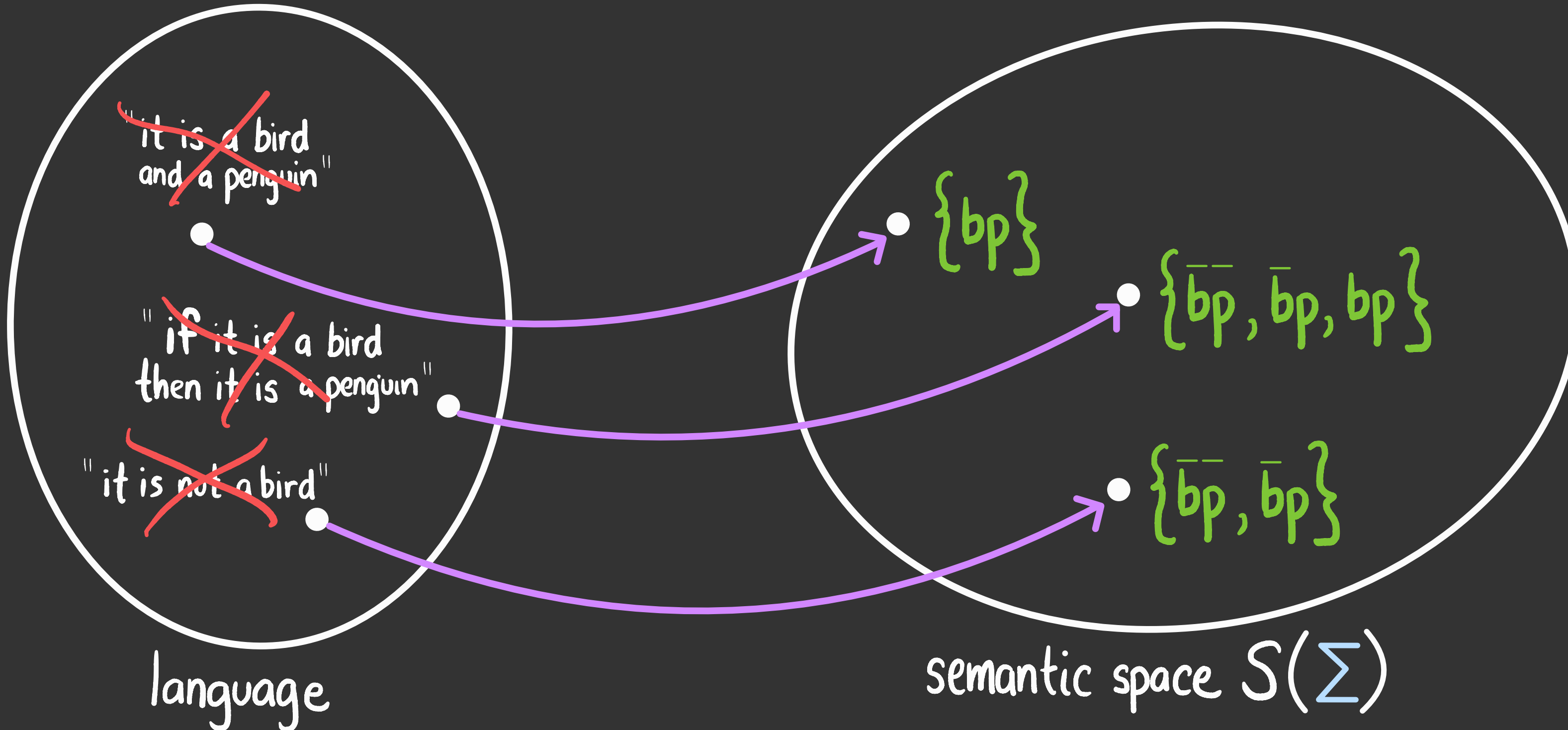


next we construct a language to represent elements of the semantic space



next we construct a language to represent elements of the semantic space



but we won't use english
english is overly general

also english is ambiguous

english	propositional logic	
"it is not a bird"	$\neg B$	not B
"it is a bird or a penguin"	$B \vee P$	B or P
"it is a bird and a penguin"	$B \wedge P$	B and P
"if it is a bird then it is a penguin"	$B \Rightarrow P$	B implies P

english	propositional logic
"it is not a bird"	$\neg B$
"it is a bird or a penguin"	$B \vee P$
"it is a bird and a penguin"	$B \wedge P$
"if it is a bird then it is a penguin"	$B \Rightarrow P$

let's define this language

the propositional language $L(\Sigma)$ for signature Σ is the smallest set of strings such that:

- $\top \in L(\Sigma)$ and $\bot \in L(\Sigma)$
- if $\sigma \in \Sigma$, then $\sigma \in L(\Sigma)$
- if $\alpha \in L(\Sigma)$, then $\neg\alpha \in L(\Sigma)$
- if $\alpha \in L(\Sigma)$ and $\beta \in L(\Sigma)$, then:
  - $(\alpha \wedge \beta) \in L(\Sigma)$
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  - $(\alpha \Rightarrow \beta) \in L(\Sigma)$

example strings in $L(\{B, P\})$

$\top$ $\qquad$ $\bot$

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example strings in $L(\{B, P\})$

$\top$	$\bot$
B	P

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    - $(\alpha \land \beta) \in L(\Sigma)$
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example strings in $L(\{B, P\})$

$\top$	$\bot$
B	P
$\neg B$	$\neg P$

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example strings in $L(\{B, P\})$

$\top$	$\bot$
B	P
$\neg B$	$\neg P$

anything else?

the propositional language $L(\Sigma)$ for signature Σ is the smallest set of strings such that:

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example strings in $L(\{B, P\})$

$\top \qquad \bot$

$B \qquad P$

$\neg\top \quad \neg B \quad \neg\neg B \quad \neg P \quad \neg\bot$

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example strings in $L(\{B, P\})$

$\top$ $\quad$ $\bot$

B $\quad$ P

$\neg\top$ $\quad$ $\neg B$ $\quad$ $\neg\neg B$ $\quad$ $\neg P$ $\quad$ $\neg\bot$

$(\neg B \land P)$ $\quad$ $((\neg B \land P) \land \bot)$

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example strings in $L(\{B, P\})$

$\top$ $\bot$

B P

$\neg\top$ $\neg B$ $\neg\neg B$ $\neg P$ $\neg\bot$

$(\neg B \wedge P)$ $((\neg B \wedge P) \wedge \bot)$

$(\neg B \vee P)$ $((\neg B \wedge P) \vee \bot)$

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example strings in $L(\{B, P\})$

$\top$ $\quad$ $\bot$

B $\quad$ P

$\neg\top$ $\quad$ $\neg B$ $\quad$ $\neg\neg B$ $\quad$ $\neg P$ $\quad$ $\neg\bot$

$(\neg B \land P)$ $\quad$ $((\neg B \land P) \land \bot)$

$(\neg B \lor P)$ $\quad$ $((\neg B \land P) \lor \bot)$

$(\neg B \Rightarrow P)$ $\quad$ $((\neg B \lor P) \Rightarrow \bot)$

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which of these strings are in $L(\{B, P\})$?

the empty string	✓ / ✗
B	✓ / ✗
$B \vee P$	✓ / ✗
$(B \wedge \neg B)$	✓ / ✗
$(F \Rightarrow B)$	✓ / ✗
$(B \wedge P \wedge \neg B)$	✓ / ✗
$\neg(\neg\neg P \Rightarrow (B \vee \neg P))$	✓ / ✗

the propositional language $L(\Sigma)$ for signature Σ is the smallest set of strings such that:

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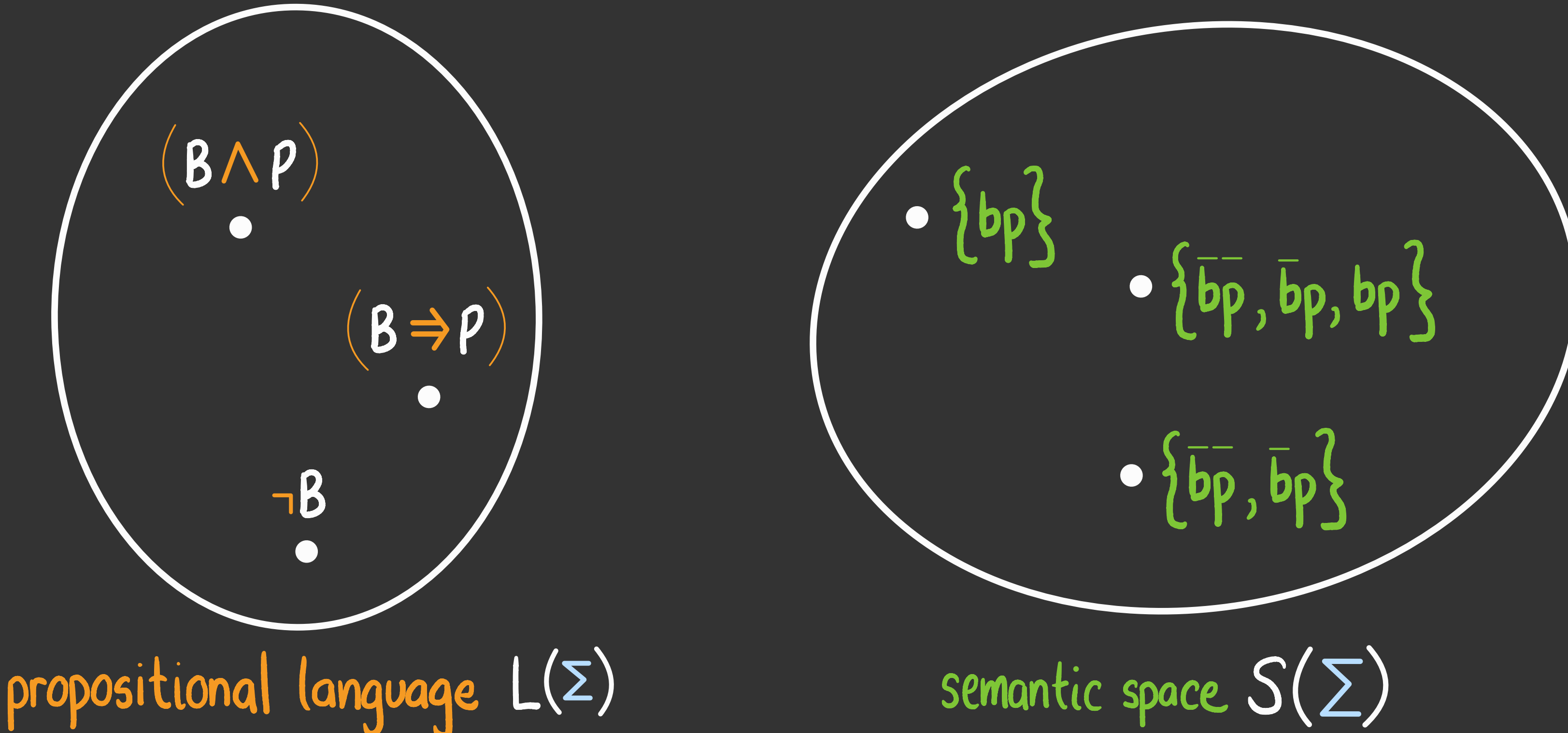
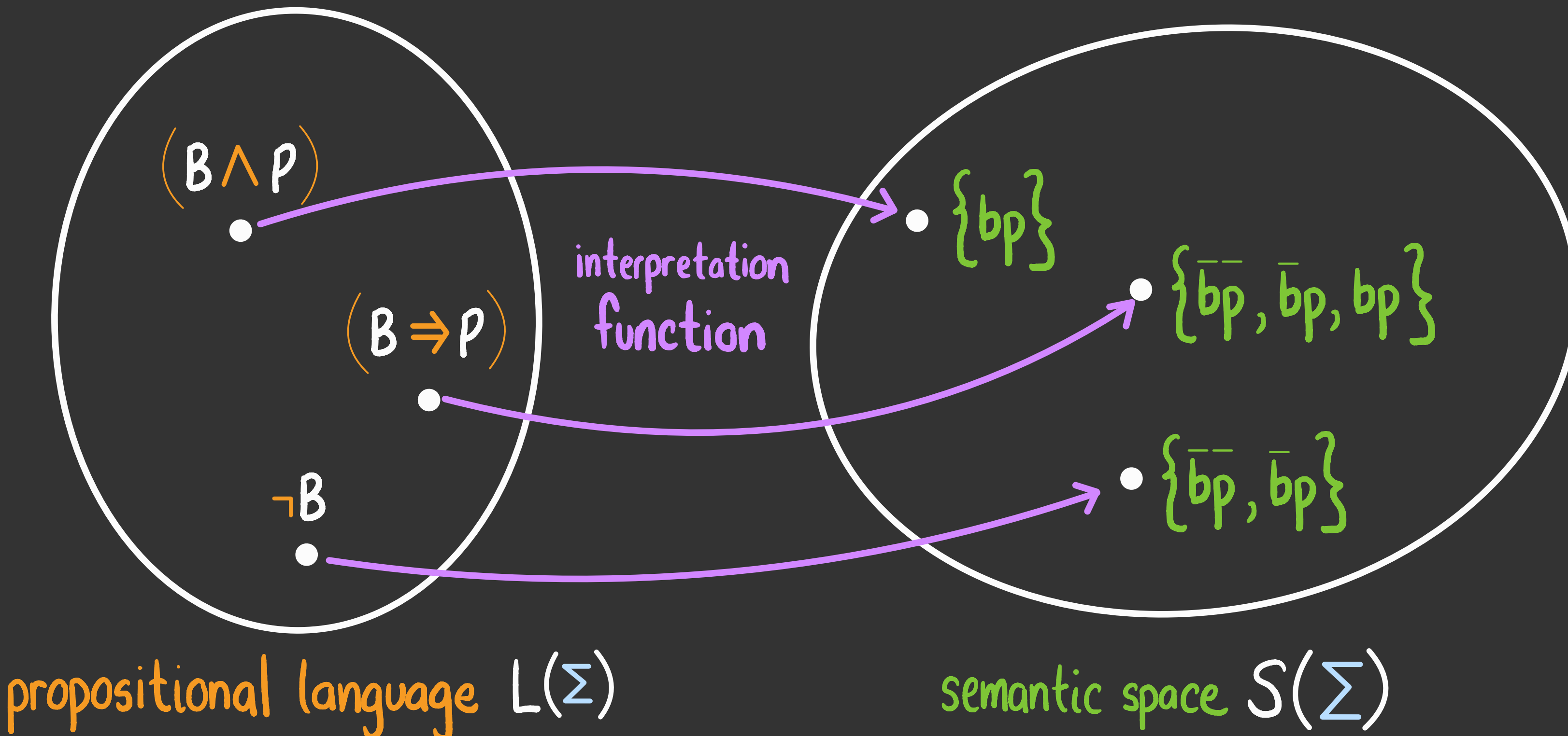
which of these strings are in $L(\{B, P\})$?

the empty string	✗
B	✓
$B \vee P$	✗
$(B \wedge \neg B)$	✓
$(F \Rightarrow B)$	✗
$(B \wedge P \wedge \neg B)$	✗
$\neg(\neg\neg P \Rightarrow (B \vee \neg P))$	✓



we've defined the language and the meanings



the final step is to assign a meaning to each string of the language

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$$\top \xrightarrow{I} \{\ ?\ \}$$

$$\bot \xrightarrow{I} \{\ ?\ \}$$

$$P \xrightarrow{I} \{\ ?\ \}$$

$$\neg B \xrightarrow{I} \{\ ?\ \}$$

$$(P \wedge \neg B) \xrightarrow{I} \{\ ?\ \}$$

$$(P \vee \neg B) \xrightarrow{I} \{\ ?\ \}$$

$$(B \Rightarrow P) \xrightarrow{I} \{\ ?\ \}$$

propositional language

$L(\Sigma)$

semantic space

$S(\Sigma)$

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$$\top \xrightarrow{I} \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}$$

$$\bot \xrightarrow{I} \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}$$

$$p \xrightarrow{I} \{\ ?\ \}$$

$$\neg B \xrightarrow{I} \{\ ?\ \}$$

$$(p \wedge \neg B) \xrightarrow{I} \{\ ?\ \}$$

$$(p \vee \neg B) \xrightarrow{I} \{\ ?\ \}$$

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$$p \xrightarrow{\mathcal{I}} \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}$$

$$\neg B \xrightarrow{\mathcal{I}} \{\ ?\ \}$$

$$(p \wedge \neg B) \xrightarrow{\mathcal{I}} \{\ ?\ \}$$

$$(p \vee \neg B) \xrightarrow{\mathcal{I}} \{\ ?\ \}$$

$$(B \Rightarrow p) \xrightarrow{\mathcal{I}} \{\ ?\ \}$$

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$$\neg b \xrightarrow{I} \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}$$

$$(p \wedge \neg b) \xrightarrow{I} \{\ ?\ \}$$

$$(p \vee \neg b) \xrightarrow{I} \{\ ?\ \}$$

$$(b \Rightarrow p) \xrightarrow{I} \{\ ?\ \}$$

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$$\top \xrightarrow{I} \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}$$

$$\bot \xrightarrow{I} \{{\color{gray}\bar{b}\bar{p}}, {\color{gray}\bar{b}p}, {\color{gray}b\bar{p}}, {\color{gray}bp}\}$$

$$p \xrightarrow{I} \{{\color{gray}\bar{b}\bar{p}}, \bar{b}p, {\color{gray}b\bar{p}}, bp\}$$

$$\neg B \xrightarrow{I} \{\bar{b}\bar{p}, \bar{b}p, {\color{gray}b\bar{p}}, {\color{gray}bp}\}$$

$$(p \wedge \neg B) \xrightarrow{I} \{{\color{gray}\bar{b}\bar{p}}, \bar{b}p, {\color{gray}b\bar{p}}, {\color{gray}bp}\}$$

$$(p \vee \neg B) \xrightarrow{I} \{\ ?\ \}$$

$$(B \Rightarrow p) \xrightarrow{I} \{\ ?\ \}$$

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$$
\begin{array}{lcl}
\top & \xrightarrow{I} & \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\} \\
\bot & \xrightarrow{I} & \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\} \\
p & \xrightarrow{I} & \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\} \\
\neg B & \xrightarrow{I} & \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\} \\
(p \wedge \neg B) & \xrightarrow{I} & \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\} \\
(p \vee \neg B) & \xrightarrow{I} & \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\} \\
(B \Rightarrow p) & \xrightarrow{I} & \{\ ?\ \}
\end{array}
$$

propositional language

$L(\Sigma)$

semantic space

$S(\Sigma)$

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$$\top \xrightarrow{I} \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}$$

$$\bot \xrightarrow{I} \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}$$

$$p \xrightarrow{I} \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}$$

$$\neg B \xrightarrow{I} \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}$$

$$(p \wedge \neg B) \xrightarrow{I} \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}$$

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propositional language

$L(\Sigma)$

semantic space

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$$(b \Rightarrow p) \xrightarrow{I} \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}$$

propositional language

$L(\Sigma)$

semantic space

$S(\Sigma)$

$$\top \xrightarrow{I} ?$$

$$\bot \xrightarrow{I} ?$$

$$\sigma \xrightarrow{I} ?$$

$$\neg \alpha \xrightarrow{I} ?$$

$$(\alpha \wedge \beta) \xrightarrow{I} ?$$

$$(\alpha \vee \beta) \xrightarrow{I} ?$$

$$(\alpha \Rightarrow \beta) \xrightarrow{I} ?$$

propositional language

$L(\Sigma)$

semantic space

$S(\Sigma)$

$$\top \xrightarrow{I} \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}$$
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$$p \xrightarrow{I} \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}$$
$$\neg B \xrightarrow{I} \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}$$
$$(p \land \neg B) \xrightarrow{I} \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}$$
$$(p \lor \neg B) \xrightarrow{I} \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}$$
$$(B \Rightarrow p) \xrightarrow{I} \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}$$

$$\top \xrightarrow{I} M(\Sigma)$$
$$\bot \xrightarrow{I} \{\}$$
$$\sigma \xrightarrow{I} ?$$
$$\neg \alpha \xrightarrow{I} ?$$
$$(\alpha \land \beta) \xrightarrow{I} ?$$
$$(\alpha \lor \beta) \xrightarrow{I} ?$$
$$(\alpha \Rightarrow \beta) \xrightarrow{I} ?$$

propositional language

$L(\Sigma)$

semantic space

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propositional language

$L(\Sigma)$

semantic space

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$$\top \xrightarrow{I} \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}$$

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$$(B \Rightarrow p) \xrightarrow{I} \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}$$

$$\top \xrightarrow{I} M(\Sigma)$$

$$\bot \xrightarrow{I} \{\}$$

$$\sigma \xrightarrow{I} \{m \mid m \in M(\Sigma), m(\sigma) = 1\}$$

$$\neg\alpha \xrightarrow{I} ?$$

$$(\alpha \land \beta) \xrightarrow{I} ?$$

$$(\alpha \lor \beta) \xrightarrow{I} ?$$

$$(\alpha \Rightarrow \beta) \xrightarrow{I} ?$$

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$L(\Sigma)$

semantic space

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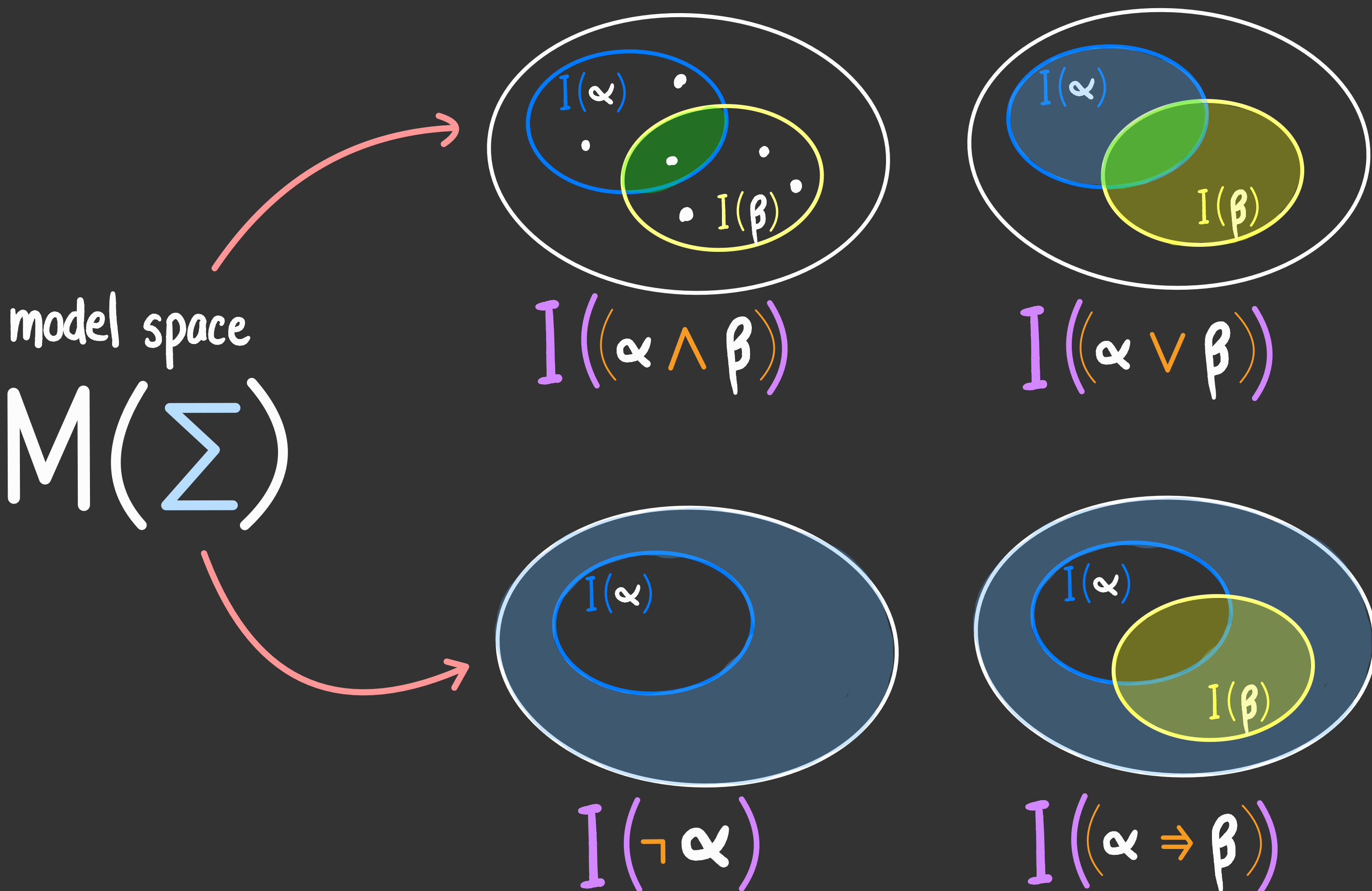
semantic space

$S(\Sigma)$


model space
M(Σ)
I(α)
I(β)
I((α ∧ β))
I((α ∨ β))
I(¬α)
I((α ⇒ β))

$$\top \xrightarrow{I} \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}$$

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$$(B \Rightarrow p) \xrightarrow{I} \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}$$

$$\top \xrightarrow{I} M(\Sigma)$$

$$\bot \xrightarrow{I} \{\}$$

$$\sigma \xrightarrow{I} \{m \mid m \in M(\Sigma), m(\sigma) = 1\}$$

$$\neg\alpha \xrightarrow{I} \overline{I(\alpha)}$$

$$(\alpha \wedge \beta) \xrightarrow{I} ?$$

$$(\alpha \vee \beta) \xrightarrow{I} ?$$

$$(\alpha \Rightarrow \beta) \xrightarrow{I} ?$$

propositional language

$L(\Sigma)$

semantic space

$S(\Sigma)$

propositional language

$L(\Sigma)$

semantic space

$S(\Sigma)$

$$\top \xrightarrow{I} \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}$$

$$\bot \xrightarrow{I} \{{\color{gray}\bar{b}\bar{p}}, {\color{gray}\bar{b}p}, {\color{gray}b\bar{p}}, {\color{gray}bp}\}$$

$$p \xrightarrow{I} \{{\color{gray}\bar{b}\bar{p}}, \bar{b}p, {\color{gray}b\bar{p}}, bp\}$$

$$\neg b \xrightarrow{I} \{\bar{b}\bar{p}, \bar{b}p, {\color{gray}b\bar{p}}, {\color{gray}bp}\}$$

$$(p \wedge \neg b) \xrightarrow{I} \{{\color{gray}\bar{b}\bar{p}}, \bar{b}p, {\color{gray}b\bar{p}}, {\color{gray}bp}\}$$

$$(p \vee \neg b) \xrightarrow{I} \{\bar{b}\bar{p}, \bar{b}p, {\color{gray}b\bar{p}}, bp\}$$

$$(b \Rightarrow p) \xrightarrow{I} \{\bar{b}\bar{p}, \bar{b}p, {\color{gray}b\bar{p}}, bp\}$$

propositional language

$$L(\Sigma)$$

semantic space

$$S(\Sigma)$$

$$\top \xrightarrow{I} M(\Sigma)$$

$$\bot \xrightarrow{I} \{\}$$

$$\sigma \xrightarrow{I} \{m \mid m \in M(\Sigma), m(\sigma) = 1\}$$

$$\neg \alpha \xrightarrow{I} \overline{I(\alpha)}$$

$$(\alpha \wedge \beta) \xrightarrow{I} I(\alpha) \cap I(\beta)$$

$$(\alpha \vee \beta) \xrightarrow{I} ?$$

$$(\alpha \Rightarrow \beta) \xrightarrow{I} ?$$

propositional language

$$L(\Sigma)$$

semantic space

$$S(\Sigma)$$

$$\top \xrightarrow{I} \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}$$

$$\bot \xrightarrow{I} \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}$$

$$p \xrightarrow{I} \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}$$

$$\neg b \xrightarrow{I} \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}$$

$$(p \wedge \neg b) \xrightarrow{I} \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}$$

$$(p \vee \neg b) \xrightarrow{I} \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}$$

$$(b \Rightarrow p) \xrightarrow{I} \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}$$

$$\top \xrightarrow{I} M(\Sigma)$$

$$\bot \xrightarrow{I} \{\}$$

$$\sigma \xrightarrow{I} \{m \mid m \in M(\Sigma), m(\sigma) = 1\}$$

$$\neg \alpha \xrightarrow{I} \overline{I(\alpha)}$$

$$(\alpha \wedge \beta) \xrightarrow{I} I(\alpha) \cap I(\beta)$$

$$(\alpha \vee \beta) \xrightarrow{I} I(\alpha) \cup I(\beta)$$

$$(\alpha \Rightarrow \beta) \xrightarrow{I} ?$$

propositional language

$L(\Sigma)$

semantic space

$S(\Sigma)$

propositional language

$L(\Sigma)$

semantic space

$S(\Sigma)$

$$
\begin{array}{lcl}
\top & \xrightarrow{I} & \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\} \\
\bot & \xrightarrow{I} & \{{\color{gray}\bar{b}\bar{p}}, {\color{gray}\bar{b}p}, {\color{gray}b\bar{p}}, {\color{gray}bp}\} \\
p & \xrightarrow{I} & \{{\color{gray}\bar{b}\bar{p}}, \bar{b}p, {\color{gray}b\bar{p}}, bp\} \\
\neg b & \xrightarrow{I} & \{\bar{b}\bar{p}, \bar{b}p, {\color{gray}b\bar{p}}, {\color{gray}bp}\} \\
(p \wedge \neg b) & \xrightarrow{I} & \{{\color{gray}\bar{b}\bar{p}}, \bar{b}p, {\color{gray}b\bar{p}}, {\color{gray}bp}\} \\
(p \vee \neg b) & \xrightarrow{I} & \{\bar{b}\bar{p}, \bar{b}p, {\color{gray}b\bar{p}}, bp\} \\
(b \Rightarrow p) & \xrightarrow{I} & \{\bar{b}\bar{p}, \bar{b}p, {\color{gray}b\bar{p}}, bp\}
\end{array}
$$

$$
\begin{array}{lcl}
\top & \xrightarrow{I} & M(\Sigma) \\
\bot & \xrightarrow{I} & \{\} \\
\sigma & \xrightarrow{I} & \{m \mid m \in M(\Sigma), m(\sigma) = 1\} \\
\neg \alpha & \xrightarrow{I} & \overline{I(\alpha)} \\
(\alpha \wedge \beta) & \xrightarrow{I} & I(\alpha) \cap I(\beta) \\
(\alpha \vee \beta) & \xrightarrow{I} & I(\alpha) \cup I(\beta) \\
(\alpha \Rightarrow \beta) & \xrightarrow{I} & \overline{I(\alpha)} \cup I(\beta)
\end{array}
$$

propositional language

$L(\Sigma)$

semantic space

$S(\Sigma)$

propositional language

$L(\Sigma)$

semantic space

$S(\Sigma)$

the interpretation of a string of the propositional language $L(\Sigma)$ is defined:

- $I(\top) = M(\Sigma)$
- $I(\bot) = \{\}$
- if $\sigma \in \Sigma$, then $I(\sigma) = \{m \mid m \in M(\Sigma), m(\sigma) = 1\}$
- if $\alpha \in \Sigma$, then $I(\neg\alpha) = \overline{I(\alpha)}$
- if $\alpha \in L(\Sigma)$ and $\beta \in L(\Sigma)$, then:

$$I((\alpha \wedge \beta)) = I(\alpha) \cap I(\beta)$$
$$I((\alpha \vee \beta)) = I(\alpha) \cup I(\beta)$$
$$I((\alpha \Rightarrow \beta)) = \overline{I(\alpha)} \cup I(\beta)$$

$$\top \xrightarrow{I} M(\Sigma)$$
$$\bot \xrightarrow{I} \{\}$$
$$\sigma \xrightarrow{I} \{m \mid m \in M(\Sigma), m(\sigma) = 1\}$$
$$\neg\alpha \xrightarrow{I} \overline{I(\alpha)}$$
$$(\alpha \wedge \beta) \xrightarrow{I} I(\alpha) \cap I(\beta)$$
$$(\alpha \vee \beta) \xrightarrow{I} I(\alpha) \cup I(\beta)$$
$$(\alpha \Rightarrow \beta) \xrightarrow{I} \overline{I(\alpha)} \cup I(\beta)$$

propositional language	semantic space
$L(\Sigma)$	$S(\Sigma)$

let signature $\Sigma = \{B, P\}$.

what is the interpretation of

$$(P \Rightarrow (\neg P \vee B))?$$

$$\top \xrightarrow{I} M(\Sigma)$$

$$\bot \xrightarrow{I} \{\}$$

$$\sigma \xrightarrow{I} \{m \mid m \in M(\Sigma), m(\sigma) = 1\}$$

$$\neg\alpha \xrightarrow{I} \overline{I(\alpha)}$$

$$(\alpha \wedge \beta) \xrightarrow{I} I(\alpha) \cap I(\beta)$$

$$(\alpha \vee \beta) \xrightarrow{I} I(\alpha) \cup I(\beta)$$

$$(\alpha \Rightarrow \beta) \xrightarrow{I} \overline{I(\alpha)} \cup I(\beta)$$

propositional language

$L(\Sigma)$

semantic space

$S(\Sigma)$

$I((p \Rightarrow (\neg p \vee B)))$

$$\top \xrightarrow{I} M(\Sigma)$$

$$\bot \xrightarrow{I} \{\}$$

$$\sigma \xrightarrow{I} \{m \mid m \in M(\Sigma),\ m(\sigma) = 1\}$$

$$\neg\alpha \xrightarrow{I} \overline{I(\alpha)}$$

$$(\alpha \wedge \beta) \xrightarrow{I} I(\alpha) \cap I(\beta)$$

$$(\alpha \vee \beta) \xrightarrow{I} I(\alpha) \cup I(\beta)$$

$$(\alpha \Rightarrow \beta) \xrightarrow{I} \overline{I(\alpha)} \cup I(\beta)$$

propositional language	semantic space
$L(\Sigma)$	$S(\Sigma)$

$$I\big((p \Rightarrow (\neg p \vee B))\big)$$

$$(\alpha \Rightarrow \quad \beta \quad)$$

$$= \overline{I(p)} \cup I\big((\neg p \vee B)\big)$$

$$\top \xrightarrow{I} M(\Sigma)$$

$$\bot \xrightarrow{I} \{\}$$

$$\sigma \xrightarrow{I} \{m \mid m \in M(\Sigma),\ m(\sigma) = 1\}$$

$$\neg\alpha \xrightarrow{I} \overline{I(\alpha)}$$

$$(\alpha \wedge \beta) \xrightarrow{I} I(\alpha) \cap I(\beta)$$

$$(\alpha \vee \beta) \xrightarrow{I} I(\alpha) \cup I(\beta)$$

$$(\alpha \Rightarrow \beta) \xrightarrow{I} \overline{I(\alpha)} \cup I(\beta)$$

propositional language

$L(\Sigma)$

semantic space

$S(\Sigma)$

$$I((p \Rightarrow (\neg p \vee B)))$$

$$(\alpha \Rightarrow \beta)$$

$$= \overline{I(p)} \cup I((\neg p \vee B))$$

$$(\alpha \vee \beta)$$

$$= \overline{I(p)} \cup I(\neg p) \cup I(B)$$

$$\top \xrightarrow{I} M(\Sigma)$$

$$\bot \xrightarrow{I} \{\}$$

$$\sigma \xrightarrow{I} \{m \mid m \in M(\Sigma),\ m(\sigma) = 1\}$$

$$\neg \alpha \xrightarrow{I} \overline{I(\alpha)}$$

$$(\alpha \wedge \beta) \xrightarrow{I} I(\alpha) \cap I(\beta)$$

$$(\alpha \vee \beta) \xrightarrow{I} I(\alpha) \cup I(\beta)$$

$$(\alpha \Rightarrow \beta) \xrightarrow{I} \overline{I(\alpha)} \cup I(\beta)$$

propositional language

$$L(\Sigma)$$

semantic space

$$S(\Sigma)$$

$$I\big(\underbrace{(p \Rightarrow (\neg p \vee B))}_{(\alpha \Rightarrow \beta)}\big)$$

$$= \overline{I(p)} \cup I\big(\underbrace{(\neg p \vee B)}_{(\alpha \vee \beta)}\big)$$

$$= \overline{I(p)} \cup I(\overset{\neg\alpha}{\neg p}) \cup I(B)$$

$$= \overline{I(p)} \cup \overline{I(p)} \cup I(B)$$

$$\top \xrightarrow{I} M(\Sigma)$$

$$\bot \xrightarrow{I} \{\}$$

$$\sigma \xrightarrow{I} \{m \mid m \in M(\Sigma),\ m(\sigma) = 1\}$$

$$\neg\alpha \xrightarrow{I} \overline{I(\alpha)}$$

$$(\alpha \wedge \beta) \xrightarrow{I} I(\alpha) \cap I(\beta)$$

$$(\alpha \vee \beta) \xrightarrow{I} I(\alpha) \cup I(\beta)$$

$$(\alpha \Rightarrow \beta) \xrightarrow{I} \overline{I(\alpha)} \cup I(\beta)$$

propositional language

$$L(\Sigma)$$

semantic space

$$S(\Sigma)$$

$$I\big((p \Rightarrow (\neg p \vee B))\big)$$

$$(\alpha \Rightarrow \beta)$$

$$= \overline{I(p)} \cup I\big((\neg p \vee B)\big)$$

$$(\alpha \vee \beta)$$

$$= \overline{I(p)} \cup I(\neg p) \cup I(B)$$

$$\neg\alpha$$

$$= \overline{I(p)} \cup \overline{I(p)} \cup I(B)$$

$$= \overline{I(p)} \cup I(B)$$

$$\top \xrightarrow{I} M(\Sigma)$$

$$\bot \xrightarrow{I} \{\}$$

$$\sigma \xrightarrow{I} \{m \mid m \in M(\Sigma),\ m(\sigma) = 1\}$$

$$\neg\alpha \xrightarrow{I} \overline{I(\alpha)}$$

$$(\alpha \wedge \beta) \xrightarrow{I} I(\alpha) \cap I(\beta)$$

$$(\alpha \vee \beta) \xrightarrow{I} I(\alpha) \cup I(\beta)$$

$$(\alpha \Rightarrow \beta) \xrightarrow{I} \overline{I(\alpha)} \cup I(\beta)$$

propositional language

$$L(\Sigma)$$

semantic space

$$S(\Sigma)$$

$$I((p \Rightarrow (\neg p \vee b)))$$

$$(\alpha \Rightarrow \beta)$$

$$= \overline{I(p)} \cup I((\neg p \vee b))$$

$$(\alpha \vee \beta)$$

$$= \overline{I(p)} \cup I(\neg p) \cup I(b)$$

$$\neg \alpha$$

$$= \overline{I(p)} \cup \overline{I(p)} \cup I(b)$$

$$= \overline{I(p)} \cup I(b)$$

$$= \overline{\{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}} \cup \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}$$

$$\top \xrightarrow{I} M(\Sigma)$$

$$\bot \xrightarrow{I} \{\}$$

$$\sigma \xrightarrow{I} \{m \mid m \in M(\Sigma), m(\sigma) = 1\}$$

$$\neg \alpha \xrightarrow{I} \overline{I(\alpha)}$$

$$(\alpha \wedge \beta) \xrightarrow{I} I(\alpha) \cap I(\beta)$$

$$(\alpha \vee \beta) \xrightarrow{I} I(\alpha) \cup I(\beta)$$

$$(\alpha \Rightarrow \beta) \xrightarrow{I} \overline{I(\alpha)} \cup I(\beta)$$

propositional language

$$L(\Sigma)$$

semantic space

$$S(\Sigma)$$

$$I((p \Rightarrow (\neg p \vee b)))$$
$$(\alpha \Rightarrow \beta)$$
$$= \overline{I(p)} \cup I((\neg p \vee b))$$
$$(\alpha \vee \beta)$$
$$= \overline{I(p)} \cup I(\neg p) \cup I(b)$$
$$\neg\alpha$$
$$= \overline{I(p)} \cup \overline{I(p)} \cup I(b)$$
$$= \overline{I(p)} \cup I(b)$$
$$= \overline{\{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}} \cup \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}$$
$$= \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\} \cup \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}$$

$$\top \xrightarrow{I} M(\Sigma)$$
$$\bot \xrightarrow{I} \{\}$$
$$\sigma \xrightarrow{I} \{m \mid m \in M(\Sigma), m(\sigma) = 1\}$$
$$\neg\alpha \xrightarrow{I} \overline{I(\alpha)}$$
$$(\alpha \wedge \beta) \xrightarrow{I} I(\alpha) \cap I(\beta)$$
$$(\alpha \vee \beta) \xrightarrow{I} I(\alpha) \cup I(\beta)$$
$$(\alpha \Rightarrow \beta) \xrightarrow{I} \overline{I(\alpha)} \cup I(\beta)$$

propositional language

$$L(\Sigma)$$

semantic space

$$S(\Sigma)$$

$$I((p \Rightarrow (\neg p \vee b)))$$

$$(\alpha \Rightarrow \beta)$$

$$= \overline{I(p)} \cup I((\neg p \vee b))$$

$$(\alpha \vee \beta)$$

$$= \overline{I(p)} \cup I(\neg p) \cup I(b)$$

$$\neg\alpha$$

$$= \overline{I(p)} \cup \overline{I(p)} \cup I(b)$$

$$= \overline{I(p)} \cup I(b)$$

$$= \overline{\{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}} \cup \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}$$

$$= \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\} \cup \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}$$

$$= \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}$$

$$\top \xrightarrow{I} M(\Sigma)$$

$$\bot \xrightarrow{I} \{\}$$

$$\sigma \xrightarrow{I} \{m \mid m \in M(\Sigma), m(\sigma) = 1\}$$

$$\neg\alpha \xrightarrow{I} \overline{I(\alpha)}$$

$$(\alpha \wedge \beta) \xrightarrow{I} I(\alpha) \cap I(\beta)$$

$$(\alpha \vee \beta) \xrightarrow{I} I(\alpha) \cup I(\beta)$$

$$(\alpha \Rightarrow \beta) \xrightarrow{I} \overline{I(\alpha)} \cup I(\beta)$$

propositional language

$$L(\Sigma)$$

semantic space

$$S(\Sigma)$$

let signature $\Sigma = \{B, P\}$.

what is the interpretation of

$$(B \wedge (B \Rightarrow P))?$$

$$\top \xrightarrow{I} M(\Sigma)$$

$$\bot \xrightarrow{I} \{\}$$

$$\sigma \xrightarrow{I} \{m \mid m \in M(\Sigma),\ m(\sigma) = 1\}$$

$$\neg\alpha \xrightarrow{I} \overline{I(\alpha)}$$

$$(\alpha \wedge \beta) \xrightarrow{I} I(\alpha) \cap I(\beta)$$

$$(\alpha \vee \beta) \xrightarrow{I} I(\alpha) \cup I(\beta)$$

$$(\alpha \Rightarrow \beta) \xrightarrow{I} \overline{I(\alpha)} \cup I(\beta)$$

propositional language

$L(\Sigma)$

semantic space

$S(\Sigma)$

$$I((B \land (B \Rightarrow P)))$$

$$\top \xrightarrow{I} M(\Sigma)$$

$$\bot \xrightarrow{I} \{\}$$

$$\sigma \xrightarrow{I} \{m \mid m \in M(\Sigma), m(\sigma) = 1\}$$

$$\neg\alpha \xrightarrow{I} \overline{I(\alpha)}$$

$$(\alpha \land \beta) \xrightarrow{I} I(\alpha) \cap I(\beta)$$

$$(\alpha \lor \beta) \xrightarrow{I} I(\alpha) \cup I(\beta)$$

$$(\alpha \Rightarrow \beta) \xrightarrow{I} \overline{I(\alpha)} \cup I(\beta)$$

propositional language

$$L(\Sigma)$$

semantic space

$$S(\Sigma)$$

$$I((B \land (B \Rightarrow P)))$$
$$= I(B) \cap I((B \Rightarrow P))$$

$$\top \xrightarrow{I} M(\Sigma)$$
$$\bot \xrightarrow{I} \{\}$$
$$\sigma \xrightarrow{I} \{m \mid m \in M(\Sigma),\ m(\sigma) = 1\}$$
$$\neg\alpha \xrightarrow{I} \overline{I(\alpha)}$$
$$(\alpha \land \beta) \xrightarrow{I} I(\alpha) \cap I(\beta)$$
$$(\alpha \lor \beta) \xrightarrow{I} I(\alpha) \cup I(\beta)$$
$$(\alpha \Rightarrow \beta) \xrightarrow{I} \overline{I(\alpha)} \cup I(\beta)$$

propositional language $L(\Sigma)$

semantic space $S(\Sigma)$

$$I((B \wedge (B \Rightarrow P)))$$
$$= I(B) \cap I((B \Rightarrow P))$$
$$= I(B) \cap \left(I(\neg B) \cup I(P)\right)$$

$$\top \xrightarrow{I} M(\Sigma)$$
$$\bot \xrightarrow{I} \{\}$$
$$\sigma \xrightarrow{I} \{m \mid m \in M(\Sigma), m(\sigma) = 1\}$$
$$\neg\alpha \xrightarrow{I} \overline{I(\alpha)}$$
$$(\alpha \wedge \beta) \xrightarrow{I} I(\alpha) \cap I(\beta)$$
$$(\alpha \vee \beta) \xrightarrow{I} I(\alpha) \cup I(\beta)$$
$$(\alpha \Rightarrow \beta) \xrightarrow{I} \overline{I(\alpha)} \cup I(\beta)$$

propositional language

$L(\Sigma)$

semantic space

$S(\Sigma)$

$$\mathrm{I}((B \land (B \Rightarrow P)))$$
$$= \mathrm{I}(B) \cap \mathrm{I}((B \Rightarrow P))$$
$$= \mathrm{I}(B) \cap (\mathrm{I}(\neg B) \cup \mathrm{I}(P))$$
$$= \mathrm{I}(B) \cap (\overline{\mathrm{I}(B)} \cup \mathrm{I}(P))$$

$$\top \xrightarrow{\mathrm{I}} M(\Sigma)$$
$$\bot \xrightarrow{\mathrm{I}} \{\}$$
$$\sigma \xrightarrow{\mathrm{I}} \{m \mid m \in M(\Sigma), m(\sigma) = 1\}$$
$$\neg\alpha \xrightarrow{\mathrm{I}} \overline{\mathrm{I}(\alpha)}$$
$$(\alpha \land \beta) \xrightarrow{\mathrm{I}} \mathrm{I}(\alpha) \cap \mathrm{I}(\beta)$$
$$(\alpha \lor \beta) \xrightarrow{\mathrm{I}} \mathrm{I}(\alpha) \cup \mathrm{I}(\beta)$$
$$(\alpha \Rightarrow \beta) \xrightarrow{\mathrm{I}} \overline{\mathrm{I}(\alpha)} \cup \mathrm{I}(\beta)$$

propositional language

$L(\Sigma)$

semantic space

$S(\Sigma)$

$$
\begin{aligned}
&I\big(B \land (B \Rightarrow p)\big) \\
&= I(B) \cap I\big(B \Rightarrow p\big) \\
&= I(B) \cap \big(I(\neg B) \cup I(p)\big) \\
&= I(B) \cap \big(\overline{I(B)} \cup I(p)\big) \\
&= \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\} \cap \big(\{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\} \cup \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}\big)
\end{aligned}
$$

$$\top \xrightarrow{I} M(\Sigma)$$

$$\bot \xrightarrow{I} \{\}$$

$$\sigma \xrightarrow{I} \{m \mid m \in M(\Sigma), m(\sigma) = 1\}$$

$$\neg\alpha \xrightarrow{I} \overline{I(\alpha)}$$

$$(\alpha \land \beta) \xrightarrow{I} I(\alpha) \cap I(\beta)$$

$$(\alpha \lor \beta) \xrightarrow{I} I(\alpha) \cup I(\beta)$$

$$(\alpha \Rightarrow \beta) \xrightarrow{I} \overline{I(\alpha)} \cup I(\beta)$$

propositional language

$$L(\Sigma)$$

semantic space

$$S(\Sigma)$$

$$
\begin{aligned}
& I\big((B \wedge (B \Rightarrow P))\big) \\
&= I(B) \cap I\big((B \Rightarrow P)\big) \\
&= I(B) \cap \big(I(\neg B) \cup I(P)\big) \\
&= I(B) \cap \big(\overline{I(B)} \cup I(P)\big) \\
&= \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\} \cap \big(\{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\} \cup \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}\big) \\
&= \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\} \cap \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}
\end{aligned}
$$

$$\top \xrightarrow{I} M(\Sigma)$$

$$\bot \xrightarrow{I} \{\}$$

$$\sigma \xrightarrow{I} \{m \mid m \in M(\Sigma), m(\sigma) = 1\}$$

$$\neg\alpha \xrightarrow{I} \overline{I(\alpha)}$$

$$(\alpha \wedge \beta) \xrightarrow{I} I(\alpha) \cap I(\beta)$$

$$(\alpha \vee \beta) \xrightarrow{I} I(\alpha) \cup I(\beta)$$

$$(\alpha \Rightarrow \beta) \xrightarrow{I} \overline{I(\alpha)} \cup I(\beta)$$

propositional language

$L(\Sigma)$

semantic space

$S(\Sigma)$

$$
\begin{aligned}
& I\big((B \land (B \Rightarrow P))\big) \\
&= I(B) \cap I\big((B \Rightarrow P)\big) \\
&= I(B) \cap \big(I(\neg B) \cup I(P)\big) \\
&= I(B) \cap \big(\overline{I(B)} \cup I(P)\big) \\
&= \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\} \cap \big(\{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\} \cup \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\}\big) \\
&= \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\} \cap \{\bar{b}\bar{p}, \bar{b}p, b\bar{p}, bp\} \\
&= \{bp\}
\end{aligned}
$$

$$\top \xrightarrow{I} M(\Sigma)$$

$$\bot \xrightarrow{I} \{\}$$

$$\sigma \xrightarrow{I} \{m \mid m \in M(\Sigma),\ m(\sigma) = 1\}$$

$$\neg\alpha \xrightarrow{I} \overline{I(\alpha)}$$

$$(\alpha \land \beta) \xrightarrow{I} I(\alpha) \cap I(\beta)$$

$$(\alpha \lor \beta) \xrightarrow{I} I(\alpha) \cup I(\beta)$$

$$(\alpha \Rightarrow \beta) \xrightarrow{I} \overline{I(\alpha)} \cup I(\beta)$$

propositional language

$$L(\Sigma)$$

semantic space

$$S(\Sigma)$$

we will refer to a string of propositional language $L(\Sigma)$

as a **sentence**.

there are a few helpful shorthands:

	example sentence	example shorthand
drop exterior parens	$(A \wedge (B \Rightarrow C))$	$A \wedge (B \Rightarrow C)$
drop parens around adjacent ands	$(A \wedge (\neg B \wedge C))$	$A \wedge \neg B \wedge C$
drop parens around adjacent ors	$(A \vee (\neg B \vee C))$	$A \vee \neg B \vee C$

how are these justified?

	example sentence	example shorthand
drop parens around adjacent ands	$(A \land (\neg B \land C))$	$A \land \neg B \land C$
drop parens around adjacent ors	$(A \lor (\neg B \lor C))$	$A \lor \neg B \lor C$

$$\top \xrightarrow{I} M(\Sigma)$$
$$\bot \xrightarrow{I} \{\}$$
$$\sigma \xrightarrow{I} \{m \mid m \in M(\Sigma), m(\sigma) = 1\}$$
$$\neg\alpha \xrightarrow{I} \overline{I(\alpha)}$$
$$(\alpha \wedge \beta) \xrightarrow{I} I(\alpha) \cap I(\beta)$$
$$(\alpha \vee \beta) \xrightarrow{I} I(\alpha) \cup I(\beta)$$
$$(\alpha \Rightarrow \beta) \xrightarrow{I} \overline{I(\alpha)} \cup I(\beta)$$

show associativity:

$$I((A \wedge (B \wedge C))) = I(((A \wedge B) \wedge C))$$

	example sentence	example shorthand
drop parens around adjacent ands	$(A \wedge (B \wedge C))$	$A \wedge B \wedge C$
drop parens around adjacent ors	$(A \vee (\neg B \vee C))$	$A \vee \neg B \vee C$

$$\top \xrightarrow{I} M(\Sigma)$$
$$\bot \xrightarrow{I} \{\}$$
$$\sigma \xrightarrow{I} \{m \mid m \in M(\Sigma),\ m(\sigma)=1\}$$
$$\neg\alpha \xrightarrow{I} \overline{I(\alpha)}$$
$$(\alpha \wedge \beta) \xrightarrow{I} I(\alpha) \cap I(\beta)$$
$$(\alpha \vee \beta) \xrightarrow{I} I(\alpha) \cup I(\beta)$$
$$(\alpha \Rightarrow \beta) \xrightarrow{I} \overline{I(\alpha)} \cup I(\beta)$$

$$\begin{aligned} I((A \wedge (B \wedge C))) &= I(A) \cap I((B \wedge C)) \\ &= I(A) \cap (I(B) \cap I(C)) \\ &= (I(A) \cap I(B)) \cap I(C) \\ &= I((A \wedge B)) \cap I(C) \\ &= I(((A \wedge B) \wedge C)) \end{aligned}$$

	example sentence	example shorthand
drop parens around adjacent ands	$(A \wedge (B \wedge C))$	$A \wedge B \wedge C$
drop parens around adjacent ors	$(A \vee (\neg B \vee C))$	$A \vee \neg B \vee C$

$$\top \xrightarrow{I} M(\Sigma)$$
$$\bot \xrightarrow{I} \{\}$$
$$\sigma \xrightarrow{I} \{m \mid m \in M(\Sigma), m(\sigma) = 1\}$$
$$\neg\alpha \xrightarrow{I} \overline{I(\alpha)}$$
$$(\alpha \wedge \beta) \xrightarrow{I} I(\alpha) \cap I(\beta)$$
$$(\alpha \vee \beta) \xrightarrow{I} I(\alpha) \cup I(\beta)$$
$$(\alpha \Rightarrow \beta) \xrightarrow{I} \overline{I(\alpha)} \cup I(\beta)$$

$$\begin{aligned} I((A \wedge (B \wedge C))) &= I(A) \cap I((B \wedge C)) \\ &= I(A) \cap (I(B) \cap I(C)) \\ &= (I(A) \cap I(B)) \cap I(C) \\ &= I((A \wedge B)) \cap I(C) \\ &= I(((A \wedge B) \wedge C)) \end{aligned}$$

drop parens around adjacent ands

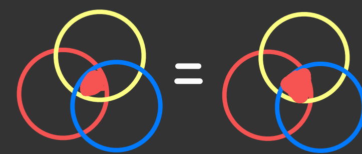
drop parens around adjacent ors

example sentence	example shorthand
$(A \wedge (B \wedge C))$	$A \wedge B \wedge C$
$(A \vee (\neg B \vee C))$	$A \vee \neg B \vee C$

$$\top \xrightarrow{I} M(\Sigma)$$
$$\bot \xrightarrow{I} \{\}$$
$$\sigma \xrightarrow{I} \{m \mid m \in M(\Sigma),\ m(\sigma) = 1\}$$
$$\neg\alpha \xrightarrow{I} \overline{I(\alpha)}$$
$$(\alpha \land \beta) \xrightarrow{I} I(\alpha) \cap I(\beta)$$
$$(\alpha \lor \beta) \xrightarrow{I} I(\alpha) \cup I(\beta)$$
$$(\alpha \Rightarrow \beta) \xrightarrow{I} \overline{I(\alpha)} \cup I(\beta)$$

$$\begin{aligned} I((A \land (B \land C))) &= I(A) \cap I((B \land C)) \\ &= I(A) \cap (I(B) \cap I(C)) \\ &= (I(A) \cap I(B)) \cap I(C) \\ &= I((A \land B)) \cap I(C) \\ &= I(((A \land B) \land C)) \end{aligned}$$

intersection is associative



drop parens around adjacent ands

drop parens around adjacent ors

example sentence	example shorthand
$(A \land (B \land C))$	$A \land B \land C$
$(A \lor (\neg B \lor C))$	$A \lor \neg B \lor C$

$\top$	$\xrightarrow{I}$	$M(\Sigma)$
$\bot$	$\xrightarrow{I}$	$\{\}$
σ	$\xrightarrow{I}$	$\{m \mid m \in M(\Sigma), m(\sigma) = 1\}$
$\neg\alpha$	$\xrightarrow{I}$	$\overline{I(\alpha)}$
$(\alpha \land \beta)$	$\xrightarrow{I}$	$I(\alpha) \cap I(\beta)$
$(\alpha \lor \beta)$	$\xrightarrow{I}$	$I(\alpha) \cup I(\beta)$
$(\alpha \Rightarrow \beta)$	$\xrightarrow{I}$	$\overline{I(\alpha)} \cup I(\beta)$

$$\begin{aligned} I((A \land (B \land C))) &= I(A) \cap I((B \land C)) \\ &= I(A) \cap (I(B) \cap I(C)) \\ &= (I(A) \cap I(B)) \cap I(C) \\ &= I((A \land B)) \cap I(C) \\ &= I(((A \land B) \land C)) \end{aligned}$$

drop parens around **adjacent ands**

example sentence	example shorthand
$(A \land (B \land C))$	$A \land B \land C$
$(A \lor (\neg B \lor C))$	$A \lor \neg B \lor C$

drop parens around **adjacent ors**

$$\top \xrightarrow{I} M(\Sigma)$$
$$\bot \xrightarrow{I} \{\}$$
$$\sigma \xrightarrow{I} \{m \mid m \in M(\Sigma),\ m(\sigma)=1\}$$
$$\neg\alpha \xrightarrow{I} \overline{I(\alpha)}$$
$$(\alpha \wedge \beta) \xrightarrow{I} I(\alpha) \cap I(\beta)$$
$$(\alpha \vee \beta) \xrightarrow{I} I(\alpha) \cup I(\beta)$$
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$$\begin{aligned} I((A \wedge (B \wedge C))) &= I(A) \cap I((B \wedge C)) \\ &= I(A) \cap (I(B) \cap I(C)) \\ &= (I(A) \cap I(B)) \cap I(C) \\ &= I((A \wedge B)) \cap I(C) \\ &= I(((A \wedge B) \wedge C)) \end{aligned}$$

	example sentence	example shorthand
drop parens around adjacent ands	$(A \wedge (B \wedge C))$	$A \wedge B \wedge C$
drop parens around adjacent ors	$(A \vee (\neg B \vee C))$	$A \vee \neg B \vee C$

$$
\begin{aligned}
\top &\xrightarrow{I} M(\Sigma)\\
\bot &\xrightarrow{I} \{\}\\
\sigma &\xrightarrow{I} \{m \mid m \in M(\Sigma),\ m(\sigma)=1\}\\
\neg\alpha &\xrightarrow{I} \overline{I(\alpha)}\\
(\alpha \wedge \beta) &\xrightarrow{I} I(\alpha) \cap I(\beta)\\
(\alpha \vee \beta) &\xrightarrow{I} I(\alpha) \cup I(\beta)\\
(\alpha \Rightarrow \beta) &\xrightarrow{I} \overline{I(\alpha)} \cup I(\beta)
\end{aligned}
$$

$$
\begin{aligned}
I((A \vee (B \vee C))) &= I(A) \cup I((B \vee C))\\
&= I(A) \cup (I(B) \cup I(C))\\
&= (I(A) \cup I(B)) \cup I(C)\\
&= I((A \vee B)) \cup I(C)\\
&= I(((A \vee B) \vee C))
\end{aligned}
$$

	example sentence	example shorthand
drop parens around adjacent ands	$(A \wedge (B \wedge C))$	$A \wedge B \wedge C$
drop parens around adjacent ors	$(A \vee (\neg B \vee C))$	$A \vee \neg B \vee C$